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On the Maunder Type Theorems in the Ex-homotopy Category

By

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Abstract

We study the homotopy theory of comma category and define the cell structure and Postnikov system in the ex-homotopy category. By using these structures we give the four types of spectral sequences and show that Maunder type theorems hold for these spectral sequences.

Introduction

In [5], C. R. F. Maunder defined the cohomology spectral sequence associated with the Postnikov decomposition of Ω-spectrum of target object and showed that his spectral sequence coincides with Atiyah-Hirzebruch spectral sequence. In [4], T. Matumoto proved Maunder's theorem in the equivariant homotopy category. We remarked in [7] that Maunder's theorem holds also in the category of functor complexes. In [6], we studied the unstable version of Maunder's theorem and applied them to the theory of phantom maps. Thus it is interesting to know whether Maunder type theorem holds in a homotopy category. In this paper, we define homotopy spectral sequences associated with cell structure and Postnikov system and prove Maunder type theorems in the ex-homotopy category.

In § 1, we study the homotopy theory of comma category and obtain results analogous to the ones of the ordinary homotopy theory (e. g. J. H. C. Whitehead's theorem). In § 2, we define the cell structure and Postnikov system in the ex-homotopy category and obtain the duality between them. In § 3, we define the homotopy spectral sequences associated with the cell decomposition of a source object and the Postnikov decomposition of a target object by the same way as [6]. In this paper, we shall show that these homotopy spectral sequences are isomorphic as exact couples. Moreover analogously we define the homotopy spectral sequences associated with the anti-skeleton filtration and anti-Postnikov decomposition defined in § 3. We also prove that these are isomorphic as exact couples.

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§ 1. The Homotopy Theory of Comma Category

We review the abstract homotopy theory defined in [7]. In this paper, we shall use the results in [7] and terminologies and notations in S. MacLane [3].

Definition 1.1. We call a category $\mathcal{C}$ a *pre-homotopy category* if it satisfies the following axioms (A1–3).

(A1) $\mathcal{C}$ is closed under finite limits and finite colimits; hence it has the initial object ϕ and the terminal object 1.

(A2) There are given covariant functors $I, P: \mathcal{C}\to\mathcal{C}$ with a natural isomorphism $\mathcal{C}(IA, B)\cong\mathcal{C}(A, PB)$ for any objects A, B of $\mathcal{C}$ ($\mathcal{C}(-, -)$ is hom-set in $\mathcal{C}$). We call these the cylinder and path functors respectively.

(A3) Moreover there are three natural transformations $k_*: Id\to I$ ($k=0, 1$) and $\tau: I\to Id$ with $\tau 0_*=Id=\tau 1_*$. Here Id means the identity functor or identity natural transformation. 0_*, 1_* and τ are called the top-face, bottom-face and projection transformations respectively.

Let I^n be n-time composed functor of I ($I^0=Id$); and define the natural transformations $d_j^k=I^{n-j}k_*I^j: I^n\to I^{n+1}$ and $s_j=I^{n-j}\tau I^j: I^{n+1}\to I^n$ for $(j, k)\in[n]\times[1]$ ($[m]=\{0, 1, \cdots, m\}$). We call these the face and degeneracy operators respectively. These operators d_j^k, s_j satisfy the cubical simplicial relations (cf. Lemma 1.3 in [7]). Let $(i_0, k_0)\in[n]\times[1]$. Then by patching the $2n+1$-faces $d_i^k: I^n\to I^{n+1}$ for $(i, k)\neq(i_0, k_0)$ according to the cubical simplicial relations, we have the functors $J^n=J^n(i_0, k_0)$ and the natural transformation $\lambda: J^n\to I^{n+1}$. We use the letter J^n for any (i_0, k_0) ($J^0=Id$).

Now we consider the following *extension condition* and the *natural homotopy axioms* for a pre-homotopy category $\mathcal{C}$:

(E.C) For any morphism $f: J^nX\to Y$, there is a morphism $F: I^{n+1}X\to Y$ with $F\lambda=f$.

(NHA 1) There is a natural transformation $\mu: I^n\to J^{n-1}$ with $\mu\lambda=Id$ for all $n>0$, that is, (EC) holds naturally by taking $F\mu=f$.

(NHA 2) There is a natural transformation $\mu: I^n\to J^{n-1}$ with $\tau'_n\mu=\tau_n$ and $\mu\lambda=Id$ for all $n>0$, where $\tau_n: I^n\to Id$ and $\tau'_n: J^{n-1}\to Id$ are defined by compositions of projections τ.

Let $\mathcal{C}$ be a pre-homotopy category. We call $\mathcal{C}$ an *abstract homotopy category* if it satisfies (NHA 2). The category CGH of compactly generated Hausdorff spaces and continuous mappings becomes our abstract homotopy category and so does the pointed category CGH_* of CGH (cf. Example 1.7 in [7]). We say that two morphisms $f_0, f_1: X\to Y$ are homotopic (relative $j: A\to X$), if there is a morphism $f: IX\to Y$ with $f_k=fk_*$ for $k=0, 1$ (and $fIj=f_0j\tau$); and then we write $f_0\simeq f_1$ (rel j) and call f a homotopy of f_0 and f_1. When $f, g: IX\to Y$ are homotopies with $f1_*=g0_*$, we can define a sum $f\oplus g$ of homotopies f and g as usual which is unique up to homotopy relative $\dot{I}=\{0_*\amalg 1_*\}$ (the terminal faces).

Here we note on the dual considerations. By using the unit $\eta: Id \to PI$ and the counit $\varepsilon: IP \to Id$, we have the following axiom (A3*) which is dual and equivalent to (A3) by defining $k^* = \varepsilon k_* P$ ($k=0, 1$) and $\sigma = P(\tau)\eta$:

(A3*) There are three natural transformations $k^*: P \to Id$ ($k=0, 1$) and $\sigma: Id \to P$ with $0^*\sigma = Id = 1^*\sigma$, called the top-coface, bottom-coface and injection transformations respectively.

By using (A3*) and dual constructions, we can obtain the duality principle in our abstract homotopy (cf. [7]).

Let $\mathcal{C}$ be a pre-homotopy category. We say that $j: A \to X$ in $\mathcal{C}$ (resp. $p: Y \to B$) has the *relative HEP* (resp. *relative HLP*), if any commutative square

$$\begin{tikzcd} I^n A \cup J^{n-1} X \arrow[r] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ I^n X \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

has a dotted morphism to obtain two commutative triangles for any $p: Y \to B$ with HLP (resp. $j: A \to X$ with HEP) and all $n > 0$. Note: When $Z = 1$, j is called HEP. When $A = \phi$, p is called HLP. HEP (resp. HLP) is known as homotopy extension (resp. lifting) property, and called a cofibration (resp. fibration) in CGH. Here $X \cup_A B$ (abbr. $X \cup B$) means the pushout of diagram $X \leftarrow A \to B$.

We consider the *comma category* $\mathcal{C}_B^A$ for fixed objects A, B and a fixed morphism $a: A \to B$ in $\mathcal{C}$ whose objects is any diagram $A \xrightarrow{x} X \xrightarrow{p} B$ in $\mathcal{C}$ with $px = a$ and whose morphism $f: (A \xrightarrow{x} X \xrightarrow{p} B) \to (A \xrightarrow{y} Y \xrightarrow{q} B)$ is any morphism $f: X \to Y$ in $\mathcal{C}$ with $fx = y$ and $qf = p$. For $A \xrightarrow{x} X \xrightarrow{p} B$, I, P, k_* and τ in $\mathcal{C}$ give us the diagrams

$$\begin{tikzcd} IA \arrow[r, "Ix"] \arrow[d, "\tau"'] \arrow[dr, phantom, "PO"] & IX \arrow[d, "\tau'"] & \\ A \arrow[r, "\bar{x}"'] & \bar{I}X \arrow[r, shift left, "\bar{\tau}"] & X \arrow[l, shift left, "\bar{k}_*"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, shift left, "\tilde{\sigma}"] & \tilde{P}X \arrow[l, shift left, "\tilde{k}^*"] \arrow[r, "\tilde{p}"] \arrow[d, "\sigma'"'] \arrow[dr, phantom, "PB"] & B \arrow[d, "\sigma"] \\ & PX \arrow[r, "Pp"'] & PB \end{tikzcd}$$

with $\bar{\tau}\bar{x} = x$, $\bar{\tau}\tau' = \tau$, $\bar{k}_* = \tau' k_*$, $\tilde{p}\tilde{\sigma} = p$, $\sigma'\tilde{\sigma} = \sigma$ and $\tilde{k}^* = k^*\sigma'$. Hence we have $A \xrightarrow{\bar{x}} \bar{I}X \xrightarrow{\bar{p}} B$ ($\bar{p} = p\bar{\tau}$) and $A \xrightarrow{\tilde{x}} \tilde{P}X \xrightarrow{\tilde{p}} B$ ($\tilde{x} = \tilde{\sigma}x$) in $\mathcal{C}_B^A$, the functors $\bar{I}$, $\tilde{P}$: $\mathcal{C}_B^A \to \mathcal{C}_B^A$ and the natural transformations $\bar{\tau}: \bar{I} \to Id$, $\bar{k}_*: Id \to \bar{I}$, satisfying (A2-3). Thus we have a theorem: If $\mathcal{C}$ is a pre-homotopy category, then so is the comma category $\mathcal{C}_B^A$. Moreover if $\mathcal{C}$ satisfies NHA 2, then so does the comma category $\mathcal{C}_B^A$ (cf. Theorem 1.9 in [7]).

When $B=1$ (resp. $A=\phi$), we write simply $(A\to X)$ (resp. $(X\to B)$) for any object in $\mathcal{C}_1^A$ (resp. $\mathcal{C}_B^\phi$). When $A=B$ and $px=id_B$, this comma category is called the *ex-homotopy category* (cf. [2]) and noted by $\mathcal{C}_B$. This category has the zero object $(B\xrightarrow{id}B\xrightarrow{id}B)$. We write $\mathcal{C}_B^A[-;-]$ (resp. $\mathcal{C}_B[-;-]$) for the homotopy set in $\mathcal{C}_B^A$ (resp. $\mathcal{C}_B$).

In our abstract homotopy category $\mathcal{C}$, *mapping cylinder* $M(f)$, *cone* CX and *suspension* ΣX are defined by pushouts of diagrams $IX\overset{1*}{\leftarrow}X\overset{f}{\rightarrow}Y$, $IX\overset{1*}{\leftarrow}X\to 1$ and $CX\leftarrow X\to 1$ respectively. D. Puppe's theorem (Theorem III. 6. 11 in [8]) and the homotopical invariance of induced (co)fibrations etc. hold also in $\mathcal{C}$ (cf. [7; §2]). Note that the suspension functor Σ_B has the right adjoint functor Ω_B (loop functor) in the ex-homotopy category, because it has the zero object. Generally this fact is not true for $px\neq id_B$.

The following result is well-known, but we give a proof under our abstract homotopy theory (cf. [1, 8]).

Lemma 1.2. *Let $\mathcal{C}$ be a pre-homotopy category satisfying* (EC) *and A a fixed object in $\mathcal{C}$. For a given morphism $f:(A\xrightarrow{x}X)\to(A\xrightarrow{y}Y)$ in $\mathcal{C}_1^A$ where x and y have* HEP, *if $f: X\to Y$ is a homotopy equivalence in $\mathcal{C}$, then so is f in $\mathcal{C}_1^A$.*

Proof. By assumptions, there is a homotopy inverse $g: Y\to X$ of f in $\mathcal{C}$ with $gy=x$. Let H (resp. K) be a homotopy of Id and gf (resp. Id and fg) and $L: M(y)=IA\cup Y\to X$ a morphism defined by $HIx: IA\to X$ and $g: Y\to X$. Since $M(y)$ is a retract of IY by $y\in$HEP, there is a morphism $\bar{L}: IY\to X$ which is the composition of the above retraction and L. Set $g'=\bar{L}0_*$ which satisfies $g'y=x$. Let $M=\bar{L}I(f)\oplus H^{-1}: IX\to X$ be the sum of $\bar{L}I(f)$ and a reverse homotopy H^{-1} of H which is a homotopy of $g'f$ and Id. Clearly homotopies $MI(x)$ and $x\tau: IA\to X$ are homotopic relative $\dot{I}$ (i.e. the terminal faces), there is a homotopy $\tilde{M}: I^2A\to X$ of $MI(x)$ and $x\tau$. Since $Ix: IA\to IX$ has HEP, there is $N: I^2X\to X$ with $\tilde{M}=NI^2x$, $N(0_*I)=M$, $N(I0_*)=g'f\tau$ and $N(I1_*)=\tau$. Hence $N(1_*I): IX\to X$ gives us a homotopy relative x of $g'f$ and Id. Analogously we have $f': X\to Y$ constructed as above with $f'g'\simeq Id$ relative y. Since $f\simeq(f'g')f=f'(g'f)\simeq f'$(rel x), then we have $f\simeq f': X\to Y$ relative x. Thus we have the result.

Corollary 1.3. *For a given morphism $f:(A\xrightarrow{x}X\xrightarrow{p}B)\to(A\xrightarrow{y}Y\xrightarrow{q}B)$ in* CGH_B^A *where x and y have* HEP *(i.e.* NDR *pairs in* CGH*) and p and q have* HLP *(i.e. fibration in* CGH*), if f is a homotopy equivalence in* CGH, *then so is f in* CGH_B^A.

Proof. By the above lemma, f is a homotopy equivalence in CGH_1^A. By the covering homotopy extension theorem (cf. Theorem I. 7. 16 in [8]) $p:(A\xrightarrow{x}X)\to(A\xrightarrow{px}B)$ and $q:(A\xrightarrow{y}Y)\to(A\xrightarrow{qy}B)$ have HLP in CGH_B^A. Hence by the dual one of the above lemma, f is a homotopy equivalence in CGH_B^A.

Proposition 1.4. (Corollary 1.1.8. in [1]) *Let $\mathcal{C}$ be a pre-homotopy category satisfying* (EC). *Consider the commutative diagram in $\mathcal{C}$ where $(A \xrightarrow{x} X)$ and $(A' \xrightarrow{x'} X')$ have* HEP *and $(Y \xrightarrow{q} B)$ and $(Y' \xrightarrow{q'} B')$ have relative* HLP *and h_i, k_i $(i=1, 2)$ are homotopy equivalences in $\mathcal{C}$.*

$$\begin{tikzcd} A' \arrow[r, "h_1"] \arrow[d, "x'"'] & A \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[r, "k_1"] \arrow[d, "q"] & Y' \arrow[d, "q'"] \\ X' \arrow[r, "h_2"'] & X \arrow[r, "g"'] & B \arrow[r, "k_2"'] & B' \end{tikzcd}$$

Then there hold the equalities

$$\mathcal{C}_B^{A'}[X'; Y] \cong \mathcal{C}_B^A[X; Y] \cong \mathcal{C}_{B'}^A[X; Y'].$$

Proof. When left-hand square is pushout, the first equality holds by the universality of pushout. Hence we may assume $h_1 = Id$ by taking the pushout of $A \xleftarrow{h_1} A' \xrightarrow{x'} X$. Since $(A \xrightarrow{x} X)$ and $(A \xrightarrow{x'} X')$ are homotopy equivalent in $\mathcal{C}_1^A$ by Lemma 1.2, we can reduce $A = \phi$ by considering in $\mathcal{C}_1^A$, and may assume $g = id$ and $q: Y \to X \in \mathrm{HLP}$ by considering the induced fibration through g. Then the first equality follows from the homotopical uniqueness of the induced fibrations (cf. Theorem 2.3 in [7]). By the duality and definition of relative HLP, we obtain the second equality.

Proposition 1.5. *Let $\mathcal{C}$ be a pre-homotopy category with the zero object satisfying* (EC). *Consider the diagram where the upper horizontal sequence is a cofiber sequence and the lower one is a fiber sequence and the right-hand square is homotopy commutative and $\psi^\wedge$ is adjoint of ψ. Then there exists a morphism $\gamma: B \to F_p$ which makes the diagram homotopy commutative.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\psi^\wedge"'] & B \arrow[r, "f_1"] \arrow[d, dashed, "\gamma"] & C(f) \arrow[r, "d"] \arrow[d, "\phi"] & \Sigma A \arrow[d, "\psi"] \\ \Omega Y \arrow[r, "j"'] & F_p \arrow[r, "i"'] & X \arrow[r, "p"'] & Y \end{tikzcd}$$

Proof. The above theorem is reduced to the following situation where f (resp. p) have HEP (resp. HLP) and $F_p = p^{-1}(*)$ (i.e. a fiber of p), $C(f) = B \cup CA$, $\Sigma A = CA/A$, $p\phi = \psi d$ by the mapping cylinder property (cf. Lemma 2.2 in [7]). Clearly there exist $\gamma: B \to F_p$ with $i\gamma = \phi f_1$. Now, we must prove $\gamma f \simeq j\psi^\wedge$. Let η be adjoint of $\phi|_{CA}: CA \to C(f) \to X$. Then we can define $\psi' = (\eta, \gamma f): A \to$

$LX\times_X F_p=LX\times_Y *$ which is homotopy equivalent to $\phi^\wedge: A\to LY\times_Y *$ by composing $LX\times_X F_p\xrightarrow{\simeq} LY\times_Y *$. By the commutativity $\gamma f=j'\phi'$ where $j': LX\times_X F_p \to F_p$ and $j: LY\times_Y * \xleftarrow{\simeq} LX\times_X F_p\to F_p$, we have the result.

Definition 1.6. (1) The *connectivity* of $f: X\to Y$ in CGH is the maximal integer in $\{n\,;\,\pi_i(M(f), X)=0 \text{ for } 0\leqq i\leqq n\}$.

(2) The *coconnectivity* of $f: X\to Y$ in CGH is the minimal integer in $\{n\,;\,\pi_i(M(f), X)=0 \text{ for } i\geqq n\}$.

Lemma 1.7. *Consider the commutative diagram* $(*)$ *in* CGH *where* (X, A) *is a relative* CW *complex and* p *is a fibration, and* $p=qj$ *is the factorization of mapping cylinder with* $qk=Id$.

$$(*)\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "p"] \arrow[dr, "j"] & \\ X \arrow[r, "g"'] \arrow[ur, dashed, "h"] & B \arrow[r, shift right, "k"'] & M(p) \arrow[l, shift right, "q"'] \end{tikzcd}$$

If kg deform into Y relative A (e.g. $\dim(X, A)\leqq$ *connectivity of p or connectivity of $(X, A)\geqq$(coconnectivity of p)-1), then there exists $h: X\to Y$ with $hi=f$ and $ph=g$.*

Proof. Since a homotopy H of kgi and jf can be constructed by using the cylinder IY of Y, there is $\bar{H}: IX\to M(p)$ with $H=\bar{H}I(i)$ and $\bar{H}0_*=kg$ by $i\in$HEP. Thus we have $\bar{g}=\bar{H}1_*: X\to M(p)$ with $\bar{g}i=jf$ and $kg\simeq\bar{g}$. Hence g and $q\bar{g}$ are homotopic relative A. By assumptions, there is $\tilde{g}: X\to Y$ with $\tilde{g}i=f$ and $j\tilde{g}\simeq\bar{g}$ relative i. Therefore $p\tilde{g}$ and g are homotopic relative A by $p\tilde{g}=qj\tilde{g}\simeq q\bar{g}$ (rel i) $\simeq qkg=g$ (rel i). By the covering homotopy extension theorem, we can choose $h: X\to Y$ with $hi=f$ and $ph=g$.

Proposition 1.8. *Let $f:(A\xrightarrow{x}X\xrightarrow{p}B)\to(A\xrightarrow{y}Y\xrightarrow{q}B)$ be a morphism in CGH^A_B where p and q are fibrations and f is n-connected in* CGH, *and $(A\xrightarrow{z}Z\xrightarrow{r}B)$ an object in CGH^A_B where (Z, A) is a relative* CW *complex. Then the induced map*

(1) $f_*: \mathrm{CGH}^A_B[Z\,;\,X]\to\mathrm{CGH}^A_B[Z\,;\,Y]$ *is bijective, if* $\dim(Z, A)\leqq n-1$.

(2) $f_*: \mathrm{CGH}^A_B[Z\,;\,X]\to\mathrm{CGH}^A_B[Z\,;\,Y]$ *is surjective, if* $\dim(Z, A)\leqq n$.

Proof. We factorize f as $f=gh: X\xrightarrow{h}X'\xrightarrow{g}Y$ where g is a fibration in CGH and h is a homotopy equivalence in CGH. For $h:(A\xrightarrow{x}X\xrightarrow{p}B)\to(A\xrightarrow{hx}X'\xrightarrow{qg}B)$, Proposition 1.8 is true by Proposition 1.4. Hence we may assume f is a fibration in CGH and apply Lemma 1.7 for $A\xrightarrow{z}Z$ and $X\xrightarrow{f}Y$. Thus we obtain the result.

Proposition 1.9. *Let* $f:(A\xrightarrow{x}X\xrightarrow{p}B)\to(A\xrightarrow{y}Y\xrightarrow{q}B)$ *be a morphism in* CGH_B^A *where* p *and* q *are fibrations and* f *is* n*-coconnected in* CGH, *and* $(A\xrightarrow{z}Z\xrightarrow{r}B)$ *an object in* CGH_B^A *where* (Z, A) *is a relative* CW *complex and* c*-connected. Then the induced map*

(1) $f_*:\mathrm{CGH}_B^A[Z;X]\to\mathrm{CGH}_B^A[Z;Y]$ *is bijective, if* $c\geqq n-1$.

(2) $f_*:\mathrm{CGH}_B^A[Z;X]\to\mathrm{CGH}_B^A[Z;Y]$ *is injective, if* $c\geqq n-2$.

Proof. We may take $(\bar{Z}, A)$ which is homotopy equivalent relative A to (Z, A) and has no i-cells for $0\leqq i\leqq c$ by the relative CW approximation theorem. By the same way as Proposition 1.8, we may assume f is a fibration in CGH. Thus we obtain the result by Proposition 1.4 and Lemma 1.7.

Since the proofs of the following propositions are analogous to the ones above, hence we omit them.

Proposition 1.10. *Let* $f:(A\xrightarrow{x}X\xrightarrow{p}B)\to(A\xrightarrow{y}Y\xrightarrow{q}B)$ *be a morphism in* CGH_B^A *where* (X, A) *and* (Y, A) *are relative* CW *complexes and* f *is* n*-connected in* CGH, *and* $(A\xrightarrow{z}Z\xrightarrow{r}B)$ *an object in* CGH_B^A *where* r *is a fibration and* c*-coconnected. Then the induced map*

(1) $f^*:\mathrm{CGH}_B^A[Y;Z]\to\mathrm{CGH}_B^A[X;Z]$ *is bijective, if* $n\geqq c-1$.

(2) $f^*:\mathrm{CGH}_B^A[Y;Z]\to\mathrm{CGH}_B^A[X;Z]$ *is injective, if* $n\geqq c-2$.

Proposition 1.11. *Let* $f:(A\xrightarrow{x}X\xrightarrow{p}B)\to(A\xrightarrow{y}Y\xrightarrow{q}B)$ *be a morphism in* CGH_B^A *where* (X, A) *and* (Y, A) *are relative* CW *complexes and* f *is* n*-dimensional* (*i.e.* $\dim(M(f), X)\leqq n$) *in* CGH, *and* $(A\xrightarrow{z}Z\xrightarrow{r}B)$ *an object in* CGH_B^A *where* r *is a fibration and* c*-connected. Then the induced map*

(1) $f^*:\mathrm{CGH}_B^A[Y;Z]\to\mathrm{CGH}_B^A[X;Z]$ *is bijective, if* $n\leqq c-1$.

(2) $f^*:\mathrm{CGH}_B^A[Y;Z]\to\mathrm{CGH}_B^A[X;Z]$ *is surjective, if* $n\leqq c$.

§ 2. Cell Structure and Postnikov System

Definition 2.1. (1) An object $(A\xrightarrow{x}X\xrightarrow{p}B)$ in CGH_B^A is called a cell complex in CGH_B^A, if $A\xrightarrow{x}X$ is a relative CW complex. The n-skeleton of $(A\xrightarrow{x}X\xrightarrow{p}B)$ is defined by the restriction $(A\xrightarrow{x_n}X^n\xrightarrow{p_n}B)$ ($x_n: A\to X^n$, $j_n: X^n\to X$, $j_nx_n=x$, $pj_n=p_n$) on the n-skeleton X^n of X.

(2) Let $(A\xrightarrow{x}X\xrightarrow{p}B)$ be a cell complex in CGH_B^A. Then $(A\xrightarrow{x_m^n}X_m^n\xrightarrow{p_m^n}B)$ $=(A\xrightarrow{j_m^n x_n}X^n/X^m\xrightarrow{p_m^n}B)$ is defined by the pushout diagram ($0\leqq m<n\leqq\infty$)

$$\begin{tikzcd}
A \arrow[r, "x_m"] \arrow[dr, "a"'] & X^m \arrow[r, "i^n_m"] \arrow[d, "p_m"] \arrow[dr, phantom, "\mathrm{PO}"] & X^n \arrow[d, "j^n_m"] \arrow[dr, "p_n"] & \\
& B \arrow[r] & X^n/X^m \arrow[r, "p^n_m"'] & B
\end{tikzcd}$$

In particular, i^n_{n-1} (resp. j^n_{n-1}) is abbreviated as i^n (resp. j^n). $(X^n/X^m, B)$ is a relative CW complex with i-cells for $m<i\leqq n$ and $(X^n/X^{n-1}, B)$ is a relative CW complex with n-cells $(B\cup_j e^n_j, B)$, where X^n/X^{n-1} is homotopy equivalent to a wedge sum $B\vee_j e^n_j$ in CGH.

Remark 2.2. The cone $C(B\to Z\to B)=(B\to C_BZ\to B)$ of $(B\to Z\to B)$ is homotopy equivalent to $(B\xrightarrow{id}B\xrightarrow{id}B)$ by Lemma 2.2 in [7], in particular C_BZ is homotopy equivalent to B in CGH. Hence X^n/X^m is homotopy equivalent to $X^n\cup C_BX^m$ in CGH by Theorem 2.5 in [7]. For the classification problem in Proposition 1.4 (when $A=B$), we may replace $(B\to X^n/X^m)$ by $(B\to X^n\cup C_BX^m)$ in the left-hand side.

Theorem 2.3. (Theorem 6.4 in [8]). *Let $q:Y\to B$ be a fibration with a connected fiber F. If $\pi_1(Y)$ acts simply on $\pi_n(M(q), Y)$, q admits a principal Postnikov system.*

$$\begin{tikzcd}
& & Y \arrow[dl, "q^n"'] \arrow[d, "q^{n-1}"] \arrow[drrr, "q"] & & & \\
\arrow[r] & Y_n \arrow[r, "q_n"'] \arrow[d, "k_{n+1}"] & Y_{n-1} \arrow[r, dash] \arrow[d, "k_n"] & \arrow[r] & Y_1 \arrow[r] & Y_0=B \\
& BK_{n+1} & BK_n & & &
\end{tikzcd}$$

where $q_n: Y_n\to Y_{n-1}$ is a fibration induced by $k_n: Y_{n-1}\to K(\pi_n(F), n+1)$ from the standard path fibration on BK_n $(K_n=K(\pi_n(F), n)$, $BK_n=K(\pi_n(F), n+1))$, and $q^n: Y\to Y_n$ is $(n+1)$-connected.

By using a fibration $q_n: Y_n\to Y_{n-1}$ and induction on Y_n, $\pi_i(Y_n)$ is equal to $\pi_i(Y)$ for $i\leqq n$ and $\pi_i(B)$ for $i\geqq n+2$ and $q_1\cdots q_n: Y_n\to B$ is $(n+2)$-coconnected. Moreover there is an exact sequence

$$0\longrightarrow \pi_{n+1}(Y_n)\longrightarrow \pi_{n+1}(B)\longrightarrow \pi_n(F)\longrightarrow \pi_n(Y_n)\longrightarrow \pi_n(Y_{n-1})\longrightarrow 0.$$

Proposition 2.4. *Let $(A\xrightarrow{x}X\xrightarrow{p}B)$ be a cell complex in CGH^A_B and $(A\xrightarrow{y}Y\xrightarrow{q}B)$ an object in CGH^A_B where q admit a principal Postnikov system and $q_n: Y_n\to B$ the Postnikov n-stage of q. Then there holds a natural isomorphim:*

$$\mathrm{CGH}^A_B[X; Y_n]\cong \textit{Image of } \mathrm{CGH}^A_B[X^{n+1}; Y]\longrightarrow \mathrm{CGH}^A_B[X^n; Y].$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \mathrm{CGH}_B^A[X^{n+1};\, Y] & & \xrightarrow{\quad (1) \quad} & & \mathrm{CGH}_B^A[X^{n+1};\, Y_n] \\ \downarrow & & & & \cong\uparrow(4) \\ \mathrm{CGH}_B^A[X^n;\, Y] & \xrightarrow[(2)]{\cong} & \mathrm{CGH}_B^A[X^n;\, Y_n] & \xleftarrow[(3)]{} & \mathrm{CGH}_B^A[X;\, Y_n] \end{array}$$

Since $q_n: Y \to Y_n$ is $(n+1)$-connected and dimension of (X^m, A) is m ($m=n$, $n+1$), (1) is surjective and (2) is isomorphic by Proposition 1.8. Since $q_1 \cdots q_n$: $Y_n \to B$ is $(n+2)$-coconnected and (X, X^m) is m-connected, (3) is injective and (4) is isomorphic by Proposition 1.10.

If q has a cross section, $\pi_i(Y_n)$ is isomorphic to $\pi_i(Y)$ for $i \leqq n$ and $\pi_i(B)$ for $i \geqq n+1$, and there holds an exact sequence $0 \to \pi_n(F) \to \pi_n(Y) \to \pi_n(B) \to 0$.

Since an Eilenberg-MacLane space $K(\pi, n)$ (π; abelian group, $n \geqq 1$) can be considered as a topological abelian group, $(B \xrightarrow{\langle id, 0\rangle} B \times K(\pi, n) \xrightarrow{pr} B)$ is homotopy equivalent to $(B \xrightarrow{\langle id, r\rangle} B \times K(\pi, n) \xrightarrow{pr} B)$ ($r: B \to K(\pi, n)$ a continuous map) in CGH_B by using $h: B \times K(\pi, n) \to B \times K(\pi, n)$ defined by $h(b, x) = (b, x + r(b))$.

Definition 2.5. Let $(B \xrightarrow{y} Y \xrightarrow{q} B)$ be an object in CGH_B where q is a fibration with a connected fiber F and admit a principal Postnikov system.

(1) An object $(B \xrightarrow{\langle id, r\rangle} B \times K(\pi, n) \xrightarrow{pr} B)$ is called an Eilenberg-MacLane object in CGH_B.

(2) $k_{n+1} = \langle q_1 \cdots q_n,\, k_{n+1}\rangle : (B \xrightarrow{q^n y} Y_n \xrightarrow{q_1 \cdots q_n} B) \to (B \xrightarrow{\langle id, r\rangle} B \times BK_{n+1} \xrightarrow{pr} B)$ is called the k-invariant of $(B \xrightarrow{y} Y \xrightarrow{q} B)$ in CGH_B.

(3) Let $LK_n \to BK_n$ be a standard path fibration with a fiber K_n. Then $(B \xrightarrow{\langle id, 0\rangle} B \times LK_n \xrightarrow{pr} B) \to (B \xrightarrow{\langle id, 0\rangle} B \times BK_n \xrightarrow{pr} B)$ becomes the standard path fibration with a fiber $(B \xrightarrow{\langle id, 0\rangle} B \times K_n \xrightarrow{pr} B)$ in CGH_B.

(4) An object $(B \xrightarrow{q^n y} Y_n \xrightarrow{q_1 \cdots q_n} B)$ is called the Postnikov n-stage of $(B \xrightarrow{y} Y \xrightarrow{q} B)$ in CGH_B. There are morphisms $q_n: (B \xrightarrow{q^n y} Y_n \xrightarrow{q_1 \cdots q_n} B) \to (B \xrightarrow{q^{n-1} y} Y_{n-1} \xrightarrow{q_1 \cdots q_{n-1}} B)$ and $q^n: (B \xrightarrow{y} Y \xrightarrow{q} B) \to (B \xrightarrow{q^n y} Y_n \xrightarrow{q_1 \cdots q_n} B)$.

(5) The homotopy fiber in CGH_B of $q^n: (B \xrightarrow{y} Y \xrightarrow{q} B) \to (B \xrightarrow{q^n y} Y_n \xrightarrow{q_1 \cdots q_n} B)$ defines $(B \xrightarrow{y[n+1]} Y^{(n+1)} \xrightarrow{q[n+1]} B)$ where $\pi_i(Y^{(n+1)})$ is isomorphic to $\pi_i(B)$ for $i \leqq n$ and $\pi_i(Y)$ for $i \geqq n+1$. Note that CGH_B has the zero object $(B \xrightarrow{id} B \xrightarrow{id} B)$.

(6) $(B \xrightarrow{y_m^n} Y(m, n) \xrightarrow{q_m^n} B)$ is defined by the homotopy fiber of $q_m \cdots q_n$: $(B \xrightarrow{q^n y} Y_n \xrightarrow{q_1 \cdots q_n} B) \to (B \xrightarrow{q^{m-1} y} Y_{m-1} \xrightarrow{q_1 \cdots q_{m-1}} B)$ in CGH_B. $\pi_i(Y(m, n))$ is isomorphic to

$\pi_i(Y)$ for $m \leqq i \leqq n$ and $\pi_i(B)$ for $i<m$ or $n<i$. We interpret $Y(-\infty, n)$, $Y(n, \infty)$ as Y_n, $Y^{(n)}$ respectively. Clearly $(\Omega_B Y)(m, n)$ is homotopy equivalent to $\Omega_B(Y(m+1, n+1))$ and $(B \to \Omega_B(B \times K) \to B)$ is homotopy equivalent to $(B \to B \times \Omega K \to B)$ in CGH_B.

Proposition 2.6. *Let $(B \xrightarrow{x} X \xrightarrow{p} B)$ be a cell complex in* CGH_B *and* $(B \xrightarrow{y} Y \xrightarrow{q} B)$ *an object in* CGH_B *where q is a fibration with a connected fiber F and admit a principal Postnikov system. Then there is a natural isomorphism:*

$$\mathrm{CGH}_B[X;\, Y^{(n+1)}] \cong \textit{Image of } \mathrm{CGH}_B[X/X^n;\, Y] \longrightarrow \mathrm{CGH}_B[X/X^{n-1};\, Y].$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& \mathrm{CGH}_B[X/X^n;\, Y^{(n+1)}] \arrow[dl, "(1)"'] \arrow[r, "(3)", "\cong"'] \arrow[dd] & \mathrm{CGH}_B[X/X^n;\, Y] \arrow[dd] \\
\mathrm{CGH}_B[X;\, Y^{(n+1)}] & & \\
& \mathrm{CGH}_B[X/X^{n-1};\, Y^{(n+1)}] \arrow[ul, "\cong", "(2)"'] \arrow[r, tail, "(4)"'] & \mathrm{CGH}_B[X/X^{n-1};\, Y]
\end{tikzcd}$$

Since $Y^{(n+1)} \to B$ is $(n+1)$-connected and $X \to X/X^m$ is $m+1$-dimensional $(m=n, n-1)$ (cf. Remark 2.2 and Proposition 1.4.) (1) is surjective and (2) is isomorphic by Proposition 1.11. Since $Y^{(n+1)} \to Y$ is $(n+1)$-coconnected and $B \to X/X^m$ is m-connected $(m=n-1, n)$, (3) is isomorphic and (4) is injective by Proposition 1.9.

Corollary 2.7. *Under the assumptions of Proposition* 2.6, *there is a natural isomorphism:*

$$\mathrm{CGH}_B[X;\, Y(m, n)] \cong \textit{Image of } \mathrm{CGH}_B[X^{n+1}/X^{m-1};\, Y] \longrightarrow \mathrm{CGH}_B[X^n/X^{m-2};\, Y].$$

§ 3. Maunder Type Theorems

In this section we work in the ex-homotopy category CGH_B and assume that all space in CGH is 0-connected and has the homotopy type of CW complex.

Lemma 3.1. *Let $(B \xrightarrow{x} B \cup e^n \xrightarrow{p} B)$ be a cell complex in* CGH_B *and* $(B \xrightarrow{y} Y \xrightarrow{q} B)$ *an object in* CGH_B *where q is a fibration with a fiber F and admit a principal Postnikov system. Then* $\mathrm{CGH}_B[B \cup e^n;\, Y]$ *is identified with $\pi_n(F)$. If $f: (B \xrightarrow{x} B \cup e^n \xrightarrow{p} B) \to (B \xrightarrow{x'} B \cup e^n \xrightarrow{p'} B)$ is a map of degree m (i.e. degree of $H_n(B \cup e^n, B) \to H_n(B \cup e^n, B)$), then the induces map f^*:* $\mathrm{CGH}_B[B \cup e^n, Y] \to \mathrm{CGH}_B[B \cup e^n, Y]$ *is a map of degree m.*

Proof. The homotopy set $\mathrm{CGH}_B[B \cup e^n;\, Y]$ is classified by the relative cross sections of the induced fibration $p^*(q)$ over $B \cup e^n$. Hence it is classified by the relative cross sections of the induced fibration over the n-disk D^n. Since a fibration over D^n is fiber homotopy equivalent to the projection $D^n \times F \to D^n$

and a cross section over S^{n-1} is given, we have the result by assigning the difference cochain $d \in H_n(D^n, S^{n-1}; \pi_n F) = \pi_n F$. For the second part, let $f: (D^n, S^{n-1}) \to (D^n, S^{n-1})$ be a map of degree m. Then the induced map of fibrations is fiber homotopy equivalent to $f^\sim: D^n \times F \to D^n \times F$, $f^\sim(x, y) = (f(x), y)$. Thus $f: (D^n, S^{n-1}) \to (D^n, S^{n-1})$ induces a map of multiple $m: \pi_n(F) \to \pi_n(F)$ by using the correspondence $(D^n, S^{n-1}) \xrightarrow{f} (D^n, S^{n-1}) \xrightarrow{s} F$ (s: cross section).

Let $(B \xrightarrow{x} X \xrightarrow{p} B)$ be a cell complex in CGH_B and $(B \xrightarrow{y} Y \xrightarrow{q} B)$ an object in CGH_B where q is a fibration with a fiber F and admit a principal Postnikov system. Hereafter we write simply total space X for an object $(B \xrightarrow{x} X \xrightarrow{p} B)$, unless there happen confusions.

The Puppe sequence in CGH_B

$$X^{s-1} \xrightarrow{i^s} X^s \xrightarrow{j^s} X^s/X^{s-1} \xrightarrow{h^s} \Sigma_B X^{s-1} \longrightarrow \Sigma_B X^s \longrightarrow \Sigma_B X^s/X^{s-1} \longrightarrow$$

gives the exact couple

$$\begin{aligned} &\{D_1^{s,t}, E_1^{s,t}, \alpha_1, \beta_1, \gamma_1\} \\ (3.2) \qquad &D_1^{s,t} = \mathrm{CGH}_B[\Sigma_B{}^{t-s} X^s; Y] \quad (t \geqq s \geqq 0) \\ &E_1^{s,t} = \mathrm{CGH}_B[\Sigma_B{}^{t-s} X^s/X^{s-1}; Y] \\ &\qquad = C^s(X, B; \pi_t(F)) \quad (t \geqq s \geqq 0) \end{aligned}$$

where α_1, β_1 and γ_1 are induced by $i^s: X^{s-1} \to X^s$, $h^s: X^s/X^{s-1} \to \Sigma_B X^{s-1}$, $j^s: X^s \to X^s/X^{s-1}$ respectively. The bidegrees α_1, β_1 and γ_1 are $(-1, -1)$, $(1, 0)$ and $(0, 0)$ respectively. Let $\{D_r{}^{s,t}, E_r{}^{s,t}, \alpha_r, \beta_r, \gamma_r\}$ be the derived couple of (3.2). Since the differential of (3.2) is induced by $X^{s+1}/X^s = B \cup_i e_i^{s+1} \to \Sigma_B X^s \to \Sigma_B X^s/X^{s-1} = B \cup_j e_j^{s+1}$, E_2-term is described by the following formula in the light of Lemma 3.1.

$$(3.3) \qquad E_2{}^{s,t} = H^s(X, B; \pi_t(F)) \quad (t \geqq s \geqq 0).$$

When $t-s=0$, E_2-term is not obtained by E_1-term. But we may use (3.3) for the E_2-term by Lemma 3.8 below.

Now let us consider the Postnikov system of $(B \xrightarrow{y} Y \xrightarrow{q} B)$. By using the fiber mapping sequence in CGH_B:

$$\longrightarrow \Omega_B Y_{n-1} \longrightarrow \Omega_B(B \times BK_n) \xrightarrow{l_n} Y_n \xrightarrow{q_n} Y_{n-1} \xrightarrow{k_n} B \times BK_n$$

we define the exact couple.

$$\begin{aligned} &\{\bar{D}_2{}^{s,t}, \bar{E}_2{}^{s,t}, \bar{\alpha}_2, \bar{\beta}_2, \bar{\gamma}_2\} \\ (3.4) \qquad &\bar{D}_2{}^{s,t} = \mathrm{CGH}_B[X; \Omega_B{}^{t-s} Y_t] \cong \mathrm{CGH}_B[\Sigma_B{}^{t-s} X; Y_t] \quad (t \geqq s \geqq 0) \\ &\bar{E}_2{}^{s,t} = \mathrm{CGH}_B[X; \Omega_B{}^{t-s+1}(B \times BK_t)] \cong H^s(X, B; \pi_t(F)) \quad (t+1 \geqq s \geqq 0) \end{aligned}$$

where $\bar{\alpha}_2$, $\bar{\beta}_2$ and $\bar{\gamma}_2$ are induced by $q_n: Y_n \to Y_{n-1}$, $k_n: Y_{n-1} \to B \times BK_n$ and $l_n: B \times K_n \to Y_n$ in CGH_B. The bidegrees of $\bar{\alpha}_2$, $\bar{\beta}_2$ and $\bar{\gamma}_2$ are $(-1, -1)$, $(2, 1)$ and $(0, 0)$ respectively. Let $\{\bar{D}_r^{s,t}, \bar{E}_r^{s,t}, \bar{\alpha}_r, \bar{\beta}_r, \bar{\gamma}_r\}$ be the derived couple of (3.4).

Theorem 3.5. *Let $(B \xrightarrow{x} X \xrightarrow{p} B)$ be a cell complex in CGH_B where (X, B) has no 0-cells and $(B \xrightarrow{y} Y \xrightarrow{q} B)$ an object in CGH_B where q is a fibration with a fiber F and admit a principal Postnikov system. Then, for $t-s \geqq 0$, there exist isomorphisms*

$$\phi: D_r^{s,t} \longrightarrow \bar{D}_r^{s,t}, \qquad \psi: E_r^{s,t} \longrightarrow \bar{E}_r^{s,t}$$

which commute $\alpha_r, \beta_r, \gamma_r$ and $\bar{\alpha}_r, \bar{\beta}_r, \bar{\gamma}_r$.

The proof of this theorem proceeds by the same way as [6]. Let consider the diagram

(3.6)
$$\begin{tikzcd}
D_2^{s+1,t+1} \arrow[r, "\alpha"] \arrow[dd, "\phi"] \arrow[ddr, phantom, "(3)"] & D_2^{s,t} \arrow[r, "\beta"] \arrow[dr, "\tilde{\beta}"'] \arrow[dd, "\phi"] \arrow[ddr, phantom, "(4)"] & E_2^{s+2,t+1} \arrow[r, "\gamma"] \arrow[d, "\psi_1"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & D_2^{s+2,t+1} \arrow[dd, "\phi"] \\
 & & \tilde{E}_2^{s+2,t+1} \arrow[ur, "\tilde{\gamma}"'] \arrow[d, "\psi_2"] \arrow[r, phantom, "(5)"] & \\
\bar{D}_2^{s+1,t+1} \arrow[r, "\bar{\alpha}"'] & \bar{D}_2^{s,t} \arrow[r, "\bar{\beta}"'] & \bar{E}_2^{s+2,t+1} \arrow[r, "\bar{\gamma}"'] & \bar{D}_2^{s+2,t+1}
\end{tikzcd}$$

We shall define the group $\tilde{E}_2^{s,t} (t-s \geqq 1)$, the isomorphisms ϕ, ψ_1, ψ_2, $\psi_2\psi_1 = \psi$ and homomorphisms $\tilde{\beta}_2$ and $\tilde{\gamma}_2$ which make the diagram commutative.

Let $Y^{(k)}$ be the homotopy fiber in CGH_B of $q_{k-1}: Y \to Y_{k-1}$, $i^{(t)}: Y^{(t)} \to Y$ the canonical inclusion and $q^{(t)}: Y^{(t)} \to B \times K_t$ the natural morphism in CGH_B. By the proof of Proposition 2.4, q_t induces an isomorphism (cf. [6]):

(3.7) $\qquad \phi: D_2^{s,t} \longrightarrow \bar{D}_2^{s,t}$ is an isomorphism for $t-s \geqq 0$.

We shall define $\tilde{E}_2^{s,t}$ for $t-s \geqq 0$.

$$\begin{aligned}
\tilde{E}_2^{s,t} &= \text{Image of } \mathrm{CGH}_B[\Sigma_B^{t-s} X^{s+1}; Y^{(t)}] \longrightarrow \mathrm{CGH}_B[\Sigma_B^{t-s} X^s; Y^{(t)}] \quad \text{for } t-s \geqq 0 \\
&= \text{Ker of } h^{s+1*}: \mathrm{CGH}_B[\Sigma_B^{t-s} X^s; Y^{(t)}] \longrightarrow \mathrm{CGH}_B[\Sigma_B^{t-s-1} X^{s+1}/X^s; Y^{(t)}] \\
&\qquad\qquad \text{for } t-s \geqq 1.
\end{aligned}$$

Lemma 3.8. (1) *There exists a natural isomorphism $\psi_2: \tilde{E}_2^{s,t} \to \bar{E}_2^{s,t}$ for $t-s \geqq 0$.*

(2) *The map $j^{s*}: \mathrm{CGH}_B[\Sigma_B^{t-s} X^s/X^{s-1}; Y^{(t)}] \to \mathrm{CGH}_B[\Sigma_B^{t-s} X^s; Y^{(t)}]$ induces an isomorphism $\psi_1: E_2^{s,t} \to \tilde{E}_2^{s,t}$ for $t-s \geqq 0$.*

Proof. (1) is proved by Proposition 2.4. (2) By the following diagram we have the result for $t-s \geqq 1$.

$$\begin{tikzcd}[column sep=small]
\mathrm{CGH}_B[\Sigma_B^{t-s}X^{s+1};\ Y^{(t)}] \arrow[r] & \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s};\ Y^{(t)}] \arrow[r] & \mathrm{CGH}_B[\Sigma_B^{t-s-1}X^{s+1}/X^{s},\ Y^{(t)}] \\
 & & =C^{s+1}(X,\ B\,;\ \pi_t(F)) \\
 & \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s}/X^{s-1};\ Y^{(t)}]=C^{s}(X,\ B\,;\ \pi_t(F)] \arrow[uu] \arrow[uur, "\delta"] & \\
C^{s-1}(X,\ B\,;\ \pi_t(Y))=\mathrm{CGH}_B[\Sigma_B^{t-s+1}X^{s-1}/X^{s-2};\ Y^{(t)}] \arrow[ur, "\delta"] \arrow[rr, two heads] & & \mathrm{CGH}_B[\Sigma_B^{t-s+1}X^{s-1};\ Y^{(t)}] \arrow[ul]
\end{tikzcd}$$

For $t-s=0$, the map $j^{s}*\colon \mathrm{CGH}_B[X^s/X^{s-1};\ Y^{(s)}]\to\mathrm{CGH}_B[X^s;\ Y^{(s)}]$ induces the isomorphism $\phi_1\colon E_2{}^{s,s}\to\tilde{E}_2{}^{s,s}$ by (1) and Proposition 2.4.

Now, for $t-s\geqq 1$, define

$$\tilde{\beta}_2=j^{s+2}*(i_*{}^{(t+1)})^{-1}h^{s+2}*(i^{s+1}*)^{-1}\colon D_2{}^{s,t}\twoheadleftarrow \mathrm{CGH}_B[\Sigma_B{}^{t-s}X^{s+1}\,;Y\,]\longrightarrow$$
$$\mathrm{CGH}_B[\Sigma_B{}^{t-s-1}X^{s+2}/X^{s+1}\,;\ Y]=\mathrm{CGH}_B[\Sigma_B{}^{t-s-1}X^{s+2}/X^{s+1}\,;\ Y^{(t+1)}]\longrightarrow \tilde{E}_2{}^{s+2,t+1}$$

$$\tilde{\gamma}_2\colon \tilde{E}_2{}^{s+2,t+1}\subset \mathrm{CGH}_B[\Sigma_B{}^{t-s-1}X^{s+2}\,;\ Y^{(t+1)}]\longrightarrow \mathrm{CGH}_B[\Sigma_B{}^{t-s-1}X^{s+2}\,;\ Y]\supset D_2{}^{s+2,t+1}.$$

We shall prove the commutativity of the diagram (3.6). The commutativities of (1), (2), (3) and (5) are obtained by the naturality and the definitions of maps involved. So we omit the proofs.

For the commutativity of (4), consider the diagram

(3.9)

$$\begin{tikzcd}[column sep=small]
D_2^{s,t}\subset \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s};Y] \arrow[dddd, "\tilde{\beta}_2"'] \arrow[r, phantom, "="] & \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s};Y_t] & \mathrm{CGH}_B[\Sigma_B^{t-s}X;Y_t]=\bar{D}_2^{s,t} \arrow[l, tail] \arrow[dddd, "\bar{\beta}_2"] \arrow[d, equal] \\
\mathrm{CGH}_B[\Sigma_B^{t-s}X^{s+1};Y] \arrow[u] \arrow[d] \arrow[r] & \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s+1};Y_t] \arrow[u] \arrow[r, phantom, "="] & \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s+2};Y_t] \arrow[d, equal] \arrow[dddll, "\partial"'] \\
\mathrm{CGH}_B[\Sigma_B^{t-s-1}X^{s+2}/X^{s+1};Y]\ (*) & & \mathrm{CGH}_B[\Sigma_B^{t-s}X^{s+2};\Omega_B Y_t] \arrow[ddl, "\partial"] \\
\mathrm{CGH}_B[\Sigma_B^{t-s-1}X^{s+2}/X^{s+1};Y^{(t+1)}] \arrow[u, "\cong"] \arrow[d] & & \\
\tilde{E}_2^{s+2,s+1}\subset \mathrm{CGH}_B[\Sigma_B^{t-s-1}X^{s+2};Y^{(t+1)}] \arrow[r, phantom, "="] & \mathrm{CGH}_B[\Sigma_B^{t-s-1}X^{s+2};B\times F_{t+1}] & \supset \bar{E}_2^{s+2,t+1}
\end{tikzcd}$$

The commutativity of the part $(*)$ is proved by applying Proposition 1.5 to the next diagram.

$$\begin{tikzcd}
\Sigma_B{}^{t-s-1}X^{s+1} \arrow[r] \arrow[d, "\hat{\phi}"] & \Sigma_B{}^{t-s-1}X^{s+2} \arrow[r] \arrow[d, "\hat{\phi}"] & \Sigma_B{}^{t-s-1}X^{s+2}/X^{s+1} \arrow[r] \arrow[d, "\eta"] & \Sigma_B{}^{t-s}X^{s+1} \arrow[r] \arrow[d, "\phi"] & \Sigma_B{}^{t-s}X^{s+2} \arrow[d, "\phi"] \\
\Omega_B Y \arrow[r] & \Omega_B Y_t \arrow[r] & Y^{(t+1)} \arrow[r] & Y \arrow[r] & Y_t
\end{tikzcd}$$

The other commutativities of (3.9) are obtained by the definitions and naturalities. Thus we proved the commutativity of (4). This complete the proof of Theorem 3.5.

Now we shall investigate the spectral sequence associated with the ex-anti-skeleton filtration. By using the Puppe sequence

$$X^s/X^{s-1} \xrightarrow{f_s} X/X^{s-1} \xrightarrow{g_s} X/X^s \xrightarrow{h_s} \Sigma_B X^s/X^{s-1} \longrightarrow \Sigma_B X/X^{s-1}$$

we define the exact couple as follow:

$$\begin{aligned} &\{D_1'^{s,t}, E_1'^{s,t}, \alpha_1', \beta_1', \gamma_1'\} \\ &D_1'^{s,t} = \mathrm{CGH}_B[\Sigma_B{}^{t-s-1}X/X^s ; Y] \quad (t \geqq s \geqq 0) \qquad (3.10) \\ &E_1'^{s,t} = \mathrm{CGH}_B[\Sigma_B{}^{t-s}X^s/X^{s-1} ; Y] \quad (t \geqq s \geqq 0) \end{aligned}$$

where α_1', β_1', γ_1' are induced by g_s: $X/X^{s-1} \to X/X^s$, f_s: $X^s/X^{s-1} \to X/X^{s-1}$ and h_s: $X/X^{s-1} \to \Sigma X^s/X^{s-1}$ and bidegrees α_1', β_1', γ_1' are $(-1, -1)$, $(1, 0)$, $(0, 0)$ respectively.

Theorem 3.11. *Let* $(B \xrightarrow{y} Y \xrightarrow{q} B)$ *be an object in* CGH_B *where* q *is a fibration with a fiber* F *and admit a principal Postnikov system. Then there exist a following system called anti-Postnikov system.*

$$\begin{tikzcd} B \arrow[r] & Y^{(s+1)} \arrow[r, "q_{s+1}'"] \arrow[d, "k_{s+1}'"] & Y^{(s)} \arrow[r] \arrow[d, "k_s'"] & \cdots \arrow[r] & Y \\ & B \times K_{s+1} & B \times K_s & & \end{tikzcd} \qquad (3.12)$$

where $Y^{(s+1)}$ *is induced by* k_s' *from the standard path fibration on* $B \times K_s$.

Proof. This is clear from Theorem 2.3 and Definition 2.5.

Similarly we can define the spectral sequence associated with the anti-Postnikov system (3.12). We define the exact couple:

$$\begin{aligned} &\{D_2''^{s,t}, E_2''^{s,t}, \alpha_2'', \beta_2'', \gamma_2''\} \\ &D_2''^{s,t} = \mathrm{CGH}_B[X ; \Omega_B{}^{t-s-1}Y^{(t+1)}] \quad (t \geqq s \geqq 0) \qquad (3.13) \\ &E_2''^{s,t} = \mathrm{CGH}_B[X ; \Omega_B{}^{t-s}(B \times K_t)] \quad (t \geqq s \geqq 0) \end{aligned}$$

where α_2'', β_2'' and γ_2'' are induced by q_{s+1}': $Y^{(s+1)} \to Y^{(s)}$, k_s': $Y^{(s)} \to B \times K_s$ and h_s': $\Omega_B(B \times K_s) \to Y^{(s+1)}$ and bidegrees α_2'', β_2'', γ_2'' are $(-1, -1)$, $(2, 1)$, $(0, 0)$ respectively.

By using Proposition 2.6, we have the following result by the same way as Theorem 3.5.

Theorem 3.14. *Under the assumptions of Theorem 3.5, there exist natural isomorphisms:*

$$\phi \colon D_r'^{s,t} \to D_r''^{s,t}, \qquad \phi \colon E_r'^{s,t} \to E_r''^{s,t}$$

which commute with α_r', β_r', γ_r' *and* α_r'', β_r'', γ_r''.

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