On the Weak Admissibility of Singular Perturbations in Cauchy Problems

By

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1. Introduction

In [3], we have studied conditions on operators for the convergence of solutions of singulary perturbed non-characteristic Cauchy problems. The convergence of solutions depends on the Cauchy data. In this paper, we shall mainly study necessary and sufficient conditions on the Cauchy data for the convergence of solutions of a given one-parameter family of singulary perturbed Cauchy problems.

Let $P_1(D)$ and $P_2(D)$ be linear differential operators with constant coefficients. Let the order of P_1 be m and that of P_2 be m'. Assume that $m > m'$. For $x = (x_1, \cdots, x_n)$ in $\boldsymbol{R}^n$, put $x' = (x_2, \cdots, x_n)$. Let us consider the following one-parameter family of Cauchy problems:

$$(1.1) \quad \begin{cases} (\varepsilon^{m-m'} \cdot P_1(D) + P_2(D))u(x) = 0, & \text{in } \boldsymbol{R}^n; \\ D_1^{j-1} u(x)|_{x_1=0} = \phi_j(x'), & j = 1, \cdots, m, \end{cases}$$

where ε is a small positive parameter. In [2], we have studied that if the solutions u_ε of (1.1) converge in a suitable topology, then the limit satisfies the following reduced problem:

$$(1.2) \quad \begin{cases} P_2(D)u = 0, & \text{in } \boldsymbol{R}^n; \\ D_1^{j-1} u|_{x_1=0} = \phi_j, & j = 1, \cdots, m'. \end{cases}$$

Let the symbols of $P_1(D)$ and $P_2(D)$ be

$$(1.3) \quad P_1(\xi) = \sum_{j=0}^{m} p_{1,j}(\xi') \cdot \xi_1^{m-j},$$

$$(1.4) \quad P_2(\xi) = \sum_{j=0}^{m'} p_{2,j}(\xi') \cdot \xi_1^{m'-j}.$$

Assume that $p_{1,0}$ is a non-zero constant and ord $p_{1,j}(\xi')\leqq j, j=1, \cdots, m$ and that $p_{2,0}$ is a non-zero constant and ord $p_{2,j}(\xi')\leqq j, j=1, \cdots, m'$. Such an operator satisfying the above conditions is said to be *kowalewskian*. Denote the characteristic roots of $\varepsilon^{m-m'} \circ P_1(\xi)+P_2(\xi)=0$ with respect to ξ_1 by $\tau_j(\varepsilon, \xi'), j=1, \cdots, m$, those of $P_1(\xi)=0$ by $\upsilon_j(\xi'), j=1, \cdots, m$, and those of $P_2(\xi)=0$ by $\sigma_j(\xi')$, $j=1, \cdots, m'$.

Assumption 1.1. There exists a point ξ'_0 in $\mathbf{R}^{n-1}$ such that for any j, k, $1\leqq j<k\leqq m'$

$$\sigma_j(\xi'_0)\neq\sigma_k(\xi'_0) .$$

Remark. If Assumption 1.1 is satisfied, then there exists an open ball $B_0=B(r_0; \xi'_0)$ of radius r_0 with the centre ξ'_0 such that all $\sigma_j(\xi')$ are distinct on the closure of B_0.

Condition 1.2. $(m-m'=2$ and $p_{2,0}/p_{1,0}<0)$ or $(m-m'=1$ and $p_{2,0}/p_{1,0}$ is real).

Remark. In [3], we have studied that the characteristic roots $\tau_j(\varepsilon, \xi')$ are classified into two classes as follows. One consists of all $\tau_j(\varepsilon, \xi')$ which converge uniformly on every compact subset of $\mathbf{R}_{\xi'}^{n-1}$ to one of $\sigma_j(\xi'), j=1, \cdots, m'$ when $\varepsilon\downarrow 0$ and the other consists of all $\tau_j(\varepsilon, \xi')$ which diverge for every fixed ξ' when $\varepsilon\downarrow 0$. Condition 1.2 is equivalent to the condition that the leading terms with respect to ε of all the diverging characteristic roots are real.

Denote $\Phi'=(\phi_1, \cdots, \phi_{m'})$, $\Phi''=(\phi_{m'+1}, \cdots, \phi_m)$, and $\Phi=(\Phi', \Phi'')$. Let the Cauchy problems (1.1) be given and the Cauchy data space $\mathcal{A}$ consisting of Φ be so chosen that for every sufficiently small ε the Cauchy problem (1.1) with $\mathcal{A}$ is uniquely solvable. If the solutions u_ε of the Cauchy problems (1.1) for Φ in $\mathcal{A}$ converge in a suitable topology when $\varepsilon\downarrow 0$, then we may say that the one-parameter family (1.1) is a singular perturbation. This is the reason why we introduced the notion of "admissibility" of singular perturbations in Cauchy problems in [3]. Let the Cauchy problem (1.2) be given and the Cauchy data space $\mathcal{A}'$ consisting of Φ' be so chosen that the Cauchy problem (1.2) with $\mathcal{A}'$ is uniquely solvable. We choose the extra Cauchy data space $\mathcal{A}''$ consisting of Φ'' so that for every sufficiently small ε the Cauchy problem (1.1) with $\mathcal{A}=\mathcal{A}'\times\mathcal{A}''$ can be solved uniquely. If for every Φ' in $\mathcal{A}'$ there exists Φ'' in $\mathcal{A}''$ such that the solutions u_ε of the Cauchy problems (1.1) for $\Phi=(\Phi', \Phi'')$ converge when $\varepsilon\downarrow 0$, then the solution of the

Cauchy problem (1.2) for Φ' in $\mathcal{A}'$ can be represented as the limit of the solutions u_ε of the Cauchy problems (1.1) for a datum Φ in $\mathcal{A}$. This is the reason why we shall introduce the notion of "weak admissibility" of singular perturbations in Cauchy problems.

Let U' be a domain of $\boldsymbol{R}_{x'}^{n-1}$ and δ be a positive number.

Definition 1.3. The uniquely solvable Cauchy problems (1.1) in $(-\delta, \delta) \times U'$ with the Cauchy data space $\mathcal{A}$ are said to be *C-admissible* in $(-\delta, \delta) \times U'$ as a singular perturbation with respect to a given uniquely solvable Cauchy problem (1.2) if for every Cauchy datum Φ in $\mathcal{A}$, the solutions of (1.1) converge to that of (1.2) in $C((-\delta, \delta) \times U')$ and said to be *C-weakly admissible* in $(-\delta, \delta) \times U'$ if for every Cauchy datum Φ' in $\mathcal{A}'$ there exists a datum Φ'' in $\mathcal{A}''$ such that the solutions of (1.1) converge to that of (1.2) in $C((-\delta, \delta) \times U')$.

Remark. Relacing $C((-\delta, \delta) \times U')$ by $L^2((-\delta, \delta) \times U')$, we can also define the L^2-admissibility and the L^2-weak admissibility.

Denote by $\mathcal{O}(\boldsymbol{C}^{n-1})$ the space of entire functions defined in $\boldsymbol{C}^{n-1}$ and by $A(\boldsymbol{R}^{n-1})$ the space of real analytic functions defined in $\boldsymbol{R}^{n-1}$. When $\mathcal{A} = \mathcal{O}(\boldsymbol{C}^{n-1})^m$, the Cauchy-Kowalewski theorem assures the analytic unique solvability in $\boldsymbol{R}^n$ of (1.1) for sufficiently small ε. But even when $\mathcal{A} = A(\boldsymbol{R}^{n-1})^m$, we can not prove that there exists a region independent of ε such that the analytic solutions of (1.1) exist in the region. Hence we must choose $\mathcal{A}$ so as to be included in $\mathcal{O}(\boldsymbol{C}^{n-1})^m$. On the other hand, we shall use the Fourier transformation. Therefore, we shall only study the C-weak admissibility in $\boldsymbol{R}^n$ when $\mathcal{A} = F^{-1}(C_0^\infty(B_0))^m$, where F^{-1} denotes the inverse Fourier transformation and B_0 is the open ball in Remark to Assumption 1.1.

The main result in [3] is that Condition 1.2 is necessary and sufficient for the C-admissibility of the Cauchy problems (1.1) in $\boldsymbol{R}^n$ with $F^{-1}(C_0^\infty(B_0))^m$. The main result in this paper is the following:

Main Theorem. *Assume that Condition 1.2 is not satisfied. Let Assumption 1.1 be satisfied and B_0 be the open ball in Remark to Assumption 1.1. Denote $m'' = m - m'$. Let $\mathcal{A}' = F^{-1}(C_0^\infty(B_0))^{m'}$ and $\mathcal{A}'' = F^{-1}(C_0^\infty(B_0))^{m''}$. Then the Cauchy problems (1.1) in $\boldsymbol{R}^n$ with $\mathcal{A}$ are C-weakly admissible in $\boldsymbol{R}^n$ if and only if $P_1(\xi)$ is divided by $P_2(\xi)$ in the polynomial ring $\boldsymbol{C}[\xi]$. In this case, the solutions $u_\varepsilon(x; \Phi)$ of (1.1) are identically equal to the solution $u_0(x; \Phi')$ of (1.2).*

In this paper, we shall always assume that Condition 1.2 is not satisfied. The proof of Main Theorem can be found in § 5.

When the Cauchy data space $\mathcal{A}$ is $\{(0, \cdots, 0)\}$, all the Cauchy problems (1.1) are C-admissible. Our another purpose is to seek the Cauchy data spaces $\mathcal{A}$ included in $F^{-1}(C_0^\infty(B_0))^m$ with which the Cauchy problems (1.1) are C-admissible. The results are stated in Theorem A and B in §4. Theorem A will be used in the proof of Main Theorem.

In §2, we shall list up notation and study algebraic lemmas. In §3, we shall study a necessary and sufficient condition on the Cauchy data for the convergence of the solutions, but this condition depends on ε. We shall show that the convergence of the solutions implies that Φ'' is uniquely determined by Φ'. In §4, we shall remove the dependency on ε from this condition and distinguish the Cauchy data with which the solutions converge in the following two special cases.

(1) Theorem A. The case when m' or $m'-1$ characteristic roots of $F_2(\xi)=0$ are equal to the characteristic roots of $P_1(\xi)=0$ in B_0.

(2) Theorem B. The case when $m'=1$ or ($m'=2$ and $m\geq 4$).

Roughly speaking, when $P_1(\xi)$ is divided by $P_2(\xi)$ in the polynomial ring $C[\xi]$, Φ' can be free but otherwise Φ' is restricted.

The same results as for $\mathcal{A}=F^{-1}(C_0^\infty(B_0))^m$ remain true for $\mathcal{A}=F^{-1}(C_0(B^0))^m$ but fail for $\mathcal{A}=F^{-1}(\mathcal{E}'(B_0))^m$ in general.

2. Preliminaries

In this section, we shall state algebraic lemmas which will be needed. Similar calculation can be found in computation of the characters of the classical groups. See Chapter VII, [4].

Notation 2.1. Let X_j, $j=1, \cdots, m$ be variables and $S_k(X_1, \cdots, X_m)$, $k=1, \cdots, m$ be the elementary symmetric polynomials of X_j $j=1, \cdots, m$, that is, for $k=1, \cdots, m$

$$S_k(X_1, \cdots, X_m) = \sum_{i_1<i_2<\cdots<i_k} X_{i_1}X_{i_2} \cdots X_{i_k} .$$

Put

$$S_0(X_1, \cdots, X_m) = 1$$

and for $k\leq -1$ or $k\geq m+1$

$$S_k(X_1, \cdots, X_m) = 0 .$$

Denote

$$D_0(X_1, \cdots, X_m) = \begin{vmatrix} X_1^0 & \cdots\cdots & X_m^0 \\ \vdots & & \vdots \\ X_1^{m-1} & \cdots & X_m^{m-1} \end{vmatrix},$$

for $k=1, \cdots, m$ and for functions $\phi_j, j=1, \cdots, m$

$$D_k = D_k(X_1, \cdots, X_m; \phi_1, \cdots, \phi_m)$$

$$= \begin{vmatrix} X_1^0 & \cdots\cdots & X_{k-1}^0 & \phi_1 & X_{k+1}^0 & \cdots\cdots & X_m^0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ X_1^{m-1} & \cdots & X_{k-1}^{m-1} & \phi_m & X_{k+1}^{m-1} & \cdots & X_m^{m-1} \end{vmatrix},$$

and for $k=0, \cdots, m-1$

$$D^{m-1-k} = D^{m-1-k}(X_1, \cdots, X_{m-1}) = \begin{vmatrix} X_1^0 & \cdots\cdots & X_{m-1}^0 \\ \vdots & & \vdots \\ X_1^{k-1} & \cdots & X_{m-1}^{k-1} \\ X_1^{k+1} & \cdots & X_{m-1}^{k+1} \\ \vdots & & \vdots \\ X_1^{m-1} & \cdots & X_{m-1}^{m-1} \end{vmatrix}.$$

Denote $\zeta = \exp\dfrac{2\pi i}{m-m'}$ and $\tau_j' = \zeta^{j-m'-1}, j=m'+1, \cdots, m$. Denote by θ the argument of $-p_{2,0}/p_{1,0}$ satisfying $0 \leqq \theta < 2\pi$ and put $\Theta = \exp\dfrac{i\theta}{m-m'}$.

Lemma 2.2. *Assume that* $X_i \neq X_j \ 1 \leqq i < j \leqq m$. *Then, for* $j=1, \cdots, m-1$

$$(2.1) \qquad D^j(X_1, \cdots, X_{m-1})/D^0(X_1, \cdots, X_{m-1}) = S_j(X_1, \cdots, X_{m-1}),$$

and for $k=1, \cdots, m$

$$(2.2) \qquad D_k(X_1, \cdots, X_m; \phi_1, \cdots, \phi_m)/D_0(X_1, \cdots, X_m)$$

$$= \frac{\sum_{j=0}^{m-1} (-1)^j \cdot S_j(X_1, \cdots, X_{k-1}, X_{k+1}, \cdots, X_m) \cdot \phi_{m-j}}{\sum_{j=0}^{m-1} (-1)^j \cdot S_j(X_1, \cdots, X_{k-1}, X_{k+1}, \cdots, X_m) \cdot X_k^{m-1-j}}.$$

Proof. Expand the Vandermonde determinant $D_0(X_1, \cdots, X_m)$ with respect to the last column and divide it by the Vandermonde determinant $D_0(X_1, \cdots, X_{m-1}) = D^0(X_1, \cdots, X_{m-1})$. Then

$$(2.3) \qquad D_0(X_1, \cdots, X_m)/D_0(X_1, \cdots, X_{m-1})$$

$$= \sum_{j=0}^{m-1} (-1)^{m+(m-1-j+1)} \cdot D^j \cdot X_m^{m-1-j}/D^0$$

$$= \sum_{j=0}^{m-1} (-1)^j \cdot (D^j/D^0) \cdot X_m^{m-1-j}.$$

Since the Vandermonde determinants are represented as the difference products, we have

$$(2.4) \qquad D_0(X_1, \cdots, X_m)/D_0(X_1, \cdots, X_{m-1}) = \prod_{j=1}^{m-1} (X_m - X_j)$$
$$= \sum_{j=0}^{m-1} (-1)^j \cdot S_j(X_1, \cdots, X_{m-1}) \cdot X_m^{m-1-j} .$$

Comparing the coefficients of (2.3) and those of (2.4), we have (2.1).

Expand $D_m(X_1, \cdots, X_m; \phi_1, \cdots, \phi_m)$ with respect to the last column and divide it by $D_0(X_1, \cdots, X_{m-1})$. Then

$$(2.5) \qquad D_m(X_1, \cdots, X_m; \phi_1, \cdots, \phi_m)/D_0(X_1, \cdots, X_{m-1})$$
$$= \sum_{j=0}^{m-1} (-1)^j \cdot S_j(X_1, \cdots, X_{m-1}) \cdot \phi_{m-j} .$$

Thus (2.4) and (2.5) imply (2.2) when $k=m$. Since

$$(2.6) \qquad D_k(X_1, \cdots, X_m; \phi_1, \cdots, \phi_m)/D_0(X_1, \cdots, X_m)$$
$$= \frac{D_m(X_1, \cdots, X_{k-1}, X_{k+1}, \cdots, X_m, X_k; \phi_1, \cdots, \phi_m)}{D_0(X_1, \cdots, X_{k-1}, X_{k+1}, \cdots, X_m, X_k)} ,$$

we have (2.2) when $k=1, \cdots, m-1$.   Q.E.D.

Let $m = m' + m''$, where $1 \leq m' \leq m-1$. When $m'' \geq 2$, we put $Y_j = X_{m'+j}$. $j=1, \cdots, m''-1$. When $m''=1$, we put

$$S_0(Y_1, \cdots, Y_{m''-1}) = 1;$$
$$S_k(Y_1, \cdots, Y_{m''-1}) = 0, \quad \text{when} \quad k \neq 0 .$$

Then we have the following lemma.

Lemma 2.3.   *For* $k=1, \cdots, m-1$,

$$(2.7) \qquad S_k(X_1, \cdots, X_{m-1}) = S_k(X_1, \cdots, X_{m'}, Y_1, \cdots, Y_{m''-1})$$
$$= \sum_{j=0}^{m''-1} S_{k-j}(X_1, \cdots, X_{m'}) \cdot S_j(Y_1, \cdots, Y_{m''-1}) .$$

Proof.   When $m''=1$, (2.7) is trivial. We may assume that $m'' \geq 2$. We can write

$$S_k(X_1, \cdots, X_{m-1}) = \sum_{i_1 < i_2 < \cdots < i_k} X_{i_1} X_{i_2} \cdots X_{i_k}$$
$$= \sum_{j=0}^{m''-1} \sum_{i_1 < i_2 < \cdots < i_{k-j} \leq m' < i_{k-j+1} < i_{k-j+2} < \cdots < i_k} X_{i_1} X_{i_2} \cdots X_{i_k} .$$

Here if $k-j \leq -1$ or $k-j \geq m'+1$ in the suffixes, then we ignore all such terms. Put $i_{k-j+l} = i'_l$, $l=1, \cdots, j$. Then

$$S_k(X_1, \cdots, X_{m'}, Y_1, \cdots, Y_{m''-1})$$
$$= \sum_{j=0}^{m''-1} \sum_{i_1 < \cdots < i_{k-j} \leq m', i'_1 < \cdots < i'_j \leq m''-1} X_{i_1} \cdots X_{i_{k-j}} Y_{i'_1} \cdots Y_{i'_j} .$$

Since

$$\sum_{i_1 < \cdots < i_{k-j} \leq m', i'_1 < \cdots < i'_j \leq m''-1} X_{i_1} \cdots X_{i_{k-j}} \, Y_{i'_1} \cdots Y_{i'_j}$$

$$= \left(\sum_{i_1 < \cdots < i_{k-j} \leq m'} X_{i_1} \cdots X_{i_{k-j}}\right)\left(\sum_{i'_1 < \cdots < i'_j \leq m''-1} Y_{i'_1} \cdots Y_{i'_j}\right)$$

$$= S_{k-j}(X_1, \cdots, X_{m'}) \cdot S_j(Y_1, \cdots, Y_{m''-1}),$$

it implies that (2.7). Q.E.D.

Lemma 2.4. *For $k=0, \cdots, m-m'-1$ and $l=m'+1, \cdots, m$,*

(2.8)
$$D^{(m-m'-1)-k}(\tau'_{m'+1}, \cdots, \tau'_{m-1})$$
$$= \prod_{0 \leq i < j \leq m-m'-1, i \neq k, j \neq k}(\zeta^j - \zeta^i)$$

and

(2.9)
$$D^{(m-m'-1)-k}(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)$$
$$= (-1)^{(m-l)^2} \cdot \zeta^{k(m-l)} \cdot D^{(m-m'-1)-k}(\tau'_{m'+1}, \cdots, \tau'_{m-1}).$$

In particular, (2.8)$\neq 0$ and (2.9)$\neq 0$.

Proof. Put

$$a_j = (\tau'^j_{m'+1}, \cdots, \tau'^j_{m-1}), \quad 0 \leq j \leq m-m'-1 \quad \text{and} \quad j \neq k.$$

Since

$$a_j = ((\zeta^0)^j, \cdots, (\zeta^{m-m'-2})^j) = ((\zeta^j)^0, \cdots, (\zeta^j)^{m-m'-2}),$$

we have

$$D^{(m-m'-1)-k}(\tau'_{m'+1}, \cdots, \tau'_{m-1}) = D_0(\zeta^0, \cdots, \zeta^{k-1}, \zeta^{k+1}, \cdots, \zeta^{m-m'-1}),$$

which equals the right-hand side of (2.8).

Multiply the rows $(\tau'^j_{m'+1}, \cdots, \tau'^j_{l-1}, \tau'^j_{l+1}, \cdots, \tau'^j_m)$ of the left-hand side of (2.9) by $\zeta^{(m-l)j}$, $j=1, \cdots, m-m'-1$ and $j \neq k$. Then

$$\zeta^{(m-l)\{(m-m'-1)(m-m')/2-k\}} \cdot D^{(m-m'-1)-k}(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)$$
$$= D^{(m-m'-1)-k}(\tau'_{m+m'-l+1}, \cdots, \tau'_{m-1}, \tau'_{m'+1}, \cdots, \tau'_{m+m'-l})$$
$$= (-1)^{(l-m'-1)(m-l)} \cdot D^{(m-m'-1)-k}(\tau'_{m'+1}, \cdots, \tau'_{m-1}).$$

Since

$$\zeta^{(m-l)(m-m'-1)(m-m')/2} = (-1)^{(m-l)(m-m'-1)}$$

and

$$(l-m'-1)(m-l)-(m-l)(m-m'-1) = -(m-l)^2 \equiv (m-l)^2 \pmod{2},$$

it implies that (2.9). Q.E.D..

Under Assumption 1.1, we have studied the following asymptotic properties of the characteristic roots in [2]. Denote the characteristic roots of $\varepsilon^{m-m'} \cdot$ $P_1(\xi) + P_2(\xi) = 0$ with respect to ξ_1 by $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ and those of $P_2(\xi) = 0$ by $\sigma_j(\xi')$, $j = 1, \cdots, m'$.

Lemma 2.5. *Let Assumption* 1.1 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 1.1. *If the suffixes* $\{j\}$ *of the characteristic roots* $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ *are properly chosen, then there exists a positive number* ε_0 *such that* $\varepsilon \cdot \tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ *are analytic functions of* (ε, ξ') *in* $\{|\varepsilon| < \varepsilon_0\} \times B_0$ *and* $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ *have the following representations*: *For* $j = 1, \cdots, m'$

$$(2.10) \qquad \tau_j(\varepsilon, \xi') = \sigma_j(\xi') + \sum_{k=2}^{\infty} s_{j,k}(\xi') \, \varepsilon^{(k-1)(m-m')} ,$$

where $\partial_1 = \dfrac{\partial}{\partial \xi_1}$ *and*

$$(2.11) \qquad s_{j,2} = -P_1(\sigma_j, \xi') \cdot \partial_1 P_2(\sigma_j, \xi')^{-1} .$$

For $j = m'+1, \cdots, m$

$$(2.12) \qquad \tau_j(\varepsilon, \xi') = \Theta \tau_j' \cdot \frac{1}{\varepsilon} + \sum_{k=2}^{\infty} t_{j,k}(\xi') \cdot \varepsilon^{k-2} .$$

Remark. If $s_{j,2}(\xi') \equiv 0$ in B_0, then $P_1(\sigma_j(\xi'), \xi') \equiv 0$ in B_0. This implies that $\tau_j(\varepsilon, \xi') \equiv \sigma_j(\xi')$ in B_0.

3. The Condition for the Convergence of the Solutions

In this section, we shall study a necessary and sufficient condition for the convergence when Condition 1.2 is not satisfied. The main result in this section is the following:

Proposition. *Assume that Condition* 1.2 *is not satisfied. Let Assumption* 1.1 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 1.1. *Let the Cauchy data* $\Phi = (\phi_1, \cdots, \phi_m)$ *of* (1.1) *belong to* $F^{-1}(C_0^{\infty}(B_0))^m$. *The solutions* $u_\varepsilon(x; \Phi)$ *of* (1.1) *converge in* $C(\mathbf{R}^n)$ *if and only if the Cauchy data* Φ *satisfy*

$$(3.1) \qquad \sum_{k=0}^{m'}(-1)^k \cdot S_k(\tau_1, \cdots, \tau_{m'}) \cdot \hat{\phi}_{m'+j-k} \equiv 0, j = 1, \cdots, m-m' .$$

Here $S_k(\tau_1, \cdots, \tau_{m'})$, $k = 0, \cdots, m'$ *are the elementary symmetric polynomials of the characteristic roots* $\tau_j = \tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m'$ *mentioned in Notation* 2.1. *For the Cauchy data satisfying* (3.1), *the partial Fourier transforms of solutions* $u_\varepsilon(x; \Phi)$ *of* (1.1) *are*

$$(3.2) \qquad \hat{u}_\varepsilon(x_1, \xi') = \sum_{j=1}^{m'} \frac{D_j(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m)}{D_0(\tau_1, \cdots, \tau_m)} \cdot \exp i\tau_j(\varepsilon, \xi') x_1 .$$

Remark. Denote $\hat{\Phi} = (\hat{\phi}_1, \cdots, \hat{\phi}_m)$. Let β belong to $C^\infty(B_0)$. If $\hat{\Phi}$ satisfies (3.1), then $\beta \cdot \hat{\Phi}$ also satisfies (3.1). Hence if there exists a Cauchy datum Φ_0 such that the solutions $u_\varepsilon(x; \Phi_0)$ converge, then for all the Cauchy data Φ satisfying $\hat{\Phi} = \beta \cdot \hat{\Phi}_0$, where β belongs to $C^\infty(B_0)$, the solutions $u_\varepsilon(x; \Phi)$ converge.

In order to prove Proposition, we need the following lemma. Denote $L = \{l; \operatorname{Im}\Theta\tau'_l \neq 0, l = m'+1, \cdots, m\}$, where Θ and τ'_l, $l = m'+1, \cdots, m$ are mentioned in Notation 2.1.

Lemma 3.1. *Let the same assumptions as in Proposition be satisfied. The solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(\mathbf{R}^n)$ if and only if the Cauchy data Φ satisfy*

$$(3.3) \quad \varepsilon^{m-1} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot S_j(\tau_1, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m) \cdot \hat{\phi}_{m-j}$$
$$= \sum_{j=0}^{m-1} (-1)^j \cdot \varepsilon^{m-1-j} \cdot S_j(\varepsilon\tau_1, \cdots, \varepsilon\tau_{l-1}, \varepsilon\tau_{l+1}, \cdots, \varepsilon\tau_m) \cdot \hat{\phi}_{m-j} \equiv 0$$

as analytic functions of ε for all l in L.

Proof. The partial Fourier transform with respect to x' of (1.1) is

$$(3.4) \qquad \begin{cases} (\varepsilon^{m-m'} \cdot P_1(D_1, \xi') + P_2(D_1, \xi')) \, \hat{u}(x_1, \xi') = 0 ; \\ D_1^{j-1} \hat{u}(0, \xi') = \hat{\phi}_j(\xi'), j = 1, \cdots, m . \end{cases}$$

For fixed ξ', (3.4) is a one-parameter family of Cauchy problems of ordinary differential equations. For $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$ satifying Lemma 2.5 and for Φ in $F^{-1}(C_0^\infty(B_0))^m$, put

$$(3.5) \qquad C_k(\varepsilon, \xi'; \Phi) = D_k(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m)/D_0(\tau_1, \cdots, \tau_m) ,$$

$k = 1, \cdots, m$. Since (3.5) are analytic in ε for sufficiently small ε, (3.5) have power series representations as

$$C_k(\varepsilon, \xi'; \Phi) = \sum_{j=0}^\infty c_{k,j}(\xi'; \Phi) \cdot \varepsilon^j, k = 1, \cdots, m .$$

The solution $v_\varepsilon(x_1, \xi')$ of (3.4) is represented as

$$(3.6) \qquad v_\varepsilon(x_1, \xi') = \sum_{k=1}^m C_k(\varepsilon, \xi'; \Phi) \cdot \exp i\tau_k(\varepsilon, \xi') x_1 .$$

Denote $u_\varepsilon(x) = F^{-1}(v_\varepsilon(x_1, \xi'))$. By the same argument as in [3], we can prove that if there exist integers l in L and j with $j \geq 0$ such that $c_{l,j}(\xi'; \Phi) \not\equiv 0$, then $u_\varepsilon(x)$ can not converge in $C(\mathbf{R}^n)$. When Condition 1.2 is not satisfied, L is not void. Therefore it is necessary for the convergence of $u_\varepsilon(x)$ in $C(\mathbf{R}^n)$ that

$c_{l,j}(\xi'; \Phi) \equiv 0$ for all l in L and all $j \geqq 0$, that is, $C_l(\varepsilon, \xi'; \Phi) \equiv 0$ for all l in L and sufficiently small ε, which implies the convergence of solutions. Since

$$\tau_j(\varepsilon, \xi') \neq \tau_k(\varepsilon, \xi'), \quad 1 \leqq j < k \leqq m$$

in B_0 for sufficiently small ε, we can apply Lemma 2.2 to $C_l(\varepsilon, \xi'; \Phi)$ for $X_j = \tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$. Then

$$C_l = \frac{\varepsilon^{m-1} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot S_j(\tau_1, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m) \cdot \hat{\phi}_{m-j}}{\varepsilon^{m-1} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot S_j(\tau_1, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m) \cdot \tau_l^{m-1-j}}.$$

The denominators are equal to

$$\sum_{j=0}^{m-1} (-1)^j \cdot S_j(\varepsilon\tau_1, \cdots, \varepsilon\tau_{l+1}, \varepsilon\tau_{l+1}, \cdots, \varepsilon\tau_m) \cdot (\varepsilon\tau_l)^{m-1-j}$$
$$= \prod_{1 \leqq j \leqq m, j \neq l}(\varepsilon\tau_l - \varepsilon\tau_j)$$

and analytic in ε and not zero in B_0 for sufficiently small ε. This implies that $C_l(\varepsilon, \xi'; \Phi) \equiv 0$ if and only if the numerators

$$\varepsilon^{m-1} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot S_j(\tau_1, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m) \cdot \hat{\phi}_{m-j}$$
$$= \sum_{j=0}^{m-1} (-1)^j \cdot \varepsilon^{m-1-j} \cdot S_j(\varepsilon\tau_1, \cdots, \varepsilon\tau_{l-1}, \varepsilon\tau_{l+1}, \cdots, \varepsilon\tau_m) \cdot \hat{\phi}_{m-j}$$

are zero as analytic functions of ε. Thus we come to the conclusion.   Q.E.D.

Proof of Proposition. Fix l in L and apply Lemma 2.3 to (3.3) for

$$X_j = \tau_j(\varepsilon, \xi'), \qquad j = 1, \cdots, m'.$$
$$Y_{j-m'} = \tau_j(\varepsilon, \xi'), \qquad j = m'+1, \cdots, l-1,$$
$$Y_{j-m'-1} = \tau_j(\varepsilon, \xi'), \quad j = l+1, \cdots, m.$$

Then the convergence of solutions is equivalent to

$$(3.7) \qquad \varepsilon^{m-1} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot S_j(\tau_1, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m) \cdot \hat{\phi}_{m-j}$$
$$= \varepsilon^{m-1} \cdot \sum_{j=0}^{m-1} (-1)^j \cdot (\sum_{k=0}^{m-m'-1} S_{j-k}(\tau_1, \cdots, \tau_{m'})$$
$$\cdot S_k(\tau_{m'+1}, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m) \cdot \hat{\phi}_{m-j} \equiv 0.$$

Substitute (2.10) and (2.13) for $\tau_j(\varepsilon, \xi')$, $j = 1, \cdots, m$. Then we can write for $j = 1, \cdots, m-1$ and $k = 0, \cdots, m-m'-1$

$$(3.8) \qquad S_{j-k}(\tau_1, \cdots, \tau_{m'}) = \sum_{i=0}^{\infty} S_{j-k,i}^1(\xi') \cdot \varepsilon^{i(m-m')},$$

$$(3.9) \qquad \varepsilon^k \cdot S_k(\tau_{m'+1}, \cdots, \tau_{l-1}, \tau_{l+1}, \cdots, \tau_m)$$
$$= S_k(\varepsilon\tau_{m'+1}, \cdots, \varepsilon\tau_{l-1}, \varepsilon\tau_{l+1}, \cdots, \varepsilon\tau_m) = \sum_{i=0}^{\infty} S_{k,i}^2(\xi') \cdot \varepsilon^i.$$

Here (3.8) and (3.9) are analytic and converge absolutely for sufficiently small

ε in B_0 and

$$(3.10) \qquad S^1_{j-k,0}(\xi') = S_{j-k}(\sigma_1, \cdots, \sigma_{m'}),$$

$$(3.11) \qquad S^2_{k,0}(\xi') = \Theta^k \cdot S_k(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \sigma'_m).$$

Put for $\kappa = 0, 1, \cdots$

$$(3.12) \quad T_{1,\kappa} = \sum_{j=0}^{m-1} (-1)^j \cdot \sum_{k=0}^{m-m'-1} S^1_{j-k,\kappa} \cdot \varepsilon^{\kappa(m-m')} \cdot S^2_{k,0} \cdot \varepsilon^{m-1-k} \cdot \hat{\phi}_{m-j},$$

$$T_{2,\kappa} = \sum_{j=0}^{m-1} (-1)^j \cdot \sum_{k=0}^{m-m'-1} S^1_{j-k,\kappa} \cdot \varepsilon^{\kappa(m-m')} \cdot (\sum_{i=1}^{\infty} S^2_{\kappa,i} \cdot \varepsilon^{m-1-k+i}) \cdot \hat{\phi}_{m-j},$$

and

$$T_{3,\kappa} = \sum_{j=0}^{m-1} (-1)^j \cdot \sum_{k=0}^{m-m'-1} (\sum_{i=\kappa+1}^{\infty} S^1_{j-k,i} \cdot \varepsilon^{i(m-m')}) \cdot (\sum_{i=0}^{\infty} S^2_{k,i} \cdot \varepsilon^{m-1-k+i}) \cdot \hat{\phi}_{m-j}.$$

Since $T_{2,\kappa}$ and $T_{3,\kappa}$ converge absolutely for sufficiently small ε, we can change the order of summations. Then (3.7) is rewritten as $T_{1,0} + T_{2,0} + T_{3,0} \equiv 0$. The least order with respect to ε of $T_{1,0}$ is m', that of $T_{2,0}$ is $m'+1$, and that of $T_{3,0}$ is m. The coefficient of $\varepsilon^{m'}$ in $T_{1,0}$ is

$$(3.13) \qquad \sum_{j=0}^{m-1} (-1)^j \cdot S^1_{j-(m-m'-1),0} \cdot S^2_{m-m'-1,0} \cdot \hat{\phi}_{m-j}$$

$$= S^2_{m-m'-1,0} \cdot \sum_{j=m-m'-1}^{m-1} (-1)^j \cdot S^1_{j-(m-m'-1),0} \cdot \hat{\phi}_{m-j}$$

and this must be zero. Lemma 2.2 implies that

$$S^2_{m-m'-1,0} = \Theta^{m-m'-1} \cdot S_{m-m'-1}(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)$$

$$= \Theta^{m-m'-1} \times \frac{D^{(m-m'-1)-0}(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)}{D^0(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)}.$$

Lemma 2.4 implies that

$$\frac{D^{(m-m'-1)-0}(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)}{D^0(\tau'_{m'+1}, \cdots, \tau'_{l-1}, \tau'_{l+1}, \cdots, \tau'_m)} \neq 0.$$

Hence $S^2_{m-m'-1,0} \neq 0$. Dividing (3.13) by $S^2_{m-m'-1,0}$, we have

$$(3.14) \qquad \sum_{j=m-m'-1}^{m-1} (-1)^j \cdot S^1_{j-(m-m'-1),0} \cdot \hat{\phi}_{m-j}$$

$$= (-1)^{m-m'-1} \cdot \sum_{j=0}^{m'} (-1)^j \cdot S^1_{j,0} \cdot \hat{\phi}_{m'+1-j} = 0.$$

When (3.14) is satisfied, we have

$$T_{1,0} = \sum_{j=0}^{m-1} (-1)^j \cdot \sum_{k=0}^{m-m'-2} (S^1_{j-k,0} \cdot S^2_{k,0} \cdot \varepsilon^{m-1-k} \cdot \hat{\phi}_{m-j})$$

and

$$T_{2,0} = \sum_{j=0}^{m-1} (-1)^j \cdot S^1_{j-(m-m'-1),0} \cdot (\sum_{i=1}^{\infty} S^2_{m-m'-1,i} \cdot \varepsilon^{m'+i}) \cdot \hat{\phi}_{m-j}$$

$$+ \sum_{j=0}^{m-1} (-1)^j \cdot \sum_{k=0}^{m-m'-2} S^1_{j-k,0} \cdot (\sum_{i=1}^{\infty} S^2_{k,i} \cdot \varepsilon^{m-1-k+i}) \cdot \hat{\phi}_{m-j}$$

$$= \sum_{i=1}^{\infty} S^2_{m-m'-1,i} \cdot \varepsilon^{m'+i} \cdot \left(\sum_{j=0}^{m-1}(-1)^j \cdot S^1_{-(m-m'-1),0} \hat{\phi} \cdot {}_{m-j}\right)$$
$$+ \sum_{j=0}^{m-1}(-1)^j \cdot \sum_{k=0}^{m-m'-2} S^1_{j-k,0} \cdot \left(\sum_{i=1}^{\infty} S^2_{k,i} \cdot \varepsilon^{m-1-k+i}\right)\hat{\phi} \cdot {}_{m-j}$$
$$= \sum_{j=0}^{m-1}(-1)^j \cdot \sum_{k=0}^{m-m'-2} S^1_{j-k,0} \cdot \left(\sum_{i=1}^{\infty} S^2_{k,i} \cdot \varepsilon^{m-1-k+i}\right) \cdot \hat{\phi}_{m-j} .$$

Thus we can diminish by one the ranges of summations with respect to k in $T_{1,0}$ and in $T_{2,0}$. Repeat this process $m-m'-1$ times until the ranges of summations with respect to k are diminished to $\{0\}$. Then we have for $j=1, \cdots, m-m'$

(3.15) $$\sum_{k=0}^{m'}(-1)^k \cdot S^1_{k,0} \cdot \hat{\phi}_{m'+j-k} = 0$$

and $T_{1,0} \equiv T_{2,0} \equiv 0$. Hence

$$T_{1,0} + T_{2,0} + T_{3,0} = T_{3,0} = T_{1,1} + T_{2,1} + T_{3,1} \equiv 0 .$$

Repeating the above argument for $\kappa=1, 2, \cdots$, we have for $\kappa=0, 1, \cdots$

(3.16) $$\sum_{k=0}^{m'}(-1)^k \circ S^1_{k,\kappa} \cdot \hat{\phi}_{m'+j-k} = 0, \quad j=1, \cdots, m-m' ,$$

which are equivalent to (3.3). Multiply (3.16) by $\varepsilon^{\kappa(m-m')}$ and sum up from $\kappa=0$ to $\kappa=\infty$. Even if we change l in L, we have the same conclusion. Hence (3.16) are equivalent to (3.1) as far as (3.8) converge absolutely. Since (3.16) are independent of l in L, it implies that (3.3) remain true for all $l=m'+1, \cdots, m$ and $C_l(\varepsilon, \xi'; \Phi) \equiv 0$ for all $l=m'+1, \cdots, m$. Then (3.6) implies (3.2). Thus we come to the conclusion.   Q.E.D.

Put

$$\tilde{p}_{2,k}(\xi') = -p_{2,k}(\xi')/p_{2,0}, \quad k=0, \cdots, m' ,$$
$$\tilde{p}_{2,k} = 0, \quad \text{for } k \leq -1 \text{ or } k \geq m'+1 .$$

Here $\tilde{p}_{2,k}(\xi')$, $k=0, \cdots, m'$ are polynomials of ξ', because $p_{2,0}$ is a non-zero constant. Denote

$$Q(\xi') = (-\tilde{p}_{2,j-k}(\xi'); j \downarrow 1, \cdots, m-m', k \rightarrow 1, \cdots, m-m') ,$$
$$R(\xi') = (\tilde{p}_{2,m'+j-k}(\xi'); j \downarrow 1, \cdots, m-m', k \rightarrow 1, \cdots, m') .$$

Here $Q(\xi')$ is a lower triangular matrix all whose diagonal components are 1. Hence $Q^{-1}(\xi')$ is a matrix with polynomial components. Denote $\hat{\Phi}' = (\hat{\phi}_1, \cdots, \hat{\phi}_{m'})$ and $\hat{\Phi}'' = (\hat{\phi}_{m'+1}, \cdots, \hat{\phi}_m)$.

   Corollary 3.2. *Let the same assumptions as in Proposition be satisfied. Then the convergence of solutions implies that*

(3.17) $\sum_{k=0}^{m'} p_{2,k}(\xi') \cdot \hat{\phi}_{m'+j-k}(\xi') = 0$, $j = 1, \cdots, m-m'$.

Then (3.17) *are equivalent to*

(3.18) $^t\hat{\phi}'' = Q^{-1}(\xi') \cdot R(\xi') \cdot {}^t\hat{\phi}'$.

The inverse Fourier transforms of (3.18) are

(3.19) $^t\Phi'' = Q^{-1}(D') \cdot R(D') \cdot {}^t\Phi'$.

Proof.   Since

$$P_2(\xi)/p_{2,0} = \xi_1^{m'} - \sum_{j=1}^{m'} \tilde{p}_{2,j}(\xi') \cdot \xi_1^{m'-j} = \prod_{j=1}^{m'}(\xi_1 - \sigma_j) ,$$

it implies that

$$-\tilde{p}_{2,j}(\xi') = (-1)^j \cdot S_j(\sigma_1, \cdots, \sigma_{m'}) ,   j = 1, \cdots, m' .$$

Hence (3.10) and (3.15) imply that for $j=1, \cdots, m-m'$

(3.20) $\sum_{k=0}^{m'} \tilde{p}_{2,k} \cdot \hat{\phi}_{m'+j-k} = 0$,

which is equivalent to (3.17).   By (3.20), we have

$$-\sum_{k=0}^{j-1} \tilde{p}_{2,k} \cdot \hat{\phi}_{m'+j-k} = \sum_{k=j}^{m'} \tilde{p}_{2,k} \cdot \hat{\phi}_{m'+j-k} ,   j = 1, \cdots, m-m' ,$$

which are equivalent to (3.18).   Q.E.D.

Remark.   In [3], we used the function $d(\xi'; \Phi)$ to decide the divergence of solutions.   Corollary 3.2 implies that $d(\xi'; \Phi) \equiv 0$ is equivalent to $\sum_{k=0}^{m'} p_{2,k}(\xi')$ $\cdot \hat{\phi}_{m'+1-k}(\xi') \equiv 0$.   We had better use (3.18) instead of $d(\xi'; \Phi) \equiv 0$ because (3.18) are more accurate and algebraic relations between the coefficients of P_2 and the Cauchy data.

Denote

$$A = (a_{j,k}; j \downarrow 1, \cdots, m-m', k \to 1, \cdots, m')$$
$$= (\hat{\phi}_{j+m'-k}; j \downarrow 1, \cdots, m-m', k \to 1, \cdots, m') ,$$

for $i=1, \cdots, m'$

$$b_i = (b_{k,i}; k \downarrow 1, \cdots, m')$$
$$= ((-1)^{k-1} \cdot S_{k-1}(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{m'}); k \downarrow 1, \cdots, m') .$$

Here we define $S_0=1$ even when $m'=1$.   Denote

$$B = (b_i; i \to 1, \cdots, m') ,$$
$$s = (s_{i,2}; i \downarrow 1, \cdots, m') ,$$

where $s_{i,2} = -P_1(\sigma_i, \xi') \cdot \partial_1 P_2(\sigma_i, \xi')^{-1}, i = 1, \cdots, m'$. Then we have the following corollary.

Corollary 3.3. *Let the same assumptions as in Proposition be satisfied. Then* $\det B = \prod_{1 \le i < j \le m'}(\sigma_j - \sigma_i) \ne 0$ *in* B_0. *The convergence of the solutions implies that* $ABs = 0$, *which are equivalent to*

$$(3.21) \qquad \sum_{j=1}^{m'} s_{i,2} \sum_{k=0}^{m'-1}(-1)^k \cdot S_k(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{m'})$$
$$\cdot \hat{\phi}_{m'-1+j-k} = 0, \quad j = 1, \cdots, m-m'.$$

Proof. Since for $k = 1, \cdots, m'-1$ and $i = 1, \cdots, m'-1$,

$$S_k(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{m'}) = \sigma_{m'} \cdot S_{k-1}(\sigma_1, \cdots \sigma_{i-1}, \sigma_{i+1}, \cdots \sigma_{m'-1})$$
$$+ S_k(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots \sigma_{m'-1})$$

and

$$S_k(\sigma_1, \cdots, \sigma_{m'-1}) = \sigma_i \cdot S_{k-1}(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots \sigma_{m'-1})$$
$$+ S_k(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots \sigma_{m'-1}),$$

it implies that for $i = 1, \cdots, m'-1$

$$b_i - b_{m'} = (\sigma_i - \sigma_{m'})$$
$$\times ((-1)^{k-2} \cdot S_{k-2}(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots \sigma_{m'-1}); k \downarrow 1, \cdots, m-m'),$$

where $S_{-1} = 0$. Then

$$\det(b_1, \cdots, b_{m'}) = \det(b_1 - b_{m'}, \cdots, b_{m'-1} - b_{m'}, b_{m'})$$
$$= (-1)^{m'+1} \cdot \prod_{i=1}^{m'-1}(\sigma_i - \sigma_{m'}) \cdot \det(b_{k,i}; j \downarrow 1, \cdots, m'-1, k \to 1, \cdots, m'-1)$$
$$= \prod_{i=1}^{m'-1}(\sigma_{m'} - \sigma_i) \cdot \det(b_{k,i}; j \downarrow 1, \cdots, m'-1, k \to 1, \cdots, m'-1).$$

Hence $\det B$ is equal to the difference product of $\sigma_j, j = 1, \cdots, m'$. Since $\sigma_j \ne \sigma_k$, $1 \le j < k \le m'$ in B_0, it implies that $\det B \ne 0$ in B_0. Since $S_{0,1}^1 = 0$ and

$$S_{k,1}^1 = \sum_{i=1}^{m'} s_{i,2} \cdot S_{k-1}(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{m'}), k = 1, \cdots, m-m',$$

(3.16) for $\kappa = 1$ implies that

$$-\sum_{i=1}^{m'} s_{i,2} \sum_{k=1}^{m'}(-1)^{k-1} \cdot S_{k-1}(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{m'})$$
$$\times \hat{\phi}_{m'+j-k} = 0, \quad j = 1, \cdots, m-m',$$

which are equivalent to $ABs = 0$.   Q.E.D.

In the rest of this section, we shall study conditions on the Cauchy data in special forms for the convergence of the solutions.

Corollary 3.4. *Let the same assumptions as in Proposition be satisfied. Let $\Phi'=(\phi_1, \cdots, \phi_{m'})=0$. The solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(\mathbf{R}^n)$ for this Cauchy data if and only if $\Phi''=(\phi_{m'+1}, \cdots, \phi_m)=0$.*

Proof. Assume that the solutions converge. Substitute $\Phi'=0$ in (3.19). Then $\Phi''=0$. The converse is trivial. Q.E.D.

Corollary 3.5. *Let the same assumptions as in Proposition be satisfied. Assume that $m-m'\geqq m'$ and $p_{2,m'}\not\equiv 0$. Let $\Phi''=(\phi_{m'+1}, \cdots, \phi_m)=0$. The solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(\mathbf{R}^n)$ for this Cauchy data if and only if $\Phi'=(\phi_1, \cdots, \phi_{m'})=0$.*

Proof. Assume tha tthe solutions converge. Substitute $\hat{\phi}_j=0$, $j=m'+1$, $\cdots, m$ in (3.17). Then

$$\sum_{k=j}^{m'} p_{2,k}\cdot\hat{\phi}_{m'+j-k} \equiv 0, \quad j=1, \cdots, m-m'.$$

When $j=m'\leqq m-m'$, we have $p_{2,m'}\cdot\hat{\phi}_{m'}\equiv 0$. Since $p_{2,m'}$ is a polynomial with $p_{2,m'}\not\equiv 0$ and $\hat{\phi}_{m'}$ is a continuous function, it implies that $\hat{\phi}_{m'}\equiv 0$. Hence

$$\sum_{k=j+1}^{m'} p_{2,k}\cdot\hat{\phi}_{m'+j-k} \equiv 0, \quad j=1, \cdots, m-m'.$$

When $j=m'-1$, we have $p_{2,m'}\cdot\hat{\phi}_{m'-1}\equiv 0$. Since $p_{2,m'}\not\equiv 0$, it implies that $\hat{\phi}_{m'-1}\equiv 0$. Repeating this process m' times, we come to the conclusion. The converse is trivial. Q.E.D.

Corollary 3.6. *Let the same assumptions as in Proposition be satisfied. Let $\phi_j=0$, $j=1, \cdots, l-1, l+1, \cdots, m$.*
(Case 1). The case when $1\leqq l\leqq m'$. Assume that there exists a positive integer j' with $m'+1-l\leqq j'\leqq\min\{m-l, m'\}$ such that $p_{2,j'}\not\equiv 0$. Then the solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(\mathbf{R}^n)$ for this Cauchy data if and only if $\phi_l=0$.
(Case 2). The case when $m'+1\leqq l\leqq m$. The solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(\mathbf{R}^n)$ for this Cauchy data if and only if $\phi_l=0$.

Proof. Assume that the solutions converge. Let $1\leqq l\leqq m'$. Substitute $\hat{\phi}_j=0$, $j=1, \cdots, l-1, l-1, \cdots, m$ in (3.17). Then $p_{2,m'+j-l}\cdot\hat{\phi}_l\equiv 0, j=1, \cdots, m-m'$. We use the case when $j=j'+l-m'$, that is, $p_{2,j'}\cdot\hat{\phi}_l\equiv 0$. Since $p_{2,j'}\not\equiv 0$, it implies that $\hat{\phi}_l\equiv 0$. When $m'+1\leqq l\leqq m$, Corollary 3.4 implies the conclusion. The converse is trivial. Q.E.D.

4. The Admissibility of the Cauchy Data

In this section, we shall remove the dependency on ε from (3.1) and

classify the Cauchy data space $F^{-1}(C_0^\infty(B_0))^m$ into two classes in the two special cases mentioned in §1. One class is the set of all the Cauchy data Φ with which the solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(R^n)$ and the other is the complement.

Denote the characteristic roots of $\varepsilon^{m-m'} \cdot P_1(\xi) + P_2(\xi) = 0$ with respect to ξ_1 by $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$, those of $P_1(\xi) = 0$ by $\upsilon_j(\xi')$, $j=1, \cdots, m$, and those of $P_2(\xi) = 0$ by $\sigma_j(\xi')$, $j=1, \cdots, m'$. Then we have the following lemma.

Lemma 4.1. *Let B be a fixed arbitrary open set in R^{n-1}. For every integer i with $1 \leq i \leq m'$, there eixsts an integer j with $1 \leq j \leq m$ such that $\sigma_i \equiv \upsilon_j$ in B if and only if $P_1(\xi)$ is divided by $P_2(\xi)$ in the polynomial ring $C[\xi]$.*

Proof. We may assume that $\sigma_j \equiv \upsilon_j$, $j=1, \cdots, m'$ in B. Applying Lemma 2.3 for $X_i = \sigma_i$, $i=1, \cdots, m'$ and $Y_j = \upsilon_{j+m'}$, $j=1, \cdots, m''$, we have in B for $k=1, \cdots, m''$

$$S_k(\upsilon_{m'+1}, \cdots, \upsilon_m) = S_k(\sigma_1, \cdots, \sigma_{m'}, \upsilon_{m'+1}, \cdots, \upsilon_m)$$
$$-\sum_{j=0}^{k-1} S_{k-j}(\sigma_j, \cdots, \sigma_{m'}) \cdot S_j(\upsilon_{m'+1}, \cdots, \upsilon_m) .$$

Since

$$S_k(\sigma_1, \cdots, \sigma_{m'}, \upsilon_{m'+1}, \cdots, \upsilon_m) = (-1)^k \cdot p_{1,k}(\xi')/p_{1,0}, \quad k=1, \cdots, m$$

and

$$S_k(\sigma_1, \cdots, \sigma_{m'}) = (-1)^k \cdot p_{2,k}(\xi')/p_{2,0}, \quad k=1, \cdots, m'$$

are polynomials of ξ', we have inductively that $S_k(\upsilon_{m'+1}, \cdots, \upsilon_m)$, $k=1, \cdots, m''$ have polynomial representations in B. Hence

$$P_1(\xi)/P_2(\xi) = (p_{1,0}/p_{2,0}) \cdot \prod_{k=1}^{m''} (\xi_1 - \upsilon_{k+m'})$$
$$= (p_{1,0}/p_{2,0}) \cdot \sum_{k=0}^{m''} (-1)^k \cdot S_k(\upsilon_{m'+1}, \cdots, \upsilon_m) \cdot \xi_1^{m''-k}$$

has a polynomial representation in $R \times B$. Thus there exists a polynomial $P_3(\xi)$ such that $P_1(\xi)/P_2(\xi) = P_3(\xi)$ in $R \times B$, which remains true in R^n. The converse is trivial.   Q.E.D

Theorem A. *Assume that Condition 1.2 is not satisfied. Let Assumption 1.1 be satisfied and B_0 be the open ball in Remark to Assumption 1.1. Let the Cauchy data $\Phi = (\phi_1, \cdots, \phi_m)$ of (1.1) belong to $F^{-1}(C_0^\infty(B_0))^m$.*
(Case 1). The case when for every integer i with $1 \leq i \leq m'$ there exists an integer j with $1 \leq j \leq m$ such that $\sigma_i \equiv \upsilon_j$ in B_0, that is, when $P_1(\xi)$ is divided by $P_2(\xi)$ in the polynomial ring $C[\xi]$. The solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and

only if the Cauchy data Φ satisfy (3.19). *In this case, the solutions $u_\varepsilon(x; \Phi)$ are identically equal to the solution $u_0(x; \Phi')$ of* (2.1).

(*Case* 2). *The case when $\sigma_1 \not\equiv v_j$, for all $j=1, \cdots, m$ and for every integer i with $2 \leqq i \leqq m'$ there exists an integer j with $1 \leqq j \leqq m$ such that $\sigma_i \equiv v_j$ in B_0. The solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and only if the Cauchy data Φ satisfy*

$$(4.1) \qquad \sum_{k=0}^{m'-1}(-1)^k \cdot S_k(\sigma_2, \cdots, \sigma_{m'}) \cdot \hat{\phi}_{m'+j-k} = 0, \quad j=0, \cdots, m-m'.$$

Here if $m'=1$, then $S_0=1$ and (4.1) *is equivalent to $\Phi=(0, \cdots, 0)$. In this case, the solutions $u_\varepsilon(x; \Phi)$ depend on ε in general.*

In order to prove Theorem A, we need the following lemma.

Lemma 4.2. *Assume that the characteristic roots satisfy $\tau_j(\varepsilon, \xi') \equiv \sigma_j(\xi')$, $j=1, \cdots, m'$ in B_0. Then for the Cauchy data satisfying* (3.17),

$$\frac{D_l(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m)}{D_0(\tau_1, \cdots, \tau_m)} = \frac{D_l(\sigma_1, \cdots, \sigma_{m'}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'})}{D_0(\sigma_1, \cdots, \sigma_{m'})},$$

$l=1, \cdots, m'$.

Proof. Put for $l=1, \cdots, m'$ and for $k=1, \cdots, m$

$$d_{l,k} = {}^t(\tau_1^{k-1}, \cdots, \tau_{l-1}^{k-1}, \hat{\phi}_k, \tau_{l+1}^{k-1}, \cdots, \tau_m^{k-1}).$$

Since for $i=1, \cdots, m'$ and $j=1, \cdots, m-m'$

$$\sum_{k=0}^{m'} p_{2,k} \cdot \tau_i^{m'+j-k-1} = \sigma_i^{j-1} \cdot P_2(\sigma_i, \xi') = 0,$$

it implies that

$$\sum_{k=0}^{m'} p_{2,k} \cdot d_{l,m'+j-k} = {}^t(0, \cdots, 0, \tau_{m'+1}^{j-1} \cdot P_2(\tau_{m'+1}, \xi'), \cdots, \tau_m^{j-1} \cdot P_2(\tau_m, \xi')).$$

Then

$$D_l(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m) = \det {}^t(d_{l,k}; k \to 1, \cdots, m)$$

$$= \det (d_{l,1}, \cdots, d_{l,m-1}, p_{2,0}^{-1} \cdot \sum_{k=0}^{m'} p_{2,k} \cdot d_{l,m-k})$$

$$= \det (d_{l,1}, \cdots, d_{l,m'}, p_{2,0}^{-1} \cdot \sum_{k=0}^{m'} p_{2,k} \cdot d_{l,m'+1-k}, \cdots, p_{2,0}^{-1} \cdot \sum_{k=0}^{m'} p_{2,k} \cdot d_{l,m-k})$$

$$= D_l(\sigma_1, \cdots, \sigma_{m'}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'}) \cdot D_0(\tau_{m'+1}, \cdots, \tau_m) \cdot (\prod_{k=m'+1}^{m} P_2(\tau_k, \xi')) \cdot p_{2,0}^{-m+m'},$$

where we use the fact that $P_2(\tau_k, \xi') \neq 0$ in B_0, $k=m'+1, \cdots, m$ for sufficiently small ε, because $\tau_j(\varepsilon, \xi')$, $j=1, \cdots, m$ are distinct in B_0 for sufficiently small ε. By the same argument,

$$D_0(\tau_1, \cdots, \tau_m)$$

$$= D_0(\sigma_1, \cdots, \sigma_{m'}) \cdot D_0(\tau_{m'+1}, \cdots, \tau_m) \cdot (\prod_{k=m'+1}^{m} P_2(\tau_k, \xi')) \cdot p_{2,0}^{-m+m'}.$$

Since $D_0(\tau_{m'+1}, \cdots, \tau_m) \cdot (\prod_{k=m'+1}^{m} P_2(\tau_k, \xi')) \neq 0$ in B_0 for sufficiently small ε except $\varepsilon = 0$, we come to the concuusion.   Q.E.D.

Proof of Theorem A.

(Case 1).   By Remark to Lemma 2.5, we have

(4.2) $\tau_j(\varepsilon, \xi') \equiv \sigma_j(\xi')$, $j = 1, \cdots, m'$ in B_0 .

Substitute σ_j for τ_j, $j = 1, \cdots, m'$ in (3.1) and (3.2).   Then (3.1) are equivalent to (3.19) when (4.2) are satisfied.   Lemma 4.2 and (3.2) imply that $u_\varepsilon(x; \Phi) = u_0(x; \Phi')$.

(Case 2).   By Remark to Lemma 2.5, we have $s_{1,2} \not\equiv 0$ and

(4.3) $\tau_j(\varepsilon, \xi') \equiv \sigma_j(\xi')$, $j = 2, \cdots, m'$.

Substitute σ_j for τ_j $j = 2, \cdots, m'$ in (3.1) and (3.2).   Since

$$S_k(\tau_1, \sigma_2, \cdots, \sigma_{m'}) = S_k(\sigma_2, \cdots, \sigma_{m'}) + \tau_1 \cdot S_{k-1}(\sigma_2, \cdots, \sigma_{m'}) ,$$

it implies that (3.1) are equivalent to

(4.4) $\sum_{k=0}^{m'}(-1)^k \cdot S_k(\sigma_2, \cdots, \sigma_{m'}) \cdot \hat{\phi}_{m'+j-k}$
         $+ \tau_1 \cdot \sum_{k=0}^{m'}(-1)^k \cdot S_{k-1}(\sigma_2, \cdots, \sigma_{m'}) \cdot \hat{\phi}_{m'+j-k}$
         $= \sum_{k=0}^{m'-1}(-1)^k \cdot S_k(\sigma_2, \cdots, \sigma_{m'}) \cdot (\hat{\phi}_{m'+j-k} - \tau_1 \cdot \hat{\phi}_{m'+j-k-1}) \equiv 0 ,$

$j = 1, \cdots, m - m'$.   Substitute (2.10) for τ_1 in (4.4).   Then the coefficients of ε^0 in (4.4) are

(4.5) $\sum_{k=0}^{m'-1}(-1)^k \cdot S_k(\sigma_2, \cdots, \sigma_{m'}) \cdot (\hat{\phi}_{m'+j-k} - \sigma_1 \cdot \hat{\phi}_{m'+j-k-1})$

and those of $\varepsilon^{m-m'}$ are

$$-s_{1,2} \cdot \sum_{k=0}^{m'-1}(-1)^k \cdot S_k(\sigma_2, \cdots, \sigma_{m'}) \cdot \hat{\phi}_{m'+j-k-1} ,$$

which must be zero.   Since $s_{1,2} \not\equiv 0$ and $s_{1,2}$ is an analytic function in B_0, it implies that

(4.6) $\sum_{k=0}^{m'-1}(-1)^k \cdot S_k(\sigma_2, \cdots, \sigma_{m'}) \cdot \hat{\phi}_{m'+j-k-1} = 0$

in $\{\xi' \in B_0; s_{1,2}(\xi') \neq 0\}$ and therefore in B_0.   By substituting (4.6) in (4.5), we have (4.1), which are equivalent to (3.1).   Obviously (3.2) depends on ε in general.   Thus we come to the conclusion.   Q.E.D.

Remark.   When $P_1(\xi)$ is divided by $P_2(\xi)$ in $C[\xi]$, there exists a datum Φ'' for an arbitrary datum Φ' such that the solutions converge.   In fact, we may take $\phi_j = D_1^{j-1} u_0(x; \Phi')$, $j = m'+1, \cdots, m$, which belong to $F^{-1}(C_0^\infty(B_0))$ and are

independent of ε. Corollary 3.4 implies the uniqueness of such data Φ'' as the solutions converge and the Cauchy-Kowalewski theorem implies the uniqueness of the solutions for fixed ε. Hence we can prove (Case 1) without Lemma 4.2. Our aim to use Lemma 4.2 is to prove (Case 1) only with algebraic calculations.

Theorem B. *Let $m'=1$ or $(m'=2$ and $m\geqq4)$. Assume that Condition 1.2 is not satisfied. Let Assumption 1.1 be satisfied and B_0 be the open ball in Remark to Assumption 1.1. Let the Cauchy data $\Phi=(\phi_1, \cdots, \phi_m)$ of (1.1) belong to $F^{-1}(C_0^\infty(B_0))^m$.*
(Case 1). The case when $m'=1$.
(1-a) If there exists an integer j with $1\leqq j\leqq m$ such that $\sigma_1\equiv v_j$ in B_0, then the solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and only if the Cauchy data Φ satisfy

$$(4.7) \qquad \hat{\phi}_j(\xi') = \sigma_i^{j-1}(\xi')\cdot\hat{\phi}_1(\xi'), j = 2, \cdots, m .$$

(1-b) If $\sigma_1\not\equiv v_j$, $j=1, \cdots, m$ in B_0. Then the solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and only if $\Phi=(0, \cdots, 0)$.
(Case 2). The case when $m'=2$ and $m\geqq4$.
(2-a) If there exist integers j_1 and j_2 with $1\leqq j_1, j_2\leqq m$ such that $\sigma_1\equiv v_{j_1}$ and $\sigma_2\equiv v_{j_2}$ in B_0 then the solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and only if the Cauchy data Φ satisfy (3.19) for $m'=2$.
(2-b) If there exists an integer j with $1\leqq j\leqq m$ such that $\sigma_1\equiv v_j$ and $\sigma_2\not\equiv v_k$, for all $k=1, \cdots, m$ in B_0, then the solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and only if the Cauchy data Φ satisfy (4.7).
(2-c) If $\sigma_i\not\equiv v_j$, $i=1, 2$ and $j=1, \cdots, m$ in B_0. Then the solutions $u_\varepsilon(x; \Phi)$ converge in $C(R^n)$ if and only if $\Phi=(0, \cdots. 0)$.

Except (2-b), the solutions $u_\varepsilon(x; \Phi)$ are identically equal to the solution $u_0(x; \Phi')$ of (1.2).

In order to prove Theorem B, we need the following lemma.

Lemma 4.3. *Let the same assumptions as in Proposition be satisfied. Then the following three conditions are equivalent.*
(1) There exist an analytic function $r(\xi')$ in B_0 and a $C_0^\infty(B_0)$-function $\beta(\xi')$ with $\beta(\xi')\not\equiv0$ such that the solutions of (1.1) converge for the Cauchy data satisfying

$$(4.8) \qquad \hat{\phi}_j(\xi') = \beta(\xi')\cdot r(\xi')^{j-1}, \quad j = 1, \cdots, m .$$

This implies that $\hat{\phi}_1 = \beta$.

(2) *There exists integers i with $1 \leq i \leq m'$ and j with $1 \leq j \leq m$ such that*

$$(4.9) \qquad \sigma_i(\xi') \equiv \upsilon_j(\xi') \quad in \ B_0 \ .$$

Here υ_k, $k=1, \cdots, m$ are characteristic roots of $P_1(\xi)=0$ with respect to ξ_1 and σ_k, $k=1, \cdots, m'$ are those of $P_2(\xi)=0$.

(3) *There exists an analytic function $\gamma(\xi')$ in B_0 such that for all the $C_0^\infty(B_0)$-functions $\beta(\xi')$, the solutions of (1.1) converge for the Cauchy data satisfying (4.8).*

Proof.   We shall show that $(1) \to (2) \to (3) \to (1)$.

First step: $(1) \to (2)$.   Substitute (4.8) in (3.17).   Then

$$\textstyle\sum_{k=0}^{m'} p_{2,k}(\xi') \cdot \beta(\xi') \cdot \gamma(\xi')^{m'-k} \equiv 0 \ .$$

Since

$$\textstyle\sum_{k=0}^{m'} p_{2,k}(\xi') \cdot \gamma(\xi')^{m'-k} = p_{2,0} \cdot \prod_{k=1}^{m'} (\gamma(\xi') - \sigma_k(\xi')) \ ,$$

it implies that if $\beta(\xi_1') \neq 0$, then there exists a neighbourhood B_1 of ξ_1' such that $\prod_{k=1}^{m'} (\gamma(\xi') - \sigma_k(\xi')) \equiv 0$ in B_1.   Hence there exists an integer i such that $\gamma(\xi') \equiv \sigma_i(\xi')$ in B_1.   The analyticity implies that $\gamma(\xi') \equiv \sigma_i(\xi')$ in B_0.   We may assume that $i=1$.   We have for $j=1, \cdots, m'-1$,

$$(4.10) \qquad S_j(\tau_1, \cdots, \tau_{m'}) = S_j(\tau_2, \cdots, \tau_{m'}) + \tau_1 \cdot S_{j-1}(\tau_2, \cdots, \tau_{m'}) \ .$$

Then (3.1) for $j=1$ implies that

$$\begin{aligned}
(4.11) \quad &\textstyle\sum_{j=0}^{m'}(-1)^j \cdot S_j(\tau_2, \cdots, \tau_{m'}) \cdot \hat{\phi}_{m'+1-j} \\
&+ \tau_1 \cdot \textstyle\sum_{j=0}^{m'}(-1)^j \cdot S_{j-1}(\tau_2, \cdots, \tau_{m'}) \cdot \hat{\phi}_{m'+1-j} \\
&= \textstyle\sum_{j=0}^{m'-1}(-1)^j \cdot S_j(\tau_2, \cdots, \tau_{m'}) \cdot (\hat{\phi}_{m'+1-j} - \tau_1 \cdot \hat{\phi}_{m'-j}) \equiv 0 \ .
\end{aligned}$$

Since

$$\hat{\phi}_{m'+1-j} - \tau_1 \cdot \hat{\phi}_{m'-j} = -s_{1,2} \cdot \beta \cdot \sigma_1{}^{m'-1-j} \cdot \varepsilon^{m-m'} + O(\varepsilon^{2m-2m'}) \ ,$$

(4.11) implies that

$$-s_{1,2} \cdot \varepsilon^{m-m'} \cdot \beta \cdot \textstyle\sum_{j=0}^{m'-1}(-1)^j \cdot S_j(\tau_2, \cdots, \tau_{m'}) \cdot \sigma_1{}^{m'-1-j} + O(\varepsilon^{2m-2m'}) \equiv 0 \ .$$

The least order of ε is $m - m'$ and the coefficient of $\varepsilon^{m-m'}$ is

$$-s_{1,2} \cdot \beta \cdot \textstyle\sum_{j=0}^{m'-1}(-1)^j \cdot S_j(\sigma_2, \cdots, \sigma_{m'}) \cdot \sigma_1{}^{m'-1-j} \ ,$$

which must be zero.   Since $\beta(\xi') \not\equiv 0$ and

$$\sum_{j=0}^{m'-1}(-1)^j \cdot S_j(\sigma_2, \cdots, \sigma_{m'}) \cdot \sigma_1^{m'-1-j} = \prod_{j=2}^{m'}(\sigma_1 - \sigma_j) \neq 0 \text{ in } B_0,$$

the analyticity of $s_{1,2}$ in B_0 implies that $s_{1,2}(\xi') \equiv 0$ in B_0. Then $P_1(\sigma_1, \xi') \equiv 0$ in B_0, which is equivalent to (4.9) for $i=1$.

Second step: (2)→(3). If (4.9) is satisfied for $i=1$, then we may assume that $\tau_1(\varepsilon, \xi') \equiv \sigma_1(\xi') \equiv r(\xi')$ in B_0 and that $\hat{\Phi} = (\hat{\phi}_1, \cdots, \hat{\phi}_m) = \hat{\phi}_1 \cdot (1, \tau_1, \cdots \tau_1^{m-1})$, where $\hat{\phi}_1$ is an arbitrary $C_0^\infty(B_0)$-function. Hence for $l=2, \cdots, m$

$$D_l(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m) = \hat{\phi}_1 \cdot D_0(\tau_1, \cdots, \tau_{l-1}, \tau_1, \tau_{l+1}, \cdots, \tau_m) \equiv 0,$$

and

$$D_1(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m) = \hat{\phi}_1 \cdot D_0(\tau_1, \cdots, \tau_m).$$

Thus

$$u_\varepsilon(x; \Phi) = F^{-1}(\phi_1(\xi') \cdot \exp i\sigma_1(\xi') x_1),$$

which is the solution of (1.2). Since the solutions u_ε are independent of ε, the solutions u_ε converge.

Third step: (3)→(1). This is trivial. Q.E.D.

Proof of Theorem B. (Case 1). First we assume that the solutions u_ε converge. In this case, (3.18) implies (4.7). When $\hat{\phi}_1(\xi') \equiv 0$, (4.7) implies that $\Phi = (0, \cdots, 0)$ and the solutions u_ε are identically zero. When $\hat{\phi}_1(\xi') \not\equiv 0$, Lemma 4.3 implies that there exists an integer j with $1 \leq j \leq m$ such that $\sigma_1 \equiv v_j$ in B_0. Hence if there exist data Φ such that $\hat{\phi}_1(\xi') \not\equiv 0$ and the solutions converge, then it leads to (1–a). Otherwise it leads to (1–b). The converse can be proved with the same argument as in Lemma 4.3.

(Case 2). The case (2–a) is the special case of (Case 1) in Theorem A. The case (2–b) is that of (Case 2) in Theorem A, where we must exchange σ_1 for σ_2. Therefore we may assume that $s_{1,2} \not\equiv 0$ and $s_{2,2} \not\equiv 0$ in B_0. Put $s = {}^t(s_{1,2}, s_{2,2})$ and $A_1 = (\hat{\phi}_{j+2-k}; j \downarrow 1, 2, k \rightarrow 1, 2)$. Assume that the solutions converge. Then Corollary 3.3 implies that $A_1 B s = 0$. Since $\det B \neq 0$ and $s \neq 0$ in B_0, it implies that $\det A_1 = 0$, that is,

(4.12) $$\hat{\phi}_2^2 = \hat{\phi}_1 \cdot \hat{\phi}_3.$$

By Corollary 3.2, we have

(4.13) $$p_{2,0} \cdot \hat{\phi}_3 + p_{2,1} \cdot \hat{\phi}_2 + p_{2,2} \cdot \hat{\phi}_1 = 0.$$

Then (4.12) and (4.13) implies that

(4.14) $p_{2,0} \cdot \hat{\phi}_2{}^2 + p_{2,1} \cdot \hat{\phi}_{11} \cdot \hat{\phi}_2 + p_{2,2} \cdot \phi_1{}^2 = 0$.

Assume that $\phi_1 \not\equiv 0$. Divide (4.14) by ϕ_1 and put $z = \hat{\phi}_2/\hat{\phi}_1$. Then

(4,.15) $p_{2,0} \cdot z^2 + p_{2,1} \cdot z + p_{2,2} = P_2(z, \xi') = 0$.

Hence there exists an open ball B_1 with centre ξ_1' included in B_0 such that $\hat{\phi}_1(\xi')$ $\not= 0$ in B_1 and $z \equiv \sigma_1(\xi')$ or $z \equiv \sigma_2(\xi')$ in B_1. We may assume that $z \equiv \sigma_1(\xi')$, that is, $\hat{\phi}_2 = \sigma_1 \cdot \hat{\phi}_1$ in B_1. Since (3.17) are equivalent to

(4.16) $\hat{\phi}_{j+2} - (\sigma_1 + \sigma_2) \cdot \hat{\phi}_{j+1} + \sigma_1 \cdot \sigma_2 \cdot \hat{\phi}_j$

$\qquad = \hat{\phi}_{j+2} - \sigma_1 \cdot \hat{\phi}_{j+1} - \sigma_2 \cdot (\hat{\phi}_{j+1} - \sigma_1 \cdot \hat{\phi}_j) = 0$, $j = 1, \cdots, m-2$,

it implies that

$$\hat{\phi}_{j+2} - \sigma_1 \cdot \hat{\phi}_{j+1} = 0 , \quad j = 0, \cdots, m-2 \text{ in } B_1 .$$

Choose β_1 in $C_0^\infty(B_1)$ satisfying $\beta_1(\xi_1') \not= 0$ and put $\beta = \beta_1 \cdot \hat{\phi}_1$ and $\gamma = \sigma_1$. Then Remark to Proposition implies that the solutions converge for the Cauchy datum Φ_1 satisfying

$$\hat{\Phi}_1 = \beta_1 \cdot \hat{\phi}_1 \cdot (1, \sigma_1, \cdots, \sigma_1{}^{m-1}) .$$

By applying Lemma 4.3, we have $\sigma_1(\xi') \equiv \tau_1(\varepsilon, \xi') \equiv 0$ in B_0, which contradicts that $s_{1,2} \not\equiv 0$. Hence $\hat{\phi}_1 \equiv 0$. Then (4.12) and (4.13) implies that $\hat{\phi}_2 \equiv \hat{\phi}_3 \equiv 0$. Thus (4.16) implies that $\Phi = (0, \cdots, 0)$.   Q.E.D.

5. The Weak Admissibility

In this section, we shall prove Main Theorem.

Proof of Main Theorem. Assume that for every Φ' in $\mathcal{A}'$ there exists Φ'' in $\mathcal{A}''$ such that the solutions $u_\varepsilon(x; \Phi)$ of (1.1) converge in $C(R^n)$. We have only to show that it leads to (Case 1) in Theorem A. Assume that the condition in (Case 1) in Theorem A is not satisfied. We may assume that $s = (s_{i,2}; i \downarrow 1, \cdots, m') \not\equiv 0$. We can apply Corollary 3.3. Since $\det B \not= 0$, it implies that $Bs \not\equiv 0$. By (3.21) for $j=1$, we have $(\hat{\phi}_{m'}, \cdots, \hat{\phi}_1) \cdot Bs \equiv 0$. Hence Φ' can not be arbitrary, which contradicts the C-weak admissibility. The converse is proved by (Case 1) in Theorem A.   Q.E.D.

Obvoously the C-admissibility implies the C-weak admissibility. When the Cauchy problems (1.1) are not C-weakly admissible, (1.1) are not suitable as a singular perturbation. Because the Cauchy data Φ' of the reduced pro-

blem (1.2) are restricted. One of the reasons why (1.2) does not admit adequate Cauchy data $\varPhi'$ is that the Cauchy data $\varPhi''$ are independent of ε. If we allow that the Cauchy data $\varPhi''$ change as ε, then we can find a sequence of solutions of (1.1) whose limit is the solution of (1.2). In fact, we may take the solutions satisfying

$$\hat{u}_\varepsilon(x_1, \xi') = \sum_{j=0}^{m'} \frac{D_j(\tau_1, \cdots, \tau_{m'}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'})}{D_0(\tau_1, \cdots, \tau_{m'})} \cdot \exp i\tau_j(\varepsilon, \xi') \, x_1 \, .$$

When the Cauchy problems (1.1) are not C-admissible but C-weakly admissible, (1.1) are trivial as a singular perturbation. Because the solutions $u_\varepsilon(x; \varPhi)$ of (1.1) are identically equal to the solution $u_0(x; \varPhi')$ of (1.2). Thus the interesting cases are limited to the cases satisfying Condition 1.2.

When $\mathcal{A}'$ is wider, there is a difficulty in the choise of $\mathcal{A}''$. But the proof of Corollary 3.2 suggests that $\varPhi''$ is uniquely determined by $\varPhi'$ for wider data. There is another difficulty. For example, when $\hat{\phi}$ belongs to $\mathcal{E}'(B_0)^m$, the situation is so delicate that we can not analyze with algebraic methods only. In fact, we used the fact that the product of an analytic function not identically zero and a continuous function can not be identically zero except that the continuous function is identically zero.

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