

Whispering Gallery Waves in a Neighborhood of a Higher Order Zero of the Curvature of the Boundary

By

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§ 1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be a domain with a smooth boundary Γ . Suppose Ω is a media where some wave propagates with a speed, say 1 for simplicity. When Ω is strictly convex, there is a high frequency wave well known by the name of whispering gallery waves which propagates along Γ . Mathematically, it can be described as an asymptotic solution u to the reduced wave equation

$$(1.1) \quad \Delta u + \omega^2 u = 0 \quad \text{in } \Omega$$

with the Dirichlet boundary condition

$$(1.2) \quad u|_{\Gamma} = 0$$

whose energy is concentrated near Γ .

Here $\omega \gg 1$ is the frequency of the wave, and u satisfies (1.1) asymptotically as $\omega \rightarrow \infty$. By using the boundary layer method, Babich and Kirpichnikova [1] constructed an asymptotic solution u which describes the whispering gallery wave.

However, if we merely assume that Ω is convex, their asymptotic solution is ineffective in a neighborhood of a point $P \in \Gamma$ at which the curvature K_0 of Γ is zero, i. e. one of its first order derivatives blows up at P .

On the other hand, Popov [4] considered the case $K_0(P) = 0$, $K'_0(P) \neq 0$. He constructed an asymptotic solution u to (1.1) and (1.2) in a relatively open neighborhood $G \subset \bar{\Omega}$ of P with the following property. Namely, the dominant term of u converges in the L^2 sense to that of whispering gallery wave constructed in [1] as $\omega \rightarrow \infty$ in a relatively open neighborhood $G_+ \subset G$ of $\Gamma_+ =$

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$\{Q \in \Gamma \cap G; K_0(Q) > 0\}$. (In §2 we will give the precise definition of G_+ .) Moreover, he showed that, by appropriate coordinate transformation and scaling transformation, the dominant term of u is determined by a certain Dirichlet boundary value problem for a certain type of Schrödinger equation with an unbounded potential, and constructed its formal solution on the image of G_+ under the composite mapping of these transformations.

In this paper, we have extended the results of Popov under the condition

$$(1.3) \quad \Gamma_+ \neq \emptyset, \quad K_0^{(j)}(P) = 0 \quad (j < \kappa), \quad K_0^{(\kappa)}(P) \neq 0$$

for some positive integer κ . Moreover, we have improved his result in the following two points. (1) The dominant term ψ (cf. (2.12) below) of the asymptotic solution u to (1.1) and (1.2) is smooth up to the boundary of G_+ . (2) ψ converges to the dominant term ψ_0 (cf. (2.9) below) of the asymptotic solution of the whispering gallery wave constructed in [1] with respect to a higher regularity norm as $\omega \rightarrow \infty$ on G_+ .

Although we have restricted our study to improving Popov's result, our analysis is a starting point for constructing a local paramatrix which describes the gliding wave generated by an incident wave grazing at an inflection point at the boundary.

Recently, we have noticed that Babich and Smyshlyaev [2] obtained the same result for the special case $\kappa=1$. However, their method is quite different from ours.

The rest of this paper is organized as follows. After some preparations, we state the results of [1], [4] and our results in §2. §3 and §4 are devoted to the proof of our main theorem.

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§2. Some Preparations and Results

Let s be the arclength of Γ measured from a fixed point P_0 of Γ , and Γ is described by $\vec{r} = \vec{r}(s)$. Let $\vec{n}(s)$ be the unit normal vector at $\vec{r}(s)$ pointing inward to Ω and q be the length measured along $\vec{n}(s)$. Then we can take (s, q) as a coordinate in a relatively open neighborhood $G \subset \bar{\Omega}$ of Γ . Thus each point $M = M(x, y) \in G$ with a cartesian coordinate (x, y) is represented in the form

$$M = M(s, q) = \vec{r}(s) + q\vec{n}(s).$$

When the wave propagates with speed $c(x, y)$, the corresponding Dirichlet boundary value problem becomes

$$(2.1) \quad \begin{cases} \Delta u + \omega^2 c^{-2}(x, y)u = 0 & \text{in } \Omega \\ u|_{\Gamma} = 0. \end{cases}$$

Now by taking the center of curvature in Ω , let $K_0(s)$ be the curvature of Γ at s . Then we define the effective curvature $K(s)$ by

$$(2.2) \quad K(s) = K_0(s) + c^{-1}(s, 0) \cdot \partial_q c(s, q)|_{q=0}.$$

Clearly, $K(s) = K_0(s)$ if the speed $c(s, q)$ of the wave is constant. For the wave with non constant speed, we have to replace the condition (1.3) by

$$(2.3) \quad \begin{aligned} \Gamma_+ = \{s \in \Gamma \cap G; K(s) > 0\} \neq \emptyset, \quad K^{(j)}(0) = 0 \quad (j < \kappa), \\ K^{(\kappa)}(0) \neq 0. \end{aligned}$$

Here we have taken $P = P_0 = (0, 0)$ and identified $Q \in \Gamma$ with s . Hence $K(s) = \delta s^\kappa + O(s^{\kappa+1})$ ($s \rightarrow +0$) for some $\delta > 0$. Moreover we define G_+ by $G_+ = \{(s, q) \in G; K(s) > 0\}$. In terms of the coordinate (s, q) , (2.1) becomes

$$(2.4) \quad \begin{cases} (1 - qK_0(s))^{-1} [\partial_s \{(1 - qK_0(s))^{-1} \partial_s\} + \partial_q \{(1 - qK_0(s)) \partial_q\}] U \\ \quad \quad \quad + \omega^2 c^{-2}(s, q)U = 0 & \text{in } G \\ U|_{q=0} = 0. \end{cases}$$

Hereafter, we mainly use (2.4) instead of (2.1) and we construct an asymptotic solution $U (=u)$ of (2.4) with the properties stated in §1.

Let

$$v(x) = \int_0^\infty \cos(t^3/3 + xt) dt \quad (x \in \mathbf{R})$$

be one of the Airy functions which is rapidly decreasing as $x \rightarrow +\infty$ together with all of its derivatives and has the zeros only on the negative real axis. Now let $-\nu < 0$ be a zero of $v(x)$. Then corresponding to this zero, there is an asymptotic solution U to (2.4) on G_+ which describes the whispering gallery wave. More precisely we have the following. (see pp. 37~47 in [1])

Theorem 2.1. *For any sufficiently small $\varepsilon > 0$, there is an asymptotic solution U of (2.4) in $G_+(\varepsilon) = \{(s, q) \in G; K(s) > \varepsilon\}$ with the following properties.*

1. U admits the following asymptotic expansion as $\omega \rightarrow \infty$:

$$(2.5) \quad U(s, q) \sim \exp \left\{ i\omega \int_{s_0}^s c^{-1}(s, 0) ds + i\omega^{1/3} h(s) \right\} \cdot \sum_{n=0}^\infty \omega^{-n/3} U_n(s, q)$$

where

$$(2.6) \quad \begin{aligned} h(s) &= -\nu \int_{s_0}^s K^{2/3}(s) [2c(s, 0)]^{-1/3} ds, \\ U_0 &= [2K(s)c(s, 0)]^{1/6} \cdot v(\omega^{2/3} q \eta(s) - \nu), \end{aligned}$$

$$\eta(s)=[2K(s)]^{1/3} \cdot c^{-2/3}(s, 0),$$

and $s_0 \in \Gamma$ with $K(s_0) > \varepsilon$.

2. $U_n (n \geq 1)$ can be obtained successively by solving a certain recursive equation. In particular, U_1 is given by

$$U_1=[(-i/2)c^{1/2}(s, 0)(2K(s))^{-1/2} \{(2/3)(2K(s))^{-1}K'(s) - (2/3)c^{-1}(s, 0)\partial_s c(s, 0)\} y^2 + A_1(s)]v(y-\nu)$$

where

$$y = \omega^{2/3} q \eta(s)$$

and $A_1(s)$ is a C^∞ function in s .

Remark. Each $U_n (n \geq 2)$ is a linear combination of $v(y-\nu)$ and $v'(y-\nu)$ whose coefficients are C^∞ function in s . By the asymptotic behavior of $v(x)$ as $x \rightarrow +\infty$ and the definition of $-\nu$, we can easily see that the asymptotic solution $U(s, q)$ decays exponentially as $\omega \rightarrow \infty$ if $q > 0$ and $U(s, q)$ satisfies the Dirichlet boundary condition. Thus this asymptotic solution corresponds to the whispering gallery wave. From (2.6), the asymptotic solution $U(s, q)$ is no longer valid as $s \rightarrow 0$.

Our next aim is to construct an asymptotic solution U to (2.4) in G which still has a meaning at $s=0$ and behaves like the whispering gallery wave in G_+ . As we had already assumed at the beginning of this section, note that

$$(2.7) \quad K(s) = \delta s^\kappa + O(s^{\kappa+1}) \quad (s \rightarrow +0)$$

for some $\delta > 0$ and positive integer κ . Introduce the scaling transformation

$$(2.8) \quad \begin{cases} t = s\omega^{1/(2\kappa+3)} [\delta^2 c^{-1}]^{1/(2\kappa+3)} \\ x = q\omega^{(\kappa+2)/(2\kappa+3)} [\delta c^{-\kappa-2}]^{1/(2\kappa+3)} \end{cases}$$

where $c=c(0, 0)$, and U_0 is the one given by (2.6). Then

$$(2.9) \quad \exp(i\omega^{1/3}h(s))U_0 = \phi_0(t, x)(1 + O(\omega^{-1/(2\kappa+3)})) \quad (\omega \rightarrow \infty)$$

where

$$(2.10) \quad \begin{aligned} \phi_0(t, x) &= D 2^{1/6} t^{\kappa/6} \exp\{-i\nu(3/2\kappa+3)2^{-1/3}t^{(2\kappa+3)/3}\} v(2^{1/3}t^{\kappa/3}x-\nu) \\ D &= C^{(\kappa+1)/(4\kappa+6)} \delta^{1/(4\kappa+6)} \omega^{-\kappa/(12\kappa+18)} \exp(-i\omega^{1/3}\nu \int_{s_0}^0 K^{2/3}(s)(2c(s, 0))^{-1/3} ds). \end{aligned}$$

The following theorem is a simple extension of Popov's result (cf. [4]).

Theorem 2.2. *There is an asymptotic solution U to (2.4) in a relatively open neighborhood $G \subset \bar{\Omega}$ of $(s, q) = (0, 0)$ with the following properties.*

- (1) U admits the following asymptotic expansion as $\omega \rightarrow \infty$:

$$U = \exp\left(i\omega \int_{s_0}^s c^{-1}(s, 0) ds\right) V(s, q),$$

$$(2.11) \quad V(s, q) \sim \sum_{n=0}^{\infty} \omega^{-n/(2\kappa+3)} V_n(t, x).$$

(2) $V_0(t, x) = \phi(t, x) \in L^2(\mathbf{R} \times \mathbf{R}^+)$ is uniquely determined by the following conditions:

$$(2.12) \quad \begin{cases} L\phi = (i\partial_t + (1/2)\partial_x^2 - xt^\kappa)\phi = 0 & (t \in \mathbf{R}, x \geq 0), \\ \phi|_{x=0} = 0 & (t \in \mathbf{R}), \\ \|\phi(t, \cdot) - \phi_0(t, \cdot)\|_{L^2(\mathbf{R}^+)} \longrightarrow 0 & (t \rightarrow +\infty). \end{cases}$$

Here $\phi_0(t, x)$ is the one given by (2.10).

(3) (2.12) admits the following formal solution $\phi_\infty(t, x)$ as $t \rightarrow +\infty$.

$$(2.13) \quad \phi_\infty(t, x) = D 2^{1/6} t^{\kappa/6} \exp\{-i\nu(3/(2\kappa+3))2^{-1/3} t^{(2\kappa+3)/3}\} \sum_{n=0}^{\infty} t^{-(2\kappa+3)n/3} \Phi_n(X)$$

where

$$X = 2^{1/3} t^{\kappa/3} x, \quad \Phi_0(X) = v(X - \nu),$$

$$\Phi_n(X) = P_{2n}(X)v(X - \nu) + Q_{2n-1}(X)v'(X - \nu) \quad (n \geq 1)$$

for some polynomials $P_{2n}(X)$ and $Q_{2n-1}(X)$ ($Q_{2n-1}(0) = 0$) of respective orders $2n$ and $2n-1$.

(4) For each nonnegative integer N , let ϕ_N be the truncated sum of $\phi_\infty(t, x)$ up to $n=N$ and $f_N(t, x) = L\phi_N$. Then for any nonnegative integer k, l, m, n , m , n satisfying

$$(4\kappa+6)N+5-\kappa \geq 6k+4\kappa m+2\kappa n,$$

$$(2.14) \quad \sup_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n f_N(t, x)| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

holds.

Remarks. 1. When $\kappa=1$, Theorem 2.2 is exactly the same as Popov's result (cf. [4]). Since the proof of Theorem 2.2 can be done in the same way as that of Popov's result (cf. [4]), we omit the proof.

2. Since $s > 0$ in G_+ , $t \rightarrow +\infty$ as $\omega \rightarrow \infty$ in G_+ . Noting this and (2.9) (2.12), the dominant term of the asymptotic solution U given in Theorem 2.2 approaches that of the whispering gallery wave with respect to the L^2 norm as $\omega \rightarrow \infty$ in G_+ .

3. $\phi_0(t, x)$ is the first term in the formal expansion (2.13) of ϕ .

Let $\chi(t) \in C^\infty(\overline{\mathbf{R}^+})$ satisfy

$$(2.15) \quad \chi(t) = 0 \quad (0 \leq t \leq 1), \quad \chi(t) = 1 \quad (t \geq 2).$$

Then we can easily see that, for any nonnegative integer k, l, m, n, j ,

$$(2.16) \quad |t^k x^l \partial_t^m \partial_x^n \{t^{-(2\kappa+3)j/3} \Phi_j(X) \chi(\varepsilon_j t)\}| \leq C_{lmnj} t^{k-\kappa l/3-m+\kappa n/3-(2\kappa+3)j/3}$$

where $C_{lmnj} > 0$ is a constant which does not depend on $1 > \varepsilon_j > 0$. Now take ε_j ($j=0, 1, \dots$) such that

$$(2.17) \quad C_{lmnj} \varepsilon_j \leq 2^{-j} \quad \text{for any } l, m, n, j \quad (l+m+n \leq j)$$

and define $\phi_N(t, x)$, $\tilde{\phi}_N(t, x)$ ($N=0, 1, \dots$), $\tilde{\phi}(t, x)$ by the following equations

$$\begin{aligned} \phi_N(t, x) &= D2^{1/6} t^{\kappa/6} \exp\{-i\nu(3/(2\kappa+3))2^{-1/3} t^{(2\kappa+3)/3}\} \sum_{j=0}^N t^{-(2\kappa+3)j/3} \Phi_j(X) \\ \tilde{\phi}_N(t, x) &= D2^{1/6} t^{\kappa/6} \exp\{-i\nu(3/(2\kappa+3))2^{-1/3} t^{(2\kappa+3)/3}\} \sum_{j=0}^N t^{-(2\kappa+3)j/3} \Phi_j(X) \chi(\varepsilon_j t) \\ \tilde{\phi}(t, x) &= D2^{1/6} t^{\kappa/6} \exp\{-i\nu(3/(2\kappa+3))2^{-1/3} t^{(2\kappa+3)/3}\} \sum_{j=0}^{\infty} t^{-(2\kappa+3)j/3} \Phi_j(X) \chi(\varepsilon_j t). \end{aligned}$$

From (2.16) and (2.17) we can easily see

$$(2.18) \quad \text{Sup}_{x \geq 0} \left| t^k x^l \partial_t^m \partial_x^n \sum_{j=N+1}^{\infty} \{t^{-(2\kappa+3)j/3} \Phi_j(X) \chi(\varepsilon_j t)\} \right| \leq t^{k-\kappa l/3-m+\kappa n/3-(2\kappa+3)N/3+1}$$

for nonnegative integers k, l, m, n ($l+m+n \leq N$). Moreover we have

$$(2.19) \quad \text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n \{t^{-(2\kappa+3)j/3} \Phi_j(X) (\chi(t) - \chi(\varepsilon_j t))\}| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

for nonnegative integers k, l, m, n .

Our next result gives a more detailed information about the asymptotic behavior of the first term ϕ of the asymptotic solution given in Theorem 2.2.

Theorem 2.3. (Main result). *There exists*

$$\phi(t, x) \in L^2(\mathbf{R} \times \mathbf{R}^+) \cap C^\infty(\bar{\mathbf{R}}^+ \times \bar{\mathbf{R}}^+) \cap (\bigcap_{\delta > 0} \mathcal{S}(\bar{\mathbf{R}}^+ \times [\delta, \infty)))$$

with the following properties.

(1) ϕ is a solution to

$$(2.20) \quad \begin{cases} L\phi = i\partial_t \phi + (1/2)\partial_x^2 \phi - xt^\kappa \phi = 0 & (t \in \mathbf{R}, x \geq 0). \\ \phi|_{x=0} = 0. \end{cases}$$

(2) For each fixed $t \geq 0$, $\phi(t, \cdot) \in \mathcal{S}(\bar{\mathbf{R}}^+)$.

(3) Given a nonnegative integer N , we have

$$\text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n (\phi - \phi_N)(t, x)| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

for any nonnegative integer k, l, m, n with $3k+2\kappa m+\kappa n \leq (2\kappa+3)N+2\kappa+2$.

(4) Moreover

$$\text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n (\phi - \tilde{\phi})(t, x)| \longrightarrow 0 \quad (t \rightarrow +\infty).$$

for any nonnegative integer k, l, m, n .

§ 3. Proof of Main Theorem (Part 1)

Let N be a nonnegative integer and $h(t, x) \in \bigcap_{\delta > 0} \mathcal{S}(\bar{\mathbf{R}}^+ \times [\delta, \infty))$ satisfy $h(t, 0) = 0$ and the estimate

$$(3.1) \quad \text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n h(t, x)| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

for any nonnegative integer k, l, m, n with

$$k + l + m + n \leq N.$$

Consider the following unique solution $w(t, x) \in L^2(\mathbf{R} \times \mathbf{R}^+)$ to

$$(3.2) \quad \begin{cases} Lw = h(t, x) & (t \in \mathbf{R}, x \geq 0) \\ w|_{x=0} = 0 \\ w(t, \cdot) \longrightarrow 0 & (t \rightarrow +\infty) \text{ in } L^2(\mathbf{R}^+). \end{cases}$$

Here we claim the following. Claim: There is a function $\rho(N)$ such that $\rho(N) \rightarrow \infty$ as $N \rightarrow \infty$ and

$$(3.3) \quad \text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n w(t, x)| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

for any nonnegative integer k, l, m, n with

$$k + l + m + n \leq \rho(N).$$

We postpone the proof of this claim for a while and show that the properties (3) (4) of Theorem 2.3 follows from the claim.

As for (4), let $h = -L(\tilde{\phi})$ and $\chi(t) \in C^\infty(\bar{\mathbf{R}}^+)$ be the function already given by (2.15). If we decompose h into the form

$$h = -L(\chi(t)\phi_N) - L(\tilde{\phi}_N - \chi(t)\phi_N) - L(\tilde{\phi} - \tilde{\phi}_N)$$

and remained the estimates (2.14), (2.19), (2.18), we have (3.1) for any nonnegative integer k, l, m, n . Then (4) immediately follows from the claim.

As for (3), fix nonnegative integers N, l and let k, m, n be nonnegative integers which satisfy $3k + 2\kappa m + \kappa n \leq (2\kappa + 3)N + 2\kappa + 2$. Take nonnegative integers $N' \geq N$ and M so that

$$\rho(M) \geq l + (2\kappa + 3)N + 2\kappa + 2$$

and

$$\text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n L(\chi(t)\phi_{N'})| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

for any nonnegative integer k, l, m, n with

$$k + l + m + n \leq M.$$

Since $w = \phi - \chi(t)\phi_{N'}$ is the solution to (3.2) with $h = -L(\chi(t)\phi_{N'})$ and

$$\sup_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n (\phi_{N'} - \phi_N)| \longrightarrow 0 \quad (t \rightarrow +\infty)$$

for any nonnegative integer k, l, m, n with

$$3k + 2\kappa m + \kappa n \leq (2\kappa + 3)N + 2\kappa + 2,$$

(3) immediately follows from the claim.

Now let us give the outline of the proof of the claim. By Duhamel's principle, $w(t, x)$ is given by

$$(3.4) \quad w(t, x) = (1/i) \int_t^\infty u(t, x; T) dT$$

if $u(t, x; T)$ is the strong L^2 solution to the following problem:

$$(3.5) \quad \begin{cases} i\partial_t u + (1/2)\partial_x^2 u - xt^\kappa u = 0 & (t \in \mathbf{R}, x \geq 0) \\ u|_{x=0} = 0 \\ u|_{t=T} = h(T, x). \end{cases}$$

Hence it is enough to show the existence, regularity and decay of $u(t, x; T)$ ($0 \leq t \leq T, x \geq 0$).

In this section, we prove the existence and the uniqueness of the strong L^2 solution $u(t, x; T)$ to (3.5) in each finite time interval $[T^*, T]$. Then in the next section, we extend $u(t, x; T)$ to $t \geq 0$ and prove its regularity and decay by using certain energy inequalities.

Let T^*, T satisfy $0 < T^* < T$ and fix them. Applying Cauchy's method, we will construct the unique solution to the problem

$$(3.6) \quad \begin{cases} i\partial_t u + (1/2)\partial_x^2 u - xt^\kappa u = 0 & (T^* < t < T, x > 0) \\ u|_{x=0} = 0 & (T^* \leq t \leq T) \\ u|_{t=T} = h(T, x) & (x \geq 0). \end{cases}$$

For this purpose, let

$$\Delta: T^* = t_m < t_{m-1} < \dots < t_1 < t_0 = T$$

be a partition of $[T^*, T]$ and define $u_\Delta(t, x)$ by

$$u_\Delta(t, x) = u_\Delta^{(k)}(t, x) \quad (k=0, 1, \dots, m-1; t_{k+1} \leq t \leq t_k)$$

where each $u_\Delta^{(k)}(t, x)$ is the unique solution to the problem

$$(3.7.k) \quad \begin{cases} i\partial_t u_\Delta^{(k)} + (1/2)\partial_x^2 u_\Delta^{(k)} - xt_k^\kappa u_\Delta^{(k)} = 0 & (t_{k+1} < t < t_k, x > 0) \\ u_\Delta^{(k)}|_{x=0} = 0 & (t_{k+1} \leq t \leq t_k) \\ u_\Delta^{(k)}|_{t=t_k} = u_\Delta^{(k-1)}(t_k, x) & (x \geq 0) \end{cases}$$

such that

$$u_{\Delta}^{-1}(t_0, x) = h(T, x).$$

Now let z_n be the n -th negative zero of the Airy function $v(z)$ and set

$$(3.8) \quad \begin{aligned} \lambda_n(t_k) &= -z_n 2^{-1/3} t_k^{2\kappa/3} \\ \phi_n(t_k, x) &= -2^{1/6} t_k^{\kappa/6} v(z) / (v'(z)|_{x=0}) \end{aligned}$$

where $z = 2^{1/3} t_k^{-2\kappa/3} (t_k^{\kappa} x - \lambda_n(t_k))$. Then according to Titchmarsh ([5] pp. 90~92), for each k ($0 \leq k \leq m-1$), $\{\phi_n(t_k, x)\}_{n=0}^{\infty}$ are eigenfunctions of

$$(3.9) \quad H(t_k) = (-1/2)\partial_x^2 + x t_k^{\kappa}$$

with eigenvalues $\{\lambda_n(t_k)\}_{n=0}^{\infty}$ satisfying the Dirichlet boundary condition and form an orthonormal system of $L^2(\mathbf{R}^+)$.

Next we define several norms and function spaces associated to these norms. Let $H_0^1(\mathbf{R}^+)$ be the completion of $C_0^\infty(\mathbf{R}^+)$ with respect to the Sobolev norm $\|\cdot\|_{W_{\frac{1}{2}}(\mathbf{R}^+)}$ and K be Sobolev's constant such that

$$\|u'(0)\| \leq K(\|u\|_{L^2(\mathbf{R}^+)} + \|u'\|_{L^2(\mathbf{R}^+)} + \|u''\|_{L^2(\mathbf{R}^+)}).$$

Definition 3.1. For $u(x) = \sum_{n=0}^{\infty} u_n \phi_n(x, t_k) \in \mathcal{S}(\bar{\mathbf{R}}^+) \cap H_0^1(\mathbf{R}^+)$, set

$$\begin{aligned} \|u\|_{1, t_k}^2 &= \sum_{n=0}^{\infty} \lambda_n(t_k) |u_n|^2 & \|u\|_{2, t_k}^2 &= \sum_{n=0}^{\infty} \lambda_n(t_k)^2 |u_n|^2 \\ \|u\|_{3, t_k}^2 &= \sum_{n=0}^{\infty} \lambda_n(t_k)^3 |u_n|^2 + t_k^{2\kappa} \sum_{n=0}^{\infty} |u_n|^2 \\ &+ \{(K+1)t_k^{\kappa}/2\} \sum_{n=0}^{\infty} (1 + \lambda_n(t_k) + \lambda_n(t_k)^2) |u_n|^2. \end{aligned}$$

Moreover we define the space $\Phi_3(t_k)$ as the completion of $\mathcal{S}(\bar{\mathbf{R}}^+) \cap H_0^1(\mathbf{R}^+)$ with respect to $\|\cdot\|_{3, t_k}$.

Definition 3.2. For $u(x) \in \mathcal{S}(\bar{\mathbf{R}}^+) \cap H_0^1(\mathbf{R}^+)$, set

$$\begin{aligned} [u]_{1, t_k}^2 &= \int_0^\infty (2^{-1} |\partial_x u|^2 + x t_k^{\kappa} |u|^2) dx \\ [u]_{2, t_k}^2 &= \int_0^\infty (4^{-1} |\partial_x^2 u|^2 + x t_k^{\kappa} |\partial_x u|^2 + x^2 t_k^{2\kappa} |u|^2) dx \\ [u]_{3, t_k}^2 &= \int_0^\infty (8^{-1} |\partial_x^3 u|^2 + (3/4) x t_k^{\kappa} |\partial_x^2 u|^2 + (3/2) x^2 t_k^{2\kappa} |\partial_x u|^2 \\ &+ x^3 t_k^{3\kappa} |u|^2) dx + (t_k^{\kappa}/2) \{(K+1)(\|u\|_{2, t_k}^2 + \|u\|_{1, t_k}^2 \\ &+ \|u\|_{2, t_k}^2 - |\partial_x u(t, 0)|^2)\}. \end{aligned}$$

Moreover we define the space $\Phi_3(t_k)$ as the completion of $\mathcal{S}(\bar{\mathbf{R}}^+) \cap H_0^1(\mathbf{R}^+)$ with

respect to $[\cdot]_{3,t}$.

Lemma 3.3. *The two spaces $(\Phi(t_k), \|\cdot\|_{3,t_k})$ and $(\Phi'_3(t_k), [\cdot]_{3,t_k})$ are isometric to each other. Thus from now on we also denote $(\Phi'_3(t_k), [\cdot]_{3,t_k})$ by $(\Phi_3(t_k), \|\cdot\|_{3,t_k})$.*

Proof. Set $u^{(N)}(x) = \sum_{n=0}^N u_n \phi_n(t_k, x)$ where $u_n \in C$ ($0 \leq n \leq N$). Then we can easily see

$$\begin{aligned} \|u^{(N)}\|_{3,t_k}^2 &= (H(t_k)^2 u^{(N)}, H(t_k) u^{(N)})_{L^2(\mathbb{R}^+)} + t_k^{2k} \|u^{(N)}\|_{L^2(\mathbb{R}^+)}^2 \\ &\quad + 2^{-1}(K+1)t_k^k \{ \|u^{(N)}\|_{L^2(\mathbb{R}^+)}^2 + (H(t_k) u^{(N)}, u^{(N)})_{L^2(\mathbb{R}^+)} \} \\ &\quad + \|H(t_k) u^{(N)}\|_{L^2(\mathbb{R}^+)}^2. \end{aligned}$$

Since $\phi_n(0, t_k) = \partial_x^2 \phi_n(0, t_k) = 0$, integrating the right hand side of the above yields $\|u^{(N)}\|_{3,t_k}^2 = [u^{(N)}]_{3,t_k}^2$. Then Lemma 3.3 immediately follows from this and the fact $\{\phi_n(x, t_k)\}_{n=0}^\infty$ is an orthonormal system in $L^2(\mathbb{R}^+)$. Q. E. D.

Next we return to the existence of $u_\Delta(t, x)$.

Lemma 3.4. *For any partition Δ of $[T^*, T]$, there exists $u_\Delta(t, x) \in L^2([T^*, T], \Phi_3(T))$. These $\{u_\Delta(t, x)\}$ are bounded in $L^2([T^*, T], \Phi_3(T))$.*

Proof. First we note that each $u_\Delta^{(k)}$ has the representation

$$\begin{aligned} (3.10) \quad u_\Delta^{(k)}(t, x) &= \sum_{n=0}^\infty \exp\{-\lambda_n(t_k)(t-t_k)\} d_n^{(k)} \phi_n(t_k, x) \\ d_n^{(k)} &= \int_0^\infty u_\Delta^{(k-1)}(t_k, x) \phi_n(t_k, x) dx. \end{aligned}$$

Combining this with Lemma 3.3, $u_\Delta^{(k)}(t, \cdot) \in \Phi_3(t_k)$ and

$$(3.11) \quad \|u_\Delta^{(k)}(t_k, \cdot)\|_{3,t_k} = \|u_\Delta^{(k)}(t, \cdot)\|_{3,t_k} \quad \text{for any } t \in [t_{k+1}, t_k].$$

Now according to Definition 3.2 and Lemma 3.3

$$(3.12) \quad \|u_\Delta^{(k)}(t_{k+1}, \cdot)\|_{3,t_{k+1}} \leq \|u_\Delta^{(k)}(t_{k+1}, \cdot)\|_{3,t_k}.$$

Moreover, by the definition of $u_\Delta^{(k)}$

$$(3.13) \quad \|u_\Delta^{(k)}(t_{k+1}, \cdot)\|_{3,t_{k+1}} = \|u_\Delta^{(k+1)}(t_{k+1}, \cdot)\|_{3,t_{k+1}}.$$

Hence combining (3.11)~(3.13) and taking account of $\|\cdot\|_{q,T} \sim \|\cdot\|_{q,T}$, there is a constant $c > 0$ such that

$$(3.14) \quad \|u_\Delta(t, \cdot)\|_{3,T} \leq c \|h(T, \cdot)\|_{3,T} \quad (T^* \leq t \leq T) \quad \text{for any } \Delta.$$

Q. E. D.

Lemma 3.5. *There is a constant $c > 0$ such that*

$$(3.15) \quad \|u_\Delta(t, \cdot)\|_{3,T} \geq c \|u_\Delta(t, \cdot)\|_{w_3^2(\mathbf{R}^+)} \quad (T^* \leq t \leq T) \quad \text{for any } \Delta.$$

Proof. This is almost clear from Definition 3.2.

Theorem 3.6. *There is a solution $u(t, x) \in L^2([T^*, T], \Phi_3(T)) \cap \mathcal{E}_{L^2}^{1,3}((T^*, T) \times \mathbf{R}^+)$ to (3.6).*

Here

$$\mathcal{E}_{L^2}^{p,q}((T^*, T) \times \mathbf{R}^+) = \{u \in \mathcal{D}'((T^*, T) \times \mathbf{R}^+); \partial_t^j \partial_x^k u(t, x) \in L^2((T^*, T) \times \mathbf{R}^+) \\ (j \leq p, k \leq q - 2j)\}$$

for nonnegative integers p, q ($2p < q$) and we will see later in Corollary 4.4 that this $u(t, x)$ is unique.

Proof of Theorem 3.6. By Lemma 3.4, $\{u_\Delta(t, x)\}$ is uniformly bounded in the Hilbert space $L^2([T^*, T], \Phi_3(T))$. Hence there exists

$$(3.16) \quad u(t, x) \in L^2([T^*, T], \Phi_3(T))$$

and a sequence of functions $\{u_j(t, x)\} \subset \{u_\Delta(t, x)\}$ such that $u_j(t, x) \rightarrow u(t, x)$ ($j \rightarrow \infty$) weakly in $L^2([T^*, T], \Phi_3(T))$. On the other hand, using the definition of $u_\Delta(t, x)$, each $u_j(t, x)$ is the approximate solution to the equation of (3.6). Hence $u(t, x)$ satisfies the same equation in the distribution sense.

Now by Lemma 3.5, the argument which led to (3.16) yields

$$(3.17) \quad u(t, x) \in \mathcal{E}_{L^2}^{0,3}((T^*, T) \times \mathbf{R}^+).$$

Combining (3.16), (3.17) with the equation of (3.6) and reminding the definition of $\Phi_3(T)$, we have

$$u(t, x) \in L^2([T^*, T], \Phi_3(T)) \cap \mathcal{E}_{L^2}^{1,3}((T^*, T) \times \mathbf{R}^+).$$

Thus $u(t, x)$ satisfies the equation in the L^2 strong sense and it has the traces to $t=0$ and $x=0$.

Finally, to see that $u(t, x)$ satisfies the boundary condition and the initial condition of (3.6) is a routine argument (see Mizohata [3] p. 333). Q. E. D.

§ 4. Proof of Main Theorem (part 2)

In this section, we study the regularity and decay at infinity of the principal term $V_0 = \phi(t, x)$ of the expansion (2.11).

To begin with, we generalize (3.6) as follows:

$$(4.1) \quad \begin{cases} i\partial_t u = (-1/2)\partial_x^2 u + xt^r u + f(t, x) & \text{in } (T^*, T) \times \mathbf{R}^+ \\ u|_{x=0} = 0 \\ u|_{t=T} = u_0(x). \end{cases}$$

Lemma 4.1. *Let $f(t, x) \in L^2([T^*, T], \Phi_3(T))$ and $u_0(x) \in \mathcal{S}(\bar{\mathbf{R}}^+) \cap H_1^1(\mathbf{R}^+)$ then there exists a solution*

$$u(t, x) \in L^2([T^*, T], \Phi_3(T)) \cap \mathcal{E}_{L^2}^{1,3}([T^*, T] \times \mathbf{R}^+)$$

to (4.1) which satisfies

$$\begin{aligned} \left[\int_{T^*}^T (\|u(t, \cdot)\|_{\Phi_3(T)})^2 dt \right]^{1/2} &= \|u(t, x)\|_{L^2([T^*, T], \Phi_3(T))} \\ &\leq C \{ (T - T^*)^{1/2} \|u_0\|_{\Phi_3(T)} + (T - T^*) \|f(t, x)\|_{L^2([T^*, T], \Phi_3(T))} \} \end{aligned}$$

where C is a constant depending only on T^*, T .

Proof. As in part 1, we apply Cauchy’s method. Take a partition $\Delta: T^* = t_m < t_{m-1} < \dots < t_1 < t_0 = T$ and define $u_\Delta(t, x) = u_\Delta^{(k)}(t, x)$, ($k=0, 1, \dots, m-1; t_{k+1} \leq t \leq t_k$) by

$$\begin{cases} i\partial_t u_\Delta^{(k)} = -(1/2)\partial_x^2 u_\Delta^{(k)} + xt_k^\kappa u_\Delta^{(k)} + f(t, x), & (t_{k+1} < t < t_k, x > 0) \\ u_\Delta^{(k)}|_{x=0} = 0 & (t_{k+1} \leq t \leq t_k) \\ u_\Delta^{(k)}(t_k, x) = u_\Delta^{(k-1)}(t_k, x) & (x \geq 0) \end{cases}$$

such that $u_\Delta^{(-1)}(t_0, x) = u_0(x)$.

Then, using the eigenfunctions $\{\phi_n(t_k, x)\}_{n=0}^\infty$, and eigenvalues $\{\lambda_n(t_k)\}_{n=0}^\infty$ of the operator $H(t_k) = -(1/2)\partial_x^2 + xt_k^\kappa$, we have

$$\begin{aligned} u_\Delta^{(k)}(t, x) &= \sum_{n=0}^\infty \exp\{-i\lambda_n(t_k)(t-t_k)\} d_n^{(k)} \phi_n(t_k, x) \\ &\quad + \sum_{n=0}^\infty \int_t^{t_k} \exp\{-i\lambda_n(t_k)(t-s)\} i f_n^{(k)}(s) ds \phi_n(t_k, x) \end{aligned}$$

where

$$d_n^{(k)} = \int_0^\infty u_\Delta^{(k-1)}(t_k, x) \phi_n(t_k, x) dx, \quad u_\Delta^{(-1)}(t_0, x) = u_0(x)$$

and $f_n^{(k)}(t) = \int_0^\infty f(t, x) \phi_n(t_k, x) dx$. By applying Minkowski’s inequality, we can easily see

$$\|u_\Delta^{(k)}(t, x)\|_{3, t_k} \leq \|u_\Delta^{(k)}(t_k, x)\|_{3, t_k} + \int_t^{t_k} \|f(s, \cdot)\|_{3, t_k} ds.$$

Now following the same argument as in the proof of Lemma 3.4, we have

$$(4.2) \quad \|u_\Delta(t, x)\|_{3, t_k} \leq \|u_0(x)\|_{\Phi_3(T)} + \int_t^T \|f(s, \cdot)\|_{\Phi_3(T)} ds.$$

Thus

$$\begin{aligned} (4.2)' \quad \left[\int_{T^*}^T \|u_\Delta(t, x)\|_{\Phi_3(T)}^2 dt \right]^{1/2} &\leq C \left\{ (T - T^*)^{1/2} \|u_0(x)\|_{\Phi_3(T)} \right. \\ &\quad \left. + (T - T^*) \left[\int_{T^*}^T \|f(s, \cdot)\|_{\Phi_3(T)}^2 ds \right]^{1/2} \right\}. \end{aligned}$$

The rest of the proof is the same as that of Theorem 3.6. Q. E. D.

Remark. Later in Corollary 4.4, we show that this solution is unique.

To derive the energy inequality for (4.1), we need the following lemma.

Lemma 4.2. *Let $u(t, x) \in L^2([T^*, T], \Phi_\delta(T))$ be the solution to (4.1), where $f(t, x) \in C^0([T^*, T], L^2(\mathbf{R}^+)) \cap L^2([T^*, T], \Phi_\delta(T))$ and $u_0(x) \in \mathcal{S}(\bar{\mathbf{R}}^+)$. Then $\|u(t, \cdot)\|, \|xu(t, \cdot)\|, \|\partial_x^2 u(t, \cdot)\|, \|u'(t, \cdot)\| \in C^0([T^*, T])$.*

Notation. Hereafter, for simplicity, we denote the $L^2(\mathbf{R}^+)$ norm by $\|\cdot\|$, and the differential with respect to t by “ $'$ ”.

Proof of Lemma 4.2. From the assumptions, $\partial_t u(t, x) \in L^2([T^*, T] \times \mathbf{R}^+)$. Then by Radon Nikodym’s theorem

$$u(t, x) - u(t', x) = \int_{t'}^t \partial_s u(s, x) ds \quad \text{for a. e. } x.$$

So we have $u(t, x) \in C^0([T^*, T], L^2(\mathbf{R}^+))$. Therefore

$$\|u(t, \cdot)\| \in C^0([T^*, T]).$$

To prove $\|xu(t, \cdot)\| \in C^0([T^*, T])$, put $(u)_\delta(t, x) = (\phi_\delta^{(\epsilon)} * u)(t, x)$ where ϕ_δ is a mollifier such that

$$\begin{aligned} \phi_\delta(t) &= \frac{1}{\delta} \phi(t/\delta), \quad \phi(t) \geq 0, \quad \phi(t) \in C_0^\infty(\mathbf{R}), \\ \text{supp } \phi &\subset \{t \mid |t| \leq 1\}, \quad \int_{\mathbf{R}} \phi(t) dt = 1. \end{aligned}$$

Now let $u_\Delta(t, x)$ be the one defined in the proof of Lemma 4.1. Then by integration by parts,

$$\begin{aligned} &\left| \frac{d}{dt} \|x(u_\Delta)_\delta(t, \cdot)\|^2 \right| \\ &\leq 2 \|\partial_x (u_\Delta)_\delta(t, \cdot)\| \cdot \|x(u_\Delta)_\delta(t, \cdot)\| + 2 \|xf_\delta(t, \cdot)\| \cdot \|x(u_\Delta)_\delta(t, \cdot)\|. \end{aligned}$$

Thus

$$\begin{aligned} &\left\| \frac{d}{dt} \|x(u_\Delta)_\delta(t, \cdot)\| \right\|_{L^2([T^*, T])} \\ &\leq \|\partial_x (u_\Delta)_\delta(t, x)\|_{L^2([T^*, T] \times \mathbf{R}^+)} + \|xf_\delta(t, x)\|_{L^2([T^*, T] \times \mathbf{R}^+)} \\ &\leq \|\partial_x u_\Delta(t, x)\|_{L^2([T^*, T] \times \mathbf{R}^+)} + \|xf(t, x)\|_{L^2([T^*, T] \times \mathbf{R}^+)} \leq M, \end{aligned}$$

where M is a constant independent of Δ . On the other hand,

$$\int_{T^*}^T (\|x(u_\Delta)_\delta(t, \cdot)\| - \|xu_\Delta(t, \cdot)\|)^2 dt \longrightarrow 0 \quad (\delta \rightarrow 0).$$

Therefore taking an appropriate subsequence $\{\delta'\} \subset \{\delta\}$,

$$\frac{d}{dt} \|x(u_\Delta)_{\delta'}(t, \cdot)\| \longrightarrow \frac{d}{dt} \|xu_\Delta(t, \cdot)\| \quad (\delta' \rightarrow 0) \quad \text{in } \mathcal{D}'([T^*, T])$$

and

$$(4.3) \quad \left\| \frac{d}{dt} \|xu_\Delta(t, \cdot)\| \right\|_{L^2([T^*, T])} \leq M \quad \text{for any } \Delta.$$

Here d/dt is the distribution derivative. Next, we will show

$$(4.4) \quad \|xu_\Delta(t, \cdot)\| \longrightarrow \|xu(t, \cdot)\| \quad (|\Delta| \rightarrow 0) \quad \text{in } \mathcal{D}'([T^*, T]).$$

From (4.2), $\|x^{1/2}u_\Delta(t, \cdot)\|$ is uniformly bounded on $[T^*, T]$ with respect to Δ . By (4.2)', $\partial_t x^{1/2}u_\Delta(t, x)$ is uniformly bounded in $L^2([T^*, T] \times \mathbb{R}^+)$ with respect to Δ , and by Radon Nikodym's theorem we have

$$\|x^{1/2}u_\Delta(t', \cdot) - x^{1/2}u_\Delta(t, \cdot)\|^2 \leq |t' - t| \int_{T^*}^T \|\partial_s x^{1/2}u_\Delta(s, \cdot)\|^2 ds.$$

Thus $x^{1/2}u_\Delta(t, x)$ is uniformly bounded and equi-continuous in $C^0([T^*, T], L^2(\mathbb{R}^+))$ with respect to Δ . So for an appropriate sequence $\{\Delta'\} \subset \{\Delta\}$,

$$\text{Sup}_{T^* \leq t \leq T} \|x^{1/2}u_{\Delta'}(t, \cdot) - x^{1/2}u(t, \cdot)\| \longrightarrow 0 \quad (|\Delta'| \rightarrow 0)$$

and

$$\begin{aligned} & \left| \int_{T^*}^T (\|xu_{\Delta'}(t, \cdot)\| - \|xu(t, \cdot)\|) \cdot \phi(t) dt \right| \\ & \leq \left(\int_{T^*}^T |\phi(t)|^2 dt \right)^{1/2} \left\{ \int_{T^*}^T \int_{\mathbb{R}^+} x^{1/2}(u_{\Delta'}(t, x) - u(t, x)) x^{3/2} \right. \\ & \quad \left. \cdot (\overline{u_{\Delta'}(t, x)} - \overline{u(t, x)}) dx dt \right\}^{1/2} \\ & \leq \left(\int_{T^*}^T |\phi(t)|^2 dt \right)^{1/2} \left\{ \int_{T^*}^T \int_{\mathbb{R}^+} |x^{1/2}(u_{\Delta'}(t, x) - u(t, x))|^2 dx dt \right. \\ & \quad \left. \times \int_{T^*}^T \int_{\mathbb{R}^+} |x^{3/2}(u_{\Delta'}(t, x) - u(t, x))|^2 dx dt \right\}^{1/4} \\ & \leq \left(\int_{T^*}^T |\phi(t)|^2 dt \right)^{1/2} (T - T^*)^{1/4} \{ \text{Sup} \|x^{1/2}u_{\Delta'}(t, \cdot) - x^{1/2}u(t, \cdot)\|^2 \}^{1/4} \\ & \quad \times \left\{ 2 \int_{T^*}^T \int_{\mathbb{R}^+} |x^{3/2}u_{\Delta'}|^2 + |x^{3/2}u|^2 dx dt \right\}^{1/4} \xrightarrow{(|\Delta'| \rightarrow 0)} 0, \end{aligned}$$

for any $\phi \in C_0^\infty([T^*, T])$. Hence we have proved (4.4). Since (4.3) and (4.4) imply

$$\frac{d}{dt} \|xu(t, \cdot)\| \in L^2([T^*, T]),$$

we have $\|xu(t, \cdot)\| \in C^0([T^*, T])$. The same argument can also be applied to

prove $\|\partial_x^2 u(t, \cdot)\| \in C^0([T^*, T])$. Finally, by using the equation of (4.1), $\|u'(t, \cdot)\| \in C^0([T^*, T])$. Q. E. D.

Relying on Lemma 4.2, we will give the following energy estimate for the solution to (4.1).

Lemma 4.3 (Energy Inequality). *Let $f(t, x) \in L^2([T^*, T], \Phi_3(T))$, $\partial_t f(t, x) \in L^2([T^*, T], L^2(\mathbf{R}^+))$, $u_0(x) = u(T, x) \in S(\mathbf{R}^+)$ and let $u(t, x) \in L^2([T^*, T], \Phi_3(T))$ be the solution to (4.1). Then we have the following energy inequality:*

$$\begin{aligned} & \|u(t, \cdot)\| + t^\epsilon \|xu(t, \cdot)\| + \|u'(t, \cdot)\| \\ & \leq \|u(T, \cdot)\| + 2(T^2/t)^\epsilon \|xu(T, \cdot)\| + (T/t)^\epsilon \|\partial_x^2 u(T, \cdot)\| \\ & \quad + 2(T/t)^\epsilon \|f(T, \cdot)\| + \|f(t, \cdot)\| + \int_t^T \{\|f(s, \cdot)\| \\ & \quad + 2\kappa(1/s)(s/t)^\epsilon \|f(s, \cdot)\| + 2(s/t)^\epsilon \|f'(s, \cdot)\|\} ds \quad \text{for a. e. } t \in [T^*, T]. \end{aligned}$$

Proof. Since

$$\frac{d}{dt} \|u(t, \cdot)\|^2 = 2 \operatorname{Im} \int_0^\infty f(t, x) \overline{u(t, x)} dx,$$

we have $|(d/dt)\|u(t, \cdot)\| \leq \|f(t, \cdot)\|$. Thus

$$(4.5) \quad \|u(t, \cdot)\| \leq \|u(T, \cdot)\| + \int_t^T \|f(s, \cdot)\| ds.$$

By integrating by parts,

$$\begin{aligned} & \int_0^\infty i \partial_t u \cdot x \bar{u} dx = \int_0^\infty \{(-1/2) \partial_x^2 u + xt^\epsilon u + f\} x \bar{u} dx \\ & = \int_0^\infty (1/2) \partial_x u \cdot x \bar{\partial_x u} dx + \int_0^\infty (1/2) \partial_x u \cdot \bar{u} dx + \int_0^\infty |xu|^2 t^\epsilon dx + \int_0^\infty f x \bar{u} dx. \end{aligned}$$

Hence,

$$\begin{aligned} & (1/2) \int_0^\infty x |\partial_x u|^2 dx + \int_0^\infty |xu|^2 t^\epsilon dx \\ & = \int_0^\infty i \partial_t u \cdot x \bar{u} dx - \int_0^\infty (1/2) \partial_x u \cdot \bar{u} dx - \int_0^\infty f x \bar{u} dx. \end{aligned}$$

Taking the real parts of the right hand side, we have

$$t^\epsilon \int_0^\infty |xu|^2 dx \leq \|\partial_t u(t, \cdot)\| \|xu(t, \cdot)\| + \|f(t, \cdot)\| \|xu(t, \cdot)\|.$$

Thus

$$(4.6) \quad t^\epsilon \|xu(t, \cdot)\| \leq \|\partial_t u(t, \cdot)\| + \|f(t, \cdot)\|.$$

Now let $\phi_\delta(t)$ be the mollifier used in the proof of Lemma 5.3 and put $u_\delta(t, x) = (\phi_\delta^{(\iota)} * u)(t, x)$. Then

$$i\partial_t u_\delta = -(1/2)\partial_x^2 u_\delta + xt^\kappa u_\delta + \tilde{f}(t, x)$$

where $\tilde{f}(t, x) = x[\phi_{\delta^*}, t^\kappa]u + f_\delta(t, x)$, $[\phi_{\delta^*}, t^\kappa]u = \phi_{\delta^*} t^\kappa u - t^\kappa u_\delta$. Since

$$i\partial_t u_\delta' = -(1/2)\partial_x^2 u_\delta' + xt^\kappa u_\delta' + \kappa xt^{\kappa-1} u_\delta + \tilde{f}'(t, x),$$

we have

$$\begin{aligned} \frac{d}{dt} \|u_\delta'\|^2 &= 2 \operatorname{Re} \int_0^\infty \partial_t u_\delta' \cdot \bar{u}_\delta' dx \\ &= 2 \operatorname{Im} \int_0^\infty \{-(1/2)\partial_x^2 u_\delta' + xt^\kappa u_\delta' + \kappa xt^{\kappa-1} u_\delta + \tilde{f}'\} \bar{u}_\delta' dx \\ &= 2 \operatorname{Im} \int_0^\infty (\kappa xt^{\kappa-1} u_\delta + \tilde{f}') \bar{u}_\delta' dx. \end{aligned}$$

So

$$\left| \frac{d}{dt} \|u_\delta'(t, \cdot)\| \right| \leq \|\kappa xt^{\kappa-1} u_\delta(t, \cdot)\| + \|\tilde{f}'(t, \cdot)\|.$$

From (4.6)

$$t^\kappa \|x u_\delta(t, \cdot)\| \leq \|\partial_t u_\delta(t, \cdot)\| + \|\tilde{f}(t, \cdot)\|.$$

Thus we have

$$\left| \frac{d}{dt} \|u_\delta'(t, \cdot)\| \right| \leq (\kappa/t) \|u_\delta'(t, \cdot)\| + (\kappa/t) \|\tilde{f}(t, \cdot)\| + \|\tilde{f}'(t, \cdot)\|.$$

Applying the comparison theorem, we have

$$\begin{aligned} (4.7)_\delta \quad \|u_\delta'(t, \cdot)\| &\leq (T/t)^\kappa \|u_\delta'(T, \cdot)\| + \int_t^T (1/s)(s/t)^\kappa (\kappa \|\tilde{f}(s, \cdot)\| + s \|\tilde{f}'(s, \cdot)\|) ds \\ &\leq (T/t)^\kappa \|u_\delta'(T, \cdot)\| + \int_t^T (1/s)(s/t)^\kappa \cdot \{\kappa \|x[\phi_{\delta^*}, s^\kappa]u\| \\ &\quad + \kappa \|f_\delta(s, \cdot)\| + s(\|x\partial_s[\phi_{\delta^*}, s^\kappa]u\| + \|\partial_s f_\delta(s, \cdot)\|)\} ds. \end{aligned}$$

Since

$$x[\phi_{\delta^*}, s^\kappa]u(s, \cdot) = \phi_{\delta^*}(s^\kappa x u) - s^\kappa \phi_{\delta^*} x u = \int_{\mathbf{R}} \phi_\delta(s - \sigma)(\sigma^\kappa - s^\kappa) x u(\sigma, x) d\sigma,$$

we have

$$\begin{aligned} \|x[\phi_{\delta^*}, s^\kappa]u(s, \cdot)\| &\leq \left\{ \int_{\mathbf{R}} \phi_\delta(s - \sigma) |\sigma^\kappa - s^\kappa|^2 \|x u(\sigma, \cdot)\|^2 d\sigma \right\}^{1/2} \\ &\leq C(T, T^*) \cdot \left(\delta^2 \int_{\mathbf{R}} \phi(t) t^2 \|x u(s - \delta t, \cdot)\|^2 dt \right)^{1/2} \\ &\longrightarrow 0 \quad (\delta \rightarrow 0) \text{ uniformly with respect to } s \in [t, T]. \end{aligned}$$

Similarly, we have

$$\int_t^T \|x\partial_s[\phi_{\delta^*}, s^\kappa]u(s, \cdot)\| ds \longrightarrow 0 \quad (\delta \rightarrow 0).$$

Moreover, taking account of the assumption and Lemma 4.2, it is clear

$$\|u'_\delta(t, \cdot)\| \longrightarrow \|u'(t, \cdot)\| \quad (\delta \rightarrow 0) \quad \text{for a.e. } t \in [T^*, T].$$

$$\|u'_\delta(T, \cdot)\|^2 \leq \int_{\mathbf{R}^+} \phi_\delta(T - \tau) \|u'(\tau, \cdot)\|^2 d\tau \longrightarrow \|u'(T, \cdot)\|^2 \quad (\delta \rightarrow 0),$$

and $\int_t^T \|f_\delta(s, \cdot) - f(s, \cdot)\| ds \rightarrow 0 \quad (\delta \rightarrow 0)$. Hence by letting $\delta \rightarrow 0$, (4.7) _{δ} implies

$$\begin{aligned} (4.7) \quad \|u'(t, \cdot)\| &\leq (T/t)^\kappa \|u'(T, \cdot)\| + \int_t^T (1/s)(s/t)^\kappa (\kappa \|f(s, \cdot)\| + s \|\partial_s f(s, \cdot)\|) ds \\ &\leq (T/t)^\kappa ((1/2) \|\partial_x^2 u(T, \cdot)\| + \|x T^\kappa u(T, \cdot)\| + \|f(T, \cdot)\|) \\ &\quad + \int_t^T (1/s)(s/t)^\kappa (\kappa \|f(s, \cdot)\| + s \|f'(s, \cdot)\|) ds. \end{aligned}$$

Combining (4.5), (4.6) and (4.7), we have the desired inequality. Q. E. D.

Next Corollary 4.4 follows from (4.5).

Corollary 4.4 (Uniqueness). *Let $u(t, x) \in L^2([T^*, T] \times \mathbf{R}^+)$ be the solution to (4.1) such that $\partial_x^2 u(t, x), xu(t, x) \in L^2([T^*, T] \times \mathbf{R}^+)$. Then $u(t, x)$ is a unique solution to (4.1).*

Next using Lemma 4.3, we will prove the higher order regularity of the solution $u(t, x)$ to (3.5) with respect to t . But, for a while, we restrict the the domain of t to the bounded interval $[T^*, T]$.

Now, taking a positive integer m large enough, we seek the solution $\check{u}(t, x)$ to (3.6) in the form

$$\begin{aligned} (4.8) \quad \check{u}(t, x) &= u_0(x) + (t-T)u_1(x) + \dots + \frac{(t-T)^{m-1}}{(m-1)!} u_{m-1}(x) \\ &\quad + \int_t^T \frac{(t-s)^{m-1}}{(m-1)!} v(s, x) ds \quad ((t, x) \in [T^*, T] \times \bar{\mathbf{R}}^+). \end{aligned}$$

Then we have the following.

Lemma 4.5. *In order that (4.8) is the solution to (3.6), it is necessary and sufficient that the following conditions (4.9) and (4.10) hold. Namely,*

$$(4.9) \quad \begin{cases} u_0(x) = h(T, x) \\ u_1(x) = (1/i) \{ (-1/2) \partial_x^2 u_0(x) + x T^\kappa u_0(x) \} \\ u_p(x) = (1/i) \left\{ (-1/2) \partial_x^2 u_{p-1}(x) + \sum_{l=0}^{\gamma} \binom{p-1}{l} x \partial_T^\kappa T^\kappa u_{p-l-1}(x) \right\} \\ \quad \quad \quad 2 \leq p \leq m-1, \quad \gamma = \min(p-1, \kappa) \\ u_p(0) = 0 \quad 0 \leq p \leq m-1 \end{cases}$$

and $g'(t, x) = \partial_t g(t, x) \in L^2([T^*, T], L^2(\mathbf{R}^+))$ exists for almost everywhere $t \in [T^*, T]$.

Let $\psi_n(T, x) = \psi_n(x)$ $n=0, 1, \dots$ be the orthonormal system defined by (3.8) and expand $g(t, x)$ as follows:

$$g(t, x) = \sum_{n=0}^{\infty} g_n(t) \psi_n(x), \quad \text{where } g_n(t) = \int_0^{\infty} g(t, x) \psi_n(x) dx.$$

Let $g_N(t, x)$ be its N -finite sum:

$$g_N(t, x) = \sum_{n=0}^N g_n(t) \psi_n(x).$$

Then it follows that

$$(4.12) \quad \begin{cases} g_N(t, x) \longrightarrow g(t, x) & \text{in } L^2([T^*, T] \times \mathbf{R}^+) \\ \partial_t g_N(t, x) \longrightarrow \partial_t g(t, x) & \text{in } L^2([T^*, T] \times \mathbf{R}^+) \text{ as } N \rightarrow \infty \end{cases}$$

and $g_N(t, x) \in L^2([T^*, T], \Phi_s(T))$.

Now we rewrite (4.11.j) as follows:

$$(4.13) \quad \begin{cases} i\partial_t u = (-1/2)\partial_x^2 u + xt^\kappa u + g & \text{in } (T^*, T) \times \mathbf{R}^+ \\ u|_{x=0} = 0, \quad u|_{t=T} = u_m(x) \\ u_m(x) \in \mathcal{S}(\bar{\mathbf{R}}^+) \\ \text{and} \\ g \in C^0([T^*, T], L^2(\mathbf{R}^+)) \text{ satisfies (4.12) for some} \\ \{g_N(t, x)\} \subset L^2([T^*, T], \Phi_s(T)). \end{cases}$$

If we can find the solution $u(t, x)$ to (4.13), the lemma is proved.

By Lemma 4.1, there is a solution $u_N(t, x) \in L^2([T^*, T], \Phi_s(T))$ to

$$\begin{cases} i\partial_t u_N = (-1/2)\partial_x^2 u_N + xt^\kappa u_N + g_N \\ u_N|_{x=0} = 0, \quad u_N|_{t=T} = u_m(x) \end{cases} \quad \text{for each } N (N=0, 1, 2, \dots).$$

From the energy inequality (cf. Lemma 4.3), we have

$$(4.14) \quad \begin{aligned} & \|u_N(t, \cdot)\| + t^\kappa \|xu_N(t, \cdot)\| + \|\partial_x^2 u_N(t, \cdot)\| \\ & \leq C \left[\|u_N(T, \cdot)\| + (T/t)^\kappa \|\partial_x^2 u_N(T, \cdot)\| + (T^2/t)^\kappa \|xu_N(T, \cdot)\| \right. \\ & \quad \left. + (T/t)^\kappa \|g_N(T, \cdot)\| + \|g_N(t, \cdot)\| + \int_t^T \{\|g_N(s, \cdot)\| \right. \\ & \quad \left. + \kappa(1/s)(s/t)^\kappa \|g_N(s, \cdot)\| + (s/t)^\kappa \|g'_N(s, \cdot)\| \} ds \right]. \end{aligned}$$

On the other hand, since

$$\begin{cases} i\partial_t(u_N - u_M) = (-1/2)\partial_x^2(u_N - u_M) + xt^\kappa(u_N - u_M) + g_N - g_M \\ (u_N - u_M)|_{x=0} = 0, \quad (u_N - u_M)|_{t=T} = 0, \end{cases}$$

we have

$$\begin{aligned} & \|u_N(t, \cdot) - u_M(t, \cdot)\| + t^\kappa \|xu_N(t, \cdot) - xu_M(t, \cdot)\| + \|\partial_x^2 u_N(t, \cdot) - \partial_x^2 u_M(t, \cdot)\| \\ & \leq C \left\{ (T/t)^\kappa \|g_N(T, \cdot) - g_M(T, \cdot)\| + \|g_N(t, \cdot) - g_M(t, \cdot)\| \right. \\ & \quad + \int_t^T (\|g_N(s, \cdot) - g_M(s, \cdot)\| + \kappa(1/s)(s/t)^\kappa \|g_N(s, \cdot) - g_M(s, \cdot)\| \\ & \quad \left. + (s/t)^\kappa \|g'_N(s, \cdot) - g'_M(s, \cdot)\|) ds \right\}. \end{aligned}$$

Hence applying Schwarz's inequality, we have

$$\begin{aligned} & \int_{T^*}^T \|u_N(t, \cdot) - u_M(t, \cdot)\|^2 dt + (T^*)^{2\kappa} \int_{T^*}^T \|xu_N(t, \cdot) - xu_M(t, \cdot)\|^2 dt \\ & \quad + \int_{T^*}^T \|\partial_x^2 u_N(t, \cdot) - \partial_x^2 u_M(t, \cdot)\|^2 dt \\ & \leq C \left[(T/T^*)^{2\kappa} (T - T^*) \|g_N(T, \cdot) - g_M(T, \cdot)\|^2 + \int_{T^*}^T \|g_N(t, \cdot) - g_M(t, \cdot)\|^2 dt \right. \\ & \quad + (1/2)(T - T^*)^2 \left\{ \int_{T^*}^T \|g_N(\tau, \cdot) - g_M(\tau, \cdot)\|^2 d\tau \right. \\ & \quad + \kappa^2(1/T^2)(T/T^*)^{2\kappa} \int_{T^*}^T \|g_N(\tau, \cdot) - g_M(\tau, \cdot)\|^2 d\tau \\ & \quad \left. \left. + (T/T^*)^{2\kappa} \int_{T^*}^T \|g'_N(\tau, \cdot) - g'_M(\tau, \cdot)\|^2 d\tau \right\} \right]. \end{aligned}$$

Then it is clear that the limit

$$u(t, x) = \lim_{N \rightarrow \infty} u_N(t, x) \in L^2([T^*, T], \Phi_2(T))$$

is the unique solution to (4.13), and Lemma 4.6 is proved.

Q. E. D.

Lemma 4.7. *There exists a solution $v(t, x) \in L^2([T^*, T], \Phi_2(T))$ to (4.10).*

Proof. By (4.14) and Minkowski's inequality, we have

$$\begin{aligned} & \int_t^T \|u_N(\tau, \cdot)\|^2 d\tau + (T^*)^{2\kappa} \int_t^T \|xu_N(\tau, \cdot)\|^2 d\tau + \int_t^T \|\partial_x^2 u_N(\tau, \cdot)\|^2 d\tau \\ & \leq C \left[(T-t) \|u_N(T, \cdot)\|^2 + (T/T^*)^{2\kappa} (T-t) \|\partial_x^2 u_N(T, \cdot)\|^2 \right. \\ & \quad \left. + (T^2/T^*)^{2\kappa} (T-t) \|xu_N(T, \cdot)\|^2 + (T/T^*)^{2\kappa} (T-t) \|g_N(T, \cdot)\|^2 \right] \end{aligned}$$

$$\begin{aligned}
 & + \int_t^T \|g_N(\tau, \cdot)\|^2 d\tau + (1/2)(T-t)^2 \left\{ \int_t^T \|g_N(\tau, \cdot)\|^2 d\tau \right. \\
 & \left. + \kappa^2 (1/T)^2 (T/T^*)^{2\kappa} \int_t^T \|g_N(\tau, \cdot)\|^2 d\tau \right\} \\
 & \left\{ \begin{aligned}
 & + (1/2)(T/T^*)^{2\kappa} (T-t)^2 \int_t^T \|g'_N(\tau, \cdot)\|^2 d\tau \\
 & + (T/T^*)^{2\kappa} (T-t) \int_t^T \int_s^T \|g'_N(\tau, \cdot)\|^2 d\tau ds \end{aligned} \right\}.
 \end{aligned}$$

Letting $N \rightarrow \infty$, and applying Lebesgue's dominated convergence theorem, this implies

$$\begin{aligned}
 (4.15) \quad & \int_t^T \|u(\tau, \cdot)\|^2 d\tau + (T^*)^{2\kappa} \int_t^T \|xu(\tau, \cdot)\|^2 d\tau + \int_t^T \|\partial_x^2 u(\tau, \cdot)\|^2 d\tau \\
 & \leq C \left[(T-t) \|u(T, \cdot)\|^2 + (T/T^*)^{2\kappa} (T-t) \|\partial_x^2 u(T, \cdot)\|^2 \right. \\
 & \quad + (T^2/T^*)^{2\kappa} (T-t) \|xu(T, \cdot)\|^2 + (T/T^*)^{2\kappa} (T-t) \|g(T, \cdot)\|^2 \\
 & \quad + \int_t^T \|g(\tau, \cdot)\|^2 d\tau + (1/2)(T-t)^2 \left\{ \int_t^T \|g(\tau, \cdot)\|^2 d\tau \right. \\
 & \quad \left. + \kappa^2 (1/T)^2 (T/T^*)^{2\kappa} \int_t^T \|g(\tau, \cdot)\|^2 d\tau \right\} \\
 & \left. \left\{ \begin{aligned}
 & + (1/2)(T/T^*)^{2\kappa} (T-t)^2 \int_t^T \|g'(\tau, \cdot)\|^2 d\tau \\
 & + (T/T^*)^{2\kappa} (T-t) \int_t^T \int_s^T \|g'(\tau, \cdot)\|^2 d\tau ds \end{aligned} \right\} \right].
 \end{aligned}$$

Therefore, we have proved that (4.13) admits a unique solution $u(t, x) \in L^2([T^*, T], \Phi_2(T))$ with the energy inequality (4.15) whenever $g(t, x) \in C^0([T^*, T], L^2(\mathbf{R}^+))$ and $\partial_t g(t, x) \in L^2([T^*, T], L^2(\mathbf{R}^+))$ exists for almost everywhere $t \in [T^*, T]$. From (4.11.j),

$$\begin{aligned}
 & i\partial_t(v_{j+1}-v_j) + (1/2)\partial_x^2(v_{j+1}-v_j) - xt^k(v_{j+1}-v_j) \\
 & = \sum_{l=1}^j \binom{m}{l} x(\partial_t^l t^k) \int_t^T \frac{(t-s)^{l-1}}{(l-1)!} (v_j - v_{j-1})(s, x) ds \\
 & (v_{j+1}-v_j)(T, x) = 0, \quad (v_{j+1}-v_j)(t, 0) = 0, \quad v_0(t, x) \equiv 0.
 \end{aligned}$$

Applying (4.15) to the above,

$$\begin{aligned}
 & \int_t^T \|(v_{j+1}-v_j)(\tau, \cdot)\|^2 d\tau + (T^*)^{2\kappa} \int_t^T \|x(v_{j+1}-v_j)(\tau, \cdot)\|^2 d\tau \\
 & \quad + \int_t^T \|\partial_x^2(v_{j+1}-v_j)(\tau, \cdot)\|^2 d\tau \\
 & \leq C(T, T^*, m, \kappa) \int_t^T \int_\tau^T \|x(v_j - v_{j-1})(s, \cdot)\|^2 ds d\tau.
 \end{aligned}$$

Here we have used the estimate:

$$\begin{aligned} & \int_t^T \left\| \sum_{l=1}^j \binom{m}{l} (\partial_{\tau}^l \tau^{\kappa}) \int_{\tau}^T \frac{(\tau-s)^{l-1}}{(l-1)!} x(v_j - v_{j-1})(s, \cdot) ds \right\|^2 d\tau \\ & \leq \int_t^T \left\{ \int_{\tau}^T \sum_{l=1}^j \binom{m}{l} (\partial_{\tau}^l \tau^{\kappa}) \frac{(\tau-s)^{l-1}}{(l-1)!} \|x(v_j - v_{j-1})(s, \cdot)\| ds \right\}^2 d\tau \\ & \leq \int_t^T \left\{ \int_{\tau}^T \left| \sum_{l=1}^j \binom{m}{l} (\partial_{\tau}^l \tau^{\kappa}) \frac{(\tau-s)^{l-1}}{(l-1)!} \right|^2 ds \right\} \left\{ \int_{\tau}^T \|x(v_j - v_{j-1})(s, \cdot)\|^2 ds \right\} d\tau. \end{aligned}$$

Now by Lemma 4.1,

$$\int_{\tau}^T \|x(v_1 - v_0)(s, \cdot)\|^2 ds \leq M.$$

Hence by putting

$$\begin{aligned} w_j(t) &= \int_t^T \|(v_{j+1} - v_j)(\tau, \cdot)\|^2 d\tau + (T^*)^{2\kappa} \int_t^T \|x(v_{j+1} - v_j)(\tau, \cdot)\|^2 d\tau \\ &\quad + \int_t^T \|\partial_x^2(v_{j+1} - v_j)(\tau, \cdot)\|^2 d\tau \end{aligned}$$

we have

$$w_1(t) \leq C \int_t^T M d\tau = CM(T-t), \quad \text{and} \quad w_{j+1}(t) \leq C \int_t^T w_j(\tau) d\tau.$$

So

$$w_j(t) \leq M \frac{\{C(T-t)\}^j}{j!} \leq M \frac{\{C(T-T^*)\}^j}{j!}$$

which tends to zero as $j \rightarrow \infty$. Therefore $\{v_j(t, x)\}_{j=0}^{\infty}$ is a Cauchy sequence in $L^2([T^*, T], \Phi_2(T))$, and there exist a solution $v(t, x) \in L^2([T^*, T], \Phi_2(T))$ to (4.10) such that

$$v_j(t, x) \longrightarrow v(t, x) \quad \text{as } j \rightarrow \infty.$$

This completes the proof of Lemma 4.7.

Q. E. D.

From the above arguments, we have constructed $\tilde{u}(t, x)$ of the form (4.8) satisfying (3.6). This shows that

$$\tilde{u}(t, x) \in C^m([T^*, T], L^2(\mathbf{R}^+)),$$

and

$$\partial_x^2 \tilde{u}(t, x), x \tilde{u}(t, x) \in C^{m-1}([T^*, T], L^2(\mathbf{R}^+)).$$

Also, by Corollary 4.4, $\tilde{u}(t, x) = u(t, x)$ where $u(t, x)$ is the solution to (3.6). Summing up, we have proved the following result.

Proposition 4.8. *Let $u(t, x)$ be the solution to (3.6). Then*

$$u(t, x) \in C^m([T^*, T], L^2(\mathbf{R}^+)),$$

and

$$\partial_x^2 u(t, x), \quad xu(t, x) \in C^{m-1}([T^*, T], L^2(\mathbf{R}^+)),$$

that is

$$u(t, x) \in C^{m-1}([T^*, T], \Phi_2(T)).$$

Next, by using this result, we show that if $t \in [T^*, T]$, $u(t, x)$ has higher order derivatives with respect to x , and is rapidly decreasing as $x \rightarrow +\infty$.

Before going further we introduce some notation which will be frequently used in the subsequent arguments.

Notation. We define the Φ_n -norm $\|u(t, \cdot)\|_{\Phi_n}$ ($n \geq 0$) of $u(t, \cdot)$ by $\|u(t, \cdot)\|_{\Phi_n} = \sum_{p=0}^n \|x^{p/2} \partial_x^{n-p} u(t, \cdot)\|$, and call the set $\{x^{p/2} \partial_x^{n-p} u(t, x), 0 \leq p \leq n\}$ Φ_n -class of $u(t, x)$.

Lemma 4.9. *Let $u(t, x) \in C^m([T^*, T], L^2(\mathbf{R}^+)) \cap C^{m-1}([T^*, T], \Phi_2(T))$ be the solution to (3.6). Then $x^{p/2} \partial_x^{2m-p} u(t, x) \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$, $0 \leq p \leq 2m$.*

Proof. We will show that the Φ_{2n} -norm ($n \leq m$) of $u(t, \cdot)$ is in $L^\infty([T^*, T])$. The proof is done by induction on n . By Proposition 4.8, this is true for the case $n=1$. Suppose that the Φ_{2k} -norm ($k \leq n$) is in $L^\infty([T^*, T])$. If we integrate $\int_0^\infty x \partial_t^n u \cdot x \overline{\partial_t^{n-1} u} dx < \infty$ ($t \in [T^*, T]$) by parts using (3.6), $\|x^{(2+p)/2} \partial_x^{2n-1-p} u\|^2$ $0 \leq p \leq 2n-1$ appear as its positive parts, and the other terms are bounded by the Φ_{2k} -norms ($k \leq n$). So we have $x^{(2+p)/2} \partial_x^{2n-1-p} u \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$ $0 \leq p \leq 2n-1$. Furthermore, $\partial_t^2 \partial_x u \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$ implies $\partial_x^{2n+1} u \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$, and from $\int_0^\infty \partial_x^{2n+1} u \cdot x \overline{\partial_x^{2n-1} u} dx < \infty$ ($t \in [T^*, T]$), we have $x^{1/2} \partial_x^{2n} u \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$. Thus the Φ_{2n+1} -norm of $u(t, \cdot)$ is in $L^\infty([T^*, T])$. In the same way, integrating $\int_0^\infty x \partial_t^n u \cdot x \overline{\partial_t^n u} dx < \infty$ ($t \in [T^*, T]$) by parts, it follows

$$x^{(2+p)/2} \partial_x^{2n-p} u \in L^\infty([T^*, T], L^2(\mathbf{R}^+)) \quad 0 \leq p \leq 2n,$$

and $\partial_t^n \partial_x^2 u \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$ implies $\partial_x^{2n+2} u \in L^\infty([T^*, T], L^2(\mathbf{R}^+))$. Moreover, from $\int_0^\infty \partial_x^{2n+2} u \cdot x \overline{\partial_x^{2n} u} dx < \infty$ ($t \in [T^*, T]$), it follows

$$x^{1/2} \partial_x^{2n+1} u \in L^\infty([T^*, T], L^1(\mathbf{R}^+)).$$

Thus the Φ_{2n+2} -norm of $u(t, \cdot)$ is in $L^\infty([T^*, T])$.

Q. E. D.

Finally, we show Lemma 4.9 still holds for $T^*=0$, and also prove the existence of a function $\rho(N)$ satisfying (3.1), (3.2) and (3.3).

Lemma 4.10. *Suppose that*

$$\text{Sup}_{x \geq 0} |t^k x^l \partial_t^m \partial_x^n h(t, x)| \rightarrow 0 \quad (t \rightarrow +\infty)$$

for any nonnegative integer k, l, m, n with

$$k+l+m+n \leq \kappa(N^2+2N)+3N+2$$

and let $u(t, x)$ be the solution to (3.5). Then for each nonnegative integer k, l, m, n such that $2k+2l+2m+n \leq 2N$ $\|t^k x^l \partial_t^m \partial_x^n u(t, \cdot)\|$ is bounded for all t ($0 \leq t < \infty$).

Proof. First we prove that each Φ_m -norm of $u(t, x)$ ($m=0, 1, 2, \dots, 2N$) is bounded as $t \rightarrow +0$. We prove this by induction on m .

Since (4.5) implies $\|u(t, \cdot)\| \leq \|h(T, \cdot)\|$, this is true for $m=0$. Now assume that Φ_m -norm ($m \leq 2n$) is bounded as $t \rightarrow +0$. From (3.5) we can easily derive for $n=0, 1, 2, \dots, N-1$

$$(4.16) \quad \begin{cases} i\partial_t(x^n \partial_x u) = (-1/2)\partial_x^2(x^n \partial_x u) + xt^\kappa(x^n \partial_x u) + \dots, \\ i\partial_t(x^{n+1} u) = (-1/2)\partial_x^2(x^{n+1} u) + xt^\kappa(x^{n+1} u) + x^n \partial_x u + \dots, \\ i\partial_t(x^{n-k-1} \partial_x \partial_t^{1+k} u) = (-1/2)\partial_x^2(x^{n-k-1} \partial_x \partial_t^{1+k} u) \\ \quad + xt^\kappa(x^{n-k-1} \partial_x \partial_t^{1+k} u) + c\kappa t^{\kappa-1} x^{n-k} \partial_x \partial_t^k u + \dots \text{ for } k=0, 1, \dots, n-1, \\ i\partial_t(x^{n-k} \partial_t^{1+k} u) = (-1/2)\partial_x^2(x^{n-k} \partial_t^{1+k} u) + xt^\kappa(x^{n-k} \partial_t^{1+k} u) \\ \quad + c\kappa t^{\kappa-1} x^{n+1-k} \partial_t^k u + c x^{n-1-k} \partial_x \partial_t^{k+1} u + \dots \text{ for } k=0, 1, 2, \dots, n. \end{cases}$$

Here \dots means the terms of the functions in Φ_m ($m \leq 2n$) class. Using these equations and L^2 estimate (4.5) repeatedly, we find that the Φ_m -norm ($m \leq 2n+2$) of $u(t, x)$ is bounded as $t \rightarrow +0$. This completes the induction.

Next, by roughly estimating the number of t included in each term of (4.16), each $\|x^l \partial_t^m \partial_x^n u(t, \cdot)\|$ ($2l+2m+n \leq 2N$) is estimated from above by $\|T^k x^l \partial_T^m \partial_x^n h(T, \cdot)\|$ ($k \leq \kappa(N^2+2N)$, $2l+2m+n \leq 2N$). Moreover from

$$i\partial_t(t^n u) + (1/2)\partial_x^2(t^n u) - xt^\kappa(t^n u) = int^{n-1} u \quad (n=1, 2, \dots).$$

$\|t^k u(t, \cdot)\|$ is estimated from above by $\|T^k h(T, \cdot)\|$. Summing up, each $\|t^k x^l \partial_t^m \partial_x^n u(t, \cdot)\|$ ($2k+2l+2m+n \leq 2N$) is estimated by $\|T^k x^l \partial_t^m \partial_x^n h(T, \cdot)\|$ ($k+l+m+n \leq \kappa(N^2+2N)+3N$). This completes the proof. Q.E.D.

Finally we remark about the proof of the existence of $\rho(N)$. First note that

$$\|T^k x^l \partial_t^m \partial_x^n h(T, \cdot)\| \quad (k+l+m+n \leq \kappa(N^2+2N)+3N)$$

is integrable with respect to T on $[0, \infty)$ if

$$\text{Sup}_{T \geq 0} \|T^k x^l \partial_t^m \partial_x^n h(T, \cdot)\| < \infty \quad (k+l+m+n \leq \kappa(N^2+2N)+3N+2).$$

Then using Duhamel's principle and the estimate stated in the proof of Lemma 4.10, we have

$$\|t^k x^l \partial_t^m \partial_x^n u(t, \cdot)\| < \infty \quad (2k+2l+2m+n \leq 2N, t \geq 0).$$

This shows the existence of a function $\rho(N)$ such that

$$\rho(N) \longrightarrow \infty \quad \text{as } N \rightarrow \infty .$$

Thus we have proved Theorem 2.3.

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