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New L^p-Estimates for Solutions to the Schrödinger Equations and Time Asymptotic Behavior of Observables

By

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Introduction

We consider the Schrödinger operator $H=H_0+V$ in the Hilbert space $L^2(\boldsymbol{R}^n)$, $n\geqq 1$, where H_0 is the self-adjoint realization of $-\Delta$ in $L^2(\boldsymbol{R}^n)$ and V is a symmetric operator with H_0-bound less than one. This paper is mainly devoted to obtaining detailed informations about the asymptotic behavior in time of the following quantities:

$$\|e^{-itH}\phi\|_{L^p(\boldsymbol{R}^n)}, \qquad 1\leqq p\leqq\infty, \tag{1}$$

$$(e^{itH}Ve^{-itH}\phi, \phi)_{L^2(\boldsymbol{R}^n)}, \tag{2}$$

$$\|(x/2it+\nabla)e^{-itH}\phi\|_{L^2(\boldsymbol{R}^n)}, \tag{3}$$

where $\phi\in L^2(\boldsymbol{R}^n)$ is an appropriate initial datum. We obtained some new estimates for (1)-(3).

For a nice initial datum ϕ, it is reasonable to expect that $e^{-itH}\phi\in L^p(\boldsymbol{R}^n)$ for all $t\in\boldsymbol{R}$ and that the map $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p(\boldsymbol{R}^n)$ is continuous even when $p\neq 2$. By taking the asymptotic behavior of the Schrödinger free evolution group into account, it is natural to think that for scattering solutions (i.e., $e^{-itH}\phi$ with ϕ orthogonal to any eigenvector of H), (1)-(3) decay as $t\to\pm\infty$ provided $2<p\leqq\infty$. The local L^2-decay of scattering solutions has been extensively studied by many mathematicians (see [14][15][16][17][18][20][24][25][26][29]). The L^p-decay estimate, however, has not been fully investigated (see [2][3][32] for some special results in three space dimensions). Concerning scattering solutions, it is known that the expectation of the potential energy (2) and the difference between the momentum of classical mechanics and that of quantum mechanics (3) decay as $t\to\pm\infty$ under suitable conditions (see [4][6][27]). Time asymptotic behavior of observables under the Schrödinger group has been of importance in proving asymptotic completeness of the wave operators

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(see [5][6][27][33]).

The contents of this paper are as follows. In § 1 we give some preliminary lemmas. In § 2 we study the Schrödinger evolution group in the weighted Sobolev spaces. In § 3 we state the pseudo-conformal conservation law for the Schrödinger equation. The pseudo-conformal conservation law for some class of non-linear Schrödinger equations was first observed by Ginibre & Velo [8]. Then, the non-linear theories have been developed by many authors (see [9][10][11][37]). In § 4 we establish the general existence theory of L^p-solutions to the Schrödinger equations. Various L^p-estimates will be obtained. We use these estimates in § 5 and § 6. We investigate scattering solutions in § 5. Sufficient conditions for the quantities (1)-(3) to decay in time will be given. The purpose of § 6 is to determine the rate of decay of (1)-(3) for short-range potentials and repulsive potentials.

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§ 1. Preliminaries

In this section we collect some preliminary lemmas. We begin with making some notational conventions. Throughout the paper we *always* assume that V is a symmetric operator of multiplication in the Hilbert space L^2 and is H_0-bounded with H_0-bound less than one. H denotes the self-adjoint operator defined by the operator sum $H=H_0+V$. $\mathcal{H}_{ac}(H)$ and $\mathcal{H}_c(H)$ denote the absolutely continuous spectral subspace of H and the continuous spectral subspace of H, respectively. For any real function we denote by the same symbol the operator of multiplication by that function when this causes no confusion.

$\boldsymbol{N}$ denotes the set of positive integers. $\binom{j}{k}=j!/k!(j-k)!$ for $j, k\in\boldsymbol{N}\cup\{0\}$, $j\geqq k$. For $s\in\boldsymbol{R}$, we denote by $[s]$ the largest integer less than or equal to s. p' denotes the conjugate exponent to $p\in[1, \infty]$. $\delta_n(p)=n/2-n/p$. Let E and F be Banach spaces. $\mathcal{L}(E\,;F)$ denotes the Banach space of continuous linear operators from E to F. We abbreviate $\mathcal{L}(E\,;E)$ by $\mathcal{L}(E)$. $C(I\,;E)$ denotes the Fréchet space of continuous functions from an interval $I\subset\boldsymbol{R}$ to E. For a linear operator T from E to F we denote by $D(T)$ its domain.

∂_j denotes the distributional derivative with respect to the j-th variable. $\nabla=$

$(\partial_1, \cdots, \partial_n)$. $\Delta=\sum_{j=1}^{n}\partial_j^2$. $\sum_{j=k}^{l}a_j=0$ if $k\geqq l$. $x=(x_1, \cdots, x_n)$, $|x|=(\sum_{j=1}^{n}x_j^2)^{1/2}$, $\omega(x)=(1+|x|^2)^{1/2}$, for $x\in\boldsymbol{R}^n$. For any multi-index $\alpha\in(\boldsymbol{N}\cup\{0\})^n$, we follow the usual conventions:

$$|\alpha|=\sum_{j=1}^{n}\alpha_j,\quad \alpha!=\prod_{j=1}^{n}\alpha!,\quad \binom{\alpha}{\beta}=\prod_{j=1}^{n}\binom{\alpha_j}{\beta_j},\quad \partial^\alpha=\prod_{j=1}^{n}\partial_j^{\alpha_j},\quad x^\alpha=\prod_{j=1}^{n}x_j^{\alpha_j}.$$

$\mathcal{S}$ denotes the Fréchet space of rapidly decreasing C^∞-functions from $\boldsymbol{R}^n$ to $\boldsymbol{C}$. $\mathcal{S}'$ denotes the dual of $\mathcal{S}$.

L^p denotes the Lebesgue space $L^p(\boldsymbol{R}^n)$ or $L^p(\boldsymbol{R}^n)\otimes\boldsymbol{C}^n$, with the norm denoted by $\|\cdot\|_p$, $1\leqq p\leqq\infty$. $L^p\cap L^q$ denotes the Banach space with the norm $\|\cdot\|_{L^p\cap L^q}=\|\cdot\|_p+\|\cdot\|_q$, $1\leqq p<q\leqq\infty$. For $m, s\in\boldsymbol{R}$, the weighted Sobolev space is defined by

$$H^{m,s}=\{\phi\in\mathcal{S}';\ \|\phi\|_{m,s}=\|(1-\Delta)^{m/2}\phi\|_2+\|\omega^s\phi\|_2<\infty\}.$$

$(\cdot, \cdot)$ denotes the L^2-scalar product and various pairings between L^p and $L^{p'}$ $(1<p<\infty)$, $H^{m,0}$ and $H^{-m,0}$ $(m\in\boldsymbol{R})$, $H^{0,s}$ and $H^{0,-s}$ $(s\in\boldsymbol{R})$, $\mathcal{S}$ and $\mathcal{S}'$. e.t.c.
$D_j=\bigcap_{m+|\alpha|\leqq j}D(x^\alpha H^m)$ denotes the Banach space with the norm

$$\|\phi\|_{D_j}=\sup_{m+|\alpha|\leqq j}\|x^\alpha H^m\phi\|_2,\qquad j\in\boldsymbol{N}.$$

A denotes the generator of dilations: $A=(1/2i)(x\cdot\nabla+\nabla\cdot x)$.
$\mathcal{F}$ denotes the Fourier transform defined according to the normalization:

$$(\mathcal{F}\phi)(\xi)=(2\pi)^{-n/2}\int e^{-ix\cdot\xi}\phi(x)dx,\qquad \xi\in\boldsymbol{R}^n.$$

Different positive constants might be denoted by the same letter C. If necessary, by $C(*, \cdots, *)$, we indicate the dependence of the constants on the quantities appearing in the parentheses.

Lemma 1.1. *Let $q, r\in[1, \infty]$. Let $j, m\in\boldsymbol{N}\cup\{0\}$ satisfy $0\leqq j<m$. Let p and a satisfy*

$$1/p=j/n+a(1/r-m/n)+(1-a)/q;$$

$$j/m\leqq a<1\quad \text{if } m-j-n/r\in\boldsymbol{N}\cup\{0\},\ j/m\leqq a\leqq 1 \text{ otherwise}.$$

Let $\phi\in L^q$ satisfy $\partial^\beta\phi\in L^r$, $|\beta|=m$. Then $\partial^\alpha\phi\in L^p$, $|\alpha|=j$, and moreover,

$$\sum_{|\alpha|=j}\|\partial^\alpha\phi\|_p\leqq C(n, m, j, q, r)\sum_{|\beta|=m}\|\partial^\beta\phi\|_r^a\|\phi\|_q^{1-a}.$$

For Lemma 1.1, see, e.g., A. Friedman [7].

Lemma 1.2 (C.S. Lin [22]). (1) *Let $j\in\boldsymbol{N}$. Let $k, l\in\boldsymbol{N}\cup\{0\}$ satisfy $k+l=j$. Let $\phi\in H^{j,j}$. Then $x^\alpha\partial^\beta\phi\in L^2$, $|\alpha|=k$, $|\beta|=l$, and moreover,*

$$\sum_{\substack{|\alpha|=k\\|\beta|=l}}\|x^\alpha\partial^\beta\phi\|_2\leqq C(n, j)\||x|^j\phi\|_2^{k/j}\|\phi\|_{j,0}^{l/j}.$$

(2) *Let* $s\geqq 0$, $m\in\boldsymbol{R}$. *Let* $\psi\in H^{0,s}$ *satisfy* $\omega^s\partial^\alpha\psi\in L^2$, $|\alpha|=m$. *Then* $\omega^s\partial^\beta\psi\in L^2$, $|\beta|\leqq m-1$, *and moreover,*

$$\sum_{|\beta|=k}\|\omega^s\partial^\beta\psi\|_2\leqq C(n, m, s)\sum_{|\alpha|=m}\|\omega^s\partial^\alpha\psi\|_2^{k/m}\|\psi\|_{0,s}^{1-k/m}, \qquad 1\leqq k\leqq m-1.$$

Lemma 1.3 (W. Hunziker [12]). *Let* $j\in\boldsymbol{N}$. *Then*:

(1)
$$e^{-itH}(D_j)\subset D_j, \qquad t\in\boldsymbol{R},$$
$$\|e^{-itH}\phi\|_{D_j}\leqq C(n, j)(1+|t|)^j\|\phi\|_{D_j}, \qquad t\in\boldsymbol{R},\ \phi\in D_j.$$

(2) *For any* $\phi\in D_j$ *the map* $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in D_j$ *is continuous.*

(3) *For any* $\phi\in D_j$,

$$x^\alpha e^{-itH}\phi=e^{-itH}x^\alpha\phi+i\int_0^t e^{-i(t-\tau)H}[H, x^\alpha]e^{-i\tau H}\phi\,d\tau, \qquad |\alpha|\leqq j,$$

where the commutator is defined as $[H, x^\alpha]=-2(\nabla x^\alpha)\cdot\nabla-(\Delta x^\alpha)$.

Lemma 1.4 (W. Hunziker [12]). *Let* $\zeta_\varepsilon(x)=\exp(-|\varepsilon x|^2)$, $\varepsilon>0$, $x\in\boldsymbol{R}^n$. *Let* $u\in C(\boldsymbol{R}; L^2)$. *Then*:

(1) $\zeta_\varepsilon u\to u$ *in* $C(\boldsymbol{R}; L^2)$ *as* $\varepsilon\to+0$.

(2) *For* $j\in\boldsymbol{N}$, $\omega^j\zeta_\varepsilon u\to 0$ *in* $C(\boldsymbol{R}; L^2)$ *as* $\varepsilon\to+0$.

§2. Some Properties of the Time Evolution e^{-itH} in the Weighted Sobolev Spaces

In this section we describe some properties of e^{-itH} in the weighted Sobolev spaces. For this purpose we use the square root $|H|^{1/2}$ of the operator $|H|$. It is well known that $D(H)=D(H_0)=H^{2,0}$ and that there exist constants $0<a<1$ and $b>0$ such that

$$|(V\psi, \psi)|\leqq a\|\nabla\psi\|_2^2+b\|\psi\|_2^2, \qquad \psi\in D(H). \tag{2.1}$$

This implies $(H\psi, \psi)\geqq -b\|\psi\|_2^2$, $\psi\in D(H)$, so that for $\lambda>\lambda_0\equiv 1+b$, $H+\lambda$ is a positive operator in L^2. Indeed, $\|(H+\lambda)\psi\|_2\|\psi\|_2\geqq((H+\lambda)\psi, \psi)\geqq(\lambda-b)\|\psi\|_2^2$, $\psi\in D(H)$. Moreover,

$$\begin{aligned} C\|(H_0+1)\psi\|_2&\leqq\|H\psi\|_2+\|\psi\|_2\leqq\|(H+\lambda)\psi\|_2+(1+\lambda)\|\psi\|_2 \\ &\leqq(1+(1+\lambda)(\lambda-b)^{-1})\|(H+\lambda)\psi\|_2, \qquad \psi\in D(H). \end{aligned}\tag{2.2}$$

On the other hand,

$$\|(H+\lambda)\psi\|_2\leqq C\|(H_0+1)\psi\|_2, \qquad \psi\in H^{2,0}, \tag{2.3}$$

By the Heinz-Kato theorem [35], we obtain from (2.2) and (2.3) that

$$D((H+\lambda)^{1/2})=D((H_0+1)^{1/2})=H^{1,0},$$

$$C(\lambda)\|\psi\|_{1,0}\leqq\|(H+\lambda)^{1/2}\psi\|_2\leqq C'(\lambda)\|\psi\|_{1,0}, \qquad \psi\in H^{1,0}.$$

Similarly, from the relation (see T. Kato [19; p. 335])

$$\||H|\psi\|_2=\|H\psi\|_2\leqq\|(H+\lambda)\psi\|_2\leqq\|(|H|+\lambda)\psi\|_2, \quad \psi\in D(|H|)=D(H), \tag{2.4}$$

we obtain $D(|H|^{1/2})=D((H+\lambda)^{1/2})$,

$$\begin{aligned}\||H|^{1/2}\psi\|_2&\leqq\|(H+\lambda)^{1/2}\psi\|_2\leqq\|(|H|+\lambda)^{1/2}\psi\|_2\\&=(\||H|^{1/2}\psi\|_2^2+\lambda\|\psi\|_2^2)^{1/2},\\&\leqq\lambda\|(|H|^{1/2}+i)\psi\|_2, \qquad \psi\in D(|H|^{1/2}).\end{aligned} \tag{2.5}$$

We have thus proved that for $j=1, 2$,

$$D(|H|^{j/2})=H^{j,0}, \tag{2.6}$$

$$C\|\psi\|_{j,0}\leqq\|(|H|^{j/2}+i)\psi\|_2\leqq C'\|\psi\|_{j,0}, \qquad \psi\in H^{j,0}. \tag{2.7}$$

It follows from (2.1) that

$$|(V\phi, \psi)|\leqq C\|\phi\|_{1,0}\|\psi\|_{1,0}, \qquad \phi, \psi\in D(H),$$

and therefore V extends to a bounded operator from $H^{1,0}$ to $H^{-1,0}$, which will be also denoted by V in the sequel. Now we have:

Theorem 2.1. (1) *Let $j=1$ or 2. Then:*

$$e^{-itH}(H^{j,0})\subset H^{j,0}, \qquad t\in\boldsymbol{R},$$

$$\|e^{-itH}\phi\|_{j,0}\leqq C\|\phi\|_{j,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{j,0}. \tag{2.8}$$

For any $\phi\in H^{j,0}$ the map $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in H^{j,0}$ is continuous.

(2)
$$e^{-itH}(H^{1,1})\subset H^{0,1}, \qquad t\in\boldsymbol{R},$$

$$\|e^{-itH}\phi\|_{0,1}\leqq C\|\phi\|_{0,1}+C|t|\,\|\phi\|_{1,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{1,1}. \tag{2.9}$$

For any $\phi\in H^{1,1}$ the map $\boldsymbol{R}\ni t\to e^{-itH}\phi\in H^{0,1}$ is continuous, $\boldsymbol{R}\ni t\mapsto e^{itH}xe^{-itH}\phi\in L^2$ is continuously differentiable, and

$$\frac{d}{dt}(e^{itH}xe^{-itH}\phi)=-2ie^{itH}\nabla e^{-itH}\phi, \qquad t\in\boldsymbol{R}. \tag{2.10}$$

(3)
$$Ve^{-itH}(H^{1,0})\subset H^{-1,0}, \qquad t\in\boldsymbol{R},$$

$$\|Ve^{-itH}\phi\|_{-1,0}\leqq C\|\phi\|_{1,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{1,0}.$$

Remark 1.1. Estimates (2.8) and (2.9) have been proved by C. Radin & B. Simon [30]. But they have not studied the continuity properties in t of e^{-itH}, which will be needed in the proof of Theorem 3.1.

Proof of Theorem 2.1. (1) Let $\phi\in H^{j,0}$. It follows from (2.5) that

$$\begin{aligned}\|e^{-itH}\phi\|_{j,0}&\leqq C\|(|H|^{j/2}+i)e^{-itH}\phi\|_2\\&=C\|e^{-itH}(|H|^{j/2}+i)\phi\|_2=C\|(|H|^{j/2}+i)\phi\|_2\leqq C\|\phi\|_{j,0},\\ \|e^{-itH}\phi-e^{-isH}\phi\|_{j,0}&\leqq C\|(|H|^{j/2}+i)(e^{-itH}-e^{-isH})\phi\|_2\\&=C\|(e^{-itH}-e^{-isH})(|H|^{j/2}+i)\phi\|_2, \qquad t,\ s\in\boldsymbol{R}.\end{aligned}$$

These inequalities prove part (1).

(2) Let $\phi\in H^{1,1}$ and let $\{\phi_j\}$ be a sequence in $H^{2,1}=D_1$ such that $\phi_j\to\phi$ in $H^{1,1}$ as $j\to\infty$. Let $u(t)=e^{-itH}\phi$ and let $u_j(t)=e^{-itH}\phi_j$, $t\in\boldsymbol{R}$, $j\in N$. By part (2) of Lemma 1.3, $u_j\in C(\boldsymbol{R}; H^{2,1})$ and

$$xu_j(t)=e^{-itH}x\phi_j-2i\int_0^t e^{-i(t-\tau)H}\nabla u_j(\tau)d\tau, \qquad t\in\boldsymbol{R}, \tag{2.11}$$

By (2.8) and (2.11),

$$\begin{aligned}\|xu_j(t)-xu_k(t)\|_2&\leqq\|x\phi_j-x\phi_k\|_2+C|t|\sup_{|\tau|\leq|t|}\|\nabla u_j(t)-\nabla u_k(t)\|_2\\&\leqq\|\phi_j-\phi_k\|_{0,1}+C|t|\,\|\phi_j-\phi_k\|_{1,0}\longrightarrow 0 \quad \text{as } j,\ k\to\infty.\end{aligned}\tag{2.12}$$

On the other hand,

$$\sup_{t\in\boldsymbol{R}}\|u_j(t)-u(t)\|_2=\|\phi_j-\phi\|_2\longrightarrow 0 \qquad \text{as } j\to\infty.$$

Since the multiplication operator x is closed, we find $xu(t)\in L^2$ and $xu_j(t)\to xu(t)$ in L^2 as $j\to\infty$. In the same way as in (2.12), we see that the R. H. S. (=right hand side) of (2.11) tends to

$$e^{-itH}\phi-2i\int_0^t e^{-i(t-\tau)H}\nabla u(\tau)d\tau$$

in L^2 uniformly on compact t-intervals as $j\to\infty$. Therefore, $xu\in C(\boldsymbol{R}; L^2)$ and

$$xu(t)=e^{-itH}x\phi-2i\int_0^t e^{-i(t-\tau)H}\nabla u(\tau)d\tau, \qquad t\in\boldsymbol{R}. \tag{2.13}$$

It follows from part (1) that $\boldsymbol{R}\ni t\mapsto e^{itH}\nabla u(t)\in L^2$ is continuous. (2.10) now follows from (2.13). (2.9) follows by estimating the R. H. S. of (2.13) in the L^2-norm.

(3) Since we already know that $V\in\mathcal{L}(H^{1,0}; H^{-1,0})$, part (3) reduces to part (1).

Q. E. D.

Theorem 2.2. $e^{-itH}(H^{2,2})\subset H^{0,2}$, $t\in\boldsymbol{R}$,

$$\|e^{-itH}\phi\|_{0,2}\leqq C\|\phi\|_{0,2}+C|t|^2\|\phi\|_{2,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{2,2}. \tag{2.14}$$

For any $\phi\in H^{2,2}$, *the map* $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in H^{0,2}$ *is continuous,* $\boldsymbol{R}\ni t\mapsto e^{itH}|x|^2e^{-itH}\phi\in L^2$ *is continuously differentiable, and*

(2.15) $$\frac{d}{dt}(e^{itH}|x|^2e^{-itH}\phi)=4e^{itH}Ae^{-itH}\phi\,, \qquad t\in\boldsymbol{R}.$$

Although a formal proof of (2.14) proceeds exactly as in [30], we give here a rigorous proof for the sake of completeness. The proof proceeds in several steps. We first approximate $u(t)=e^{-itH}\phi$ by $u_\lambda(t)=i\lambda(H+i\lambda)^{-1}u(t)$ and prove that $Au_\lambda\in C(\boldsymbol{R}; L^2)$. We next show that $\{|x|^2u_\lambda(t); \lambda>1\}$ is bounded in L^2. This in turn implies that $|x|^2u(t)\in L^2$ and $Au\in C(\boldsymbol{R}; L^2)$. We then prove that $u\in C(\boldsymbol{R}; H^{0,2})$ and (2.15) by obtaining an identity for $|x|^2u(t)$. The proof of (2.14) uses a differential inequality associated with $\|(|x|^2+i)u(t)\|_2$. To this end we start with the following

Lemma 2.1. *Let $\lambda\in\boldsymbol{R}\setminus[-1, 1]$, $1\leqq j\leqq n$. Then:*

(1) $$\|\partial_j(H+i\lambda)^{-1}\|_{\mathcal{L}(L^2)}\leqq C|\lambda|^{-1/2}.$$

(2) $$(H+i\lambda)^{-1}(H^{0,1})\subset H^{0,1},$$

(2.16) $$x_j(H+i\lambda)^{-1}\phi=(H+i\lambda)^{-1}x_j\phi-2(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\phi\,, \qquad \phi\in H^{0,1}.$$

(3) $$\partial_j(H+i\lambda)^{-1}(H^{0,1})\subset H^{0,1},$$

(2.17) $$\begin{aligned} x_j\partial_j(H+i\lambda)^{-1}\phi=&\partial_j(H+i\lambda)^{-1}x_j\phi-(H+i\lambda)^{-1}\phi\\ &-2\partial_j(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\phi\,, \qquad \phi\in H^{0,1}.\end{aligned}$$

(4) $$(H+i\lambda)^{-1}(H^{0,2})\subset H^{0,2},$$

(2.18) $$\begin{aligned} x_j^2(H+i\lambda)^{-1}\phi=&(H+i\lambda)^{-1}x_j^2\phi+2(H+i\lambda)^{-2}\phi\\ &-4(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}x_j\phi\\ &+8(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\phi\,, \qquad \phi\in H^{0,2}.\end{aligned}$$

Proof. (1) For $\phi\in D(H)$, we have

$$\||H|^{1/2}\phi\|_2^2\leqq\||H|\phi\|_2\|\phi\|_2=\|H\phi\|_2\|\phi\|_2\,.$$

Therefore it follows from (2.7) that

$$\begin{aligned}\|\partial_j\phi\|_2^2&\leqq C\||H|^{1/2}\phi\|_2^2+C\|\phi\|_2^2\\ &\leqq C\|H\phi\|_2\|\phi\|_2+C|\lambda|\|\phi\|_2^2\\ &\leqq C|\lambda|^{-1}\|H\phi\|_2^2+C|\lambda|\|\phi\|_2^2\\ &\leqq C|\lambda|^{-1}(\|H\phi\|_2^2+\lambda^2\|\phi\|_2^2)=C|\lambda|^{-1}\|(H+i\lambda)\phi\|_2^2\,,\end{aligned}$$

form which part (1) follows.

(2) Let ζ_ε be as in Lemma 1.4. We have on L^2

$$\begin{aligned} x_j\zeta_\varepsilon &= x_j\zeta_\varepsilon(H+i\lambda)(H+i\lambda)^{-1} \\ &= ((H+i\lambda)x_j\zeta_\varepsilon+[x_j\zeta_\varepsilon,\ H+i\lambda])(H+i\lambda)^{-1} \\ &= (H+i\lambda)x_j\zeta_\varepsilon(H+i\lambda)^{-1}+[x_j\zeta_\varepsilon,\ H_0](H+i\lambda)^{-1}, \end{aligned}$$

so that

$$\begin{aligned} (2.19)\qquad (H+i\lambda)^{-1}x_j\zeta_\varepsilon &= x_j\zeta_\varepsilon(H+i\lambda)^{-1}+2(H+i\lambda)^{-1}\zeta_\varepsilon\partial_j(H+i\lambda)^{-1} \\ &\quad -2(H+i\lambda)^{-1}((n+2-2|\varepsilon x|^2)\varepsilon^2x_j\zeta_\varepsilon+2\varepsilon^2x_j\zeta_\varepsilon(x\cdot\nabla))(H+i\lambda)^{-1}. \end{aligned}$$

Let $\psi\in H^{0,1}$. By Lemma 1.4,

$$x_j\zeta_\varepsilon\psi \longrightarrow x_j\psi\,,\qquad \zeta_\varepsilon\partial_j(H+i\lambda)^{-1}\psi \longrightarrow \partial_j(H+i\lambda)^{-1}\psi\,,$$

$$((n+2-2|\varepsilon x|^2)\varepsilon^2x_j\zeta_\varepsilon+2\varepsilon^2x_j\zeta_\varepsilon(x\cdot\nabla))(H+i\lambda)^{-1}\psi \longrightarrow 0\,,$$

in L^2 as $\varepsilon\to+0$. Therefore from (2.19) we see that

$$x_j\zeta_\varepsilon(H+i\lambda)^{-1}\psi \longrightarrow (H+i\lambda)^{-1}x_j\psi-2(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\psi$$

in L^2 as $\varepsilon\to+0$. Since the multiplication operator x_j is closed, we obtain part (2).

(3) Let $\psi\in H^{0,1}$. By part (1), $(H+i\lambda)^{-1}x_j\psi$, $(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\psi\in H^{1,0}$ and thus, by (2.16), $x_j(H+i\lambda)^{-1}\psi\in H^{1,0}$. Consequently,

$$x_k\partial_j(H+i\lambda)^{-1}\psi=\partial_j(x_k(H+i\lambda)^{-1}\psi)-\delta_{jk}(H+i\lambda)^{-1}\psi\in L^2\,.$$

Part (3) now follows from part (2) and the equality above.

(4) Let $\psi\in H^{0,2}$. Then $x_j\psi$, $\partial_j(H+i\lambda)^{-1}\psi\in H^{0,1}$. By part (2), $(H+i\lambda)^{-1}x_j\psi$, $(H+i\lambda)^{-1}\partial_j(H+i\lambda)^{-1}\psi\in H^{0,1}$. By (2.16), $x_j(H+i\lambda)^{-1}\psi\in H^{0,1}$. We have thus proved that $(H+i\lambda)^{-1}$ keeps $H^{0,2}$ invariant. (2.18) follows by iterative use of (2.16) and (2.17). Q.E.D.

Proof of Theorem 2.2. Let $\phi\in H^{2,2}$ and let $\phi_\lambda=i\lambda(H+i\lambda)^{-1}\phi$, $\lambda>1$. Without loss of generality we assume that $\phi\neq0$. We define u and u_λ by $u(t)=e^{-itH}\phi$ and $u_\lambda(t)=e^{-itH}\phi_\lambda=i\lambda(H+i\lambda)^{-1}u(t)$, $t\in\boldsymbol{R}$, $\lambda>1$, respectively. Theorem 2.1 implies that u, $u_\lambda\in C(\boldsymbol{R};H^{2,1})$. Moreover, Lemma 2.1 shows that $Au_\lambda\in C(\boldsymbol{R};L^2)$. Since $i\lambda(H+i\lambda)^{-1}\to1$ strongly in $\mathcal{L}(L^2)$, we see that

$$\sup_{t\in\boldsymbol{R}}\|u_\lambda(t)-u(t)\|_2=\|\phi_\lambda-\phi\|_2 \longrightarrow 0 \qquad \text{as } \lambda\to\infty.$$

We now prove that $u(t)\in H^{0,2}$ for each $t\in\boldsymbol{R}$. Let ζ_ε be as in Lemma 1.4. Noting that $|x|^2\zeta_\varepsilon u_\lambda\in C^1(\boldsymbol{R};L^2)\cap C(\boldsymbol{R};H^{2,0})$,

$$i\frac{d}{dt}|x|^2\zeta_\varepsilon u_\lambda=|x|^2\zeta_\varepsilon Hu_\lambda,\quad \|(|x|^2\zeta_\varepsilon+i)u_\lambda(t)\|_2^2=\||x|^2\zeta_\varepsilon u_\lambda(t)\|_2^2+\|\phi_\lambda\|_2^2\,,$$

we have

$$\begin{aligned}\frac{d}{dt}\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2^2 = \frac{d}{dt}\||x|^2\zeta_\varepsilon u_\lambda(t)\|_2^2 \\ &= 2\,\mathrm{Re}\left(\frac{d}{dt}|x|^2\zeta_\varepsilon u_\lambda,\ |x|^2\zeta_\varepsilon u_\lambda\right) \\ &= 2\,\mathrm{Im}\,(|x|^2\zeta_\varepsilon H_0 u_\lambda,\ |x|^2\zeta_\varepsilon u_\lambda) \\ &= 2\,\mathrm{Im}\,([|x|^2\zeta_\varepsilon,\ H_0]u_\lambda,\ |x|^2\zeta_\varepsilon u_\lambda) \\ &= 8\,\mathrm{Im}\,((1-|\varepsilon x|^2)\zeta_\varepsilon x\cdot\nabla u_\lambda,\ |x|^2\zeta_\varepsilon u_\lambda) \\ &= -4\,\mathrm{Re}\,((1-|\varepsilon x|^2)\zeta_\varepsilon A u_\lambda,\ |x|^2\zeta_\varepsilon u_\lambda)\,.\end{aligned}$$

The Lipschitz continuous function $\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2$ is differentiable *a.e.* (=almost everywhere) and

$$\frac{d}{dt}\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2^2 = 2\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2\cdot\frac{d}{dt}\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2 \quad a.e.$$

Consequently,

$$\begin{aligned}\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2\cdot\left|\frac{d}{dt}\|(|x|^2\zeta_\varepsilon+i)u_\lambda(t)\|_2\right| \\ &\leqq 2\|(1-|\varepsilon x|^2)\zeta_\varepsilon A u_\lambda\|_2\||x|^2\zeta_\varepsilon u_\lambda\|_2 \\ &\leqq 2\|(1-|\varepsilon x|^2)\zeta_\varepsilon A u_\lambda\|_2\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2 \quad a.e.\end{aligned}$$

We devide both sides of the above inequality by $\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2$ $(\geqq\|\phi_\lambda\|_2>0)$ and integrate the resulting inequality with respect to t to get

$$\begin{aligned}(2.20)\qquad &\|(|x|^2\zeta_\varepsilon+i)u_\lambda\|_2 \\ &\leqq \|(|x|^2\zeta_\varepsilon+i)\phi_\lambda\|_2+2\left|\int_0^t\|(1-|\varepsilon x|^2)Au_\lambda(\tau)\|_2 d\tau\right|,\quad t\in\boldsymbol{R}.\end{aligned}$$

By Lemma 1.4, the R.H.S. of (2.20) tends to

$$(2.21)\qquad \|(|x|^2+i)\phi_\lambda\|_2+2\left|\int_0^t\|Au_\lambda(\tau)\|_2 d\tau\right|$$

as $\varepsilon\to+0$, since we already know from Lemma 2.1 that $\phi_\lambda\in H^{0,2}$, $Au_\lambda\in C(\boldsymbol{R};L^2)$. Fatou's lemma then shows that $(|x|^2+i)u_\lambda(t)$ is in L^2 and is estimated in the L^2-norm by (2.21). We estimate the integrand in (2.21). By part (1) of Lemma 1.2 and (2.8),

$$\begin{aligned}\|Au_\lambda(\tau)\|_2 &\leqq C\|(|x|^2+i)u_\lambda(\tau)\|_2^{1/2}\|u_\lambda(\tau)\|_{2,0}^{1/2} \\ &\leqq C\|(|x|^2+i)u_\lambda(\tau)\|_2^{1/2}\|\phi_\lambda\|_{2,0}^{1/2}.\end{aligned}$$

By (2.2) and (2.3), we have for $\lambda_0=1+b$ (see (2.1)),

$$\begin{aligned}\|\phi_\lambda\|_{2,0} &\leqq C\|(H+\lambda_0)\phi_\lambda\|_2 = C\|i\lambda(H+i\lambda)^{-1}(H+\lambda_0)\phi\|_2 \\ &\leqq C\|(H+\lambda_0)\phi\|_2\leqq C\|\phi\|_{2,0}.\end{aligned}$$

Combining these estimates, we obtain

$$\|(|x|^2+i)u_\lambda(t)\|_2 \leqq \|\phi_\lambda\|_{0,2}+C|t|\,\|\phi\|_{2,0}+C\left|\int_0^t \|(|x|^2+i)u_\lambda(\tau)\|_2 d\tau\right|$$

Gronwall's lemma now gives

$$\|(|x|^2+i)u_\lambda(t)\|_2 \leqq C\exp(C|t|)(\|\phi_\lambda\|_{0,2}+\|\phi\|_{2,0}), \qquad t\in\boldsymbol{R}.$$

where C is independent of $\lambda>1$ and $t\in\boldsymbol{R}$. By Lemma 2.1, $\sup_{\lambda>1}\|\phi_\lambda\|_{0,2}\leqq C\|\phi\|_{0,2}$. Therefore, again by Fatou's lemma, $(|x|^2+i)u(t)$ is in L^2 and is estimate in the L^2-norm by $C\exp(C|t|)\|\phi\|_{2,2}$, as was to be shown. We next prove that $Au\in C(\boldsymbol{R};L^2)$. This follows from the $H^{2,0}$-continuity and $H^{0,2}$-boundedness of u. Indeed, by part (1) of Lemma 1.2,

$$\begin{aligned}\|Au(t)-Au(s)\|_2 &\leqq C\|u(t)-u(s)\|_{0,2}^{1/2}\|u(t)-u(s)\|_{2,0}^{1/2}\\ &\leqq C(\|u(t)\|_{0,2}+\|u(s)\|_{0,2})^{1/2}\|u(t)-u(s)\|_{2,0}^{1/2}\\ &\longrightarrow 0 \qquad \text{as } t\to s\,.\end{aligned}$$

Now the rest of the proof proceeds with some modifications of the arguments in [12] and [30]. We compute

$$\begin{aligned}\frac{d}{dt}(e^{itH}|x|^2\zeta_\varepsilon u(t)) &= ie^{itH}[H,\ |x|^2\zeta_\varepsilon]u(t)\\ &= -2ie^{itH}((n-(n+4)|\varepsilon x|^2+2|\varepsilon x|^4)\zeta_\varepsilon+2(1-|\varepsilon x|^2)\zeta_\varepsilon x\cdot\nabla)u(t)\,.\end{aligned}$$

This leads to

$$\begin{aligned}(2.22)\qquad |x|^2\zeta_\varepsilon u(t) &= e^{-itH}|x|^2\zeta_\varepsilon\phi\\ &\quad -2i\int_0^t e^{-i(t-\tau)H}\Big((n-(n+4)|\varepsilon x|^2+2|\varepsilon x|^4)\zeta_\varepsilon+2(1-|\varepsilon x|^2)\zeta_\varepsilon x\cdot\nabla\Big)u(\tau)d\tau\,.\end{aligned}$$

The L. H. S. of (2.22) tends to $|x|^2u(t)$ in L^2 as $\varepsilon\to+0$ for each $t\in\boldsymbol{R}$, while by Lemma 1.4, the R. H. S. of (2.22) tends to

$$e^{-itH}|x|^2\phi-4\int_0^t e^{-i(t-\tau)H}Au(\tau)d\tau$$

in L^2 as $\varepsilon\to+0$ uniformly on compact t-intervals. This proves $u\in C(\boldsymbol{R};H^{0,2})$ and (2.15). We turn to (2.14). In the same way as before, we have

$$\begin{aligned}\frac{d}{dt}\|(|x|^2\zeta_\varepsilon+i)u(t)\|_2^2 &= 8\,\mathrm{Im}\,((1-|\varepsilon x|^2)\zeta_\varepsilon x\cdot\nabla u(t),\ |x|^2\zeta_\varepsilon u(t))\\ &= 8\,\mathrm{Im}\,(x\cdot\nabla(\zeta_\varepsilon u(t))-|\varepsilon x|^2\zeta_\varepsilon x\cdot\nabla u(t),\ |x|^2\zeta_\varepsilon u(t))\,,\end{aligned}$$

and hence

$$\begin{aligned}&\left|\frac{d}{dt}\|(|x|^2\zeta_\varepsilon+i)\,u\|_2\right|\\ &\leqq 4\|x\cdot\nabla(\zeta_\varepsilon u)\|_2+4\|\,|\varepsilon x|^2\zeta_\varepsilon x\cdot\nabla u\|_2\end{aligned}$$

$$\leqq C\||x|^2\zeta_\varepsilon u\|_2^{1/2}\|\zeta_\varepsilon u\|_{2,0}^{1/2}+4\||\varepsilon x|^2\zeta_\varepsilon x\cdot\nabla u\|_2$$

$$\leqq C\|(|x|^2\zeta_\varepsilon+i)u\|_2^{1/2}\|\zeta_\varepsilon u\|_{2,0}^{1/2}+4\||\varepsilon x|^2\zeta_\varepsilon x\cdot\nabla u\|_2$$

$$\leqq C\|(|x|^2\zeta_\varepsilon+i)u\|_2^{1/2}(\|\zeta_\varepsilon u\|_{2,0}^{1/2}+\|\phi\|_2^{-1/2}\||\varepsilon x|^2\zeta_\varepsilon x\cdot\nabla u\|_2) \qquad a.e.$$

Noting that $(d/dt)\|(|x|^2\zeta_\varepsilon+i)u\|_2=2\|(|x|^2\zeta_\varepsilon+i)u\|_2^{1/2}\cdot(d/dt)\|(|x|^2\zeta_\varepsilon+i)u\|_2^{1/2}$ *a.e.*, we devide both sides of the above inequality by $\|(|x|^2\zeta_\varepsilon+i)u\|_2^{1/2}$ ($\geqq\|u\|_2^{1/2}=\|\phi\|_2^{1/2}>0$) and integrate the resulting inequality with respect to t to obtain

$$\|(|x|^2\zeta_\varepsilon+i)u(t)\|_2^{1/2}\leqq\|(|x|^2\zeta_\varepsilon+i)\phi\|_2^{1/2}$$
$$+C\left|\int_0^t(\|\zeta_\varepsilon u(\tau)\|_{2,0}^{1/2}+\|\phi\|_2^{-1/2}\||\varepsilon x|^2\zeta_\varepsilon x\cdot\nabla u(\tau)\|_2)d\tau\right|, \qquad t\in\boldsymbol{R}.$$

Taking the limit $\varepsilon\to+0$ in the inequality above, we find

$$\|(|x|^2+i)u(t)\|_2^{1/2}\leqq\|(|x|^2+i)\phi\|_2^{1/2}+C\left|\int_0^t\|u(\tau)\|_{2,0}^{1/2}d\tau\right|, \qquad t\in\boldsymbol{R}.$$

This together with (2.8) yields (2.14). Q.E.D.

Theorem 2.3. *Let $j\in\boldsymbol{N}$. Then*:

(1) $$e^{-itH_0}(H^{j,0})\subset H^{j,0}, \qquad t\in\boldsymbol{R},$$

(2.23) $$\|e^{-itH_0}\phi\|_{j,0}=\|\phi\|_{j,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{j,0},$$

For any $\phi\in H^{j,0}$, the map $\boldsymbol{R}\ni t\mapsto e^{-itH_0}\phi\in H^{j,0}$ is continuous.

(2) $$e^{-itH_0}(H^{j,j})\subset H^{0,j}, \qquad t\in\boldsymbol{R},$$

(2.24) $$\|e^{-itH_0}\phi\|_{0,j}\leqq C\|\phi\|_{0,j}+C|t|^j\|\phi\|_{j,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{j,j},$$

For any $\phi\in H^{j,j}$, the map $\boldsymbol{R}\ni t\mapsto e^{-itH_0}\phi\in H^{0,j}$ is continuous, $\boldsymbol{R}\ni t\mapsto e^{itH_0}x^\alpha e^{-itH_0}\phi\in L^2$, $|\alpha|\leqq j$, is continuously differentiable, and

(2.25) $$\frac{d}{dt}(e^{itH_0}x^\alpha e^{-itH_0}\phi)=-ie^{itH_0}((\Delta x^\alpha)+2(\nabla x^\alpha)\cdot\nabla)e^{-itH_0}\phi.$$

Proof. We use the following relations:

(2.26) $$(1-\Delta)^{j/2}e^{-itH_0}\phi=e^{-itH_0}(1-\Delta)^{j/2}\phi, \qquad \phi\in H^{j,0},$$

(2.27) $$xe^{-itH_0}\phi=e^{-itH_0}(x-2it\nabla)\phi, \qquad \phi\in H^{1,1}.$$

Part (1) is an immediate consequence of (2.26). We turn to part (2). We first prove by induction on $j\in\boldsymbol{N}$ that if $\phi\in H^{j,j}$, then $\boldsymbol{R}\ni t\mapsto e^{-itH_0}\phi\in H^{0,j}$ is continuous. The case $j=1$ follows from (2.27) and is also the special case of part (2) of Theorem 2.1. Let $j\geqq 2$ and suppose that if $\phi\in H^{j-1,j-1}$, then $\boldsymbol{R}\ni t\mapsto e^{-itH_0}\phi\in H^{0,j-1}$ is continuous. Let $\phi\in H^{j,j}$. By part (1) of Lemma 1.2, we have $x\phi$, $\nabla\phi\in H^{j-1,j-1}$. It follows from the induction hypothesis that

$$x|x|^{j-1}e^{-itH_0}\phi=|x|^{j-1}xe^{-itH_0}\phi-2it|x|^{j-1}e^{-itH_0}\nabla\phi$$

defines a continuous map from $\boldsymbol{R}$ to L^2, as required. We next prove (2.25). Let $\phi\in H^{j,j}$ and let $u(t)=e^{-itH_0}\phi$, $t\in\boldsymbol{R}$. Since $u\in C(\boldsymbol{R};\ H^{j,j})$, we see from part (1) of Lemma 1.2 that $x^\beta\partial^\gamma u\in C(\boldsymbol{R};\ L^2)$, $|\beta+\gamma|\leqq j$, and therefore the R.H.S. of (2.25) defines a continuous map from $\boldsymbol{R}$ to L^2. Let ζ_ε be as in Lemma 1.4. We compute

$$\begin{aligned}&\frac{d}{dt}(e^{itH_0}x^\alpha\zeta_\varepsilon u(t))=ie^{itH_0}[H_0,\ x^\alpha\zeta_\varepsilon]u(t)\\&=-ie^{itH_0}(((\Delta x^\alpha)-2(n+2)\varepsilon^2x^\alpha+4\varepsilon^4x^\alpha|x|^2)\zeta_\varepsilon+2((\nabla x^\alpha)-2\varepsilon^2x^\alpha x)\cdot\zeta_\varepsilon\nabla)u(t)\,.\end{aligned}$$

In the same way as in the proof of (2.15), this leads to

$$x^\alpha u(t)=e^{-itH_0}x^\alpha\phi-i\int_0^t e^{-i(t-\tau)H_0}((\Delta x^\alpha)+2(\nabla x^\alpha)\cdot\nabla)u(\tau)d\tau\,,$$

which in turn implies (2.25). It remains to prove (2.24). Let $\phi\in H^{j,j}\setminus\{0\}$. Let u and ζ_ε be as above. In the same way as in the proof of (2.14), we have for $|\alpha|=j$

$$\begin{aligned}(2.28)\qquad\frac{d}{dt}\|(x^\alpha\zeta_\varepsilon+i)u\|_2^2&=\frac{d}{dt}\|x^\alpha\zeta_\varepsilon u\|_2^2\\&=2\,\mathrm{Im}\,([x^\alpha\zeta_\varepsilon,\ H_0]u(t),\ x^\alpha\zeta_\varepsilon u(t))\\&=4\,\mathrm{Im}\,(\nabla x^\alpha\cdot\nabla(\zeta_\varepsilon u(t))-2\varepsilon^2x^\alpha\zeta_\varepsilon x\cdot\nabla u(t),\ x^\alpha\zeta_\varepsilon u(t))\,.\end{aligned}$$

Using part (1) of Lemma 1.2, we estimate the R.H.S. of (2.28) by

$$\begin{aligned}(2.29)\qquad&C(\||x|^{j-1}\nabla(\zeta_\varepsilon u)\|_2+\|\varepsilon^2x^\alpha\zeta_\varepsilon x\cdot\nabla u\|_2)\|x^\alpha\zeta_\varepsilon u\|_2\\&\leqq C(\||x|^j\zeta_\varepsilon u\|_2^{(j-1)/j}\|\zeta_\varepsilon u\|_{j,0}^{1/j}+\|\varepsilon^2|x|^{j+1}\zeta_\varepsilon\nabla u\|_2)\||x|^j\zeta_\varepsilon u\|_2\\&\leqq C\|(|x|^j\zeta_\varepsilon+i)u\|_2^{2-1/j}\|\zeta_\varepsilon u\|_{j,0}^{1/j}+C\|\varepsilon^2|x|^{j+1}\zeta_\varepsilon\nabla u\|_2\|(|x|^j\zeta_\varepsilon+i)u\|_2\,.\end{aligned}$$

Since

$$\|(|x|^j\zeta_\varepsilon+i)u\|_2^2=\sum_{|\alpha|=j}\frac{j!}{\alpha!}\|x^\alpha\zeta_\varepsilon u\|_2^2+\|u\|_2^2\,,$$

from (2.28) and (2.29) we obtain

$$\begin{aligned}(2.30)\qquad&\left|\frac{d}{dt}\|(|x|^j\zeta_\varepsilon+i)u\|_2^{1/j}\right|\\&=(2j)^{-1}\|(|x|^j\zeta_\varepsilon+i)u\|_2^{1/j-2}\cdot\left|\frac{d}{dt}\|(|x|^j\zeta_\varepsilon+i)\,u\|_2^2\right|\\&\leqq C\|\zeta_\varepsilon u\|_{j,0}^{1/j}+C\|\phi\|_2^{1/j-1}\||\varepsilon x|^2\zeta_\varepsilon|x|^{j-1}\nabla u\|_2\qquad a.e.\end{aligned}$$

Integrating (2.30) with respect to t, taking the limit $\varepsilon\to+0$ in the resulting inequality and using (2.23), we have

$$\|(|x|^j+i)u(t)\|_2^{1/j}\leqq\|(|x|^j+i)\phi\|_2^{1/j}+C|t|\|\phi\|_{j,0}^{1/j}\,,\qquad t\in\boldsymbol{R},$$

from which (2.25) follows. Q. E. D.

§ 3. The Pseudo-conformal Conservation Law

In this section we provide the pseudo-conformal conservation law, which will be one of the main tools in this paper. For this purpose we introduce some notations and assumptions on H. We denote by F the function defined as

$$F(t)=\|xe^{itH_0}e^{-itH}\phi\|_2^2+4t^2(e^{itH}Ve^{-itH}\phi,\ \phi), \qquad t\in\boldsymbol{R}.$$

If $\phi\in H^{1,1}$, then Theorem 2.1 shows that $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in H^{1,1}$ is continuous, so that we may write

$$xe^{itH_0}e^{-itH}\phi=e^{itH_0}(x+2it\nabla)e^{-itH}\phi\,,$$

$$(e^{itH}Ve^{-itH}\phi,\ \phi)=(Ve^{-itH}\phi,\ e^{-itH}\phi)\,.$$

Then $F\in C(\boldsymbol{R}\,;\boldsymbol{R})$ and $F(t)$ satisfies

$$F(t)=\|(x+2it\nabla)e^{-itH}\phi\|_2^2+4t^2(Ve^{-itH}\phi,\ e^{-itH}\phi)\,.$$

We denote by $i[A,\ H]$ the symmetric form on $D(H)\cap D(A)$ defined as

$$(i[A,\ H]\phi,\ \psi)=i(H\phi,\ A\psi)-i(A\phi,\ H\psi)\,, \qquad \phi,\ \psi\in D(H)\cap D(A)\,.$$

We consider the following assumptions (Aj), $j=1, 2$:

(Aj) *$i[A,\ H]$ extends to a bounded operator $B\in\mathcal{L}(H^{j,0}\,;H^{-j,0})$.*

Such assumptions are variants of those of E. Mourre [23] and P. Perry, I. M. Sigal & B. Simon [28]. Since $i[A,\ H_0]=-2H_0$ as forms on $D(A)\cap D(H)$, (Aj) is equivalent to the following

(Aj)$^{\#}$ *The form $i[A,\ V]$ on $D(A)\cap D(H)$, defined by*

$$(i[A,\ V]\phi,\ \psi)=i(V\phi,\ A\psi)-i(A\phi,\ V\psi)\,, \qquad \phi,\ \psi\in D(A)\cap D(H)\,,$$

extends to a bounded operator $V^{\#}\in\mathcal{L}(H^{j,0}\,;H^{-j,0})$.

Lemma 3.1. *Let $j=1$ or 2. Then:*

(1) *H extends to a bounded operator $H^{(j)}\in\mathcal{L}(H^{2-j,0}\,;H^{-j,0})$. Moreover, $H\subset H^{(1)}\subset H^{(2)}$.*

(2) *If $\psi\in L^2$ and $H^{(2)}\psi\in L^2$, then $\psi\in H^{2,0}$ and $H^{(2)}\psi=H\psi$.*

(3) *If $\psi\in L^2$ and $H^{(2)}\psi\in H^{-1,0}$, then $\psi\in H^{1,0}$ and $H^{(1)}\psi=H\psi$.*

(4) *For any $\phi,\ \psi\in H^{1,0}$, $(H^{(1)}e^{-itH}\phi,\ e^{-itH}\psi)=(H^{(1)}\phi,\ \psi)$, $t\in\boldsymbol{R}$.*

Proof. (1) By (2.4), we have

$$|(H\psi, \psi)| \leqq (|H|\psi, \psi) = \||H|^{1/2}\psi\|_2^2 \leqq C\|\psi\|_{1,0}^2, \qquad \psi \in D(H).$$

Therefore,

$$|(H\psi, \phi)| \leqq C\|\psi\|_{1,0}\|\phi\|_{1,0}, \qquad \psi, \phi \in D(H),$$

from which it follows that H extends to a bounded operator $H^{(1)} \in \mathcal{L}(H^{1,0}; H^{-1,0})$. On the other hand,

$$|(H\psi, \phi)| = |(\psi, H\phi)| \leqq C\|\psi\|_2\|\phi\|_{2,0}, \qquad \psi, \phi \in D(H),$$

which shows that H extends to a bounded operator $H^{(2)} \in \mathcal{L}(H^{2,0}; H^{-2,0})$. If $\psi \in H^{1,0}$, then there exists a sequence $\{\psi_k\}$ in $\mathcal{S}$ such that $\psi_k \to \psi$ in $H^{1,0}$ as $k \to \infty$. Consequently, $H\psi_k = H^{(j)}\psi_k \to H^{(j)}\psi$ in $H^{-j,0}$ as $k \to \infty$. This implies that $H^{(2)}\psi = H^{(1)}\psi$, and hence $H^{(1)} \subset H^{(2)}$.

(2) Let $\psi \in L^2$ satisfy $H^{(2)}\psi \in L^2$. Put $\psi_\lambda = i\lambda(H+i\lambda)^{-1}\psi$, $\lambda > 1$. We easily see that $H\psi_\lambda = i\lambda(H+i\lambda)^{-1}H^{(2)}\psi$ and $(\psi_\lambda, H\phi) = (i\lambda(H+i\lambda)^{-1}H^{(2)}\psi, \phi)$ for any $\phi \in D(H)$. Letting $\lambda \to \infty$ in this equality, we have $(\psi, H\phi) = (H^{(2)}\psi, \phi)$ for any $\phi \in D(H)$. This proves that $\psi \in D(H)$ and $H\psi = H^{(2)}\psi$.

(3) Let $\psi \in L^2$ satisfy $H^{(2)}\psi \in H^{-1,0}$. Put $\psi_\lambda = i\lambda(H+i\lambda)^{-1}\psi$, $\lambda > 1$. Let $\lambda_0 = 1+b$ (see (2.1)). We easily see that

$$\begin{aligned}
&(H+\lambda_0)\psi_\lambda = i\lambda(H+i\lambda)^{-1}(H^{(2)}+\lambda_0)\psi, \\
&(H+\lambda_0)^{-1/2}\psi_\lambda = i\lambda(H+i\lambda)^{-1}(H+\lambda_0)^{-1/2}(H^{(2)}+\lambda_0)\psi, \\
&\|(H+\lambda_0)^{1/2}(\psi_\lambda - \psi_\mu)\|_2^2 \\
&\quad = ((H+\lambda_0)(\psi_\lambda - \psi_\mu), \psi_\lambda - \psi_\mu) \\
&\quad = ((i\lambda(H+i\lambda)^{-1} - i\mu(H+i\mu)^{-1})(H^{(2)}+\lambda_0)\psi, \psi_\lambda - \psi_\mu) \\
&\quad = ((i\lambda(H+i\lambda)^{-1} - i\mu(H+i\mu)^{-1})(H+\lambda_0)^{-1/2}(H^{(2)}+\lambda_0)\psi, (H+\lambda_0)^{1/2}(\psi_\lambda - \psi_\mu)),
\end{aligned}$$

and therefore

$$\|(H+\lambda_0)^{1/2}(\psi_\lambda - \psi_\mu)\|_2 \leqq \|(i\lambda(H+i\lambda)^{-1} - i\mu(H+i\mu)^{-1})(H+\lambda_0)^{-1/2}(H^{(2)}+\lambda_0)\psi\|_2 \longrightarrow 0 \quad \text{as } \lambda, \mu \to \infty.$$

This implies that $\psi \in D((H+\lambda_0)^{1/2}) = H^{1,0}$ and $H^{(1)}\psi = H^{(2)}\psi$.

(4) If $\phi \in H^{2,0}$ and $\psi \in L^2$, then the result is immediate. If $\phi, \psi \in H^{1,0}$, then we approximate ϕ (resp. ψ) by a sequence $\{\phi_k\}$ (resp. $\{\psi_k\}$) in $\mathcal{S}$ in the $H^{1,0}$-norm and obtain the desired result by taking the limit $k \to \infty$ in the corresponding result for ϕ_k and ψ_k. Q. E. D.

Let $j=1$ or 2. If (Aj) holds, then we have the identity $H^{(2-j)} + (1/2)B = V + (1/2)V^{\#}$ as operators in $\mathcal{L}(H^{j,0}; H^{-j,0})$, where $H^{(0)} = H$. Concerning sufficient conditions for H to satisfy (Aj), we have the following:

Proposition 3.1. *Let $j=1$ or 2. Suppose that V can be decomposed as $V=V_1+V_2$, with $x\cdot\nabla V_1$, $V_2\in\mathcal{L}(H^{j,0}; H^{-j,0})$ and $xV_2\in\mathcal{L}(H^{j,0}; H^{1-j,0})$. Then* (Aj) *holds. Moreover,*

$$(V^{\#}\phi, \psi)=((x\cdot\nabla V_1)\phi, \psi)-(xV_2\phi, \nabla\psi)-(\nabla\phi, xV_2\psi)-n(V_2\phi, \psi), \quad \phi, \psi\in H^{j,0}.$$

Proof. The proof is parallel to that of [28; Proposition 1.3]. Q.E.D.

Proposition 3.2. *Let W be a function on $\boldsymbol{R}^n$. Suppose that W can be decomposed as $W=W^{(q)}+W^{(\infty)}$, where $W^{(q)}\in L^q$ with $q\geqq 1$, $W^{(\infty)}\in L^\infty$. Then:*

(1) *$W\in\mathcal{L}(H^{1,0}; H^{-1,0})$ for all q such that $q\geqq 1$ for $n=1$, $q>1$ for $n=2$, and $q\geqq n/2$ for $n\geqq 3$.*

(2) *$W\in\mathcal{L}(H^{2,0}; H^{-2,0})$ for all q such that $q\geqq 1$ for $n\leqq 3$, $q>1$ for $n=4$, and $q\geqq n/5$ for $n\geqq 5$.*

(3) *$W\in\mathcal{L}(H^{1,0}; L^2)$ for all q such that $q\geqq 2$ for $n=1$, $q>2$ for $n=2$, and $q\geqq n$ for $n\geqq 3$.*

(4) *$W\in\mathcal{L}(H^{2,0}; H^{-1,0})$ for all q such that $q\geqq 1$ for $n=1$, $q>1$ for $n=2$, $q\geqq 6/5$ for $n=3$, $q>4/3$ for $n=4$, and $q\geqq n/3$ for $n\geqq 5$.*

Proof. We only prove part (1), since the other parts can be proved analogously. It suffices to obtain the estimate

$$|(W^{(q)}\phi, \psi)|\leqq C\|\phi\|_{1,0}\|\psi\|_{1,0}, \qquad \phi, \psi\in H^{1,0}.$$

By Hölder's inequality,

$$|(W^{(q)}\phi, \psi)|\leqq\|W^{(q)}\|_q\|\phi\|_p\|\psi\|_p,$$

where $1/q+2/p=1$. By Lemma 2.1 ($\delta_n(p)=n/2-n/p$, see § 1.),

$$\|\phi\|_p\leqq C\|\nabla\phi\|_2^{\delta_n(p)}\|\phi\|_2^{1-\delta_n(p)}, \tag{3.1}$$

where p ranges over $2\leqq p\leqq\infty$ for $n=1$, $2\leqq p<\infty$ for $n=2$, and $2\leqq p\leqq 2n/(n-2)$ for $n\geqq 3$. Combining these estimates, we obtain the required one. Q.E.D.

Theorem 3.1. *Let $j=1$ or 2. Suppose that* (Aj) *holds. Let $\phi\in H^{j,1}$. Then $F\in C(\boldsymbol{R};\boldsymbol{R})$ and F satisfies*

$$\begin{aligned}\frac{d}{dt}F(t)&=8t((H^{(2-j)}+(1/2)B)e^{-itH}\phi, e^{-itH}\phi)\\&=8t((V+(1/2)V^{\#})e^{-itH}\phi, e^{-itH}\phi), \qquad t\in\boldsymbol{R}.\end{aligned}$$

Remark 3.1. The pseudo-conformal conservation law for the nonlinear Schrödinger equation of the form $i\partial_t u=-\Delta u+\lambda|u|^{p-1}u$ is expressed as follows

$$\frac{d}{dt}(\|(x+2it\nabla)u(t)\|_2^2+(8\lambda/(p+1))t^2\|u(t)\|_{p+1}^{p+1})$$
$$=(4\lambda(n+4-np)/(p+1))t\|u(t)\|_{p+1}^{p+1}.$$

See J. Ginibre & G. Velo [8], Y. Tsutsumi [37], N. Hayashi, K. Nakamitsu & M. Tsutsumi [9][10] for related results.

The proof of Theorem 3.1 is devided into several steps. We first summarize the properties of the operator A. For $\lambda\in\boldsymbol{R}\setminus\{0\}$, we define $R_\lambda=i\lambda(A+i\lambda)^{-1}$ and $A_\lambda=AR_\lambda$.

Lemma 3.2. (1) *A defines a bounded operator (also denoted by A) from $H^{1,0}$ to $H^{0,-1}$. Moreover, A satisfies*

$$(A\phi, \psi)=-i(\nabla\phi, x\psi)-i(n/2)(\phi, \psi), \qquad \phi\in H^{1,0},\ \psi\in H^{0,1}, \tag{3.2}$$

$$(A\phi, \phi)=\mathrm{Im}\,(\nabla\phi, x\phi), \qquad \phi\in H^{1,1}. \tag{3.3}$$

(2) *Let $m\in\boldsymbol{R}$. Then $R_\lambda\in\mathcal{L}(H^{m,0}; H^{m,0})$ for $|\lambda|>m$,*

$$\sup_{|\lambda|\geqq|m|+1}\|R_\lambda\|_{\mathcal{L}(H^{m,0};H^{m,0})}\leqq|m|+1, \tag{3.4}$$

$$R_\lambda \longrightarrow 1 \text{ strongly in } \mathcal{L}(H^{m,0}; H^{m,0}) \quad \text{as } |\lambda|\to\infty. \tag{3.5}$$

(3) *Let $m\in\boldsymbol{R}$. Then $A_\lambda\in\mathcal{L}(H^{m,0}; H^{m,0})$ for $|\lambda|>m$. Moreover, $A_\lambda\in\mathcal{L}(H^{1,0}; H^{0,-1})$ for $\lambda\neq1$,*

$$A_\lambda \longrightarrow A \text{ strongly in } \mathcal{L}(H^{1,0}; H^{0,-1}) \quad \text{as } |\lambda|\to\infty. \tag{3.6}$$

Proof. (1) If $\phi, \psi\in\mathcal{S}$, then (3.2) holds and thus

$$|(A\phi, \psi)|\leqq C\|\phi\|_{1,0}\|\psi\|_{0,1}, \qquad \phi, \psi\in\mathcal{S}.$$

Therefore $A\in\mathcal{L}(H^{1,0}; H^{0,-1})$ and (3.2) extends to the case where $\phi\in H^{1,0}$, $\psi\in H^{0,1}$. If $\phi, \psi\in H^{1,1}$, then $(A\phi, \psi)=(A\psi, \phi)$, so that $(A\phi, \phi)=\mathrm{Re}\,(A\phi, \phi)=\mathrm{Im}\,(\nabla\phi, x\phi)$.

(2) (Compare with [28; Lemma 6.2]). We use the representation

$$(A+i\lambda)^{-1}=\begin{cases} -i\displaystyle\int_0^\infty e^{-\lambda s}e^{isA}ds & \text{if } \lambda>0, \\ i\displaystyle\int_{-\infty}^0 e^{-\lambda s}e^{isA}ds & \text{if } \lambda<0. \end{cases}$$

From now on we consider the case $\lambda>0$, since the case $\lambda<0$ can be treated similarly. Let $\psi\in H^{m,0}$. From the relation $(e^{isA}\psi)(x)=e^{ns/2}\psi(e^s x)$, $s\in\boldsymbol{R}$, $x\in\boldsymbol{R}^n$, it follows that $\mathcal{F}e^{isA}=e^{-isA}\mathcal{F}$. Put $\omega_s(\xi)=(1+e^{2s}|\xi|^2)^{1/2}$, $s\in\boldsymbol{R}$, $\xi\in\boldsymbol{R}^n$. Then we have in L^2

$$(1-\Delta)^{m/2}e^{isA}\psi=\mathcal{F}^{-1}\omega^m\mathcal{F}e^{isA}\psi=\mathcal{F}^{-1}\omega^m e^{-isA}\mathcal{F}\psi=\mathcal{F}^{-1}e^{-isA}\omega_s^m\mathcal{F}\psi$$
$$=e^{isA}\mathcal{F}^{-1}\omega_s^m\mathcal{F}\psi .$$

Thus, $\boldsymbol{R}\ni s\mapsto(1-\Delta)^{m/2}e^{isA}\psi\in L^2$ is continuous and

$$\|(1-\Delta)^{m/2}e^{isA}\psi\|_2\leqq\|\omega_s^m\mathcal{F}\psi\|_2\leqq\max(1,\ e^{ms})\|\omega^m\mathcal{F}\psi\|_2\leqq\max(1,\ e^{ms})\|\psi\|_{m,0} .$$

Therefore, if $\lambda>m$, then $(0,\ \infty)\ni s\mapsto e^{-\lambda s}\|(1-\Delta)^{m/2}e^{isA}\psi\|_2\in\boldsymbol{R}$ is integrable,

$$\int_0^\infty e^{-\lambda s}\|(1-\Delta)^{m/2}e^{isA}\psi\|_2 ds\leqq\max((\lambda-m)^{-1},\ \lambda^{-1})\|\psi\|_{m,0} ,$$

and hence, $(0,\ \infty)\ni s\mapsto e^{-\lambda s}(1-\Delta)^{m/2}e^{isA}\psi\in L^2$ is integrable. Since $(1-\Delta)^{m/2}$ is closed, we see that

$$(A+i\lambda)^{-1}\psi\in D((1-\Delta)^{m/2})=H^{m,0} ,$$
$$(1-\Delta)^{m/2}(A+i\lambda)^{-1}\psi=-i\int_0^\infty e^{-\lambda s}(1-\Delta)^{m/2}e^{isA}\psi ds$$
$$=-i\int_0^\infty e^{-\lambda s}e^{isA}\mathcal{F}^{-1}\omega_s\mathcal{F}\psi ds ,$$
$$\|R_\lambda\psi\|_{m,0}\leqq\max(\lambda(\lambda-m)^{-1},\ 1)\|\psi\|_{m,0} ,$$

from which (3.4) follows. We next prove (3.5). We write

$$(1-\Delta)^{m/2}(1-R_\lambda)\psi=\lambda\int_0^\infty e^{-\lambda s}(1-\Delta)^{m/2}(1-e^{isA})\psi ds$$
$$=\lambda\int_0^\infty e^{-\lambda s}(\mathcal{F}^{-1}\omega^m\mathcal{F}-e^{isA}\mathcal{F}^{-1}\omega_s^m\mathcal{F})\psi ds$$
$$=\int_0^\infty e^{-\tau}(\mathcal{F}^{-1}\omega^m-e^{i(\tau/\lambda)A}\mathcal{F}^{-1}\omega_{\tau/\lambda}^m)\mathcal{F}\psi d\tau .$$

The above integrals converge in L^2 uniformly in $\lambda\in(\max(0,\ m)+1,\ \infty)$. Therefore (3.5) will follow if we can show that for each $\tau>0$,

$$\|(\mathcal{F}^{-1}\omega^m-e^{i(\tau/\lambda)A}\mathcal{F}^{-1}\omega_{\tau/\lambda}^m)\mathcal{F}\psi\|_2 \longrightarrow 0 \qquad \text{as } \lambda\to\infty .$$

This follows from Lebesgue's dominated convergence theorem. Indeed,

$$\begin{aligned}&\|(\mathcal{F}^{-1}\omega^m-e^{i(\tau/\lambda)A}\mathcal{F}^{-1}\omega_{\tau/\lambda}^m)\mathcal{F}\psi\|_2\\&=\|(1-e^{i(\tau/\lambda)A})\mathcal{F}^{-1}\omega^m\mathcal{F}\psi+e^{i(\tau/\lambda)A}\mathcal{F}^{-1}(\omega^m-\omega_{\tau/\lambda}^m)\mathcal{F}\psi\|_2\\&\leqq\|(1-e^{i(\tau/\lambda)A})(1-\Delta)^{m/2}\psi\|_2+\|(\omega^m-\omega_{\tau/\lambda}^m)\mathcal{F}\psi\|_2 \longrightarrow 0 \qquad \text{as } \lambda\to\infty .\end{aligned}$$

Q. E. D.

Lemma 3.3. *Let $j=1$ or 2. Suppose that* (Aj) *holds. Let $\phi,\ \psi\in H^{j,1}$. Then $\boldsymbol{R}\ni t\mapsto(Ae^{-itH}\phi,\ e^{-itH}\psi)\in\boldsymbol{C}$ is continuously differentiable, and*

$$\frac{d}{dt}(Ae^{-itH}\phi,\ e^{-itH}\psi)=-(Be^{-itH}\phi,\ e^{-itH}\psi) ,\qquad t\in\boldsymbol{R} . \tag{3.7}$$

Proof. Let $\phi_\mu=i\mu(H+i\mu)^{-1}\phi$, $\psi_\mu=i\mu(H+i\mu)^{-1}\psi$, $\mu>1$. Let $u_\mu(t)=e^{-itH}\phi_\mu$, $v_\mu(t)=e^{-itH}\psi_\mu$, $u(t)=e^{-itH}\phi$, $v(t)=e^{-itH}\psi$, $t\in\boldsymbol{R}$. It follows from Theorem 2.1, (2.7), and Lemma 2.1 that $u_\mu, v_\mu\in C(\boldsymbol{R}; H^{2,1})$, $Au_\mu, Av_\mu\in C(\boldsymbol{R}; L^2)$, and that $u_\mu\to u$, $v_\mu\to v$ in $C(\boldsymbol{R}; H^{2,1})$ as $\mu\to\infty$. We first show that

$$(A_\lambda u_\mu(t), v_\mu(t))=(A_\lambda\phi_\mu, \psi_\mu)-\int_0^t(R_\lambda BR_\lambda u_\mu(\tau), v_\mu(\tau))d\tau\,. \tag{3.8}$$

For this purpose we compute

$$\frac{d}{dt}(A_\lambda u_\mu, v_\mu)=-i(A_\lambda Hu_\mu, v_\mu)+i(A_\lambda u_\mu, Hv_\mu)\,.$$

We consider the first term on the R. H. S. of the equality above. By Lemma 3.2, $(A-i\lambda)^{-1}Av_\mu=v_\mu+iR_{-\lambda}v_\mu\in H^{2,0}\cap D(A)=D(H)\cap D(A)$. We write $-i(A_\lambda Hu_\mu, v_\mu)$ as

$$\begin{aligned}
&-i(u_\mu, HR_{-\lambda}Av_\mu)\\
&=-i(A(A+i\lambda)^{-1}u_\mu, HR_{-\lambda}Av_\mu)-i(R_\lambda u_\mu, HR_{-\lambda}Av_\mu)\\
&=-i(AR_\lambda u_\mu, H(A-i\lambda)^{-1}Av_\mu)-i(R_\lambda u_\mu, HR_{-\lambda}Av_\mu)\\
&=i([A, H]R_\lambda u_\mu, (A-i\lambda)^{-1}Av_\mu)-i(HR_\lambda u_\mu, A(A-i\lambda)^{-1}Av_\mu)-i(R_\lambda u_\mu, HR_{-\lambda}Av_\mu)\\
&=(BR_\lambda u_\mu, (A-i\lambda)^{-1}Av_\mu)-i(HR_\lambda u_\mu, Av_\mu)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}(A_\lambda u_\mu, v_\mu)&=(BR_\lambda u_\mu, (A-i\lambda)^{-1}Av_\mu)-i(HR_\lambda u_\mu, Av_\mu)+i(AR_\lambda u_\mu, Hv_\mu)\\
&=(BR_\lambda u_\mu, (A-i\lambda)^{-1}Av_\mu)-(BR_\lambda u_\mu, v_\mu)\\
&=-(BR_\lambda u_\mu, R_{-\lambda}v_\mu)=-(R_\lambda BR_\lambda u_\mu, v_\mu)\,,
\end{aligned}$$

which shows (3.8). Since $A_\lambda\in\mathcal{L}(L^2)$ and $R_\lambda BR_\lambda\in\mathcal{L}(H^{j,0}; H^{-j,0})$, we take the limit $\mu\to\infty$ in (3.8) to obtain

$$(A_\lambda u(t), v(t))=(A_\lambda\phi, \psi)-\int_0^t(R_\lambda BR_\lambda u(\tau), v(\tau))d\tau\,. \tag{3.9}$$

By (3.4) and (2.8),

$$\begin{aligned}
\sup_{|\lambda|\geqq 3}|(R_\lambda BR_\lambda u(t), v(t))|&\leqq C\|B\|_{\mathcal{L}(H^{j,0};H^{-j,0})}\|u(t)\|_{j,0}\|v(t)\|_{j,0}\\
&\leqq C\|B\|_{\mathcal{L}(H^{j,0};H^{-j,0})}\|\phi\|_{j,0}\|\psi\|_{j,0}\,,\qquad t\in\boldsymbol{R}.
\end{aligned}$$

By (3.5) and (3.6), for each $t\in\boldsymbol{R}$,

$$\lim_{|\lambda|\to\infty}(R_\lambda BR_\lambda u(t), v(t))=(Bu(t), v(t))\,,$$

$$\lim_{|\lambda|\to\infty}(A_\lambda u(t), v(t))=(Au(t), v(t))\,.$$

Therefore we apply Lebesgue's bounded convergence theorem to the integral in

(3.9) to get

$$(Au(t),\ v(t))=(A\phi,\ \psi)-\int_0^t(Bu(\tau),\ v(\tau))d\tau\,,$$

which implies (3.7), since $Bu\in C(\boldsymbol{R};H^{-j,0})$. Q.E.D.

Lemma 3.4. *Let $\phi, \psi\in H^{1,1}$. Then $\boldsymbol{R}\ni t\mapsto(xe^{-itH}\phi,\ xe^{-itH}\psi)\in\boldsymbol{C}$ is continuously differentiable, and*

$$\frac{d}{dt}(xe^{-itH}\phi,\ xe^{-itH}\psi)=4(Ae^{-itH}\phi,\ e^{-itH}\psi)\,,\qquad t\in\boldsymbol{R}\,.$$

Proof. Let $\{\phi_k\}$ be a sequence in $H^{2,2}$ such that $\phi_k\to\phi$ in $H^{1,1}$ as $k\to\infty$. Let $u_k(t)=e^{-itH}\phi_k$, $u(t)=e^{-itH}\phi$, $v(t)=e^{-itH}\psi$. By Theorem 2.2,

$$\begin{aligned}\frac{d}{dt}(xu_k(t),\ xv(t))&=\frac{d}{dt}(e^{itH}|x|^2e^{-itH}\phi_k,\ \psi)\\&=4(e^{itH}Ae^{-itH}\phi_k,\ \psi)=4(Au_k(t),\ v(t))\end{aligned}$$

so that

$$(xu_k(t),\ xv(t))=(x\phi_k,\ x\psi)+4\int_0^t(Au_k(\tau),\ v(\tau))d\tau\,.$$

It follows from (2.8) and (2.9) that $u_k\to u$ in $C(\boldsymbol{R};H^{1,1})$ as $k\to\infty$, which when combined with part (1) of Lemma 3.2 shows that $(Au_k,\ v)\to(Au,\ v)$ in $C(\boldsymbol{R};\boldsymbol{C})$ as $k\to\infty$. Therefore we have

$$(xu(t),\ v(t))=(x\phi,\ x\psi)+4\int_0^t(Au(\tau),\ v(\tau))d\tau\,.$$

This proves the lemma. Q.E.D.

Proof of Theorem 3.1. We write

$$\begin{aligned}F(t)&=\|xe^{-itH}\phi\|_2^2-4t(Ae^{-itH}\phi,\ e^{-itH}\phi)+4t^2(H^{(2-j)}e^{-itH}\phi,\ e^{-itH}\phi)\\&=\|xe^{-itH}\phi\|_2^2-4t(Ae^{-itH}\phi,\ e^{-itH}\phi)+4t^2(H^{(2-j)}\phi,\ \phi)\,.\end{aligned}$$

By Lemmas 3.3 and 3.4, we obtain $F\in C(\boldsymbol{R};\boldsymbol{R})$ and

$$\begin{aligned}\frac{d}{dt}F(t)&=4t(Be^{-itH}\phi,\ e^{-itH}\phi)+8t(H^{(2-j)}\phi,\ \phi)\\&=8t((H^{(2-j)}+(1/2)B)e^{-itH}\phi,\ e^{-itH}\phi)\,.\end{aligned}$$

Q.E.D.

Theorem 3.2. *Let $q\geqq1$ for $n=1$ and let $q>n/2$ for $n\geqq2$. Suppose that* (A2)$^{\#}$ *holds and that $V, V^{\#}\in L^q+L^\infty$. Then*:

(1) $$e^{itH_0}e^{-itH}(H^{0,1})\subset H^{0,1},\qquad t\in\boldsymbol{R},$$

(3.10) $$\|xe^{itH_0}e^{-itH}\phi\|_2\leqq C\|x\phi\|_2+C|t|\,\|\phi\|_2\,,\qquad t\in\boldsymbol{R},\ \phi\in H^{0,1}.$$

For any $\phi\in H^{0,1}$, the map $\boldsymbol{R}\ni t\mapsto e^{itH_0}e^{-itH}\phi\in H^{0,1}$ is continuous.

(2) $$e^{itH}Ve^{-itH}(H^{0,1})\subset H^{0,-1}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

(3.11) $$\|e^{itH}Ve^{-itH}\phi\|_{0,-1}\leqq C(1+|t|^{-n/q})\|\phi\|_{0,1}, \qquad t\in\boldsymbol{R}\setminus\{0\},\ \phi\in H^{0,1}.$$

For any $\phi, \psi\in H^{0,1}$, *the map* $\boldsymbol{R}\setminus\{0\}\ni t\mapsto(e^{itH}Ve^{-itH}\phi, \psi)\in\boldsymbol{C}$ *is continuous.*

(3) *For any* $\phi\in H^{0,1}$, *the map* $\boldsymbol{R}\setminus\{0\}\ni t\mapsto F(t)\in\boldsymbol{R}$ *is continuously differentiable and satisfies*

(3.12) $$\frac{d}{dt}F(t)=t(e^{itH}(V+(1/2)V^{\#})e^{-itH}\phi, \phi), \qquad t\in\boldsymbol{R}\setminus\{0\}.$$

Moreover, F is absolutely continuous on $\boldsymbol{R}$.

Remark 3.2. Part (2) with V replaced by $V^{\#}$ also holds. This can we proved if we replace V by $V^{\#}$ in the proof below.

Remark 3.3. Let $1\leqq q<p<\infty$. If $V\in L^p+L^\infty$, then $V\in\bigcap_{q\leqq r\leqq p}L^r+L^\infty$. This follows by cutting the L^p-part of V into two pieces. See H. Isozaki [13].

Remark 3.4. See A. Jensen [14] and N. Hayashi & T. Ozawa [11] for related results.

Proof of Theorem 3.2. We begin with some preliminary estimates. Let $\psi_k\in\mathcal{S}$, $k=1, 2$. Put $\varphi_k=\exp(|x|^2/4it)\psi_k$, $t\in\boldsymbol{R}\setminus\{0\}$. Let W be a real function such that $W\in L^q+L^\infty$, where q is as in the theorem. From (3.1) we obtain

(3.13)
$$\begin{aligned}|(W\psi_1, \psi_2)|&=|(W\varphi_1, \varphi_2)|\\ &\leqq C\prod_{k=1}^{2}\|\nabla\varphi_k\|_2^{n/q}\|\varphi_k\|_2^{1-n/q}+C\prod_{k=1}^{2}\|\varphi_k\|_2\\ &\leqq C\prod_{k=1}^{2}\|(x/2it+\nabla)\psi_k\|_2^{n/q}\|\psi_k\|_2^{1-n/q}+C\prod_{k=1}^{2}\|\psi_k\|_2\\ &=C\prod_{k=1}^{2}\|(x/2it)e^{itH_0}\psi_k\|_2^{n/q}\|\psi_k\|_2^{1-n/q}+C\prod_{k=1}^{2}\|\psi_k\|_2\,.\end{aligned}$$

In the same way as in the proof of [11; Lemma 3.5], by using Theorem 3.1 and (3.13), we obtain

(3.14) $$\|(x+2it\nabla)e^{-itH}\psi\|_2\leqq C\|x\psi\|_2+C|t|\|\psi\|_2, \qquad t\in\boldsymbol{R},\ \psi\in\mathcal{S}.$$

Let $\phi\in H^{0,1}$ and let $\{\phi_j\}$ be a sequence in $\mathcal{S}$ such that $\phi_j\to\phi$ in $H^{0,1}$ as $j\to\infty$. By (3.14), we see that for each $t\in\boldsymbol{R}$ $\{xe^{itH_0}e^{-itH}\phi_j\}$ is a Cauchy sequence in L^2. On the other hand,

$$\sup_{t\in R}\|e^{itH_0}e^{-itH}\phi_j-e^{itH_0}e^{-itH}\phi\|_2=\|\phi_j-\phi\|_2\longrightarrow 0 \qquad \text{as } j\to\infty\,.$$

Since the multiplication operator x is closed,

$$xe^{itH_0}e^{-itH}\phi\in L^2, \tag{3.15}$$

$$xe^{itH_0}e^{-itH}\phi_j \longrightarrow xe^{itH_0}e^{-itH}\phi \quad \text{in } L^2 \qquad \text{as } j\to\infty. \tag{3.16}$$

(3.10) now follows (3.14) and (3.16). We next prove that for each $t\in \boldsymbol{R}\setminus\{0\}$, $e^{itH}Ve^{-itH}$ extends to a bounded operator from $H^{0,1}$ to $H^{0,-1}$ and has the estimate (3.15). It is enough to show

$$|(e^{itH}Ve^{-itH}\psi_1, \psi_2)|\leqq C(1+|t|^{-n/q})\prod_{k=1}^{2}\|\psi_k\|_{0,1}, \qquad \psi_k\in\mathcal{S}. \tag{3.17}$$

This follows immediately from (3.13) and (3.14). We now prove that for any $\phi, \psi\in H^{0,1}$, $\boldsymbol{R}\setminus\{0\}\ni t\mapsto(e^{itH}Ve^{-itH}\phi, \psi)\in \boldsymbol{C}$ is continuous. Let $\{\phi_j\}$ (resp. $\{\psi_j\}$) be a sequence in $\mathcal{S}$ such that $\phi_j\to\phi$ (resp. $\psi_j\to\psi$) in $H^{0,1}$ as $j\to\infty$. It suffices to prove that for $T>0$,

$$\sup_{|t|\leqq T}|t|^{n/q}|(e^{itH}Ve^{-itH}\phi_j, \psi_j)-(e^{itH}Ve^{-itH}\phi, \psi)|\longrightarrow 0$$

as $j\to\infty$. This follows if we write

$$\begin{aligned}&(e^{itH}Ve^{-itH}\phi_j, \psi_j)-(e^{itH}Ve^{-itH}\phi, \psi)\\ &=(e^{itH}Ve^{-itH}(\phi_j-\phi), \psi_j-\psi)+(e^{itH}Ve^{-itH}\phi, \psi_j-\psi)\\ &\quad+(e^{itH}Ve^{-itH}(\phi_j-\phi), \psi)\end{aligned}$$

and use (3.10) and (3.16). The rest of the theorem can be proved in the same way as in the proof of [11; Proposition 3.2]. Q.E.D.

Theorem 3.3. *Let $q\geqq 1$ for $n=1$ and let $q>n/2$ for $n\geqq 2$. Suppose that $V\in L^q+L^\infty$ and that* (A2) *holds. If in addition, there exists a constant $\lambda\in\boldsymbol{R}$ such that*

$$H+(1/2)B\leqq\lambda V \qquad \textit{as forms on } H^{2,0}, \tag{3.18}$$

then the conclusions of Theorem 3.2 *hold.*

Proof. Let $\phi\in H^{0,1}$ and let $\{\phi_j\}$ be a sequence in $H^{2,1}$ such that $\phi_j\to\phi$ in $H^{0,1}$ as $j\to\infty$. It follows from Theorem 3.1 and (3.18) that

$$\begin{aligned}&\|xe^{itH_0}e^{-itH}\phi_j\|_2^2+4t^2(e^{itH}Ve^{-itH}\phi_j, \phi_j)\\ &=\|x\phi_j\|_2^2+8\int_0^t\tau((H+(1/2)B)e^{-i\tau H}\phi_j, e^{-i\tau H}\phi_j)d\tau\\ &\leqq\|x\phi_j\|_2^2+8\lambda\int_0^t\tau(Ve^{-i\tau H}\phi_j, e^{-i\tau H}\phi_j)d\tau\end{aligned}$$

and therefore

$$(3.19)\qquad \|xe^{itH_0}e^{-itH}\phi_j\|_2^2 \leqq 4t^2|(e^{itH}Ve^{-itH}\phi_j,\ \phi_j)|+\|x\phi_j\|_2^2+8\lambda\int_0^t \tau(e^{i\tau H}Ve^{-i\tau H}\phi_j,\ \phi_j)d\tau$$

The conclusions of Theorem 3.2 follow by the same argument as in the proof of Theorem 3.2 if we use (3.19) instead of Theorem 3.1. Q.E.D.

§ 4. L^p-Estimates of the Schrödinger Equations I

In this section we study sufficient conditions for the existence of L^p-solutions.

Theorem 4.1. (1) *Let* $2\leqq p\leqq\infty$ *for* $n=1$, $2\leqq p<\infty$ *for* $n=2$, *and let* $2\leqq p\leqq 2n/(n-2)$ *for* $n\geqq 3$. *Then*

$$e^{-itH}(H^{1,0})\subset L^p, \qquad t\in \boldsymbol{R},$$

$$\|e^{-itH}\phi\|_p\leqq C\|\phi\|_{1,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{1,0}.$$

For any $\phi\in H^{1,0}$, *the map* $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ *is continuous.*

(2) *Let* $2\leqq p\leqq\infty$ *for* $n\leqq 3$, $2\leqq p<\infty$ *for* $n=4$, *and let* $2\leqq p\leqq 2n/(n-4)$ *for* $n\geqq 5$. *Then*

$$e^{-itH}(H^{2,0})\subset L^p, \qquad t\in \boldsymbol{R},$$

$$\|e^{-itH}\phi\|_p\leqq C\|\phi\|_{2,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{2,0}.$$

For any $\phi\in H^{2,0}$, *the map* $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ *is continuous.*

(3) *Let* $m\in\boldsymbol{N}$. *Suppose that* H *satisfies* $D(|H|^{m/2})=H^{m,0}$. *Let* $2\leqq p\leqq\infty$ *for* $n\leqq 2m-1$, $2\leqq p<\infty$ *for* $n=2m$, *and let* $2\leqq p\leqq 2n/(n-2m)$ *for* $n\geqq 2m+1$. *Then*

$$e^{-itH}(H^{m,0})\subset L^p, \qquad t\in \boldsymbol{R},$$

$$\|e^{-itH}\phi\|_p\leqq C\|\phi\|_{m,0}, \qquad t\in\boldsymbol{R},\ \phi\in H^{m,0}.$$

For any $\phi\in H^{m,0}$, *the map* $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ *is continuous.*

Proof. In view of (2.6), parts (1) and (2) reduce to part (3). We prove part (3). By the assumption $D(|H|^{m/2})=H^{m,0}=D(H_0^{m/2})$ and by the closed graph theorem, we obtain

$$(4.1)\qquad C\|(H_0^{m/2}+i)\phi\|_2\leqq\|(|H|^{m/2}+i)\phi\|_2\leqq C'\|(H_0^{m/2}+i)\phi\|_2, \qquad \phi\in H^{m,0}.$$

Lemma 1.1 implies

$$(4.2)\qquad \|e^{-itH}\phi\|_p\leqq C\sum_{|\alpha|=m}\|\partial^\alpha e^{-itH}\phi\|_2^a\cdot\|e^{-itH}\phi\|_2^{1-a},$$

where $a=\delta_n(p)/m$. By (4.1), we estimate the first factor $\sum_{|\alpha|=m}\|\partial^\alpha e^{-itH}\phi\|_2=\sum_{|\alpha|=m}\|\xi^\alpha\mathcal{F}e^{-itH}\phi\|_2$ on the R.H.S. of (4.2) as

$$(4.3)\qquad \begin{aligned} C\||\xi|^m\mathcal{F}e^{-itH}\phi\|_2 &\leqq C\|H_0^{m/2}e^{-itH}\phi\|_2 \\ &\leqq C\|(|H|^{m/2}+i)e^{-itH}\phi\|_2 \\ &\leqq C\|(|H|^{m/2}+i)\phi\|_2 \leqq C\|(H_0^{m/2}+i)\phi\|_2 . \end{aligned}$$

Combining (4.2) and (4.3), we have

$$\|e^{-itH}\phi\|_p \leqq C\|\phi\|_{m,0}^a\|e^{-itH}\phi\|_2^{1-a} \leqq C\|\phi\|_{m,0}, \qquad t\in\boldsymbol{R}.$$

Similarly,

$$\begin{aligned} \|e^{-itH}\phi-e^{-isH}\phi\|_p &\leqq C\sum_{|\alpha|=m}\|\partial^\alpha(e^{-itH}\phi-e^{-isH}\phi)\|_2^a\|e^{-itH}\phi-e^{-isH}\phi\|_2^{1-a} \\ &\leqq C\|(e^{-itH}-e^{-isH})(|H|^{m/2}+i)\phi\|_2^a\|(e^{-itH}-e^{-isH})\phi\|_2^{1-a} \\ &\longrightarrow 0, \qquad \text{as } t\to s. \end{aligned}$$

Q. E. D.

We study sufficient conditions for V to ensure $D(|H|^{m/2})=H^{m,0}$. For this purpose we prepare the following

Lemma 4.1. *Let $k\in\boldsymbol{N}$ and let $V_1,\cdots,V_k\in H_{\mathrm{loc}}^{2(k-1)}$. Suppose that V_j and their distributional derivatives up to the $2(k-1)$-th order are H_0-bounded, for all $j=1,\cdots,k$. Let a_j be the H_0-bound of V_j. Then, $\prod_{j=1}^k V_j$ is H_0^k-bounded with H_0^k-bound$\leqq\prod_{j=1}^k a_j$.*

Proof. The proof is essentially the same as that of M. Arai [1; Lemma 5]. Therefore, we omit the details. Q. E. D.

Proposition 4.1. *Let $m\in\boldsymbol{N}$. Let a be the H_0-bound of V. Suppose that $a<2^{1/([(m-1)/2]+1)}-1$. When $m\geqq3$, assume in addition that the distributional derivatives up to $(m-2)$-th order are H_0-bounded. Then, $D(|H|^{m/2})=H^{m,0}$,*

$$(4.4)\qquad C(m,a)\|\phi\|_{m,0}\leqq\|(|H|^{m/2}+i)\phi\|_2\leqq C(m)\|\phi\|_{m,0}, \qquad \phi\in H^{m,0}.$$

Remark 4.1. $2^{1/([(m-1)/2]+1)}-1=1$ when $m\leqq2$.

Proof of Proposition 4.1 (See also [1; Lemma 4]). Let $k=[m/2]$. We first remark that $\prod_{\mu=1}^l\partial^{\alpha_\mu}V\in\mathcal{S}'$, provided $l\leqq k$ and multi-indices α_μ $(1\leqq\mu\leqq l)$ satisfy $|\alpha_\mu|\leqq2(k-l)$. Indeed, by Lemma 4.1, we have for $\phi\in\mathcal{S}$

$$\begin{aligned} \Big|\Big(\prod_{\mu=1}^l\partial^{\alpha_\mu}V,\phi\Big)\Big| &\leqq\Big\|\Big(\prod_{\mu=1}^l\partial^{\alpha_\mu}V\Big)(1+|x|^2)^{[n/4]+1}\phi\Big\|_2\|(1+|x|^2)^{-[n/4]-1}\|_2 \\ &\leqq C\|(1+|x|^2)^{[n/4]+1}\phi\|_{2l,0} \\ &\leqq C\sum_{|\alpha|\leqq2l}\sup_{x\in\boldsymbol{R}^n}(1+|x|^2)^{[n/2]+2}|\partial^\alpha\phi(x)|. \end{aligned}$$

Let $j\in N$ satisfy $j\leqq k$ and let $\phi\in\mathcal{S}$. The Leibniz rule $\nabla(f\cdot g)=(\nabla f)g+f(\nabla g)$, $f\in\mathcal{S}$, $g\in\mathcal{S}'$, then gives

$$(-\Delta+V)^j\phi=(-\Delta)^j\phi+L_j\phi\,, \tag{4.5}$$

where

$$L_j\phi=\sum_{l=0}^{j-1}\binom{j}{l}V^{j-l}(-\Delta)^l\phi+\sum_{l=1}^{j-1}\sum_{\substack{|\beta|\leqq 2l-1\\ |\alpha_1+\cdots+\alpha_{j-l}+\beta|=2l}}C(\alpha_1,\cdots,\alpha_{j-l},\beta)\Big(\prod_{\mu=1}^{j-l}\partial^{\alpha_\mu}V\Big)\partial^\beta\phi.$$

This formula follows by induction on j and makes sense in $\mathcal{S}'$. Nevertheless, Lemma 4.1 implies that the R. H. S. of (4.5) is in L^2. Therefore, $\mathcal{S}\subset D(H^k)$ and the L. H. S. of (4.5) is equal to $H^j\phi$. Using (4.5) and Lemma 4.1, we have for $\varepsilon>0$

$$\begin{aligned}\|(H^k-H_0^k)\phi\|_2&\leqq\sum_{l=0}^{k-1}\binom{k}{l}\|V^{k-l}H_0^l\phi\|_2+C\sum_{l=1}^{k-1}\sum_{\substack{|\beta|\leqq 2l-1\\ |\alpha_1+\cdots+\alpha_{k-j}+\beta|=2l}}\Big\|\Big(\prod_{\mu=1}^{k-l}\partial^{\alpha_\mu}V\Big)\partial^\beta\phi\Big\|_2\\
&\leqq\sum_{l=0}^{k-1}\binom{k}{l}(a+\varepsilon)^{k-l}\|H_0^k\phi\|_2+C(\varepsilon)\|\phi\|_2+C\sum_{l=1}^{k-1}\sum_{|\beta|\leqq 2l-1}\|H_0^{k-l}\partial^\beta\phi\|_2\\
&\leqq((a+1+\varepsilon)^k-1)\|H_0^k\phi\|_2+C(\varepsilon)\|\phi\|_2+C\sum_{|\beta|\leqq 2k-1}\|\partial^\beta\phi\|_2\,.\end{aligned}$$

We estimate the last term by using the following inequality.

$$\begin{aligned}\sum_{|\alpha|=l}\|\partial^\alpha\phi\|_2&\leqq C\||\xi|^l\mathcal{F}\phi\|_2\leqq C\||\xi|^k\mathcal{F}\phi\|_2^{l/k}\|\mathcal{F}\phi\|_2^{(l-k)/k}\\
&\leqq\varepsilon\|H_0^k\phi\|_2+C\varepsilon^{-(k-l)/k}\|\phi\|_2\,,\qquad l\leqq k-1,\ \varepsilon>0\,.\end{aligned} \tag{4.6}$$

Collecting these estimates, we obtain

$$\|(H^k-H_0^k)\phi\|_2\leqq((a+1+\varepsilon)^k-1+\varepsilon)\|H_0^k\phi\|_2+C(\varepsilon)\|\phi\|_2\,.$$

When m is even, we have $k=m/2=[(m-1)/2]+1$, and hence $(a+1+\varepsilon)^k-1+\varepsilon<1$ for $\varepsilon>0$ sufficiently small. Since $H_0^{m/2}$ is essentially self-adjoint on $\mathcal{S}$, we apply the symmetrized version of the Kato-Rellich theorem (T. Kato [19; Theorem IX-4.5], M. Reed-B. Simon [31; Theorem X. 13]) to the inequality above to conclude that $H^{m/2}$ is essentially self-adjoint on $\mathcal{S}$ and $D(\overline{H^{m/2}\restriction\mathcal{S}})=H^{m,0}$, where the bar denotes the closure. This proves the theorem for m even. We turn to the case where m is odd. In this case, $k+1/2=m/2$ and $k+1=[(m-1)/2]+1$. Let $\phi\in\mathcal{S}$. The preceding argument shows

$$\begin{aligned}C\sum_{|\gamma|=1}(\|H_0^k\partial^\gamma\phi\|_2+\|\partial^\gamma\phi\|_2)&\leqq\sum_{|\gamma|=1}(\|H^k\partial^\gamma\phi\|_2+\|\partial^\gamma\phi\|_2)\\
&\leqq C'\sum_{|\gamma|=1}(\|H_0^k\partial^\gamma\phi\|_2+\|\partial^\gamma\phi\|_2)\,.\end{aligned} \tag{4.7}$$

We see from (4.5) that

$$[H^k, \partial^\gamma]\phi=[L_k, \partial^\gamma]\phi$$
$$=\sum_{l=0}^{k-1}\binom{k}{l}(\partial^\gamma(V^{k-l}))H_0^l\phi+\sum_{l=1}^{k-1}\sum_{\substack{|\beta|\leqq 2l-1\\ |\alpha_1+\cdots+\alpha_{k-l}+\beta|=2l}} C(\alpha_1, \cdots, \alpha_{k-l}, \beta)\Big(\partial^\gamma\Big(\sum_{\mu=1}^{k-l}\partial^{\alpha_\mu}V\Big)\Big)\partial^\beta\phi.$$

By applying Lemma 4.1 to the R. H. S. of the last equality, we find

$$\sum_{|\gamma|=1}\|[H^k, \partial^\gamma]\phi\|_2\leqq C\sum_{|\beta|\leqq 2k}\|\partial^\beta\phi\|_2\,. \tag{4.8}$$

By (4.7), (4.8) and (4.4) for m even, we obtain

$$\sum_{|\gamma|\leqq 1}\|\partial^\gamma H^k\phi\|_2\leqq C\sum_{|\beta|\leqq m}\|\partial^\beta\phi\|_2\,.$$

This together with (2.6), (2.7) and (2.4), implies that $H^k\phi\in D(|H|^{1/2})$ and

$$\begin{aligned}\||H|^{m/2}\phi\|_2&=\|H^k|H|^{1/2}\phi\|_2=\||H|^{1/2}H^k\phi\|_2\leqq C\sum_{|\gamma|\leqq 1}\|\partial^\gamma H^k\phi\|_2\\ &\leqq C\sum_{|\beta|\leqq m}\|\xi^\beta\mathcal{F}\phi\|_2\leqq C\|(|\xi|^m+i)\mathcal{F}\phi\|_2=C\|(H_0^{m/2}+i)\phi\|_2\,,\end{aligned} \tag{4.9}$$

and therefore the second inequality in (4.4) follows if $\phi\in\mathcal{S}$. If $\phi\in H^{m,0}$, then there exists a sequence $\{\phi_j\}$ in $\mathcal{S}$ such that $\phi_j\to\phi$ in $H^{m,0}$ as $j\to\infty$. By the inequality which we have just proved, we find that $\{(|H|^{m/2}+i)\phi_j\}$ is a Cauchy sequence in L^2. Since $|H|^{m/2}$ is closed, we have $\phi\in D(|H|^{m/2})$. Thus, $H^{m,0}\subset D(|H|^{m/2})$. We next prove that $D(|H|^{m/2})\subset H^{m,0}$. Let $\phi\in D(|H|^{m/2})$. By the first equalities in (4.9), we see that $H^k\phi\in D(|H|^{1/2})=H^{1,0}$, and moreover, by the moment inequality [35],

$$\begin{aligned}\sum_{|\gamma|\leqq 1}\|\partial^\gamma H^k\phi\|_2&\leqq C\||H|^{1/2}H^k\phi\|_2+C\|H^k\phi\|_2\\ &\leqq C\||H|^{m/2}\phi\|_2+C\||H|^k\phi\|_2\\ &\leqq C\||H|^{m/2}\phi\|_2+C\||H|^{m/2}\phi\|_2^{2k/m}\|\phi\|_2^{1-2k/m}\\ &\leqq C\||H|^{m/2}\phi\|_2+C\|\phi\|_2\,.\end{aligned} \tag{4.10}$$

Let $\phi\in D(H^k)$. Since $D(|H|^{m/2})\subset D(H^k)=H^{2k,0}=D(\overline{H^k\restriction\mathcal{S}})$, there exists a sequence $\{\psi_j\}$ (resp. $\{\phi_j\}$) such that $(H^k+i)\psi_j\to(H^k+i)\psi$ (resp. $(H^k+i)\phi_j\to(H^k+i)\phi$) in L^2 as $j\to\infty$. By the preceding step where m is even, we see that $\psi_j\to\psi$, $\phi_j\to\phi$ in $H^{2k,0}$ as $j\to\infty$. We write for $|\gamma|=1$,

$$\begin{aligned}(\partial^\gamma\psi_j, H^k\phi_j)&=(H^k\partial^\gamma\psi_j, \phi_j)=(\partial^\gamma H^k\psi_j, \phi_j)+([H^k, \partial^\gamma]\psi_j, \phi_j)\\ &=-(H^k\psi_j, \partial^\gamma\phi_j)+([H^k, \partial^\gamma]\psi_j, \phi_j)\,,\end{aligned}$$

so that by (4.8),

$$|(\partial^\gamma\psi_j, H^k\phi_j)|\leqq|(H^k\psi_j, \partial^\gamma\phi_j)|+C\sum_{|\beta|\leqq 2k}\|\partial^\beta\psi_j\|_2\|\phi_j\|_2\,.$$

Letting $j\to\infty$, we obtain

$$|(\partial^\gamma\psi, H^k\phi)|\leqq|(H^k\psi, \partial^\gamma\phi)|+C\sum_{|\beta|\leqq 2k}\|\partial^\beta\psi\|_2\|\phi\|_2\,.$$

By (4.10),

$$|(\partial^\gamma\phi,\ H^k\phi)|\leqq C(\||H|^{m/2}\phi\|_2+\|\phi\|_{2k,0})\|\phi\|_2, \qquad \phi\in D(H^k).$$

This implies that $\partial^\gamma\phi\in D((H^k)^*)=D(H^k)=H^{2k,0}$, $|\gamma|=1$, and hence $\phi\in H^{m,0}$, as required. We have thus proved that $D(|H|^{m/2})=H^{m,0}=D((1-\Delta)^{m/2})$. (4.4) then follows from the closed graph theorem. Q. E. D.

Proposition 4.2. *Let V be as in Proposition 4.1. Then for any $\lambda\geqq\lambda_0\equiv1+b$ (see (2.1).), $D((H+\lambda)^{m/2})=H^{m,0}$,*

$$C(a,\ \lambda,\ m)\|\phi\|_{m,0}\leqq\|(H+\lambda)^{m/2}\phi\|_2\leqq C(\lambda,\ m)\|\phi\|_{m,0}, \qquad \phi\in H^{m,0}.$$

Proof. Notice that $(H+\lambda)^{m/2}$ is a positive operator in L^2 if $\lambda\geqq\lambda_0$. Since $\sup_{\mu\geqq-b}(||\mu|^{m/2}+i|(\mu+\lambda)^{-m/2}+(\mu+\lambda)^{m/2}||\mu|^{m/2}+i|^{-1})<\infty$, an operator calculus shows that $D((H+\lambda)^{m/2})=D(|H|^{m/2})$, $(|H|^{m/2}+i)(H+\lambda)^{-m/2}\in\mathcal{L}(L^2)$, $(H+\lambda)^{m/2}\cdot(|H|^{m/2}+i)^{-1}\in\mathcal{L}(L^2)$. Therefore the result is obtained from Proposition 4.1. Q. E. D.

In the case $1\leqq p<2$, we have the following results (see §1 for notations).

Theorem 4.2. (1) *Let $1\leqq p<2$ for $n=1$ and let $2n/(n+2)<p<2$ for $n\geqq2$. Then*

$$e^{-itH}(H^{1,1})\subset L^p, \qquad t\in\boldsymbol{R},$$

$$\|e^{-itH}\phi\|_p\leqq C\|\phi\|_{0,1}, \qquad t\in\boldsymbol{R},\ \phi\in H^{1,1}.$$

For any $\phi\in H^{1,1}$, the map $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ is continuous.

(2) *Let $1\leqq p<2$ for $n\leqq3$ and let $2n/(n+4)<p<2$ for $n\geqq4$. Then*

$$e^{-itH}(H^{2,2})\subset L^p, \qquad t\in\boldsymbol{R},$$

$$\|e^{-itH}\phi\|_p\leqq C\|\phi\|_{2,2}, \qquad t\in\boldsymbol{R},\ \phi\in H^{2,2}.$$

For any $\phi\in H^{2,2}$, the map $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ is continuous.

(3) *Let $j\in\boldsymbol{N}$. Let $1\leqq p<2$ for $n\leqq2j-1$ and let $2n/(n+2j)<p<2$ for $n\geqq2j$. Then*

$$e^{-itH}(D_j)\subset L^p, \qquad t\in\boldsymbol{R},$$

$$\|e^{-itH}\phi\|_p\leqq C(1+|t|)^{-\delta_n(p)}\|\phi\|_{D_j}, \qquad t\in\boldsymbol{R},\ \phi\in D_j.$$

For any $\phi\in D_j$, the map $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ is continuous.

(4) *Let $j\in\boldsymbol{N}$. Let p be as in part (3). Then*

$$e^{-itH_0}(H^{j,j})\subset L^p, \qquad t\in\boldsymbol{R},$$

$$\|e^{-itH_0}\phi\|_p\leqq C(1+|t|)^{-\delta_n(p)}\|\phi\|_{j,j}, \qquad t\in\boldsymbol{R},\ \phi\in H^{j,j}.$$

For any $\phi\in H^{j,j}$, the map $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi\in L^p$ is continuous.

Proof. We first prove part (4). Let $u(t)=e^{-itH}\phi$. Let q satisfy $1/q=1/p-1/2$. Noting that $jq>n$, we obtain by Hölder's inequality.

$$\|u(t)\|_p \leqq \left(\int_{|x|\leqq 1+|t|} |u(t)|^2 dx\right)^{1/2} \left(\int_{|x|\leqq 1+|t|} dx\right)^{1/q}$$
$$+\left(\int_{|x|>1+|t|} |x|^{2j} |u(t)|^2 dx\right)^{1/2} \left(\int_{|x|>1+|t|} |x|^{-jq} dx\right)^{1/q}$$
$$\leqq C(1+|t|)^{n/q}\|\phi\|_2 + C(1+|t|)^{n/q-j}\| |x|^j u(t)\|_2 .$$

By virtue of Theorem 2.3, we have

$$\| |x|^j u(t)\|_2 \leqq C(1+|t|)^j \|\phi\|_{j,j}$$

and thus,

$$\|u(t)\|_p \leqq C(1+|t|)^{n/q}\|\phi\|_{j,j} .$$

Similarly,

$$\|u(t)-u(s)\|_p \leqq \left(\int_{|x|\leqq 1+|s|} |u(t)-u(s)|^2 dx\right)^{1/2} \left(\int_{|x|\leqq 1+|s|} dx\right)^{1/q}$$
$$+\left(\int_{|x|>1+|s|} |x|^{2j} |u(t)-u(s)|^2 dx\right)^{1/2} \left(\int_{|x|>1+|s|} |x|^{-jq} dx\right)^{1/q}$$
$$\leqq C(1+|s|)^{n/q}\|u(t)-u(s)\|_2 + C(1+|s|)^{n/q-j}\|u(t)-u(s)\|_{0,j} .$$

Again by Theorem 2.3, the R. H. S. of the last inequality tends to zero as $t\to s$. This proves part (4). Parts (1), (2) and (3) can be treated analogously if we use Theorems 2.1, 2.2 and Lemma 1.3, respectively, instead of Theorem 2.3.

Q. E. D.

Remark 4.2. It is well known (see [8]) that for $p\in[1, 2)$,

$$\|e^{-itH_0}\phi\|_{p'} \leqq (4\pi|t|)^{\delta_n(p)}\|\phi\|_p , \qquad t\in \boldsymbol{R}\setminus\{0\},\ \phi\in L^p.$$

From this estimate and Theorem 4.2 (4), we have

$$C|t|^{-\delta_n(p)}\|\phi\|_{p'} \leqq \|e^{-itH_0}\phi\|_p \leqq C(1+|t|)^{-\delta_n(p)}\|\phi\|_{j,j} , \qquad t\in \boldsymbol{R},\ \phi\in H^{j,j}.$$

where p and j are as in part (3) of Theorem 4.2.

Theorem 4.3. *Let $q\geqq 1$ for $n=1$ and let $q>n/2$ for $n\geqq 2$. Suppose that $(A2)^{\#}$ holds and that $V, V^{\#}\in L^q+L^\infty$. Let $2\leqq p\leqq\infty$ for $n=1$, $2\leqq p<\infty$ for $n=2$, and let $2\leqq p\leqq 2n/(n-2)$ for $n\geqq 3$. Then:*

$$e^{-itH}(H^{0,1})\subset L^p , \qquad t\in \boldsymbol{R}\setminus\{0\},$$

$$\|e^{-itH}\phi\|_p \leqq C(1+|t|^{-\delta_n(p)})\|\phi\|_{0,1} , \qquad t\in \boldsymbol{R}\setminus\{0\},\ \phi\in H^{0,1}. \tag{4.11}$$

For any $\phi\in H^{0,1}$, the map $\boldsymbol{R}\setminus\{0\}\ni t\mapsto e^{itH}\phi\in L^p$ is continuous.

Proof. Let $\{\phi_j\}$ be a sequence in $H^{2,1}$ such that $\phi_j\to\phi$ in $H^{0,1}$ as $j\to\infty$.

We already know from Theorem 4.1 that $\boldsymbol{R}\ni t\mapsto e^{-itH}\phi_j\in L^p$ is continuous. We prove that $\{e^{-itH}\phi_j\}$ is a Cauchy sequence in L^p if $t\neq 0$. Let $a=\delta_n(p)$. Let $S(t)=\exp(|x|^2/4it)$. By (3.1) and (3.10),

$$
\begin{aligned}
(4.12)\qquad \|e^{-itH}(\phi_j-\phi_k)\|_p &= \|S(t)e^{-itH}(\phi_j-\phi_k)\|_p \\
&\leqq C|t|^{-a}\|(x+2it\nabla)e^{-itH}(\phi_j-\phi_k)\|_2^a\cdot\|\phi_j-\phi_k\|_2^{1-a} \\
&\leqq C|t|^{-a}(1+|t|)^a\|\phi_j-\phi_k\|_{0,1}\longrightarrow 0 \qquad \text{as } j,\ k\to\infty.
\end{aligned}
$$

Therefore, $\{e^{-itH}\phi_j\}$ is convergent in $L^p\cap L^2$. This implies $e^{-itH}\phi\in L^p$ and $e^{-itH}\phi_j\to e^{-itH}\phi$ in L^p as $j\to\infty$, provided $t\neq 0$. (4.11) now follows by letting $j\to\infty$ in the following inequality:

$$\|e^{-itH}\phi_j\|_p\leqq C(1+|t|^{-a})\|\phi_j\|_{0,1}, \qquad t\neq 0.$$

We next prove that $\boldsymbol{R}\setminus\{0\}\ni t\mapsto e^{-itH}\phi\in L^p$ is continuous. Let $v_j(t)=e^{itH_0}xe^{-itH_0}e^{-itH}\phi_j$. Again by (3.1).

$$
\begin{aligned}
(4.13)\qquad & \|S(t)e^{-itH}\phi_j-S(s)e^{-isH}\phi_j\|_p \\
&\leqq C\|S(t)(2it)^{-1}(x+2it\nabla)e^{-itH}\phi_j-S(s)(2is)^{-1}(x+2is\nabla)e^{-isH}\phi_j\|_2^a \\
&\quad\cdot\|S(t)e^{-itH}\phi_j-S(s)e^{-isH}\phi_j\|_2^{1-a} \\
&\leqq C\|S(t)(2it)^{-1}v_j(t)-S(s)(2is)^{-1}v_j(s)\|_2^a \\
&\quad\cdot\|S(t)e^{-itH}\phi_j-S(s)e^{-itH}\phi_j\|_2^{1-a}.
\end{aligned}
$$

Put $u(t)=e^{-itH}\phi$, $v(t)=e^{itH_0}xe^{-itH_0}u(t)$. Taking the limit $j\to\infty$ in (4.13), we obtain from (3.10) that

$$
\begin{aligned}
(4.14)\qquad & \|S(t)u(t)-S(s)u(s)\|_p \\
&\leqq C\|t^{-1}S(t)v(t)-s^{-1}S(s)v(s)\|_2^a\|S(t)u(t)-S(s)u(s)\|_2^{1-a}.
\end{aligned}
$$

Since we already know from Theorem 3.2 that $v\in C(\boldsymbol{R};L^2)$, we conclude from (4.14) that $Su\in C(\boldsymbol{R}\setminus\{0\};L^p)$ and therefore $u\in C(\boldsymbol{R}\setminus\{0\};L^p)$, as was to be shown. Q. E. D.

Remark 4.3. If we assume that the hypotheses of Theorem 3.3 hold, then we have the same conclusions as those of Theorem 4.3. The proof is exactly the same.

§ 5. L^p-Estimates of the Schrödinger Equations II

In this section we consider some classes of potentials V and of initial data ϕ, for which the quantities (1)-(3) in the introduction tend to zero as $t\to\pm\infty$. We list the additional assumptions on V.

(A3) *V is compact from $H^{2,0}$ to $H^{-2,0}$.*

(A4) *The H_0-bound of V is equal to zero.*

(A5) *$V^{\#}$ is compact from $H^{2,0}$ to $H^{-2,0}$.*

(A6) *$V^{\#}\leqq 0$ as forms on $H^{2,0}$.*

Theorem 5.1. *Suppose that* (A2)$^{\#}$, (A3) *and* (A5) *hold. Then*:

(1) *For any* $\phi\in\mathcal{H}_{ac}\cap H^{1,0}$, $\lim_{t\to\pm\infty}(e^{itH}Ve^{-itH}\phi, \phi)=0$.

(2) *For any* $\phi\in\mathcal{H}_{ac}\cap H^{1,1}$, $\lim_{t\to\pm\infty}\|(x/t)e^{itH_0}e^{-itH}\phi\|_2=0$.

(3) *For any* $\phi\in\mathcal{H}_{ac}\cap H^{1,1}$, $\lim_{t\to\pm\infty}\|e^{-itH}\phi\|_p=0$,

for all p such that $2<p\leqq\infty$ for $n=1$, $2<p<\infty$ for $n=2$, and $2<p\leqq 2n/(n-2)$ for $n\geqq 3$.

(4) *If in addition, $D(|H|^{m/2})=H^{m,0}$ for some $m\in\boldsymbol{N}$, then for any $\phi\in\mathcal{H}_{ac}\cap H^{m,1}$, $\lim_{t\to\pm\infty}\|e^{-itH}\phi\|_p=0$,*

for all p such that $2<p\leqq\infty$ for $n\leqq 2m-1$, $2<p<\infty$ for $n=2m$, and $2<p\leqq 2n/(n-2m)$ for $n\geqq 2m+1$.

Remark 5.1. For the proof of part (1), we need (A3) only (see the proof below).

Remark 5.2. See also V. Enss [5][6] and P. Perry [28] for related results.

Proof of Theorem 5.1. (1) Let $\phi\in\mathcal{H}_{ac}\cap H^{2,0}$. It follows that $e^{-itH}\phi\in H^{2,0}$, $e^{-itH}\phi=(H+i)^{-1}e^{-itH}(H+i)\phi$ and $(H+i)\phi\in\mathcal{H}_{ac}$. We estimate $(e^{itH}Ve^{-itH}\phi, \phi)$ as

$$
\begin{aligned}
(5.1)\qquad |(e^{itH}Ve^{-itH}\phi, \phi)| &= |(V(H+i)^{-1}e^{-itH}(H+i)\phi, e^{-itH}\phi)| \\
&\leqq \|V(H+i)^{-1}e^{-itH}(H+i)\phi\|_{-2,0}\cdot\|e^{-itH}\phi\|_{2,0}.
\end{aligned}
$$

Since $e^{-itH}(H+i)\phi\to 0$ weakly in L^2 as $t\to\pm\infty$, (A5) implies that the first factor of the R. H. S. of the inequality in (5.1) tends to zero as $t\to\pm\infty$. On the other hand, by (2.8), the second factor is estimated by $C\|\phi\|_{2,0}$. Thus, for $\phi\in\mathcal{H}_{ac}\cap H^{2,0}$,

$$(5.2)\qquad |(e^{itH}Ve^{-itH}\phi, \phi)|\longrightarrow 0 \quad \text{as } t\to\pm\infty.$$

Let $\phi\in\mathcal{H}_{ac}\cap H^{2,0}$ and let $\phi_\lambda=i\lambda(H+i\lambda)^{-1}\phi$, $\lambda\in\boldsymbol{R}\setminus\{0\}$. Then $\phi_\lambda\in\mathcal{H}_{ac}\cap H^{2,0}$. We estimate $(e^{itH}Ve^{-itH}\phi, \phi)$ as

$$
\begin{aligned}
(5.3)\qquad & |(e^{itH}Ve^{-itH}\phi, \phi)| \\
& \leqq |(e^{itH}Ve^{-itH}(\phi-\phi_\lambda), \phi)| + |(e^{itH}Ve^{-itH}\phi_\lambda, \phi-\phi_\lambda)| \\
& \quad + |(e^{itH}Ve^{-itH}\phi_\lambda, \phi_\lambda)| \\
& \leqq C\|\phi-\phi_\lambda\|_{1,0}\cdot(\|\phi\|_{1,0}+\|\phi_\lambda\|_{1,0}) + |(e^{itH}Ve^{-itH}\phi_\lambda, \phi_\lambda)|,
\end{aligned}
$$

where we have used part (3) of Theorem 2.1. It follows from (5.2) that the second term of the R.H.S. of the final inequality in (5.3) tends to zero as $t\to\pm\infty$. Part (1) of Lemma 2.1 shows that $\phi_\lambda\to\phi$ in $H^{1,0}$ as $\lambda\to\pm\infty$. This proves part (1).

(2) Let $\phi\in\mathcal{H}_{ac}\cap H^{2,1}$. In the same way as above,

$$\lim_{t\to\pm\infty}(e^{itH}V^{\#}e^{-itH}\phi, \phi)=0\,.$$

Noting that

$$\sup_{t\in R}|(e^{itH}(V+(1/2)V^{\#})e^{-itH}\phi, \phi)|\leqq C\|\phi\|_{2,0}^2\,,$$

we obtain

$$
\begin{aligned}
& \left|t^{-2}\int_0^t \tau(e^{i\tau H}(V+(1/2)V^{\#})e^{-i\tau H}\phi, \phi)d\tau\right| \\
& \quad \leqq \left|t^{-1}\int_0^t |(e^{i\tau H}(V+(1/2)V^{\#})e^{-i\tau H}\phi, \phi)|\,d\tau\right| \longrightarrow 0 \quad \text{as } t\to\pm\infty.
\end{aligned}
$$

By using Theorem 3.1, we see that $\|(x/t)e^{itH_0}e^{-itH}\phi\|_2\to 0$ as $t\to\pm\infty$. Let $\phi\in\mathcal{H}_{ac}\cap H^{1,1}$ and let $\phi_\lambda=i\lambda(H+i)^{-1}\phi$. By Lemma 2.1, $\phi_\lambda\to\phi$ in $H^{1,1}$ as $\lambda\to\pm\infty$. We estimate $\|(x/t)e^{itH_0}e^{-itH}\phi\|_2$ as

$$
\begin{aligned}
\|(x/t)e^{itH_0}e^{-itH}\phi\|_2 & \leqq \|(x/t)e^{itH_0}e^{-itH}(\phi-\phi_\lambda)\|_2 + \|(x/t)e^{itH_0}e^{-itH}\phi_\lambda\|_2 \\
& \leqq C\|\phi_\lambda-\phi\|_2 + \|(x/t)e^{itH_0}e^{-itH}\phi_\lambda\|_2\,, \qquad |t|\geqq 1,
\end{aligned}
$$

where we have used Theorem 2.1. This proves part (2), since $\phi_\lambda\in\mathcal{H}_{ac}\cap H^{2,1}$.

(3) Part (3) follows from part (2) and the following inequality

$$\|\phi\|_p\leqq C\|(x/t)e^{itH_0}\phi\|_2^{\delta_n(p)}\|\phi\|_2^{1-\delta_n(p)}\,,$$

which can be derived from (3.1).

(4) When $m=1$, part (4) reduces to part (3). Let $m\geqq 2$. We distinguish between two cases:

$$\text{(i)}\quad n\geqq 3,\ p\geqq 2n/(n-2). \qquad \text{(ii)}\quad n=2,\ p=\infty.$$

(i) When $n\geqq 3$ and $p\geqq 2n/(n-2)$, we use the following inequality which can be derived from Lemma 1.1

(5.4) $$\|\phi\|_p \leqq C \sum_{|\alpha|=m} \|\partial^\alpha \phi\|_2^a \|\phi\|_{2n/(n-2)}^{1-a},$$

where $a=(n/(m-1))(1/2+1/n-1/p)$; $2n/(n-2)\leqq p\leqq\infty$ for $n\leqq 2m-1$; $2n/(n-2)\leqq p<\infty$ for $n=2m$; $2n/(n-2)\leqq p\leqq 2n/(n-2m)$ for $n\geqq 2m+1$. (4.3) and (5.4) yield

$$\|e^{-itH}\phi\|_p \leqq C\|\phi\|_{m,0}^a \|e^{-itH}\phi\|_{2n/(n-2)}^{1-a},$$

so that the result follows from part (3).

(ii) When $n=2$ and $p=\infty$, we use the following inequality which can be derived from Lemma 1.1

(5.5) $$\|\phi\|_\infty \leqq C \sum_{|\alpha|=m} \|\partial^\alpha \phi\|_2^a \|\phi\|_q^{1-a},$$

where $2\leqq q<\infty$, $a=2/((m-1)q+2)$. In the same way as in the case (i), the result follows from (4.3), (5.5), and part (3). Q.E.D.

Theorem 5.2. *Let $q\geqq 1$ for $n=1$ and let $q>n/2$ for $n\geqq 2$. Suppose that* (A2)$^\#$, (A3) *and* (A5) *hold and that V, $V^\# \in L^q+L^\infty$. Then*:

(1) *For any $\phi\in\mathcal{H}_{ac}\cap H^{0,1}$, $\lim_{t\to\pm\infty}(e^{itH}Ve^{-itH}\phi, \phi)=0$.*

(2) *For any $\phi\in\mathcal{H}_{ac}\cap H^{0,1}$, $\lim_{t\to\pm\infty}\|(x/t)e^{itH_0}e^{-itH}\phi\|_2=0$.*

(3) *For any $\phi\in\mathcal{H}_{ac}\cap H^{0,1}$, $\lim_{t\to\pm\infty}\|e^{-itH}\phi\|_p=0$,*

for all p such that $2<p\leqq\infty$ for $n=1$, $2<p<\infty$ for $n=2$, and $2<p\leqq 2n/(n-2)$ for $n\geqq 3$.

Proof. Let $\phi\in\mathcal{H}_{ac}\cap H^{0,1}$ and let $\phi_\lambda=i\lambda(H+i)^{-1}\phi$, $\lambda\in \boldsymbol{R}\setminus\{0\}$. By Lemma 2.1, $\phi_\lambda\in H^{2,1}$ and $\phi_\lambda\to\phi$ in $H^{0,1}$ as $\lambda\to\pm\infty$. Theorem 5.1 shows that

$$\lim_{t\to\pm\infty}(e^{itH}Ve^{-itH}\phi_\lambda, \phi_\lambda)=0,$$

$$\lim_{t\to\pm\infty}\|(x/t)e^{itH_0}e^{-itH}\phi_\lambda\|_2=0,$$

$$\lim_{t\to\pm\infty}\|e^{-itH}\phi_\lambda\|_p=0.$$

Moreover, we find from (3.10), (3.11) and (4.7) that for $|t|\geqq 1$,

$$|(e^{itH}Ve^{-itH}(\phi-\phi_\lambda), \phi_\lambda)+(e^{itH}Ve^{-itH}\phi, \phi-\phi_\lambda)|$$
$$\leqq C\|\phi-\phi_\lambda\|_{0,1}\cdot(\|\phi\|_{0,1}+\|\phi_\lambda\|_{0,1}),$$

$$\|(x/t)e^{itH_0}e^{-itH}(\phi-\phi_\lambda)\|_2\leqq C\|\phi-\phi_\lambda\|_{0,1},$$

$$\|e^{-itH}(\phi-\phi_\lambda)\|_p\leqq C\|\phi-\phi_\lambda\|_{0,1}.$$

Therefore we have the assertion. Q.E.D.

Theorem 5.3. *Let* $n \geqq 3$. *Suppose that* (A2)$^{\#}$, (A3), (A4) *and* (A6) *hold. Then*:

(1) *For any* $\phi \in H^{1,0}$, $\lim_{t\to\pm\infty}(e^{itH}Ve^{-itH}\phi, \phi)=0$.

(2) *For any* $\phi \in H^{1,1}$, $\lim_{t\to\pm\infty}\|(x/t)e^{itH_0}e^{-itH}\phi\|_2=0$.

(3) *For any* $\phi \in H^{1,1}$, $\lim_{t\to\pm\infty}\|e^{-itH}\phi\|_p=0$,

for all p *such that* $2<p\leqq 2n/(n-2)$.

(4) *If in addition*, $D(|H|^{m/2})=H^{m,0}$ *for some* $m\in \boldsymbol{N}$, *then for any* $\phi\in H^{m,1}$,

$$\lim_{t\to\pm\infty}\|e^{-itH}\phi\|_p=0,$$

for all p *such that* $2<p\leqq\infty$ *for* $n\leqq 2m-1$, $2<p<\infty$ *for* $n=2m$, *and* $2<p\leqq 2n/(n-2m)$ *for* $n\geqq 2m+1$.

Proof. By virtue of the Lavine-Arai Theorem (see R. Lavine [21], M. Arai [1], M. Reed & B. Simon [31]), H is absolutely continuous under the assumptions (A4), (A6) and $n\geqq 3$. Therefore all of the arguments in the proof of Theorem 5.1 work, except that we use Theorem 3.3 with $\lambda=1$ in place of Theorem 3.1.
Q. E. D.

Theorem 5.4. *Let* $n\geqq 3$. *Let* $q\geqq 2$ *for* $n=3$ *and let* $q>n/2$ *for* $n\geqq 4$. *Suppose that* $V\in L^q+L^\infty$ *and that* (A2)$^{\#}$, (A3) *and* (A6) *hold. Then*:

(1) *For any* $\phi\in H^{0,1}$, $\lim_{t\to\pm\infty}(e^{itH}Ve^{-itH}\phi, \phi)=0$.

(2) *For any* $\phi\in H^{0,1}$, $\lim_{t\to\pm\infty}\|(x/t)e^{itH_0}e^{-itH}\phi\|_2=0$.

(3) *For any* $\phi\in H^{0,1}$, $\lim_{t\to\pm\infty}\|e^{-itH}\phi\|_p=0$,

for all p *such that* $2<p\leqq 2n/(n-2)$.

Proof. We first note that (A4) follows from the assumption that $V\in L^q+L^\infty$. The proof now proceeds from Theorem 5.3 in the same way as in the derivation of Theorem 5.2 from Theorem 5.1. Q. E. D.

§ 6. L^p-Estimates for the Schrödinger Equations III

In this section we study the decay rate of the scattering solutions in the L^p-norm. We need the following function spaces. For $m, s\in \boldsymbol{R}$, $\tilde{H}^{m,s}$ denotes the Hilbert space

$$\tilde{H}^{m,s}=\{\phi\in\mathcal{S}' ; |||\phi|||_{m,s}=\|\omega^s(1-\Delta)^{m/2}\phi\|_2<\infty\}$$

with the scalar product

$$(\psi, \phi)_{m,s}=(\omega^s(1-\Delta)^{m/2}\psi, \omega^s(1-\Delta)^{m/2}\phi).$$

By M. Tsutsumi's theorem [36; Theorem 2.3], an equivalent norm on $\tilde{H}^{m,s}$ is given by

$$|||\psi|||_{m,s}^{(1)}=\|(1-\Delta)^{m/2}\omega^s\psi\|_2=\|\omega^s\psi\|_{m,0},$$

so that $\tilde{H}^{m,s}\hookrightarrow\tilde{H}^{m',s'}$ if $m'\leqq m$, $s'\leqq s$. Part (2) of Lemma 1.2 shows that if $m\in\boldsymbol{N}$, $s\geqq 0$, then the following norms are equivalent norms on $\tilde{H}^{m,s}$:

$$|||\psi|||_{m,s}^{(2)}=\sum_{|\alpha|\leqq m}\|\partial^\alpha(\omega^s\psi)\|_2,$$

$$|||\psi|||_{m,s}^{(3)}=\sum_{|\alpha|=m}\|\partial^\alpha(\omega^s\psi)\|_2+\|\psi\|_{0,s},$$

$$|||\psi|||_{m,s}^{(4)}=\sum_{|\alpha|\leqq m}\|\omega^s\partial^\alpha\psi\|_2,$$

$$|||\psi|||_{m,s}^{(5)}=\sum_{|\alpha|=m}\|\omega^s\partial^\alpha\psi\|_2+\|\psi\|_{0,s}.$$

In this case, part (2) of Lemma 1.2 also implies that

$$|||\psi|||_{k,s}\leqq C|||\psi|||_{m,s}^{k/m}|||\psi|||_{0,s}^{1-k/m},\tag{6.1}$$

where $k\in\boldsymbol{N}$ satisfy $k\leqq m$.

Lemma 6.1. *Let* $s\geqq 0$. *Then*:

(1) $\tilde{H}^{2,s}=\{\psi\in H^{2,s};\ H\psi\in H^{0,s}\}$,

$$C|||\psi|||_{2,s}\leqq\|H\psi\|_{0,s}+\|\psi\|_{0,s}\leqq C'|||\psi|||_{2,s},\qquad \psi\in\tilde{H}^{2,s}.\tag{6.2}$$

(2) *For any* $\mu\geqq\lambda_0\equiv 1+b$ *(see (2.1)) and any* $\lambda\in\boldsymbol{C}$ *with* $\mathrm{Re}\,\lambda\geqq 0$, $\partial^\alpha(H+\lambda+\mu)^{-1}\in\mathcal{L}(H^{0,s})$, $|\alpha|\leqq 1$.

(3) *There exists a constant* $\lambda_1\geqq\lambda_0$ *depending only on* s, n, a *and* b *such that for any* $\mu\geqq\lambda_1$ *and any* $\lambda\in\boldsymbol{C}$ *with* $\mathrm{Re}\,\lambda\geqq 0$,

$$\|(H+\mu+\lambda)^{-1}\|_{\mathcal{L}(H^{0,s})}\leqq(\mathrm{Re}\,\lambda+1)^{-1}.\tag{6.3}$$

(4) *For any* $\lambda\in\boldsymbol{C}$ *with* $\mathrm{Re}\,\lambda\geqq\lambda_1$, $\{\psi\in H^{2,0};\ (H+\lambda)\psi\in H^{0,s}\}=\tilde{H}^{2,s}$,

$$C(\lambda, s)|||\psi|||_{2,s}\leqq\|(H+\lambda)\psi\|_{0,s}\leqq C'(\lambda, s)|||\psi|||_{2,s},\qquad \psi\in\tilde{H}^{2,s}.\tag{6.4}$$

Proof. (1) (See also W. Hunziker [12; Lemma 1]). It follows from the preceding argument and (2.5) that the norm $|||\cdot|||_{2,s}^{(6)}$, defined by $|||\psi|||_{2,s}^{(6)}=\|H\omega^s\psi\|_2+\|\omega^s\psi\|_2$, is an equivalent norm on $\tilde{H}^{2,s}$. Thus, part (1) will follow if we can show that the norm $|||\cdot|||_{2,s}^{(7)}$, defined by $|||\psi|||_{2,s}^{(7)}=\|\omega^s H\psi\|_2+\|\omega^s\psi\|_2$, is equivalent to $|||\cdot|||_{2,s}^{(6)}$. Let $\psi\in H^{2,s}$ satisfy $H\psi\in H^{0,s}$. Then for any $\phi\in\mathcal{S}$ we have

$$\begin{aligned}(\omega^s\psi, H\phi)&=(\omega^s H\psi, \phi)-(\psi, [H, \omega^s]\phi)\\&=(\omega^s H\psi, \phi)+2\sum_{j=1}^n(\psi, \partial_j(\partial_j\omega^s\cdot\phi))+(\psi, \Delta\omega^s\cdot\phi).\end{aligned}$$

We estimate the first and the last terms on the R. H. S. of the last equality as

$$|(\omega^s H\psi,\ \phi)|\leqq\|\omega^s H\psi\|_2\|\phi\|_2\leqq|||\psi|||_{2,s}^{(7)}\|\phi\|_2\,,$$

$$|(\psi,\ \Delta\omega^s\cdot\phi)|\leqq C\|\omega^{s-2}\psi\|_2\|\phi\|_2\leqq C|||\psi|||_{2,s}^{(7)}\|\phi\|_2\,.$$

If $0\leqq s\leqq 1$, then the middle term is estimated as

$$|2\sum_{j=1}^{n}(\psi,\ \partial_j(\partial_j\omega^s\cdot\phi))|\leqq 2s\sum_{j=1}^{n}\|\omega^{s-1}\partial_j\psi\|_2\|\phi\|_2\leqq C\|\nabla\psi\|_{2,0}\|\phi\|_2 \tag{6.5}$$

$$\leqq C\|\psi\|_{2,0}\|\phi\|_2\leqq C|||\psi|||_{2,0}^{(7)}\|\phi\|_2\,,$$

and therefore we obtain

$$|(\omega^s\psi,\ H\phi)|\leqq C|||\psi|||_{2,s}^{(7)}\|\phi\|_2\,. \tag{6.6}$$

Since (6.6) extends to $\phi\in D(H)$, we conclude that $\omega^s\psi\in D(H^*)=D(H)$ and $|||\psi|||_{2,s}^{(6)}\leqq C|||\psi|||_{2,s}^{(7)}$, provided $0\leqq s\leqq 1$. Let $\psi\in H^{2,s}$ satisfy $\omega^s\psi\in D(H)$. Since $|||\cdot|||_{2,s}^{(6)}$ is an equivalent norm on $\widetilde{H}^{2,s}$, we have $\psi\in\widetilde{H}^{2,s}$. If $0\leqq s\leqq 1$, then in the same way as above, we see that $H\psi\in D(\omega^s)=H^{0,s}$ and $|||\psi|||_{2,s}^{(7)}\leqq C|||\psi|||_{2,s}^{(6)}$. We have thus proved part (1) in the case $0\leqq s\leqq 1$. Let $k\in\boldsymbol{N}$ and assume that part (1) holds for $s\in(k-1,\ k]$. Let $\psi\in H^{2,s}$ satisfy $H\psi\in H^{0,s}$. We perform the same procedure as in the case $0\leqq s\leqq 1$, so that we only have to estimate the L. H. S. of (6.5) in the case $k<s\leqq k+1$. Now

$$\|\omega^s\partial_j\psi\|_2\leqq C|||\psi|||_{2,s-1}^{(4)}\leqq C|||\psi|||_{2,s-1}^{(6)}\leqq C|||\psi|||_{2,s-1}^{(7)}\,,$$

where we have used the induction hypothesis at the last inequality. Therefore, $|||\psi|||_{2,s}^{(6)}\leqq C|||\psi|||_{2,s}^{(7)}$. Similarly, we have $|||\psi|||_{2,s}^{(7)}\leqq C|||\psi|||_{2,s}^{(6)}$, $\psi\in\widetilde{H}^{2,s}$. This proves part (1).

(2) For any $\lambda\in\boldsymbol{C}$ with $\operatorname{Re}\lambda\geqq 0$, we obtain, by an operator calculus and (2.7),

$$\|(H+\lambda_0+\lambda)^{-1}\|_{\mathcal{L}(L^2)}\leqq\sup_{\mu\geqq -b}|(\mu+\lambda_0+\lambda)^{-1}|\leqq((1+\operatorname{Re}\lambda)^2+(\operatorname{Im}\lambda)^2)^{-1/2} \tag{6.7}$$

$$\leqq C(1+|\lambda|)^{-1}\,,$$

$$\sum_{|\alpha|=1}\|\partial^\alpha(H+\lambda_0+\lambda)^{-1}\|_{\mathcal{L}(L^2)}\leqq C\|(|H|^{1/2}+i)(H+\lambda_0+\lambda)^{-1}\|_{\mathcal{L}(L^2)} \tag{6.8}$$

$$\leqq C\cdot\sup_{\mu\geqq -b}|(|\mu|^{1/2}+i)(\mu+\lambda_0+\lambda)^{-1}|$$

$$\leqq C\lambda_0(1+|\lambda|)^{-1/2}\,.$$

This proves part (2) for $s=0$. Let ζ_ε be as in Lemma 1.4. For $\psi\in H^{0,s}$, we have

$$\begin{aligned}&\zeta_\varepsilon\omega^s(H+\lambda_0+\lambda)^{-1}\psi \\ &=(H+\lambda_0+\lambda)^{-1}\zeta_\varepsilon\omega^s\psi-(H+\lambda_0+\lambda)^{-1}(\Delta(\zeta_\varepsilon\omega^s))(H+\lambda_0+\lambda)^{-1}\psi \\ &\quad-2(H+\lambda_0+\lambda)^{-1}(\nabla(\zeta_\varepsilon\omega^s))\cdot\nabla(H+\lambda_0+\lambda)^{-1}\psi\,,\end{aligned} \tag{6.9}$$

(6.10) $$\zeta_\varepsilon\omega^s\partial_j(H+\lambda_0+\lambda)^{-1}\psi$$
$$=\partial_j(H+\lambda_0+\lambda)^{-1}\zeta_\varepsilon\omega^s\psi-\partial_j(H+\lambda_0+\lambda)^{-1}(\Delta(\zeta_\varepsilon\omega^s))(H+\lambda_0+\lambda)^{-1}\psi$$
$$-2\partial_j(H+\lambda_0+\lambda)^{-1}(\nabla(\zeta_\varepsilon\omega^s))\cdot\nabla(H+\lambda_0+\lambda)^{-1}\psi-(\partial_j(\zeta_\varepsilon\omega^s))(H+\lambda_0+\lambda)^{-1}\psi.$$

If $0<s\leqq 1$, then for any nonzero multi-index β, $\partial^\beta(\zeta_\varepsilon\omega^s)\in\mathcal{L}(L^2)$, $\partial^\beta(\zeta_\varepsilon\omega^s)\to\partial^\beta\omega^s$ strongly in $\mathcal{L}(L^2)$ as $\varepsilon\to+0$, and hence, in the same way as in the proof of Lemma 2.1, it follows from (6.7), (6.8), (6.9) and (6.10) that $\partial^\alpha(H+\lambda_0+\lambda)^{-1}\psi\in H^{0,s}$, $|\alpha|\leqq 1$,

(6.11) $$\omega^s(H+\lambda_0+\lambda)^{-1}\psi$$
$$=(H+\lambda_0+\lambda)^{-1}\omega^s\psi-(H+\lambda_0+\lambda)^{-1}(\Delta\omega^s)(H+\lambda_0+\lambda)^{-1}\psi$$
$$-2(H+\lambda_0+\lambda)^{-1}(\nabla\omega^s)\cdot\nabla(H+\lambda_0+\lambda)^{-1}\psi,$$

(6.12) $$\omega^s\partial_j(H+\lambda_0+\lambda)^{-1}\psi$$
$$=\partial_j(H+\lambda_0+\lambda)^{-1}\omega^s\psi-\partial_j(H+\lambda_0+\lambda)^{-1}(\Delta\omega^s)(H+\lambda_0+\lambda)^{-1}\psi$$
$$-2\partial_j(H+\lambda_0+\lambda)^{-1}(\nabla\omega^s)\cdot\nabla(H+\lambda_0+\lambda)^{-1}\psi-(\partial_j\omega^s)(H+\lambda_0+\lambda)^{-1}\psi.$$

By (6.7), (6.8), (6.11) and (6.12), we obtain

(6.13) $$\begin{cases}\|(H+\lambda_0+\lambda)^{-1}\|_{\mathcal{L}(H^{0,s})}\leqq C(1+|\lambda|)^{-1}, \\ \sum_{|\alpha|=1}\|\partial^\alpha(H+\lambda_0+\lambda)^{-1}\|_{\mathcal{L}(H^{0,s})}\leqq C(1+|\lambda|)^{-1/2},\end{cases}$$

provided $0<s\leqq 1$. Let $k\in\boldsymbol{N}$ and assume that (6.13) holds for $s\in(k-1, k]$. We prove (6.13) for $s\in(k, k+1]$. Let $\psi\in H^{0,s}$. Then by the induction hypothesis, the R.H.S. of (6.9) (resp. (6.10)) tends to the R.H.S. of (6.11) (resp. (6.12)) in L^2 as $\varepsilon\to+0$. Therefore, $\partial^\alpha(H+\lambda_0+\lambda)^{-1}\psi\in H^{0,s}$, $|\alpha|\leqq 1$, and the equalities (6.11) and (6.12) hold for $s\in(k, k+1]$. (6.13) for $s\in(k, k+1]$ then follow from (6.11), (6.12) and (6.13) for $s\in(k-1, k]$.

(3) Let $\psi\in\mathcal{S}$ and let $z\in\boldsymbol{C}$. By the Schwarz inequality,

$$\|(H+z)\psi\|_{0,s}\|\psi\|_{0,s}\geqq|(\omega^s(H+z)\psi, \omega^s\psi)|$$
$$\geqq\operatorname{Re}(\omega^s(H+z)\psi, \omega^s\psi)$$
$$=-\operatorname{Re}(\omega^s\Delta\psi, \omega^s\psi)+(V\omega^s\psi, \omega^s\psi)+(\operatorname{Re} z)\|\psi\|_{0,s}.$$

By (2.1),

$$|(V\omega^s\psi, \omega^s\psi)|$$
$$\leqq a\|\nabla(\omega^s\psi)\|_2^2+b\|\omega^s\psi\|_2^2$$
$$=a(\|\omega^s\nabla\psi\|_2^2+2\operatorname{Re}(\omega^s\nabla\psi, (\nabla\omega^s)\psi)+\|(\nabla\omega^s)\psi\|_2^2)+b\|\psi\|_{0,s}^2,$$

while

$$2\operatorname{Re}(\omega^s\nabla\psi, (\nabla\omega^s)\psi)=(1/2)(\nabla\omega^{2s}, \nabla|\psi|^2)=-(1/2)(\Delta\omega^{2s}, |\psi|^2).$$

Now

$$-\mathrm{Re}\,(\omega^s\Delta\psi,\ \omega^s\psi)=-(1/2)(\omega^{2s},\ \Delta|\psi|^2)+(\omega^{2s},\ |\nabla\psi|^2)$$
$$=-(1/2)(\Delta\omega^{2s},\ |\psi|^2)+\|\omega^s\nabla\psi\|_2^2\,.$$

Collecting everything, we obtain

$$\begin{aligned}&\|(H+z)\psi\|_{0,s}\|\psi\|_{0,s}\\ &\geqq(1-a)\|\omega^s\nabla\psi\|_2^2-((1-a)/2)(\Delta\omega^{2s},\ |\psi|^2)-a\|(\nabla\omega^s)\psi\|_2^2+(\mathrm{Re}\,z-b)\|\psi\|_{0,s}^2\\ &\geqq-C(s,\ n,\ a)\|\psi\|_{0,s-1}^2+(\mathrm{Re}\,z-b)\|\psi\|_{0,s}^2\\ &\geqq(\mathrm{Re}\,z-b-C(s,\ n,\ a))\|\psi\|_{0,s}^2\,.\end{aligned}$$

Therefore there exists $\lambda_1\geqq\lambda_0$ such that for any $\mu\geqq\lambda_1$ and any $\lambda\in\boldsymbol{C}$ with $\mathrm{Re}\,\lambda\geqq0$,

$$\|(H+\mu+\lambda)\psi\|_{0,s}\geqq(\mathrm{Re}\,\lambda+1)\|\psi\|_{0,s}\,,\qquad \psi\in\mathcal{S}\,. \tag{6.14}$$

We now prove (6.3). Let $\phi\in H^{0,s}$ and put $\psi=(H+\mu+\lambda)^{-1}\phi$. By part (2), $\psi\in H^{0,s}\cap H^{2,0}=H^{2,s}$. Moreover, by part (1), $\psi\in\widetilde{H}^{2,s}$, since $H\psi=\phi-(\mu+\lambda)\psi\in H^{0,s}$. There exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\psi_j\to\psi$ in $\widetilde{H}^{2,s}$ as $j\to\infty$. Put $\phi_j=(H+\mu+\lambda)\psi_j$. By (6.2), $\phi_j\to(H+\mu+\lambda)\psi=\phi$ in $H^{0,s}$ as $j\to\infty$. Applying (6.14) to $\psi_j\in\mathcal{S}$ and letting $j\to\infty$ in the resulting inequality, we obtain $\|\phi\|_{0,s}\geqq(\mathrm{Re}\,\lambda+1)\|\psi\|_{0,s}$, as required.

(4) Let $\psi\in\widetilde{H}^{2,s}$. By (6.2),

$$\|(H+\lambda)\psi\|_{0,s}\leqq\|H\psi\|_{0,s}+|\lambda|\,\|\psi\|_{0,s}\leqq C(1+|\lambda|)\|\psi\|_{2,s}\,.$$

By (6.7), we have

$$\|\psi\|_{0,s}\leqq(1+|\lambda-\lambda_0|)^{-1}\|(H+\lambda)\psi\|_{0,s}$$

so that

$$\begin{aligned}\|H\psi\|_{0,s}+\|\psi\|_{0,s}&\leqq\|(H+\lambda)\psi\|_{0,s}+(|\lambda|+1)\|\psi\|_{0,s}\\ &\leqq(1+(|\lambda|+1)(1+|\lambda-\lambda_0|)^{-1})\|(H+\lambda)\psi\|_{0,s}\leqq C\lambda_0\|(H+\lambda)\psi\|_{0,s}\,.\end{aligned}$$

Combining this with (6.2), we obtain (6.4). If $\psi\in H^{2,0}$ satisfy $(H+\lambda)\psi\in H^{0,s}$, then by (6.3) we have $\psi\in H^{0,s}$. Part (1) then implies $\psi\in\widetilde{H}^{2,s}$. Q.E.D.

We define the operator $\hat{H}$ in the Hilbert space $\widetilde{H}^{0,s}=H^{0,s}$ as follows: $D(\hat{H})=\widetilde{H}^{2,s}$, $\hat{H}\psi=-\Delta\psi+V\psi$, $\psi\in D(\hat{H})$. If $\psi\in D(\hat{H})$, then $H\psi=\hat{H}\psi\in H^{0,s}$. If $\psi\in H^{0,s}$, then in the same way as in the proof of part (3) of Lemma 6.1, $\phi\equiv(H+\mu+\lambda)^{-1}\psi\in\widetilde{H}^{2,s}=D(\hat{H})$. By the preceding argument, $H\phi=\hat{H}\phi$, and therefore $\psi=(H+\mu+\lambda)\phi=(\hat{H}+\mu+\lambda)\phi$, $(\hat{H}+\mu+\lambda)^{-1}\psi=\phi=(H+\mu+\lambda)^{-1}\psi$ for all $\lambda\geqq0$. Thus (6.3) becomes

$$\|(\hat{H}+\mu+\lambda)^{-1}\|_{\mathcal{L}(H^{0,s})}\leqq(\mathrm{Re}\,\lambda+1)^{-1}\qquad \mathrm{Re}\,\lambda\geqq0,\ \ \mu\geqq\lambda_1. \tag{6.15}$$

And (6.4) becomes

(6.16) $$C|||\psi|||_{2,s}\leqq\|(\hat{H}+\lambda)\psi\|_{0,s}\leqq C'|||\psi|||_{2,s}, \qquad \psi\in D(\hat{H}),\ \mathrm{Re}\,\lambda\geqq\lambda_1.$$

It follows from (6.15) that $\hat{H}+\mu$ is maximal accretive [35] in $\tilde{H}^{0,s}$ for $\mu\geqq\lambda_1$, $s\geqq 0$. In this case we can define the fractional power $(\hat{H}+\mu)^\alpha$ of $\hat{H}+\mu$ for any $\alpha\in\boldsymbol{R}$. With these notations, we have the following Lemmas 6.2, 6.3 and Proposition 6.1.

Lemma 6.2. *Let $\mu\geqq\lambda_1$ and let $s\geqq 0$. Then:*

(1) $(\hat{H}+\mu)^{-1/2}\in\mathcal{L}(H^{0,s})$.

(2) $D((\hat{H}+\mu)^{1/2})=\tilde{H}^{1,s}$,

(6.17) $$C|||\psi|||_{1,s}\leqq\|(\hat{H}+\mu)^{1/2}\psi\|_{0,s}\leqq C'|||\psi|||_{1,s}, \qquad \psi\in\tilde{H}^{1,s}.$$

Proof. (1) Let $\psi\in H^{0,s}$ and let $\phi=(\hat{H}+\mu)^{-1}\psi$. By the moment inequality [35], we have

$$\|(\hat{H}+\mu)^{1/2}\phi\|_{0,s}\leqq C\|(\hat{H}+\mu)\phi\|_{0,s}^{1/2}\|\phi\|_{0,s}^{1/2}.$$

This and (6.15) yield

$$\|(\hat{H}+\mu)^{-1/2}\psi\|_{0,s}\leqq C\|\psi\|_{0,s}^{1/2}\|(\hat{H}+\mu)^{-1}\psi\|_{0,s}^{1/2}\leqq C\|\psi\|_{0,s},$$

as required.

(2) (6.16) is written as

$$C\|(1-\Delta)\psi\|_{0,s}\leqq\|(\hat{H}+\mu)\psi\|_{0,s}\leqq C'\|(1-\Delta)\psi\|_{0,s}.$$

Since $\hat{H}+\mu$ and $1-\Delta$ are maximal accretive in $\tilde{H}^{0,s}$, it follows from the Heinz-Kato theorem [35] that $D((\hat{H}+\mu)^{1/2})=\tilde{H}^{1,s}$, $(1-\Delta)^{1/2}(\hat{H}+\mu)^{-1/2}\in\mathcal{L}(\tilde{H}^{0,s})$, and $(\hat{H}+\mu)^{1/2}(1-\Delta)^{-1/2}\in\mathcal{L}(\tilde{H}^{0,s})$. Q.E.D.

Lemma 6.3. (1) *If $\psi\in D(\tilde{H})$, then $\hat{H}\psi=H\psi$.*

(2) *If $\psi\in H^{0,s}$, then $(\hat{H}+\mu+\lambda)^{-1}\psi=(H+\mu+\lambda)^{-1}\psi$ for all $\lambda\geqq 0$.*

(3) *If $\psi\in H^{2,0}$ and $(H+\mu)\psi\in H^{0,s}$, then $\psi\in D(\hat{H})$ and $\hat{H}\psi=H\psi$.*

(4) *Let $k\in\boldsymbol{N}$. If $\psi\in D(H^k)$ and $(H+\mu)^k\psi\in H^{0,s}$, then $\psi\in D((\hat{H}+\mu)^k)$ and $(\hat{H}+\mu)^k\psi=(H+\mu)^k\psi$.*

(5) *Let $\alpha\in(0,1)$. If $\psi\in D((H+\mu)^\alpha)$ and $(H+\mu)^\alpha\psi\in H^{0,s}$, then $\psi\in D((\hat{H}+\mu)^\alpha)$ and $(\hat{H}+\mu)^\alpha\psi=(H+\mu)^\alpha\psi$.*

Proof. Parts (1) and (2) are restatements of the results preceded by the definition of $\hat{H}$. Part (3) is an immediate consequence of part (3) of Lemma 6.1 and part (1). We turn to part (4). The case $k=1$ reduces to part (3). Let $k\geqq 2$ and let $\psi\in D(H^k)$ satisfy $(H+\mu)^k\psi\in H^{0,s}$. We deduce, from the statement of part (4) with k replaced by $k-1$, that $\psi\in D((\hat{H}+\mu)^k)$ and $(\hat{H}+\mu)^k\psi=(H+\mu)^k\psi$.

We have $(H+\mu)\psi\in D(H^{k-1})$, $(H+\mu)^{k-1}(H+\mu)\psi\in H^{0,s}$ and therefore by the induction hypothesis, $(H+\mu)\psi\in D((\hat{H}+\mu)^{k-1})$, $(\hat{H}+\mu)^{k-1}(H+\mu)\psi=(H+\mu)^k\psi$. By (6.3) and part (3), $\psi\in D(\hat{H})$ and $(\hat{H}+\mu)\psi=(H+\mu)\psi$. This proves $\psi\in D((\hat{H}+\mu)^k)$ and $(H+\mu)^k\psi=(H+\mu)^k\psi$, as was to be shown. We finally prove part (5). Let $\psi\in D((H+\mu)^\alpha)$ satisfy $(H+\mu)^\alpha\psi\in H^{0,s}$. Put $\phi=(H+\mu)^\alpha\psi$. By part (2), $(\hat{H}+\mu+\lambda)^{-1}\phi=(H+\mu+\lambda)^{-1}\phi$ for all $\lambda\geqq 0$. Now

$$\psi=(H+\mu)^{-\alpha}\phi=\pi^{-1}\sin\pi\alpha\int_0^\infty\lambda^{-\alpha}(H+\mu+\lambda)^{-1}\phi\,d\lambda$$

$$=\pi^{-1}\sin\pi\alpha\int_0^\infty\lambda^{-\alpha}(\hat{H}+\mu+\lambda)^{-1}\phi\,d\lambda=(\hat{H}+\mu)^{-\alpha}\phi\,,$$

where the integrals converge in $H^{0,s}$ by virtue of (6.3) and (6.15). This proves $\psi\in D((\hat{H}+\mu)^\alpha)$ and $(\hat{H}+\mu)^\alpha\psi=\phi$, as required. Q.E.D.

Proposition 6.1. *Let $m\in\boldsymbol{N}$ and $s\geqq 0$. Let V be as in Proposition 4.1. Then, $D((\hat{H}+\mu)^{m/2})=\tilde{H}^{m,s}$,*

$$C(\mu,m,s,a)|||\psi|||_{m,s}\leqq\|(\hat{H}+\mu)^{m/2}\psi\|_{0,s}\leqq C(\mu,m,s)|||\psi|||_{m,s}\,,\quad \psi\in\tilde{H}^{m,s}. \tag{6.18}$$

Proof. We assume that $s>0$, since the case $s=0$ reduces to Proposition 4.2. It follows from Proposition 4.2 that the norm $|||\cdot|||_{m,s}^{(8)}$, defined by $|||\psi|||_{m,s}^{(8)}=\|(H+\mu)^{m/2}\omega^s\psi\|_2$, is an equivalent norm on $\tilde{H}^{m,s}$. Therefore, if we can show that the norm $|||\cdot|||_{m,s}^{(9)}$, defined by $|||\psi|||_{m,s}^{(9)}=\|\omega^s(H+\mu)^{m/2}\psi\|_2$, is equivalent to $|||\cdot|||_{m,s}^{(8)}$, then the result will follow from Lemma 6.3. We treat the cases $m=2k$ and $m=2k+1$, $k\in\boldsymbol{N}$, separately.

(i) When $m=2k$, $k\in\boldsymbol{N}$, we prove by induction on $l\in\boldsymbol{N}$ that $|||\cdot|||_{m,s}^{(8)}$ and $|||\cdot|||_{m,s}^{(9)}$ are equivalent for any $s\in(l-1,l]$. Let $\phi\in\mathcal{S}$. Replacing V by $V+\mu$ in (4.5), we see from the remark just after (4.5) that $\phi\in D(H^k)$ and

$$(H+\mu)^k\phi=(-\Delta)^k\phi+\sum_{j=0}^{k-1}\sum_{\substack{|\beta|\leq 2j\\ |\alpha_1+\cdots+\alpha_{k-j}+\beta|=2j}}W(\{\alpha_\nu\},\beta)\cdot\partial^\beta\phi\,, \tag{6.19}$$

where $W(\{\alpha_\nu\},\beta)=C(\alpha_1,\cdots,\alpha_{k-j},\beta)\cdot\prod_{\nu=1}^{k-j}\partial^{\alpha_\nu}(V+\mu)$.

Therefore,

$$[(H+\mu)^k,\omega^s]\phi=[(-\Delta)^k,\omega^s]\phi+\sum_{j=0}^{k-1}\sum_{\substack{|\beta|\leq 2j\\ |\alpha_1+\cdots+\alpha_{k-j}+\beta|=2j}}W(\{\alpha_\nu\},\beta)[\partial^\beta,\omega^s]\phi\,. \tag{6.20}$$

We write

$$[(-\Delta)^k,\omega^s]\phi=\sum_{|\beta|=k}(-1)^k\frac{k!}{\beta!}[\partial^{2\beta},\omega^s]\phi$$

$$=\sum_{|\beta|=k}\sum_{\beta'<2\beta}(-1)^k\frac{k!}{\beta!}\binom{2\beta}{\beta'}(\partial^{2\beta-\beta'}\omega^s)\partial^{\beta'}\phi$$

$$= \sum_{|\beta|=k} \sum_{\beta'<2\beta} \sum_{\gamma\leqq\beta'} (-1)^{k+|\gamma|} \frac{k!}{\beta!} \binom{2\beta}{\beta'} \binom{\beta'}{\gamma} \partial^{\beta'-\gamma}((\partial^{\gamma+2\beta-\beta'}\omega^s)\phi),$$

$$W(\{\alpha_\nu\}, \beta)[\partial^\beta, \omega^s]\phi$$

$$= \sum_{\beta'<\beta} \sum_{\gamma\leqq\beta'} (-1)^{|\gamma|} \binom{\beta}{\beta'} \binom{\beta'}{\gamma} \partial^{\beta'-\gamma}((\partial^\gamma(W(\{\alpha_\nu\}, \beta)\partial^{\beta-\beta'}\omega^s))\phi)$$

so that for $\psi\in\mathcal{S}$,

$$(\psi, [(-\Delta)^k, \omega^s]\phi)$$

$$= \sum_{|\beta|=k} \sum_{\beta'<2\beta} \sum_{\gamma\leqq\beta'} (-1)^{k+|\beta'|} \frac{k!}{\beta!} \binom{2\beta}{\beta'} \binom{\beta'}{\gamma} ((\partial^{\gamma+2\beta-\beta'}\omega^s)\partial^{\beta'-\gamma}\psi, \phi),$$

$$(\psi, W(\{\alpha_\nu\}, \beta)[\partial^\beta, \omega^s]\phi)$$

$$= \sum_{\beta'<\beta} \sum_{\gamma\leqq\beta'} (-1)^{|\beta'|} \binom{\beta}{\beta'} \binom{\beta'}{\gamma} ((\partial^\gamma(W(\{\alpha_\nu\}, \beta)\partial^{\beta-\beta'}\omega^s))\partial^{\beta'-\gamma}\psi, \phi).$$

We note here that

$$|(\psi, [(-\Delta)^k, \omega^s]\phi)| \leqq C|||\psi|||_{m-1, s-1}\|\phi\|_2, \qquad \psi\in\mathcal{S}, \tag{6.21}$$

and that Lemma 4.1 implies

$$|(\psi, W(\{\alpha_\nu\}, \beta)[\partial^\beta, \omega^s]\phi)| \leqq C|||\psi|||_{m-1, s-1}\|\phi\|_2, \qquad \psi\in\mathcal{S}, \tag{6.22}$$

provided $|\beta|\leqq 2j$, $|\alpha_1+\cdots+\alpha_{k-j}+\beta|=2j$ and $j\leqq k$. Let $0<s\leqq 1$. If $\psi\in H^{0,s}$ satisfies $\omega^s\psi\in D((H+\mu)^k)$, then by Proposition 4.2, there exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\psi_j\to\psi$ in $H^{0,s}$, $(H+\mu)^k\psi_j\to(H+\mu)^k\psi$ in L^2 as $j\to\infty$. By (6.19), (6.20), (6.21) and (6.22), we obtain for any $\phi\in\mathcal{S}$,

$$|((H+\mu)^k\psi_j, \omega^s\phi)| \leqq |(\omega^s\psi_j, (H+\mu)^k\phi)| + C\|\psi_j\|_{m-1,0}\|\phi\|_2. \tag{6.23}$$

Letting $j\to\infty$ in (6.23), we obtain for any $\phi\in\mathcal{S}$,

$$|((H+\mu)^k\psi, \omega^s\phi)| \leqq |(\omega^s\psi, (H+\mu)^k\phi)| + C\|\psi\|_{m-1,0}\|\phi\|_2$$

$$\leqq (\|(H+\mu)^k\omega^s\psi\|_2 + C\|\psi\|_{m-1,0})\|\phi\|_2 \leqq C|||\psi|||^{(8)}_{m,s}\|\phi\|_2.$$

This implies that $(H+\mu)^k\psi\in H^{0,s}$ and $|||\psi|||^{(9)}_{m,s}\leqq C|||\psi|||^{(8)}_{m,s}$. If $\psi\in D((H+\mu)^k)$ satisfies $(H+\mu)^k\psi\in H^{0,s}$, $0<s\leqq 1$, then in the same way as above, we find that $\omega^s\psi\in D((H+\mu)^k)$ and $|||\psi|||^{(8)}_{m,s}\leqq C|||\psi|||^{(9)}_{m,s}$. Let $l\in\boldsymbol{N}$ and assume that $|||\cdot|||^{(8)}_{m,s}$ and $|||\cdot|||^{(9)}_{m,s}$ are equivalent for $s\in(l-1, l]$. We perform the same procedure as in the case $0<s\leqq 1$ to prove that the equivalence still holds for $s\in(l, l+1]$. We use the induction hypothesis only in obtaining estimates of the R. H. S. of (6.21) or of (6.22) through the known quantities. We omit the details since the proof is similar to that of the case $0<s\leqq 1$.

(ii) When $m=2k+1$, $k\in\boldsymbol{N}$, we prove the proposition by showing that $D((\hat{H}+\mu)^{m/2})=\tilde{H}^{m,s}$. (6.18) will then follow by the closed graph theorem applied to the closed operators $(\hat{H}+\mu)^{m/2}$ and $(1-\Delta)^{m/2}$ in the Hilbert space $\tilde{H}^{0,s}$. Let

$\psi\in\mathcal{S}$. We see from the preceding argument that

$$C\sum_{|\gamma|\leqq 1}|||\partial^\gamma\psi|||_{2k,s}\leqq\sum_{|\gamma|\leqq 1}\|(\hat{H}+\mu)^k\partial^\gamma\psi\|_{0,s}\leqq C'\sum_{|\gamma|\leqq 1}|||\partial^\gamma\psi|||_{2k,s} \tag{6.24}$$

and that (6.19) with $(H+\mu)^k$ (resp. ϕ) replaced by $(\hat{H}+\mu)^k$ (resp. ψ) makes sense as an identity in $\tilde{H}^{0,s}$. Moreover, from the relation

$$[(\hat{H}+\mu)^k,\partial^\gamma]\phi=\sum_{j=0}^{k-1}\sum_{\substack{|\beta|\leqq 2j\\|\alpha_1+\cdots+\alpha_{k-j}+\beta|=2j}}\partial^\gamma W(\{\alpha_\nu\},\beta)\cdot\partial^\beta\phi\,,\qquad \phi\in\mathcal{S},$$

we obtain by Lemma 4.1 that

$$\sum_{|\gamma|=1}\|[(\hat{H}+\mu)^k,\partial^\gamma]\omega^s\psi\|_2\leqq C\sum_{|\beta|\leqq 2k}\|\partial^\beta(\omega^s\psi)\|_2\leqq C|||\psi|||_{2k,s}\,. \tag{6.25}$$

Similarly, from the relation

$$[[(\hat{H}+\mu)^k,\partial^\gamma],\omega^s]\psi=\sum_{j=0}^{k-1}\sum_{\substack{|\beta|\leqq 2j\\|\alpha_1+\cdots+\alpha_{k-j}+\beta|=2j}}\partial^\gamma W(\{\alpha_\nu\},\beta)\cdot[\partial^\beta,\omega^s]\psi\,,$$

we obtain

$$\sum_{|\gamma|=1}\|[[(\hat{H}+\mu)^k,\partial^\gamma],\omega^s]\psi\|_2\leqq C|||\psi|||_{2k,s}\,. \tag{6.26}$$

By (6.24), (6.25) and (6.26),

$$\begin{aligned}&\sum_{|\gamma|\leqq 1}\|\partial^\gamma(\hat{H}+\mu)^k\psi\|_{0,s}\\&=\sum_{|\gamma|\leqq 1}\|\omega^s(\hat{H}+\mu)^k\partial^\gamma\psi-[(\hat{H}+\mu)^k,\partial^\gamma]\omega^s\psi+[[(\hat{H}+\mu)^k,\partial^\gamma],\omega^s]\psi\|_2\\&\leqq C\sum_{|\gamma|\leqq 1}|||\partial^\gamma\psi|||_{2k,s}\leqq C|||\psi|||_{m,s}\,.\end{aligned} \tag{6.27}$$

By part (2) of Lemma 6.2, we find from (6.27) that $(H+\mu)^k\psi\in D((\hat{H}+\mu)^{1/2})$ and that

$$\|(\hat{H}+\mu)^{m/2}\psi\|_{0,s}\leqq C|||\psi|||_{m,s}\,. \tag{6.28}$$

If $\psi\in\tilde{H}^{m,s}$, then there exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\psi_j\to\psi$ in $\tilde{H}^{m,s}$ as $j\to\infty$. Since the multiplication operator ω^s is closed in L^2, we have $(H+\mu)^{m/2}\psi\in H^{0,s}$ and $(\hat{H}+\mu)^{m/2}\psi_j\to(H+\mu)^{m/2}\psi$ in $H^{0,s}$ as $j\to\infty$. By parts (4) and (5) of Lemma 6.3, we conclude that $\psi\in D((\hat{H}+\mu)^{m/2})$. If $\psi\in D((\hat{H}+\mu)^{m/2})$, then by Proposition 4.2 and the case (i), there exists a sequence $\{\psi_j\}$ in φ such that $\psi_j\to\psi$ in $\tilde{H}^{2k,s}$ and in $H^{m,0}$ as $j\to\infty$. For $|\gamma|=1$ and $\phi\in\mathcal{S}$, we write

$$\begin{aligned}&((H+\mu)^k\partial^\gamma\psi_j,\omega^s\phi)\\&=-(\psi_j,\partial^\gamma(H+\mu)^k\omega^s\phi)\\&=-((H+\mu)^k\psi_j,\omega^s\partial^\gamma\phi)+([\partial^\gamma,\omega^s](H+\mu)^k\psi_j,\phi)-(\omega^s[(H+\mu)^k,\partial^\gamma]\psi_j,\phi)\,.\end{aligned}$$

In the same way as in case (i), we obtain

$$|((H+\mu)^k\partial^\gamma\psi_j,\omega^s\phi)|\leqq|((\hat{H}+\mu)^k\psi_j,\omega^s\partial^\gamma\phi)|+C|||\psi_j|||_{2k,s}\|\phi\|_2\,. \tag{6.29}$$

Letting $j\to\infty$ in (6.29) and using part (2) of Lemma 6.2 and the results in case (i), we obtain

$$(6.30)\qquad \begin{aligned}|((H+\mu)^k\partial^\gamma\psi,\ \omega^s\phi)| &\leqq |((\hat{H}+\mu)^k\psi,\ \omega^s\partial^\gamma\phi)|+C|||\psi|||_{2k,s}\|\phi\|_2\\ &\leqq(\|\partial^\gamma\omega^s(\hat{H}+\mu)^k\psi\|_2+C|||\psi|||_{2k,s})\|\phi\|_2\\ &\leqq C(|||(\hat{H}+\mu)^k\psi|||_{1,s}+|||\psi|||_{2k,s})\|\phi\|_2\\ &\leqq C\|(\hat{H}+\mu)^{m/2}\psi\|_{0,s}\|\phi\|_2\,,\qquad \phi\in\mathcal{S}.\end{aligned}$$

It follows from (6.30) that $\partial^\gamma\psi\in D((\hat{H}+\mu)^k)=\tilde{H}^{2k,s}$, $|\gamma|=1$, and hence $\psi\in\tilde{H}^{m,s}$. We have thus proved that $\tilde{H}^{m,s}=D((\hat{H}+\mu)^{m/2})$, as required. Q.E.D.

Lemma 6.4. *Let $q\geqq 2$ for $n\leqq 3$, let $q>2$ for $n=4$, and let $q>n/2$ for $n\geqq 5$. Let $p\in[2, q]$. Suppose that $V\in L^q+L^\infty$. Then:*

(1) *For any $\psi\in L^p$ with $\Delta\psi\in L^p$, we have $V\psi\in L^p$. Moreover,*

$$(6.31)\qquad \|V\psi\|_p\leqq C\|(1-\Delta)\psi\|_p^{n/2q}\|\psi\|_p^{1-n/2q}\,.$$

(2) *For any $\psi\in L^2\cap L^p$ with $\Delta\psi\in L^2\cap L^p$,*

$$(6.32)\qquad C(\|\Delta\psi\|_p+\|\psi\|_p)\leqq\|H\psi\|_p+\|\psi\|_p\leqq C'(\|\Delta\psi\|_p+\|\psi\|_p)\,.$$

(3) *There exists $\lambda_2\geqq\lambda_1$ such that for any $\psi\in\mathcal{S}$ and any $z\in\boldsymbol{C}$ with* $\operatorname{Re} z\geqq\lambda_2$,

$$(6.33)\qquad \|(H+z)\psi\|_p\geqq(\operatorname{Re} z-\lambda_2)\|\psi\|_p\,.$$

Proof. (1) We first consider the case $q>n/2$. V can be decomposed as $V=V^{(q)}+V^{(\infty)}$ with $V^{(q)}\in L^q$, $V^{(\infty)}\in L^\infty$. Let $r\in(2,\infty]$ satisfy $1/r=1/p-1/q$. Then by Hölder's inequality, Lemma 1.1 and the L^p-boundedness of the Riesz transform (see, e.g., E.M. Stein [34; Chapter V]), we obtain

$$\|V\psi\|_p\leqq\|V^{(q)}\|_q\|\psi\|_r+\|V^{(\infty)}\|_\infty\|\psi\|_p\,,$$

$$\|\psi\|_r\leqq C\|\Delta\psi\|_p^{n/2q}\|\psi\|_p^{1-n/2q}\,,$$

so that part (1) holds.

(2) Part (2) is an immediate consequence of part (1).

(3) We consider only the case $p>2$, since the case $p=2$ can be proved in the same way as in the derivation of (2.2). Let $\psi\in\mathcal{S}$. By part (2), $(H+z)\psi\in L^p$. Since V is H_0-bounded with H_0-bound $=0$, it follows from (2.1) that for any $\varepsilon>0$,

$$|(V\phi,\ \phi)|\leqq\varepsilon\|\nabla\phi\|_2^2+C(\varepsilon)\|\phi\|_2^2\,,\qquad \phi\in H^{2,0}.$$

This implies

$$\begin{aligned}|(V,\ |\psi|^p)| &\leqq\varepsilon\|\nabla|\psi|^{p/2}\|_2^2+C(\varepsilon)\||\psi|^{p/2}\|_2^2\\ &\leqq\varepsilon(p^2/4)(|\nabla\psi|^2,\ |\psi|^{p-2})+C(\varepsilon)\|\psi\|_p^p\,.\end{aligned}$$

By Hölder's inequality and integration by parts, we have

$$\begin{aligned}
&\|(H+z)\phi\|_p\|\phi\|_p^{p-1}\\
&\geqq|((H+z)\phi,\ |\phi|^{p-2}\phi)|\\
&\geqq\mathrm{Re}\,((H+z)\phi,\ |\phi|^{p-2}\phi)\\
&=\mathrm{Re}\,(\nabla\phi,\ \nabla(|\phi|^{p-2}\phi))+\mathrm{Re}\,((V+z)\phi,\ |\phi|^{p-2}\phi)\\
&=(|\nabla\phi|^2,\ |\phi|^{p-2})+(p-2)(|\mathrm{Re}\,(\mathrm{sgn}\,\bar{\phi})\nabla\phi|^2,\ |\phi|^{p-2})+\mathrm{Re}\,z\|\phi\|_p^p+(V,\ |\phi|^p),
\end{aligned}$$

where sgn denotes the function on $\boldsymbol{C}$ defined by $\mathrm{sgn}\,z=z/|z|$ for $z\neq 0$, $\mathrm{sgn}\,0=0$. Combining these estimates, we have

$$\|(H+z)\phi\|_p\|\phi\|_p^{p-1}\geqq(1-\varepsilon(p^2/4))(|\nabla\phi|^2,\ |\phi|^{p-2})+(\mathrm{Re}\,z-C(\varepsilon))\|\phi\|_p^p.$$

(6.33) then follows by putting $\varepsilon=4/p^2$ and $\lambda_2=\max(\lambda_1,\ C(4/p^2))$. Q.E.D.

Let V and p be as in Lemma 6.4. We consider the differential operator $K=(-\Delta+V)\restriction\mathcal{S}$. K is closable in L^p. Indeed, if a sequence $\{\phi_j\}$ in $\mathcal{S}$ and $\phi\in L^p$ satisfy $\phi_j\to 0$, $K\phi_j\to\phi$ in L^p as $j\to\infty$, then for any $f\in\mathcal{S}\subset L^{p'}$,

$$(\phi,\ f)=\lim_{j\to\infty}(K\phi_j,\ f)=\lim_{j\to\infty}(\phi_j,\ Kf)=0$$

and therefore $\phi=0$. We denote by $\widetilde{H}$ the closure in L^p of K. (6.33) then extends to

$$\|(\widetilde{H}+z)\phi\|_p\geqq(\mathrm{Re}\,z-\lambda_2)\|\phi\|_p, \qquad \text{for } \mathrm{Re}\,z\geqq\lambda_2,\ \phi\in D(\widetilde{H}). \tag{6.34}$$

Put $\mu=\lambda_2+1$, $\widetilde{H}_\mu=\widetilde{H}+\mu$. Then (6.34) yields

$$\|(\widetilde{H}_\mu+z)^{-1}\|_{\mathcal{L}(L^p)}\leqq(\mathrm{Re}\,z+1)^{-1}, \qquad \text{for } \mathrm{Re}\,z\geqq 0, \tag{6.35}$$

since $z\in\rho(\widetilde{H}_\mu)$. By virtue of (6.35) we can define the fractional power $\widetilde{H}_\mu^\alpha$ of $\widetilde{H}_\mu$, for any $\alpha\in\boldsymbol{R}$. With the notations above, we have the following Lemmas 6.5, 6.6 and 6.7, and Proposition 6.2.

Lemma 6.5. (1) $D(\widetilde{H})=\{\phi\in L^p\,;\ \Delta\phi\in L^p\}$,

$$\widetilde{H}\phi=-\Delta\phi+V\phi, \qquad \phi\in D(\widetilde{H}), \tag{6.36}$$

$$C(\|\Delta\phi\|_p+\|\phi\|_p)\leqq\|\widetilde{H}\phi\|_p+\|\phi\|_p\leqq C'(\|\Delta\phi\|_p+\|\phi\|_p), \qquad \phi\in D(\widetilde{H}). \tag{6.37}$$

(2) *If* $V\in L^\infty$, *then* $D(\widetilde{H}^*)=\{\phi\in L^{p'}\,;\ \Delta\phi\in L^{p'}\}$,

$$\widetilde{H}^*\phi=-\Delta\phi+V\phi, \qquad \phi\in D(\widetilde{H}^*).$$

Proof. (1) Let $\phi\in D(\widetilde{H})$. There exists a sequence $\{\phi_j\}$ in $\mathcal{S}$ such that $\phi_j\to\phi$, $\widetilde{H}\phi_j\to\widetilde{H}\phi$ in L^p as $j\to\infty$. It follows from (6.32) that $\{\Delta\phi_j\}$ is a Cauchy sequence in L^p. Therefore, $\Delta\phi\in L^p$ and $\Delta\phi_j\to\Delta\phi$ in L^p as $j\to\infty$, so that

$\tilde{H}\psi_j=-\Delta\psi_j+V\psi_j\to-\Delta\psi+V\psi$ in L^p as $j\to\infty$. This proves $\tilde{H}\psi=-\Delta\psi+V\psi$. Conversely, if $\psi\in L^p$ satisfies $\Delta\psi\in L^p$, then there exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\psi_j\to\psi$, $\Delta\psi_j\to\Delta\psi$ in L^p as $j\to\infty$. This implies $\tilde{H}\psi_j=-\Delta\psi_j+V\psi_j\to -\Delta\psi+V\psi$ in L^p as $j\to\infty$. Since $\tilde{H}$ is closed in L^p, we find $\psi\in D(\tilde{H})$ and $\tilde{H}\psi= -\Delta\psi+V\psi$. (6.37) follows from (6.36) and (6.31).

(2) If $\phi\in D(\tilde{H}^*)$, then for any $\psi\in\mathcal{S}$,

$$|(\phi, \Delta\psi)|=|(\tilde{H}^*\phi, \psi)-(\phi, V\psi)|\leqq(\|\tilde{H}^*\phi\|_{p'}+\|V\|_\infty\|\phi\|_{p'})\|\psi\|_p,$$

and therefore $\Delta\phi\in L^{p'}$. If $\phi\in L^{p'}$ satisfies $\Delta\phi\in L^{p'}$, then by a density argument, we see that $(\phi, \tilde{H}\psi)=(-\Delta\phi+V\phi, \psi)$, $\psi\in D(\tilde{H})$. This proves that $\phi\in D(\tilde{H}^*)$ and $\tilde{H}^*\phi=-\Delta\phi+V\phi$. Q.E.D.

Lemma 6.6. (1) *If $\psi\in D(\tilde{H})\cap L^2$ and $\tilde{H}\psi\in L^2$, then $\psi\in H^{2,0}$ and $\tilde{H}\psi=H\psi$.*

(2) *If $\phi\in L^2\cap L^p$, then $(\tilde{H}_\mu+\lambda)^{-1}\phi=(H+\mu+\lambda)^{-1}\phi$ for all $\lambda\geqq0$.*

(3) *If $\psi\in H^{2,0}\cap L^p$ and $H\psi\in L^p$, then $\psi\in D(\tilde{H})$ and $\tilde{H}\psi=H\psi$.*

(4) *Let $k\in\boldsymbol{N}$. If $\psi\in D(H^k)\cap L^p$ and $(H+\mu)^k\psi\in L^p$, then $\psi\in D(\tilde{H}_\mu^k)$ and $\tilde{H}_\mu^k\psi= (H+\mu)^k\psi$.*

(5) *Let $0<\alpha<1$. If $\psi\in D((H+\mu)^\alpha)\cap L^p$ and $(H+\mu)^\alpha\psi\in L^p$, then $\psi\in D(\tilde{H}_\mu^\alpha)$ and $\tilde{H}_\mu^\alpha\psi=(H+\mu)^\alpha\psi$.*

Proof. (1) If $\psi\in D(\tilde{H})\cap L^2$ and $\tilde{H}\psi\in L^2$, then there exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\psi_j\to\psi$ in $L^2\cap L^p$, $H\psi_j\to\tilde{H}\psi$ in L^p as $j\to\infty$. For any $\phi\in\mathcal{S}$ we have

$$(\psi, H\phi)=\lim_{j\to\infty}(\psi_j, H\phi)=\lim_{j\to\infty}(H\psi_j, \phi)=(\tilde{H}\psi, \phi).$$

This implies that $\psi\in D(H)=H^{2,0}$ and $H\psi=\tilde{H}\psi$.

(2) Let $\phi\in L^2\cap L^p$ and let $\{\phi_j\}$ be a sequence in $\mathcal{S}$ such that $\phi_j\to\phi$ in $L^2\cap L^p$ as $j\to\infty$. By (6.35) with $p=2$, $\{(\tilde{H}_\mu+\lambda)^{-1}\phi_j\}$ is a Cauchy sequence in L^2. Since $(\tilde{H}_\mu+\lambda)^{-1}\phi_j\to(\tilde{H}_\mu+\lambda)^{-1}\phi$ in L^p as $j\to\infty$, we have $(\tilde{H}_\mu+\lambda)^{-1}\phi\in L^2$. Put $\psi= (\tilde{H}_\mu+\lambda)^{-1}\phi$. Then $\psi\in D(\tilde{H})\cap L^2$ and $\tilde{H}\psi=(\tilde{H}_\mu+\lambda)\psi-(\mu+\lambda)\psi=\phi-(\mu+\lambda)\psi\in L^2$. By part (1), $\tilde{H}\psi=H\psi$. Thus $\phi=\tilde{H}\psi+(\mu+\lambda)\psi$ is equal to $(H+\mu+\lambda)\psi$ and hence $\psi=(H+\mu+\lambda)^{-1}\phi$, as was to be shown.

(3) If $\psi\in H^{2,0}\cap L^p$ and $H\psi\in L^p$, then we put $\phi=(H+\mu)\psi\in L^2\cap L^p$. By part (2), $\psi=(H+\mu)^{-1}\phi=\tilde{H}_\mu^{-1}\phi\in D(\tilde{H})$. By part (1) of Lemma 6.5, we have $\tilde{H}\psi= -\Delta\psi+V\psi=H\psi$.

Part (4) (resp. (5)) follows in the same way as in the proof of part (4) (resp. (5)) of Lemma 6.3. Q.E.D.

Lemma 6.7. (1) *For any* $\lambda\geqq 0$ *and* $\psi\in L^p$,

$$\|\Delta(\tilde{H}_\mu+\lambda)^{-1}\psi\|_p\leqq C\|\psi\|_p\,, \tag{6.38}$$

$$\sum_{|\alpha|=1}\|\partial^\alpha(\tilde{H}_\mu+\lambda)^{-1}\psi\|_p\leqq C(\lambda+1)^{-1/2}\|\psi\|_p\,, \tag{6.39}$$

$$\|\tilde{H}_\mu(\tilde{H}_\mu+\lambda)^{-1}\psi\|_p\leqq 2\|\psi\|_p\,, \tag{6.40}$$

$$\|\tilde{H}_\mu^{1/2}(\tilde{H}_\mu+\lambda)^{-1}\psi\|_p\leqq C(\lambda+1)^{-1/2}\|\psi\|_p\,. \tag{6.41}$$

(2) $$D(\tilde{H}_\mu^{1/2})=\{\psi\in L^p\,;\ \partial^\alpha\psi\in L^p,\ |\alpha|=1\}=\{\psi\in L^p\,;\ (-\Delta)^{1/2}\psi\in L^p\},$$
$$=\{\psi\in L^p\,;\ (1-\Delta)^{1/2}\psi\in L^p\},$$

$$C\sum_{|\alpha|\leqq 1}\|\partial^\alpha\psi\|_p\leqq\|\tilde{H}_\mu^{1/2}\psi\|_p\leqq C'\sum_{|\alpha|\leqq 1}\|\partial^\alpha\psi\|_p\,,\qquad \psi\in D(\tilde{H}_\mu^{1/2}). \tag{6.42}$$

Let

$$\|\psi\|_1^{(1)}=\sum_{|\alpha|\leqq 1}\|\partial^\alpha\psi\|_p\,,$$

$$\|\psi\|_1^{(2)}=\|\tilde{H}_\mu^{1/2}\psi\|_p\,,$$

$$\|\psi\|_1^{(3)}=\|(-\Delta)^{1/2}\psi\|_p+\|\psi\|_p\,,$$

$$\|\psi\|_1^{(4)}=\|(1-\Delta)^{1/2}\psi\|_p\,.$$

All these norms are equivalent norms on $D(\tilde{H}_\mu^{1/2})$.

Proof. (1) Let $\psi\in L^p$ and let $\phi=(\tilde{H}_\mu+\lambda)^{-1}\psi$, $\lambda\geqq 0$. By part (1) of Lemma 6.5, $\Delta\phi\in L^p$ and $\Delta\phi=-\tilde{H}\phi+V\phi=-\psi+(V+\mu+\lambda)\phi$. From (6.31) and (6.35), we get

$$\|\Delta\phi\|_p\leqq\|\psi\|_p+(1/2)\|\Delta\phi\|_p+C\|\phi\|_p$$
$$\leqq\|\psi\|_p+(1/2)\|\Delta\phi\|_p+C(\lambda+1)^{-1}\|\psi\|_p\,,$$

from which we get (6.38). Since $\tilde{H}_\mu\phi=\psi-\lambda\phi$, we obtain, similarly as above,

$$\|\tilde{H}_\mu\phi\|_p\leqq\|\psi\|_p+\lambda(\lambda+1)^{-1}\|\psi\|_p\leqq 2\|\psi\|_p\,,$$

which is exactly (6.40). By the L^p-boundedness of the Riesz transform, $\Delta\phi\in L^p$ implies $\partial^\alpha\phi\in L^p$, $|\alpha|=2$, and

$$\sum_{|\alpha|=2}\|\partial^\alpha\phi\|_p\leqq C\|\Delta\phi\|_p\,. \tag{6.43}$$

By Lemma 1.1,

$$\sum_{|\beta|=1}\|\partial^\beta\phi\|_p\leqq C\sum_{|\alpha|=2}\|\partial^\alpha\phi\|_p^{1/2}\|\phi\|_p^{1/2}\,. \tag{6.44}$$

Combining (6.35), (6.38), (6.43) and (6.44), we obtain (6.39). By the moment inequality [35],

$$\|\tilde{H}_\mu^{1/2}\phi\|_p\leqq C\|\tilde{H}_\mu\phi\|_p^{1/2}\|\phi\|_p^{1/2}\,,$$

which together with (6.35), (6.40) leads to (6.41).

(2) By Calderón's theorem [34; Chapter V, Theorem 2], $\|\cdot\|_1^{(1)}$ and $\|\cdot\|_1^{(4)}$ are equivalent. By Stein's theorem [34; Chapter V, Lemma 2], $\|\cdot\|_1^{(3)}$ and $\|\cdot\|_1^{(4)}$ are equivalent. Therefore it suffices to prove

$$\sum_{|\alpha|=1}\|\partial^\alpha\phi\|_p\leqq C\|\phi\|_1^{(2)} \qquad \text{for } \phi\in D(\tilde{H}_\mu^{1/2}), \tag{6.45}$$

$$\|\phi\|_1^{(2)}\leqq C\|\phi\|_1^{(4)} \qquad \text{for } \phi\in L^p \text{ satisfying } (1-\Delta)^{1/2}\phi\in L^p. \tag{6.46}$$

Let $\phi\in L^p$. We have in L^p

$$(\tilde{H}_\mu+\lambda)^{-1}\phi=(1-\Delta+\lambda)^{-1}\phi-(\tilde{H}_\mu+\lambda)^{-1}(V+\mu-1)(1-\Delta+\lambda)^{-1}\phi\,. \tag{6.47}$$

(6.34) specialized to the case $V=0$ implies that

$$\|(1-\Delta+\lambda)^{-1}\phi\|_p\leqq(\lambda+1)^{-1}\|\phi\|_p\,. \tag{6.48}$$

(6.40) specialized to the case $V=0$ implies that $\Delta(1-\Delta+\lambda)^{-1}\in\mathcal{L}(L^p)$. It follows therefore from (6.31) and (6.48) that

$$\|(V+\mu-1)(1-\Delta+\lambda)^{-1}\phi\|_p\leqq C(1+\lambda)^{-1+n/2q}\|\phi\|_p\,. \tag{6.49}$$

By (6.47),

$$\begin{aligned}\tilde{H}_\mu^{-1/2}\phi&=\pi^{-1}\int_0^\infty\lambda^{-1/2}(\tilde{H}_\mu+\lambda)^{-1}\phi\,d\lambda\\ &=\pi^{-1}\int_0^\infty\lambda^{-1/2}(1-\Delta+\lambda)^{-1}\phi\,d\lambda-\pi^{-1}\int_0^\infty\lambda^{-1/2}(\tilde{H}_\mu+\lambda)^{-1}(V+\mu-1)(1-\Delta+\lambda)^{-1}\phi\,d\lambda.\\ &=(1-\Delta)^{-1/2}\phi-\pi^{-1}\int_0^\infty\lambda^{-1/2}(\tilde{H}_\mu+\lambda)^{-1}(V+\mu-1)(1-\Delta+\lambda)^{-1}\phi\,d\lambda\,.\end{aligned} \tag{6.50}$$

By (6.35), (6.48) and (6.49), the integrals in (6.50) converge in L^p. Since $\|\cdot\|_1^{(1)}$ and $\|\cdot\|_1^{(4)}$ are equivalent,

$$\sum_{|\alpha|=1}\|\partial^\alpha(1-\Delta)^{-1/2}\phi\|_p\leqq C\|\phi\|_p\,.$$

By (6.39) and (6.49),

$$\sum_{|\alpha|=1}\|\partial^\alpha(\tilde{H}_\mu+\lambda)^{-1}(V+\mu-1)(1-\Delta+\lambda)^{-1}\phi\|_p\leqq C(\lambda+1)^{-3/2+n/2q}\|\phi\|_p\,.$$

This shows that the first derivatives of the second term on the R. H. S. of the last equality in (6.50) are in L^p, and their L^p-norms are bounded by $C\|\phi\|_p$. Therefore by (6.50), we conclude $\partial^\alpha\tilde{H}_\mu^{-1/2}\phi\in L^p$, $|\alpha|=1$, and

$$\sum_{|\alpha|=1}\|\partial^\alpha\tilde{H}_\mu^{-1/2}\phi\|_p\leqq C\|\phi\|_p\,,$$

which is exactly (6.45). In view of (6.41), (6.49) and (6.50), we see that (6.46) follows from in a way analogous to the preceding argument. Q. E. D.

Proposition 6.2. *Let $p\in[2, \infty)$ and let $m\in N$. Suppose that $\partial^\alpha V\in L^\infty$ for all $|\alpha|\leqq\max(m-2, 0)$. Then*

$$D(\tilde{H}_\mu^{m/2})=\{\phi\in L^p ; \partial^\alpha\phi\in L^p,\ |\alpha|\leqq m\}=\{\phi\in L^p ; (-\Delta)^{m/2}\phi\in L^p\}$$
$$=\{\phi\in L^p ; (1-\Delta)^{m/2}\phi\in L^p\},$$

$$C\sum_{|\alpha|\leqq m}\|\partial^\alpha\phi\|_p\leqq\|\tilde{H}_\mu^{m/2}\phi\|_p\leqq C'\sum_{|\alpha|\leqq m}\|\partial^\alpha\phi\|_p\,,\qquad \phi\in D(\tilde{H}_\mu^{m/2}). \tag{6.51}$$

Let

$$\|\phi\|_m^{(1)}=\sum_{|\alpha|\leqq m}\|\partial^\alpha\phi\|_p\,,$$
$$\|\phi\|_m^{(2)}=\|\tilde{H}_\mu^{m/2}\phi\|_p\,,$$
$$\|\phi\|_m^{(3)}=\|(-\Delta)^{m/2}\phi\|_p+\|\phi\|_p\,,$$
$$\|\phi\|_m^{(4)}=\|(1-\Delta)^{m/2}\phi\|_p\,.$$

All these norms are equivalent norms on $D(\tilde{H}_\mu^{m/2})$.

Proof. By Calderón's and Stein's theorems (see the proof of Lemma 6.6), $\|\cdot\|_m^{(j)}$, $j=1, 3, 4$, are equivalent. By Lemma 1.1,

$$\sum_{|\beta|=l}\|\partial^\beta\phi\|_p\leqq C\sum_{|\alpha|=m}\|\partial^\alpha\phi\|_p^{l/m}\|\phi\|_p^{1-l/m}\,,\qquad 1\leqq l\leqq m-1. \tag{6.52}$$

Therefore it suffices to prove

$$\|\phi\|_m^{(2)}\leqq C\|\phi\|_m^{(1)}\,,\qquad \phi\in\mathcal{S}, \tag{6.53}$$

$$D(\hat{H}_\mu^{m/2})\subset\{\phi\in L^p ; \partial^\alpha\phi\in L^p,\ |\alpha|=m\}, \tag*{(6.54)$_m$}$$

since $\tilde{H}_\mu^{m/2}$ is a closed operator in L^p. Let $\phi\in\mathcal{S}$. To prove (6.53) we distinguish between two cases:

(i) $m=2k$ with $k\in N$. (ii) $m=2k+1$ with $k\in N$.

(i) When $m=2k$ with $k\in N$, we see from Proposition 4.1 that $D(H^k)=H^{2k,0}$. Since $\partial^\alpha(V+\mu)\in L^\infty$, $|\alpha|\leqq m-2$, it follows from (6.19) that

$$\|(H+\mu)^k\phi\|_p\leqq C\sum_{|\beta|\leqq 2k}\|\partial^\beta\phi\|_p\,. \tag{6.55}$$

Since $\phi\in\mathcal{S}\subset H^{2k,0}\cap L^p=D(H^k)\cap L^p$, part (4) of Lemma 6.6 implies that $\tilde{H}_\mu^k\phi=(H+\mu)^k\phi$, and therefore (6.55) becomes (6.53).

(ii) When $m=2k+1$ with $k\in N$, we have, in the same way as in the proof of (4.8),

$$\sum_{|\gamma|=1}\|[\tilde{H}_\mu^k, \partial^\gamma]\phi\|_p\leqq C\sum_{|\beta|\leqq 2k-2}\|\partial^\beta\phi\|_p\,. \tag{6.56}$$

Since we have proved (6.53) with $m=2k$, we see from (6.56) that

$$\sum_{|\gamma|=1} \|\partial^\gamma \tilde{H}_\mu^k \phi\|_p \leqq C \sum_{|\gamma|=1} \|\partial^\gamma \phi\|_{2k}^{(1)} + C \sum_{|\beta| \leqq 2k-2} \|\partial^\beta \phi\|_p \leqq C \sum_{|\beta| \leqq 2k+1} \|\partial^\beta \phi\|_p . \tag{6.57}$$

(6.57) and part (5) of Lemma 6.6 imply that $\tilde{H}_\mu^k \phi \in D(\tilde{H}_\mu^{1/2})$,

$$\|\tilde{H}_\mu^{k+1/2} \phi\|_p \leqq C \sum_{|\beta| \leqq 2k+1} \|\partial^\beta \phi\|_p , \tag{6.58}$$

which is exactly (6.53). We now prove $(6.54)_m$. To this end we again treat the cases $m=2k$ and $m=2k+1$ separately.

(iii) When $m=2k$, $k \in \boldsymbol{N}$, we prove $(6.54)_{2k}$ by induction on $k \in \boldsymbol{N}$. The case $k=1$ follows from part (1) of Lemma 6.5 and the L^p-boundedness of the Riesz transform. Let $k \geqq 2$ and suppose that $(6.54)_{2j}$ holds for all $j \leqq k-1$. We prove $(6.54)_{2k}$. Let $\phi \in \mathcal{S}$. By making use of part (2) of Lemma 6.5, we find in the same way as in the case (i) that $\phi \in D((\tilde{H}_\mu^*)^k)$ and that $(\tilde{H}_\mu^*)^k \phi$ is equal to the R.H.S. of (6.19). Moreover, for any $\psi \in \mathcal{S}$,

$$|(\psi, W(\{\alpha_\nu\}, \beta) \partial^\beta \phi)| \leqq C \|\psi\|_{2k-2}^{(1)} \|\phi\|_{p'} , \tag{6.59}$$

provided $|\beta| \leqq 2j$, $|\alpha_1 + \cdots + \alpha_{k-j} + \beta| = 2j$ and $j \leqq k$. The proof of (6.59) is similar to, and in fact, easier than, that of (6.22). Let $\psi \in D(\tilde{H}_\mu^k)$. By the induction hypothesis, there exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\|\psi_j - \psi\|_{2k-2}^{(1)} \to 0$ as $j \to \infty$. By (6.59),

$$|(\psi_j, (-\Delta)^k \phi)| \leqq |(\psi_j, (\tilde{H}_\mu^*)^k \phi)| + C \|\psi_j\|_{2k-2}^{(1)} \|\phi\|_{p'} . \tag{6.60}$$

Letting $j \to \infty$ in (6.60), we have

$$|(\psi, (-\Delta)^k \phi)| \leqq C(\|\tilde{H}_\mu^k \psi\|_p + \|\psi\|_{2k-2}^{(1)}) \|\phi\|_{p'} , \tag{6.61}$$

and therefore $\Delta^k \psi \in L^p$. $(6.54)_{2k}$ now follows from the L^p-boundedness of the Riesz transform.

(iv) When $m=2k+1$, we prove $(6.54)_m$ by showing that $\psi \in D(\tilde{H}_\mu^{m/2})$ implies $\partial^\gamma \psi \in D(\tilde{H}_\mu^k)$ for any $|\gamma|=1$. There exists a sequence $\{\psi_j\}$ in $\mathcal{S}$ such that $\sum_{l=1}^{2} \|\psi_j - \psi\|_{2k}^{(l)} \to 0$ as $j \to \infty$. For $|\gamma|=1$ and $\phi \in \mathcal{S}$, we write

$$(\partial^\gamma \psi_j, (\tilde{H}_\mu^*)^k \phi) = -(\tilde{H}_\mu^k \psi_j, \partial^\gamma \phi) + ([\tilde{H}_\mu^k, \partial^\gamma] \psi_j, \phi) . \tag{6.62}$$

By (6.56),

$$|([\tilde{H}_\mu^k, \partial^\gamma] \psi_j, \phi)| \leqq C \|\psi_j\|_{2k-2}^{(1)} \|\phi\|_{p'} . \tag{6.63}$$

By (6.42),

$$|(\tilde{H}_\mu^k \psi, \partial^\gamma \phi)| = |(\partial^\gamma \tilde{H}_\mu^k \psi, \phi)| \leqq C \|\tilde{H}_\mu^{m/2} \psi\|_p \|\phi\|_p \tag{6.64}$$

Letting $j \to \infty$ in (6.62) and using (6.63) and (6.64), we have

$$|(\partial^\gamma\psi, (\tilde{H}_\mu^*)^k\phi)| \leqq C\|\psi\|_m^{(2)}\|\phi\|_{p'}, \qquad \phi\in\mathcal{S}.$$

This proves $\partial^\gamma\psi\in D(\tilde{H}_\mu^k)$, as required. Q.E.D.

Lemma 6.8. (1) *There exists a constant C depending only on n such that*

$$\|\psi\|_p \leqq Cp^{1/2}\|(-\Delta)^{n/4}\psi\|_2^{1-2/p}\|\psi\|_2^{2/p}, \qquad \psi\in H^{n/2,0},\ p\in[2,\infty). \tag{6.65}$$

(2) *There exists a constant C depending only on n such that*

$$\|\psi\|_\infty \leqq C\|\psi\|_{n/2+1,0}^{n/(p+n)}\|\psi\|_p^{p/(p+n)}, \qquad \psi\in H^{n/2+1,0},\ p\in[2,\infty). \tag{6.66}$$

Proof. (See also M. Reed & B. Simon [31; Theorem IX. 28] and J.Q. Yao [38]).

(1) It suffices to prove (6.1) in the case $p\in(2,\infty)$ and $\psi\in\mathcal{S}$. Let $\lambda>0$. By the Hausdorff-Young and Hölder inequalities, we have

$$\begin{aligned}\|\psi\|_p &\leqq (2\pi)^{-\delta_n(p)}\|\mathcal{F}\psi\|_{p'}\\ &\leqq (2\pi)^{-\delta_n(p)}\|(\lambda^n+|\xi|^n)^{1/2}\mathcal{F}\psi\|_2\|(\lambda^n+|\xi|^n)^{-1/2}\|_{2p/(p-2)}.\end{aligned}$$

Moreover,

$$\begin{aligned}\|(\lambda^n+|\xi|^n)^{1/2}\mathcal{F}\psi\|_2 &= (\lambda^n\|\mathcal{F}\psi\|_2^2+\||\xi|^{n/2}\mathcal{F}\psi\|_2^2)^{1/2}\\ &= (\lambda^n\|\psi\|_2^2+\|(-\Delta)^{n/4}\psi\|_2^2)^{1/2},\end{aligned}$$

$$\|(\lambda^n+|\xi|^n)^{-1/2}\|_{2p/(p-2)} = (\pi^n(p-2)/(n\Gamma(n/2)))^{1/2-1/p}\cdot\lambda^{-n/p},$$

where Γ denotes the gamma function. Thus $\|\psi\|_p$ is estimated by

$$\begin{aligned}&((p-2)/(n\Gamma(n/2)))^{1/2-1/p}2^{n/p}\cdot\lambda^{-n/p}(\lambda^n\|\psi\|_2^2+\|(-\Delta)^{n/4}\psi\|_2^2)^{1/2}\\ &\leqq Cp^{1/2}\cdot\lambda^{-n/p}(\lambda^n\|\psi\|_2^2+\|(-\Delta)^{n/4}\psi\|_2^2)^{1/2},\end{aligned}$$

where C is independent of p. If we choose

$$\lambda=(\|(-\Delta)^{n/4}\psi\|_2+\varepsilon)^{2/n}(\|\psi\|_2+\varepsilon)^{-2/n}$$

with $\varepsilon>0$, then we have

$$\|\psi\|_p \leqq Cp^{1/2}(\|(-\Delta)^{n/4}\psi\|_2+\varepsilon)^{1-2/p}(\|\psi\|_2+\varepsilon)^{2/p}.$$

Since $\varepsilon>0$ is arbitrary, this proves (6.65).

(2) Let $\psi\in H^{n/2+1,0}$. By Lemma 1.1, we obtain $\psi\in L^2\cap L^\infty=\bigcap_{2\leq p\leq\infty}L^p$ and (6.66) with C replaced by some constant which might depend on p. It is therefore sufficient to prove (6.66) by assuming that $\|\psi\|_{n/2+1,0}>0$ and $\|\psi\|_p>0$ for all $p\geqq 2$. Let $f(p)=\|\psi\|_p$, $p\geqq 2$. f is a continuous function on the interval $[2,\infty)$, since $[2,\infty)\ni p\mapsto\log(f(1/p))\in\mathbf{R}$ is convex. Furthermore, $\lim_{p\to\infty}f(p)=\|\psi\|_\infty$, since $\psi\in L^2\cap L^\infty$. We now consider the function defined by

$g(p)=\|\phi\|_\infty\|\phi\|_{n/2+1,0}^{-n/(p+n)}, f(p)^{-p/(p+n)}$, $p\geqq 2$. g is a continuous function on $[2, \infty)$ and satisfies $\lim_{p\to\infty} g(p)=1$. This proves part (2). Q.E.D.

We collect here some results concerning the spectral properties of Schrödinger operators and the decay properties of the Schrödinger evolution groups. See M. Murata [26], A. Jensen & T. Kato [18] and A. Jensen [15][16] for the proof.

Theorem A. (1) *Let $\varepsilon(n)=0$ for n odd and $\varepsilon(n)=1$ for n even, $\sigma>-1/2$ and $s>\max(\sigma+1, 2\sigma+2-n/2)$. Then we have the following expansion in $\mathcal{L}(H^{0,s}; \tilde{H}^{2,-s})$ as $z\to 0$ with* $\operatorname{Im} z$, $\operatorname{Im} z^{1/2}$, $\operatorname{Im}\log z\geqq 0$:

$$(H_0-z)^{-1}=\sum_{j=0}^{[\sigma+1-n/2]} F_j z^{n/2-1-j}(\log z)^{\varepsilon(n)}+\sum_{j=0}^{[\sigma]} G_j z^j+o(z^\sigma)$$

where F_j, $G_j\in\mathcal{L}(H^{0,s}; H^{0,-s})$, and these operators can be computed explicitly (*see the references cited above*).

(2) *Assume that there exist $\rho>\max(2, 4-n)$ and $m<1$ such that V is a compact operator from $H^{m,s}$ to $H^{0,\rho+s}$ for any $s\in\boldsymbol{R}$. We define the generalized eigenspace M by*

$$M=\{\phi\in\bigcap_{r<n/2-2}\tilde{H}^{2,r}\,;\,(1+G_0V)\phi=0\}\qquad \text{for } n\geqq 3,$$

$$M=\{\phi\in\bigcap_{r<-n/2}\tilde{H}^{2,r}\,;\,(1+G_0V)\phi\in\operatorname{Range}(F_0),\ F_0V\phi=0\}\qquad \text{for } n\leqq 2.$$

Let P be the orthogonal projection onto $\operatorname{Ker}(H)$. *Then, $P(\mathcal{H})\subset M$ and $P(\mathcal{H})=M$ for $n\geqq 5$.*

(3) *Let V be as in part* (2). *Assume that $M=\{0\}$ and there are no eigenvalues in $[0, \infty)$. Let $\rho>3$ for $n=1$ and $\rho>2$ for $n\geqq 2$; $1/2<\gamma<\min(\rho-1, (\rho+n)/2-1)$; $s>\max(\gamma, 2\gamma-n/2)$. Let Π_s be the orthogonal projection onto $\mathcal{H}_c$. Then we have the following expansion in $\mathcal{L}(H^{0,s}; H^{0,-s})$ as $t\to+\infty$:*

(i) *For n odd,*

$$e^{-itH}\Pi_s=\sum_{j=\max(n/2-1/2,1)}^{[\gamma-1/2]} t^{-j-1/2}\pi^{-1}(-i)^{j-1/2}\Gamma(j+1/2)B_{2j-1}+o(t^{-\gamma}),$$

where $B_{2j-1}\in\mathcal{L}(H^{0,s}; H^{0,-s})$.

(ii) *For $n\geqq 4$ even,*

$$e^{-itH}\Pi_s=\sum_{j=n/2}^{[\gamma]}\sum_{k=0}^{N(j)} t^{-j}(\log t)^k\sum_{l=k+1}^{N(j)+1} a_{jkl}B_{j-1,l}+o(t^{-\gamma}),$$

where $B_{j-1,l}\in\mathcal{L}(H^{0,s}; H^{0,-s})$, $N(j)=[(j-1)/(n/2-1)]-1$ and $a_{jkl}\in\boldsymbol{C}$.

(3) *For $n=2$,*

$$e^{-itH}\Pi_s=\sum_{j=1}^{[\gamma]} t^{-j}\Phi_j(t)+o(t^{-\gamma-\varepsilon})$$

for some $\varepsilon>0$, *where the* Φ_j *are* $\mathcal{L}(H^{0,s}; H^{0,-s})$*-valued smooth functions such that for any* $\nu>0$,

$$\Phi_j(t)=\sum_{k=M(j)}^{\nu}(\log t)^{-k}\sum_{l=-j+1}^{k-1}a_{jkl}B_{j-1,l}+O((\log t)^{-\nu-1})$$

as $t\to+\infty$, *where* $M(1)=2$, $M(j)=-j+2$ *for* $j\geqq 2$. B_{2j-1}, $B_{j-1,l}$ *and* a_{jkl} *can be computed explicitly* (*see the references cited above*). *Analogous results also hold in the case* $t\to-\infty$.

Theorem 6.1. *Let* $\alpha>\max(2, 4-n)$ *and let* $\omega^{\alpha}V\in L^{\infty}$. *Suppose that* (A2)$^{\#}$ *holds and that there exists a constant* $C\geqq 0$ *satisfying* $V+(1/2)V^{\#}\leqq C\omega^{-\alpha}$ *as forms on* $H^{2,0}$. *When* $n\leqq 4$, *suppose in addition that* $M=\{0\}$. *Let* $\max(2, 4-n)<\rho<\alpha$. *Then*:

(1) *For any* $\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$,

$$\|xe^{itH_0}e^{-itH}\phi\|_2\leqq C\|\phi\|_{0,\rho/2}, \qquad t\in\boldsymbol{R}.$$

(2) *There exists* $\delta>0$ *such that for* $n\neq 2$ *and any* $\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$,

$$|(e^{itH}Ve^{-itH}\phi, \phi)|\leqq C(1+|t|)^{-2(1+\delta)}\|\phi\|_{0,\rho/2}^2, \qquad t\in\boldsymbol{R}.$$

For $n=2$ *and any* $\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$,

$$|(e^{itH}Ve^{-itH}\phi, \phi)|\leqq C(1+|t|)^{-2}(1+\log(1+|t|))^{-4}\|\phi\|_{0,\rho/2}^2, \qquad t\in\boldsymbol{R}.$$

(3) *For any* $\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$,

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-\delta_n(p)}\|\phi\|_{0,\rho/2}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where p *satisfies* $2\leqq p\leqq\infty$ *if* $n=1$, $2\leqq p<\infty$ *if* $n=2$, *and* $2\leqq p\leqq 2n/(n-2)$ *if* $n\geqq 3$.

(4) *Let* $n\geqq 3$ *and let* $m\in\boldsymbol{N}$. *If* $\partial^{\beta}V\in L^{\infty}$ *for all* $|\beta|\leqq\max(m-2, 0)$, *then for any* $\phi\in\mathcal{H}_c\cap\widetilde{H}^{m,\rho/2}$,

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-1}|||\phi|||_{m,\rho/2}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where p *satisfies* $2n/(n-2)\leqq p\leqq\infty$ *if* $n\leqq 2m+1$, $2n/(n-2)\leqq p<\infty$ *if* $n=2m+2$, *and* $2n/(n-2)\leqq p\leqq 2n/(n-2-2m)$ *if* $n\geqq 2m+3$.

(5) *Let* $n=2$. *For any* $\phi\in\mathcal{H}_c\cap H^{2,\rho/2}$,

$$\|e^{-itH}\phi\|_{\infty}\leqq C(1+|t|)^{-1}(1+\log(1+|t|))^{1/2}\|\phi\|_{2,\rho/2}, \qquad t\in\boldsymbol{R}.$$

Proof. By the Kato-Agmon-Simon theorem (see M. Reed & B. Simon [31]), H has no eigenvalues in $(0, \infty)$. Without loss of generality we assume that H satisfies the assumptions of parts (2) and (3) of Theorem A, since $\mathcal{H}_c$ and $\mathrm{Ker}(H)$ are invariant under e^{-itH}. We prove the theorem only for the case $t>0$, since the case $t<0$ can be treated analogously. We apply part (3) of Theorem A to

$\gamma=1+(\rho-\max(2, 4-n))/8$ and $s=\rho/2$ to conclude that there exist constants $\delta>0$, $C>0$ and $t_0>3$ such that for any $\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$ and any $t\geqq t_0$,

$$\|e^{-itH}\phi\|_{0,-\rho/2}\leqq\begin{cases} Ct^{-1-\delta}\|\phi\|_{0,\rho/2} & \text{for } n\neq 2, \\ Ct^{-1}(\log t)^{-2}\|\phi\|_{0,\rho/2} & \text{for } n=2. \end{cases} \tag{6.67}$$

(6.67) gives

$$|(e^{itH}Ve^{-itH}\phi, \phi)|\leqq\begin{cases} Ct^{-2(1+\delta)}\|\phi\|_{0,\rho/2}^2 & \text{for } n\neq 2, \\ Ct^{-2}(\log t)^{-4}\|\phi\|_{0,\rho/2}^2 & \text{for } n=2. \end{cases} \tag{6.68}$$

Since $V\in L^\infty$, we have

$$\sup_{t\in\boldsymbol{R}}|(e^{itH}Ve^{-itH}\phi, \phi)|\leqq\|V\|_\infty\|\phi\|_2^2.$$

This proves part (2). We next prove part (1). Let $\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$ and let $\phi_\lambda=\lambda(\lambda+H)^{-1}\phi$, $\lambda\geqq\lambda_0$ (see Lemma 6.1). Then, by Lemma 6.1, $\phi_\lambda\in\mathcal{H}_c\cap H^{2,\rho/2}$ and $\phi_\lambda\to\phi$ in $H^{0,\rho/2}$ as $\lambda\to\infty$. (6.67) implies

$$\begin{aligned}&((V+(1/2)V^{\#})e^{-itH}\phi_\lambda, e^{-itH}\phi_\lambda)\\ &\qquad\leqq\begin{cases} Ct^{-2(1+\delta)}\|\phi_\lambda\|_{0,\rho/2}^2 & \text{for } n\neq 2,\ t\geqq t_0, \\ Ct^{-2}(\log t)^{-4}\|\phi\|_{0,\rho/2}^2 & \text{for } n=2,\ t\geqq t_0. \end{cases}\end{aligned}$$

From Theorem 3.2, (6.67) and (6.68), we obtain

$$\begin{aligned}&\|xe^{itH_0}e^{-itH}\phi_\lambda\|_2^2\\ &\leqq 4t^2|(e^{itH}Ve^{-itH}\phi_\lambda, \phi_\lambda)|+4t_0^2|(e^{it_0H}Ve^{-it_0H}\phi_\lambda, \phi_\lambda)|\\ &\qquad+\|xe^{it_0H_0}e^{-it_0H}\phi_\lambda\|_2^2+\int_{t_0}^t\tau((V+(1/2)V^{\#})e^{-i\tau H}\phi_\lambda, e^{-i\tau H}\phi_\lambda)d\tau\\ &\leqq Ct_0^2\|\phi_\lambda\|_{0,\rho/2}^2, \qquad t\geqq t_0.\end{aligned} \tag{6.69}$$

Letting $\lambda\to\infty$ in (6.69), we obtain

$$\|xe^{itH_0}e^{-itH}\phi\|_2^2\leqq Ct_0^2\|\phi\|_{0,\rho/2}^2, \qquad t\geqq t_0. \tag{6.70}$$

Combining (6.70) and (3.10), we obtain part (1). In the same way as in the proof of Theorem 4.3, part (3) now follows from (6.69), (6.70) and (3.10). We turn to part (4). If $\phi\in\mathcal{H}_c\cap\tilde{H}^{m,\rho/2}$, then by Proposition 6.1 and Lemma 6.3, we have for $\mu\geqq\lambda_1$ (see Lemma 6.2), $(H+\mu)^{m/2}\phi=(\hat{H}+\mu)^{m/2}\phi\in\mathcal{H}_c\cap H^{0,\rho/2}$ and therefore by part (3),

$$\begin{aligned}\|(H+\mu)^{m/2}e^{-itH}\phi\|_{2n/(n-2)}&=\|e^{-itH}(H+\mu)^{m/2}\phi\|_{2n/(n-2)}\\ &\leqq Ct^{-1}\|(H+\mu)^{m/2}\phi\|_{0,\rho/2}\leqq Ct^{-1}|||\phi|||_{m,\rho/2}.\end{aligned}$$

By Proposition 6.2 and Lemma 6.6, this leads to

$$\sum_{|\alpha|=m}\|\partial^\alpha e^{-itH}\phi\|_{2n/(n-2)}\leqq C\|\tilde{H}_\mu^{m/2}e^{-itH}\phi\|_{2n/(n-2)}\leqq Ct^{-1}|||\phi|||_{m,\rho/2},$$

for μ sufficiently large. By Lemma 1.1,

$$\|e^{-itH}\phi\|_p \leqq C \sum_{|\alpha|=m} \|\partial^\alpha e^{-itH}\phi\|_{2n/(n-2)}^a \|e^{-itH}\phi\|_{2n/(n-2)}^{1-a},$$

where $a=(n/m)(1/2-1/n-1/p)$ and p ranges over the intervals indicated as in the theorem. Combining these estimates, we obtain part (4). We finally prove part (5). The idea of the proof is essentially the same as that of J. Q. Yao [38]. Let $\phi \in \mathcal{H}_c \cap H^{2,\rho/2}$. By (6.65) and part (1),

$$\begin{aligned}
\|e^{-itH}\phi\|_p &= \|\exp(|x|^2/4it)e^{-itH}\phi\|_p \\
&\leqq Cp^{1/2}\|\nabla(\exp(|x|^2/4it)e^{-itH}\phi)\|_2^{1-2/p}\|\exp(|x|^2/4it)e^{-itH}\phi\|_2^{2/p} \\
&\leqq Cp^{1/2}\|(x/2it+\nabla)e^{-itH}\phi\|_2^{1-2/p}\|\phi\|_2^{2/p} \\
&\leqq Cp^{1/2}t^{-1+2/p}\|xe^{itH_0}e^{-itH}\phi\|_2^{1-2/p}\|\phi\|_2^{2/p} \leqq Cp^{1/2}t^{-1+2/p}\|\phi\|_{0,\rho/2}.
\end{aligned}$$

By (2.8),

$$\|e^{-itH}\phi\|_{2,0} \leqq C\|\phi\|_{2,0}.$$

We apply (6.66) to the estimates above to obtain

$$\|e^{-itH}\phi\|_\infty \leqq Cp^{p/(2(p+n))}t^{(2-p)/(p+n)}\|\phi\|_{2,\rho/2}.$$

Now taking $p=\log t$ in this inequality, we obtain part (5) for $t>0$ large. In view of Theorem 4.1, the proof is complete. Q. E. D.

Theorem 6.2. *Let $n \geqq 3$, $\alpha > 2$ and let $\omega^\alpha V \in L^\infty$. Suppose that* (A2)$^\#$ *and* (A6) *hold. When $n \leqq 4$, suppose in addition that $M=\{0\}$. Let $2<\rho<\alpha$. Then:*

(1) *For any $\phi \in H^{0,\rho/2}$,*

$$\|xe^{itH_0}e^{-itH}\phi\|_2 \leqq C\|\phi\|_{0,\rho/2}, \qquad t \in \mathbf{R}.$$

(2) *There exists $\delta>0$ such that for any $\phi \in H^{0,\rho/2}$,*

$$|(e^{itH}Ve^{-itH}\phi, \phi)| \leqq C(1+|t|)^{-2(1+\delta)}\|\phi\|_{0,\rho/2}^2, \qquad t \in \mathbf{R}.$$

(3) *For any $\phi \in H^{0,\rho/2}$,*

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-\delta_n(p)}\|\phi\|_{0,\rho/2}, \qquad t \in \mathbf{R},$$

where p satisfies $2 \leqq p \leqq 2n/(n-2)$.

(4) *Let $m \in \mathbf{N}$. If $\partial^\beta V \in L^\infty$ for all $|\beta| \leqq \max(m-2, 0)$, then for any $\phi \in \widetilde{H}^{m,\rho/2}$,*

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-1}|||\phi|||_{m,\rho/2}, \qquad t \in \mathbf{R}\setminus\{0\},$$

where p is as in part (4) *of Theorem* 6.1.

Proof. Since H is absolutely continuous (see the proof of Theorem 5.3), the result follows in the same way as in the proof of Theorem 6.1. Q. E. D.

Theorem 6.3. *Let $q\geqq 2$ for $n\leqq 3$, let $q>2$ for $n=4$, and let $q>n/2$ for $n\geqq 5$. Suppose that $V\in L^q+L^\infty$, $V\geqq 0$, and that* (A2)$^\#$ *holds. Suppose in addition that there exists $\beta\in[0, 2]$ such that $V^\#\leqq-\beta V$ as forms on $H^{2,0}$. For $n\geqq 3$, $p\geqq 2n/(n-2)$ and $q\geqq 2$, we define*:

$$a_n(p, q)=(1/2-1/n-1/p)/(1/2-1/q),\quad a_n(p, 2)=0\,.$$

$$b_n(p, q)=(1/2-1/n-1/p)/(1/2+1/n-1/q)\,,$$

$$\delta_n(p, q)=a_n(p, q)\delta_n(q)+1-a_n(p, q)\,,$$

$$\gamma_n(p, q)=b_n(p, q)\delta_n(q)+1-b_n(p, q)\,.$$

Then:

(1) *For any $\phi\in H^{0,1}$,*

$$\|xe^{itH_0}e^{-itH}\phi\|_2\leqq C(1+|t|)^{1-\beta/2}\|\phi\|_{0,1}\,,\qquad t\in\boldsymbol{R}.$$

(2) *For any $\phi\in H^{0,1}$,*

$$0\leqq(e^{itH}Ve^{-itH}\phi, \phi)\leqq C|t|^{-\beta}\|\phi\|_{0,1}^2\,,\qquad |t|\geqq 1\,.$$

(3) *For any $\phi\in H^{0,1}$,*

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-(\beta/2)\delta_n(p)}\|\phi\|_{0,1}\,,\qquad t\in\boldsymbol{R}\setminus\{0\},$$

where n and p are as in part (3) *of Theorem* 6.1.

(4) *Let $n=2$. For any $\phi\in\tilde{H}^{2,1}$,*

$$\|e^{-itH}\phi\|_\infty\leqq C(1+|t|)^{-\beta/2}(1+\log(1+|t|))^{1/2}|||\phi|||_{2,1}\,,\qquad t\in\boldsymbol{R}.$$

(5) *Let $n=3$.* (i) *If $2\leqq q<3$, then for any $\phi\in\tilde{H}^{1,1}$,*

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-(\beta/2)\delta_3(p,q)}|||\phi|||_{1,1}\,,\qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $6\leqq p\leqq 3q/(3-q)$. If $q=3$, then for any $\phi\in\tilde{H}^{1,1}$,

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-(\beta/2)(1/2+3/p)}|||\phi|||_{1,1}\,,\qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $6\leqq p<\infty$. If $3<q<6$, then for any $\phi\in\tilde{H}^{1,1}$,

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-(\beta/2)\delta_3(p,q)}|||\phi|||_{1,1}\,,\qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $6\leqq p\leqq\infty$. If $q\geqq 6$, then for any $\phi\in\tilde{H}^{1,1}$,

$$\|e^{-itH}\phi\|_p\leqq C|t|^{-\beta/2}|||\phi|||_{1,1}\,,\qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $6\leqq p\leqq\infty$.

(ii) *If $2\leqq q<6$, then for any $\phi\in\tilde{H}^{2,1}$,*

$$\|e^{-itH}\phi\|_p\leqq C(1+|t|)^{-(\beta/2)\gamma_3(p,q)}|||\phi|||_{2,1}\,,\qquad t\in\boldsymbol{R},$$

where $6\leqq p\leqq\infty$. If $q\geqq 6$, then for any $\phi\in\tilde{H}^{2,1}$,

$$\|e^{-itH}\phi\|_p \leqq C(1+|t|)^{-\beta/2}|||\phi|||_{2,1}, \qquad t\in\boldsymbol{R},$$

where $6\leqq p\leqq\infty$.

(6) *Let* $n=4$ *or* 5. (i) *If* $n/2<q<2n/(n-2)$, *then for any* $\phi\in\widetilde{H}^{1,1}$,

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-(\beta/2)\delta_n(p,q)}|||\phi|||_{1,1}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $2n/(n-2)\leqq p\leqq nq/(n-q)$. *If* $q\geqq 2n/(n-2)$, *then for any* $\phi\in\widetilde{H}^{1,1}$,

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-\beta/2}|||\phi|||_{1,1}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $4\leqq p\leqq\infty$ *for* $n=4$, $10/3\leqq p\leqq 10$ *for* $n=5$.

(ii) *If* $n/2<q<2n/(n-2)$, *then for any* $\phi\in\widetilde{H}^{2,1}$,

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-(\beta/2)\gamma_n(p,q)}|||\phi|||_{2,1}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $2n/(n-2)\leqq p\leqq\infty$. *If* $q\geqq 2n/(n-2)$, *then for any* $\phi\in\widetilde{H}^{2,1}$,

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-\beta/2}|||\phi|||_{2,1}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $2n/(n-2)\leqq p\leqq\infty$.

(7) *Let* $n\geqq 6$. *Let* $m=1$ *or* 2. *Then for any* $\phi\in\widetilde{H}^{m,1}$,

$$\|e^{-itH}\phi\|_p \leqq C|t|^{-\beta/2}|||\phi|||_{m,1}, \qquad t\in\boldsymbol{R}\setminus\{0\},$$

where $3\leqq p\leqq 6$ *for* $n=6$, $m=1$; $3\leqq p<\infty$ *for* $n=6$, $m=2$; $2n/(n-2)\leqq p<2n/(n-2-2m)$ *for* $n\geqq 7$.

Remark 6.1. T. Cazenave, J.P. Dias & M. Figueira [2] and J.P. Dias & M. Figueira [3] have obtained some estimates similar to those in Theorem 6.3. These references were brought to the attention of the author by Y. Tsutsumi.

Remark 6.2. If $3\leqq n\leqq 5$, $2<q<2n/(n-2)\leqq p\leqq\infty$, then $\delta_n(p)\geqq\gamma_n(p,q)\geqq\delta_n(p,q)\geqq\delta_n(q)>0$. In particular, $1=\delta_n(2n/(n-2))=\gamma_n(2n/(n-2),q)=\delta_n(2n/(n-2),q)>\delta_n(q)>0$.

Proof of Theorem 6.3. We consider only the case $t>0$, since the case $t<0$ can be treated analogously. Let $\phi\in H^{0,1}$ and let $\{\phi_j\}$ be a sequence in $\mathcal{S}$ such that $\phi_j\to\phi$ in $H^{0,1}$ as $j\to\infty$. For $\psi\in\mathcal{S}$ we put $F(t)=\|(x+2it\nabla)e^{-itH}\psi\|_2^2+4t^2(Ve^{-itH}\psi, e^{-itH}\psi)$. By Theorem 3.1 and the assumption that $V^{\#}\leqq-\beta V$,

$$\begin{aligned}\frac{d}{dt}F(t)&=8t((V+(1/2)V^{\#})e^{-itH}\psi, e^{-itH}\psi)\\&\leqq 4(2-\beta(t)Ve^{-itH}\psi, e^{-itH}\psi)\leqq(2-\beta)t^{-1}F(t).\end{aligned}$$

This gives $(d/dt)(t^{\beta-2}F(t))\leqq 0$, so that $F(t)\leqq t^{2-\beta}F(1)$ for $t\geqq 1$. It follows from Theorem 3.2 that $F(1)\leqq C\|\psi\|_{0,1}^2$. Therefore,

$$\|xe^{itH_0}e^{-itH}\psi\|_2^2+4t^2(e^{itH}Ve^{-itH}\psi, \psi)\leqq t^{2-\beta}\|\psi\|_{0,1}^2, \qquad t\geqq 1. \tag{6.71}$$

Substituting $\phi_j-\phi_k$ for ϕ in (6.71), we see that $\{e^{itH_0}e^{-itH}\phi_j\}$ is a Cauchy sequence in $H^{0,1}$. Therefore we obtain $e^{itH_0}e^{-itH}\phi_j \to e^{itH_0}e^{-itH}\phi$ in $H^{0,1}$ as $j\to\infty$. This together with Theorem 3.2 shows that (6.71) holds with ϕ replaced by ϕ and that parts (1) and (2) hold. Parts (3) and (4) follow in the same way as in the proof of Theorem 6.1. We turn to part (5). Let $\phi\in\tilde{H}^{1,1}$. If $2\leqq q<6$ and $p\geqq 6$, then we use Lemma 1.1 to obtain

$$\|e^{-itH}\phi\|_p \leqq C\|\nabla e^{-itH}\phi\|_q^{a_3(p,q)}\|e^{-itH}\phi\|_6^{1-a_3(p,q)}, \tag{6.72}$$

where p ranges over the intervals indicated in the theorem. If $q\geqq 6$ and $p\geqq 6$, then we use Lemma 1.1 to obtain

$$\|e^{-itH}\phi\|_p \leqq C\|\nabla e^{-itH}\phi\|_6^a\|e^{-itH}\phi\|_6^{1-a}, \tag{6.73}$$

where $a=(p-6)/(2p)$. Therefore, the results for $\phi\in\tilde{H}^{1,1}$ follow from part (3), (6.72), (6.73), Lemmas 6.2, 6.3, 6.6 and 6.7, and Proposition 6.1. We next consider the case $\phi\in\tilde{H}^{2,1}$. As in the preceding argument, we use Lemma 1.1. We estimate $\|e^{-itH}\phi\|_p$ by $C\sum_{|\alpha|=2}\|\partial^\alpha e^{-itH}\phi\|_q^{b_n(p,q)}\|e^{-itH}\phi\|_6^{1-b_n(p,q)}$ if $q<6$, and by $C\sum_{|\alpha|=2}\|\partial^\alpha e^{-itH}\phi\|_6^b\|e^{-itH}\phi\|_6^b$ if $q\geqq 6$, where $b=(p-6)/(4p)$. Here, p is in the intervals determined by the conditions of Lemma 1.1. The results for $\phi\in\tilde{H}^{2,1}$ then follows from part (3), Lemmas 6.3, 6.5, 6.6, Proposition 6.1, and part (2) of Theorem 4.1. Parts (6) and (7) can be obtained in the same way as in the proof of part (5). Q.E.D.

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