Publ. RIMS. Kyoto Univ.
25 (1989), 503–520

A Poincaré-Birkhoff-Witt Theorem for Quantized Universal Enveloping Algebras of Type A_N

By

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Introduction

In this paper, we construct an explicit basis of the quantized universal enveloping algebra $U_q(sl_{N+1}(\mathbf{C}))$.

Let $A=(a_{ij})_{1\le i,j\le N}$ be a symmetrizable generalized Cartan matrix, and $\mathscr{G}(A)$ the Kac-Moody Lie algebra of A. Motivated by studies of quantum Yang-Baxter equations, Jimbo [6] and Drinfeld [2, 3] introduced a Hopf algebra $U_q(\mathscr{G}(A))$ with a nonzero complex parameter q. This Hopf algebra, which is also called [3] a "quantum group", can be considered as a natural q-analogue of the universal enveloping algebra $U(\mathscr{G}(A))$ of $\mathscr{G}(A)$. For example, it is known that the representation theory of $U_q(\mathscr{G}(A))$ is quite analogous to that of $U(\mathscr{G}(A))$. See Lusztig [9] and Rosso [11]. The purpose of this paper is to show that, if $\mathscr{G}(A)$ is of type A_N, and $q^8\ne 1$, then $U_q(\mathscr{G}(A))$ has a Poincaré-Birkhoff-Witt type basis.

Let R be a commutative ring with 1. Denote by $sl_{N+1}(R)$, the Lie algebra of $(N+1)\times(N+1)$ matrices over R of trace 0. It has the standard R-basis consisting of the elements

$$e_{i,j}=E_{i,j},\ f_{i,j}=E_{j,i}\ (1\le i<j\le N+1),\ h_i=E_{i,i}-E_{i+1,i+1}\ (1\le i\le N)$$

($E_{i,j}$ is the matrix having 1 in (i,j) position and 0 elsewhere). By the Poincaré-Birkhoff-Witt theorem [1], the elements

$$f_{m_1,n_1}\cdots f_{m_s,n_s}h_1^{r_1}\cdots h_N^{r_N}e_{i_1,j_1}\cdots e_{i_t,j_t} \tag{$*$}$$

$(r_1,\dots,r_N\ge 0;\ (m_1,n_1)\le\cdots\le(m_s,n_s)$ and $(i_1,j_1)\le\cdots\le(i_s,j_s)$ with respect to the lexicographic order $\le$) form an R-basis of $U(sl_{N+1}(R))$. Let $U_q(sl_{N+1}(R))$ be the quantum group over R associated with the Cartan matrix of type A. (See the beginning of Section 6 for the definition of $U_q(sl_{N+1}(R))$.) Let $R^\times$ be the unit group of R. In this paper, for $q\in R^\times$ such that $q^8-1\in R^\times$, we construct an R-basis of $U_q(sl_{N+1}(R))$ which can be considered as a natural q-analogue of $(*)$(Theorem 1.1 and 6.1). Here the condition $q^8-1\in R^\times$ is essential. In fact,

Communicated by T. Kawai, April 17, 1989.
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(i) When R is a field and $q^8 = 1$, $U_q(sl_{N+1}(R))$ seems to have no basis q-analogous to $(*)$. (Even in this case, we can give an explicit basis. See Proposition 6.2.)

(ii) If q is an indeterminate and $R = \mathbb{C}[q^{\mp 1}, (q^4 - 1)^{-1}]$, then the R-module $U_q(sl_{N+1}(R))$ is not free. (See Proposition 6.3.)

To remedy this unpleasant situation, we are naturally led to introduce a new quantum group $\bar{U}_q(sl_{N+1}(R))$(Section 6), which seems to be more natural than $U_q(sl_{N+1}(R))$ in the following sense (See Theorem 6.1):

(o) $\bar{U}_q(sl_{N+1}(R)) = U_q(sl_{N+1}(R))$ if $q^8 - 1 \in R^\times$,

(i) $\bar{U}_q(sl_{N+1}(R))$ has an R-basis q-analogous to $(*)$.

We can define a filtration in $\bar{U}_q(sl_{N+1}(R))$ such that the associated graded algebra is a non-commutative analogue of a polynomial ring (Section 5 (and 6)). As a corollary of this fact, we show that if R is a Noetherian ring, then $\bar{U}_q(sl_{N+1}(R))$ is a left (right) Noetherian ring, and that, if R has no zero divisors $\neq 0$, then $\bar{U}_q(sl_{N+1}(R))$ has no zero divisors $\neq 0$ (Theorem 1.2 and 6.1).

An important step in proving our main results is to show that a quantum group $U_q(\mathscr{G}(A))$ has a "triangular decomposition"; this is done in Section 2 for a general A. We also need "(q-)commutator relations" in $U_q(sl_{N+1}(\mathbb{C}))$(Section 3), which have been communicated to the author by Professor M. Jimbo. The author is very grateful to him.

In the last section, we also give an explicit Poincaré-Birkhoff-Witt basis of $U_q(so_5(\mathbb{C}))$.

§1. Statement of the Main Results

Let $A = (a_{ij})_{1 \le i,j \le N}$ be a symmetrizable generalized Cartan matrix (see [8]); there exists a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_N)$ such that $d_i \in \mathbb{Z} \setminus \{0\}$ and $DA = {}^t(DA)$. Let F be a field, and let $q \in F^\times$ be such that $q^{4d_i} \neq 0$ $(1 \le i \le N)$. Let $U_q(\mathscr{G}(A)) = U_q(\mathscr{G}(A), D)$ be the associative F-algebra with 1 with generators $e_i, f_i, k_i^{\pm 1}$ $(1 \le i \le N)$, and relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1,\ k_i k_j = k_j k_i \tag{1.1}$$

$$k_i e_j k_i^{-1} = q^{d_i a_{ij}} e_j,\ k_i f_j k_i^{-1} = q^{-d_i a_{ij}} f_j \tag{1.2}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i^2 - k_i^{-2}}{q^{2d_i} - q^{-2d_i}} \tag{1.3}$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} e_i^{1-a_{ij}-\nu} e_j e_i^\nu = 0 \quad (i \neq j) \tag{1.4}$$

$$\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} f_i^{1-a_{ij}-\nu} f_j f_i^\nu = 0 \quad (i \neq j) \tag{1.5}$$

where, for any two integers $m \geq n \geq 0$ and an arbitrary parameter t, $\begin{bmatrix} m \\ n \end{bmatrix}_t \in \mathbf{Z}[t, t^{-1}]$ is defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \prod_{i=1}^{n} \dfrac{t^{m-i+1} - t^{-m+i-1}}{t^i - t^{-i}} & \text{if } m > n > 0, \\ 1 & \text{if } n = 0 \text{ or } m = n. \end{cases}$$

When A is the Cartan matrix of type A_N, we put $U_q(sl_{N+1}(F)) = U_q(\mathscr{G}(A), \operatorname{diag}(1, 1, \ldots, 1))$ for $q^4 \neq 1$. For $1 \leq i < j \leq N+1$, we define inductively the elements e_{ij}, f_{ij} of $U_q(sl_{N+1}(F))$ by

$$e_{i,i+1} = e_i,\ f_{i,i+1} = f_i,$$
$$e_{ij} = q e_{i,j-1} e_{j-1,j} - q^{-1} e_{j-1,j} e_{i,j-1},\ (j - i > 1) \tag{1.6}$$

and

$$f_{ij} = q f_{i,j-1} f_{j-1,j} - q^{-1} f_{j-1,j} f_{i,j-1},\ (j - i > 1).$$

(The elements e_{ij}, f_{ij} were introduced by Jimbo [7].)

Define the lexicographic order $<$ on $\mathbf{Z} \times \mathbf{Z}$ by

$$(i, j) < (m, n) \text{ if } i < m \text{ or if } i = m,\ j < n. \tag{1.7}$$

Now we can state our main theorem.

Theorem 1.1. *Let $q \in F^\times$ be such that $q^8 \neq 1$. Then the elements*

$$f_{m_1,n_1} \cdots f_{m_s,n_s} k_1^{\ell_1} \cdots k_N^{\ell_N} e_{i_1,j_1} \cdots e_{i_t,j_t}$$

($(m_1, n_1) \leq \cdots \leq (m_s, n_s)$, $(i_1, j_1) \leq \cdots \leq (i_t, j_t)$, $\ell_1, \ldots, \ell_N \in \mathbf{Z}$) *form a basis of* $U_q(sl_{N+1}(F))$.

Theorem 1.2. *If $q^8 \neq 1$, then $U_q(sl_{N+1}(F))$ is a left (right) Noetherian ring, and has no zero divisors $\neq 0$.*

Remark. If $N \geq 3$ and q is a primitive 8-th root of unity, the set of elements given in Theorem 1.1 does not span $U_q(sl_{N+1}(F))$. Even in that case we can give an explicit basis of $U_q(sl_{N+1}(F))$. See Proposition 6.2.

§2. The Triangular Decomposition of $U_q(\mathscr{G}(A))$

Let $A = (a_{ij})_{1 \leq i,j \leq N}$ be a symmetrizable generalized Cartan matrix. Let $\tilde{U}_q(\mathscr{G}(A), \operatorname{diag}(d_1, \ldots, d_N))$ be the associative F-algebra with 1 with generators e_i, f_i, $k_i^{\pm 1}$ $(1 \leq i \leq N)$, and relations (1.1), (1.2), (1.3). Let $\mathscr{X}_+$ (resp. $\mathscr{X}_-$) be the free associative F-algebra with 1 with generators $\zeta_1, \ldots, \zeta_N$ (resp. $\xi_1, \ldots, \xi_N$). Let $F[v_1^{\pm 1}, \ldots, v_N^{\pm 1}]$ be the F-algebra of Laurent polynomials in indeterminates

$v_1,\dots,v_N$. Let $\mathfrak{M}=\mathscr{X}_-\otimes_F F[v_1^{\pm 1},\dots, v_N^{\pm N}]\otimes_F\mathscr{X}_+$. Then the elements $\xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t}$ $(\ell_1,\dots,\ell_N\in\mathbf{Z},\ 1\le i_1,\dots,i_s, j_1,\dots,j_t\le N)$ form an F-basis of $\mathfrak{M}$.

Lemma 2.1. *$\mathfrak{M}$ has a left $\tilde{U}_q(A,\ \mathrm{diag}(d_1,\dots,d_N))$-module structure defined by*

$$k_r\cdot \xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t}$$
$$=q^{-d_r(a_{r,i_1}+\cdots+a_{r,i_s})}\xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_r^{\ell_r+1}\cdots u_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t} \tag{2.1}$$

$$f_r\cdot \xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t}$$
$$=\xi_r\xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t} \tag{2.2}$$

$$e_r\cdot \xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t}$$
$$=q^{-d_r(\ell_1a_{r,1}+\cdots+\ell_Na_{r,N})}\xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_r\zeta_{j_1}\cdots\zeta_{j_t}$$
$$+\frac{1}{q^{2d_r}-q^{-2d_r}}\sum_{i_u=r}\{q^{-2d_r\alpha_u}\xi_{i_1}\cdots\hat{\xi}_{i_u}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_r^{\ell_r+2}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t}$$
$$-q^{2d_r\alpha_u}\xi_{i_1}\cdots\hat{\xi}_{i_u}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_r^{\ell_r-2}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t}\} \tag{2.3}$$

where $\alpha_u=a_{r,i_{u+1}}+a_{r,i_{u+2}}+\cdots+a_{r,i_s}$, and $\hat{\xi}_{i_u}$ means that ξ_{i_u} is omitted.

This can be verified by straightforward computations.

Lemma 2.2. *The elements $f_{i_1}\cdots f_{i_s}k_1^{\ell_1}\cdots k_N^{\ell_N}e_{j_1}\cdots e_{j_t}$ $(\ell_1,\dots,\ell_N\in\mathbf{Z},$ $1\le i_1,\cdots,i_s, j_1,\cdots,j_t\le N)$ form a basis of $\tilde{U}_q(A,\ \mathrm{diag}(d_1,\cdots,d_N))$.*

Proof. Let $1_{\mathfrak{M}}\in\mathfrak{M}$ (resp. $1_{\tilde{U}}\in\tilde{U}_q(\mathscr{G}(A))$) be the unit element of $F[v_1^{\pm 1},\dots,v_N^{\pm 1}]$ (resp. $\tilde{U}_q(\mathscr{G}(A))$). By Lemma 2.1, we can define the left $\tilde{U}_q(\mathscr{G}(A))$-module homomorphisms σ: $\mathfrak{M}\to U_q(\mathscr{G}(A))$ and τ: $\tilde{U}_q(\mathscr{G}(A))\to\mathfrak{M}$ by $\sigma(\xi_{i_1}\cdots\xi_{i_s}v_1^{\ell_1}\cdots v_N^{\ell_N}\zeta_{j_1}\cdots\zeta_{j_t})=f_{i_1}\cdots f_{i_s}k_1^{\ell_1}\cdots k_N^{\ell_N}e_{j_1}\cdots e_{j_t}$ and $\tau(x)=x.\,1_{\mathfrak{M}}$. Then $\tau\circ\sigma$ is the identity map. Moreover, $\tilde{U}_q(\mathscr{G}(A))=\tilde{U}_q(\mathscr{G}(A)).\,1_{\tilde{U}}=\tilde{U}_q(\mathscr{G}(A)).\,\sigma(1_{\mathfrak{M}})$ $=\sigma(\mathfrak{M})$. Hence σ is bijective. Hence the lemma follows.

We prepare some notations which will be used hereafter.

$\cdot U_q(\mathscr{N}_+)$ (resp. $\tilde{U}_q(\mathscr{N}_+)$) is the subalgebra of $U_q(\mathscr{G}(A))$ (resp. $\tilde{U}_q(\mathscr{G}(A))$) generated by the e_i's along with 1.

$\cdot U_q(\mathscr{N}_-)$ (resp. $\tilde{U}_q(\mathscr{N}_-)$) is the subalgebra of $U_q(\mathscr{G}(A))$ (resp. $\tilde{U}_q(\mathscr{G}(A))$) generated by the f_i's along with 1.

$\cdot H$ (resp. $\tilde{H}$) is the subalgebra of $U_q(\mathscr{G}(A))$ (resp. $\tilde{U}_q(\mathscr{G}(A))$) generated by the $k_i^{\pm 1}$'s.

$\cdot\ \phi_{ij}^+$, ϕ_{ij}^- $(1\le i\ne j\le N)$ are the elements of $\tilde{U}_q(\mathscr{G}(A),\ \mathrm{diag}(d_1,\dots,d_N))$ defined by

$$\phi_{ij}^{+} = \sum_{\nu=0}^{1-a_{ij}} (-1)^{\nu} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} e_i^{1-a_{ij}-\nu} e_j e_i^{\nu},$$

$$\phi_{ij}^{-} = \sum_{\nu=0}^{1-a_{ij}} (-1)^{\nu} \begin{bmatrix} 1-a_{ij} \\ \nu \end{bmatrix}_{q^{2d_i}} f_i^{1-a_{ij}-\nu} f_j f_i^{\nu}.$$

$\cdot I_+$ (resp. I_-) is the two sided ideal of $\tilde{U}_q(\mathcal{N}_+)$ (resp. $\tilde{U}_q(\mathcal{N}_-)$) generated by the ϕ_{ij}^{+}'s (resp. the ϕ_{ij}^{-}'s).

$\cdot K$ is the two sided ideal of $\tilde{U}_q(\mathcal{G}(A))$ generated by the ϕ_{ij}^{+}'s and ϕ_{ij}^{-}'s.

Obviously, $U_q(\mathcal{G}(A), D) \simeq \tilde{U}_q(\mathcal{G}(A), D)/K$ as F-algebras. By Lemma 2.2, we have $\tilde{U}_q(\mathcal{G}(A)) \simeq \tilde{U}_q(\mathcal{N}_-) \otimes_F \tilde{H} \otimes_F \tilde{U}_q(\mathcal{N}_+)$ as vector spaces, and $\tilde{U}_q(\mathcal{N}_+) \simeq \mathcal{X}_+ \simeq \mathcal{X}_- \simeq \tilde{U}_q(\mathcal{N}_-)$ as F-algebras.

The following proposition gives a triangular decomposition of $U_q(\mathcal{G}(A))$.

Proposition 2.3. *$U_q(\mathcal{G}(A), D) \simeq U_q(\mathcal{N}_-) \otimes_F H \otimes_F U_q(\mathcal{N}_+)$ as vector spaces. $U_q(\mathcal{N}_\pm) \simeq \tilde{U}_q(\mathcal{N}_\pm)/I_\pm$ as F-algebras. The elements $k_1^{\ell_1} \cdots k_N^{\ell_N} (\ell_1, \ldots, \ell_N \in \mathbf{Z})$ form a basis of H.*

Proof. By Lemma 2.2, it suffices to prove:

$$K = (\tilde{U}_q(\mathcal{N}_-))\tilde{H} I_+ + I_- \tilde{H}(\tilde{U}_q(\mathcal{N}_+)).$$

This can be done by showing that $(\tilde{U}_q(\mathcal{N}_-))\tilde{H} I_+$ and $I_- \tilde{H}(\tilde{U}_q(\mathcal{N}_+))$ are ideals of $\tilde{U}_q(\mathcal{G}(A), D)$. We only consider $I_- \tilde{H}(\tilde{U}_q(\mathcal{N}_+))$, the argument for $(\tilde{U}_q(\mathcal{N}_+))\tilde{H} I_+$ being analogous. Let $Y = I_- \tilde{H}(\tilde{U}_q(\mathcal{N}_+))$. It is clear that $k_i^{\pm 1} Y \subset Y$, $Y k_i^{\pm 1} \subset Y$ $f_i Y \subset Y$, $Y f_i \subset Y$, $Y e_i \subset Y$. The proof of $e_i Y \subset Y$ is similar to that of [9, Lemma 2.3] and as follows. Let $e_i^{\pm}$: $\tilde{U}_q(\mathcal{N}_-) \to \tilde{U}_q(\mathcal{N}_-)$ be the two F-linear maps defined by

$$e_i^{\pm}(f_{i_1} \cdots f_{i_s}) = \sum_{i_u = i} q^{\pm 2 d_i \alpha_u} f_{i_1} \cdots \hat{f}_{i_u} \cdots f_{i_s},$$

where α_u is an in (2.3), so that

$$\begin{aligned} & e_i \cdot f_{i_1} \cdots f_{i_s} k_1^{\ell_1} \cdots k_N^{\ell_N} e_{j_1} \cdots e_{j_t} \\ &= q^{-d_i(\ell_1 a_{i,1} + \cdots + \ell_N a_{i,N})} f_{i_1} \cdots f_{i_s} k_1^{\ell_1} \cdots k_N^{\ell_N} e_i e_{j_1} \cdots e_{j_t} \\ &+ \frac{1}{q^{2d_i} - q^{-2d_i}} \sum_{i_u = i} \{ e_i^{-}(f_{i_1} \cdots f_{i_s}) k_1^{\ell_1} \cdots k_i^{\ell_i + 2} \cdots k_N^{\ell_N} e_{j_1} \cdots e_{j_t} \\ &- e^{+}(f_{i_1} \cdots f_{i_s}) k_1^{\ell_1} \cdots k_i^{\ell_i - 2} \cdots k_N^{\ell_N} e_{j_1} \cdots e_{j_t} \}. \end{aligned}$$

But we have

$$\begin{aligned} & e_i^{\pm}(f_{i_1} \cdots f_{i_p} \phi_{\ell m}^{-} f_{i_s} \cdots f_{i_{s+\ell}}) \\ &= \beta e_i^{\pm}(f_{i_1} \cdots f_{i_p}) \phi_{\ell m}^{-} f_{i_s} \cdots f_{i_{s+\ell}} \end{aligned}$$

$$+\beta' f_{i_1}\cdots f_{i_p} e_i^{\pm}(\phi_{\ell m}^-) f_{i_s}\cdots f_{i_{s+\ell}}$$
$$+ f_{i_1}\cdots f_{i_p}\phi_{\ell m}^- e_i^{\pm}(f_{i_s}\cdots f_{i_{s+\ell}})$$

(β, $\beta'\in F^\times$, $1\le \ell\ne m\le N$). Hence it is enough to show that $e_i^{\pm}(\phi_{\ell m}^-)=0$. If $i\ne\ell$, m, this is obvious. We consider the case $i=m$.

$$\begin{aligned}
&e_i^{\pm}(\phi_{\ell i}^-)\\
&=\left(\sum_{\nu=0}^{1-a_{\ell i}}(-1)^\nu\begin{bmatrix}1-a_{\ell i}\\ \nu\end{bmatrix}_{q^{2d_\ell}} q^{\pm 2d_\ell\cdot\nu a_{\ell i}}\right) f_\ell^{1-a_{\ell i}}\\
&=\left(\sum_{\nu=1}^{1-a_{\ell i}}(-1)^\nu\begin{bmatrix}1-a_{\ell i}\\ \nu-1\end{bmatrix}_{q^{2d_\ell}} q^{\pm 2d_\ell(\nu-1)(a_{\ell i}-1)}\right.\\
&\quad\left.+\sum_{\nu=0}^{-a_{\ell i}}(-1)^\nu\begin{bmatrix}1-a_{\ell i}\\ \nu\end{bmatrix}_{q^{2d_\ell}} q^{\pm 2d_\ell\cdot\nu(a_{\ell i}-1)}\right) f_\ell^{1-a_{\ell i}}\\
&=0.
\end{aligned}$$

In the above computation, we used the formulas:

$$\begin{bmatrix}m\\ n\end{bmatrix}_t = t^{\pm(m-n)}\begin{bmatrix}m-1\\ n-1\end{bmatrix}_t + t^{\pm n}\begin{bmatrix}m-1\\ n\end{bmatrix}_t \quad (m>n>1).$$

The remaining case $i=\ell$ can be verified by a direct computation.

Remark. Proposition 2.3 is an extention of [11, Prop. 2].

Corollary 2.4. *Let $A=(a_{ij})_{1\le i,j\le N}$ be the symmetrizable Cartan matrix. For $1\le M\le N$, let $A'=(a_{ij})_{1\le i,j\le M}$ be the submatrix of A. Then the subalgebra of $U_q(\mathscr{G}(A)$, $\mathrm{diag}(d_1,\dots,d_N))$ generated by $\{e_i, f_i, k_i^{\pm1}\,|\,1\le i\le M\}$ is isomorphic to $U_q(\mathscr{G}(A')$, $\mathrm{diag}(d_1,\dots,d_M))$(as Hopf algebras).*

Proof. Let $U_q(\mathscr{N}'_-)\otimes_F H\otimes_F U_q(\mathscr{N}'_+)$ be the triangular decomposition of $U_q(\mathscr{G}(A')$, $\mathrm{diag}(d_1,\dots,d_M))$. Define the two homomorphisms $i_\pm$: $U_q(\mathscr{N}'_\pm)\to U_q(\mathscr{N}_\pm)$ by $i_+(e_i)=e_i$ and $i_-(f_i)=f_i(1\le i\le M)$. By Proposition 2.3, it suffices to show that $i_\pm$ are injective. We consider i_+. By Proposition 2.3, we can define the homomorphism p_+: $U_q(\mathscr{N}_+)\to U_q(\mathscr{N}'_+)$ by $p_+(e_i)=e_i(1\le i\le M)$ and $p_+(e_i)=0(M<i\le N)$. It is clear that $p_+\circ i_+$ is the identity map. Hence it is injective.

§3. Some (q-)commutator Relations in $U_q(sl_{N+1}(F))$

From now on, until the end of §5, *we are concerned with the quantum group* $U_q(sl_{N+1}(F))$, $q^8\ne1$. For a positive integer N, put $\Lambda_N=\{(i,j)\in\mathbb{Z}\times\mathbb{Z}\,|\,1\le i<j\le N+1\}$. For (i,j), $(m,n)\in\Lambda_N$ such that $(i,j)<(m,n)$(see(1.7)), there are following six cases:

(I) $i = m < j < n$, (II) $i < m < n < j$, (III) $i < m < j = n$,

$\underline{i \quad j}$ $\qquad$ $\underline{i \qquad\quad j}$ $\qquad$ $\underline{i \qquad\quad j}$

$\underline{m \qquad n}$ $\qquad$ $\underline{m \quad n}$ $\qquad$ $\underline{m \quad n}$

(IV) $i < m < j < n$, (V) $i < j = m < n$, (VI) $i < j < m < n$.

$\underline{i \quad j}$ $\qquad$ $\underline{i \quad j}$ $\qquad$ $\underline{i \; j}$

$\underline{m \quad n}$ $\qquad$ $\underline{m \quad n}$ $\qquad$ $\underline{m \; n}$

Set

$$
\begin{aligned}
C_{(\mathrm{I})} &= \{((i, j), (m, n)) \in \Lambda_N \times \Lambda_N \mid i = m < j < n\}, \\
C_{(\mathrm{II})} &= \{((i, j), (m, n)) \in \Lambda_N \times \Lambda_N \mid i < m < n < j\}, \\
C_{(\mathrm{III})} &= \{((i, j), (m, n)) \in \Lambda_N \times \Lambda_N \mid i < m < j = n\}, \\
C_{(\mathrm{IV})} &= \{((i, j), (m, n)) \in \Lambda_N \times \Lambda_N \mid i < m < j < n\}, \\
C_{(\mathrm{V})} &= \{((i, j), (m, n)) \in \Lambda_N \times \Lambda_N \mid i < j = m < n\}, \\
C_{(\mathrm{VI})} &= \{((i, j), (m, n)) \in \Lambda_N \times \Lambda_N \mid i < j < m < n\}.
\end{aligned}
$$

When $q^8 \neq 1$, we get the following formulas. We denote by e_{ij} and f_{mn} the elements of $U_q(sl_{N+1}(F))$ defined in (1, 6), and by $[x, y]$ the usual commutator $xy - yx$.

(1)

$$
\begin{aligned}
& q^{-2}e_{ij}e_{mn} - e_{mn}e_{ij} = 0 && \text{if } ((i, j), (m, n)) \in C_{(\mathrm{I})} \cup C_{(\mathrm{III})}. \\
& [e_{ij}, e_{mn}] = 0 && \text{if } ((i, j), (m, n)) \in C_{(\mathrm{II})} \cup C_{(\mathrm{VI})}. \\
& [e_{ij}, e_{mn}] = (q^2 - q^{-2})e_{in}e_{mj} && \text{if } ((i, j), (m, n)) \in C_{(\mathrm{IV})}. \\
& q^2 e_{ij}e_{mn} - e_{mn}e_{ij} = q e_{in} && \text{if } ((i, j), (m, n)) \in C_{(\mathrm{V})}.
\end{aligned}
$$

(2) The f_{ij}'s also satisfy relations similar to (1).

(3)

$$
\begin{aligned}
& [e_{ij}, f_{mn}] = (-1)^{j-i+1} q f_{jn} k_i^2 k_{i+1}^2 \cdots k_{j-1}^2 \quad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{I})}. \\
& [e_{ij}, f_{mn}] = (-1)^{n-m+1} q k_m^2 k_{m+1}^2 \cdots k_{n-1}^2 e_{im} \quad \text{if } ((i, j)), (m, n) \in C_{(\mathrm{III})}. \\
& [e_{ij}, f_{mn}] = (-1)^{j-m+1} (q^4 - 1) f_{jn} k_m^2 k_{m+1}^2 \cdots k_{j-1}^2 e_{im} \\
& \qquad\qquad\qquad\qquad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{IV})}. \\
& [e_{ij}, f_{mn}] = 0 \quad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{II})} \cup C_{(\mathrm{V})} \cup C_{(\mathrm{VI})}.
\end{aligned}
$$

$$[e_{mn}, f_{ij}] = (-1)^{j-i} q^{-1} k_i^{-2} k_{i+1}^{-2} \cdots k_{j-1}^{-2} e_{jn} \text{ if } ((i, j), (m, n)) \in C_{(\mathrm{I})}.$$

$$[e_{mn}, f_{ij}] = (-1)^{n-m} q^{-1} f_{jm} k_m^{-2} k_{m+1}^{-2} \cdots k_{n-1}^{-2} \text{ if } ((i, j), (m, n)) \in C_{(\mathrm{III})}.$$

$$[e_{mn}, f_{ij}] = (-1)^{j-m} (1 - q^{-4}) f_{im} k_m^{-2} k_{m+1}^{-2} \cdots k_{j-1}^{-2} e_{jn} \text{ if } ((i, j), (m, n)) \in C_{(\mathrm{IV})}.$$

$$[e_{mn}, f_{ij}] = 0 \text{ if } ((i, j), (m, n)) \in C_{(\mathrm{II})} \cup C_{(\mathrm{V})} \cup C_{(\mathrm{VI})}.$$

$$[e_{ij}, f_{ij}] = \frac{(-1)^{j-i+1}}{q^2 - q^{-2}} (k_i^2 k_{i+1}^2 \cdots k_{j-1}^2 - k_i^{-2} k_{i+1}^{-2} \cdots k_{j-1}^{-2}).$$

(4)

$$k_r e_{ij} k_r^{-1} = q^{(a_{i,r}+a_{i+1,r}+\dots+a_{j-1,r})} e_{ij}.$$
$$k_r f_{ij} k_r^{-1} = q^{-(a_{i,r}+a_{i+1,r}+\dots+a_{j-1,r})} f_{ij}.$$

Among these, we just prove:

$$q^2 e_{ij} e_{mn} - e_{mn} e_{ij} = q e_{in} \qquad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{V})}, \tag{3.1}$$

$$[e_{ij}, e_{mn}] = 0 \qquad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{II})}; \tag{3.2}$$

to verify other formulas is left to the reader. To prove the formula (3.1), we use induction on $n - m$:

$$\begin{aligned} &q^2 e_{i,j} e_{m,n} - e_{m,n} e_{i,j} \\ &\quad = q^2 e_{i,m}(q e_{m,n-1} e_{n-1,n} - q^{-1} e_{n-1,n} e_{m,n-1}) \\ &\qquad - (q e_{m,n-1} e_{n-1,n} - q^{-1} e_{n-1,n} e_{m,n-1}) e_{i,m} \\ &\quad = q^2 e_{i,n-1} e_{n-1,n} - e_{n-1,n} e_{i,n-1} \\ &\quad = q e_{i,n}. \end{aligned}$$

We get the formula (3.2) by (3.1) and the following formula:

$$\begin{aligned} &[e_{i,i+3}, e_{i+1,i+2}] \\ &\quad = (q^2 e_i e_{i+1} e_{i+2} - e_{i+1} e_i e_{i+2} - e_{i+2} e_i e_{i+1} + q^{-2} e_{i+2} e_{i+1} e_i) e_{i+1} \\ &\qquad - e_{i+1}(q^2 e_i e_{i+1} e_{i+2} - e_{i+1} e_i e_{i+2} - e_{i+2} e_i e_{i+1} + q^{-2} e_{i+2} e_{i+1} e_i) \\ &\quad = q^2 e_i((q^2 + q^{-2})^{-1}(e_{i+1}^2 e_{i+2} + e_{i+2} e_{i+1}^2)) - e_{i+1} e_i e_{i+2} e_{i+1} \\ &\qquad - e_{i+2} e_i e_{i+1}^2 + q^{-2} e_{i+2}((q^2 + q^{-2})^{-1}(e_{i+1}^2 e_i + e_i e_{i+1}^2)) \\ &\qquad - q^2((q^2 + q^{-2})^{-1}(e_{i+1}^2 e_i + e_i e_{i+1}^2)) e_{i+2} + e_{i+1}^2 e_i e_{i+2} \\ &\qquad + e_{i+1} e_{i+2} e_i e_{i+1} - q^{-2}((q^2 + q^{-2})^{-1}(e_{i+1}^2 e_{i+2} + e_{i+2} e_{i+1}^2)) e_i \\ &\quad = 0. \end{aligned} \tag{3.3}$$

We note that the condition $q^8 \neq 1$ is used in proving (3.2).

§4. Proof of Theorem 1.1

Let Λ_N be as in §3. Let $\mathcal{N}$ be an F-vector space spanned by a basis $\{\mathscr{X}_{ij} | (i, j) \in \Lambda_N\}$. Let $\mathscr{X}(\mathcal{N})$ be the free associative F-algebra with 1 with generators x_{ij} $((i, j) \in \Lambda_N)$. For $(i, j), (m, n) \in \Lambda_N$ such that $(i, j) < (m, n)$, we define $\varepsilon_{ijmn} \in F^{\times}$ and $y_{ijmn} \in \mathscr{X}(\mathcal{N})$ by

$$\varepsilon_{ijmn} = \begin{cases} 1 & \text{if } ((i,j),(m,n)) \in C_{(\mathrm{II})} \cup C_{(\mathrm{IV})} \cup C_{(\mathrm{VI})} \\ q^{-2} & \text{if } ((i,j),(m,n)) \in C_{(\mathrm{I})} \cup C_{(\mathrm{III})} \\ q^2 & \text{if } ((i,j),(m,n)) \in C_{(\mathrm{V})}, \end{cases} \tag{4.1}$$

$$y_{ijmn} = \begin{cases} qx_{in} & \text{if } ((i,j),(m,n)) \in C_{(\mathrm{V})} \\ (q^2 - q^{-2})x_{in}x_{mj} & \text{if } ((i,j),(m,n)) \in C_{(\mathrm{IV})} \\ 0 & \text{if } ((i,j),(m,n)) \in C_{(\mathrm{I})} \cup C_{(\mathrm{II})} \cup C_{(\mathrm{III})} \cup C_{(\mathrm{VI})}, \end{cases} \tag{4.2}$$

where $C_{(\mathrm{I})}, \dots, C_{(\mathrm{VI})}$ are as in §3.

Let $\mathbf{I}$ (resp. $\mathbf{J}$) be the two sided ideal of $\mathscr{X}(\mathscr{N})$ generated by the elements $\varepsilon_{ijmn}x_{ij}x_{mn} - x_{mn}x_{ij} - y_{ijmn}$ (resp. $\varepsilon_{ijmn}x_{ij}x_{mn} - x_{mn}x_{ij}$) for $(i,j) < (m,n)$. Put $U_q(\mathscr{N}) = \mathscr{X}(\mathscr{N})/\mathbf{I}$ and $\mathfrak{S}_q(\mathscr{N}) = \mathscr{X}(\mathscr{N})/\mathbf{J}$. Let $\tilde{x}_{ij} = x_{ij} + \mathbf{I} \in U_q(\mathscr{N})$, $\tilde{y}_{ijmn} = y_{ijmn} + \mathbf{I} \in U_q(\mathscr{N})$ and $z_{ij} = x_{ij} + \mathbf{J} \in \mathfrak{S}_q(\mathscr{N})$.

Lemma 4.1. *If $q^8 \neq 1$, there exist isomorphisms $\varphi_\pm$: $U_q(\mathscr{N}) \to U_q(\mathscr{N}_\pm)$ such that $\varphi_+(\tilde{x}_{ij}) = e_{ij}$, $\varphi_-(\tilde{x}_{ij}) = f_{ij}$ $((i,j) \in \Lambda_N)$.*

Proof. We only consider φ_+. By the formulas in §3, φ_+ is well-defined. By the definition of $U_q(\mathscr{N})$, we have

$$\tilde{x}_{i,i+1}\tilde{x}_{j,j+1} - \tilde{x}_{j,j+1}\tilde{x}_{i,i+1} = 0 \qquad \text{for } |i-j| \geq 2,$$

and

$$\begin{aligned}&(\tilde{x}_{i,i+1})^2\tilde{x}_{i+1,i+2} - (q^2+q^{-2})\tilde{x}_{i,i+1}\tilde{x}_{i+1,i+2}\tilde{x}_{i,i+1} + \tilde{x}_{i+1,i+2}(\tilde{x}_{i,i+1})^2 \\ &= q^{-1}\tilde{x}_{i,i+1}\tilde{x}_{i,i+2} - q\tilde{x}_{i,i+2}\tilde{x}_{i,i+1} = 0.\end{aligned}$$

Similarly,

$$\begin{aligned}&(\tilde{x}_{i+1,i+2})^2\tilde{x}_{i,i+1} - (q^2+q^{-2})\tilde{x}_{i+1,i+2}\tilde{x}_{i,i+1}\tilde{x}_{i+1,i+2} \\ &\quad + \tilde{x}_{i,i+1}(\tilde{x}_{i+1,i+2})^2 \\ &= 0.\end{aligned}$$

Hence we can define the homomorphism ψ_+: $U_q(\mathscr{N}_+) \to U_q(\mathscr{N})$ by $\psi_+(e_i) = \tilde{x}_{i,i+1}$. It is obvious that $\varphi_+ \circ \psi_+$ and $\psi_+ \circ \varphi_+$ are the identity maps.

Let P_N be the set of finite sequences of elements of Λ_N. We consider the "empty sequence" (ϕ) is also an element of P_N. For $\Sigma = ((i_1,j_1), \dots, (i_t,j_t)) \in P_N$, we put $x_\Sigma = x_{i_1,j_1} \cdots x_{i_t,j_t}$; we understand that $x_{(\phi)}1$. Put $\tilde{x}_\Sigma = x_\Sigma + \mathbf{I} \in U_q(\mathscr{N})$, $z_\Sigma = x_\Sigma + \mathbf{J} \in \mathfrak{S}_q(\mathscr{N})$. Define the function η: $P_N \to \mathbf{Z}$ by $\eta(\Sigma) = i_1(j_1 - i_1) + \cdots + i_t(j_t - i_t)$ and $\eta((\phi)) = 0$. For a nonnegative integer m, let U_m (resp. S_m) be the subspace of $U_q(\mathscr{N})$(resp. $\mathfrak{S}_q(\mathscr{N})$) spanned by the elements $\tilde{x}_\Sigma$ (resp. z_Σ) such that $\eta(\Sigma) \leq m$, along with 1. Call Σ increasing if $(i_1,j_1) \leq \cdots \leq (i_t,j_t)$. (ϕ) is also considered to be increasing.

Lemma 4.2. *$U_q(\mathcal{N})$ is spanned by $\{\tilde{x}_\Sigma | \Sigma$ increasing$\}$ as a vector space.*

Proof. Assume that any element of U_{m-1} is an F-linear combination of the $\tilde{x}_\Sigma$'s such that $\eta(\Sigma) \leq m-1$ and that Σ is increasing. It suffices to show that, for $\Sigma = ((i_1, j_1), \ldots, (i_t, j_t))$ satisfying $\eta(\Sigma) = m$, we have

$$x_{i_1,j_1} \cdots x_{i_u,j_u} x_{i_{u+1},j_{u+1}} \cdots x_{i_t,j_t}$$
$$\equiv \varepsilon_{i_{u+1},j_{u+1},i_u,j_u} x_{i_1,j_1} \cdots x_{i_{u+1},j_{u+1}} x_{i_u,j_u} \cdots x_{i_t,j_t}$$
$$(\mathrm{mod}\ U_{m-1}).$$

for an integer u such that $(i_u, j_u) > (i_{u+1}, j_{u+1})$. Indeed, if $((i_{u+1}, j_{u+1}), (i_u, j_u)) \in C_{(\mathrm{I})} \cup C_{(\mathrm{II})} \cup C_{(\mathrm{III})} \cup C_{(\mathrm{VI})}$, this follows from (4.2). If it belongs to $C_{(\mathrm{V})}$(resp. $C_{(\mathrm{IV})}$), then it is obtained from (4.2) and the formula:

$$\eta\Big(((i_{u+1}, j_{u+1}), (i_u, j_u))\Big) - \eta\Big(((i_{u+1}, j_u))\Big)$$
$$= (i_u - i_{u+1})(j_u - j_{u+1}) > 0 \tag{4.3}$$

$$(\text{resp. } \eta\Big(((i_{u+1}, j_{u+1}), (i_u, j_u))\Big) - \eta\Big(((i_{u+1}, j_u), (i_u, j_{u+1}))\Big)$$
$$= (i_u - i_{u+1})(j_u - j_{u+1}) > 0). \tag{4.4}$$

Now we show that $\{\tilde{x}_\Sigma | \Sigma$ increasing$\}$ is, in fact, a basis of $U_q(\mathcal{N})$.

Lemma 4.3. *The set $\{z_\Sigma | \Sigma$ increasing$\}$ is a basis of $\mathfrak{S}_q(\mathcal{N})$.*

The proof is similar to that of Lemma 2.2; instead of Lemma 2.1, we need the following Lemma 4.4. We omit the details.

Lemma 4.4. *For $1 \leq i < j \leq n$, let $c_{ij} \in F$. We denote by $\mathcal{T}_n$ the associative F-algebra with generators $t_1, \ldots, t_n$ and relations $c_{ij} t_i t_j - t_j t_i$. Let $F[v_1, \ldots, v_n]$ be the F-algebra of polynomials in indeterminates $v_1, \ldots, v_n$. Then $F[v_1, \ldots, v_n]$ has a left $\mathcal{T}_n$-module structure defined by*

$$t_i \cdot v_1^{r_1} \cdots v_n^{r_n} = c_{1,i}^{r_1} c_{2,i}^{r_2} \cdots c_{i-1,i}^{r_{i-1}} v_1^{r_1} \cdots v_i^{r_i+1} \cdots v_n^{r_n}.$$

For $\lambda, \mu \in \Lambda_N$ and $\Sigma \in P_N$, write $\lambda \leq \Sigma$ if $\lambda \leq \mu$ for all $\mu \in \Sigma$.

Lemma 4.5. *There exists an F-bilinear map f: $\mathcal{N} \times \mathfrak{S}_q(\mathcal{N}) \to \mathfrak{S}_q(\mathcal{N})$ satisfying:*

(A) $f(x_\lambda, z_\Sigma) = z_\lambda z_\Sigma$ *for* $\lambda \leq \Sigma$.

(B) $f(x_\lambda, z_\Sigma) \equiv z_\lambda z_\Sigma\ (\mathrm{mod}\ S_{\eta(\Sigma)+\eta(\lambda)-1})$.

(C) *For all* $(i, j) < (m, n)$,

$$\varepsilon_{ijmn} f(x_{ij}, f(x_{mn}, z_T)) - f(x_{mn}, f(x_{ij}, z_T))$$

$$= \begin{cases} qf(x_{in}, z_T) & \text{if } ((i, j), (m, n)) \in C_{(\mathrm{V})} \\ (q^2 - q^{-2}) f(x_{in}, f(x_{mj}, z_T)) & \text{if } ((i, j), (m, n)) \in C_{(\mathrm{IV})} \\ 0 \quad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{II})} \bigcup C_{(\mathrm{III})} \bigcup C_{(\mathrm{VI})}. \end{cases}$$

In order to prove this, we need:

Lemma 4.6. *Let r be a positive integer. Assume an F-bilinear map* f': $\mathcal{N} \times \mathfrak{S}_q(\mathcal{N}) \to \mathfrak{S}_q(\mathcal{N})$ *satisfies the following*:

(B′) $f'(x_\lambda, z_\Sigma) \equiv z_\lambda z_\Sigma \pmod{S_{\eta(\Sigma)+\eta(\lambda)-1}}$.

(C′) *For all* $(i, j), (m, n) \in \Lambda_N$, $T \in P_N$ *such that* $(i, j) < (m, n)$ *and* $\eta(i, j) + \eta(m, n) + \eta(T) < r$,

$$\varepsilon_{ijmn} f'(x_{ij}, f'(x_{mn}, z_T)) - f'(x_{mn}, f'(x_{ij}, z_T))$$

$$= \begin{cases} qf'(x_{in}, z_T) & \text{if } ((i, j), (m, n)) \in C_{(\mathrm{V})} \\ (q^2 - q^{-2}) f'(x_{in}, f'(x_{mj}, z_T)) & \text{if } ((i, j), (m, n)) \in C_{(\mathrm{IV})} \\ 0 \quad \text{if } ((i, j), (m, n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{II})} \bigcup C_{(\mathrm{III})} \bigcup C_{(\mathrm{VI})}. \end{cases}$$

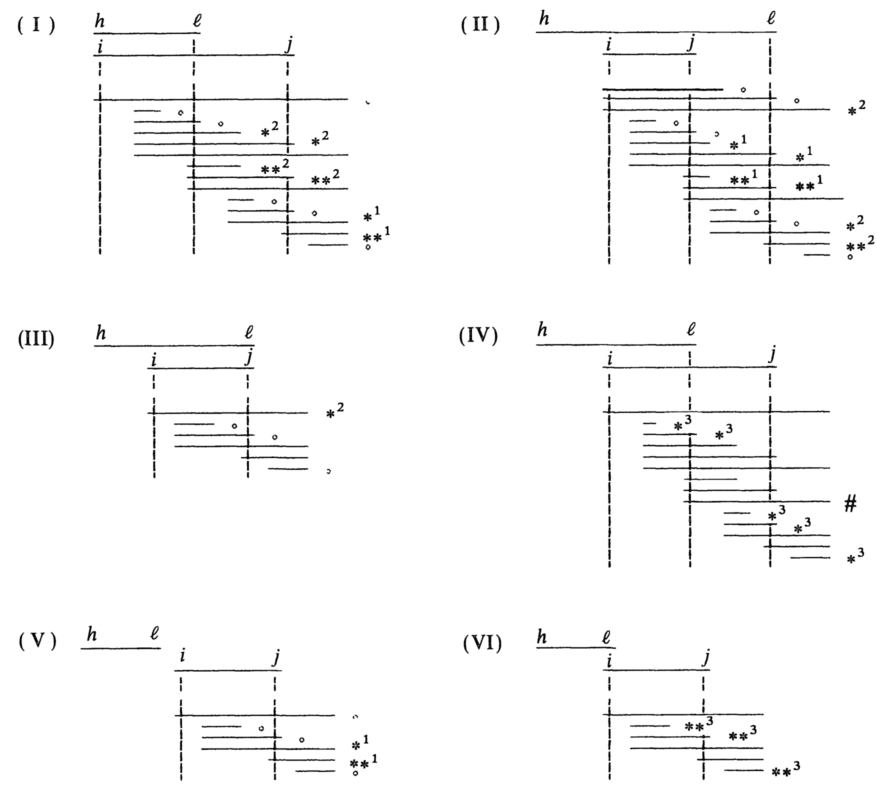
Then, for $(h, \ell) < (i, j) < (m, n)$ *and* $\Psi \in P_N$ *such that* $\eta(h, \ell) + \eta(i, j) + \eta(m, n) + \eta(\Psi) \leq r$, *it follows*:

$$\begin{aligned} &\varepsilon_{h\ell ij}\varepsilon_{h\ell mn} x_{h\ell} y_{ijmn} z_\Psi - y_{ijmn} x_{h\ell} z_\Psi \\ &\quad - \varepsilon_{ijmn} x_{ij} y_{h\ell mn} z_\Psi + \varepsilon_{h\ell ij} y_{h\ell mn} x_{ij} z_\Psi \\ &\quad + x_{mn} y_{h\ell ij} z_\Psi - \varepsilon_{ijmn}\varepsilon_{h\ell mn} y_{h\ell ij} x_{mn} z_\Psi \\ &= 0. \end{aligned} \tag{4.5}$$

(*Here we abbreviate* $f'(x_{i_1, j_1}, f'(x_{i_2,j_2}, \ldots, f'(x_{i_t,j_t}, z_\Psi).))$ *to* $x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_t,j_t} z_\Psi$.)

Sketch of the proof. In § 3, we have seen that there are 6 cases for $((i, j), (m, n)) \in \Lambda_N \times \Lambda_N$ such that $(i, j) < (m, n)$. Similarly we can see that there are 62 cases for $((h, \ell), (i, j), (m, n)) \in \Lambda_N \times \Lambda_N \times \Lambda_N$ such that $(h, \ell) < (i, j) < (m, n)$ (See Figure 1).

Figure 1 (we omit the letters m and n).



In the 20 cases labelled $\circ$, since $y_{ijmn} = y_{h\ell mn} = y_{h\ell ij} = 0$, the formula (4.5) is obvious. In the 24 cases labelled $*^r$ or $**^r$ ($r = 1, 2, 3$), we have

$$y_{ijmn} = \delta_{1r},\ y_{h\ell mn} = \delta_{2r},\ y_{h\ell ij} = \delta_{3r}.$$

In the cases of $*^r$,

the left-hand side of (4.5)

$$\begin{aligned}
&= \varepsilon_{h\ell ij}\varepsilon_{h\ell mn}x_{h\ell}\delta_{1r}(q^2 - q^{-2})x_{in}x_{mj}z_\Psi - \delta_{1r}(q^2 - q^{-2})x_{in}x_{mj}x_{h\ell}z_\Psi \\
&\quad - \varepsilon_{ijmn}x_{ij}\delta_{2r}(q^2 - q^{-2})x_{hn}x_{m\ell}z_\Psi + \varepsilon_{h\ell ij}\delta_{2r}(q^2 - q^{-2})x_{hn}x_{m\ell}x_{ij}z_\Psi \\
&\quad + x_{mn}\delta_{3r}(q^2 - q^{-2})x_{hj}x_{i\ell}z_\Psi - \varepsilon_{ijmn}\varepsilon_{h\ell mn}\delta_{3r}(q^2 - q^{-2})x_{hj}x_{i\ell}x_{mn}z_\Psi \\
&= (q^2 - q^{-2})\Big(\delta_{1r}(\varepsilon_{h\ell ij}\varepsilon_{h\ell mn} - \varepsilon_{h\ell mj}\varepsilon_{h\ell in})x_{h\ell}x_{in}x_{mj}z_\Psi \\
&\quad - \delta_{2r}(\varepsilon_{ijmn}\varepsilon_{hnij} - \varepsilon_{h\ell ij}\varepsilon_{ijm\ell})x_{hn}x_{ij}x_{m\ell}z_\Psi \\
&\quad + \delta_{3r}(\varepsilon_{hjmn}\varepsilon_{i\ell mn} - \varepsilon_{ijmn}\varepsilon_{h\ell mn})x_{hj}x_{mn}x_{i\ell}z_\Psi\Big) \\
&= 0.
\end{aligned}$$

The cases of $**^r$, can be treated similarly.

In each of the remaining 18 cases, we can get (4.5) after easy computation. For example, in the case of $h < i < \ell = m < j < n$ (labelled #),

the left-hand side of (4.5)

$$= q^2(q^2 - q^{-2})x_{h\ell}x_{in}x_{mj}z_\Psi - (q^2 - q^{-2})x_{in}x_{mj}x_{h\ell}z_\Psi$$

$$- qx_{ij}x_{hn}z_\Psi + qx_{hn}x_{ij}z_\Psi$$

$$+ (q^2 - q^{-2})x_{mn}x_{hj}x_{i\ell}z_\Psi - q^2(q^2 - q^{-2})x_{hj}x_{i\ell}x_{mn}z_\Psi$$

$$= q^2(q^2 - q^{-2})x_{h\ell}x_{in}x_{mj}z_\Psi - q^2(q^2 - q^{-2})x_{in}x_{h\ell}x_{mj}z_\Psi + q(q^2 - q^{-2})x_{in}x_{hj}z_\Psi$$

$$+ (q^2 - q^{-2})x_{hj}x_{mn}x_{i\ell}z_\Psi - (q^2 - q^{-2})^2x_{hn}x_{mj}x_{i\ell}z_\Psi - q^2(q^2 - q^{-2})x_{hj}x_{i\ell}x_{mn}z_\Psi$$

$$= q^2(q^2 - q^{-2})^2x_{hn}x_{i\ell}x_{mj}z_\Psi + q(q^2 - q^{-2})x_{in}x_{hj}z_\Psi$$

$$- q(q^2 - q^{-2})x_{hj}x_{in}z_\Psi - (q^2 - q^{-2})^2x_{hn}x_{mj}x_{i\ell}z_\Psi$$

$$= q(q^2 - q^{-2})^2x_{hn}x_{ij}z_\Psi - q(q^2 - q^{-2})^2x_{hn}x_{ij}z_\Psi = 0.$$

Proof of Lemma 4.5. To define $f(x_\lambda, z_\Sigma)$ satisfying (A) and (B), we proceed by induction on $\eta(\lambda) + \eta(\Sigma)$. If $\eta(\lambda) + \eta(\Sigma) = 1$, only the case $\lambda = (1, 2)$ and $\Sigma = (\phi)$ occurs; therefore we can put $f(x_{12}, 1) = z_{12}$. Assume that we have already defined the elements $f(x_{\lambda'}, z_{\Sigma'}) \in \mathfrak{S}_q(\mathcal{N})$ for λ', Σ' with $\eta(\lambda') + \eta(\Sigma') < \eta(\lambda) + \eta(\Sigma)$ so that they satisfy (A) and (B). We define $f(x_\lambda, z_\Sigma)$ when Σ is increasing. For the case $\lambda \le \Sigma$, we define $f(x_\lambda, z_\Sigma) = z_\lambda z_\Sigma$. If $\lambda \le \Sigma$ fails, then $\Sigma = (\mu, T)$, $\mu \le T$ and $\mu < \lambda$. Put $(i, j) = \mu$, $(m, n) = \lambda$. By (A) and (4.3 and 4), we can put:

$$f(x_{mn}, z_\Sigma) = \varepsilon_{ijmn}z_{ij}z_{mn}z_T + \varepsilon_{ijmn}f(x_{ij}, f(x_{mn}, z_T) - z_{mn}z_T) - X_{((i,j),(m,n))}$$

where

$$X_{((i,j),(m,n))} = \begin{cases} qf(x_{in}, z_T) & \text{if } ((i,j), (m,n)) \in C_{(\mathrm{V})} \\ (q^2 - q^{-2})f(x_{in}, f(x_{mj}, z_T)) & \text{if } ((i,j), (m,n)) \in C_{(\mathrm{IV})} \\ 0 & \text{if } ((i,j), (m,n)) \in C_{(\mathrm{I})} \bigcup C_{(\mathrm{II})} \bigcup C_{(\mathrm{III})} \bigcup C_{(\mathrm{VI})}. \end{cases}$$

Since $\varepsilon_{ijmn}z_{ij}z_{mn}z_T = z_{mn}z_\Sigma$, $f(x_{mn}, z_\Sigma)$ satisfies (B). By the definition of f, f satisfies (C) in the case $(i, j) \le T$. We shall consider the case when $(i, j) \le T$ fails. Write $T = ((h, \ell), \Psi)$ where $(h, \ell) \le \Psi$, $(h, \ell) < (i, j)$. By induction on $\eta(i, j) + \eta(m, n) + \eta(T)$, we show that f satisfies (C). Assume that, for each $\eta(i, j) + \eta(m, n) + \eta(T) \le r$, (C) holds. Then, for $\eta(i, j) + \eta(m, n) + \eta(T) \le r + 1$, we have:

$$x_{ij}x_{mn}x_{h\ell}z_{\Psi}$$
$$= x_{ij}(\varepsilon_{h\ell mn}x_{h\ell}x_{mn}z_{\Psi} - y_{h\ell mn}z_{\Psi})$$
$$= \varepsilon_{h\ell mn}(\varepsilon_{h\ell ij}x_{h\ell}x_{ij}x_{mn}z_{\Psi} - y_{h\ell ij}x_{mn}z_{\Psi}) - x_{ij}y_{h\ell mn}z_{\Psi}$$
$$= \varepsilon_{h\ell mn}\varepsilon_{h\ell ij}x_{h\ell}x_{ij}x_{mn}z_{\Psi} - \varepsilon_{h\ell mn}y_{h\ell ij}x_{mn}z_{\Psi} - x_{ij}y_{h\ell mn}z_{\Psi}.$$

(Here we abbreviate $f(x_{i_1,j_1}, f(x_{i_2,j_2},\dots, f(x_{i_t,j_t}, z_{\Psi}).))$ to $x_{i_1,j_1}x_{i_2,j_2}\cdots x_{i_t,j_t}z_{\Psi}$.) Similary,

$$x_{mn}x_{ij}x_{h\ell}z_{\Psi}$$
$$= \varepsilon_{h\ell ij}\varepsilon_{h\ell mn}\varepsilon_{ijmn}x_{h\ell}x_{ij}x_{mn}z_{\Psi} - \varepsilon_{h\ell ij}\varepsilon_{h\ell mn}x_{h\ell}y_{ijmn}z_{\Psi}$$
$$- \varepsilon_{h\ell ij}y_{h\ell mn}x_{ij}z_{\Psi} - x_{mn}y_{h\ell ij}z_{\Psi}.$$

Therefore, by Lemma 4.6, we get:

$$\varepsilon_{ijmn}x_{ij}x_{mn}z_T - x_{mn}x_{ij}z_T - y_{ijmn}z_T = 0.$$

Hence f satisfies (C). This completes the proof.

We can restate Lemma 4.5 as:

Lemma 4.7. *$\mathfrak{S}_q(\mathcal{N})$ has a left $U_q(\mathcal{N})$-module structure satisfying:*

(A) $\tilde{x}_\lambda z_\Sigma = z_\lambda z_\Sigma$ *for* $\lambda \le \Sigma$.

(B) $\tilde{x}_\lambda z_\Sigma = z_\lambda z_\Sigma$ $\pmod{S_{\eta(\Sigma)+\eta(\lambda)-1}}$.

As in the proof of Lemma 2.2, Lemma 4.2 and Lemma 4.7 imply the following:

Lemma 4.8. *Let $q \in F^\times$ be such that $q^8 \ne 1$. Then the set $\{z_\Sigma | \Sigma$ increasing$\}$ is a basis of $U_q(\mathcal{N})$.*

Combining Proposition 2.3, and Lemma 4.1 with Lemma 4.8, we obtain Theorem 1.1.

§5. Proof of Theorem 1.2

Let $\Sigma = ((i_1, j_1),\dots, (i_t, j_t)) \in P_N$. Define the function δ: $P_N \to \mathbb{Z}$ by $\delta(\Sigma) = (j_1 - i_1) + \cdots + (j_t - i_t)$ and $\delta((\phi)) = 0$. Denote the element $e_{i_1,j_1}\cdots e_{i_t,j_t}$ (resp. $f_{i_1,j_1}\cdots f_{i_t,j_t}$) of $U_q(sl_{N+1}(F))$ by e_Σ (resp. f_Σ). We understand $e_{(\phi)} = f_{(\phi)} = 1$. Put

$$\Omega_N = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} | m, n \ge 0,\ n \le \mathcal{M}ax\{\eta(\Sigma) | \Sigma \in P_N,\ \delta(\Sigma) = m\}\}.$$

For $(m, n) \in \Omega_N$, let $\mathfrak{A}_{(m,n)}$ be the subspace of $U_q(sl_{N+1}(F))$ spanned by the elements $f_{\Sigma'}k_1^{\ell_1}\cdots k_N^{\ell_N}e_\Sigma$, where Σ and Σ' are increasing, $(\delta(\Sigma') + \delta(\Sigma),\ \eta(\Sigma') + \eta(\Sigma)) \le (m, n)$, and $\ell_1,\dots, \ell_N \in \mathbb{Z}$. Clearly, $\mathfrak{A}_{(m',n')}\mathfrak{A}_{(m,n)} \subset \mathfrak{A}_{(m'+m,n'+n)}$, $\mathfrak{A}_{(m,n)}$

$\subset \mathfrak{A}_{(i,j)}$ if $(m, n) < (i, j)$. For $(m, n) \in \Omega_N$, let $(m, n)^\wedge \in \Omega_N$, be the element satisfying that

(1) $(m, n)^\wedge < (m, n)$.

(2) There is no element $(i, j) \in \Omega_N$ such that $(m, n)^\wedge < (i, j) < (m, n)$.

Put $\bar{\mathfrak{A}}_{(m,n)} = \mathfrak{A}_{(m,n)}/\mathfrak{A}_{(m,n)^\wedge}$, and $\bar{\mathfrak{A}}_{(0,0)} = \mathfrak{A}_{(0,0)}$. Then $\bar{\mathfrak{A}}_N = \bigoplus_{(m,n)\in\Omega_N} \bar{\mathfrak{A}}_{(m,n)^\wedge}$ has a graded algebra structure with 1 whose multiplication is defined component-wise by

$$(x + \mathfrak{A}_{(m,n)^\wedge})(y + \mathfrak{A}_{(m',n')^\wedge}) = xy + \mathfrak{A}_{(m+m',n+n')^\wedge},$$

where $x \in \mathfrak{A}_{(m,n)}$, $y \in \mathfrak{A}_{(m',n')}$. Let $\mathfrak{S}_q(\mathscr{G})$ be the associative F-algebra with 1 with generators $\tilde{e}_{ij}$, $\tilde{f}_{ij}$ $((i, j) \in \Lambda_N)$, $\tilde{k}_i^{\mp 1}$, $(1 \le i \le N)$ and relations:

$$\tilde{k}_i\tilde{k}_i^{-1} = \tilde{k}_i^{-1}\tilde{k}_i = 1,\ \tilde{k}_i\tilde{k}_j = \tilde{k}_j\tilde{k}_i$$

$$\tilde{k}_r\tilde{e}_{ij}\tilde{k}_r^{-1} = q^{(a_{i,r}+a_{i+1,r}+\cdots+a_{j-1,r})}\tilde{e}_{ij}$$

$$\tilde{k}_r\tilde{f}_{ij}\tilde{k}_r^{-1} = q^{-(a_{i,r}+a_{i+1,r}+\cdots+a_{j-1,r})}\tilde{f}_{ij}$$

$$\tilde{e}_{ij}\tilde{f}_{mn} = \tilde{f}_{mn}\tilde{e}_{ij}$$

$$\varepsilon_{ijmn}\tilde{e}_{ij}\tilde{e}_{mn} = \tilde{e}_{mn}\tilde{e}_{ij},\ \varepsilon_{ijmn}\tilde{f}_{ij}\tilde{f}_{mn} = \tilde{f}_{mn}\tilde{f}_{ij}\ ((i, j) < m, n)).$$

As in the proof of Lemma 2.2, we can show that the elements $\tilde{f}_{m_1,n_1}\cdots\tilde{f}_{m_s,n_s}\tilde{k}_1^{\ell_1}\cdots\tilde{k}_N^{\ell_N}\tilde{e}_{i_1,j_1}\cdots\tilde{e}_{i_t,j_t}$ $((m_1, n_1) \le \cdots \le (m_s, n_s)$, $(i_1, j_1) \le \cdots \le (i_t, j_t)$, $\ell_1,\ldots, \ell_N \in \mathbf{Z})$ form an F-basis of $\mathfrak{S}_q(\mathscr{G})$. As an immediate consequence of Theorem 1.1 and the formulas in §3, we get:

Lemma 5.1. *$\bar{\mathfrak{A}}_N$ is isomorphic to $\mathfrak{S}_q(\mathscr{G})$ as F-algebras.*

Remark. The above argument shows that, using our filtration, we can compute the structure constants of $U_q(sl_{N+1}(F))$ with respect to the basis given in Theorem 1.1.

Lemma 5.2. (*a*) *$\mathfrak{S}_q(\mathscr{G})$ has no zero divisors* $\neq 0$.
(*b*) *$\mathfrak{S}_q(\mathscr{G})$ is a left* (*right*) *Noetherian ring.*

Since $\mathfrak{S}_q(\mathscr{G})$ is a non-commutative analogue of polynomial rings, (*a*) and (*b*) can be proved in a way similar to the case of usual polynomial ring (e.g. see [10, Th. 1.2.10]).

By Lemma 5.1 and Lemma 5.2, we obtain Theorem 1.2. The proof is entirely similar to that of [5, Chap. 4, Theorem 4].

§6. $U_q(sl_{N+1}(R))$ over a Commutative Ring R

Let R be a commutative ring with 1, and $R^\times$ the unit group of R. Assume $q \in R^\times$ is such that $q^4 - 1 \in R^\times$. Let $U_q(sl_{N+1}(R))$ be the associative R-algebra with 1 with generators $e_i, f_i, k_i^{\pm 1}$ $(1 \le i \le N)$, and relations (1.1),..., (1.5), where $A = (a_{ij})_{1 \le i,j \le N}$ is the Cartan matrix of type A_N. For $1 \le i < j \le N+1$, we define e_{ij}, f_{ij} by (1.6). Let L be the two sided ideal of $U_q(sl_{N+1}(R))$ generated by $[e_{i,i+3}, e_{i+1}]$, $[f_{i,i+3}, f_{i+1}]$ $(1 \le i \le N-2)$. Put $\bar{U}_q(sl_{N+1}(R)) = U_q(sl_{N+1}(R))/L$. By (3.3), for a field F, if $q^8 \ne 1$, $\bar{U}_q(sl_{N+1}(F)) \simeq U_q(sl_{N+1}(F))/L$.

Theorem 6.1. (*a*) *As an R-module*, $\bar{U}_q(sl_{N+1}(R))$ *is free. The elements*

$$f_{m_1,n_1} \cdots f_{m_s,n_s} k_1^{\ell_1} \cdots k_N^{\ell_N} e_{i_1,j_1} \cdots e_{i_t,j_t} \tag{6.1}$$

($(m_1, n_1) \le \cdots \le (m_s, n_s)$, $(i_1, j_1) \le \cdots \le (i_t, j_t)$, $\ell_1, \ldots, \ell_N \in \mathbb{Z}$) *form an R-basis of* $\bar{U}_q(sl_{N+1}(R))$.

(*b*) *If R has no zero divisors* $\ne 0$, *then* $\bar{U}_q(sl_{N+1}(R))$ *has no zero divisors* $\ne 0$. *If R is a Noetherian ring*, $\bar{U}_q(sl_{N+1}(R))$ *is a left* (*right*) *Noetherian ring.*

(*c*) *If* $q^8 - 1 \in R^\times$, *then* $L = (0)$.

Proof. First we prove (*a*). Let v be an indeterminate, $\mathbb{C}(v)$ the field of rational functions. Let $\mathscr{A}$ be the $\mathbb{Z}$-subalgebra of $\mathbb{C}(v)$ generated by 1, $v^{\mp 1}$, $(v^4 - 1)^{-1}$. We define $U_{\mathscr{A}}$ to be the $\mathscr{A}$-submodule of $U_v(sl_{N+1}(\mathbb{C}(v))$ generated by the elements (6.1). From the arguments in §5, we see that $U_{\mathscr{A}}$ is an $\mathscr{A}$-algebra with a free $\mathscr{A}$-basis (6.1). We now define, for R and q, $U_{R,q} = U_{\mathscr{A}} \otimes_{\mathscr{A}} R_q$, where R_q is R, regarded as an $\mathscr{A}$-algebra with v (resp. 1) acting as multiplication by q (resp. 1). Since $U_{R,q}$ satisfies the formulas $[e_{i,i+3}, e_{i+1}] = [f_{i,i+3}, f_{i+1}] = 0$, we can define the epimorphism φ: $\bar{U}_q(sl_{N+1}(R)) \to U_{R,q}$ by $\varphi(e_i) = e_i \otimes 1$, $\varphi(f_i) = f_i \otimes 1$, $\varphi(k_i^{\mp 1}) = k_i^{\mp 1} \otimes 1$. Hence the elements (6.1) are linearly independence over R. We know that the elements e_{ij}, f_{ij} and k_i of $\bar{U}_q(sl_{N+1}(R))$ satisfies the formulas (1), (2), (3), (4) in §3. Hence, defining the filtration on $\bar{U}_q(sl_{N+1}(R))$ similar to $\{\mathfrak{A}_{(m,n)}\}_{(m,n) \in \Omega_N}$ in §5, we see that, as an R-module, $\bar{U}_q(sl_{N+1}(R))$ is generated by the elements (6.1). This completes the proof of (*a*).

We obtain (*b*) from the same argument as in §5, and (*c*) from the formula (3.3).

For $U_q(sl_{N+1}(F))$ over a field F, we have the following supplementary result:

Proposition 6.2. *Assume* $q \in F^\times$ *is a primitive* 8*-th root of unity. Then the elements*

$$f_{m_1,n_1} f_{m_2,n_2} \cdots f_{m_s,n_s} f_1^{2b_1} \cdots f_N^{2b_N} k_1^{\ell_1} \cdots k_N^{\ell_N} e_{i_1,j_1} e_{i_2,j_2} \cdots e_{i_t,j_t} e_1^{2c_1} \cdots e_N^{2c_N}$$

$$(m_p < n_{p+1} - 1\, (1 \le p \le s-1),\ i_r < j_{r+1} - 1\ (1 \le r \le t-1),$$

$$b_1, \ldots, b_N,\ c_1, \ldots, c_N \ge 0,\ \ell_1, \ldots, \ell_N \in \mathbb{Z})$$

form a basis of $U_q(sl_{N+1}(F))$.

The proof is based on Proposition 2.3 and an argument similar to the one used in the proof of Lemma 2.1, 2.2; we omit the details.

From Proposition 6.2, we obtain:

Proposition 6.3. *Assume q to be an indeterminate. Let $\mathscr{A} = F[q^{\pm 1}, (q^4 - 1)^{-1}]$. Then, if $N \geq 3$, $U_q(sl_{N+1}(\mathscr{A}))$ is not free as an $\mathscr{A}$-module.*

Proof. By the argument of (3.3), we see

$$(q^2 + q^{-2})[e_{i,i+3}, e_{i+1,i+2}] = 0.$$

Let $\zeta \in F$ be a primitive 8-th root of unity. Define the F-algebra homomorphism p: $U_q(sl_{N+1}(\mathscr{A})) \to U_\zeta(sl_{N+1}(F))$ by $p(e_i) = e'_i$, $p(f_i) = f'_i$, $p(k_i^{\pm 1}) = k'^{\pm 1}_i$, $p(q) = \zeta$ where e'_i, f'_i, $k'^{\pm 1}_i$ $(1 \leq i \leq N)$ are generators in the definition of $U_\zeta(sl_{N+1}(F))$. By Proposition 6.2, we have

$$p([e_{i,i+3}, e_{i+1,i+2}]) = e'_{i,i+3}e'_{i+1,i+2} - e'_{i+1,i+2}e'_{i,i+3} \neq 0.$$

Hence $[e_{i,i+3}, e_{i+1,i+2}] \neq 0$, which shows that $U_q(sl_{N+1}(\mathscr{A}))$ is not free.

§7. On $U_q(so_5(F))$

Let A be the Cartan matrix of type B_2, namely, $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. Put $U_q(so_5(F)) = U_q(A, \mathrm{diag}(1, 2))$ where $q^8 \neq 1$. Define the elements E_i, F_i $(1 \leq N)$ by

$$E_1 = e_1,\ E_2 = e_2,\ E_3 = e_1e_2 - q^4e_2e_1,\ E_4 = e_1E_3 - q^{-4}E_3e_1,$$

$$F_1 = f_1,\ F_2 = f_2,\ F_3 = f_1f_2 - q^4f_2f_1,\ F_4 = f_1F_3 - q^{-4}F_3f_1.$$

Proposition 7.1. (*a*) *The elements* $F_1^{m_1}F_2^{m_2}F_3^{m_3}F_4^{m_4} \cdot k_1^{\ell_1}k_2^{\ell_2} \cdot E_1^{i_1}E_2^{i_2}E_3^{i_3}E_4^{i_4}$ $(m_s, i_s \geq 0,\ \ell_1, \ell_2 \in \mathbf{Z})$ *form a basis of* $U_q(so_5(F))$.

(*b*) $U_q(so_5(F))$ *is a left* (*right*) *Noetherian ring, and has no zero divisors* $\neq 0$.

Let V_i be the subspace of $U_q(so_5(F))$ spanned by the elements $F_1^{m_1}F_2^{m_2}F_3^{m_3}F_4^{m_4} \cdot k_1^{\ell_1}k_2^{\ell_2} \cdot E_1^{i_1}E_2^{i_2}E_3^{i_3}E_4^{i_4}$ such that $2(m_1 + i_1) + (m_2 + i_2) + 2(m_3 + i_3) + 3(m_4 + i_4) \leq i$. The proofs of (*a*) and (*b*) are obtained by using the filtration $\{V_i\}_{i=0}^{\infty}$ of $U_q(so_5(F))$. This is similar to those of Theorem 1.1 and 1.2. The details are omitted.

Acknowledgements.

The author would like to express his heartfelt thanks to Professor M. Jimbo for showing him some useful formulas which were necessary in proving our theorem, and to Professor A. Gyoja for pointing out that the Noetherian property of $U_q(sl_{N+1}(F))$ follows easily from its filtration. He also thanks Y. Izumi for some very interesting discussion.

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