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A Tower of Riemann Surfaces whose Bergman Kernels Jump at the Roof

by

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Abstract

It is shown that, for any Fuchsian group Γ acting on the complex upper half plane $\mathcal{H}$ such that $\mathcal{H}/\Gamma$ is a compact hyperelliptic Riemann surface, there exists a sequence of subgroups $\Gamma_n \subset \Gamma$ $(n = 1, 2, \dots)$ satisfying $\Gamma_1 = \Gamma$ and $\bigcap_{n=1}^{\infty} \Gamma_n = \{\mathrm{id}\}$ such that the associated sequence of the Bergman kernels of $\mathcal{H}/\Gamma_n$, pulled back to $\mathcal{H}$, does not converge to the Bergman kernel of $\mathcal{H}$.

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Introduction

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and let Γ be a Fuchsian group acting on $\mathcal{H}$. In [R], J. A. Rhodes studied the asymptotic behavior of the Bergman kernels associated to the towers of compact Riemann surfaces $\{S_n\}_{n=1}^{\infty}, S_n = \mathcal{H}/\Gamma_n$, with $\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} \Gamma_n = \{\mathrm{id}\}$. He showed that, letting $\pi_n : \mathcal{H} \to S_n$ be the projections, the Bergman kernels $\mathcal{K}_n$ of S_n restricted to the diagonal satisfy

$$\lim_{n\to\infty} \pi_n^* \mathcal{K}_n = \frac{dz \otimes d\overline{z}}{4\pi (\operatorname{Im} z)^2} \tag{1}$$

and

$$\lim_{n\to\infty} \partial\overline{\partial} \log \pi_n^* \mathcal{K}_n = \frac{dz \wedge d\overline{z}}{2 (\operatorname{Im} z)^2} \tag{2}$$

uniformly on compact subsets of $\mathcal{H}$ if either the following (a) or (b) holds.

(a) Γ_n is normal in Γ for all n.

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(b) The smallest nonzero eigenvalue of the Laplace–Beltrami operator associated to the Poincaré metric on S_n is bounded away from zero as n varies.

(Cf. Theorem 1 in [R].) This result supports the validity of the following, which was first stated by D. Mumford [M] who attributed it to D. Kazhdan [K]:

Suppose $\{S_n\}_{n=1}^{\infty}$ is a sequence of compact Riemann surfaces, $S_n = \mathcal{H}/\Gamma_n$, $\Gamma_1 \supset \Gamma_2 \supset \cdots$, and $\bigcap_{n=1}^{\infty} \Gamma_n = \{\mathrm{id}\}$. Then, letting $ds^2_{B,n}$ denote the pullback of the Bergman metric of S_n to $\mathcal{H}$, with suitably chosen scalars λ_n,

$$\lim_{n\to\infty} \lambda_n ds^2_{B,n} = \frac{dz \otimes d\overline{z}}{(\operatorname{Im} z)^2}. \quad (\lambda_n = 2 \text{ in } (2).)$$

The purpose of the present note is to construct $\{S_n\}$ such that neither (1), (2), (a) nor (b) hold. This counterexample will also show that the above assertion of Mumford is valid only under some additional assumptions. The author does not know whether or not the assertion is valid under a weaker notion of convergence than pointwise.

Although the example is very simple, it may be of interest because of the following reasons:

(i) Rhodes made the following remark in [R]:

> There is no reason to think that the statement would be false if neither conditions (a) nor (b) held; they are needed only to overcome technical problems.

(ii) It explicitly gives a sequence of covering spaces of a fixed Riemann surface along which the corresponding sequence of the smallest nonzero eigenvalues of the Laplacian converges to zero.

Our $\{S_n\}$ will be constructed from any hyperelliptic Riemann surface $S = \mathcal{H}/\Gamma$. By exploiting the involution on S, we shall extend the fundamental domain of Γ in a symmetric way to obtain a desired tower consisting of hyperelliptic Riemannn surfaces. The nonconvergence phenomenon for such a tower occurs in a manner that (2) does not hold at some point. We shall say that the Bergman kernels *jumps at the roof* if such a discontinuity holds. The reason why it is the case for our $\{S_n\}$ is because $\partial\overline{\partial} \log \mathcal{K}_n$ degenerates at the Weierstrass points of S_n.

§1. Construction of a hyperelliptic tower

Let S be any compact Riemann surface which is of genus $g \geq 2$ and hyperelliptic. Let $\Gamma \subset \operatorname{Aut} \mathcal{H}$ be a Fuchsian group such that $\mathcal{H}/\Gamma = S$. Once and for all we fix any Weierstrass point $p \in S$, a point which is fixed by the involution σ of S such that $S/\{\mathrm{id}, \sigma\}$ is isomorphic to the Riemann sphere.

With respect to the Poincaré metric of S, let γ_k ($k \in \mathbb{Z}/(2g+2)\mathbb{Z}$) be the simple closed geodesics which are invariant under σ such that

(i) $S \setminus \gamma_k$ are connected,

(ii) $\#(\gamma_k \cap \gamma_{k+1}) = 1$,

(iii) $\gamma_j \cap \gamma_k = \emptyset$ if $j \neq k, k \pm 1$.

We put $\gamma^0 = \gamma_{(2g+2)\mathbb{Z}}$ and $\gamma^1 = \gamma_{1+(2g+2)\mathbb{Z}}$. We may assume that $\gamma^0 \cap \gamma^1 = \{p\}$.

By cutting S along those γ_k's which do not contain p, we obtain a fundamental domain of Γ, say Ω, in $\mathcal{H}$ whose sides are the copies of the arcs on γ_k joining two adjacent Weierstrass points. (See Figure 1.)

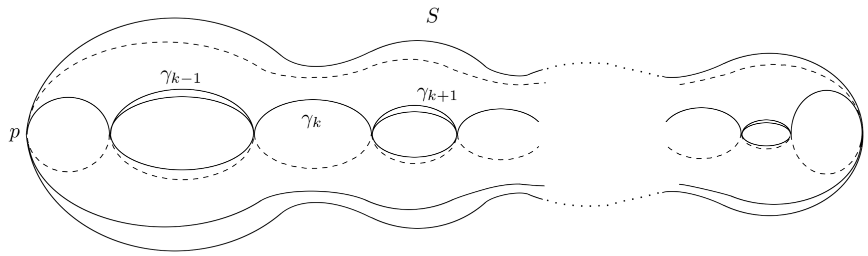

Figure 1

The point p is in the interior of Ω. Further, γ^0 and γ^1 give rise to axes, say L^0 and L^1, respectively, of antiholomorphic involutions of Ω.

Conversely, let Ω_0 be any polygon in $\mathcal{H}$ whose sides intersect adjacent ones orthogonally. Then, by making the double, say Ω_1, of Ω_0 along a side, say τ, of Ω_0, and then by making the double of Ω_1 along one of the sides consisting of two copies of a side adjacent to τ, one has the fundamental domain Ω constructed as in the above mentioned way from a hyperelliptic Riemann surface. Namely, Ω is characterized as a right polygon with two mutually orthogonal axes of reflections intersecting the boundary of Ω orthogonally.

Let σ^0 (resp. σ^1) be the reflection of Ω with axis L^0 (resp. L^1). Then it is easy to see that, for any side γ of Ω, one can extend Ω along the sides γ, $\sigma^0(\gamma)$, $\sigma^1(\gamma)$ and $\sigma^0\sigma^1(\gamma)$ symmetrically, to obtain the fundamental domain of a Riemann surface S' which is a hyperelliptic double cover over S such that the preimage of p consists of Weierstrass points of S'. (See Figures 2 and 3 of fundamental domains in the unit disk.)

By repeating this procedure one obtains, for any $n \in \mathbb{N}$, a covering $\varpi_n : S_n \to S$ such that S_n is hyperelliptic, $\varpi_n^{-1}(p)$ consists of Weierstrass points of S_n, and that the injectivity radius of S_n at some point of $\varpi_n^{-1}(p)$ is greater than n.

Let Γ_n be a decreasing sequence of subgroups of Γ satisfying $\bigcap_{n=1}^{\infty} \Gamma_n = \{\mathrm{id}\}$ such that $\mathcal{H}/\Gamma_n = S_n$ and that the images of $\sqrt{-1} \in \mathcal{H}$ in $\mathcal{H}/\Gamma_n$ are Weierstrass.

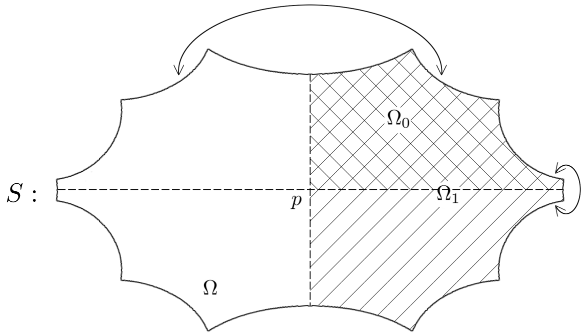

Figure 2

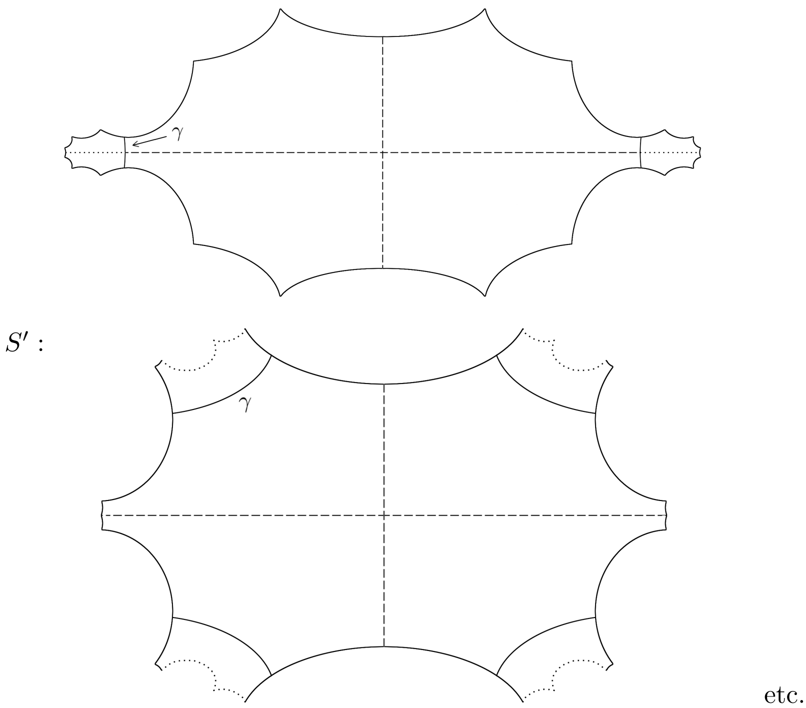

Figure 3

Identifying S_n with $\mathcal{H}/\Gamma_n$, let $\mathcal{K}_n$ be the Bergman kernel of S_n, restricted to the diagonal. Since S_n are hyperelliptic and $\pi_n(\sqrt{-1})$ are Weierstrass,

$$\partial\overline{\partial}\log\pi_n^*\mathcal{K}_n|_{z=\sqrt{-1}}=0 \tag{3}$$

holds for all n, because $\sqrt{-1}\,\partial\overline{\partial}\log\mathcal{K}_n$ is the pull-back of the Fubini–Study form by the mapping associated to the canonical linear system.

On the other hand, it is immediate from the definition of the Bergman kernel that, for any $z \in S_n$, the equality

$$\mathcal{K}_n(z) = \sup\{\varphi(z) \otimes \overline{\varphi(z)} \mid \varphi \text{ is a holomorphic 1-form on } S_n \text{ with } \|\varphi\| = 1\},$$

$\|\varphi\|$ being the L^2 norm of φ, holds and that the supremum is attained by

$$\varphi(w) = \mathcal{K}_n(z)^{-1/2}\mathcal{K}_n(w, z),$$

where $\mathcal{K}_n(w, z)$ denotes the (non-diagonalized) Bergman kernel.

From this and the Cauchy estimate, it is easy to see that, for the Bergman kernel $\mathcal{K}_n(w, z)$ on $S_n \times S_n$, $(\pi_n \times \pi_n)^*\mathcal{K}_n(w, z)$ converges to $-dw \otimes d\overline{z}/(\pi(w - \overline{z})^2)$ uniformly on compact subsets of $\mathcal{H} \times \mathcal{H}$ if (1) holds uniformly on compact subsets of $\mathcal{H}$. Here we identify w and z with local coordinates of S_n. Accordingly, since $\mathcal{K}_n(w, z)$ is holomorphic in $(w, \overline{z})$, (2) should hold if (1) is the case. But (2) contradicts (3).

Remark. What happens near the tower $\{S_n\}$ when it is extended as a family of towers associated to an analytic family of S_1? Let $f : \mathcal{S} \to \mathcal{T}$ be an analytic family of compact Riemann surfaces over a contractible and irreducible complex analytic space $\mathcal{T}$, let $\widetilde{\mathcal{S}}$ be the universal cover of $\mathcal{S}$, let $\tilde{f} : \widetilde{\mathcal{S}} \to \mathcal{T}$ be the lift of f to $\widetilde{\mathcal{S}}$, and let $f_G : \mathcal{S}_G = \mathcal{S}/G \to \mathcal{T}$ be the family induced from f and the natural action on $\widetilde{\mathcal{S}}$ of a subgroup G of the fundamental group $\pi_1(\mathcal{S}, x)$ of $\mathcal{S}$ for some fixed $x \in \mathcal{S}$. Let $G_1 \supset G_2 \supset \cdots$ be a decreasing sequence of subgroups of $\pi_1(\mathcal{S}, x)$ satisfying $G_1 = \pi_1(\mathcal{S}, x)$ and $\bigcap_{n=1}^{\infty} G_n = \{\mathrm{id}\}$. Then, in view of the above mentioned characterization of the values of the Bergman kernels as solutions to the extremal problem, from the L^2 extension theorem in [O], applied to the disjoint union of $\widetilde{\mathcal{S}}$ and $\mathcal{S}_{G_n}$ $(n = 1, 2, \dots)$ and the disjoint union of $\tilde{f}^{-1}(t)$ and $f_{G_n}^{-1}(t)$ $(n = 1, 2, \dots)$ for $t \in \mathcal{T}$, one has the following.

> *For any fixed family $f : \mathcal{S} \to \mathcal{T}$ and a sequence $G_1 \supset G_2 \supset \cdots$ as above, the set of points t in $\mathcal{T}$ such that the Bergman kernels for the tower $\{f_{G_n}^{-1}(t)\}$ jump at the roof is open.*

Hence a new question arises whether or not such a subset of $\mathcal{T}$ is dense if the jump at the roof occurs for some t.

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