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The Super-Toda Lattice Hierarchy

By

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Abstract

The super-Toda lattice (STL) hierarchy is introduced. The equivalence between the Lax representation and Zakharov-Shabat representation of the STL hierarchy is shown. Introducing the Lie superalgebra $\mathrm{osp}(\infty|\infty)$, the ortho-symplectic (OSp)-STL hierarchy is defined as well. These equations are solved through the Riemann-Hilbert decomposition of corresponding infinite dimensional Lie supergroups. An explicit representation of solutions is given by means of the super-τ field.

§ 0. Introduction

In this paper we consider the super-Toda lattice (STL) hierarchy. This paper is a complete version of our previous announcement [4], containing some new results.

The STL equations are considered in several ways. V. A. Andreyev [1] and M. A. Olshanetsky [9] classified the STL equations according to Lie superalgebras and solved them by inverse scattering method.

Inspired by the studies of the super-KP hierarchy by Yu. I. Manin and A. O. Radul [7], M. Mulase [8], K. Ueno, H. Yamada and K. Ikeda [12], [13], [14], [15], we try to investigate the STL equation by a method which is different from [1] and [9]. We extend the Toda lattice (TL) hierarchy, which is introduced by K. Ueno and K. Takasaki in [11], to a supersymmetric one. Considering the STL hierarchy, we can naturally extend several concepts in the theory of soliton equations, namely, the tau (τ) function and the reduction of solutions.

We first define the STL hierarchy through the Lax representation and show the equivalence with the Zakharov-Shabat representation. From the STL hierarchy, we derive the STL equation,

$$\tilde{D}_1^+ \tilde{D}_1^- u(s) = \exp(u(s) - u(s-1)) + \exp(u(s+1) - u(s)),$$

where $\tilde{D}_1^{\pm}$ are certain odd derivations. Introducing an infinite dimensional Lie superalgebra $\mathrm{osp}(\infty|\infty)$ (cf. [5], [6]), we define the ortho-symplectic (OSp)-STL hierarchy. The STL equation derived from the OSp-STL hierarchy, the OSp-STL equation, reduces to the STL equation studied in [1] by adding the condition of "$4N$-periodicity". Putting $N = 1$ the OSp-STL equation is simplified to the super-sine-Gordon equation. To solve a Cauchy problem of the STL hierarchy, we consider a Riemann-Hilbert (R-H) decomposition of the Lie supergroup $\mathrm{SGL}(S)$, where S is an algebra of superfields.

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Through the R-H decomposition we represent components of the wave matrices of the STL hierarchy in terms of the super-τ field. Considering the R-H decomposition for constant matrices of the Lie supergroup OSp(S), we solve the OSp-STL hierarchy. In his recent paper [10], Takasaki determines the equations satisfied by the super-τ field of the super-KP hierarchy in terms of the differential algebra generated by the coefficients of the wave operator of the super-KP hierarchy. It is an interesting problem to find out the equations for our super-τ field, applying his idea.

This paper is organized as follows. In Section 1, we review shortly the theory of the TL hierarchy according to [11]. In Section 2, we define the STL hierarchy and show that the Lax representation of the STL hierarchy is equivalent to that of Zakharov-Shabat (Z-S)'s. We discuss the STL equation derived from the STL hierarchy. We also derive the ordinary TL hierarchy by taking the body part of the STL hierarchy. In Section 3, we introduce an infinite dimensional Lie superalgebra osp($\infty | \infty$) and define the OSp-STL hierarchy. We consider several equations reduced from OSp-STL hierarchy. We derive the BTL and CTL hierarchy [11] by taking the body part of the OSp-STL hierarchy. In Section 4, we solve the STL hierarchy by the R-H decomposition and give a representation of the solutions in terms of the super-τ field. We also discuss the R-H decomposition of the OSp-STL hierarchy.

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§ 1. Review of the TL Hierarchy

Let $\Lambda = (\delta_{i+1,j})_{i,j \in \mathbb{Z}}$ be the $\mathbb{Z} \times \mathbb{Z}$ shift matrix. We denote by diag$[a(s)]$ the diagonal matrix $\begin{bmatrix} \ddots & & & \\ & a(-1) & & \\ & & a(0) & \\ & & & a(1) \\ & & & & \ddots \end{bmatrix}$. Consider a $\mathbb{Z} \times \mathbb{Z}$ matrices of the form $A = \sum_{j \in \mathbb{Z}} \mathrm{diag}[a_j(s)]\Lambda^j$. We define $(A)_+$ and $(A)_-$ by

$$(A)_+ = \sum_{j \geq 0} \mathrm{diag}\,[a_j(s)]\Lambda^j\,, \qquad (A)_- = \sum_{j<0} \mathrm{diag}[a_j(s)]\Lambda^j\,.$$

We call $(A)_+$ and $(A)_-$ the plus (+)-part and the minus (−)-part of A, respectively.

We introduce infinitely many time variables $t^+ = (t_1^+, t_2^+, \ldots)$ and $t^- = (t_1^-, t_2^-, \ldots)$. Put $\mathscr{K}$ be the quotient field of $\mathbb{C}[[t^+, t^-]]$. The Toda lattice (TL) hierarchy is defined as follows. Let L and M be $\mathbb{Z} \times \mathbb{Z}$ matrices of the form $L = \sum_{j=0}^{\infty} \mathrm{diag}[u_j(s)]\Lambda^{1-j}$, $M = \sum_{j=0}^{\infty} \mathrm{diag}[v_j(s)]\Lambda^{-1+j}$, where $u_j(s), v_j(s) \in \mathscr{K}$, $u_0(s) = 1$ and $v_0(s) \neq 0$. The TL hierarchy is the system of infinitely many Lax equations:

$$\begin{aligned} \partial_{t_n^+}L &= [B_n, L], \qquad \partial_{t_n^+}M = [B_n, M], \\ \partial_{t_n^-}L &= [C_n, L], \qquad \partial_{t_n^-}M = [C_n, M], \end{aligned} \tag{1.1}$$

where $B_n = (L^n)_+$ and $C_n = (M^n)_-$.

Theorem 1.1 ([11]). *The TL hierarchy* (1.1) *is equivalent to the following system of the Zakharov-Shabat* (*Z-S*) *equations.*

$$\begin{aligned} \partial_{t_n^+}B_m - \partial_{t_m^+}B_n &= [B_n, B_m], \\ \partial_{t_n^+}C_m - \partial_{t_m^-}B_n &= [B_n, C_m], \\ \partial_{t_n^-}C_m - \partial_{t_m^-}C_n &= [C_n, C_m]. \end{aligned} \tag{1.2}$$

△

Put $m = n = 1$ in the second equation and put $B_1 = \Lambda + \mathrm{diag}[b(s)]$, $C_1 = \mathrm{diag}[c(s)]\Lambda^{-1}$. Let $u(s)$ be a function such that $b(s) = \partial_{t_1^+}u(s)$ and $c(s) = \exp(u(s) - u(s-1))$. Then we have the two dimensional Toda lattice equation

$$\partial_{t_1^+}\partial_{t_1^-}u(s) = \exp(u(s) - u(s-1)) - \exp(u(s+1) - u(s)).$$

From (1.2), there exist

$$W_+ = \sum_{j=0}^{\infty} \mathrm{diag}[w_j^+(s)]\Lambda^{-j}, \qquad W_- = \sum_{j=0}^{\infty} \mathrm{diag}[w_j^-(s)]\Lambda^{j},$$

with $w_j^{\pm}(s) \in \mathscr{K}$, $w_0^+(s) = 1$ and $w_0^-(s) \neq 0$, satisfying

$$\begin{aligned} \partial_{t_n^+}W_+ &= B_n W_+ - W_+\Lambda^n, \qquad \partial_{t_n^+}W_- = C_n W_-, \\ \partial_{t_n^-}W_+ &= C_n W_+, \qquad \partial_{t_n^-}W_- = C_n W_- - W_-\Lambda^{-n}, \end{aligned} \tag{1.3}$$

where $B_n = (W_+\Lambda^n W_+^{-1})_+$, $C_n = (W_-\Lambda^{-n}W_-^{-1})_-$. Conversely the existence of $W_\pm$ satisfying (1.3) implies (1.2) as a compatibility condition of (1.3). We call (1.3) the Sato equations and $W_\pm$ the wave matrices of the TL hierarchy.

To solve the TL hierarchy we introduce an infinite dimensional Lie group $GL(\infty; \mathscr{K})$ as follows.

$$GL(\infty; \mathscr{K}) = \{A \in \mathrm{Mat}(\mathbf{Z} \times \mathbf{Z}; \mathscr{K}) | A \text{ is invertible.}\}.$$

Note that $W_\pm \in GL(\infty; \mathscr{K})$. Put $\Phi_+ = \Phi_+(t^+) = \exp\left(\sum_{j=1}^{\infty} t_j^+\Lambda^j\right)$ and $\Phi_-(t^-) = \exp\left(\sum_{j=1}^{\infty} t_j^-\Lambda^{-j}\right)$. We denote by $GL(\infty; \mathbf{C})$ the subgroup of $GL(\infty; \mathscr{K})$ consisting of constant matrices.

Theorem 1.2 ([11]). *Suppose that, for $A \in GL(\infty; \mathbf{C})$, the following decomposition with $W_\pm$ is satisfied*:

$$\Phi_+ A \Phi_-^{-1} = W_+^{-1}W_-. \tag{1.4}$$

Then $W_\pm$ are the wave matrices of the TL hierarchy. △

We call (1.4) the Riemann-Hilbert (R-H) decomposition. Put $H = \Phi_+ A \Phi_- =$

$(h_{ij})_{i,j\in\mathbb{Z}}$. From (1.4) we see that $(W_+H)_- = 0$. This induces the linear equation

$$(\dots w_2^+(s), w_1^+(s), 1)M(s) = (\dots 0, 0, 0)$$

for all $s \in \mathbb{Z}$, where $M(s) = (h_{s-i+1,s-j})_{i,j<0}$. Put $\tau(s) = \det(h_{s-i,s-j})_{i,j>0}$. Then we have the following explicit representation.

Theorem 1.3 ([11]). (i) $w_j^+(s) = \dfrac{p_j(-\tilde{\partial}_+)\tau(s)}{\tau(s)}$,

(ii) $w_j^-(s) = \dfrac{p_j(-\tilde{\partial}_-)\tau(s+1)}{\tau(s)}$,

where $p_j(t_1, t_2, \dots.)$ *is defined by* $\exp\left(\sum_{j=1}^{\infty} t_j k^j\right) = \sum_{j=0}^{\infty} p_j(t_1, t_2, \dots.)k^j$ *and* $\tilde{\partial}_\pm = (\partial_{t_1^\pm}, 2^{-1}\partial_{t_2^\pm}, 3^{-1}\partial_{t_3^\pm}, \dots.)$. △

We define the matrices $J = ((-)^i\delta_{i,-j})_{i,j\in\mathbb{Z}}$ and $K = AJ$. We introduce the infinite dimensional Lie algebras

$$o(\infty; \mathscr{K}) = \{A \in \mathrm{Mat}(\mathbb{Z} \times \mathbb{Z}; \mathscr{K}) | J^tA + AJ = 0\},$$

$$sp(\infty; \mathscr{K}) = \{A \in \mathrm{Mat}(\mathbb{Z} \times \mathbb{Z}; \mathscr{K}) | KA + {}^tAK = 0\}.$$

The corresponding Lie groups are

$$O(\infty; \mathscr{K}) = \{A \in \mathrm{Mat}(\mathbb{Z} \times \mathbb{Z}; \mathscr{K}) | J^tAJ = A^{-1}\},$$

$$Sp(\infty; \mathscr{K}) = \{A \in \mathrm{Mat}(\mathbb{Z} \times \mathbb{Z}; \mathscr{K}) | {}^tK^tAK = A^{-1}\}.$$

Put $t^{\pm}{}_{2n} = 0$ for $n = 1, 2, \dots$ in (1.3). By adding symmetry conditions to the wave matrices so that $W_\pm \in O(\infty; \mathscr{K})$ (resp. $Sp(\infty; \mathscr{K})$), we obtain the BTL (resp. CTL) hierarchy.

Remark. Assume that $W_\pm$ are the wave matrices of the BTL (resp. CTL) hierarchy. Then $B_n = (W_+A^nW_+^{-1})_+$ and $C_n = (W_-A^{-n}W_-^{-1})_- \in o(\infty; \mathscr{K})$ (resp. $sp(\infty; \mathscr{K})$) for odd n.

Theorem 1.4 ([11]). *Let* $t^{\pm}{}_{2n} = 0$ *for* $n = 1, 2, \dots$. *Assume that* $A \in O(\infty; \mathbb{C})$ (resp. $Sp(\infty; \mathbb{C})$) *in* (1.4). *Then* $W_\pm \in O(\infty; \mathscr{K})$ (resp. $Sp(\infty; \mathscr{K})$), *namely* $W_\pm$ *are the wave matrices of the* BTL (resp. CTL) *hierarchy.* △

§2. The STL Hierarchy

Throughout this paper, $\underline{k}$ denotes the modulo class of k by $\mathbb{Z}_2$. Let $\mathscr{A} = \mathscr{A}_{\underline{0}} \oplus \mathscr{A}_{\underline{1}}$ be an arbitrary supercommutative superalgebra. The body map is the canonical projection $\varepsilon: \mathscr{A} \to \mathscr{A}/(\mathscr{A}_{\underline{1}})$, where $(\mathscr{A}_{\underline{1}})$ is the ideal generated by $\mathscr{A}_{\underline{1}}$. For $a = a_{\underline{0}} + a_{\underline{1}} \in \mathscr{A}$, $a_{\underline{i}} \in \mathscr{A}_{\underline{i}}$, we denote $a^* = a_0 - a_1$. The body map ε and the operator $*$ can be naturally extended on the superalgebra of matrices with entries in $\mathscr{A}$. Namely, for $A = (a_{ij})_{i,j}(a_{ij} \in \mathscr{A})$, $\varepsilon(A) = (\varepsilon(a_{ij}))_{i,j}$ and $A^* = (a_{ij}^*)_{i,j}$. For $k \in \mathbb{Z}$, put $A^{*(k)} = A(\underline{k} = 0)$, $= A^*(\underline{k} = 1)$.

Moreover we put, for $n \geq 1$, $A_*^n = A^{*(n-1)} A_*^{n-1}$, $A_*^1 = A$. If D is an odd derivation acting on $\mathscr{A}$, then it satisfies the super-Leibniz rule:

$$D(ab) = (Da)b + a^*(Db), \qquad a, b \in \mathscr{A} .$$

Let V be the Grassmann algebra $\Lambda(\mathbf{C}^\infty)$ with infinitely many generators, $e_1, e_2, \dots$. Let $t^+ = (t_1^+, t_2^+, \dots)$ and $t^- = (t_1^-, t_2^-, \dots)$ be infinitely many Grassmann variables. Here $t_{2j}^{\pm}$ are even (commutative) variables and $t_{2j-1}^{\pm}$ are odd (anti-commutative) ones. Let $\mathscr{K}$ be the quotient field of the $\mathbf{C}$-algebra $\mathbf{C}[[t_2^{\pm}, t_4^{\pm}, \dots]]$. The supercommutative superalgebra $S = S_{\underline{0}} \oplus S_{\underline{1}}$ is now defined by

$$S = \mathscr{K} \otimes \mathbf{C}[[t_1^{\pm}, t_3^{\pm}, \dots]] \otimes V .$$

We define the super-vector fields, acting on S,

$$D_{2j}^{\pm} = \partial_{t_{2j}^{\pm}} , \qquad D_{2j-1}^{\pm} = \partial_{t_{2j-1}^{\pm}} + \sum_{k \geq 1} t_{2k-1}^{\pm} \partial_{t_{2j+2k-2}^{\pm}} , \qquad j \geq 1 .$$

These super-vector fields satisfy the (anti-) commutation relations

$$[D_j^{\pm}, D_k^{\pm}]_{(-)^{jk+1}} = 2\delta_{\underline{jk},\underline{1}} D_{j+k}^{\pm} , \qquad [D_j^{\pm}, D_k^{\mp}]_{(-)^{jk+1}} = 0 ,$$

where $[\ ]_1 = [\ ]_+$ (=anti-commutator) and $[\ ]_{-1} = [\ ]$ (=commutator). Now we give the definition of the super-Toda lattice (STL) hierarchy. Define the matrices L and M as follows:

$$L = \sum_{j=0}^{\infty} \operatorname{diag}[u_j(s)] \Lambda^{1-j} , \qquad \text{with} \quad u_0(s) = 1 \quad \text{and} \quad u_j(s) \in S_{\underline{j}} .$$

$$M = \sum_{j=0}^{\infty} \operatorname{diag}[v_j(s)] \Lambda^{-1+j} , \qquad \text{with} \quad \varepsilon(v_0(s)) \neq 0 \quad \text{and} \quad v_j(s) \in S_{\underline{j}} .$$

We put $B_n = (L_*^n)_+$, $C_n = (M_*^n)_-$. The STL hierarchy is a system of equations of the Lax type,

$$D_m^+ L = (-)^m B_m^* L - L^{*(m)} B_m + 2\delta_{\underline{m},\underline{1}} L_*^{m+1} , \tag{2.1}$$

$$D_m^+ M = (-)^m B_m^* M - M^{*(m)} B_m , \tag{2.2}$$

$$D_m^- L = (-)^m C_m^* L - L^{*(m)} C_m , \tag{2.3}$$

$$D_m^- M = (-)^m C_m^* M - M^{*(m)} C_m + 2\delta_{\underline{m},\underline{1}} M_*^{m+1} . \tag{2.4}$$

Theorem 2.1. *The STL hierarchy* (2.1) ~ (2.4) *is equivalent to the system of the Zakharov-Shabat* (*Z-S*) *type*:

$$D_n^+ C_m - (-)^{mn} D_m^- B_n = (-)^{mn} B_n^{*(m)} C_m - C_m^{*(n)} B_n . \tag{2.5}$$

$$D_m^+ B_n - (-)^{mn} D_n^+ B_m = (-)^{mn} B_m^{*(n)} B_n - B_n^{*(m)} B_m + 2\delta_{\underline{mn},\underline{1}} B_{m+n} , \tag{2.6}$$

$$D_m^- C_n - (-)^{mn} D_n^- C_m = (-)^{mn} C_m^{*(n)} C_n - C_n^{*(m)} C_m + 2\delta_{\underline{mn},\underline{1}} C_{m+n} . \tag{2.7}$$

Proof. First we show that the Lax type system ((2.1) ~ (2.4)) induces the Z-S type system ((2.5) ~ (2.7)). We can easily see that

$$D_m^+ L_*^n = (-)^{mn} B_m^{*(n)} L_*^n - L_*^{n*(m)} B_m + 2\delta_{\underline{mn},\underline{1}} L_*^{m+n}\,, \tag{2.8}$$

$$D_m^+ M_*^n = (-)^{mn} B_m^{*(n)} M_*^n - M_*^{n*(m)} B_m\,, \tag{2.9}$$

$$D_m^- L_*^n = (-)^{mn} C_m^{*(n)} L_*^n - L_*^{n*(m)} C_m\,, \tag{2.10}$$

$$D_m^- M_*^n = (-)^{mn} C_m^{*(n)} M_*^n - M_*^{n*(m)} C_m + 2\delta_{\underline{mn},\underline{1}} M_*^{m+n}\,. \tag{2.11}$$

$$(m, n = 1, 2, 3, \dots)$$

Taking the $(+)$-part of (2.8), we have

$$\begin{aligned} D_m^+ B_n - (-)^{mn} D_n^+ B_m = {} & 2(-)^{mn} B_m^{*(n)} B_n - 2B_n^{*(m)} B_m + (-)^{mn}(B_m^{*(n)}(L_*^n)_-)_+ \\ & - ((L_*^n)_-^{*(m)} B_m)_+ - (B_n^{*(m)}(L_*^m)_-)_+ \\ & + (-)^{mn}((L_*^m)_-^{*(n)} B_n)_+ + 4\delta_{\underline{mn},\underline{1}} B_{m+n}\,. \end{aligned} \tag{2.12}$$

Using the identity

$$L_*^{n*(m)} L_*^m - (-)^{mn} L_*^{m*(n)} L_*^n = 2\delta_{\underline{mn},\underline{1}} L_*^{m+n}\,, \tag{2.13}$$

we have

$$\begin{aligned} & (B_n^{*(m)}(L_*^m)_-)_+ + ((L_*^n)_-^{*(m)} B_m)_+ - (-)^{mn}(B_m^{*(n)}(L_*^n)_-)_+ - (-)^{mn}((L_*^m)_-^{*(n)} B_n)_+ \\ & = -B_n^{*(m)} B_m + (-)^{mn} B_m^{*(n)} B_n + 2\delta_{\underline{mn},\underline{1}} B_{m+n}\,. \end{aligned}$$

Then (2.12) reduces to (2.6). From (2.9) and (2.10), we have

$$D_m^+ M_*^n = (-)^{mn} B_m^{*(n)} M_*^n - M_*^{n*(m)} B_m\,, \tag{Υ}$$

$$D_n^- L_*^m = (-)^{mn} C_n^{*(m)} L_*^m - L_*^{m*(n)} C_n\,. \tag{$\Upsilon\Upsilon$}$$

We denote by $(\Upsilon)_-$ and $(\Upsilon\Upsilon)_+$ the $(-)$-part of (Υ) and the $(+)$-part of $(\Upsilon\Upsilon)$. By taking the difference $(\Upsilon)_- - (-)^{mn}(\Upsilon\Upsilon)_+$ we obtain (2.5). We obtain (2.7) similarly.

Now we show the converse.

Lemma 2.2. The matrices L, M satisfy the following equations:

$$\begin{aligned} & D_m^+ L_*^n - (-)^{mn} B_m^{*(n)} L_*^n + L_*^{n*(m)} B_m - 2\delta_{\underline{mn},\underline{1}} L_*^{m+n} \\ & = \sum_{l=0}^{n-1} (-)^{(l+n+1)m} L_*^{l*(m+n+l)}(D_m^+ L - (-)^m B_m^* L + L^{*(m)} B_m - 2\delta_{\underline{m},\underline{1}} L_*^{m+1})^{*(l+n+1)} L_*^{n-1-l}\,, \end{aligned} \tag{2.14}$$

$$\begin{aligned} & D_m^- L_*^n - (-)^{mn} C_m^{*(n)} L_*^n + L_*^{n*(m)} C_m \\ & = \sum_{l=0}^{n-1} (-)^{m(l+n+1)} L_*^{l*(m+n+l)}(D_m^- L - (-)^m C_m^* L + L^{*(m)} C_m)^{*(l+n+1)} L_*^{n-l-1}\,, \end{aligned} \tag{2.15}$$

$$\begin{aligned} & D_m^+ M_*^n - (-)^{mn} B_m^{*(n)} M_*^n + M_*^{n*(m)} B_m \\ & = \sum_{l=0}^{n-1} (-)^{m(l+n+1)} M_*^{l*(m+l+n)}(D_m^+ M - (-)^m B_m^* M + M^{*(m)} B_m)^{*(l+n+1)} M_*^{n-1-l}\,, \end{aligned} \tag{2.16}$$

$$\begin{aligned} & D_m^- M_*^n - (-)^{mn} C_m^{*(n)} M_*^n + M_*^{n*(m)} C_m - 2\delta_{\underline{mn},\underline{1}} M_*^{m+n} \\ & = \sum_{l=0}^{n-1} (-)^{(l+n+1)m} M_*^{l*(m+n+l)}(D_m^- M - (-)^m C_m^* M + M^{*(m)} B_m \\ & \quad - 2\delta_{\underline{m},\underline{1}} M_*^{m+1})^{*(l+n+1)} M_*^{n-1-l}\,. \end{aligned} \tag{2.17}$$

Proof. First we show (2.14) by induction. It holds trivially for $n = 1$. For $n \geq 0$,

$$\begin{aligned} D_m^+ L_*^{n+1} &- (-)^{(n+1)m} B_m^{*(n+1)} L_*^{n+1} + L_*^{n+1*(m)} B_m - 2\delta_{\underline{m(n+1)},\underline{1}} L_*^{m+n+1} \\ &= (-)^{mn}(D_m^+ L - (-)^m B_m^* L + L^{*(m)} B_m - 2\delta_{\underline{m},\underline{1}} L_*^{m+1})^{*(n)} L_*^n \\ &\quad + L^{*(m+n)}(D_m^+ L_*^n - (-)^{mn} B_m^{*(n)} L_*^n + L_*^{n*(m)} B_m - 2\delta_{\underline{mn},\underline{1}} L_*^{m+n}) \\ &\quad + 2((-)^{mn}\delta_{\underline{m},\underline{1}} + \delta_{\underline{mn},\underline{1}} - \delta_{\underline{m(n+1)},\underline{1}}) L_*^{m+n+1} . \end{aligned}$$

Notice that $(-)^{mn}\delta_{\underline{m},\underline{1}} + \delta_{\underline{mn},\underline{1}} - \delta_{\underline{m(n+1)},\underline{1}} = 0$. By the induction hypothesis we have the conclusion. We can show (2.15) $\sim$ (2.17) similarly. $\triangle$

We define ord A and coord A for $A = \sum_{j \in \mathbf{Z}} \mathrm{diag}[a_j(s)] \Lambda^j$ as follows:

$$\mathrm{ord}\ A = \inf\{ j \in \mathbf{Z} \mid \mathrm{diag}[a_i(s)] = 0, \text{ for } i > j \} ,$$

$$\mathrm{coord}\ A = \sup\{ j \in \mathbf{Z} \mid \mathrm{diag}[a_i(s)] = 0, \text{ for } i < j \} .$$

Lemma 2.3. *Suppose that* $D_m^+ L - (-)^m B_m^* L + L^{*(m)} B_m - 2\delta_{\underline{mn},\underline{1}} L_*^{m+1} = \mathrm{diag}[a(s)]\Lambda^r +$ *lower order, for some* r *and* $\mathrm{diag}[a(s)] \neq 0$. *Then*

$$\lim_{n \to \infty} \mathrm{ord}(D_m^+ L_*^n - (-)^{mn} B_m^{*(n)} L_*^n + L_*^{n*(m)} B_m - 2\delta_{\underline{mn},\underline{1}} L_*^{m+n}) = +\infty .$$

Proof. By using Lemma 2.2, we see that the highest order term of the left hand side of (2.14) is

$$\mathrm{diag}\left[\sum_{l=0}^{n-1} (-)^{(l+n+1)(1+r)} a(s+l)\right] \Lambda^{r+n-1} . \tag{2.18}$$

Fact A. *For any* $n \in \mathbf{N}$, *there exists* $n' \in \mathbf{N}$ *such that* $n' > n$ *and*

$$\mathrm{diag}\left[\sum_{l=0}^{n'-1} (-)^{(l+n'+1)(1+r)} a(s+l)\right] \neq 0 .$$

Proof of Fact A. Suppose that there exists $N \in \mathbf{N}$ such that

$$\sum_{l=0}^{N-1} (-)^{(l+N+1)(1+r)} a(s+l) = 0 \qquad \text{and}$$

$$\sum_{l=0}^{N} (-)^{(l+N)(1+r)} a(s+l) = 0 , \qquad \text{for all} \quad s \in \mathbf{Z} .$$

From this we have $a(s+N) = 0$ for all $s \in \mathbf{Z}$. This contradicts the assumption of Lemma 2.3. $\triangle$

By Fact A the highest order term of the left hand side of (2.14) never vanishes as $n \to \infty$. This completes the proof of Lemma 2.3. $\triangle$

On the other hand we see the following fact.

Fact B. *The order of the left hand side of* (2.14) *is less than* m.

Proof of Fact B. From (2.6) we see that

$$D_m^+ L_*^n - (-)^{mn} B_m^{*(n)} L_*^n + L_*^{n*(m)} B_m - 2\delta_{\underline{mn},\underline{1}} L_*^{m+n} \tag{2.19}$$
$$= D_m^+(L_*^n)_- + (-)^{mn} D_n^+ B_m - (-)^{mn} B_m^{*(n)}(L_*^n)_- + (L_*^{n*(m)})_- B_m - 2\delta_{\underline{mn},\underline{1}}(L_*^{m+n})_- \,.$$

The order of the right hand side of (2.19) is less than m. △

From Lemma 2.5 and Fact B, we can conclude that

$$D_m^+ L - (-)^m B_m^* L + L^{*(m)} B_m - 2\delta_{\underline{m},\underline{1}} L_*^{m+1} = 0\,.$$

Secondly we show (2.2).

Lemma 2.4. *Suppose that* $D_m^+ M - (-)^m B_m^* M + M^{*(m)} B_m = \mathrm{diag}[b(s)]\Lambda^r +$ *higher order terms for some* r *and* $\mathrm{diag}[b(s)] \neq 0$. *Then*

$$\lim_{n\to\infty} \mathrm{coord}(D_m^+ M_*^n - (-)^{mn} B_m^{*(n)} M_*^n + M_*^{n*(m)} B_m) = -\infty\,.$$

Proof. From Lemma 2.2 we see that the lowest order term of the left hand side of (2.16) is

$$\mathrm{diag}\left[\sum_{l=0}^{n-1} (-)^{(l+n+1)(r+1)} v_0(s)\dots v_0(s-l+1)v_0(s+r-1)\dots v_0(s-n+2+r)b(s-1)\right]\Lambda^{-n+1+r}\,.$$

Fact C. *For any* $n \in \mathbb{N}$, *there exists* $n' \in \mathbb{N}$ *such that* $n' \geq n$ *and*

$$\mathrm{diag}\left[\sum_{l=0}^{n'-1} (-)^{(l+n'+1)(r+1)} v_0(s)\dots v_0(s-l+1)v_0(s+r-1)\dots v_0(s-n'+2+r)b(s-l)\right] \neq 0\,.$$

The proof is similar to that of Fact A. By Fact C, the lowest order term of the left hand side of (2.16) never vanishes. This completes the proof of Lemma 2.4. △

Fact D. *The coorder of the left hand side of* (2.16) *is positive.*

Proof of Fact D. From (2.5) we see that

$$D_m^+ M_*^n - (-)^{mn} B_m^{*(n)} M_*^n + M_*^{n*(m)} B_m$$
$$= D_m^+(M_*^n)_+ + (-)^{mn} D_n^- B_m - (-)^{mn} B_m^{*(n)}(M_*^n)_+ + (M_*^n)_+^{*(m)} B_m\,.$$

This assures the claim of Fact D. △

From Lemma 2.4 and Fact D, we see that

$$D_m^+ M - (-)^m B_m^* M + M^{*(m)} B_m = 0\,.$$

The equations (2.3) and (2.4) can be shown similarly. Q.E.D.

We define $\tilde{D}_1^\pm$, $\tilde{B}_1$ and $\tilde{C}_1$ as the restriction of $D_1^\pm$, B_1 and C_1 to the sector $t_{2j+1}^\pm = 0, j \geq 1$, respectively. Accordingly let

$$\tilde{D}_1^\pm = \partial_{t_1^\pm} + t_1^\pm \partial_{t_2^\pm}\,.$$

From (2.5), we get

$$\tilde{D}_1^+\tilde{C}_1 + \tilde{D}_1^-\tilde{B}_1 = -\tilde{B}_1^*\tilde{C}_1 - \tilde{C}_1^*\tilde{B}_1 . \tag{2.19}$$

Substituting $\tilde{B}_1 = \Lambda + \mathrm{diag}[\tilde{b}(s)]$ and $\tilde{C}_1 = \mathrm{diag}[\tilde{c}(s)]\Lambda^{-1}$ to (2.19), we have

$$\tilde{D}_1^-\tilde{b}(s) = -\tilde{c}(s) - \tilde{c}(s+1) , \tag{2.20}$$

$$\tilde{D}_1^+\tilde{c}(s) = (\tilde{b}(s) - \tilde{b}(s+1))\tilde{c}(s) . \tag{2.21}$$

Let $u(s) \in S_{\underline{0}}$ be a superfield such that

$$\tilde{b}(s) = \tilde{D}_1^+ u(s) ,$$

$$\tilde{c}(s) = \exp(u(s) - u(s-1)) .$$

Then one can see that (2.20) and (2.21) reduce to a single equation

$$\tilde{D}_1^+\tilde{D}_1^- u(s) = \exp(u(s) - u(s-1)) + \exp(u(s+1) - u(s)) . \tag{2.22}$$

We call (2.22) the STL equation. The body part $f(s) = \varepsilon(u(s))$ satisfies the ordinary TL equation,

$$\partial_{t_2^+}\partial_{t_2^-} f(s) = \exp(f(s) - f(s-2)) - \exp(f(s+2) - f(s)) .$$

Let W_+ and W_- be matrices such that

$$W_+ = \sum_{j=0}^{\infty} \mathrm{diag}[w_j^+(s)]\Lambda^{-j} , \qquad W_- = \sum_{j=0}^{\infty} \mathrm{diag}[w_j^-(s)]\Lambda^{j} ,$$

where $w_0^+(s) = 1$, $\varepsilon(w_0^-(s)) \neq 0$ and $w_j^\pm(s) \in S_{\underline{j}}$. From (2.5) $\sim$ (2.7) we can conclude the existence of $W_\pm$ satisfying the equations

$$\begin{aligned} &D_n^+ W_+ = B_n W_+ - W_+^{*(n)}\Lambda^n , \qquad D_n^+ W_- = B_n W_- , \\ &D_n^- W_+ = C_n W_+ , \qquad D_n^- W_- = C_n W_- - W^{*(n)}\Gamma^{-n} , \end{aligned} \tag{2.23}$$

where $\Gamma = ((-)^i\delta_{i+1,j})_{i,j\in\mathbf{Z}}$. Conversely the existence of $W_\pm$ of (2.23) implies (2.5) $\sim$ (2.7) as a compatibility condition. We call (2.23) the Sato equations and $W_\pm$ the wave matrices of the STL hierarchy.

Finally we mention a relation with the ordinary TL hierarchy. With $A = (a_{i,j})_{i,j\in\mathbf{Z}}$, we associate a matrix

$$\check{A} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} , \qquad \text{where} \quad A_{ij} = (a_{\mu\nu})_{\underline{\mu}=i, \underline{\nu}=j} .$$

Put $\varepsilon(\check{W}_\pm) = \begin{bmatrix} W_{\bar{0}\bar{0}}^\pm & 0 \\ 0 & W_{\bar{1}\bar{1}}^\pm \end{bmatrix}$ for the wave matrices of the STL hierarchy. Then $W_{\bar{0}\bar{0}}^\pm$ and $W_{\bar{1}\bar{1}}^\pm$ are the wave matrices of the ordinary TL hierarchy.

§ 3. The OSp-STL Hierarchy

In this section, we investigate the OSp-STL hierarchy. Consider the Lie superalgebra $gl_{\underline{0}}(\infty|\infty) = gl_{\underline{0}}(\infty|\infty) \oplus gl_{\underline{1}}(\infty|\infty)$,

$$gl_{\underline{0}}(\infty|\infty) = \left\{ \begin{bmatrix} A_{00} & 0 \\ 0 & A_{11} \end{bmatrix}; A_{00}, A_{11} \in \mathrm{Mat}(\mathbb{Z} \times \mathbb{Z} : \mathbb{C}) \right\},$$

$$gl_{\underline{1}}(\infty|\infty) = \left\{ \begin{bmatrix} 0 & A_{01} \\ A_{10} & 0 \end{bmatrix}; A_{01}, A_{10} \in \mathrm{Mat}(\mathbb{Z} \times \mathbb{Z} : \mathbb{C}) \right\}.$$

Next we define $gl(S) = gl_{\underline{0}}(S) \oplus gl_{\underline{1}}(S)$ by

$$gl_{\underline{i}}(S) = \bigoplus_{\underline{\mu+\nu=i}} gl_{\underline{\mu}}(\infty|\infty) \otimes S_{\underline{\nu}} .$$

The Lie supergroup SGL(S) is defined by

$$\mathrm{SGL}(S) = \left\{ A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}; A_{ij} = (a_{ij}^{\mu\nu})_{i,j \in \mathbb{Z}}, a_{ij}^{\mu\nu} \in S_{\underline{\mu+\nu}}, \varepsilon(A_{00}) \text{ and } \varepsilon(A_{11}) \text{ are invertible} \right\}.$$

Substituting V for S, we can similarly define SGL(V). We define two operators "st" and "$\tilde{s}t$" on $gl(\infty|\infty)$ and $gl(S)$ respectively by

$${}^{st}A = {}^{st}\begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} = \begin{bmatrix} {}^{t}A_{00} & {}^{t}A_{10} \\ -{}^{t}A_{01} & {}^{t}A_{11} \end{bmatrix},$$

$${}^{\tilde{s}t}B = {}^{\tilde{s}t}\begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} {}^{t}B_{00} & (-)^{i}\,{}^{t}B_{10} \\ (-)^{i+1}\,{}^{t}B_{01} & {}^{t}B_{11} \end{bmatrix},$$

for $A \in gl(\infty|\infty)$ and $B \in gl_{\underline{i}}(S)$. Note that

$${}^{st}(AB) = (-)^{ij}\,{}^{st}B\,{}^{st}A , \qquad {}^{\tilde{s}t}(CD) = (-)^{ij}\,{}^{\tilde{s}t}D\,{}^{\tilde{s}t}C$$

for $A \in gl_{\underline{i}}(\infty|\infty)$, $B \in gl_{\underline{j}}(\infty|\infty)$, $C \in gl_{\underline{i}}(S)$ and $D \in gl_{\underline{j}}(S)$. We introduce a Lie superalgebra osp$(\infty|\infty)$ (cf [5], [6]). Put $P = \begin{bmatrix} J & 0 \\ 0 & -K \end{bmatrix} \in gl(\infty|\infty)$, with J and K defined in §1. The Lie superalgebra osp$(\infty|\infty)$ is introduced as

$$\mathrm{osp}(\infty|\infty) = \{A \in gl(\infty|\infty);\ {}^{st}P\,{}^{st}AP = -A\} ,$$

with the $\mathbb{Z}_2$-gradation of $gl(\infty|\infty)$. Define $\mathrm{osp}(S) = \mathrm{osp}_{\underline{0}}(S) \oplus \mathrm{osp}_{\underline{1}}(S)$ by

$$\mathrm{osp}_{\underline{i}}(S) = \bigoplus_{\underline{\mu+\nu=i}} \mathrm{osp}_{\underline{\mu}}(\infty|\infty) \otimes S_{\underline{\nu}} .$$

We introduce a Lie supergroup OSp(S), which is generated by $\exp(A)$, $A \in \mathrm{osp}_{\underline{0}}(S)$, as follows:

$$\mathrm{OSp}(S) = \{A \in \mathrm{SGL}(S);\ {}^{\tilde{s}t}P\,{}^{\tilde{s}t}AP = A^{-1}\} .$$

We define OSp(V) similarly. In the rest of this section we impose the restriction $t_j^{\pm} = 0$ for $j \equiv 0, 3 \pmod 4$. Put $\tilde{S} = Sl_{t_j^{\pm}=0,\, j \equiv 0,3 \,(\mathrm{mod}\, 4)}$.

The OSp-STL hierarchy is a system of the Sato equations with a condition of symmetry:

$$\mathring{D}_n^+ W_+ = B_n W_+ - W_+^{*(n)} \Lambda^n , \qquad \mathring{D}_n^+ W_- = C_n W_-$$
$$\mathring{D}_n^- W_+ = C_n W_+ , \qquad \mathring{D}_n^- W_- = C_n W_- - W_-^{*(n)} \Gamma^{-n} , \tag{3.1}$$
$$\check{W}_{\pm} \in \mathrm{OSp}(\tilde{S}) , \qquad n \equiv 1, 2 \pmod 4 ,$$

where $W_{\pm} = \sum_{j=0}^{\infty} \mathrm{diag}[w_j^{\pm}(s)] \Lambda^{\mp j}$, $w_j^{\pm}(s) \in \tilde{S}_{\underline{j}}$, $w_0^+(s) = 1$, $\varepsilon(w_0^-(s)) \neq 0$ and $\mathring{D}_n^{\pm} = D_n^{\pm}|_{\tilde{S}}$.

Theorem 3.1. *If $W_{\pm}$ are the wave matrices of the OSp-STL hierarchy, then $\check{B}_n$, $\check{C}_n \in \mathrm{osp}_{\underline{n}}(\tilde{S})$ for $n \equiv 1, 2 \pmod 4$.*

Proof. We see that

$$^{\tilde{s}t}P\, ^{\tilde{s}t}(W_+^{*(n)} \Lambda^n W_+^{-1}) P = (-)^{n(n-1)/2} (W_+^{*(n)} \Lambda^n W_+^{-1})^{*(n)} \tag{3.2}$$

by an easy calculation. Notice that $^{\tilde{s}t}(A^{*(a)}) = (-)^a\, {}^{st}A$, where $A \in gl(S)_{\underline{a}}$. Hence we have

$$^{st}P\, ^{st}(W_+^{*(n)} \Lambda^n W_+^{-1}) P = (-)^{n(n+1)/2} (W_+^{*(n)} \Lambda^n W_+^{-1}) . \tag{3.3}$$

Taking the $(+)$-part of (3.3), we see that $B_n \in \mathrm{osp}(\tilde{S})_{\underline{n}}$ for $n \equiv 1, 2 \pmod 4$. For C_n the proof is similar. Q.E.D.

Let $\varepsilon(\check{W}_{\pm}) = \begin{bmatrix} W_{\bar{0}\bar{0}}^{\pm} & 0 \\ 0 & W_{\bar{1}\bar{1}}^{\pm} \end{bmatrix}$. Then $W_{\bar{0}\bar{0}}^{\pm}$ and $W_{\bar{1}\bar{1}}^{\pm}$ are the wave matrices of the BTL hierarchy and the CTL hierarchy respectively for time evolutions $\mathring{D}_{4n+2}^{\pm} (n \geq 0)$.

Put $\tilde{B}_1 = B_1|_{t_j^{\pm}=0, j>2}$ and $\tilde{C}_1 = C_1|_{t_j^{\pm}=0, j>2}$. Since $\check{B}_1$, $\check{C}_1 \in \mathrm{osp}(\tilde{S})_{\underline{1}}$, one has $\check{\tilde{B}}_1$, $\check{\tilde{C}}_1 \in \mathrm{osp}(\tilde{S})_{\underline{1}}$. Therefore the solution $u(s)$ of the STL equation (2.22) can be accompanied with the constraint

$$u(s) = -u(-s) + \log(-1)^s , \tag{3.4}$$

where $\exp(\log(-1)^s) = (-1)^s$. We call the STL equation with the symmetry (3.4) the OSp-STL equation. Furthermore imposing the constraint $u(s + 4N) = u(s)$, (2.22) reduces to the following equations:

$$\tilde{D}_1^+ \tilde{D}_1^- u(1) = \exp(u(1)) + \exp(u(2) - u(1)) ,$$
$$\tilde{D}_1^+ \tilde{D}_1^- u(s) = \exp(u(s) - u(s-1)) + \exp(u(s+1) - u(s)) , \qquad 2 \leq s \leq 2N-2 ,$$
$$\tilde{D}_1^+ \tilde{D}_1^- u(2N-1) = \exp(u(2N-1) - u(2N-2)) + \exp(-u(2N-1)) . \tag{3.5}$$

These equations coincide with the supper-Toda lattice equations corresponding to the Lie superalgebra $\mathrm{su}(2N|2N+1)$ which are discussed in [1]. Putting $N = 1$ in (3.5), we obtain the super-sine-Gordon equation

$$\tilde{D}_1^+ \tilde{D}_1^- u(1) = 2 \cosh u(1) . \tag{3.6}$$

The body part $f(1) = \varepsilon(u(1))$ satisfies the ordinary sine-Gordon equation

$$\partial_{t_2^+} \partial_{t_2^-} f(1) = -2 \sinh 2f(1) .$$

§4. An Explicit Representation of Solutions of the STL Hierarchy

Put $\Phi_+ = \exp\left(\sum_{j=0}^{\infty} t_j^+ \Lambda^j\right)$ and $\Phi_- = \exp\left(\sum_{j=0}^{\infty} t_j^- \Gamma^{-j}\right)$.

Proposition 4.1. *According to the R-H decomposition*

$$\Phi_+ A \Phi_-^{-1} = W_+^{-1} W_- \tag{4.1}$$

for $A \in SGL(V)$, the STL hierarchy (2.23) *is described by*

$$B_n = (W_+^{*(n)} \Lambda^n W_+^{-1})_+ \qquad \textit{and} \qquad C_n = (W_-^{*(n)} \Gamma^{-n} W_-^{-1})_- .$$

Proof. From (4.1), we have

$$W_- = W_+ \Phi_+ A \Phi_-^{-1} . \tag{4.2}$$

Note that $\Phi_\pm$ satisfy the equations

$$D_n^+ \Phi_+ = \Lambda^n \Phi_+ \qquad \text{and} \qquad D_n^- \Phi_- = \Gamma^{-n} \Phi_- .$$

Operate D_n^+ on (4.2). Then we have

$$D_n^+ W_- = (D_n^+ W_+) \Phi_+ A \Phi_-^{-1} + W_+^{*(n)} \Lambda^n \Phi_+ A \Phi_-^{-1} . \tag{4.3}$$

Multiply W_-^{-1} each hand side of (4.3) from the right. Then we have

$$(D_n^+ W_-) W_-^{-1} = (D_n^+ W_+) W_+^{-1} + W_+^{*(n)} \Lambda^n W_+^{-1} . \tag{4.4}$$

Taking the $(-)$-part of (4.4), we obtain $D_n^+ W_+ = B_n W_+ - W_+^{*(n)} \Lambda^n$. We can get other equations of (2.23) similarly. △

Consider a matrix

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

where $A_{ij} \in \mathrm{Mat}(\mathbf{N}^c \times \mathbf{N}^c : S_{\overline{i+j}})$, and A_{00} and A_{11} are invertible. Recall the definition of the superdeterminants of A:

$$\mathrm{s}\det A = \det(A_{00} - A_{01} A_{11}^{-1} A_{10}) / \det A_{11} ,$$

$$\mathrm{s}^{-1} \det A = \det(A_{11} - A_{10} A_{00}^{-1} A_{01}) / \det A_{00} .$$

It is known that $(\mathrm{s}\det A)(\mathrm{s}^{-1}\det A) = 1$ (cf. [3]).

Theorem 4.2. *Put $H = \Phi_+ A \Phi_-^{-1} = (h_{i,j})_{i,j \in \mathbf{Z}}$ for $A \in \mathrm{SGL}(V)$. And put $\tau(s) = \mathrm{s}\det(H(s))$, where $H(s) = (h_{i,j})_{i,j<s}$. Then, for the solution of the R-H decomposition* (4.1), *we have*

(i) $w_1^+(s) = D_1^+ \log \tau(s)$,

(ii) $w_0^-(s) = 1/(\tau(s)\tau(s+1))$,

(iii) $w_1^-(s) = (-)^{s+1} (D_1^- \tau(s+1)) / \tau(s+1)^2 \tau(s)$.

Proof. Let take the $(-)$-part of (4.2). Then we have $(W_+ H)_- = 0$. From this we get the linear algebraic equation

$${}^t\vec{w}_+(s) H(s) = -(\ldots h_{s,s-2}, h_{s,s-1}) , \tag{4.5}$$

where ${}^t\vec{w}_+(s) = (\dots w_2^+(s), w_1^+(s))$. Take the check (ˇ) of (4.5). Then we have

$$(\vec{w}_e^+(s), \vec{w}_o^+(s))\begin{pmatrix} H_{00}(s) & H_{01}(s) \\ H_{10}(s) & H_{11}(s) \end{pmatrix} = -(\vec{h}_e(s), \vec{h}_o(s)), \tag{4.6}$$

where ${}^t\vec{w}_e^+(s) = (\dots w_4^+(s), w_2^+(s))$, ${}^t\vec{w}_o^+(s) = (\dots w_3^+(s), w_1^+(s))$, ${}^t\vec{h}_e(s) = (\dots h_{s,s-4}, h_{s,s-2})$, ${}^t\vec{h}_o(s) = (\dots h_{s,s-3}, h_{s,s-1})$ and $H_{\mu\nu}(s) = (h_{s-i,s-j})_{\substack{i=\mu \\ j=\nu i, j>0}}$. Multiplying $\begin{pmatrix} 1 & -H_{00}^{-1}(s)H_{01}(s) \\ 0 & 1 \end{pmatrix}$ to both sides of (4.6) from the right, we have

$${}^t\vec{w}_o(s)V(s) = -{}^t(\alpha_{0,j}(s))_{j<0}, \tag{4.7}$$

where

$$V(s) = (\alpha_{ij}(s))_{i,j<0} = H_{11}(s) - H_{10}(s)H_{00}^{-1}(s)H_{01}(s)$$

and

$${}^t(\alpha_{0j}(s))_{j<0} = {}^t\vec{h}_o(s) - {}^t\vec{h}_e(s)H_{00}^{-1}(s)H_{01}(s).$$

By Cramer's formula, we obtain a solution of (4.7),

$$w_1^+(s) = -\sigma_1(s)/\sigma(s), \tag{4.8}$$

where $\sigma_1(s) = \det\begin{pmatrix} (\alpha_{ij}(s))_{\substack{i<-1 \\ j<0}} \\ {}^t(\alpha_{0j}(s))_{j<0} \end{pmatrix}$ and $\sigma(s) = \det V(s)$. We rewrite (4.8) in such a way that

$$w_1^+(s) = \frac{-(\sigma_1(s)/\det H_{00}(s))}{(\sigma(s)/\det H_{00}(s))}. \tag{4.9}$$

Lemma 4.4.

$$\sigma_1(s)/\det H_{00}(s) = D_1^+(\sigma(s)/\det H_{00}(s)). \tag{4.10}$$

Proof. From the construction of H

$$D_1^+ H = \Lambda H, \tag{4.11}$$

we have the following relations:

$$D_1^+ H_{00}(s) = H_{10}(s),$$

$$D_1^+ H_{01}(s) = H_{11}(s),$$

$$D_1^+ H_{10}(s) = \Lambda_{\mathbf{N}^c} H_{00}(s) + \begin{pmatrix} 0 \\ {}^t\vec{h}_e(s) \end{pmatrix},$$

$$D_1^+ H_{11}(s) = \Lambda_{\mathbf{N}^c} H_{01}(s) + \begin{pmatrix} 0 \\ {}^t\vec{h}_o(s) \end{pmatrix}, \tag{4.12}$$

where $\Lambda_{\mathbf{N}^c} = (\delta_{i+1,j})_{i,j<0}$. Notice that $\sigma(s)/\det H_{00}(s) = \det(H_{00}^{-1}(s)V(s))$. From (4.12) we have

$$D_1^+(H_{00}^{-1}(s)V(s)) = H_{00}^{-1}(s)\begin{pmatrix} 0 \\ {}^t(\alpha_{0j}(s))_{j<0} \end{pmatrix}. \tag{4.13}$$

Put $(\zeta_{ij}(s))_{i,j<0} = H_{00}^{-1}(s)$. Then one gets

$$D_1^+ \det (H_{00}^{-1}(s)V(s)) = \sum_{k<0} \det \gamma_k(s)\,, \tag{4.14}$$

where

$$\gamma_k(s) = \begin{pmatrix} {}^t(\zeta_{k-1}(s)\alpha_{0j}(s))_{j<0} \\ {}^t\left(\sum_{\rho<0} \zeta_{i\rho}(s)\alpha_{\rho j}(s)\right)_{j<0} \end{pmatrix} \begin{matrix} \cdots k\text{-th row} \\ \cdots i(\neq k)\text{-th row} \end{matrix}$$

Expanding each determinant of (4.14) along the k-th row, we have

$$D_1^+(H_{00}^{-1}(s)V(s)) = \sum_{k<0}\sum_{j<0} (-)^{k+j}\zeta_{k-1}(s)\alpha_{0j}(s)\Delta_{kj}(s)\,, \tag{4.15}$$

where $\Delta_{kj}(s)$ is a (k, j)-th minor determinant of $\gamma_k(s)$. One sees that

$$\begin{aligned}
&\sum_{k<0}\sum_{j<0} (-)^{k+j}\zeta_{k-1}(s)\alpha_{0j}(s)\Delta_{kj}(s) \\
&= \sum_{j<0} \alpha_{0j}(s) \det \begin{bmatrix} \left(\sum_{l<0} \zeta_{il}(s)\alpha_{ln}(s)\right)_{i<0} & (\zeta_{i-1}(s))_{i<0} \\ n(\neq j)\text{-th column} & j\text{-th column} \end{bmatrix} \\
&= \sum_{j<0} \alpha_{0j}(s) \det \begin{bmatrix} \left(\sum_{l<-1} \zeta_{il}(s)\alpha_{ln}(s)\right)_{i<0} & (\zeta_{i-1}(s))_{i<0} \\ n(\neq j)\text{-th column} & j\text{-th column} \end{bmatrix} \\
&= \sum_{j<0} \alpha_{0j}(s) \det(\zeta_{\mu\nu}(s))_{\mu,\nu<0} \det \begin{bmatrix} \begin{pmatrix} (\alpha_{mn}(s))_{m<-1} \\ 0 \end{pmatrix}, & (\delta_{i,-1})_{i<0} \\ n(\neq j)\text{-th column} & j\text{-th column} \end{bmatrix} \\
&= \det(\zeta_{\mu\nu}(s))_{\mu,\nu<0} \sum_{j<0} (-)^{j+1}\alpha_{0j}(s) \det(\alpha_{mn}(s))_{m,n<-1\, n\neq j} \\
&= \det(\zeta_{ij}(s))_{i,j<0} \det(\alpha_{ij}(s))_{\substack{i<1, i\neq -1 \\ j<0}}\,.
\end{aligned}$$

This completes the proof of Lemma 4.4. △

Let us return to the proof of Theorem 4.3. We see that

$$w_1^+(s) = \frac{D_1^+(\det V(s)/\det H_{00}(s))}{(\det V(s)/\det H_{00}(s))}\,.$$

Noting that $\tau(s) = (\det V(s)/\det H_{00}(s))^{-1}$, we get (i).

Let us show (ii). From (4.2) we have $(HW_-^{-1})_+ = 1_{\mathbf{Z}}$, where $1_{\mathbf{Z}}$ is the unit matrix. We denote $W_-^{-1} = \sum_{j=0}^{\infty} \operatorname{diag}[u_j^-(s)]\Lambda^j$. Then one gets the linear equation

$$V(s+1)\vec{u}_e^-(s) = (\delta_{i,-1})_{i<0}, \tag{4.16}$$

where $\vec{u}_e^-(s) = (u_{-2j-2}^-(s+2j+2))_{j<0}$. By Cramer's formula, we get

$$u_0^-(s) = \frac{\tilde{\sigma}(s+1)/\det H_{00}(s+1)}{\det(\alpha(s+1)_{ij})_{i,j<0}/\det H_{00}(s+1)}, \tag{4.17}$$

where $\tilde{\sigma}(s+1) = \det((\alpha_{ij}(s+1))_{\substack{i<0\\ j<-1}} (\delta_{i,-1})_{i<0})$. We can easily verify that $\tilde{\sigma}(s+1)/\det H_{00}(s+1) = \tau(s)$. From the fact that

$$\det V(s+1)/\det H_{00}(s+1) = 1/\tau(s+1),$$

we have $u_0^-(s) = \tau(s)\tau(s+1)$. This completes the proof of (ii).

For a matrix $A = \sum_{j\in\mathbf{Z}} \mathrm{diag}[a_j(s)]\Lambda^j$, we put $(A)_{>0} = \sum_{j>0} \mathrm{diag}[a_j(s)]\Lambda^j$. From (4.2) one gets $(HW_-^{-1})_{>0} = 0$. Then we have the linear equation

$$V(s)\vec{u}_o^-(s) = -u_0^-(s)(\vec{h}_o(s) - H_{10}(s)H_{00}^{-1}(s)\vec{h}_e(s)), \tag{4.18}$$

where $\vec{h}_o = (h_{s-2i+1,s})_{i>0}$ and $\vec{h}_e(s) = (h_{s-2i,s})_{i>0}$. We note that $(\alpha_{i0}(s))_{i<0} = \vec{h}_o(s) - H_{10}(s)H_{00}^{-1}(s)\vec{h}_e(s)$. By Cramer's formula we see that

$$u_1^-(s-1) = \frac{-u_0(s)(\tilde{\sigma}_1(s)/\det H_{00}(s))}{\sigma(s)/\det H_{00}(s)}, \tag{4.19}$$

where $\tilde{\sigma}_1(s) = \det((\alpha_{ij}(s))_{\substack{i<0\\ j<-1}} \underset{(-1)\text{-th column}}{(\alpha_{i0}(s))_{i<0}})$. To show (iii), we prove the following lemma.

Lemma 4.5.

$$D_1^-(\sigma(s)/\det H_{00}(s)) = (-)^{s+1}(\tilde{\sigma}_1(s)/\det H_{00}(s)). \tag{4.20}$$

Proof. From the equation $D_1^- H = H^*\Gamma^{-1}$, one gets the following relations:

$$\begin{aligned}
D_1^- H_{00}(s) &= (-)^s H_{01}(s), \\
D_1^- H_{01}(s) &= (-)^s(H_{00}(s)\Lambda_{\mathbf{N}^c}^{-1} + [0, \vec{h}_e(s)]), \\
D_1^- H_{10}(s) &= (-)^{s+1}H_{11}(s), \\
D_1^- H_{11}(s) &= (-)^{s+1}(H_{10}(s)\Lambda_{\mathbf{N}^c}^{-1} + [0, \vec{h}_o(s)]).
\end{aligned} \tag{4.21}$$

Notice that $\sigma(s)/\det H_{00}(s) = \det(V(s)H_{00}^{-1}(s))$. Then we have the relation

$$D_1^-(V(s)H_{00}^{-1}(s)) = (-)^{s+1}[0, (\alpha_{i0}(s))_{i<0}]H_{00}^{-1}(s). \tag{4.22}$$

Let us calculate $D_1^- \det\{(V(s)H_{00}^{-1}(s)\}$. Let $(\beta_{i0}(s))_{i<0} = (-)^{s+1}(\alpha_{i0}(s))_{i<0}$. Then we have

$$D_1^- \det\{V(s)H_{00}^{-1}(s)\} = \sum_{k<0} \det\begin{bmatrix} \left(\sum_{\mu<0} \alpha_{i\mu}(s)\zeta_{\mu j}(s)\right)_{i<0}, & (\beta_{i0}(s)\zeta_{-1,k}(s))_{i<0} \\ \vdots & \vdots \\ j(\neq k)\text{-th column} & k\text{-th column} \end{bmatrix}. \tag{4.23}$$

Expand each determinant of (4.23) along the k-th column. Then we have

$$\text{the right hand side of (4.23)} = \sum_{k<0} \zeta_{-1,k}(s) \sum_{i<0} (-)^{i+k} \beta_{i0}(s) \Delta_{ik}(s)$$

$$= \sum_{i<0} \beta_{i0}(s) \det \begin{bmatrix} {}^t\left(\sum_{\mu<0} \alpha_{k\mu}(s)\zeta_{\mu j}(s)\right)_{j<0} \\ {}^t(\zeta_{-1j}(s))_{j<0} \end{bmatrix} \begin{matrix} \cdots k(\neq i)\text{-th row} \\ \cdots i\text{-th row} \end{matrix}$$

$$= \sum_{i<0} \beta_{i0}(s) \det \begin{bmatrix} {}^t\left(\sum_{\mu<-1} \alpha_{k\mu}(s)\zeta_{\mu j}(s)\right)_{j<0} \\ {}^t(\zeta_{-1j}(s))_{j<0} \end{bmatrix} \begin{matrix} \cdots k(\neq i)\text{-th row} \\ \cdots i\text{-th row} \end{matrix}$$

$$= \sum_{i<0} (-)^{i+1} \beta_{i0}(s) \det(\alpha_{kl}(s))_{\substack{k<0\, k\neq i \\ l<-1}} \det(\zeta_{\mu\nu}(s))_{\mu,\nu<0}$$

$$= \det[(\alpha_{ij}(s))_{\substack{i<0 \\ j<-1}}, (\beta_{i0}(s))_{i<0}] \det(\zeta_{ij}(s))_{i,j<0}\,.$$

This completes the proof of Lemma 4.5. △

By Lemma 4.5, $u_1(s-1)$ is represented as

$$u_1(s-1) = \frac{(-)^{s+1} u_0(s)(D_1^-(\sigma(s)/\det H_{00}(s))}{\sigma(s)/\det H_{00}(s)}\,.$$

Noting that $u_1^-(s) = -w_1^-(s)/(w_0^-(s)w_0^-(s+1))$, we have (iii). Q.E.D.

Proposition 4.6. *Put $t_j^{\pm} = 0$ for $j \equiv 0, 3 \pmod 4$ and let $\check{A} \in \mathrm{OSp}(V)$ be decomposed as* (4.1). *Then $W_{\pm}$ in the right hand side are the wave matrices of the OSp-STL hierarchy.*

Proof. Note that $\exp\left(\sum_{j\equiv 1,2 \pmod 4} t_j^+ \check{\Lambda}^j\right)$, $\exp\left(\sum_{j\equiv 1,2 \pmod 4} t_j^- \check{\Gamma}^{-j}\right) \in \mathrm{OSp}(\tilde{S})$. From the assumption of the Proposition 4.5, we have

$$({}^{\mathrm{st}}P\, {}^{\mathrm{st}}\check{W}_+^{-1} P)({}^{\mathrm{st}}P\, {}^{\mathrm{st}}\check{W}_- P) = \exp\left(\sum_{j\equiv 1,2 \pmod 4} t_j^+ \check{\Lambda}^j\right) A \exp\left(\sum_{j\equiv 1,2 \pmod 4} t_j^- \check{\Gamma}^{-j}\right).$$

By the uniqueness of the R-H decomposition, we have $\check{W}_{\pm} \in \mathrm{OSp}(\tilde{S})$. △

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