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Extended Affine Lie Algebras and their Vertex Representations

By

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§ 0. Introduction

The concept of extended affine root systems was introduced by K. Saito [6] to construct a flat structure for the space of the universal deformation of a simple elliptic singularity. An extended affine root system is by definition an extension of an affine root system by one dimensional radical (see Definition 1.2). It is a natural problem to construct a Lie algebra associated with the root system.

In [7], P. Slodowy constructed a Lie algebra for an arbitray extended affine root system in such a way that the set of its real roots coincides with the root system. For instance, in the case of $A_l^{(1,1)}$, $D_l^{(1,1)}$ or $E_l^{(1,1)}$, they may be expressed in the form $\mathfrak{g}\otimes\boldsymbol{C}[\lambda_1^{\pm1}, \lambda_2^{\pm1}]$. Here $\mathfrak{g}$ is a finite dimensional simple Lie algebra of type A_l, D_l or E_l and the commutation relations are defined by the formula

$$[x\otimes\lambda_1^m\lambda_2^n,\ y\otimes\lambda_1^k\lambda_2^l]=[x,\ y]\otimes\lambda_1^{m+k}\lambda_2^{n+l}\qquad \text{for all}\quad x,\ y\in\mathfrak{g}.$$

Independently, M. Wakimoto also constructed in [8] Lie algebras associated with the extended affine root systems. In the case of $A_l^{(1,1)}$, $D_l^{(1,1)}$ or $E_l^{(1,1)}$, they may be expressed in the form $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\boldsymbol{C}[\lambda_1^{\pm1}, \lambda_2^{\pm1}]\otimes\boldsymbol{C}c\oplus\boldsymbol{C}d_1\oplus\boldsymbol{C}d_2$, whose commutation relations are defined by the formulae

$$\begin{cases}[x\otimes\lambda_1^m\lambda_2^n,\ y\otimes\lambda_1^k\lambda_2^l]=[x,\ y]\otimes\lambda_1^{m+k}\lambda_2^{n+l}\oplus(m+n)\delta_{m+n,0}\delta_{k+l,0}\cdot(x\,|\,y)c\\ [d_i,\ x\otimes\lambda_1^{m_1}\lambda_2^{m_2}]=m_i x\otimes\lambda_1^{m_1}\lambda_2^{m_2}\end{cases}$$

where c is the center. Further he constructed their Hermitian representation such that the center of $\tilde{\mathfrak{g}}$ acts trivially.

For an application to the deformation theory of simple elliptic singularities, it should be important to construct vertex representations of the Lie algebras. In this paper, using vertex operators, we shall construct a Lie algebra which has the extended affine root system $A_l^{(1,1)}$, $D_l^{(1,1)}$ or $E_l^{(1,1)}$ as the set of real roots, following the idea of I. B. Frenkel [1], I. B. Frenkel-V. G. Kac [2] and

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P. Goddard-D. Olive [3]. They may be expressed in the form $\tilde{\mathfrak{g}}(R)=\mathfrak{g}(R_a)\otimes \boldsymbol{C}[\lambda, \lambda^{-1}]\oplus \boldsymbol{C}d_1\oplus \boldsymbol{C}d_2$ where $\mathfrak{g}(R_a)$ is the affine Lie algebra of type $A_l^{(1)}$, $D_l^{(1)}$ or $E_l^{(1)}$ (see Theorem 2.5). Furthermore we shall consider the Weyl group W_R of the Lie algebra $\tilde{\mathfrak{g}}(R)$ (see Proposition 3.4). The Weyl group W_R is important for the theory of simple elliptic singularities since the coordinate ring of the base space of the deformation is the W_R-invariant functions (θ-functions) on an affine subspace of the Cartan subalgebra of $\tilde{\mathfrak{g}}(R)$.

Let us give a brief view on the contents of this paper. In § 1, following K. Saito, we shall describe the structure of an extended affine root system with a marking (see Proposition 1.7). In § 2, for any extended affine root system whose elements are all of length 2, we shall construct a Lie algebra using a vertex operator (see Theorem 2.5). In § 3, we shall consider the Weyl group of the Lie algebra (see Proposition 3.4).

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§ 1. Marked Extended Affine Root Systems

In this section, following K. Saito [6], we shall describe the marked extended affine root systems.

Let us start with the definition of general root systems. Let F be a finite dimensional vector space over $\boldsymbol{R}$ with a metric $(\cdot|\cdot)$ of signature (l_+, l_0, l_-), i. e., l_+, l_0 or l_- is the number of positive, zero or negative eigenvalues of $(\cdot|\cdot)$ respectively.

Definition 1.1. A subset R of F is called a root system belonging to $(\cdot|\cdot)$ if it satisfies the following conditions (R.1)$\sim$(R.5):

(R.1) Let $Q(R)$ be the $\boldsymbol{Z}$-submodule of F generated by R. Then $Q(R)$ is a full lattice of F, i. e., $Q(R)\otimes_{\boldsymbol{Z}}\boldsymbol{R}=F$.

(R.2) For any $\alpha\in R$, $(\alpha|\alpha)\neq 0$.

(R.3) We define the reflection $r_\alpha\in GL(F)$ for any non-isotropic vector $\alpha\in F$ by

$$r_\alpha(\lambda):=\lambda-\frac{2(\lambda|\alpha)}{(\alpha|\alpha)}\alpha \qquad \text{for any} \quad \lambda\in F.$$

Then for any $\alpha\in R$, $r_\alpha(R)=R$.

(R.4) For any α, $\beta\in R$, $\dfrac{2(\alpha|\beta)}{(\beta|\beta)}\in\boldsymbol{Z}$.

(R.5) (Irreducibility) If $R=R_1\cup R_2$ and $R_1\perp R_2$ with respect to $(\cdot|\cdot)$, then either $R_1=\phi$ or $R_2=\phi$ holds.

Assume that the metric $(\cdot|\cdot)$ is positive semi-definite, i. e., $l_-=0$. If $l_0=0$, then R is a finite root system. If $l_0=1$, then R is an affine root system.

Definition 1.2. R is called an extended affine root system if $l_0=2$ and $l_-=0$. (In general, R is called a k-extended affine root system if $l_0=k\geqq 3$ and $l_-=0$.)

From now on, we investigate extended affine root systems only. Namely, we assume that the metric $(\cdot|\cdot)$ has a signature $(l, 2, 0)$.

Definition 1.3. A linear subspace G of F is said to be *defined over* $\boldsymbol{Q}$, if $G\cap Q(R)$ is a full lattice of G.

We define a subspace of F by

$$\mathrm{Rad}(\cdot|\cdot):=\{\lambda\in F\,|\,(\lambda|\gamma)=0 \text{ for any } \gamma\in F\}.$$

Then $\mathrm{Rad}(\cdot|\cdot)$ is clearly a 2-dimensional subspace of F defined over $\boldsymbol{Q}$, since there exists a non-zero constant $c\in\boldsymbol{R}$ such that $c(\cdot|\cdot)$ is an integral bilinear form on $Q(R)\times Q(R)$ and that the equations $c(\lambda|\gamma)=0$ for all $\gamma\in Q(R)$ have rational coefficients.

Definition 1.4. A 1-dimensional subspace G of $\mathrm{Rad}(\cdot|\cdot)$ defined over $\boldsymbol{Q}$ is called a marking for an extended affine root system R.

Let G be a marking for R. Then the pair (R, G) is called a *marked extended affine root system* belonging to $(\cdot|\cdot)$. Note that there may be (at most) two different marked extended affine root systems for an extended affine root system. For example, following K. Saito's classification of marked extended affine root systems, $G_2^{(1,3)}$ and $G_2^{(3,1)}$ are isomorphic as extended affine root systems (see [6]).

We denote by $R_f(R_a)$ the image set of R in $F_f=F/\mathrm{Rad}(\cdot|\cdot)$ (resp. $F_a=F/G$) and by $(\cdot|\cdot)_f((\cdot|\cdot)_a)$ the metric on F_f (resp. F_a) induced by $(\cdot|\cdot)$. Then $R_f(R_a)$ is a finite (resp. affine) root system belonging to $(\cdot|\cdot)_f$ (resp. $(\cdot|\cdot)_a$). In fact, we can easily see that $R_f(R_a)$ satisfies the axioms of a root system (R.1)$\sim$(R.5) since $\mathrm{Rad}(\cdot|\cdot)$ (resp. G) is a subspace defined over $\boldsymbol{Q}$. Let $\{\alpha_0, \alpha_1, \cdots, \alpha_l\}$ be a fundamental root system of the affine root system R_a and

$$A=\left(\frac{2(\alpha_\nu|\alpha_\mu)_a}{(\alpha_\mu|\alpha_\mu)_a}\right)_{\mu,\nu=0,1,\cdots,l}$$

its generalized Cartan matrix. Following M. Wakimoto [8], let us define counting weights of $\{\alpha_0, \alpha_1, \cdots, \alpha_l\}$ as follows:

Definition 1.5. An $(l+1)$-tuple $(k_0, k_1, \cdots, k_l)$ of positive integers is called a set of counting weights of $\{\alpha_0, \cdots, \alpha_l\}$ if the matrix

$$\begin{pmatrix} k_0 & & \\ & \ddots & \\ & & k_l \end{pmatrix} A \begin{pmatrix} k_0 & & \\ & \ddots & \\ & & k_l \end{pmatrix}^{-1}$$

is again a generalized Cartan matrix and G. C. D. $(k_0, \cdots, k_l)=1$.

Let Γ_a be the affine Dynkin diagram of the affine root system R_a. Then assigning a counting weight k_μ, $\mu=0, 1, \cdots, l$, to the Γ_a for each vertex α_μ, $\mu=0, 1, \cdots, l$, we get a weighted affine Dynkin diagram $(\Gamma_a, (k_\mu))$ (see [6] and [8]). To get the marked extended affine Dynkin diagram (cf. Saito [6]) from the weighted affine Dynkin diagram $(\Gamma_a, (k_\mu))$, we do the following operation: Let I be the set of μ $(0\leqq\mu\leqq l)$ such that

$$a_\nu^\vee/k_\nu \leqq a_\mu^\vee/k_\mu \qquad \text{for every} \quad 0\leqq\nu\leqq l,$$

where $\alpha^\vee=(\alpha_0^\vee, \cdots, \alpha_l^\vee)$ is the set of positive mutually prime integers such that $\alpha^\vee A=0$. Then extending the affine Dynkin diagram Γ_a by adding new vertices $\{\alpha_\mu^* \mid \mu\in I\}$ under the rule in [6], we obtain a marked extended affine Dynkin diagram. In this way, the weighted affine Dynkin diagrams $(\Gamma_a, (k_\mu))$ are in one-to-one correspondence with Saito's extended affine Dynkin diagrams. The following definition is necessary to describe the Weyl group of (R, G) (see §3 Lemma 3.1).

Definition 1.6. The set $\{\alpha_0, \alpha_1, \cdots, \alpha_l\}\cup\{\alpha_\mu^* \mid \mu\in I\}$ is called the *generator system* of (R, G).

Now we describe the structure of a marked extended affine root system (R, G). Since R_a is an affine root system, R_a is decomposed into a finite number of orbits of the affine Weyl group W_a and each orbit contains some simple root α_μ $(0\leqq\mu\leqq l)$. Hence we can define a mapping k of R_a into the set of natural numbers $\boldsymbol{N}$ as follows:

$$k(w(\alpha_\mu)) := k_\mu \qquad \text{for any} \quad w\in W_a \text{ and } \mu=0, 1, \cdots, l.$$

Then we get the following proposition:

Proposition 1.7. ([6]) *Let (R, G) be a marked extended affine root system and δ_2 be a $\boldsymbol{Z}$-basis of G. Then we have*

$$R=\{\alpha+mk(\alpha)\delta_2 \mid \alpha\in R_a, m\in\boldsymbol{Z}\}.$$

§2. Construction of Lie Algebras Associated with Marked Extended Affine Root Systems

In this section, following the ideas of I. B. Frenkel-V. G. Kac [2], I. B. Frenkel [1] and P. Goddard-D. Olive [3], we shall associate a Lie algebra with a marked extended affine root system such that all the elements are of length 2, by using a vertex operator.

Let $\hat{F}$ be an $(l+4)$-dimensional vector space over $\boldsymbol{R}$ with a metric $(\cdot|\cdot)$ whose signature is $(l+2, 0, 2)$. We fix a maximal isotropic subspace L in $\hat{F}$:

(i) $(\Lambda|\Lambda')=0$ for any $\Lambda, \Lambda' \in L$.

(ii) $\dim L=2$.

Let F be an orthogonal complement of L in $\hat{F}$. Then F is an $(l+2)$-dimensional vector space over $\boldsymbol{R}$ with a metric $(\cdot|\cdot)$ whose signature is $(l, 2, 0)$ where $(\cdot|\cdot)$ is the metric induced by $(\cdot|\cdot)$. Let R be an extended affine root system with a marking G belonging to $(\cdot|\cdot)$. Let $\{\alpha_0, \alpha_1, \cdots, \alpha_l, \delta_2, \Lambda_1, \Lambda_2\}$ be a basis of $\hat{F}$ which satisfies the following conditions:

(2.1)
- (i) $\{\Lambda_1, \Lambda_2\}$ is a basis of L.
- (ii) $\{\alpha_0, \alpha_1, \cdots, \alpha_l\}$ is a fundamental root system of R_a and $A=\left(\frac{2(\alpha_\nu|\alpha_\mu)}{(\alpha_\mu|\alpha_\mu)}\right)_{\nu,\mu=0,\cdots,l}$ is a generalized Cartan matrix of R_a.
- (iii) δ_2 is a $\boldsymbol{Z}$-basis of $G \cap Q(R)$.
- (iv) We set $\delta_1=\alpha_0-\theta$, where θ is the highest root of R_f. Then $(\delta_i|\Lambda_j)=\delta_{ij}$ for $i, j=1, 2$.

From now on, we assume that (R, G) is a marked extended affine root system of the following type:

$$X_l^{(1,1)}=\{\alpha_f+m\delta_1+n\delta_2 \mid \alpha_f \in R_f,\ m, n \in \boldsymbol{Z}\} \tag{2.2}$$

where $X_l=A_l$, D_l or E_l and R_f is a finite root system of type X_l. Note that all the elements of $X_l^{(1,1)}$ are of length 2.

Now we introduce an infinite set of creation and annihilation operators $p^\mu(m)$, where m is a non-zero integer and $\mu=1, \cdots, l$, and a finite set of operators x^μ, p^μ, x_+^i, x_-^1, p_-^i and p_+^1, $\mu=1, \cdots, l$, $i=1, 2$. We assume that they satisfy the following commutation relations:

(2.3)
- (i) $[p^\mu(m), p^\nu(n)]=m\delta_{m+n,0}(\alpha_\mu|\alpha_\nu)$,
- (ii) $[x^\mu, p^\nu]=\sqrt{-1}(\alpha_\mu|\alpha_\nu)$,
- (iii) $[x_+^i, p_-^j]=\sqrt{-1}(\delta_i|\Lambda_j)$,
- (iv) $[x_-^1, p_+^1]=\sqrt{-1}(\delta_1|\Lambda_1)$ and
- (v) the others$=0$.

Note that we do not define operators p_+^2 and x_-^2.

Define a $\boldsymbol{Z}$-sublattice $\tilde{Q}(R)$ in $\hat{F}$ by

$$\tilde{Q}(R) := Q(R) \oplus \boldsymbol{Z}\Lambda_1.$$

Here we treat Λ_1 and Λ_2 *asymmetrically*. Define the subspace $\tilde{F}$ of $\hat{F}$ by

$$\tilde{F} := \{x \in \hat{F} \mid (\delta_2 \mid x) = 0\}$$

and let $(\cdot|\cdot)$ be the metric on $\tilde{F}$ induced by $(\cdot|\cdot)$. The pair $(\tilde{F}, (\cdot|\cdot))$ is called a hyperbolic extension of $(F, (\cdot|\cdot))$ in [6]. Note that G is a radical of $(\cdot|\cdot)$ and that $\tilde{Q}(R)$ is a full lattice in $\tilde{F}$.

Let $\boldsymbol{C}[\tilde{Q}(R)]$ be a group algebra of $\tilde{Q}(R)$ and $\mathcal{H}_-$ be the infinite dimensional vector space over $\boldsymbol{C}$ spanned by $p^{\mu}(m)$, $m<0$ and $\mu=1, \cdots, l$:

$$\mathcal{H}_- = \sum_{\mu=1}^{l} \sum_{m<0} \boldsymbol{C} p^{\mu}(m).$$

We denote by $S(\mathcal{H}_-)$ the symmetric algebra generated by $\mathcal{H}_-$.

Now let the above operators act on $V = S(\mathcal{H}_-) \otimes \boldsymbol{C}[\tilde{Q}(R)]$ as follows:

$$(2.4)\quad \begin{cases} \text{(i)} & \text{for } m<0,\ p^{\mu}(m)(v\otimes e^{\alpha}) := (p^{\mu}(m)v)\otimes e^{\alpha}, \\ \text{(ii)} & \text{for } m<0, \text{ inductively,} \\ & p^{\mu}(m)(1\otimes e^{\alpha}) := 0, \\ & p^{\mu}(m)(p^{\nu}(-n)\otimes e^{\alpha}) := m\delta_{m,n}(\alpha_{\mu}|\alpha_{\nu})1\otimes e^{\alpha}, \\ & p^{\mu}(m)(p^{\nu}(-n)v\otimes e^{\alpha}) := (p^{\mu}(m)p^{\nu}(-n))v\otimes e^{\alpha} + p^{\nu}(-n)(p^{\mu}(m)v)\otimes e^{\alpha} \\ \text{(iii)} & p^{\mu}(v\otimes e^{\alpha}) := (\alpha_{\mu}|\alpha)v\otimes e^{\alpha}, \\ & p_-^{i}(v\otimes e^{\alpha}) := (\Lambda_i|\alpha)v\otimes e^{\alpha}, \\ & p_+^{1}(v\otimes e^{\alpha}) := (\delta_1|\alpha)v\otimes e^{\alpha}, \\ \text{(iv)} & \exp\{\sqrt{-1}x^{\mu}\}(v\otimes e^{\alpha}) := v\otimes e^{\alpha+\alpha_{\mu}}, \\ & \exp\{\sqrt{-1}x_+^{i}\}(v\otimes e^{\alpha}) := v\otimes e^{\alpha+\delta_i}, \\ & \exp\{\sqrt{-1}x_-^{1}\}(v\otimes e^{\alpha}) := v\otimes e^{\alpha+\Lambda_1}. \end{cases}$$

We set for $z \in \boldsymbol{C}$,

$$(2.5)\quad \begin{cases} \text{(i)} & Q^{\mu}(z) := x^{\mu} - \sqrt{-1}\log z\, p^{\mu} + \sqrt{-1}\sum_{n\neq 0}\frac{z^{-n}}{n}p^{\mu}(n), \\ \text{(ii)} & Q_{(0)}^{i,\pm}(z) := x_{\pm}^{i} - \sqrt{-1}\log z\, p_{\pm}^{i}, \text{ where } x_-^{2} = p_+^{2} = 0, \\ \text{(iii)} & Q_{(+)}^{\mu}(z) := \sqrt{-1}\sum_{n>0}\frac{z^{-n}}{n}p^{\mu}(n), \\ \text{(iv)} & Q_{(-)}^{\mu}(z) := -\sqrt{-1}\sum_{n>0}\frac{z^{n}}{n}p^{\mu}(-n), \text{ and} \\ \text{(v)} & Q_{(0)}^{\mu}(z) := x^{\mu} - \sqrt{-1}\log z\, p^{\mu}. \end{cases}$$

For any $\alpha = \sum_{\mu=1}^{l} a_{\mu}\alpha_{\mu} + m\delta_1 + n\delta_2 \in R$, we define a vertex operator of "momentum" α by

(2.6) $$X(\alpha, z):=\exp\{\sqrt{-1}\langle\alpha, Q_{(-)}(z)\rangle\}\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(0)}(z)\rangle\}\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\},$$

where

$$\langle\alpha, Q_{(\pm)}(z)\rangle=\sum_{\mu=1}^{l}\alpha_{\mu}Q_{(\pm)}^{\mu}(z),$$

$$\langle\alpha, Q_{(0)}(z)\rangle=\sum_{\mu=1}^{l}a_{\mu}Q_{(0)}^{\mu}(z)+mQ_{(0)}^{1,+}+nQ_{(0)}^{2,+}.$$

Generally speaking, $X(\alpha, z)$ maps V into the space $\overline{V}=\{\sum_{n\in Z}z^{n}v_{n}\mid v_{n}\in V\}$. However, the homogeneous components of $X(\alpha, z)$ are well-defined operators on V. We define

(2.7) $$E(\alpha):=\frac{1}{2\pi\sqrt{-1}}\oint\frac{dz}{z}X(\alpha, z)\qquad\text{for any}\quad\alpha\in R$$

with the integration contour positively encircling $z=0$. To compute the commutation relations of the operators $E(\alpha)$ for $\alpha\in R$, we need the following lemma:

Lemma 2.1. *For any $\alpha, \beta\in R$ and $|z|>|\zeta|$, the following holds*:

(i) $$[\langle\alpha, Q_{(+)}(z)\rangle, \langle\beta, Q_{(-)}(\zeta)\rangle]=-(\alpha\mid\beta)\log\left(1-\frac{\zeta}{z}\right).$$

(ii) $$[\langle\alpha, Q_{(0)}(z)\rangle, \langle\beta, Q_{(0)}(\zeta)\rangle]=(\alpha\mid\beta)\log\frac{\zeta}{z}.$$

(iii) $$\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\}\cdot\exp\{\sqrt{-1}\langle\beta, Q_{(-)}(\zeta)\rangle\}$$
$$=\left(\frac{z-\zeta}{z}\right)^{(\alpha\mid\beta)}\exp\{\sqrt{-1}\langle\beta, Q_{(-)}(\zeta)\rangle\}\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\}.$$

(iv) $$\exp\{\sqrt{-1}\langle\alpha, Q_{(0)}(z)\rangle\}\cdot\exp\{\sqrt{-1}\langle\beta, Q_{(0)}(\zeta)\rangle\}$$
$$=\left(\frac{z}{\zeta}\right)^{(\alpha\mid\beta)/2}\exp\{\sqrt{-1}\langle\beta, Q_{(0)}(\zeta)\rangle\}\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(0)}(z)\rangle\}.$$

Proof. *Proof of* (i). From (2.3) and (2.5), we have

$$[\langle\alpha, Q_{(+)}(z)\rangle, \langle\beta, Q_{(-)}(\zeta)\rangle]$$
$$=\sum_{\mu,\nu=1}^{l}a_{\mu}b_{\nu}[Q_{(+)}^{\mu}(z), Q_{(-)}^{\nu}(\zeta)]$$
$$=\sum_{\mu,\nu=1}^{l}a_{\mu}b_{\nu}\sum_{m,n>0}\frac{z^{-m}\zeta^{n}}{m\cdot n}[p^{\mu}(m), p^{\nu}(-n)]$$
$$=\sum_{\mu,\nu=1}^{l}a_{\mu}b_{\nu}\sum_{m,n>0}\frac{z^{-m}\zeta^{n}}{m\cdot n}m\delta_{m,n}(\alpha_{\mu}\mid\alpha_{\nu})$$

$$=(\alpha|\beta)\sum_{m>0}\frac{1}{m}\left(\frac{\zeta}{z}\right)^{m}$$

$$-(\alpha|\beta)\log\left(1-\frac{\zeta}{z}\right).$$

Proof of (ii). By (2.3) and (2.5), we have

$$[\langle\alpha, Q_{(0)}(z)\rangle, \langle\beta, Q_{(0)}(\zeta)\rangle]$$

$$=\left[\sum_{\mu=1}^{l} a_{\mu}Q_{(0)}(z)+\sum_{i=1}^{2} m_{i}Q_{(0)}^{i,+}(z),\ \sum_{\nu=1}^{l} b_{\nu}Q_{(0)}(\zeta)+\sum_{i=1}^{2} n_{i}Q_{(0)}^{i,+}(\zeta)\right]$$

$$=\sum_{\mu,\nu=1}^{l} a_{\mu}b_{\nu}[x^{\mu}-\sqrt{-1}\log z p^{\mu},\ x^{\nu}-\sqrt{-1}\log\zeta p^{\nu}]$$

$$=\sum_{\mu,\nu=1}^{l} a_{\mu}b_{\nu}(\alpha_{\mu}|\beta_{\nu})(\log z-\log\zeta)$$

$$=(\alpha|\beta)\log\frac{\zeta}{z}.$$

Proof of (iii). From the Campbell-Hausdorff formula and (i), we have

$$\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\}\cdot\exp\{\sqrt{-1}\langle\beta, Q_{(-)}(\zeta)\rangle\}$$

$$=\exp\{-[\langle\alpha, Q_{(+)}(z)\rangle, \langle\beta, Q_{(-)}(\zeta)\rangle]\}$$

$$\cdot\exp\{\sqrt{-1}\langle\beta, Q_{(-)}(\zeta)\rangle\}\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\}$$

$$=\exp\left\{(\alpha|\beta)\log\left(1-\frac{\zeta}{z}\right)\right\}\cdot\exp\{\sqrt{-1}\langle\beta, Q_{(-)}(\zeta)\rangle\}$$

$$\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\}$$

$$=\left(\frac{z-\zeta}{z}\right)^{(\alpha|\beta)}\cdot\exp\{\sqrt{-1}\langle\beta, Q_{(-)}(\zeta)\rangle\}\cdot\exp\{\sqrt{-1}\langle\alpha, Q_{(+)}(z)\rangle\}.$$

Proof of (iv). From the Campbell-Hausdorff formula and (ii), we can prove (iv) similarly to (iii). □

Proposition 2.2. *For any* $\alpha, \beta\in R$, *one has the following*:

(i) $X(\alpha, z)X(\beta, \zeta)=(z-\zeta)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha, \beta; z, \zeta)$ *for* $|z|>|\zeta|$,

where

$$X(\alpha, \beta; z, \zeta)=\exp\{\sqrt{-1}(\langle\alpha, Q_{(-)}(z)\rangle+\langle\beta, Q_{(-)}(\zeta)\rangle)\}$$

$$\cdot\exp\{\sqrt{-1}(\langle\alpha, Q_{(0)}(z)\rangle+\langle\beta, Q_{(0)}(\zeta)\rangle\}$$

$$\cdot\exp\{\sqrt{-1}(\langle\alpha, Q_{(+)}(z)\rangle+\langle\beta, Q_{(+)}(\zeta)\rangle)\}.$$

(ii)
$$E(\alpha)E(\beta)-(-1)^{(\alpha|\beta)}E(\beta)E(\alpha)$$
$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^2\oint\frac{dz}{z}\oint\frac{d\zeta}{\zeta}(z-\zeta)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha,\beta;z,\zeta),$$

where the z integral on a contour positively encircling ζ, excluding $z=0$ and the ζ integral is then taken positively encircling $\zeta=0$.

Proof. We can easily check (i) by (2.6) and Lemma 2.1. Here we prove (ii) only. Let $\Gamma_0=\{\zeta\in\boldsymbol{C}\,|\,|\zeta|=r\}$ and $\Gamma_i=\{z\in\boldsymbol{C}\,|\,|z|=r_i\}(i=1, 2)$ for $r_2<r_0<r_1$. Then, by (i) and (2.7), we have

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^2\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\left\{\oint_{\Gamma_1}\frac{dz}{z}(z-\zeta)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha,\beta;z,\zeta)\right.$$
$$\left.-(-1)^{(\alpha|\beta)}\oint_{\Gamma_2}\frac{dz}{z}(\zeta-z)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha,\beta;z,\zeta)\right\}$$
$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^2\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\left\{\oint_{\Gamma_1}\frac{dz}{z}(z-\zeta)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha,\beta;z,\zeta)\right.$$
$$\left.-\oint_{\Gamma_2}\frac{dz}{z}(z-\zeta)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha,\beta;z,\zeta)\right\}$$
$$=\left(\frac{1}{2\pi\sqrt{-1}}\right)^2\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\oint_{\Gamma_1-\Gamma_2}\frac{dz}{z}(z-\zeta)^{(\alpha|\beta)}(z\zeta)^{-(\alpha|\beta)/2}X(\alpha,\beta;z,\zeta).\quad\square$$

The integrand in (ii) of Proposition 2.2 is non-singular if $(\alpha|\beta)\geqq 0$, it has a simple pole if $(\alpha|\beta)=-1$ and a double pole if $(\alpha|\beta)=-2$. Thus we obtain the following:

Corollary 2.3. *For any $\alpha,\beta\in R$, one has the following:*
(i) *If $(\alpha|\beta)\geqq 0$, then*
$$E(\alpha)E(\beta)-(-1)^{(\alpha|\beta)}E(\beta)E(\alpha)=0.$$
(ii) *If $(\alpha|\beta)=-1$, then $\alpha+\beta\in R$ and*
$$E(\alpha)E(\beta)-(-1)^{(\alpha|\beta)}E(\beta)E(\alpha)=E(\alpha+\beta).$$
(iii) *If $(\alpha|\beta)=-2$, then $\alpha+\beta\equiv 0$ mod $\mathrm{Rad}(\cdot|\cdot)$ and*
$$E(\alpha)E(\beta)-(-1)^{(\alpha|\beta)}E(\beta)E(\alpha)=\frac{1}{2\pi\sqrt{-1}}\oint\frac{dz}{z}\langle\alpha, P(z)\rangle X(\alpha+\beta, z)$$
where $P(z)=\sqrt{-1}\,z\dfrac{d}{dz}Q(z)$.

Remark 2.4. In Corollary 2.3 (iii), since $\alpha+\beta\equiv 0$ mod $\mathrm{Rad}(\cdot|\cdot)$, we have $\alpha+\beta=k\delta_1+l\delta_2$ for some $k, l\in\boldsymbol{Z}$. Hence from (2.6) it follows that
$$X(\alpha+\beta, z)=\exp\{\sqrt{-1}\langle k\delta_1+l\delta_2, Q_{(0)}(z)\rangle\}$$
$$=\exp\{\sqrt{-1}(kQ^{1+}_{(0)}(z)+lQ^{2+}_{(0)}(z))\}$$

This operator commutes with operators $X(\alpha, z)$ for all $\alpha\in R$.

Proof of Corollary 2.3. First of all, we set

$$F(z\,;\,\zeta):=(z-\zeta)^{(\alpha|\beta)}(z\cdot\zeta)^{-(\alpha|\beta)/2-1}X(\alpha,\,\beta\,;\,z,\,\zeta).$$

Proof of (i). If $(\alpha|\beta)\geqq 0$, $F(z\,;\,\zeta)$ is a holomorphic function with respect to z at $z=\zeta$. Hence we obtain (i) from Proposition 2.2 (ii).

Proof of (ii). If $(\alpha|\beta)=-1$, $F(z\,;\,\zeta)$ has a simple pole at $z=\zeta$ as a function of z. Therefore we have

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\oint dz\,F(z\,;\,\zeta)&=\lim_{z\to\zeta}(z-\zeta)F(z\,;\,\zeta)\\&=\lim_{z\to\zeta}(z\,;\,\zeta)^{-1/2}X(\alpha,\,\beta\,;\,z,\,\zeta)\\&=\frac{1}{\zeta}X(\alpha+\beta\,;\,\zeta).\end{aligned}$$

Hence by Proposition 2.2 (ii) and (2.7), we obtain

$$\begin{aligned}E(\alpha)E(\beta)-(-1)^{(\alpha|\beta)}E(\beta)E(\alpha)&=\frac{1}{2\pi\sqrt{-1}}\oint\frac{d\zeta}{\zeta}X(\alpha+\beta\,;\,\zeta)\\&=E(\alpha+\beta).\end{aligned}$$

Proof of (iii). If $(\alpha|\beta)=-2$, $F(z,\,\zeta)$ has a double pole at $z=\zeta$ as a function of z. Hence we have

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\oint dz\,F(z\,;\,\zeta)&=\lim_{z\to\zeta}\frac{d}{dz}(z-\zeta)^2F(z\,;\,\zeta)\\&=\lim_{z\to\zeta}\Big\{\Big\langle\alpha,\,\sqrt{-1}\frac{d}{dz}Q_{(-)}(z)\Big\rangle X(\alpha,\,\beta\,;\,z,\,\zeta)\\&\quad+\Big\{\Big\langle\alpha,\,\sqrt{-1}\frac{d}{dz}Q_{(0)}(z)\Big\rangle X(\alpha,\,\beta\,;\,z,\,\zeta)\\&\quad+X(\alpha,\,\beta\,;\,z,\,\zeta)\Big\langle\alpha,\,\sqrt{-1}\frac{d}{dz}Q_{(+)}(z)\Big\rangle\Big\}\\&=\frac{1}{\zeta}\Big\langle\alpha,\,\sqrt{-1}\,\zeta\frac{d}{d\zeta}Q(\zeta)\Big\rangle X(\alpha+\beta\,;\,\zeta).\end{aligned}$$

The last equality follows from the fact $\alpha+\beta\equiv 0$ mod Rad$(\cdot|\cdot)$. Therefore we obtain (iii). □

Now we want to modify the equations in Corollary 2.3 so that their left hand sides become commutators. To this end, following [2], we introduce a 2-cocycle ε of the root lattice $Q(R_f)$ of R_f: there exists a $\boldsymbol{Z}$-bilinear form ε: $Q(R_f)\times Q(R_f)\to\{\pm 1\}$ such that

$$(2.8)\quad \begin{cases} \text{(i)} & \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha)=(-1)^{(\alpha|\beta)} \quad \text{for any} \quad \alpha, \beta\in Q(R_f), \\ \text{(ii)} & \varepsilon(\alpha, \alpha)=-1 \quad \text{for any} \quad \alpha\in R_f. \end{cases}$$

Moreover we define $\tilde{\varepsilon}$: $\tilde{Q}(R)\times\tilde{Q}(R)\to\{\pm1\}$ by

$$(2.9)\qquad \tilde{\varepsilon}(\alpha, \beta):=\varepsilon(\alpha_f, \beta_f) \quad \text{for any} \quad \alpha, \beta\in\tilde{Q}(R),$$

where α_f and β_f are orthogonal projections of F_f of α and β, respectively. Clearly $\tilde{\varepsilon}$ satisfies the following equations:

$$\tilde{\varepsilon}(m\delta_1+n\delta_2+k\Lambda_1, \alpha)=1 \quad \text{for any} \quad \alpha\in\tilde{Q}(R),$$

since $\varepsilon(0, \alpha)=1$. We define $\tilde{\varepsilon}_\alpha$: $\boldsymbol{C}[\tilde{Q}(R)]\to\boldsymbol{C}[\tilde{Q}(R)]$ for any $\alpha\in\tilde{Q}(R)$ by

$$(2.10)\qquad \tilde{\varepsilon}_\alpha(e^\beta):=\tilde{\varepsilon}(\alpha, \beta)e^\beta \quad \text{for any} \quad \beta\in\tilde{Q}(R).$$

Now to modify equations in Corollary 2.3, we introduce the following operators:

$$(2.11)\quad \begin{cases} \text{(i)} & \tilde{e}(\alpha)=e_{\alpha_f}(m, n):=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\oint\frac{dz}{z}X(\alpha\,;\,z)\tilde{\varepsilon}_\alpha \\ \text{(ii)} & \tilde{h}(\alpha)=h_{\alpha_f}(m, n):=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\oint\frac{dz}{z}\langle\alpha_f, P(z)\rangle X(m\delta_1+n\delta_2, z) \\ & \qquad\text{for any} \quad \alpha=\alpha_f+m\delta_1+n\delta_2\in R, \\ \text{(iii)} & \tilde{c}(m):=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\oint\frac{dz}{z}\langle\delta_1, P(z)\rangle X(m\delta_2, z) \quad \text{for any} \quad m\in\boldsymbol{Z}, \\ \text{(iv)} & \tilde{d}_i:=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\oint\frac{dz}{z}\langle\Lambda_i, P(z)\rangle \quad \text{for} \quad i=1, 2. \end{cases}$$

Let us define an infinite dimensional vector space $\tilde{\mathfrak{g}}(R)$ over $\boldsymbol{C}$ and its subspace $\tilde{\mathfrak{h}}(R)$ as follows:

$$(2.12)\quad \begin{cases} \tilde{\mathfrak{g}}(R):=\displaystyle\sum_{\alpha\in R}\boldsymbol{C}\tilde{e}(\alpha)\oplus\sum_{\mu=1}^{l}\sum_{m,n\in\boldsymbol{Z}}\boldsymbol{C}h_{\alpha_\mu}(m, n)\oplus\sum_{m\in\boldsymbol{Z}}\boldsymbol{C}\tilde{c}(m)\oplus\boldsymbol{C}\tilde{d}_1\oplus\boldsymbol{C}\tilde{d}_2, \\ \tilde{\mathfrak{h}}(R):=\displaystyle\sum_{\mu=1}^{l}\boldsymbol{C}h_{\alpha_\mu}(0, 0)\oplus\boldsymbol{C}\tilde{c}(0)\oplus\boldsymbol{C}\tilde{d}_1\oplus\boldsymbol{C}\tilde{d}_2. \end{cases}$$

Then we obtain the following:

Theorem 2.5. (I) *$\tilde{\mathfrak{g}}(R)$ has the following Lie algebra structure: For any $\alpha=\alpha_f+m_1\delta_1+m_2\delta_2$, $\beta=\beta_f+n_1\delta_1+n_2\delta_2\in R$,*

$$\text{(i)}\quad [\tilde{e}(\alpha), \tilde{e}(\beta)]=\begin{cases} 0 & \textit{if} \quad (\alpha|\beta)\geqq0 \\ \tilde{\varepsilon}(\alpha, \beta)\tilde{e}(\alpha+\beta) & \textit{if} \quad (\alpha|\beta)=-1 \\ -\{\tilde{h}(\alpha)+m_1\delta_{m_1+n_1,0}\tilde{c}(m_2+n_2)\} & \textit{if} \quad (\alpha|\beta)=-2, \end{cases}$$

$$\text{(ii)}\quad [\tilde{h}(\alpha), \tilde{e}(\beta)]=(\alpha|\beta)\tilde{e}(\beta+m_1\delta_1+n_1\delta_2),$$

$$\text{(iii)}\quad [\tilde{h}(\alpha), \tilde{h}(\beta)]=m_1\delta_{m_1+n_1,0}(\alpha|\beta)\tilde{c}(m_2+n_2),$$

(iv) $[\tilde{d}_i, \tilde{e}(\alpha)]=(\Lambda_i|\alpha)\tilde{e}(\alpha)$

$[\tilde{d}_i, \tilde{h}(\alpha)]=(\Lambda_i|\alpha)\tilde{h}(\alpha)$

$[\tilde{d}_i, \tilde{c}(m)]=m(\Lambda_i|\delta_2)\tilde{c}(m)$,

(v) *the other commutation relations are trivial.*

(II) *Let* $\mathfrak{g}(R_a)$ *be the affine Lie algebra associated with the affine root system* R_a, *then we have*

$$\tilde{\mathfrak{g}}(R)\cong\mathfrak{g}(R_a)\otimes \boldsymbol{C}[\lambda, \lambda^{-1}]\oplus \boldsymbol{C}d_1\oplus \boldsymbol{C}d_2$$

where d_1 *is the scaling element of* $\mathfrak{g}(R_a)$ *and* $d_2=\lambda(d/d\lambda)$.

Proof. We can easily check (II) from (I), and so we here show (I) only. By (2.4) (iv), (2.6) and (2.10), we have

$$\tilde{\varepsilon}_\alpha X(\beta, z)=\tilde{\varepsilon}(\alpha, \beta)X(\beta, z)\tilde{\varepsilon}_\alpha \quad \text{for any} \quad \alpha, \beta\in R.$$

Thus (i) follows from Corollary 2.3. We can prove (ii), (iii), (iv) and (v) similarly to Proposition 2.2. Here we prove (iii) only. For any $\alpha_f, \beta_f\in R_f$ and $|z|>|\zeta|$,

$$\langle\alpha_f, P(z)\rangle\langle\beta_f, P(\zeta)\rangle=:\langle\alpha_f, P(z)\rangle\langle\beta_f, P(\zeta)\rangle:+(\alpha_f|\beta_f)\frac{z\cdot\zeta}{(z-\zeta)^2}$$

where: $*$: is the normal ordering defined by

$$:p^\mu(m)p^\nu(n):\ :=p^\mu(m)p^\nu(n)-m\delta_{m+n,0}(\alpha_\mu|\alpha_\nu)y_+(m),$$

where $y_+(m):=\begin{cases}1 & m>0\\ 0 & m\leqq 0\end{cases}$. Hence, by (2.11) (ii) and Remark 2.4, we have

$$\begin{aligned}
&[\tilde{h}(\alpha), \tilde{h}(\beta)]\\
&=\Big(\frac{1}{2\pi\sqrt{-1}}\Big)^2\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\Big\{\oint_{\Gamma_1}\frac{dz}{z}\langle\alpha_f, P(z)\rangle\langle\beta_f, P(\zeta)\rangle X(m_1\delta_1+m_2\delta_2, z)\\
&\qquad\qquad\times X(n_1\delta_1+n_2\delta_2, \zeta)\\
&\quad-\oint_{\Gamma_2}\frac{dz}{z}\langle\beta_f, P(\zeta)\rangle\langle\alpha_f, P(z)\rangle X(m_1\delta_1+m_2\delta_2, z)X(n_1\delta_1+n_2\delta_2, \zeta)\Big\}\\
&=\Big(\frac{1}{2\pi\sqrt{-1}}\Big)^2\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\oint_{\Gamma_1-\Gamma_2}\frac{dz}{z}(\alpha_f|\beta_f)\frac{z\cdot\zeta}{(z-\zeta)^2}X(m_1\delta_1+m_2\delta_2, z)\\
&\qquad\qquad\times X(n_1\delta_1+n_2, \delta_2, \zeta)\\
&=\frac{1}{2\pi\sqrt{-1}}(\alpha_f|\beta_f)\oint_{\Gamma_0}\frac{d\zeta}{\zeta}\Big(z\frac{d}{dz}X(m_1\delta_1+m_2\delta_2, z)\Big)_{z=\zeta}X(n_1\delta_1+n_2\delta_2, \zeta)\\
&=\frac{1}{2\pi\sqrt{-1}}(\alpha_f|\beta_f)\oint_{\Gamma_0}\frac{d\delta}{\zeta}\langle m_1\delta_1, P(\zeta)\rangle X((m_1+n_1)\delta_1+(m_2+n_2)\delta_2, \zeta).
\end{aligned}$$

Therefore (iii) follows from the facts:

$$\oint_{\Gamma_0} \frac{d\zeta}{\zeta} \langle \delta_1, P(\zeta) \rangle X(m\delta_1 + n\delta_2, \zeta) = \delta_{m,0} c(n),$$

$$(\alpha_f | \beta_f) = (\alpha | \beta). \qquad \square$$

By Theorem 2.5, $\tilde{\mathfrak{h}}(R)$ is a commutative subalgebra of $\tilde{\mathfrak{g}}(R)$ and is called Cartan subalgebra. From this theorem and (2.12) it follows that we have the root space decomposition of $\tilde{\mathfrak{g}}(R)$ with respect to $\tilde{\mathfrak{h}}(R)$:

$$\tilde{\mathfrak{g}}(R) = \bigoplus_{\alpha \in Q(R)} \tilde{\mathfrak{g}}_\alpha$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}}(R) | [h, x] = \langle h, \alpha \rangle x \text{ for all } h \in \tilde{\mathfrak{h}}(R)\}$. Note that

$$\begin{cases} \tilde{\mathfrak{g}}_\alpha = \boldsymbol{C}\tilde{e}(\alpha) \quad \text{for any} \quad \alpha \in R, \\ \mathfrak{g}_{m\delta_1 + n\delta_2} = \begin{cases} \sum_{\mu=1}^{l} \boldsymbol{C} h_{\alpha_\mu}(m, n) \oplus \boldsymbol{C}\tilde{c}(n) & \text{if} \quad n \neq 0 \\ \sum_{\mu=1}^{l} \boldsymbol{C} h_{\alpha_\mu}(m, n) & \text{if} \quad n = 0, \end{cases} \\ \text{and} \\ \tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{h}}(R). \end{cases}$$

Here the set of real roots, i.e., roots with non-zero length, coincides with the extended affine root system R. Let us call $\tilde{\mathfrak{g}}(R)$ the extended affine Lie algebra associated with (R, G).

§ 3. Weyl Group of an Extended Affine Lie Algebra

In this section, we describe the Weyl group of an extended affine Lie algebra $\tilde{\mathfrak{g}}(R)$.

Let $\hat{F}$ be an $(l+4)$-dimensional vector space over $\boldsymbol{R}$ with a metric $(\cdot|\cdot)$ of signature $(l+2, 0, 2)$ and we take a basis $\{\alpha_1, \cdots, \alpha_l, \delta_1, \delta_2, \Lambda_1, \Lambda_2\}$ as in (2.1). Let $\hat{F}^*$ be the dual space of $\hat{F}$. We define a basis $\{\alpha_1^\vee, \cdots, \alpha_l^\vee, c_1, c_2, d_1, d_2\}$ of $\hat{F}^*$ by

$$(3.1) \qquad \begin{cases} \text{(i)} \quad \langle \alpha_\mu^\vee, \alpha_\nu \rangle := \dfrac{2(\alpha_\nu | \alpha_\mu)}{(\alpha_\mu | \alpha_\mu)} \quad \mu, \nu = 1, \cdots, l \\ \text{(ii)} \quad \langle c_i, \Lambda_j \rangle = \langle d_i, \delta_j \rangle = \delta_{ij} \qquad i, j = 1, 2, \\ \text{(iii)} \quad \text{the others} = 0 \end{cases}$$

where $\langle\ ,\ \rangle$ is the pairing of $\hat{F}^*$ and $\hat{F}$. Since the metric $(\cdot|\cdot)$ on $\hat{F}$ and the pairing $\langle\ ,\ \rangle$ are non-degenerate, we obtain an isomorphism $\varphi: \hat{F}^* \to \hat{F}$ defined by

$$(3.2) \qquad (\varphi(h) | \alpha) := \langle h, \alpha \rangle \qquad \text{for any} \quad h \in \hat{F}^* \text{ and } \alpha \in \hat{F}.$$

We denote by $(\cdot|\cdot)$ the metric on $\hat{F}^*$ defined by

$$(h | h') := \langle h, \varphi(h') \rangle \qquad \text{for any} \quad h, h' \in \hat{F}^*.$$

Then the metric $(\cdot|\cdot)$ on $\hat{F}^*$ is a non-degenerate one with sign $(l+2, 0, 2)$. From (3.2), we have

$$\begin{cases} \varphi(\alpha_\mu^\vee)=\dfrac{2}{(\alpha_\mu|\alpha_\mu)}\alpha_\mu, & \mu=1, \cdots, l, \\ \varphi(c_i)=\delta_i,\ \varphi(d_i)=\Lambda_i, & i=1, 2. \end{cases} \tag{3.3}$$

Let $\tilde{\mathfrak{g}}(R)$ be the Lie algebra associated with a marked extended affine root system (R, G) and $\tilde{h}(R)$ its Cartan subalgebra. Note that $\mathfrak{h}(R)$ is identified with the following subspace of $\hat{F}_C^*=\hat{F}^*\otimes C$:

$$\mathfrak{h}(R)=\{h\in\hat{F}_C^*\,|\,\langle h, \Lambda_2\rangle=0\}.$$

From this fact it follows that $\{\alpha_1^\vee, \cdots, \alpha_l^\vee, c_1, d_1, d_2\}$ is a basis of $\tilde{\mathfrak{h}}(R)$ and we denote by $(\cdot|\cdot)$ the induced metric on $\tilde{\mathfrak{h}}(R)$, which is degenerate.

For any $\alpha\in R$, we define $\alpha^\vee\in\tilde{\mathfrak{h}}(R)$ by

$$\alpha^\vee:=\frac{2}{(\alpha|\alpha)}\varphi^{-1}(\alpha) \bmod Cc_2. \tag{3.4}$$

It is clear from (3.3) and (3.4) that

$$(\alpha_\mu+m\delta_1+n\delta_2)^\vee=\alpha_\mu^\vee+\frac{2m}{(\alpha_\mu|\alpha_\mu)}c_1.$$

Let $e(\alpha)$ be a basis of $\tilde{\mathfrak{g}}_\alpha$ for any $\alpha\in R$ satisfying the following conditions:

$$\begin{cases} \text{(i)} & [e(\alpha), e(-\alpha)]=\alpha^\vee, \\ \text{(ii)} & [\alpha^\vee, e(\beta)]=\langle\alpha^\vee, \beta\rangle e(\beta). \end{cases}$$

Now we return to the description of the Weyl group of the extended affine Lie algebra $\tilde{\mathfrak{g}}(R)$. Since the adjoint representation of $\tilde{\mathfrak{g}}(R)$ is integrable (i.e. for any $x, y\in\tilde{\mathfrak{g}}(R)$, there exists a positive integer N such that $(ad(x))^N y=0$), we can define an automorphism r_α of $\tilde{\mathfrak{g}}(R)$ for any $\alpha\in R$ as follows:

$$r_\alpha:=\exp\{ad(e(\alpha))\}\exp\{-ad(e(-\alpha))\}\exp\{ad(e(\alpha))\}. \tag{3.5}$$

Then we can easily check by (3.5) that

$$r_\alpha(h)=h-\langle h, \alpha\rangle\alpha^\vee \quad \text{for any} \quad h\in\tilde{\mathfrak{h}}(R). \tag{3.6}$$

This is the reflection of $\tilde{\mathfrak{h}}(R)$ with respect to $\alpha^\vee$. Now we define a reflection group by

$$W_R:=\text{the subgroup of } O(\tilde{\mathfrak{h}}, (\cdot|\cdot)) \text{ generated by } r_\alpha,\ \alpha\in R. \tag{3.7}$$

We call this reflection group Weyl group of the extended affine Lie algebra $\tilde{\mathfrak{g}}(R)$.

Denote by W_f the subgroup of W_R generated by $r_{\alpha_1}, \cdots, r_{\alpha_l}$. As $r_{\alpha_\mu}(c_1)=c_1$ and $r_{\alpha_\mu}(d_i)=d_i$, $i=1, 2$, we deduce that W_f operates trivially on $Cc_1\oplus Cd_1\oplus Cd_2$. We conclude that W_f operates faithfully on $\mathfrak{h}_f$ which is the subspace of $\tilde{\mathfrak{h}}(R)$

spanned by $\alpha_1^\vee, \cdots, \alpha_l^\vee$ and we can identify W_f with the Weyl group of the Lie algebra $\mathfrak{g}_f$ which is the finite dimensional Lie algebra associated with R_f.

Since we consider marked extended affine root systems of type $X_l^{(1,1)}$ (see (2.2)) where $X_l=A_l$, D_l or E_l exclusively, the generator system π of (R, G) (Definition 1.6) is as follows:

$$\pi=\{\alpha_0, \alpha_1, \cdots, \alpha_l\}\cup\{\alpha_\mu^*\mid\mu\in I\}$$

where $\alpha_0=\delta_1-\theta$ and $\alpha_\mu^*=\delta_2+\alpha_\mu$ for $\mu\in I$. Here θ is the highest root of the root system R_f.

Lemma 3.1. *Let $\pi=\{\alpha_0, \cdots, \alpha_l\}\cup\{\alpha_\mu^*\mid\mu\in I\}$ be the generator system of (R, G). Then W_R is generated by $r_{\alpha_0}, \cdots, r_{\alpha_l}, r_{\alpha_\mu^*}, \mu\in I$.*

Proof. Let W be the subgroup of W_R generated by $r_{\alpha_0}, \cdots, r_{\alpha_l}, r_{\alpha_\mu^*}$ for $\mu\in I$. Then one can easily check that

$$R=\Big(\bigcup_{\mu=0}^{l} W(\alpha_\mu)\Big)\cup\Big(\bigcup_{\mu\in I} W(\alpha_\mu^*)\Big).$$

Therefore for any $\alpha\in R$, there exists an element $w\in W$ such that $\alpha=w(\alpha_\mu)$ or $w(\alpha_\nu^*)$ for $\mu=0, \cdots, l$, $\nu\in I$. Then $r_\alpha=r_{w(\alpha_\mu)}=w\cdot r_{\alpha_\mu}\cdot w^{-1}$ or $=w\cdot r_{\alpha_\nu^*}\cdot w^{-1}$. From (3.7), W_R coincides with W. □

Lemma 3.2. (i) *Let θ be the highest root of R_f. Then*

$$r_{\alpha_0}\cdot r_\theta(h)=h+\langle h, \delta_1\rangle\theta^\vee-\Big((h\mid\theta^\vee)+\frac{1}{2}(\theta^\vee\mid\theta^\vee)\langle h, \delta_1\rangle\Big)c_1$$

for any $h\in\tilde{\mathfrak{h}}(R)$.

(ii) *For any α_μ^*, $\mu\in I$ and $h\in\tilde{\mathfrak{h}}(R)$, $r_{\alpha_\mu^*}\cdot r_{\alpha_\mu}(h)=h+\langle h, \delta_2\rangle\alpha_\mu^\vee$. Note that $\alpha_0=\delta_1-\theta$ and $\alpha_\mu^*=\delta_2+\alpha_\mu$.*

Proof. By (3.6), for any $h\in\tilde{\mathfrak{h}}(R)$, we have

$$\begin{aligned}r_{\delta_1-\theta}\cdot r_\theta(h)&=r_{\delta_1-\theta}(h-\langle h, \theta\rangle\theta^\vee)\\&=h-\langle h, \theta\rangle\theta^\vee-\langle h, \delta_1-\theta\rangle(\delta_1-\theta)^\vee-\langle h, \theta\rangle\langle\theta^\vee, \theta\rangle(\delta_1-\theta)^\vee\\&=h+\langle h, \delta_1\rangle\theta^\vee-((h\mid\theta^\vee)+2/(\theta\mid\theta)\langle h, \delta_1\rangle)c_1,\end{aligned}$$

which implies (i) from the fact $2/(\theta\mid\theta)=(\theta^\vee\mid\theta^\vee)/2$. We can prove (ii) similarly to (i). □

Motivated by these formulae, we introduce the following endomorphism t_α and p_β of $\tilde{\mathfrak{h}}(R)$ for any $\alpha, \beta\in Q(R_f)$:

$$
(3.8)\quad \begin{cases} \text{(i)} & t_\alpha(h):=h+\langle h,\delta_1\rangle\varphi^{-1}(\alpha) \\ & \qquad -((h\mid\varphi^{-1}(\alpha))+\frac{1}{2}(\varphi^{-1}(\alpha)\mid\varphi^{-1}(\alpha))\langle h,\delta_1\rangle)c_1 \\ \text{(ii)} & p_\beta(h):=h+\langle h,\delta_2\rangle\varphi^{-1}(\beta). \end{cases}
$$

We denote by H^{2l+1} the subgroup of $\mathrm{GL}(\tilde{\mathfrak{h}}(R))$ generated by t_α and p_β for all $\alpha, \beta\in Q(R_f)$.

Lemma 3.3. *H^{2l+1} is a Heisenberg group satisfying the following formulae: For any $\alpha, \beta\in Q(R_f)$,*

(i) $t_\alpha\cdot t_\beta=t_{\alpha+\beta},\qquad p_\alpha\cdot p_\beta=p_{\alpha+\beta},$

(ii) $(t_\alpha\cdot p_\beta)\cdot(p_\beta\cdot t_\alpha)^{-1}=-\lambda\cdot(\alpha\mid\beta)c_1$

where $\lambda(h)=\langle h,\delta_2\rangle$ for any $h\in\tilde{\mathfrak{h}}(R)$.

Proof. One can easily prove (i) by using (3.8). Here we prove (ii) only. By (3.8), we obtain

$$
\begin{aligned}
t_\alpha\cdot p_\beta(h)&=h+\langle h,\delta_2\rangle\varphi^{-1}(\beta)+\langle h,\delta_1\rangle\varphi^{-1}(\alpha)\\
&\quad-\Big((h\mid\varphi^{-1}(\alpha))+\frac{1}{2}(\alpha\mid\alpha))\langle h,\delta_1\rangle+\langle h,\delta_2\rangle(\alpha\mid\beta)\Big)c_1,\\
p_\beta\cdot t_\alpha(h)&=h+\langle h,\delta_1\rangle\varphi^{-1}(\alpha)+\langle h,\delta_2\rangle\varphi^{-1}(\beta)\\
&\quad-\Big((h\mid\varphi^{-1}(\alpha))+\frac{1}{2}(\alpha\mid\alpha)\langle h,\delta_1\rangle\Big)c_1,
\end{aligned}
$$

which prove (ii). □

We can also obtain

$$
(3.9)\quad \begin{cases} \text{(i)} & t_{w(\alpha)}=w\cdot t_\alpha\cdot w^{-1} \\ \text{(ii)} & p_{w(\alpha)}=w\cdot p_\alpha\cdot w^{-1} \end{cases}
$$

for any $\alpha\in Q(R_f)$ and $w\in W_R$. Indeed, for any $h\in\tilde{\mathfrak{h}}$, we have

$$
\begin{aligned}
w\cdot t_\alpha\cdot w^{-1}(h)=w\Big\{&w^{-1}(h)+\langle w^{-1}(h),\delta_1\rangle\varphi^{-1}(\alpha)\\
&-\Big((w^{-1}(h)\mid\varphi^{-1}(\alpha))+\frac{1}{2}(\alpha\mid\alpha)\langle w^{-1}(h),\delta_1\rangle\Big)c_1\Big\}
\end{aligned}
$$

and

$$
w\cdot p_\beta\cdot w^{-1}(h)=w\{w^{-1}(h)+\langle w^{-1}(h),\delta_2\rangle\varphi^{-1}(\beta)\}.
$$

Hence (i) and (ii) hold since $W_R(\delta_i)=\delta_i$ for $i=1, 2$, and $(\cdot\mid\cdot)$ is W_R-invariant. Now we can prove the following proposition:

Proposition 3.4. $W_R=W_f\ltimes H^{2l+1}$.

Proof. Since $\alpha_0=\delta_1-\theta$ and $\alpha_\mu^*=\delta_2+\alpha_\mu$, $\mu\in I$ are contained in R, we have t_θ, $p_{\alpha_\mu}\in W_R$. Hence $t_{w(\theta)}$, $p_{w(\alpha_\mu)}\in W_R$ for any $w=W_f$ by (3.9). Now by Lemma 3.3 and (3.9), H^{2l+1} is a normal subgroup of W_R. Since W_f is a finite subgroup and H^{2l+1} is a Heisenberg group, we have $W_f\cap H^{2l+1}=1$. Finally, since $r_{\alpha_0}=t_\theta\cdot r_\theta^{-1}$ and $r_{\alpha_\mu^*}=p_{\alpha_\mu}\cdot r_{\alpha_\mu}^{-1}$, it follows from Lemma 3.1 that W_R coincides with the subgroup generated by W_f and H^{2l+1}. □

It should be remarked that for any marked extened affine root system, K. Saito [6] proved Proposition 3.4. This proposition is important for the theory of simple elliptic singularities since the coordinate ring of the base space of the deformation is the ring of W_R-invariant functions (θ-functions) on an affine subspace of the Cartan subalgebra $\tilde{\mathfrak{h}}(R)$ (see [4], [5] and [6]).

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