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Lower L^p-Bounds for Scattering Solutions of the Schrödinger Equations

By

Tohru OZAWA*

Abstract

In this article, the asymptotic behavior in time of scattering solutions to the Schrödinger equation

$$i\partial_t u = Hu, \qquad (t, x) \in \boldsymbol{R} \times \boldsymbol{R}^n, \qquad H = -\Delta + V,$$
$$u(0, x) = \phi(x), \qquad x \in \boldsymbol{R}^n,$$

is investigated.

Under rather natural assumptions, $L^p(\boldsymbol{R}^n)$-lower bound estimates of the form

$$\liminf_{t \to \pm\infty} |t|^{n/2-n/p} \| e^{-itH}\phi \|_{L^p(\boldsymbol{R}^n)} > 0 \qquad (1 \leqq p \leqq +\infty)$$

for $\phi \in \mathscr{H}_{\mathrm{cont}}(H)$ with $\phi \neq 0$ are established, where $\mathscr{H}_{\mathrm{cont}}(H)$ denotes the continuous spectral subspace of H.

This shows that the estimates obtained by the author in [10] are optimal.

Introduction

We study the asymptotic behavior in time of scattering L^p-solutions to the Cauchy problem for the equation

$$i\partial_t u = Hu, \qquad (t, x) \in \boldsymbol{R} \times \boldsymbol{R}^n \quad (n \geqq 1),$$
$$u(0, x) = \phi(x), \; x \in \boldsymbol{R}^n,$$

where H is a self-adjoint operator in the Hilbert space $L^2(\boldsymbol{R}^n)$. A scattering solution means a solution of the form $e^{-itH}\phi$ with ϕ in the continuous spectral subspace $\mathscr{H}_{\mathrm{cont}} = \mathscr{H}_{\mathrm{cont}}(H)$ of H. Our main attention is on the case where H takes the form

$$H = H_0 + V, \; H_0 = -\Delta$$

with a potential V which should satisfy some conditions specified later. Roughly speaking, for a short-range potential V satisfying the conditions

$$|V(x)| \leqq C(1+|x|)^{-\alpha} \quad \text{and} \quad V(x) + (1/2)x \cdot \nabla V(x) \leqq C(1+|x|)^{-\alpha}$$

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* Department of Mathematics, Nagoya University, Nagoya 464, Japan.

for almost every $x\in \boldsymbol{R}^n$, where $\alpha>\max(2, 4-n)$ and $C>0$, it is known that every scattering solution $e^{-itH}\phi$ with a nice initial datum ϕ decays in the L^p-norm if $p>2$. In fact, the following estimate holds:

$$\|e^{-itH}\phi\|_{L^p(R^n)}\leqq C|t|^{-\delta_n(p)}, \quad |t|\geqq 1, \tag{0.1}$$

where $\delta_n(p)=n/2-n/p$. For $1\leqq p\leqq 2$, we also have the same estimate as (0.1) with more general potentials. But they do not imply the decay of solutions.

The above results have been obtained by the author in [10].

We shall show that these estimates are really optimal. To be more precise, under certain hypotheses it is shown that for any $\phi\in \mathscr{H}_{\text{cont}}\setminus\{0\}$,

$$\liminf_{t\to\pm\infty}|t|^{\delta_n(p)}\|e^{-itH}\phi\|_{L^p(R^n)}>0 \qquad (1\leqq p\leqq\infty)$$

holds. We deduce from this lower bound estimate the result that any scattering solution with its L^p-norm decaying faster than $O(|t|^{-\delta_n(p)})$ $(2\leqq p\leqq\infty)$ or growing slower than $O(|t|^{-\delta_n(p)})$ $(1\leqq p<2)$ vanishes. We shall prove these facts by mainly using Strauss' argument in [13].

§ 1. Preliminaries

We use the following notations. For an open subset $\Omega\subset\boldsymbol{R}^n$ and $p\in[1, \infty]$, we denote by $L^p(\Omega)$ the usual Lebesgue space of p-th integrable functions on Ω. The associated norm is denoted by $\|\cdot\|_{L^p(\Omega)}$. We abbreviate $L^p(\boldsymbol{R}^n)$ by L^p. L^p_{loc} denotes the space of locally p-th integrable functions on $\boldsymbol{R}^n$. For $x=(x_1, \cdots, x_n)\in\boldsymbol{R}^n$, we write $|x|=\left(\sum_{j=1}^n x_j^2\right)^{1/2}$, $\omega(x)=(1+|x|^2)^{1/2}$. For m, $s\in\boldsymbol{R}$, the weighted Sobolev spaces $H^{m,s}$ and $\tilde{H}^{m,s}$ are defined respectively by

$$H^{m,s}=\{\phi\in\mathcal{S}'\,;\ \|\phi\|_{m,s}=\|(1-\Delta)^{m/2}\phi\|_2+\|\omega^s\phi\|_2<\infty\}$$

and

$$\tilde{H}^{m,s}=\{\phi\in\mathcal{S}'\,;\ |||\phi|||_{m,s}=\|\omega^s(1-\Delta)^{m/2}\phi\|_2<\infty\}.$$

For an operator T in L^2, we denote by $D(T)$ its domain. A denotes the generator of dilations: $A=(1/2i)(x\cdot\nabla+\nabla\cdot x)$. Strauss' lemma will be used in the following form:

Lemma (Strauss [13]). *For any $\phi\in L^2\setminus\{0\}$, there exist k, $k'>0$, $t_0>0$, and $C_0>0$ such that*

$$\int_{kt<|x|<k't}|(e^{-itH_0}\phi)(x)|^2dx\geqq C_0 \qquad \textit{for all}\quad t>t_0.$$

Proof. Although the lemma in [13] is stated in a rather restrictive form, we see that the proof in [13] with some modifications shows the lemma above. So we omit the details. Q.E.D.

§ 2. The Case $2\leqq p\leqq\infty$

We consider the following hypotheses on a self-adjoint operator H in L^2:

(H0) *H has no singular continuous spectrum.*

$(\mathrm{H1})_+$ *The wave operator*

$$W_+=\text{s-}\lim_{t\to+\infty} e^{itH}e^{-itH_0}$$

exists and is complete.

For sufficient conditions of (H0) and $(\mathrm{H1})_+$, see, e.g., [1], [2], [3], [8], [9], [11] and [12].

In this section we prove:

Theorem 2.1. *Suppose that* (H0) *and* $(\mathrm{H1})_+$ *hold. Let* $2\leqq p\leqq\infty$ *and* $\phi\in\mathcal{H}_{\mathrm{cont}}\setminus\{0\}$. *Assume that there exists* $t_0>0$ *such that*

$$e^{-itH}\phi\in L^p_{\mathrm{loc}} \quad \textit{for all} \quad t>t_0. \tag{2.1}$$

Then, there exist $k>0$ *and* $k'>k$ *such that*

$$\liminf_{t\to+\infty} t^{\delta_n(p)}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ kt<|x|<k't\})}>0. \tag{2.2}$$

Proof. Let $2\leqq p\leqq\infty$ and let $\phi\in\mathcal{H}_{\mathrm{cont}}$ satisfy (2.1). Assume $\phi\neq 0$. It suffices to prove that there exist $t_1>0$, $k'>k>0$ and $C_1>0$ such that

$$\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ kt<|x|<k't\})}\geqq C_1t^{-\delta_n(p)} \quad \text{for all} \quad t>t_1. \tag{2.3}$$

From the assumptions on H, we conclude that

$$\mathrm{Range}(W_+)=\mathcal{H}_{\mathrm{cont}} \tag{2.4}$$

and that

$$\text{s-}\lim_{t\to+\infty} e^{itH_0}e^{-itH}P_{\mathrm{cont}}=W_+^* \tag{2.5}$$

where P_{cont} is the orthogonal projection on $\mathcal{H}_{\mathrm{cont}}$ and W_+^* is the adjoint of W_+. Consequently,

$$W_+^*\phi\neq 0. \tag{2.6}$$

By virtue of Strauss' lemma, there exist $t_2>0$, $k'>k>0$ and $C_2>0$ such that

$$\int_{kt<|x|<k't}|(e^{-itH_0}W_+^*\phi)(x)|^2dx\geqq C_2 \quad \text{for all} \quad t>t_2. \tag{2.7}$$

By Hölder's inequality, we see that

$$\begin{aligned}(2.8)\qquad &\left(\int_{kt<|x|<k't}|(e^{-itH}\phi)(x)|^2dx\right)^{1/2}\\ &\leqq(\omega_n(k'^n-k^n)t^n)^{1/2-1/p}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ kt<|x|<k't\})}\end{aligned}$$

for any $t>0$, where $\omega_n=\pi^{n/2}/\Gamma(n/2+1)$ and Γ denotes the gamma function. We estimate the L. H. S. of (2.8) from below as follows:

$$(2.9)\qquad \left(\int_{kt<|x|<k't}|(e^{-itH}\phi)(x)|^2dx\right)^{1/2}$$
$$\geqq\left(\int_{kt<|x|<k't}|(e^{-itH_0}W_+^*\phi)(x)|^2dx\right)^{1/2}$$
$$-\left(\int_{kt<|x|<k't}|(e^{-itH}\phi-e^{-itH_0}W_+^*\phi)(x)|^2dx\right)^{1/2}$$
$$\geqq C_2^{1/2}-\|e^{-itH}\phi-e^{-itH_0}W_+^*\phi\|_2$$
$$=C_2^{1/2}-\|e^{itH_0}e^{-itH}\phi-W_+^*\phi\|_2 \qquad \text{for any } t>t_2.$$

Hence, there exist $t_3>0$ and $C_3>0$ such that

$$(2.10)\qquad \left(\int_{kt<|x|<k't}|(e^{-itH}\phi)(x)|^2dx\right)^{1/2}\geqq C_3 \qquad \text{for any } t>t_3.$$

(2.8) and (2.10) give

$$(2.11)\qquad \|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ kt<|x|<k't\})}$$
$$\geqq(C_3\omega_n(k'^n-k^n)^{1/p-1/2})\cdot t^{-\delta_n(p)} \qquad \text{for any } t>t_3,$$

as required. Q. E. D.

Corollary 2.1. *Suppose that* (H0) *and* $(\mathrm{H1})_+$ *hold. Let* $2\leqq p\leqq\infty$ *and* $\phi\in\mathcal{H}_{\mathrm{cont}}\setminus\{0\}$. *Assume that there exists* $t_0>0$ *such that*

$$(2.12)\qquad e^{-itH}\phi\in L^p(R^n) \qquad \textit{for all } t>t_0.$$

Then we have

$$\liminf_{t\to+\infty} t^{\delta_n(p)}\|e^{-itH}\phi\|_p>0.$$

Remark 2.1 For sufficient conditions of (2.12), see [10].

Corollary 2.2. *Suppose that* (H0) *and* $(\mathrm{H1})_+$ *hold. Let* $2\leqq p\leqq\infty$ *and let* $\phi\in\mathcal{H}_{\mathrm{cont}}$. *Assume that for each* $k'>k>0$,

$$\liminf_{t\to+\infty} t^{\delta_n(p)}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ kt<|x|<k't\})}=0.$$

Then we have $\phi=0$.

Remark 2.2 (H0) and $(\mathrm{H1})_-$:

$$(\mathrm{H1})_-\qquad W_-=\operatorname*{s-lim}_{t\to-\infty} e^{itH}e^{-itH_0} \ \textit{exists and is complete},$$

imply that the above statements for $t<0$ analogous to the case $t>0$ also hold. For example, for every $\phi\in\mathcal{H}_{\mathrm{cont}}\setminus\{0\}$ satisfying (2.12), we have

$$\liminf_{t\to-\infty} |t|^{\delta_n(p)}\|e^{-itH}\phi\|_p>0.$$

Remark 2.3. The asymptotic behavior as $t\to\pm\infty$ of the L.H.S. of (2.10) has been studied in numerous articles by different approaches (see, e.g., [3], [4], [9] and [11]).

Remark 2.4. When $H=H_0$ and $p=\infty$, some classes of initial data actually give $\lim_{t\to\pm\infty}$ instead of $\liminf_{t\to\pm\infty}$ in (2.2). Such examples can be found in [11].

§ 3. The Case $1\leqq p<2$

In the case $1\leqq p<2$, we need some more additional assumptions:

Theorem 3.1. *Suppose that* (H0) *and* $(\mathrm{H1})_+$ *hold. Let* $1\leqq p<2$ *and let* $\phi\in\mathcal{H}_{\mathrm{cont}}\setminus\{0\}$. *Assume that there exist* $t_0>0$ *such that*

$$e^{-itH}\phi\in L^\infty_{\mathrm{loc}}(\boldsymbol{R}^n) \quad \text{for all} \quad t>t_0. \tag{3.1}$$

Assume in addition that there exists an increasing function α: $\boldsymbol{R}\to[0,\infty)$ *such that for each* $k'>k>0$,

$$\sup_{t>t_0}\alpha(t)\|e^{-itH}\phi\|_{L^\infty(\{x\in R^n;\ kt<|x|<k't\})}<\infty \tag{3.2}$$

and $\alpha(t_0)>0$. *Then, there exist* $h'>h>0$ *such that*

$$\liminf_{t\to+\infty}\alpha(t)^{1-2/p}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ ht<|x|<h't\})}>0. \tag{3.3}$$

Proof. Put

$$C_{k,k'}=\sup_{t>t_0}\alpha(t)\|e^{-itH}\phi\|_{L^\infty(\{x\in R^n;\ kt<|x|<k't\})}.$$

Since

$$\begin{aligned}
&\left(\int_{kt<|x|<k't}|(e^{-itH}\phi)(x)|^2dx\right)^{1/2} \\
&\leqq\|e^{-itH}\phi\|^{1-p/2}_{L^\infty(\{x\in R^n;\ kt<|x|<k't\})}\cdot\|e^{-itH}\phi\|^{p/2}_{L^p(\{x\in R^n;\ kt<|x|<k't\})} \\
&\leqq C^{1-p/2}_{k,k'}(\alpha(t)^{1-2/p}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\ kt<|x|<k't\})})^{p/2},
\end{aligned} \tag{3.4}$$

a slight modification of the preceding proof works. We omit the details.

Q.E.D.

In view of (2.10) and (3.4), we easily obtain:

Corollary 3.1. *Suppose that* (H0) *and* $(\mathrm{H1})_+$ *hold. Let* $1\leqq p<2$, *and let* $\phi\in\mathcal{H}_{\mathrm{cont}}$. *Assume that there exists* $t_0>0$ *satisfying* (3.1). *Then, we have* $\phi=0$ *if either* (A) *or* (B) *holds:*

(A) *There exists an increasing function* α: $\boldsymbol{R}\to[0,\infty)$ *such that* $\alpha(t_0)>0$ *and* *that for each* $k'>k>0$,

$$\sup_{t>t_0} \alpha(t)\|e^{-itH}\phi\|_{L^\infty(\{x\in R^n;\, kt<|x|<k't\})}<\infty$$

and

$$\liminf_{t\to+\infty} \alpha(t)^{1-2/p}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\, kt<|x|<k't\})}=0.$$

(B) *There exists an increasing function* $\alpha: \boldsymbol{R}\to[0, \infty)$ *such that* $\alpha(t_0)>0$ *and that for each* $k'>k>0$,

$$\sup_{t>t_0} \alpha(t)^{1-2/p}\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\, kt<|x|<k't\})}<\infty$$

and

$$\liminf_{t\to+\infty} \alpha(t)\|e^{-itH}\phi\|_{L^\infty(\{x\in R^n;\, kt<|x|<k't\})}=0.$$

§4. Lower Bounds of Growth Order in Time for Scattering L^p-Solutions $(1\leqq p<2)$

We consider the following class of potentials V, which is identical to that of [10; Theorem 6.1].

(H2) *The form* $i[A, V]$ *on* $D(A)\cap D(H)$, *defined by*

$$(i[A, V]\phi, \psi)=i(V\phi, A\psi)-i(A\phi, V\psi),\ \phi, \psi\in D(A)\cap D(H),$$

extends to a bounded operator $V^\#\in\mathcal{L}(H^{2,0}; H^{-2,0})$.
(H3) *There exist* $\alpha>\max(2, 4-n)$ *and* $C>0$ *such that* $\omega^\alpha V\in L^\infty$ *and* $V+(1/2)V^\#\leqq C\omega^{-\alpha}$ *as forms on* $H^{2,0}$.

In order to describe lower bounds for growth order of scattering L^p-solutions for $1\leqq p<2$, we put

$$\beta(t)=|t|^{1/2-1/p} \quad \text{for } n=1,$$

$$\beta(t)=|t|^{1-2/p}(\log|t|)^{1/2-1/p} \quad \text{for } n=2,$$

$$\beta(t)=|t|^{1-2/p} \quad \text{for } n\geqq 3.$$

Theorem 4.1. *Suppose that* (H2) *and* (H3) *hold. Let* $\max(2, 4-n)<\rho<\alpha$. *When* $n\leqq 4$, *suppose in addition that the generalized eigenspace for zero* (*see* [10; Theorem A]) *equals* $\{0\}$. *Let* $1\leqq p<2$ *and let* $\phi\in\mathcal{H}_{\mathrm{cont}}\setminus\{0\}$ *satisfy the following regularity assumptions:*

(1) $\phi\in H^{0,\rho/2}$ *for* $n=1$.

(2) $\phi\in H^{2,\rho/2}$ *for* $n=2$.

(3) $\phi\in\tilde{H}^{(n-1)/2,\rho/2}$ *for* $n\geqq 3$.

Then there exists $k'>k>0$ *such that*

$$\liminf_{t\to\pm\infty} \beta(t)\|e^{-itH}\phi\|_{L^p(\{x\in R^n;\, k|t|<|x|<k'|t|\})}>0.$$

Proof. All we have to do is to determine the L^∞-decay rate $\alpha(t)$ in (3.1) of Theorem 3.1. See [10] for details. Q.E.D.

§5. Remarks on the Non-Linear Schrödinger Equations

In this section, we shall give some comments on lower bounds for solutions to the non-linear Schrödinger equations. We restrict our attention to the following non-linear Schrödinger equation with a single power interaction:

$$i\partial_t u=-\Delta u+|u|^{p-1}u, \qquad (t,x)\in \boldsymbol{R}\times\boldsymbol{R}^n, \tag{5.1}$$
$$u(0,x)=\phi(x), \qquad x\in\boldsymbol{R}^n,$$

where $1+2/n<p<\alpha(n)$ with $\alpha(n)=\infty$ for $n\leqq 2$ and $\alpha(n)=(n+2)/(n-2)$ for $n\geqq 3$.

We recall the following theorem of Y. Tsutsumi & K. Yajima [14]:

For any $\phi\in H^{1,1}$, there exist $u_\pm\in L^2$ such that

$$\lim_{t\to\pm\infty}\|e^{-itH_0}u_\pm-u(t)\|_2=0 \tag{5.2}$$

where $u\in C(\boldsymbol{R};\, H^{1,1})$ is the unique solution of the integral equation

$$u(t)=e^{-itH_0}\phi-i\int_0^t e^{-i(t-\tau)H_0}|u|^{p-1}u(\tau)d\tau. \tag{5.3}$$

We note that for any $\phi\in H^{1,1}$, the solution u in the theorem satisfies $u(t)\in L^q(\boldsymbol{R}^n)$ for any $t\in\boldsymbol{R}$ provided $2\leqq q<\alpha(n)+1$.

Now we have:

Theorem 5.1. *Let $2\leqq q<\alpha(n)+1$ and let $\phi\in H^{1,1}\setminus\{0\}$. Then, the unique solution of* (5.3) *has the following estimate:*

$$\liminf_{t\to\pm\infty}|t|^{\delta_n(q)}\|u(t)\|_{L^q(\{x\in R^n;\, k|t|<|x|<k'|t|\})}>0$$

for some $k'>k>0$.

Proof. We note that $\phi=0$ implies $u_\pm=0$ in (5.2). By virtue of the theorem above, almost the same argument as in the proof of Theorem 2.1 yields Theorem 5.1. Q.E.D.

Remark 5.1. For the decay estimates (from above) of the solutions of (5.3), see [5] and [6]. Theorem 5.1 tells that their results are best possible with respect to the decay order in time. See also [7] for detailed analysis of $L^\infty(\boldsymbol{R}^n)$-decay for the classical solutions of (5.1).

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