Publ. RIMS, Kyoto Univ.
25 (1989), 847–859

Lower Bounds for Order of Decay or of Growthin Time for Solutions to Linear and Non-linear Schrödinger Equations

By

Tohru OZAWA* and Nakao HAYASHI**

Abstract

We study lower bounds of decay (or of growth) order in time for solutions to the Cauchy problem for the Schrödinger equation:

$$i\partial_t u = -\Delta u + f(u),\ (t, x)\in \boldsymbol{R}\times\boldsymbol{R}^n \qquad (n\geq 1),$$
$$u(0)=\phi, \qquad x\in\boldsymbol{R}^n,$$

where f is a linear or non-linear complex-valued function.

Under some conditions on f and ϕ, it is shown that every nontrivial solution u has the estimate

$$\liminf_{t\to\pm\infty} |t|^{n/2-n/q}\|u(t)\|_{L^q(|x|<k|t|)}>0$$

for sufficiently large $k>0$ and for any $q\in[2, \infty]$.

In the previous work [12] of the first named author, we imposed on the assumption that u is *asymptotically free*. In this article, however, we shall show the assumption is, in fact, irrelevant to the results.

§ 1. Introduction

In this paper we consider the asymptotic behavior in time of solutions to the equation:

$$(1.1)\qquad \begin{cases} i\partial_t u = -\Delta u + f(u)\,,\ (t, x)\in \boldsymbol{R}\times\boldsymbol{R}^n \qquad (n\geq 1)\,, \\ u(0) = \phi \neq 0\,, \qquad x\in\boldsymbol{R}^n\,, \end{cases}$$

where f describes a linear or non-linear perturbation and ϕ is a given initial data.

Communicated by S. Matsuura, April 4, 1988.
* Department of Mathematics, Nagoya University, Nagoya 464, Japan.
** Department of Mathematics, Faculty of Engineering, Gunma University, Kiryu 376, Japan.

More precisely, we deal with the following three types of equations.

(1) Non-linear Schrödinger equation with power interaction:

$$i\partial_t u = -\Delta u + \lambda |u|^{p-1} u ,$$

where $\lambda > 0$ and $1 < p < \alpha(n)$ with $\alpha(n) = \infty$ $(n=1, 2)$, $\alpha(n) = (n+2)/(n-2)$ $(n \geq 3)$.

(2) Non-linear Schrödinger equation with non-local interaction:

$$i\partial_t u = -\Delta u + (V * |u|^2) u ,$$

where $V = V(x) = \lambda |x|^{-\gamma}$ with $\lambda > 0$ and $0 < \gamma < \min(4, n)$.

(3) Linear Schrödinger equation with long-range perturbation:

$$i\partial_t u = Hu , \quad H = H_0 + V ,$$

where H_0 is the self-adjoint realization of $-\Delta$ in the Hilbert space $L^2 = L^2(\mathbf{R}^n)$ and the long-range perturbation V is assumed to satisfy some conditions specified later.

Concerning the asymptotic behavior in time of a solution u for (1.1), it is possible to distinguish between the following two cases (a) and (b) in the category of L^2-scattering theories.

(a) There exist some (or equivalently, unique) states $u_{\pm} \in L^2$ such that

$$\lim_{t \to \pm\infty} \| u(t) - e^{-itH_0} u_{\pm} \|_{L^2} = 0 , \tag{1.2}$$

where $\{e^{-itH_0};\ t \in \mathbf{R}\}$ is the free Schrödinger evolution group. In this case, we call a solution u *asymptotically free*.

(b) There do not exist any states $u_{\pm} \in L^2$ satisfying (1.2).

In the case (a), it has been proved in [1] that every non-trivial solution u has the estimate

$$\liminf_{t \to \pm\infty} |t|^{n/2 - n/q} \| u(t) \|_{L^q(k'|t| < |x| < k|t|)} > 0 \tag{1.3}$$

for some $0 < k' < k$ and for any $q \in [2, \infty]$. In view of the proof, it is clear that k and k' in (1.3) depend on the momentum support of $u_{\pm}$, i.e., the support of the Fourier transform. Therefore, the argument in [12] only gives rather implicit relations between the pair (k', k) and the initial data ϕ.

We now state our main purpose in this paper, which is twofold.

One is to obtain similar lower-bound estimates as in (1.3) even when (1.2) do not hold. To be more specific, we show that the following estimate slightly weaker than (1.3)

$$\liminf_{t\to\pm\infty} |t|^{n/2-n/q}\|u(t)\|_{L^q(|x|<k|t|)}>0 \tag{1.4}$$

holds for non-trivial solutions to the equations (1)–(3).

The other is to give explicit lower bounds of k in (1.4) in terms of the given initial data ϕ.

As corollaries to main theorems proved in this paper, we see that L^p-decay or growth estimates obtained by many peoples (see [1], [3], [4], [7], [8], [10] and [11]) are optimal.

Finally we list some notations which will be used in the sequel.

$\sum$ denotes the Hilbert space

$$\sum = \{u\in L^2(\mathbf{R}^n);\ \partial_j u,\ x_j u\in L^2(\mathbf{R}^n) \qquad (j=1,\cdots,n)\}$$

with the norm

$$\|u\|_{\Sigma} = (\|u\|_2^2+\sum_{j=1}^n \|\partial_j u\|_2^2+\sum_{j=1}^n \|x_j u\|_2^2)^{1/2}\,.$$

$\|\cdot\|_p$ denotes the usual $L^p(\mathbf{R}^n)$-norm and $(\cdot,\cdot)$ denotes the $L^2(\mathbf{R}^n)$-scalar product. $H^1(\mathbf{R}^n)$ denotes the usual Sobolev space of order one. For an interval $I\subset\mathbf{R}$ and a Banach space E, $C(I\,;E)$ denotes the space consisting of E-valued continuous functions on I and $\|\cdot\|_{\mathcal{L}(E)}$ denotes the operator norm on the space of all bounded linear maps from E into E. For a self-adjoint operator H in the Hilbert space $L^2(\mathbf{R}^n)$, $\mathcal{H}_{\mathrm{cont}}(H)$ denotes the continuous spectral subspace of H. $-i\nabla$ and x also denote the momentum operator and the position operator acting on the Hilbert space $L^2(\mathbf{R}^n)\otimes\mathbf{C}^n$, respectively. Different positive constants might be denoted by the same letter C, if necessary, by $C(*,\cdots,*)$ in order to indicate constants depending only on the quantities appearing in parentheses.

§ 2. Non-linear Schrödinger Equations with Power Interaction

In this section we consider the equation of the form:

$$\begin{cases} i\partial_t u = -\Delta u+|u|^{p-1}u\,, & (t,x)\in\mathbf{R}\times\mathbf{R}^n \qquad (n\geq 1)\,, \\ u(0)=\phi, & x\in\mathbf{R}^n\,, \end{cases} \tag{2.1}$$

where $1<p<\alpha(n)$. By a mild solution of (2.1), we mean a function $u\in C(\mathbf{R};L^2)$ satisfying the integral equation

$$u(t) = e^{-itH_0}\phi - i\int_0^t e^{-i(t-\tau)H_0}(|u(\tau)|^{p-1}u(\tau))d\tau \quad \text{in} \quad L^2 \tag{2.2}$$

for any $t \in \boldsymbol{R}$. We summarize the results concerning mild solutions of (2.1).

Lemma 1. *Let $\phi \in \Sigma$. Then there exists a unique mild solution $u \in C(\boldsymbol{R}; \Sigma)$ of* (2.1) *satisfying*

$$||u(t)||_2 = ||\phi||_2 \tag{2.3}$$

and

$$||\nabla u(t)||_2^2 + \frac{2}{p+1}||u(t)||_{p+1}^{p+1} = ||\nabla\phi||_2^2 + \frac{2}{p+1}||\phi||_{p+1}^{p+1} \tag{2.4}$$

for $t \in \boldsymbol{R}$. Furthermore, u satisfies

$$\begin{aligned} &||(x+2it\nabla)u(t)||_2^2 + \frac{8}{p+1}t^2||u(t)||_{p+1}^{p+1} \\ &= ||(x+2is\nabla)u(s)||_2^2 + \frac{8}{p+1}s^2||u(s)||_{p+1}^{p+1} \\ &\quad + \frac{4(n+4-np)}{p+1}\int_s^t \tau||u(\tau)||_{p+1}^{p+1}d\tau, \qquad t, s \in \boldsymbol{R}, \end{aligned} \tag{2.5}$$

$$||u(t)||_{p+1} \leq C(n, p, ||\phi||_\Sigma)\cdot(1+|t|)^{-\theta(p)}, \qquad t \in \boldsymbol{R}, \tag{2.6}$$

and

$$||(x+2it\nabla)u(t)||_2 \leq C(n, p, ||\phi||_\Sigma)\cdot(1+|t|)^{a(p)}, \qquad t \in \boldsymbol{R}, \tag{2.7}$$

where $\theta(p)=n(p-1)/2(p+1)$ and $a(p)=1-n(p-1)/4$ if $1<p\leq\gamma(n)=(n+2+(n^2+12n+4)^{1/2})/2n$, $a(p)=0$ if $\gamma(n)<p<\alpha(n)$.

For Lemma 1, see, e.g., [1], [3] and [15].

We now have:

Theorem 1. *Let $\phi \in \Sigma\backslash\{0\}$ and let $u \in C(\boldsymbol{R}; \Sigma)$ be the solution given by Lemma* 1. *Then for any*

$$k > k_0 = 2\left(||\nabla\phi||_2^2 + \frac{2}{p+1}||\phi||_{p+1}^{p+1}\right)^{1/2} / ||\phi||_2, \tag{2.8}$$

we have

$$\liminf_{t\to\pm\infty} \int_{|x|<k|t|} |u(t, x)|^2 dx > 0. \tag{2.9}$$

Proof. We assume

$$\liminf_{t\to\infty}\int_{|x|<kt}|u(t,x)|^2dx=0 \tag{2.10}$$

for some $k>k_0$ and we deduce a contradiction. From the assumption (2.10), there exist a sequence $\{t_j;j\geq 1\}$ in $\boldsymbol{R}$ such that $0<t_1<t_2<\cdots<t_j\uparrow\infty$ as $j\to\infty$ and

$$\lim_{j\to\infty}\int_{|x|<kt_j}|u(t_j,x)|^2dx=0\,. \tag{2.11}$$

(2.7) and (2.11) give

$$\begin{aligned}&\int_{|x|<kt_j}|\nabla u(t_j,x)|^2dx\\&\leq 2\int_{|x|<kt_j}\left|\frac{x}{2it_j}u(t_j,x)\right|^2dx+2\int_{|x|<kt_j}\left|\left(\frac{x}{2it_j}+\nabla\right)u(t_j,x)\right|^2dx\\&\leq\frac{k^2}{2}\int_{|x|<kt_j}|u(t_j,x)|^2dx+2\left\|\left(\frac{x}{2it_j}+\nabla\right)u(t_j)\right\|_2^2\to 0\qquad(j\to\infty)\,,\end{aligned}$$

from which we get

$$\begin{aligned}\lim_{j\to\infty}&\int_{|x|>kt_j}|\nabla u(t_j,x)|^2dx\\&=\lim_{j\to\infty}\|\nabla u(t_j)\|_2^2\\&=\lim_{j\to\infty}\left(\|\nabla\phi\|_2^2+\frac{2}{p+1}\|\phi\|_{p+1}^{p+1}-\frac{2}{p+1}\|u(t_j)\|_{p+1}^{p+1}\right)\\&=\|\nabla\phi\|_2^2+\frac{2}{p+1}\|\phi\|_{p+1}^{p+1}\,.\end{aligned} \tag{2.12}$$

Here we have used (2.4) and (2.6). Similarly, by (2.3) and (2.11) we have

$$\lim_{j\to\infty}\int_{|x|>kt_j}|u(t_j,x)|^2dx=\|\phi\|_2^2\,. \tag{2.13}$$

A simple calculation leads to

$$\begin{aligned}&\left(\int_{|x|>kt_j}|\nabla u(t_j,x)|^2dx\right)^{1/2}\\&\geq\frac{k}{2}\left(\int_{|x|>kt_j}|u(t_j,x)|^2dx\right)^{1/2}-\left\|\left(\frac{x}{2it_j}+\nabla\right)u(t_j)\right\|_2\,.\end{aligned} \tag{2.14}$$

We take the limit $j\to\infty$ in (2.14) and apply (2.12)–(2.13) to (2.14) to conclude

$$\left(\|\nabla\phi\|_2^2+\frac{2}{p+1}\|\phi\|_{p+1}^{p+1}\right)^{1/2}\geq\frac{k}{2}\|\phi\|_2\,.$$

This contradicts the fact that $k>k_0$.
The case $t<0$ can be treated similarly. Q.E.D.

Remark 1. (2.9) gives a propagation property of quantum particles obeying non-linear Schrödinger equations with power interaction. We also have from (2.6) that for any $R>0$,

$$\lim_{t\to\pm\infty}\int_{|x|<R}|u(t,x)|^2dx=0\,.$$

Compare (2.9).

Corollary 1. *Under the assumptions of Theorem* 1, *the unique mild solution* $u\in C(\mathbb{R};\Sigma)$ *has the estimate*

$$\liminf_{t\to\pm\infty}|t|^{n/2-n/q}\|u(t)\|_{L^q(|x|<k|t|)}>0 \tag{2.15}$$

for any $k>k_0$ *and* $q\in[2,\infty]$.

Proof. (2.15) is an easy consequence of (2.9) and the Hölder inequality. Q.E.D.

Remark 2. Lower-bound estimates for the case $1+2/n<p<\alpha(n)$ have been obtained in [12].

§ 3. Non-linear Schrödinger Equations with Non-local Interaction

This section deals with the following Hartree type equation:

$$\begin{cases} i\partial_t u=-\Delta u+(V*|u|^2)u, & (t,x)\in\mathbb{R}\times\mathbb{R}^n \quad (n\geq 1)\,,\\ u(0)=\phi\,, & x\in\mathbb{R}^n\,, \end{cases} \tag{3.1}$$

where $V=V(x)=|x|^{-\gamma}$ with $0<\gamma<\min(4,n)$.

By a mild solution of (3.1), we mean a function $u\in C(\mathbb{R};L^2)$ satisfying the integral equation in L^2 associated with (3.1).

We state the results corresponding to Lemma 1.

Lemma 2. *Let* $\phi\in\Sigma$. *Then, there exists a unique mild solution* $u\in C(\mathbb{R};\Sigma)$ *of* (3.1) *satisfying*

$$\|u(t)\|_2=\|\phi\|_2 \tag{3.2}$$

and

$$\|\nabla u(t)\|_2^2+P(u(t))=\|\nabla\phi\|_2^2+P(\phi) \tag{3.3}$$

for $t\in\mathbb{R}$, *where*

$$(3.4)\qquad P(\phi)=\frac{1}{2}\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}\frac{|\phi(x)|^2|\phi(y)|^2}{|x-y|^\gamma}dxdy\,.$$

Furthermore, u satisfies

$$(3.5)\qquad \begin{aligned}&\|(x+2it\nabla)u(t)\|_2^2+4t^2P(u(t))\\&=\|(x+2is\nabla)u(s)\|_2^2+4s^2P(u(s))\\&\quad+4(2-\gamma)\int_s^t \tau P(u(\tau))d\tau\,,\qquad t,\,s\in\boldsymbol{R}\,,\end{aligned}$$

$$(3.6)\qquad P(u(t))\le C(n,\,\gamma,\,\|\phi\|_\Sigma)\cdot(1+|t|)^{-\gamma}\,,\qquad t\in\boldsymbol{R}\,,$$

and

$$(3.7)\qquad \|(x+2it\nabla)u(t)\|_2\le C(n,\,\gamma,\,\|\phi\|_\Sigma)\cdot(1+|t|)^{b(\gamma)}\,,\qquad t\in\boldsymbol{R}\,,$$

where $b(\gamma)=1-\gamma/2$, *if* $0<\gamma\le 4/3$ $(n\ge 2)$ *or* $0<\gamma<1$ $(n=1)$, *and* $b(\gamma)=0$ *if* $4/3<\gamma<\min(4,\,n)$ $(n\ge 2)$.

Proof. For the case $4/3<\gamma<\min(4,n)$, see [4], [5] and [8]. For the cases $0<\gamma\le 4/3$ $(n\ge 2)$ and $0<\gamma<1$ $(n=1)$, see Appendix in § 5. Q.E.D.

Since we have Lemma 2, in the same way as in the proof of Theorem 1 we obtain:

Theorem 2. *Let* $\phi\in\Sigma\setminus\{0\}$ *and let* $u\in C(\boldsymbol{R};\,\Sigma)$ *be the solution given by Lemma* 2. *Then for any*

$$k>k_1=2(\|\nabla\phi\|_2^2+P(\phi))^{1/2}/\|\phi\|_2\,,$$

we have

$$(3.8)\qquad \liminf_{t\to\pm\infty}\int_{|x|<k|t|}|u(t,\,x)|^2dx>0\,.$$

Remark 3. When $n=3$ and $\gamma=1$, Glassey [6] has proved that for any $R>0$,

$$(3.9)\qquad \lim_{t\to\pm\infty}\int_{|x|<R}|u(t,\,x)|^2dx=0\,.$$

Since we easily obtain L^p-decay $(2<p<\alpha(n)+1)$ estimates by applying the Gagliardo-Nirenberg inequality to (3.7), we find that (3.9) holds when $n\ge 1$ and $0<\gamma<\min(4,\,n)$.

Corollary 2. *Under the assumptions of Theorem* 2, *the unique mild solution* $u\in C(\boldsymbol{R};\,\Sigma)$ *has the estimate*

$$\liminf_{t\to\pm\infty} |t|^{n/2-n/q}\|u(t)\|_{L^q(|x|<k|t|)}>0$$

for any $k>k_1$ *and* $q\in[2, \infty]$.

Remark 4. From the Theorem 3.1 in [8] and the same argument as in [12], we have the following assertion:

Suppose $n\geq 2$ *and* $1<\gamma<\min(4, n)$. *Assume* $\phi\in\Sigma\setminus\{0\}$. *Let* $u\in C(\mathbf{R}; \Sigma)$ *be the solution given by Lemma* 2. *Then there exist* $0<k'<k$ *satisfying*

$$\liminf_{t\to\pm\infty}\int_{k'|t|<|x|<k|t|} |u(t, x)|^2dx>0$$

and

$$\liminf_{t\to\pm\infty} |t|^{n/2-n/q}\|u(t)\|_{L^q(k'|t|<|x|<k|t|)}>0$$

for any $q\in[2, \infty]$.

Corollary 3. *Under the assumptions of Theorem 2,* $P(u(t))$ *has the estimate*

$$\liminf_{t\to\pm\infty} |t|^{\gamma}P(u(t))>0\,. \tag{3.10}$$

Proof. We first prove (3.10) in the case $1\leq\gamma<\min(4, n)$. Let $k>k_1$. Then, $|x|$, $|y|<k|t|$ implies $|x-y|^{\gamma}<(2k|t|)^{\gamma}$. Consequently,

$$P(u(t))\geq\frac{1}{2}\left(\frac{1}{2k|t|}\right)^{\gamma}\left(\int_{|x|<k|t|}|u(t, x)|^2dx\right)^2$$

from which (3.10) follows.

We next assume $0<\gamma<1$. Let $k>2k_1$ and $k|t|>1$. We estimate $P(u(t))$ from below as follows:

$$\begin{aligned}
&P(u(t))\\
&\geq\frac{1}{2}\iint_{1<|x-y|<k|t|}\frac{|u(t, x)|^2|u(t, y)|^2}{|x-y|^{\gamma}}dx\,dy\\
&\quad+\frac{1}{2}\iint_{|x-y|<1}\frac{|u(t, x)|^2|u(t, y)|^2}{|x-y|^{\gamma}}dx\,dy\\
&\geq\frac{1}{2}\left(\frac{1}{k|t|}\right)^{\gamma}\iint_{|x-y|<k|t|}|u(t, x)|^2|u(t, y)|^2dx\,dy\\
&\geq\frac{1}{2}\left(\frac{1}{k|t|}\right)^{\gamma}\left(\int_{|x|<k|t|/2}|u(t, x)|^2dx\right)^2.
\end{aligned}$$

Since $k/2>k_1$, we obtain (3.10) for $0<\gamma<1$. Q.E.D.

Remark 5. (3.6) and (3.10) completely characterize the large time behavior of the so-called *direct potential energy* $P(u(t))$.

§ 4. Linear Schrödinger Equations with Long-range Perturbation

In this section we freely use the operator theoretic language (see, e.g., [9] and [14]). We consider the symmetric form

$$h = h_0 + h_1 \text{ (as a form sum)}, \tag{4.1}$$

where h_0 is defined as $h_0[\phi, \psi] = (\nabla\phi, \nabla\psi)$ with form domain $Q(h_0) = H^1(\boldsymbol{R}^n)$ and h_1 is assumed to be a closed symmetric form relatively bounded with h_0-bounded less than one.

By the KLMN theorem [14] we see that h is a lower-semibounded closed symmetric form with domain $Q(h) = Q(h_0)$ and that h has a unique self-adjoint operator H with domain $D(H)$ satisfying

$$D(H) \subset Q(h) \tag{4.2}$$

and

$$\begin{aligned}(H\psi, \phi) &= h[\psi, \phi] \\ &= (\nabla\psi, \nabla\phi) + h_1[\psi, \phi], \qquad \text{for} \quad \psi \in D(H) \text{ and } \phi \in Q(h).\end{aligned} \tag{4.3}$$

Moreover, for some $j > 0$,

$$Q(h) = D((H+j)^{1/2}) \tag{4.4}$$

and

$$h[\psi, \phi] = ((H+j)^{1/2}\psi, (H+j)^{1/2}\phi) - j(\psi, \phi), \ \psi, \phi \in Q(h). \tag{4.5}$$

Thus, we conclude by the closed graph theorem that $(H_0+j)^{1/2}(H+j)^{-1/2}$ is a bounded operator defined on L^2.

We need the following lemma.

Lemma 3. *Let H be as above. Then we have:*
(1) e^{-itH} *maps* $H^1(\boldsymbol{R}^n)$ *into* $H^1(\boldsymbol{R}^n)$ *continuously and furthermore, there exists a constant* $a > 0$ *such that*

$$\|e^{-itH}\phi\|_{H^1(\boldsymbol{R}^n)} \le a\|\phi\|_{H^1(\boldsymbol{R}^n)}, \qquad (t, \phi) \in \boldsymbol{R} \times H^1(\boldsymbol{R}^n).$$

(2) e^{-itH} *maps* $\sum$ *into* $\sum$ *continuously and furthermore, there exists a constant* $b > 0$ *such that*

$$\|e^{-itH}\phi\|_{\Sigma} \le b(1+|t|)\|\phi\|_{\Sigma}, \qquad (t, \phi) \in \boldsymbol{R} \times \sum.$$

(3) $$h[\phi, \phi] = h[e^{-itH}\phi, e^{-itH}\phi], \qquad (t, \phi) \in \boldsymbol{R} \times H^1(\boldsymbol{R}^n).$$

Proof. (1) and (2) have been proved by Radin and Simon [13]. We prove (3). Let $\phi \in H^1(\boldsymbol{R}^n) = Q(h_0) = Q(h)$ and fix $t \in \boldsymbol{R}$. We set $R_\lambda = i\lambda(H+i\lambda)^{-1}$ for $\lambda > 0$. It follows that

$$R_\lambda e^{-itH}\phi, \qquad R_\lambda \phi \in D(H)$$

and that

$$\begin{aligned} h[R_\lambda e^{-itH}\phi, R_\lambda e^{-itH}\phi] &= (HR_\lambda e^{-itH}\phi, R_\lambda e^{-itH}\phi) \\ &= (He^{-itH}R_\lambda\phi, e^{-itH}R_\lambda\phi) = (HR_\lambda\phi, R_\lambda\phi) \\ &= h[R_\lambda\phi, R_\lambda\phi]. \end{aligned}$$

Since h is a lower-semibounded closed form, we have the assertion if we prove that for any $\psi \in H^1(\boldsymbol{R}^n)$, $\{h[R_\lambda\psi, R_\lambda\psi]; \lambda > 0\}$ is a Cauchy sequence in $\boldsymbol{R}$. From the assumption on h_1, it suffices to show

$$\|\nabla(R_\lambda\psi - R_\mu\psi)\|_2 \to 0 \qquad \text{as} \quad \lambda, \mu \to \infty.$$

We now do this. Let $j > 0$ be as in (4.4)–(4.5). Then,

$$\begin{aligned} &\|\nabla(R_\lambda\psi - R_\mu\psi)\|_2^2 \\ &= \|(H_0+j)^{1/2}(R_\lambda\psi - R_\mu\psi)\|_2^2 - j\|R_\lambda\psi - R_\mu\psi\|_2^2 \\ &\leq \|(H_0+j)^{1/2}(H+j)^{-1/2}\|_{\mathcal{L}(L^2)}^2 \cdot \|(H+j)^{1/2}(R_\lambda\psi - R_\mu\psi)\|_2^2 \\ &= \|(H_0+j)^{1/2}(H+j)^{-1/2}\|_{\mathcal{L}(L^2)}^2 \cdot \|(R_\lambda - R_\mu)(H+j)^{1/2}\psi\|_2^2 \to 0 \\ &\qquad\qquad (\lambda, \mu \to \infty), \end{aligned}$$

which proves our claim. Q.E.D.

We now state the assumption on H.

(H) For any $\phi \in \Sigma \cap \mathcal{H}_{\text{cont}}(H)$, we have

(4.6) $$\lim_{t \to \pm\infty} \left\| \left(\frac{x}{2it} + \nabla \right) e^{-itH}\phi \right\|_2 = 0.$$

By virtue of Lemma 3, the conditions given by Enss [2] are sufficient for (4.6) to hold. They cover the case where h_1 is obtained by the following perturbation V:

V is decomposable as $V = V_s + V_l$, where V_s is a short-range potential and V_l is a multiplication operator by a continuously differentiable real-valued function V_l satisfying

$$V_l(x),\ x \cdot \nabla V_l(x) \to 0 \qquad \text{as} \quad |x| \to +\infty.$$

Theorem 3. *Let H be as above. Assume that (H) holds. Let* $\phi \in (\Sigma \cap \mathcal{H}_{\rm cont}(H)) \setminus \{0\}$. *Then for any*

$$k > k_2 := 2(h[\phi, \phi])^{1/2}/||\phi||_2 ,$$

we have

$$\liminf_{t \to \pm\infty} \int_{|x|<k|t|} |(e^{-itH}\phi)(x)|^2 dx > 0 .$$

If in addition, there exist $t_0 > 0$ *and* $q \in [2, \infty]$ *such that*

$$e^{-itH}\phi \in L^q_{\rm loc}(\mathbf{R}^n) , \qquad |t| \geq t_0 ,$$

then,

$$\liminf_{t \to \pm\infty} |t|^{n/2-n/q} ||e^{-itH}\phi||_{L^q(|x|<k|t|)} > 0 .$$

Proof. Proof is immediate since we have Lemma 3–3) and (H). Q.E.D.

§ 5. Appendix

In this appendix, we prove Lemma 2 in the cases $0 < \gamma \leq 4/3$ ($n \geq 2$) and $0 < \gamma < 1$ ($n=1$). For $T > 0$, we introduce the following Banach psace B_T by

$$B_T = C([-T, T]; \Sigma) \text{ with the norm } ||u||_{B_T} = \sup_{|t| \leq T} ||e^{itH_0}u(t)||_\Sigma$$

and the closed ball $B_T(\rho)$ ($\rho > 0$) by

$$B_T(\rho) = \{u \in B_T;\ ||u||_{B_T} \leq \rho\} .$$

Note that $||\cdot||_{B_T}$ is an equivalent norm to the usual norm on $C([-T, T]; \Sigma)$.

We are now in a position to complete the proof of Lemma 2.

Proof of Lemma 2 *in the cases* $0 < \gamma \leq 4/3$ ($n \geq 2$) *and* $0 < \gamma < 1$ ($n=1$).

Let $w \in B_T(\rho)$. We define Sw by

$$(5.1) \quad (Sw)(t) = e^{-itH_0}\phi - i \int_0^t e^{-i(t-\tau)H_0}((V * |w|^2)w(\tau)) d\tau , \qquad |t| \leq T .$$

In the same way as in the proof of Theorem 4.1 in [8] we get

$$(5.2) \quad ||e^{i\tau H_0}((V * |w|^2)w(\tau))||_\Sigma \leq C(n, \gamma) g(\tau) ||e^{i\tau H_0}w(\tau)||_\Sigma , \qquad |\tau| \leq T ,$$

where $g(\tau) = ||w(\tau)||_r^2 + ||w(\tau)||_{r+\varepsilon}^2 + ||w(\tau)||_{r-\varepsilon}^2$ with sufficiently small $\varepsilon > 0$ and $r = 2n/(n-\gamma)$.

For $0 < \gamma < \min(2, n)$, we have by the Gagliardo-Nirenberg inequality

(5.3) $$\|w(t)\|_r \le C(n,\gamma)\|w(t)\|_2^{1-\gamma/2}\|\nabla w(t)\|_2^{\gamma/2},$$

and

(5.4)$_\pm$ $$\|w(t)\|_{r\pm\varepsilon} \le C(n,\gamma)\|w(t)\|_2^{1-\gamma(\pm\varepsilon)/2}\|\nabla w(t)\|_2^{\gamma(\pm\varepsilon)},$$

where

$$\gamma(\pm\varepsilon) = \left(\gamma \pm \frac{n}{2}\left(1-\frac{\gamma}{n}\right)\varepsilon\right)\Big/\left(2\pm\left(1-\frac{\gamma}{n}\right)\varepsilon\right).$$

(5.2)-(5.4)$_\pm$ imply

(5.5) $$\begin{aligned}&\|e^{i\tau H_0}((V*|w|^2)w(\tau))\|_\Sigma\\ &\le C(n,\gamma)\rho^2\cdot\sup_{|t|\le T}\|e^{i\tau H_0}w(\tau)\|_\Sigma\\ &\le C(n,\gamma)\rho^3, \qquad t\in[-T,T],\end{aligned}$$

from which it follows that

$$Sw\in C([-T,T];\Sigma)$$

and

(5.6) $$\begin{aligned}&\|e^{itH_0}(Sw)(t)\|_\Sigma\\ &\le\|\phi\|_\Sigma+\left\|\int_0^t e^{i\tau H_0}((V*|w|^2)w(\tau))d\tau\right\|_\Sigma\\ &\le\|\phi\|_\Sigma+C(n,\gamma)\rho^3|t|, \qquad t\in[-T,T].\end{aligned}$$

This gives

(5.7) $$\|Sw\|_{B_T}\le\|\phi\|_\Sigma+C(n,\gamma)\rho^3T.$$

We also have for $w_1, w_2\in B_T$ that

(5.8) $$\|Sw_1-Sw_2\|_{B_T}\le C(n,\gamma)\rho^2T\|w_1-w_2\|_{B_T}.$$

If ρ and T are chosen to satisfy

$$\rho\ge2\|\phi\|_\Sigma \quad\text{and}\quad T\le1/(2C(n,\gamma)\rho^2),$$

then (5.7) and (5.8) allows us to conclude that S is a contraction mapping from $B_T(\rho)$ into itself. This implies that there exists a unique mild solution $u\in B_T(\rho)$ for sufficiently small $T>0$. Furthermore, along the line of the argument of Ginibre and Velo [4] it is easily verified that u satisfies (3.2), (3.3) and (3.5) for any $t, s\in[-T,T]$. Then, by virtue of (3.2), (3.3), (3.5) and the Gagliardo-Nirenberg inequality we have

(5.9) $$\|u(t)\|_2,\ \|\nabla u(t)\|_2\le C(n,\gamma,\|\phi\|_\Sigma),$$

and

$$\|xe^{itH_0}u(t)\|_2 \leq C(n, \gamma, \|\phi\|_\Sigma)\cdot(1+|t|) \tag{5.10}$$

for any $t\in[-T, T]$. From (5.9) and (5.10) it follows that for any $T>0$ there exists a unique mild solution $u\in B_T$ of (3.1) satisfying (3.2) and (3.3) for any $t\in\boldsymbol{R}$ and that u satisfies (3.5) for any $t, s\in\boldsymbol{R}$. Therefore we have $u\in C(\boldsymbol{R}; \Sigma)$. In the same fashion as in the proof of (5.9) and (5.10) in [8], we observe that u satisfies (3.6) and (3.7) with $b(\gamma)$ replaced by $1-\gamma/2$. This completes the proof. Q.E.D.

References

[1] Barab, J.E., Nonexistence of asymptotic free solutions for a nonlinear Schrödinger equation, *J. Math. Phys.*, **25** (1984), 3270–3273.
[2] Enss, V., Asymptotic observables on scattering states, *Comm. Math. Phys.*, **89** (1983), 245–268.
[3] Ginibre, J. and Velo, G., On a class of nonlinear Schrödinger equations I, II, *J. Funct. Anal.*, **32** (1979), 1–32, 33–71.
[4] ———, On a class of nonlinear Schrödinger equations with non local interaction, *Math. Z.*, **170** (1980), 109–136.
[5] ———, Sur une équation de Schrödinger non linéaire avec interaction non locale, in "nonlinear Partial Differential Equations and Their Applications," College de France Seminar, Vol. II, Pitman, Boston, 1981.
[6] Glassey, R.T., Asymptotic behavior of solutions to certain nonlinear Schrödinger-Hartree equations, *Comm. Math. Phys.*, **53** (1977), 9–18.
[7] Hayashi, N. and Tsutsumi, M., $L^\infty(\boldsymbol{R}^n)$-decay of classical solutions for nonlinear Schrödinger equations, *Proceedings of the Royal Society of Edinburgh*, **104A** (1986), 309–327.
[8] Hayashi, N. and Tsutsumi, Y., Scattering theory for Hartree type equations, *Ann. Inst. Henri Poincaré, Physique théorique*, **46** (1987), 187–213.
[9] Kato, T., "Perturbation Theory for Linear Operators." Second Edition, Springer-Verlag, Berlin, Heiderberg, New York, 1976.
[10] Lin, J.E. and Strauss, W.A., Decay and scattering of solutions of a nonlinear Schrödinger equation, *J. Funct. Anal.*, **30** (1978), 245–263.
[11] Ozawa, T., New L^p-estimates for solutions to the Schrödinger equations and time asymptotic behavior of observables, in Publ. RIMS, Kyoto Univ., **25** (1989), 521–577.
[12] ———, Lower L^p-bounds for scattering solutions of the Schrödinger equations, Publ. RIMS, Kyoto Univ., **25** (1989), 579–586.
[13] Radin, C. and Simon, B., Invariant domains for the time-dependent Schrödinger equation, *J. Differential Equations*, **29** (1978), 289–296.
[14] Reed, M. and Simon, B., "Methods of Modern Mathematical Physics," I: Functional Analysis (1972), II: Fourier Analysis, Self-adjointness (1975), Academic Press, New York.
[15] Tsutsumi, Y., "Global existence and asymptotic behavior of nonlinear Schrödinger equations," Doctoral Thesis, University of Tokyo.