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Lifting Problem of η and Mahowald's Element η_j

Dedicated to Professor Shôrô Araki on his 60-th birthday

By

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§1. Introduction and Statements of Results

In this paper we consider the following lifting problem: for which n and k is there a lift $\tilde{\eta}$ making the following diagram (1) or (2) commute up to stable homotopy?

Diagram (1)

$$\begin{array}{ccc} & & Q^n_{n-k+1} \\ & \overset{\tilde{\eta}}{\nearrow} & \downarrow p \\ S^{4n} & \xrightarrow{\ \eta\ } & S^{4n-1} \end{array}$$

Diagram (2)

$$\begin{array}{ccc} & & HP^n_{n-k+1} \\ & \overset{\tilde{\eta}}{\nearrow} & \downarrow p \\ S^{4n+1} & \xrightarrow{\ \eta\ } & S^{4n} \end{array}$$

Here and throughout this paper we use the following notations;

Notation

HP^n: the quaternionic n-dimensional projective space.
Q^n: the quaternionic quasi-projective space of dimension $4n$-1.
$HP^n_{n-k+1} = HP^n/HP^{n-k}$.

$Q^n_{n-k+1} = Q^n/Q^{n-k}$.

p is the canonical collapsing map.
η is the non-trivial element of $\pi_1^s(S^0)$.

These problems are natural 'next' questions after the stable James number problem (For example, see [3]). Since Q^n is a stable retract of $Sp(n)$ [5] and since HP^n is a stable retract of $\Omega(U(2n+2)/Sp(n+1))$ [2], these problems are closely related to the unstable lifting problem of η in the canonical Stiefel

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bundles: the problem with respect to Diagram (1) is related to the lifting problem of η to the quaternionic Stiefel bundle $X_{n,k} \to S^{4n-1}$, and the problem with respect to Diagram (2) is related to the relative complex-quaternionic Stiefel fibration [5] $\Gamma_{n+1,k} \to \Omega S^{4n+1}$, where $\Gamma_{n,k}$ is the homotopy fiber of the inclusion $X_{n,k} \to W_{2n,2k}$. For the lifting problem of η to the real Stiefel bundle $V_{n,k} \to S^{n-1}$, the complete answer is known by M.C.Crabb and W.A.Sutherland[4] and the complex case is easy.

Theorem A. *If $n = 2^i$, then for any $k \leqq n$ there exists a lift $\tilde{\eta}$ so that the diagram* (1) *commutes up to stable homotopy.*

Theorem B. *There exists a lift $\tilde{\eta}$ in diagram* (2) *if and only if one of the following conditions is satisfied;*

(1) $k = 1$ *or* 2,

(2) $k = 3$ *or* 4, *and* $n \equiv 2 \bmod 4$.

Theorem C. *Let $i \geqq 1$. Let $n = 2^i a$ for some odd integer $a > 1$. If there is a lift $\tilde{\eta}$: $S^{4n} \to Q^n_{n-2^i+1}$, then the composite*

$$S^{4n} \xrightarrow{\tilde{\eta}} Q^n_{n-2^i+1} \xrightarrow{\partial} S^{4(n-2^i)}$$

is non-trivial, where the map ∂ is the usual one in the following usual cofiber sequence;

$$S^{4(n-2^i)-1} \xrightarrow{i} Q^n_{n-2^i} \xrightarrow{p} Q^n_{n-2^i+1} \xrightarrow{\partial} S^{4(n-2^i)}.$$

Therefore there is no lift for $k = 2^i + 1$ when $n = 2^i a$ (a is odd).

In fact the above composite is detected by the secondary operation associated to the following relation;

$$Sq^{2^{i+2}+1}Sq^1 + Sq^2Sq^{2^{i+2}} + Sq^4Sq^{2^{i+2}-2} + Sq^{2^{i+2}}Sq^2 = 0.$$

Therefore we have a family of the stable homotopy groups closely related to what Mahowald constructed in [8]. If we choose a specific lift, we get precisely Mahowald's element $\eta_{5,i+2}$ constructed in [9]. This fact follows from the construction and the result due to Mann-Miller[10] or Mann-Miller-Miller[11]. From Theorems A and C we get the following corollaries.

Corollary D. *There exists a stable lift $\tilde{\eta}$: $S^{4n} \to Q^n$ if and only if $n = 2^t$ for some t.*

Remark. There is no unstable lift of η to the usual bundle projection $Sp(n) \to S^{4n-1}$, because $\pi_{4n}(Sp(n)) \cong \pi_{4n}(Sp)$ is $Z/2$ or 0 according as n is odd or even and because the generator of $\pi_{8m+4}(Sp)$ comes from $Sp(1)$.

Corollary E. *Let $i \geqq 1$. The Mahowald's elements $\eta_{5,i+2}$ as above referred are in the image of the S^3-transfer homomorphism t: $\pi_*^s(Q^\infty) \to \pi_*^s(S^0)$.*

The following theorem is a partial result about the lifting problem in Diagram (1) in case that k is small.

Theorem F. *Let $k \leqq 6$. Then in Diagram* (1) *there exists a stable lift $\tilde{\eta}$ for k, if and only if one of the following conditions is satisfied.*

(1) $k = 1$ *or* 2,
(2) $k = 3$ *or* 4 *and* $n \equiv 0 \bmod 4$,
(3) $k = 5$ *or* 6 *and* $n \equiv 0 \bmod 8$.

§2. Proof of Theorem A

Throughout this paper, homology and cohomology are assumed to be with $Z/2$-coefficients.

For the proof of Theorem A we need the following lemmas;

Lemma 2.1.

(i) $H_*(\Omega^2 S^5) = Z/2[x_1, x_2, x_3, \dots]$,

$x_i = Q_1 Q_1 Q_1 \cdots Q_1(x_1)$ *and the dimension of* $x_i = 2^{i+1} - 1$.

(ii) (S. Kochman[7]) *In* $H_*(Sp) = \Lambda_{Z/2}(\gamma_1, \gamma_2, \gamma_3, \dots)$, $Q_1(\gamma_n) = \gamma_{2n}$,

where Q_1 is the Dyer-Lashof (*subscripted*) *homology operation.*

Let α: $S^3 \to Sp$ be the representative of a generator of $\pi_3(Sp) \cong Z$. Since Sp is an infinite loop space, we have a canonical extension $\bar{\alpha}$: $\Omega^2 S^5 \to Sp$ of the map α. Let θ: $Sp \to \Omega^\infty \Sigma^\infty Q^\infty$ be the *James splitting*[5]. Taking the adjoint of the composite $\theta \circ \bar{\alpha}$, we have a stable map, say, g: $\Omega^2 S^5 \to Q^\infty$.

Lemma 2.2. *Let g_*: $H_*(\Omega^2 S^5) \to H_*(Q^\infty)$ be the homology induced homomorphism of g. Then,*

$$g_*(x_i) = \gamma_{2^{i-1}},$$

where $\gamma_i \in H_{4i-1}(Q^\infty)$ is the standard generator.

Proof. Let σ: $H_*(\Omega^\infty \Sigma^\infty Q^\infty) \to H_*(Q^\infty)$ be the homology suspension. Then $\sigma\theta_*(\gamma_i) = \gamma_i$ and $\sigma\theta_*(\text{decomposables}) = 0$[5]. Now consider the following commutative diagram;

$$\begin{array}{ccccc} H_*(\Omega^2 S^5) & \xrightarrow{\bar{\alpha}_*} & H_*(Sp) & \xrightarrow{\theta_*} & H_*(\Omega^\infty \Sigma^\infty Q^\infty) \\ & \searrow^{g_*} & & & \downarrow \sigma \\ & & & & H_*(Q^\infty). \end{array}$$

So it is enough to show that $\bar{\alpha}_*(x_i) = \gamma_{2^{i-1}}$. When $i = 1$ it is obviously true. Since $\bar{\alpha}$ is a double loop map, $\bar{\alpha}_*$ commutes with Q_1- operations. Therefore the cases $i \geqq 2$ follow from Lemma 2.1.

Recall, by Snaith decomposition [16], that the suspension spectrum of $\Omega^2 S^5$ is a wedge of spectra, say, D_k for $k \geqq 1$. Homologically, $H_*(D_k)$ corresponds to the submodule of height k in $H_*(\Omega^2 S^5)$. Here the height h is defined as $h(x_i) = 2^{i-1}$. Thus D_{2^i} is stably $3 \cdot 2^i - 1$ connected and of dimension $2^{i+2} - 1$ complex: the bottom cell corresponds to $x_1^{2^i} \in H_{3 \cdot 2^i}(\Omega^2 S^5) \cong Z/2$ and the top to $x_{i+1} \in H_{2^{i+2}-1}(\Omega^2 S^5) \cong Z/2$. According to Mahowald [8], Brown and Peterson [1], D_k is homotopy equivalent to the Brown-Gitler spectrum $\Sigma^{3k} B\left(\left[\frac{k}{2}\right]\right)$. Mahowald [8] proved that there is a stable map g_i: $S^{2^{i+2}} \to D_{2^i}$ such that the composite;

$$S^{2^{i+2}} \longrightarrow D_{2^i} \longrightarrow D_{2^i}/D_{2^i}^{(2^{i+2}-2)} = S^{2^{i+2}-1}$$

is η. Thus by Lemma 2.2 the stable map $g \circ g_i$ gives the desired lift of η. This completes the proof of Theorem A.

§3. Proof of Theorem C

Let $y_i \in H^{4i-1}(Q^\infty)$ be the dual basis of $\gamma_i \in H_{4i-1}(Q^\infty)$. The following lemma easily follows by using the cofiber sequence;

$$CP^\infty \longrightarrow HP^\infty \longrightarrow Q^\infty \longrightarrow \Sigma CP^\infty.$$

Lemma 3.1. $Sq^{4j}(y_i) = \binom{2i-1}{2j} y_{i+j}$, *where* Sq^k *is the Steenrod operation.*

Now the proof of Theorem C follows by standard arguments. However, for my own safety I give the details. Let $n = 2^i a$ for some odd integer $a > 1$. If there is a lift $\tilde{\eta}$: $S^{4n} \to Q^n_{n-2^i+1}$, then we denote the composite

$$S^{4n} \xrightarrow{\tilde{\eta}} Q^n_{n-2^i+1} \xrightarrow{\partial} S^{4(n-2^i)}$$

by $h_i \in \pi^s_{2^{i+2}}(S^0)$. For convenience we denote the normalized spectrum of the mapping cone of h_i, say c_{h_i}, by $X_i \cong S^0 \cup_{h_i} e^{4 \cdot 2^i + 1}$. Let $u \in H^0(X_i)$ be the bottom generator. All we have to do is to calculate the secondary composition associated to the following sequence;

$$X_i \xrightarrow{u} K(0) \xrightarrow{f} K(1) \times K(2^{i+2}) \times K(2^{i+2} - 2) \times K(2) \xrightarrow{g} K(2^{i+2} + 2),$$

where $f = Sq^1 \times Sq^{2^{i+2}} \times Sq^{2^{i+2}-2} \times Sq^2$, $g = Sq^{2^{i+2}+1} + Sq^2 + Sq^4 + Sq^{2^{i+2}}$ and $K(m)$ is the m-fold suspension of the Eilenberg-MacLane spectrum

$HZ/2$. By the definition there is a cofibration;

$$C_{\tilde{\eta}} \longrightarrow C_{h_i} \xrightarrow{w} \Sigma Q^n_{n-2^i}.$$

Let $v \in H^0(\Sigma^{4(2^i-n)+1} Q^n_{n-2^i})$ be the bottom generator. Then there is a commutative (up to stable homotopy) diagram;

$$\begin{array}{ccccccc}
X_i & \xrightarrow{u} & K(0) & \xrightarrow{f} & K(1) \times K(2^{i+2}) \times K(2^{i+2}-2) \times K(2) & \xrightarrow{g} & K(2^{i+2}+2) \\
\| & & \uparrow v & & \| & & \| \\
X_i & \xrightarrow{w} & \Sigma^{4(2^i-n)+1} Q^n_{n-2^i} & \xrightarrow{fv} & K(1) \times K(2^{i+2}) \times K(2^{i+2}-2) \times K(2) & \xrightarrow{g} & K(2^{i+2}+2).
\end{array}$$

So it is enough to compute the bracket $\langle g, fv, w \rangle$. From Lemma 3.1 it is easy to see that $\langle g, f, u \rangle = \langle g, fv, w \rangle \neq 0$ without indeterminacy. This completes the proof of Theorem C.

Now we shall prove Corollaries. First, let M_k = the order of $J(\xi_k)$, where ξ_k is the canonical symplectic line bundle over HP^{k-1} and J is the classical J-homomorphism. Then by James periodicity and by Theorem A, we see that there is a lift $\tilde{\eta}$ in diagram (1) for $n = 2^i + M_{2^i}$ and $k = 2^i$. In this case, since $n = 2^i a$ for some odd integer a (see Sigrist and Suter [15]), by Theorem C we get a non-trivial family $h_i \in \pi^s_{2^i+2}(S^0)$. Now according to B. M. Mann and E. Y. Miller [10] or B. M. Mann, E. Y. Miller and H. Miller [11], there is a commutative diagram up to homotopy;

$$\begin{array}{ccc}
Sp & \xrightarrow{\theta} & \Omega^\infty \Sigma^\infty Q^\infty \\
\downarrow & & \downarrow t \\
SO & \xrightarrow{J} & \Omega^\infty S^\infty.
\end{array}$$

Here t is the representative as an infinite loop map of the S^3-transfer homomorphism. Note [6][14] that the S^3-transfer homomorphism $t: \pi^s_k(Q^\infty) \to \pi^s_k(S^0)$ for $k \leqq 4l+1$ is induced by the map ∂: $Q^{Ml+l}_{Ml+1} \to S^{4Ml}$ using James periodicity [5]. Thus if we take the lift as in the proof of Theorem A, then from the constructions of Mahowald's element $\eta_{5,i+2}$ [8] [9] and our element h_i, we see that our h_i coincides to the Mahowald element $\eta_{5,i+2}$. This proves Corollary E.

§4. Proof of Theorem B and F

First we prove Theorem B. For $k = 1$ or 2, it is trivial. So there is a lift $\tilde{\eta}$: $S^{4n+1} \to HP^n_{n-1}$. Consider the cofibration;

$$S^{4(n-k)} \xrightarrow{i_k} HP^n_{n-k} \xrightarrow{p_k} HP^n_{n-k+1} \xrightarrow{\partial_k} S^{4(n-k)+1},$$

where i_k is the bottom inclusion and p_k is the collapsing map. Let $k=2$. Consider the composite $\partial_2 \circ \tilde{\eta}$. Then we have

Lemma 4.1.

$$\partial_2 \circ \tilde{\eta} = \begin{cases} \bar{\nu} & \text{if } n \equiv 0 \bmod 4 \\ \eta\sigma & \text{if } n \equiv 1 \bmod 4 \\ 0 & \text{if } n \equiv 2 \bmod 4 \\ \varepsilon & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

The above lemma has been known [12][13], but here we give a very simple (at least, theoretically) proof. Recall $\pi_8^s(S^0) \cong Z/2 \oplus Z/2$ generated by $\bar{\nu}$ and ε. Put

$$\partial_2 \circ \tilde{\eta} = a\bar{\nu} + b\varepsilon.$$

Note that the integer a and b are independent of choice of $\tilde{\eta}$. By using e-invariant methods or the Hurewicz homomorphism of Im J theory, occasionally denoted by h^A, we see that $b=0$ if and only if n is even. Here the symbol A means A-theory, which is defined as the fiber spectrum of $\Phi^3 - 1$: $ko \to kspin$, where ko (resp. $kspin$) is the connective (resp. 2-connected) cover of (2)-localized KO-theory. On the other hand, by using the well-known structure of $H^*(HP^n_{n-k})$ as a module over the Steenrod algebra, we see that $\partial_2 \circ \tilde{\eta}$ is detected by the secondary operation cited ($i=1$) in §2 if and only if $n \equiv 0$ or 1 mod 4. This implies that $a \neq 0$ if and only if $n \equiv 0$ or 1 mod 4. This proves Lemma 4.1.

Thus from the above lemma we see that for $k=3$ there is a lift of η if and only if $n \equiv 2$ mod 4. Since $\pi_{12}^s(S^0) = 0$, we see that for $k=4$ there is a lift of η if and only if $n \equiv 2$ mod 4. Now we shall prove that there is no lift of η for $k \geqq 5$. For this purpose we use KO theory and Adams operation. Assume that there exists a map f: $S^{4n+1} \to HP^n_{n-k+1}$ such that the following diagram commutes;

$$\begin{array}{ccc} KO^*(S^{4n+1}) & \xleftarrow{\eta^*} & KO^*(S^{4n}) \\ & \nwarrow^{f^*} & \downarrow p^* \\ & & KO^*(HP^n_{n-k+1}) \end{array}$$

Recall that $KO^*(HP^n_{n-k+1}) \cong KO^*(S^0)\{x^s \mid n-k+1 \leqq s \leqq n\}$, where $x^s \in KO^{4s}(HP^n_{n-k+1})$. Let $\alpha_k \in KO^{-4k-1}(S^0)$ be the element such that

$$f^*(x^s) = \alpha_{n-s} \cdot \iota_{4n+1},$$

where $\iota_m \in KO^m(S^m)$ is the standard generator. Note that $p^*(x^n) = \iota_{4n}$ and that $\alpha_0 \neq 0$. Let Φ^3 be the stable Adams operation in KO-theory. It is not difficult to show that

$$\Phi^3(x^s) \equiv \sum_{i=0}^{[\frac{n-s}{2}]} \binom{s}{i} y^i x^{2i+s} \mod 2,$$

in $KO^*(HP^n_{n-k+1})$, where $y \in KO^{-8}(S^0)$ is the standard generator. From the commutativity between Adams operation and an induced homomorphism, we see that, for any s such that $n-k+1 \leqq s \leqq n$, the following relations hold

$$\sum_{i=1}^{[\frac{n-s}{2}]} \binom{s}{i} y^i \alpha_{n-2i-s} = 0.$$

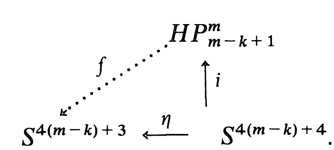
Also note that $\alpha_{odd} = 0$. Let $k = 5$. Then, applying the above equation, we have

$$(n-4)y\alpha_2 + \binom{n-4}{2} y^2 \alpha_0 = 0.$$

Since n must be even if $k \geqq 3$, we get that $\binom{n-4}{2} \equiv 0 \mod 2$. Thus we see that $n \equiv 0 \mod 4$. But this contradicts the condition that $n \equiv 2 \mod 4$ for $k = 4$. Therefore there is no lift of η for $k = 5$. This completes the proof of Theorem B.

Now we shall study necessary conditions for the existence of a lift of η with respect to Diagram (1). For convenience, we take the S-dual of Diagram (1). Then we get the following diagram for some integer m;

Diagram (3)



Recall that A-theory is defined as the fiber spectrum of $\Phi^3 - 1$: $ko \to kspin$, where ko (resp. $kspin$) is the connective (resp. 2-connected) cover of KO. Then by similar consideration, using A-theory, as in the proof of Theorem B, we get the following necessary condition;

$$\left[\frac{k-1}{2}\right] < 2^{\nu_2(m)},$$

where $\nu_2(m)$ is the exponent of 2 in the prime decomposition of m. Thus taking S-dual again, we see that the following condition is necessary for the existence of a lift in Diagram (1);

$$(*) \qquad \left[\frac{k-1}{2}\right] < 2^{\nu_2(n)}.$$

Remark that the condition obtained from Theorem C is more restrictive than this condition. This implies that the essential obstruction of co-extending η is not in the image of the classical J-homomorphism. So the problem does not seem to be solved by e-invariant methods. However, for the case that k is small, by using both e-invariant and secondary operation in §2, we can solve the problem. Thus we obtain Theorem F. Details are omitted.

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