Publ. RIMS, Kyoto Univ.
25 (1989), 647-658

On the Adams Filtration of a Generator of the Free Part of $\pi_*^s(Q_n)$

Dedicated to Professor Shôrô Araki on his sixtieth birthday

By

Mitsunori IMAOKA*

§1. Introduction and Results

Let Q_n be the quaternionic quasi-projective space (cf. [11]). Then it is a 2-connected CW-complex having one $4i-1$ dim cell for each i with $1 \leqq i \leqq n$. It is known that the order of the cokernel of the stable Hurewicz homomorphism $h: \pi_{4n-1}^s(Q_n) \to H_{4n-1}(Q_n; Z)$ is equal to $a(n-1)\cdot(2n-1)!$ (see Theorem 3.10), where and throughout the paper $a(i)$ denotes 1 if i is an even integer and 2 if i is an odd integer. Then, for any generator x of the free part of $\pi_{4n-1}^s(Q_n)$, the mod p Adams filtration $F_p(x)$ is less than or equal to $\nu_p(a(n-1)\cdot(2n-1)!)$, where $\nu_p(j)$ denotes the exponent of a prime p in the prime power decomposition of an integer j. Let $G_{(p)}$ denote the tensor product $G \otimes Z_{(p)}$ for an abelian group G, where $Z_{(p)}$ is the ring of integers localized at p. Then, one of the results in this note is the following.

Theorem 1. *For $n \geqq 1$ we have an element $x_n \in \pi_{4n-1}^s(Q_n)_{(2)}$ which is a generator of the free part and whose* mod 2 *Adams filtration is equal to $\nu_2(a(n-1)\cdot(2n-1)!)$.*

In [6] M. C. Crabb and K. Knapp has proved that $F_p(\sigma^r) = \nu_p(r!)$ for the generator σ^r of the free part of $\pi_{2r}^s(CP^\infty)$, where CP^∞ is the complex projective space and σ^r is the r-fold product of the canonical generator $\sigma \in \pi_2^s(CP^\infty)$ by the H-space structure of CP^∞. Theorem 1 is an analogy of their result for the case of the quaternionic quasi-projective space. For the case of an odd prime p, the element $q_*(\sigma^{2n-1}) \in \pi_{4n-1}^s(Q_n)_{(p)}$ satisfies the corresponding properties, that is, it is a generator of the free part and satisfies $F_p(q_*(\sigma^{2n-1})) = \nu_p(a(n-1)\cdot(2n-1)!)$, where $q: \Sigma CP^{2n-1} \to Q_n$ is the map canonically defined by the definition of Q_n (cf. [20]).

Communicated by N. Shimada, January 19, 1989.
* Department of Mathematics, Faculty of Education, Wakayama University, Wakayama 640, Japan.

In [18] D. M. Segal has obtained some results about $\pi_*^s(HP^n)$ by applying the Adams spectral sequence, whose method originated with M. Mahowald [12], where HP^n denotes the quaternionic projective space. Our proof of Theorem 1 will be done as an application of Segal's method and of Crabb-Knapp's result mentioned above. Then by our method some properties concerned with the $\mathrm{Im}(J)$ classes are also shown, which is the second result of this note. To state them, we prepare some notations.

Let $\mathrm{Im}(J)_n$ be the image of J-homomorphism in $\pi_{4n-1}^s(S^0)_{(2)}$. Then it is a cyclic group of order $2^{3+\nu(n)}$ and a direct summand of $\pi_{4n-1}^s(S^0)$, where $\nu(n)=\nu_2(n)$. (Cf. [2]). We denote by α_{2n} the element of $\mathrm{Im}(J)_n$ of order 2, and by $\alpha_{2n/i}$ the element of $\mathrm{Im}(J)_n$ satisfying $2^{i-1}\alpha_{2n/i}=\alpha_{2n}$ for $i\geqq 1$. Let $i\colon S^2\to CP^n$ and $i\colon S^3\to Q_n$ be the inclusions to the bottom spheres respectively. Then we have the following theorems.

Theorem 2. *Let* $i_*\colon \pi_{4n-1}^s(S^0)\cong\pi_{4n+1}^s(S^2)\to\pi_{4n+1}^s(CP^k)$.

(i) *When* $n=2m\geqq 2$, *we have* (a) $i_*(\alpha_{4m})\neq 0$ *if* $1\leqq k\leqq 4m$, *and* (b) $i_*(\alpha_{4m})=0$ *and* $i_*(\alpha_{4m/2})\neq 0$ *if* $k\geqq 4m+1$.

(ii) *When* $n=2m+1\geqq 1$, *we have* (a) $i_*(\alpha_{4m+2})=0$ *if* $k\geqq 2$, (b) $i_*(\alpha_{4m+2/2})\neq 0$ *if* $1\leqq k\leqq 4m+2$, *and* (c) $i_*(\alpha_{4m+2/2})=0$ *and* $i_*(\alpha_{4m+2/3})\neq 0$ *if* $k\geqq 4m+3$.

Theorem 3. *Let* $i_*\colon \pi_{4n-1}^s(S^0)\cong\pi_{4n+2}^s(S^3)\to\pi_{4n+2}^s(Q_k)$.

(i) *When* $n=2m\geqq 2$, $i_*(\alpha_{4m})\neq 0$ *if* $k\geqq 1$.

(ii) *When* $n=2m+1\geqq 1$, (a) $i_*(\alpha_{4m+2})\neq 0$ *if* $1\leqq k\leqq 2m+1$, *and* (b) $i_*(\alpha_{4m+2})=0$ *and* $i_*(\alpha_{4m+2/3})\neq 0$ *if* $k\geqq 2m+2$.

This paper is organized as follows: In § 2, we show Proposition 2.1, which proves a part of Theorems 2 and 3, and which we need in the proof of Theorem 1. In § 3, we prove Theorem 1 by applying the Adams spectral sequences for $\pi_*^s(CP^n)$ and $\pi_*^s(Q_n)$, which also proves a part of Theorems 2 and 3. In § 4, we complete the proof of Theorems 2 and 3.

The author thanks K. Morisugi, H. Ōshima and T. Watanabe for their valuable comments.

§ 2. Estimation by J-Theory

Let $(Y_n, d)=(CP^{2n}, 2)$ or $(Q_n, 3)$, and let $i_*|_J$ be the restriction of $i_*\colon \pi_{4n-1}^s(S^0)\cong\pi_{4n+d-1}^s(S^d)\to\pi_{4n+d-1}^s(Y_n)$ to $\mathrm{Im}(J)_n$, where $i\colon S^d\to Y_n$ is the inclusion into the bottom sphere and $\mathrm{Im}(J)_n$ is the 2-primary part of the image of the J-homomorphism in $\pi_{4n-1}^s(S^0)$. In this section, we prove the following

Proposition 2.1. (i) *When* $Y_n=CP^{2n}$ *and* $n=2m\geqq 2$, $i_*|_J$ *is injective.*

(ii) *When* $Y_n=CP^{2n}$ *and* $n=2m+1\geqq 1$, $\nu_2(|i_*(\mathrm{Im}(J)_n)|)=\nu_2(|\mathrm{Im}(J)_n|)-1$, *where* $|G|$ *denotes the order of the group* G.

(iii) *When* $Y_n=Q_n$ *and* $n\geqq 1$, $i_*|_J$ *is injective.*

Let ψ^3: $KO_{(2)}\to KO_{(2)}$ be the stable Adams operation on the KO-theory localized at 2, and kO and $kSpin$ the (-1) and 3 connective cover of KO respectively. Then we have a lifting ϕ: $kO_{(2)}\to kSpin_{(2)}$ of the operation ψ^3-1: $KO_{(2)}\to KO_{(2)}$, and we define j to be the fiber spectrum of ϕ. (Cf. [7], [16]). Then j is a connected 2-local spectrum, and $\pi_0(j)\cong Z_{(2)}$. We denote by $j_i(X)$ and $j^i(X)$ the reduced homology and cohomology groups of a space X respectively. Then the j-theory is known to be closely related to the J-groups as follows, where h_j: $\pi_*^s(X)_{(2)}\to j_*(X)$ is the stable Hurewicz homomorphism.

Lemma 2.2. *The restriction of* h_j: $\pi_{4n-1}^s(S^0)_{(2)}\to j_{4n-1}(S^0)$ *to* $\mathrm{Im}(J)_n$ *is an isomorphism.*

Thus we investigate the homomorphism i_*: $j_{4n-1}(S^0)\to j_{4n+d-1}(Y_n)$ to prove Proposition 2.1. For example, to prove (i) and (iii) it is sufficient to show that i_* is injective. To make use of the known results about the KO-cohomology of Y_n, we consider the S-dual of Y_n, and denote it by V_n. Then by [4], the stable homotopy type of V_n is given as follows:

Lemma 2.3. *When* $Y_n=CP^{2n}$ *(resp.* Q_n*),* V_n *is stably homotopy equivalent to the stunted complex (resp. quaternionic) projective space* $CP_{2(N-n)-1}^{2(N-1)}=CP^{2(N-1)}/CP^{2(N-n-1)}$ *(resp.* $HP_{N-n}^{N-1}=HP^{N-1}/HP^{N-n-1}$*) for some sufficiently large integer* N *which we can take so as to satisfy* $N>n$ *and* $N\equiv 0 \bmod 8$.

We regard V_n as the stunted projective space as in Lemma 2.3. Let p: $V_n\to S^{4(N-1)}$ be the collapsing map to the top cell, and let $M=N-n-1$. Then through the S-duality, the homomorphism i_*: $j_{4n-1}(S^0)\to j_{4n+d-1}(Y_n)$ is identified with p^*: $j^{-4n+1}(S^0)\cong j^{4M+1}(S^{4(N-1)})\to j^{4M+1}(V_n)$. Thus to prove Proposition 2.1 we determine the order of the image of p^*. Let ϕ: $kO^*(X)_{(2)}\to kSpin^*(X)_{(2)}$ denote the homomorphism defined by the operation ψ^3-1 mentioned at the first of this section. Then we have the following commutative diagram;

$$\begin{array}{ccccccc} kO^{-4n}(S^0)_{(2)} & \xrightarrow{\phi} & kSpin^{-4n}(S^0)_{(2)} & \xrightarrow{\delta} & j^{-4n+1}(S^0) & \longrightarrow & 0 \\ \downarrow p^* & & \downarrow p^* & & \downarrow p^* & & \\ kO^{4M}(V_n)_{(2)} & \xrightarrow{\phi} & kSpin^{4M}(V_n)_{(2)} & \xrightarrow{\delta} & j^{4M+1}(V_n), & & \end{array} \tag{2.4}$$

where and throughout the paper the cohomology and homology groups are assumed to be reduced. Let ι_n be the generator of $kSpin^{-4n}(S^0)_{(2)}\cong Z_{(2)}$. Then $\delta(\iota_n)$ is a generator of $j^{-4n+1}(S^0)$, and thus our task is to determine the order of $p^*(\iota_n)$ in $kSpin^{4M}(V_n)_{(2)}/\mathrm{image}(\phi)$.

The group structure of $KO^*(CP^n)$ has been determined by M. Fujii [8], and that of $KO^*(HP^n)$ is classically well-known, since the Atiyah-Hirzebruch spectral sequence for $KO^*(HP^n)$ collapses. In particular both groups $KO^{4M}(CP^{2(N-1)})$ and $KO^{4M}(HP^{N-1})$ are free abelian groups of rank $N-1$. Using [8] and the elementary properties of the Adams operation, we have the following lemmas.

Lemma 2.5. *Let* $V_n=CP_{2M+1}^{2(N-1)}$. *Then we have*

(i) $kSpin^{4M}(V_n)$ *and* $kO^{4M}(V_n)$ *are identified with the following subgroups of* $KO^{4M}(CP^{2(N-1)})$ *through the collapsing map* $CP^{2(N-1)}\to V_n$: *For a suitably chosen basis* $\{X_1, X_2, \cdots, X_{N-1}\}$ *of* $KO^{4M}(CP^{2(N-1)})$,

$$kSpin^{4M}(V_n)\cong Z\{2X_{M+1}, X_{M+2}, \cdots, X_{N-1}\}$$

and

$$kO^{4M}(V_n)\cong Z\{X_{M+1}, X_{M+2}, \cdots, X_{N-1}\}.$$

(ii) $p^*(\iota_n)=X_{N-1}$ *if* n *is even, and* $p^*(\iota_n)=2X_{N-1}$ *if* n *is odd, where* p^* *and* ι_n *are those as in* (2.4).

(iii) $(\psi^3-1)(X_j)\equiv(3^{2(j-M)}-1)X_j \mod (X_{j+1}, \cdots, X_{N-1})$ *for* $M+1\leqq j\leqq N-1$.

Lemma 2.6. *Let* $V_n=HP_{N-n}^{N-1}$. *Then* $kO^{4M}(V_n)\cong kSpin^{4M}(V_n)\cong KO^{4M}(V_n)\cong Z\{Y_1, Y_2, \cdots, Y_n\}$ *for a basis* $\{Y_i\}$ *satisfying*

(i) $p^*(\iota_n)=Y_n$ *and*

(ii) $(\psi^3-1)(Y_j)=(3^{2j}-1)Y_j \mod (Y_{j+1}, \cdots, Y_n)$ *for* $1\leqq j\leqq n$.

Proof of Proposition 2.1. Let c_n be the order of $p^*(\iota_n)$ in $kSpin^{4M}(V_n)_{(2)}/\text{image}(\psi)$. Then for the case (i) and (iii), we have $\nu_2(c_n)=\nu_2(3^{2n}-1)$ by Lemmas 2.5 and 2.6 respectively. Since $\nu_2(|j_{4n-1}(S^0)|)=\nu_2(3^{2n}-1)$, $p^*: j^{-4n+1}(S^0)\to j^{4M+1}(V_n)$ is injective, and thus we have the desired results for (i) and (iii). Similarly, we have (ii), because in this case we have $\nu_2(c_n)=\nu_2(3^{2n}-1)-1$ by Lemma 2.5. Thus we have completed the proof.

§ 3. Proof of Theorem 1

In this section, we consider the classical Adams spectral sequence [1] converging to $\pi_*^s(S^0)$, $\pi_*^s(CP^n)$ and $\pi_*^s(Q_n)$ at a prime 2. Let $E_r^{s,t}(X)$ be the E_r-term of the Adams spectral sequence based on the ordinary $Z/2$-coefficient cohomology groups and converging to $\pi_*^s(X)\otimes Z_2$, where Z_2 denotes the ring of 2-adic integers, and let $d_r: E_r^{s,t}(X)\to E_r^{s+r,t+r-1}(X)$ be the differential between E_r-terms. Thus, $E_2^{s,t}(X)=\mathrm{Ext}_A^{s,t}(H^*(X; Z/2), Z/2)$, and $E_\infty^{s,t}(X)$ is the bigraded group associated with the Adams filtration of $\pi_*^s(X)$, where A denotes the mod 2 Steenrod algebra. We refer to [17] with respect to the definitions and the general properties for the Adams spectral sequences. Sometimes, we abbreviate s and t from $E_r^{s,t}(X)$ and simply denote by $E_r(X)$, and we say that $E_r^{s,t}(X)$ is in the u-stem if it satisfies $t-s=u$. For a continuous map $f: X\to Y$, $f_*: E_r^{s,t}(X)\to E_r^{s,t}(Y)$ denotes the homomorphism between the spectral sequences induced from f.

Let $i_{k,n}$: $CP^k \to CP^n$ and p_n: $CP^n \to S^{2n}$ be the inclusion map and the collapsing map respectively. If we have a relation $(p_k)_*(x)=g$ for some element $g \in E_2^{s,t}(S^{2k}) \cong E_2^{s,t-2k}(S^0)$ and $x \in E_2^{s,t}(CP^k)$, then we denote by ${}_kg$ the element $(i_{k,n})_*(x)$ for $n \geqq k$. Of course ${}_kg$ is not unique for g and k in general. We use these notations similarly for the elements of $E_2^{s,t}(Q_n)$.

Let $h_0 \in E_2^{1,1}(S^0)=Z/2$ be the generator. Then, for each $i \geqq 1$, h_0^i is a permanent cycle and represented by $2^i \in \pi_0^s(S^0)$. Let $(X_n, d)=(CP^n, 2)$ or $(\Sigma Q_n, 4)$, where $\Sigma Q_n = Q_n \wedge S^1$ denotes the reduced suspension. If we have an element ${}_nh_0^i \in E_2^{i,i+dn}(X_n)$, then it satisfies $(p_n)_*({}_nh_0^i)=h_0^i$ by definition, where $(p_n)_*$: $E_2^{i,i+dn}(X_n) \to E_2^{i,i+dn}(S^{dn})=E_2^{i,i}(S^0)$. Thus, if ${}_nh_0^i$ is a permanent cycle furthermore, then we have an element y of $\pi_{dn}^s(X_n)$ with the property that $F_2(y)=\nu_2(h(y))=i$, where $F_2(y)$ denotes the Adams filtration of y and $h(y)$ denotes the image of the stable Hurewicz homomorphism $h: \pi_{dn}^s(X_n) \to H_{dn}(X_n; Z) = Z$. Hence Theorem 1 is equivalent to the existence of an element ${}_nh_0^i \in E_\infty^{i,i+4n-1}(Q_n)$ for $i=\nu_2(a(n-1)\cdot(2n-1)!)$ by Theorem 3.10 which appears later on, and we will show the existence of such an element.

Let $\alpha(n)$ denote the number of 1 in the diadic expansion of n. We remark that $\nu_2(n!)=n-\alpha(n)$. Then Crabb-Knapp's theorem mentioned in §1 can be stated as follows:

Theorem 3.1. (M.C. Crabb-K. Knapp [6]) *There exists an element* ${}_nh_0^{n-\alpha(n)} \in E_\infty^{n-\alpha(n),3n-\alpha(n)}(CP^n)$.

In §1 we defined the element $\alpha_{2n/i}$ of $\mathrm{Im}(J)_n \subset \pi_{4n-1}^s(S^0)$. For $1 \leqq i \leqq 3$ we use the same notation $\alpha_{2n/i}$ for the element of $E_2(S^0)$ represented by $\alpha_{2n/i}$. Thus the element ${}_1\alpha_{2n/i} \in E_2(CP^k)$ for $k \geqq 1$ is the image of $\alpha_{2n/i} \in E_2(S^0)$ through i_*: $E_2^{s,t}(S^0)=E_2^{s,t+2}(S^2) \to E_2^{s,t+2}(CP^k)$. It is known that $\alpha_{2n/i}$ of $E_2(S^0)$ are in neighborhood of the vanishing line (cf. [13]), which we will describe below.

The vanishing theorem for $E_2(S^0)$ is given as follows:

Theorem 3.2. (J.F. Adams [3]) $E_2^{s,t}(S^0)=0$ *for* $0<t-s<q(s)$, *where* $q(s)=2s-\varepsilon$ *for* $\varepsilon=1, 2$ *and* 3 *if* $s \equiv 0, 1 \bmod 4$, $s \equiv 2 \bmod 4$ *and* $s \equiv 3 \bmod 4$ *respectively.*

The vanishing line is formed by the equality $t-s=q(s)$ when we treat the spectral sequence by writing each $E_2^{s,t}(S^0)$ in the plane with the $(t-s, s)$ coordinate. Then, by the periodicity theorem [3] and the calculations for the lower stems (cf. [19]), the structure of $E_2(S^0)$ is well investigated for some range near the vanishing line (cf. [17; Chapter 3]). Thus we have the following lemma which is a direct consequence of it.

Lemma 3.3. *Let* $m \geqq 1$ *and* $1 \leqq j \leqq 3$, *and let* $h_1 \in E_2^{1,2}(S^0)=Z/2$ *be the generator.*
(i) *Concerning the* $(8m-1)$*-stem, we have* $E_2^{4m+1-j,12m-j}(S^0)=Z/2\{\alpha_{4m/j}\}$ *and*

$E_2^{i,i+8m-1}(S^0)=0$ *for* $i\geqq 4m+1$, *and* $\alpha_{4m/j}$ *is not divisible by* h_1.

(ii) *Concerning the* $(8m+1)$*-stem, we have an element* α_{4m+1} *such that* $E_2^{4m+1,12m+2}(S^0)=Z/2\{\alpha_{4m+1}\}$, *and we have* $E_2^{i,i+8m+1}(S^0)=0$ *for* $i\geqq 4m+2$

(iii) *Concerning the* $(8m+2)$*-stem,* $E_2^{4m+2,12m+4}(S^0)=Z/2\{h_1\alpha_{4m+1}\}$ *and* $E_2^{i,i+8m+2}(S^0)=0$ *for* $i\geqq 4m+3$.

(iv) *Concerning the* $(8m+3)$*-stem, we have* $E_2^{4m+4-j,12m+7-j}(S^0)=Z/2\{\alpha_{4m+2/j}\}$ *and* $E_2^{i,i+8m+3}(S^0)=0$ *for* $i\geqq 4m+4$, $\alpha_{4m+2}=h_1^2\alpha_{4m+1}$, *and* $\alpha_{4m+2/j}$ *is not divisible by* h_1 *for* $j=2$ *and* 3.

(v) *Concerning other stems, we have* $E_2^{i,i+8m}(S^0)=0$ *for* $i\geqq 4m$, $E_2^{i,i+8m-2}(S^0)=0$ *for* $i\geqq 4m-1$, *and* $E_2^{i,i+8m-3}(S^0)=E_2^{i,i+8m-4}(S^0)=0$ *for* $i\geqq 4m-2$.

The cofibering $CP^{k-1}\xrightarrow{i}CP^k\xrightarrow{p}S^{2k}$ induces a long exact sequence

$$(3.4)\qquad \cdots\longrightarrow E_2^{s,t}(CP^{k-1})\xrightarrow{i_*}E_2^{s,t}(CP^k)\xrightarrow{p_*}E_2^{s,t}(S^{2k})\xrightarrow{\delta} E_2^{s+1,t}(CP^{k-1})\longrightarrow\cdots,$$

where δ is the connecting homomorphism (cf. [17; Chapter 2]). δ is compatible with the boundary homomorphisms, and $\delta\colon E_\infty^{s,t-1}(S^{2k-1})\cong E_\infty^{s,t}(S^{2k})\to E_\infty^{s+1,t}(CP^{k-1})$ is associated with $(\phi_k)_*\colon \pi_*^s(S^{2k-1})\to\pi_*^s(CP^{k-1})$, where ϕ_k is the attaching map of the top cell of CP^k. In particular, since the stable homotopy class of the composition $S^{4k-1}\xrightarrow{\phi_{2k}}CP^{2k-1}\xrightarrow{p}S^{4k-2}$ represents the nontrivial class $\eta\in\pi_1^s(S^0)\cong Z/2$, we have the following lemma.

Lemma 3.5. *The composition* $p_*\delta\colon E_2^{s,t-4k}(S^0)\cong E_2^{s,t}(S^{4k})\to E_2^{s+1,t}(CP^{2k-1})\to E_2^{s+1,t}(S^{4k-2})\cong E_2^{s+1,t-4k+2}(S^0)$ *is the multiplication by* h_1.

By Lemmas 3.3 and 3.5 and Theorem 3.1 we have the following

Lemma 3.6. (i) *For* $4\leqq k\leqq 4m$, $E_2^{i,i+8m+1}(CP^k)=0$ *if* $i\geqq 4m+1$, $=Z/2\{{}_1\alpha_{4m}\}$ *if* $i=4m$, *and* $=Z/2\{{}_1\alpha_{4m/2}\}$ *if* $i=4m-1$.

(ii) *For* $m\geqq 1$ *and* $2\leqq k\leqq 4m+2$, $E_2^{i,i+8m+5}(CP^k)=0$ *if* $i\geqq 4m+3$, $=Z/2\{{}_1\alpha_{4m+2/2}\}$ *if* $i=4m+2$, *and* $=Z/2\{{}_1\alpha_{4m+2/3}\}$ *if* $i=4m+1$.

(iii) *For* $i\geqq 2m+2\geqq 4$, $E_2^{i,i+4m}(CP^k)=0$ *if* $1\leqq k\leqq 2m-1$, *and* $E_2^{i,i+4m}(CP^k)=E_\infty^{i,i+4m}(CP^k)=Z/2\{{}_{2m}h_0^i\}$ *if* $k\geqq 2m$.

Proof. (i) Assume that $i\geqq 4m-1\geqq 3$. For $5\leqq k\leqq 4m$, since $E_2^{i,i+8m+1}(S^{2k})=0$ and $E_2^{i-1,i+8m+1}(S^{2k})=0$ by Lemma 3.3, we have an isomorphism $(i_{4,k})_*\colon E_2^{i,i+8m+1}(CP^4)\to E_2^{i,i+8m+1}(CP^k)$ by (3.4). Thus we may consider the case for CP^4. First, to investigate CP^2, we consider (3.4) for $k=2$. Then, since $\delta\colon E_2^{s,t}(S^4)\to E_2^{s+1,t}(S^2)$ multiplies each element by h_1 by Lemma 3.5, we have $\delta=0$ on $E_2^{i-1,i+8m+1}(S^4)$ by Lemma 3.3 (i). Thus we have an isomorphism $(i_{1,2})_*\colon E_2^{i,i+8m+1}(S^2)\to E_2^{i,i+8m+1}(CP^2)$ from (3.4), because $E_2^{i,i+8m+1}(S^4)=0$ by Lemma 3.3 (v). We also have $E_2^{i+1,i+8m+1}(CP^2)=0$, because $E_2^{i+1,i+8m+1}(S^2)=E_2^{i+1,i+8m+1}(S^4)$

$=0$ by Lemma 3.3 (v). Next, consider (3.4) for $k=3$ to investigate CP^3. Since $E_2^{i-1,i+8m+1}(S^6)=0$ by Lemma 3.3 (v), we have a short exact sequence

$$0 \longrightarrow E_2^{i,i+8m+1}(CP^2) \xrightarrow{i_*} E_2^{i,i+8m+1}(CP^3) \xrightarrow{p_*} E_2^{i,i+8m+1}(S^6) \longrightarrow 0 .$$

By Lemma 3.3 (iv), $E_2^{4m-1,12m}(S^6)=Z/2\{\alpha_{4m-2}\}$ and $E_2^{j,j+8m+1}(S^6)=0$ for $j\geqq 4m$. Now, consider (3.4) for $k=4$ as the last step. By Lemma 3.5, the composition $p_*\delta$: $E_2^{s-1,t}(S^8)\to E_2^{s,t}(CP^3)\to E_2^{s,t}(S^6)$ multiplies each element by h_1. Therefore we have $p_*\delta(h_1\alpha_{4m-3})=h_1^2\alpha_{4m-3}=\alpha_{4m-2}$ by Lemma 3.3 (iv). Then, using the above short exact sequence and (3.4), we have an isomorphism i_*: $E_2^{i,i+8m+1}(CP^2)\to E_2^{i,i+8m+1}(CP^4)$, because $E_2^{4m-2,12m}(S^8)=Z/2\{h_1\alpha_{4m-3}\}$ and $E_2^{j,j+8m+2}(S^8)=0$ for $j\geqq 4m-1$ by Lemma 3.3 (iii). Thus we have an isomorphism i_*: $E_2^{i,i+8m+1}(S^2)\to E_2^{i,i+8m+1}(CP^4)$, and we have the desired result by Lemma 3.3 (i).

(ii) Assume that $i\geqq 4m+1\geqq 5$. Then, considering similarly as (i), we have an isomorphism i_*: $E_2^{i,i+8m+5}(CP^2)\to E_2^{i,i+8m+5}(CP^k)$ for $3\leqq k\leqq 4m+2$. Thus, we may consider the case for CP^2, and to prove this case we consider (3.4) for $k=2$. Then, concerning the boundary δ in the sequence, we have $\delta(\alpha_{4m+1})=h_1\alpha_{4m+1}\neq 0$ and $\delta(h_1\alpha_{4m+1})=\alpha_{4m+2}$ by Lemma 3.3 (iii), (iv) and Lemma 3.5, where $\alpha_{4m+1}\in E_2^{4m+1,12m+6}(S^4)$. Since $\alpha_{4m+2/2}$ and $\alpha_{4m+2/3}$ are not divisible by h_1 by Lemma 3.3 (iv), we have an isomorphism i_*: $E_2^{i,i+8m+5}(S^2)/(\alpha_{4m+2})\to E_2^{i,i+8m+5}(CP^2)$. Thus we have the desired result by Lemma 3.3 (iv).

(iii) Assume that $i\geqq 2m+2\geqq 4$. Then we have that i_*: $E_2^{i,i+4m}(CP^{k-1})\to E_2^{i,i+4m}(CP^k)$ is an isomorphism if $k\neq 2m$ and $k\geqq 2$, by using Lemma 3.3 similarly as in (i) and (ii). Thus we have the first half of the desired result, because $E_2^{i,i+4m}(CP^1)=0$. Then p_*: $E_2^{i,i+4m}(CP^{2m})\to E_2^{i,i+4m}(S^{4m})\cong Z/2\{h_0^i\}$ is injective. But by Theorem 3.1 we have an element ${}_{2m}h_0^i$ which is a permanent cycle mapped by p_* to h_0^i by definition. Thus we have $E_2^{i,i+4m}(CP^{2m})=E_\infty^{i,i+4m}(CP^{2m})=Z/2\{{}_{2m}h_0^i\}$. Since $(i_{2m,k})_*$: $E_r^{i,i+4m}(CP^{2m})\to E_r^{i,i+4m}(CP^k)$ is an isomorphism for $k\geqq 2m$ and $r\geqq 2$, we have the latter half of the desired result. Q.E.D.

Proposition 3.7. *Let $m\geqq 1$. Then we have the following:*

(i) ${}_1\alpha_{4m}\neq 0$ *in* $E_{\alpha(m)}^{4m,12m+1}(CP^{4m+1})$,

(ii) ${}_1\alpha_{4m+2/2}\neq 0$ *in* $E_{\alpha(m)+1}^{4m+2,12m+7}(CP^{4m+3})$,

where $\alpha(m)$ denotes the number of 1 *in the diadic expansion of m.*

Proof. To treat both cases (i) and (ii), we denote by β the element ${}_1\alpha_{4m}$ for (i) and ${}_1\alpha_{4m+2/2}$ for (ii) respectively, and by l the integer $4m+1$ for (i) and $4m+3$ for (ii) respectively. First we must show that $\beta\neq 0$ in $E_2^{l-1,3l-1}(CP^l)$. By (3.4), we have the exact sequence

$$E_2^{s,t}(CP^l) \xrightarrow{p_*} E_2^{s,t}(S^{2l}) \xrightarrow{\delta} E_2^{s+1,t}(CP^{l-1}) \xrightarrow{i_*} E_2^{s+1,t}(CP^l)$$

for $s=l-2$ and $t=3l-2$. By Theorem 3.1 we have an element ${}_lh_0^{l-\alpha(l)}\in$

$E_2^{l-\alpha(l),3l-\alpha(l)}(CP^l)$, and so $h_0^{l-2}\in \mathrm{Im}\, p_*$ because $\alpha(l)\geqq 2$. Thus i_* is injective, and we have $\beta\neq 0$ in $E_2(CP^l)$ by Lemma 3.6. Since $d_r(\beta)=0$ for any $r\geqq 2$ by Lemma 3.6 (iii), we have $\beta\in E_{\alpha(l)-1}^{l-1,3l-1}(CP^l)$. Assume that $\beta=0$ in $E_{\alpha(l)-1}(CP^l)$ for some l with $\alpha(l)\geqq 4$. Then there is an integer r with $2\leqq r\leqq \alpha(l)-2$ and an element $y\in E_r^{l-r-1,3l-r-1}(CP^l)$ which satisfy that $d_r(y)=\beta\neq 0$. By Theorem 3.1 we have an element ${}_lh_0^{l-r-1}\in E_\infty(CP^l)$, and we have that $i_*\colon E_r^{i,i+2l-1}(CP^{l-1})\to E_r^{i,i+2l-1}(CP^l)$ is an isomorphism for $i\geqq l-\alpha(l)+r-1$. Then we have that y or $y-{}_lh_0^{l-r-1}$ is in the image of $i_*\colon E_r(CP^{l-1})\to E_r(CP^l)$, that is, it is equal to $i_*(z)$ for some element $z\in E_r^{l-r-1,3l-r-1}(CP^{l-1})$. Then we have $i_*d_r(z)=d_r(y)=\beta$. Since $E_r^{l-1,3l-2}(CP^{l-1})=Z/2\{\beta'\}$ by Lemma 3.6 (i) and (ii), we have $d_r(z)=\beta'$, where β' denotes the element ${}_1\alpha_{l-1}$ or ${}_1\alpha_{l-1/2}$ of $E_{\alpha(l)-1}(CP^{l-1})$ which satisfies $i_*(\beta')=\beta$. This contradicts Proposition 2.1, because β' is represented by $i_*(\alpha)\neq 0$ of $\pi_*^s(CP^{l-1})$, where $\alpha=\alpha_{4m}\in \mathrm{Im}(J)_{2m}$ for (i) and $\alpha=\alpha_{4m+2/2}\in \mathrm{Im}(J)_{2m+1}$ for (ii). Thus we have $\beta\neq 0$ in $E_{\alpha(l)-1}(CP^l)$, and we have completed the proof. Q. E. D.

Proposition 3.8. *Let k be an odd integer with $k\geqq 3$. Then we have an element ${}_kh_0^{k-\alpha(k)-1}\in E_2^{k-\alpha(k)-1,3k-\alpha(k)-1}(CP^k)$ satisfying the following property:*

(i) $h_0\cdot {}_kh_0^{k-\alpha(k)-1}\in E_\infty(CP^k)$.

(ii) ${}_kh_0^{k-\alpha(k)-1}\in E_{\alpha(k)}(CP^k)$ *and* $d_{\alpha(k)}({}_kh_0^{k-\alpha(k)-1})={}_1\alpha_{k-1}$ *if* $k\equiv 1 \bmod 4$ *and* $={}_1\alpha_{k-1/2}$ *if* $k\equiv 3 \bmod 4$.

Proof. We prove by induction on $\alpha(k)$. First, we prove the case $\alpha(k)=2$. Consider the exact sequence (3.4), and let $\delta\colon E_2^{s,t-2k}(S^0)\cong E_2^{s,t}(S^{2k})\to E_2^{s+1,t}(CP^{k-1})$ be the boundary homomorphism in it. If there is not any element ${}_kh_0^{k-3}$ in $E_2^{k-3,3k-3}(CP^k)$, then by Lemma 3.6 we have $\delta(h_0^{k-3})={}_1\alpha_{4m/2}$ and ${}_1\alpha_{4m+2/3}$ if $k=4m+1$ and $4m+3$ respectively. Then $\delta(h_0^{k-2})\neq 0$ by Lemma 3.6. But this contradicts Theorem 3.1, and thus we have an element ${}_kh_0^{k-3}$ satisfying (i). Then this element also satisfies (ii). In fact, if it does not satisfy (ii), then it is a permanent cycle by Lemma 3.6 and it contradicts Theorem 3.1. Now we assume that $\alpha(k)=u\geqq 3$ and that we have a desired element for the case $\alpha(k)\leqq u-1$. Let t be the integer such that $2^t\leqq k\leqq 2^{t+1}-1$, and $l=k-2^t$. Then $\alpha(l)=\alpha(k)-1$, and so by the inductive hypothesis we have an element ${}_lh_0^{l-\alpha(k)}$ of $E_{\alpha(k)-1}(CP^l)$ with the property (i) and (ii) for l. Let $s=2^t$. Then by Theorem 3.1 we have ${}_sh_0^{s-1}\in E_\infty(CP^s)$. Let X be the product ${}_sh_0^{s-1}\cdot {}_lh_0^{l-\alpha(k)}\in E_{\alpha(k)-1}(CP^k)$ by the H-space structure of CP^∞. Then, X is an element ${}_kh_0^{k-\alpha(k)-1}$ since the binomial coefficient $\binom{k}{s}\equiv 1 \bmod 2$, and we have $h_0\cdot X\in E_\infty(CP^k)$. If $d_{\alpha(k)-1}(X)\neq 0$, then by Lemma 3.6 it is equal to ${}_1\alpha_{k-1/2}$ if $k\equiv 1 \bmod 4$ and ${}_1\alpha_{k-1/3}$ if $k\equiv 3 \bmod 4$. Then $d_{\alpha(k)-1}(h_0\cdot X)\neq 0$ by Proposition 3.7, but this cannot occur. Thus $X\in E_{\alpha(k)}(CP^k)$ and satisfies (i). If X does not satisfy (ii), then we have $X\in E_\infty(CP^k)$. In fact, since, for $i\geqq k$, $E_2^{i,i+2k-1}(CP^{k-1})=0$ by Lemma 3.6 and

$i_*: E_2^{i,i+2k-1}(CP^{k-1}) \to E_2^{i,i+2k-1}(CP^k)$ is surjective, we have $d_r X \in E_r^{j+r,j+r+2k-1}(CP^k)=0$ for $r \geqq \alpha(k)+1$, where $j=k-\alpha(k)-1$. Then we have an element $x \in \pi_{2k}^s(CP^k)$ such that $h(x)=2^{k-\alpha(k)-1}c$ for some odd integer c, where $h: \pi_{2k}^s(CP^k) \to H_{2k}(CP^k; Z)=Z$ is the stable Hurewicz homomorphism. But this contradicts the well known fact that the image of h is generated by $k!$ and $\nu_2(k!)=k-\alpha(k)$. Thus X satisfies (ii), and we have completed the proof.

Q. E. D.

Next we consider the mod 2 Adams spectral sequence for $\pi_*^s(Q_n)$. The following lemma can be proved similarly as Lemma 3.6.

Lemma 3.9. (i) *For an odd integer* $n \geqq 3$, $E_2^{i,i+4n-2}(Q_n)=0$ *if* $i \geqq 2n-1$, *and* $=Z/2\{_1\alpha_{2(n-1)}\}$ *if* $i=2(n-1)$.

(ii) *For an even integer* $n \geqq 2$, $E_2^{i,i+4n-2}(Q_n)=0$ *if* $i \geqq 2n$, *and* $=Z/2\{_1\alpha_{2(n-1)}\}$ *if* $i=2n-1$.

In [9] B. Harris shows that the homotopy group $\pi_{4n-2}(Sp(n-1))$ of the symplectic group is isomorphic to the cyclic group of order $a(n-1)\cdot(2n-1)!$, which is a consequence of a result of R. Bott [5]. Hence the degree of q_*: $\pi_{4n-1}(Sp(n)) \to \pi_{4n-1}(Sp(n)/Sp(n-1)) \cong Z$ is equal to $a(n-1)\cdot(2n-1)!$, where q is the canonical quotient map. Let $\theta: \pi_{4n-1}(Sp(n)) \to \pi_{4n-1}^s(Q_n)$ be the homomorphism given by I. M. James (cf. [11]), and $p: Q_n \to Q_n/Q_{n-1} \approx S^{4n-1}$ denote the collapsing map. Then θ is an isomorphism onto the free part of $\pi_{4n-1}^s(Q_n)$, and q_* is identified with the composition $p_* \circ \theta: \pi_{4n-1}(Sp(n)) \to \pi_{4n-1}(S^{4n-1})$. Thus we have the following theorem which is also shown by several authors (cf. [21], [14], [15]).

Theorem 3.10. *Let* $h: \pi_{4n-1}^s(Q_n) \to H_{4n-1}(Q_n; Z)=Z$ *be the stable Hurewicz homomorphism. Then an element* x *of* $\pi_{4n-1}^s(Q_n)$ *is a generator of the free part if and only if it satisfies* $h(x)=a(n-1)\cdot(2n-1)!$, *where* $a(i)=1$ *for even* i *and* $=2$ *for odd* i.

Let $q: \Sigma CP_+^\infty \to Q_\infty$ be the map obtained by the symplectification (cf. [20]), and $q: \Sigma CP^{2n-1} \to Q_n$ the restriction of it by the cellular approximation. Then $q^*: H^*(Q_n; Z) \to H^{*-1}(CP^{2n-1}; Z)$ is the map of degree 2, that is, it maps a base of $H^{4j-1}(Q_n; Z) \cong Z$ to $\pm 2b$ for a base $b \in H^{4j-2}(CP^{2n-1}; Z) \cong Z$ for each $1 \leqq j \leqq n$. Thus q raises the mod 2 Adams filtration by 1, and we have a map $q_*: E_r^{s,t}(CP^{2n-1}) \to E_r^{s+1,t+2}(Q_n)$ between the spectral sequences.

For $t=2n-1-\alpha(2n-1)=2n-\alpha(n-1)$ and $n \geqq 3$, let $_{2n-1}h_0^{t-1} \in E_{\alpha(n-1)+1}^{t-1,t+4n-3}(CP^{2n-1})$ be the element in Proposition 3.8. Then $q_*(_{2n-1}h_0^{t-1})$ is an element $_nh_0^t \in E_{\alpha(n-1)+1}^{t,t+4n-1}(Q_n)$, and we have

Lemma 3.11. *Let* $t=2n-\alpha(n-1)$. *Then we have the following.*

(i) *For an odd integer* $n \geqq 3$, ${}_n h_0^t \in E_\infty(Q_n)$.
(ii) *For an even integer* $n \geqq 2$, $d_{\alpha(n-1)+1}({}_n h_0^t) = {}_1\alpha_{2(n-1)}$.

Proof. For an odd $n \geqq 3$, since $d_r({}_n h_0^t) \in E_r^{t+r, t+r+4n-2}(Q_n) = 0$ for $r \geqq \alpha(n-1)+1$ by Lemma 3.9 (i), we have (i). For even n, if (ii) is not true, then $d_{\alpha(n-1)+1}({}_n h_0^t) = 0$ and ${}_n h_0^t \in E_\infty(Q_n)$ by Lemma 3.9 (ii). Then there exists an element x of $\pi_{4n-1}^s(Q_n)$ which satisfies $2^t | h(x)$ but $2^{t+1} \nmid h(x)$, where h is the stable Hurewicz homomorphism. But it contradicts Theorem 3.10, because $t = \nu_2((2n-1)!)$. Thus we have the desired result. Q.E.D.

Proof of Theorem 1. We assume $n \geqq 2$, since the assertion is clear for $n=1$. Let $y_n \in E_2(Q_n)$ denote ${}_n h_0^t$ for odd $n \geqq 3$ and ${}_n h_0^{t+1}$ for even $n \geqq 2$, where $t = 2n - \alpha(n-1)$. Then y_n is an element of $E_\infty(Q_n)$ by Lemmas 3.9 and 3.11. Let $x_n \in \pi_{4n-1}^s(Q_n)$ be an element which represents y_n. Then the mod 2 Adams filtration of x_n is equal to $\nu_2(a(n-1)\cdot(2n-1)!)$, and x_n is a generator of the free part by Theorem 3.10. Thus we have the desired result.

Remark 3.8. For even n, $q_*(\sigma^{2n-1})$ is a generator of the free part of $\pi_{4n-1}^s(Q_n)$ by Theorem 3.10, where $\sigma^{2n-1} \in \pi_{4n-2}^s(CP^\infty)$ is the element mentioned in §1. Thus for even n Theorem 1 is a corollary of Crabb-Knapp's theorem [6].

§4. Proof of Theorems 2 and 3

Theorem 2 (i)(a) and (ii)(b) and Theorem 3 (i) follow from Proposition 2.1. Theorem 2 (ii)(a) follows from Lemma 3.6 (ii). The equation $i_*(\alpha_{4m}) = 0$ and $i_*(\alpha_{4m+2/2}) = 0$ in Theorem 2 (i)(b) and (ii)(c) respectively follow from Proposition 3.8 (ii). Also the equation $i_*(\alpha_{4m+2}) = 0$ in Theorem 3 (ii) follows from Lemma 3.11 (ii).

Let W_n be the stunted projective space CP^n/S^2 or Q_n/S^3, and $u = 2n$ or $4n-1$ for the respective case. When $W_n = CP^n/S^2$ (resp. Q_n/S^3), we denote by A_n (resp. B_n) the value $\nu_2(|\mathrm{Coker}\, h|)$, where $|\mathrm{Coker}\, h|$ is the order of the cokernel of the stable Hurewicz homomorphism $h: \pi_u^s(W_n) \to H_u(W_n; Z)$. By [10; Theorem II and Corollary III], we have

Proposition 4.1. (i) $A_{2m+1} = \nu_2((2m+1)!) - 1$.
(ii) $B_{2m+1} = \nu_2((4m+1)!)$, *and* $B_{2m} = \nu_2((4m-1)!) - 2$.

Consider the following diagram:

$$
\begin{array}{ccccccc}
 & & \pi_{2k}^s(S^{2k}) & & & & \\
 & & \uparrow p_* & & & & \\
\pi_{2k}^s(CP^k) & \xrightarrow{q_*} & \pi_{2k}^s(CP^k/S^2) & \xrightarrow{\partial} & \pi_{2k-1}^s(S^2) & \xrightarrow{i_*} & \pi_{2k-1}^s(CP^k) \\
 & & \uparrow i_* & & \| & & \uparrow i_* \\
 & & \pi_{2k}^s(CP^{k-1}/S^2) & \xrightarrow{\partial} & \pi_{2k-1}^s(S^2) & \xrightarrow{i_*} & \pi_{2k-1}^s(CP^{k-1}),
\end{array}
$$

where the horizontal and vertical sequences are derived from the cofiber sequences. From this diagram and Proposition 4.1, we have

(4.2) $i_*\pi_{2k}^s(CP^{k-1}/S^2)$ is the torsion part of $\pi_{2k}^s(CP^k/S^2)$, and

(4.3) $q_*x=2x_1+w$, where x and x_1 are the generators of the free parts of $\pi_{2k}^s(CP^k)$ and $\pi_{2k}^s(CP^k/S^2)$ respectively and w is some torsion element of $\pi_{2k}^s(CP^k/S^2)$.

We complete the proof of Theorem 2 by the following lemma.

Lemma 4.4. *$i_*(\alpha_{4n/2})\neq 0$ in $\pi_{8n+1}^s(CP^{4n+1})$ and $i_*(\alpha_{4n+2/3})\neq 0$ in $\pi_{8n+5}^s(CP^{4n+3})$.*

Proof. Let γ denote $\alpha_{4n/2}$ or $\alpha_{4n+2/3}$ for $k=4n+1$ or $4n+3$ respectively, and assume that $i_*\gamma=0$ in $\pi_{2k-1}^s(CP^k)$. Then in the above diagram we have an element $y\in\pi_{2k}^s(CP^k/S^2)$ satisfying $\partial y=\gamma$. Then we can put $y=tx_1+v$ for some integer t and some torsion element v. By (4.3) we have $2\gamma=2\partial(tx_1+v)=\partial(tq_*(x)+2v-tw)=\partial(2v-tw)$, and so $i_*(2\gamma)=0$ in $\pi_{2k-1}^s(CP^{k-1})$ by (4.2). But 2γ is the non zero element of $\mathrm{Im}(J)$, and $i_*(2\gamma)\neq 0$ by Proposition 2.1. Thus we have a contradiction, and we conclude $i_*\gamma\neq 0$. Q.E.D.

Similarly, by using Propositions 2.1 and 4.1 (ii), and by considering the analogous diagram for Q_n, we have $i_*(\alpha_{4m})\neq 0$ in $\pi_{8m+2}^s(Q_{2m+1})$ and $i_*(\alpha_{4m+2/3})\neq 0$ in $\pi_{8m+6}^s(Q_{2m+2})$. Thus we complete the proof of Theorem 3.

Remark. We have not determined whether $i_*(\alpha_{4m+2/2})=0$ or $\neq 0$ in $\pi_{8m+6}^s(Q_{2m+2})$. By using [10; Theorem I], we can prove that $i_*(\alpha_{4m+2/2}+\beta)=0$ for an element β which is not in $\mathrm{Im}(J)$ and the order of which is less than or equal to 4.

References

[1] Adams, J.F., On the structure and applications of the Steenrod algebra, *Comm. Math. Helv.*, **32** (1958), 180-214.
[2] ———, On the groups $J(X)$—IV, *Topology*, **5** (1966), 21-71.
[3] ———, A periodicity theorem in homological algebra, *Proc. Camb. Phil. Soc.*, **62** (1966), 365-377.
[4] Atiyah, M.F., Thom complexes, *Proc. London Math. Soc.*, **11** (1961), 291-310.
[5] Bott, R., The space of loops on a Lie group, *Michigan Math. J.*, **5** (1958), 35-61.
[6] Crabb, M.C. and Knapp, K., Vector bundles of maximal codegree, *Math. Z.*, **193** (1986), 285-296.
[7] ———, James numbers, *Math. Ann.*, **282** (1988), 395-422.
[8] Fujii, M., K_0-groups of projective spaces, *Osaka J. Math.*, **4** (1967), 141-149.
[9] Harris, B., Some calculations of homotopy groups of symmetric spaces, *Trans. Amer. Math. Soc.*, **106** (1963), 174-184.
[10] Imaoka, M. and Morisugi, K., On the stable Hurewicz image of some stunted projective spaces, II, *Publ. RIMS Kyoto Univ.*, **20** (1984), 853-866.

[11] James, I.M., *The topology of Stiefel manifolds*, London Math. Soc. Lecture note series **24**, 1976.

[12] Mahowald, M., The metastable homotopy of S^n, *Mem. Amer. Math. Soc.*, **72**(1967).

[13] ———, The order of the image of the *J*-homomorphism, *Bull. Amer. Math. Soc.*, **76** (1970), 1310-1313.

[14] Morisugi, K., Stable self maps of the quaternionic (quasi-)projective space, *Publ. RIMS Kyoto Univ.*, **20** (1984), 971-976.

[15] Mukai, J., The order of the attaching class in the suspended quaternionic quasi-projective space, *Publ. RIMS Kyoto Univ.*, **20** (1984), 717-725.

[16] Ōshima, H., Remarks on the *j*-codegree of vector bundles, *preprint*.

[17] Ravenel, D.C., *Complex cobordism and stable homotopy groups of spheres*, Mathematics 121, Academic Press, 1986.

[18] Segal, D.M., On the stable homotopy of quaternionic and complex projective spaces, *Proc. Amer. Math. Soc.*, **25** (1970), 838-841.

[19] Tangora, M.C., On the cohomology of Steenrod algebra, *Math. Z.*, **116** (1970), 18-64.

[20] Toda, H. and Kozima, K., The symplectic Lazard ring, *J. Math. Kyoto Univ.*, **22** (1982), 131-153.

[21] Walker, G., Estimates for the complex and quaternionic James numbers, *Quart. J. Math. Oxford*, (2) **32** (1981), 467-489.