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A Systematization of the L^2-well-posed Mixed Problem and its Applications to the Mixed Problem for Weakly Hyperbolic Equations of Second Order (II)

By

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Introduction

In [23], we obtained a complete systematization of the L^2-well-posed mixed problem for regularly hyperbolic equations of second order. But we could not get an applicable one under the condition (A.II$_0$) (see [23: p. 251]). In this paper, we have a useful systematization for the above problem with real boundary operators. In (P.1) in §2, the boundary operator B_1 has the following form related to L_1 in (P.1), (2.5) and (2.9),

$$B_1 = \left\{\sqrt{a_{11}(t, 0, x')}\frac{\partial}{\partial x_1} + \alpha\right\} + \beta \qquad (x' = (x_2, \cdots, x_n))$$

where $\sqrt{a_{11}}\dfrac{\partial}{\partial x_1} + \alpha$ is the natural normal operator of L_1, and α and β are first order real vector fields tangential to $\Gamma = [0, T] \times \mathbf{R}^{n-1}$. By the condition (2.1) for $b_j(t, x')$ and $c(t, x')$ in (P.1), we have the general oblique boundary operator B_1, which gives the L^2-well-posed problem.

We have two main results in our paper. One of them is that we are able to obtain a complete and applicable systematization of the L^2-well-posed mixed problem for regularly hyperbolic equations of second order with real boundary operators. Another of them is that by this systematization, we prove that a mixed problem (P.2) for weakly hyperbolic equations of second order degenerating on the initial surface, is C^∞-well-posed. To obtain C^∞-well-posedness, we use the energy estimate. By the above systematization, we reduce our mixed problem to the problem with non-negative type boundary condition for symmetric hyperbolic pseudo differential systems of first order, and obtain the energy estimate. This systematization which was discovered in [7], [19], [20: Cor.], [21] and [23], is further developed in our paper. Also, our systematization is used in [22] and [24].

As for the other systematization on the Cauchy problem and the mixed problem for hyperbolic equations, we can refer to [4], [5], [12], and [13]. Also, as for the other

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result on the mixed problem for hyperbolic equations degenerating on the initial surface, we can refer to [1], [2], [6], [8], [9], [10], [17] and [18].

An outline of this paper is as follows. In §1, we give the notation. In §2, we state the problem and the result. In §3, we discuss a systematization of the L^2-well-posed mixed problem with real boundary operators.

Seeing that a mixed problem for degenerate hyperbolic equation of second order can be derived from the result in §3 by the procedure used in [23: §3 and §4], we shall treat only the systematization of the L^2-well-posed mixed problem.

§1. Notation

(i) $\|u\|_{m,D}\dots$ the norm of the Sobolev space $H_m(D)$.

(ii) $u(t)\in\mathscr{E}_t^r(E)\dots u(t)$ is r-times continuously differentiable in t as E-valued function.

(iii) $H^\infty(D)=\bigcap_{m=0}^\infty H_m(D)$.

(iv) $(\ ,\ )\dots$ the inner product in $L^2(\mathbb{R}^n_+)$.

(v) $\langle\ ,\ \rangle\dots$ the inner product in $L^2(\mathbb{R}^{n-1})$.

(vi) $\|u\|_m^2=\|u\|_{m,\mathbb{R}^n_+}^2$, $\|u\|^2=\|u\|_0^2$.

(vii) $\langle\!\langle u\rangle\!\rangle_m^2=\|u\|_{m,\mathbb{R}^{n-1}}^2$, $\langle\!\langle u\rangle\!\rangle^2=\langle\!\langle u\rangle\!\rangle_0^2$.

(viii) $S^m=\{p(t,x_1,x',\eta')\in C^\infty([0,T]\times\overline{\mathbb{R}^1_+}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$: for any $\theta=(\theta_1,\theta_2,\theta_3,\theta_4)$, there is a constant C_θ such that

$$\left|\left(\frac{\partial}{\partial t}\right)^{\theta_1}\left(\frac{\partial}{\partial x_1}\right)^{\theta_2}\left(\frac{\partial}{\partial x'}\right)^{\theta_3}\left(\frac{\partial}{\partial \eta'}\right)^{\theta_4}p\right|\leqq C_\theta\langle\eta'\rangle^{m-|\theta_4|}$$

where $x'=(x_2,\cdots,x_n)$ and $\langle\eta'\rangle=(\sum_{j=2}^n\eta_j^2+1)^{1/2}\}$.

(ix) $\mathcal{S}^m\dots$ the set of the pseudo differential operator with respect to $x'=(x_2,\cdots,x_n)$ with its symbol $p\in S^m$.

§2. Statement of the Problem and the Result

We consider the mixed problem

$$(\text{P.1})\quad\begin{cases} L_1[u]=\dfrac{\partial^2u}{\partial t^2}-2\displaystyle\sum_{j=1}^n h_j(t,x)\frac{\partial^2u}{\partial t\partial x_j}-\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2u}{\partial x_i\partial x_j}\\ \qquad+a_0(t,x)\dfrac{\partial u}{\partial t}+\displaystyle\sum_{j=1}^n a_j(t,x)\frac{\partial u}{\partial x_j}+d(t,x)u=f(t,x)\\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x)\\ B_1[u]|_{x_1=0}=a_{11}(t,0,x')^{-1/2}\Big\{a_{11}(t,0,x')\dfrac{\partial u}{\partial x_1}\\ \qquad+\displaystyle\sum_{j=2}^n a_{1j}(t,0,x')\frac{\partial u}{\partial x_j}+h_1(t,0,x')\frac{\partial u}{\partial t}\Big\}\\ \qquad+\displaystyle\sum_{j=2}^n b_j(t,x')\frac{\partial u}{\partial x_j}-c(t,x')\left(1+\frac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{1/2} \end{cases}$$

$$\left\{\begin{array}{l} \cdot\left\{\dfrac{\partial u}{\partial t}-\left(1+\dfrac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{-1}\displaystyle\sum_{j=2}^{n}\left(h_j(t,0,x')\right.\right. \\ \left.\left.\qquad -\dfrac{h_1(t,0,x')}{a_{11}(t,0,x')}a_{1j}(t,0,x')\right)\dfrac{\partial u}{\partial x_j}\right\}+\gamma(t,x')u|_{x_1=0}=g(t,x') \\ (t,x)=(t,x_1,x')\in[0,T]\times\overline{\mathbf{R}^1_+}\times\mathbf{R}^{n-1} \end{array}\right.$$

where $x=(x_1, x_2, \cdots, x_n)$, $x'=(x_2, \cdots, x_n)$, $n\geqq 2$, T is a positive constant, the coefficients h_j, a_{ij}, a_0, a_j and d belong to $\mathscr{B}([0,T]\times\overline{\mathbf{R}^n_+})$ and are constant outside a compact set in $[0,T]\times\overline{\mathbf{R}^n_+}$, the coefficients b_j, c and γ belong to $\mathscr{B}([0,T]\times\mathbf{R}^{n-1})$ and are constant outside a compact set in $[0,T]\times\mathbf{R}^{n-1}$, and explain how to reduce the problem (P.1) to the mixed problem for symmetric hyperbolic pseudo differential systems of first order with non-negative type boundary condition.

We assume the following conditions for the problem (P.1) :

(A.I) The operator L_1 is regularly hyperbolic on $[0,T]\times\overline{\mathbf{R}^n_+}$ and $a_{11}(t,x)>0$ on $[0,T]\times\overline{\mathbf{R}^n_+}$.

(A.II) The functions b_j and c are real valued functions on $[0,T]\times\mathbf{R}^{n-1}$, and the following inequalities hold,

$$\text{(2.1)}\qquad \left\{\begin{array}{l} c(t,x')\geqq 0 \\ c(t,x')^2\geqq b(t,x',\eta')^2 \end{array}\right.$$

for all $(t,x',\eta')\in[0,T]\times\mathbf{R}^{n-1}\times(\mathbf{R}^{n-1}-\{0\})$ where

$$\text{(2.2)}\qquad \left\{\begin{array}{l} b(t,x',\eta')=\displaystyle\sum_{j=2}^{n} b_j(t,x')\eta_j/d(t,x',\eta') \\ d(t,x',\eta')=\left[\displaystyle\sum_{i,j=2}^{n} a_{ij}(t,0,x')\eta_i\eta_j-\dfrac{1}{a_{11}(t,0,x')}\right. \\ \qquad\cdot\left(\displaystyle\sum_{j=2}^{n} a_{1j}(t,0,x')\eta_j\right)^2+\left(1+\dfrac{h_1(t,0,x')^2}{a_{11}(t,0,x')}\right)^{-1} \\ \qquad\left.\cdot\left\{\displaystyle\sum_{j=2}^{n}\left(h_j(t,0,x')-\dfrac{h_1(t,0,x')}{a_{11}(t,0,x')}a_{1j}(t,0,x')\right)\eta_j\right\}^2\right]^{1/2} \\ \eta'=(\eta_2,\cdots,\eta_n)\in\mathbf{R}^{n-1}-\{0\}. \end{array}\right.$$

The result in § 3 is that we are able to obtain a complete and useful systematization of the problem (P.1) under the conditions (A.I) and (A.II). This result is one of the main results in our paper. The explicit representation of the systematization is mentioned in §3.

Remark 1. Assume the condition (A.I) and that b_j and c are real valued functions. Then, the problem (P.1) is L^2-well-posed if and only if (2.1) holds (see [15]).

Remark 2. Assume the condition (A.I). Then, the problem (P.1) is L^2-well-posed if and only if the following condition (A.II') holds, (A.II′) (i) If $|\mathrm{Re}\, p| + |\mathrm{Re}\, q| \neq 0$, the inequalities hold :

$$\text{(2.3)} \qquad \begin{cases} \mathrm{Re}\, c(t, x') \geqq 0 \\ \{\mathrm{Re}\, c(t, x')\}^2 \geqq \{\mathrm{Re}\, b(t, x', \eta')\}^2 \\ \qquad\qquad + \{\mathrm{Re}\, c(t, x')\cdot \mathrm{Im}\, b(t, x', \eta') \\ \qquad\qquad\quad - \mathrm{Im}\, c(t, x')\cdot \mathrm{Re}\, b(t, x', \eta')\}^2. \end{cases}$$

(ii) If $|\mathrm{Re}\, p| + |\mathrm{Re}\, q| = 0$, the inequality holds :

$$\text{(2.4)} \qquad 1 + \{\mathrm{Im}\, c(t, x')\}^2 > \{\mathrm{Im}\, b(t, x', \eta')\}^2$$

where b is the same function as in (2.2), $p = c(t, x') + b(t, x', \eta')$ and $q = c(t, x') - b(t, x', \eta')$ (see [15]).

Now, we explain the condition (A.II$_0$) in the introduction. The condition (A.II$_0$) is as follows,

(A.II$_0$) (i) The condition (A.II′) holds.

(ii) There is a sufficiently small positive constant δ_0 such that $\mathrm{Re}\, c(t, x') \not\equiv 0$ and $\delta_0 > \mathrm{Re}\, c(t, x') \geqq 0$.

Let M be

$$\text{(2.5)} \qquad M = \frac{\partial^2}{\partial t^2} - 2\left(\alpha_1 \frac{\partial}{\partial x_1} + \sum_{j=2}^{n} \alpha_j \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial t} - \left(\beta_{11}\frac{\partial^2}{\partial x_1^2} + 2\sum_{j=2}^{n} \beta_{1j}\frac{\partial^2}{\partial x_1 \partial x_j} + \sum_{i,j=2}^{n} \beta_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\right).$$

If M is regularly hyperbolic with respect to t, we have

$$\text{(2.6)} \qquad \begin{cases} \beta_{11} + \alpha_1^2 > 0 \\ d_*(\eta') = \sum_{i,j=2}^{n} \beta_{ij}\eta_i\eta_j + \left(\sum_{j=2}^{n} \alpha_j\eta_j\right)^2 \\ \qquad - (\beta_{11} + \alpha_1^2)^{-1}\left(\sum_{j=2}^{n} \beta_{1j}\eta_j + \alpha_1 \sum_{j=2}^{n} \alpha_j\eta_j\right)^2 > 0 \end{cases}$$

for all $\eta' = (\eta_2, \cdots, \eta_n) \in \mathbb{R}^{n-1} - \{0\}$ and

$$\text{(2.7)} \qquad \sigma(M) = \tilde{\xi}_*^2 + \tilde{d}_*(\eta')^2 - \tilde{\tau}_*^2$$

where

$$\text{(2.8)} \qquad \begin{cases} \tilde{\tau}_* = \tau - \alpha_1\xi - \sum_{j=2}^{n} \alpha_j\eta_j \\ \tilde{\xi}_* = (\beta_{11} + \alpha_1^2)^{1/2}\cdot\left\{\xi + (\beta_{11} + \alpha_1^2)^{-1}\left(\sum_{j=2}^{n} \beta_{1j}\eta_j + \alpha_1 \sum_{j=2}^{n} \alpha_j\eta_j\right)\right\} \\ \tilde{d}_*(\eta') = \{d_*(\eta')\}^{1/2}. \end{cases}$$

This representation (2.8) corresponds to the symbol of the wave equation $\xi^2 + \sum_{j=2}^{n} \eta_j^2 - \tau^2$, and is used to obtain the energy estimate for the Cauchy problem. Moreover, if $\beta_{11} > 0$, we obtain

$$\sigma(M) = \tilde{\tilde{\xi}}^2 + \tilde{\tilde{d}}(\eta')^2 - \tilde{\tilde{\tau}}^2 \tag{2.9}$$

where

$$\left\{\begin{aligned}
&\tilde{\tilde{\xi}} = \beta_{11}{}^{-1/2}\left(\beta_{11}\xi + \sum_{j=2}^{n} \beta_{1j}\eta_j + \alpha_1\tau\right)\\
&\tilde{\tilde{\tau}} = \left(1 + \frac{\alpha_1^2}{\beta_{11}}\right)^{1/2}\left\{\tau - \left(1 + \frac{\alpha_1^2}{\beta_{11}}\right)^{-1}\left(\sum_{j=2}^{n} \alpha_j\eta_j - \frac{\alpha_1}{\beta_{11}}\sum_{j=2}^{n} \beta_{1j}\eta_j\right)\right\}\\
&\tilde{\tilde{d}}(\eta') = \left[\sum_{i,j=2}^{n} \beta_{ij}\eta_i\eta_j - \frac{1}{\beta_{11}}\left(\sum_{j=2}^{n} \beta_{1j}\eta_j\right)^2\right.\\
&\qquad\left. + \left(1 + \frac{\alpha_1^2}{\beta_{11}}\right)^{-1}\left(\sum_{j=2}^{n} \alpha_j\eta_j - \frac{\alpha_1}{\beta_{11}}\sum_{j=2}^{n} \beta_{1j}\eta_j\right)^2\right]^{1/2}\\
&\tilde{\tilde{d}}(\eta')^2 = d_*(\eta').
\end{aligned}\right. \tag{2.10}$$

This representation (2.9) is used to obtain the energy estimate for the mixed problem.

Nextly, we treat the mixed problem for weakly hyperbolic equations of second order

$$\left\{\begin{aligned}
&L_2[u] = \frac{\partial^2 u}{\partial t^2} - 2t^k \sum_{j=1}^{n} h_j(t, x)\frac{\partial^2 u}{\partial t\partial x_j} - t^{2k}\sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial^2 u}{\partial x_i\partial x_j}\\
&\qquad + a_0(t, x)\frac{\partial u}{\partial t} + t^{k-1}\sum_{j=1}^{n} a_j(t, x)\frac{\partial u}{\partial x_j} + d(t, x) = f(t, x)\\
&u(0, x) = u_0(x),\ u_t(0, x) = u_1(x)\\
&B_2[u]|_s = \left\{\sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)v_j(s)\right\}^{-1/2}\\
&\qquad \cdot\left\{t^k \sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)\frac{\partial u}{\partial x_j} + \left(\sum_{j=1}^{n} h_j(t, s)v_j(s)\right)\frac{\partial u}{\partial t}\right\}\\
&\qquad + t^k \sum_{j=1}^{n} \alpha_j(t, s)\frac{\partial u}{\partial x_j} - \beta(t, s)\frac{\partial u}{\partial t} + \gamma(t, s)u|_s = g(t, s)\\
&(t, x) \in (0, T) \times \Omega
\end{aligned}\right. \tag{P.2}$$

and prove the existence and uniqueness theorem for smooth solutions where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqq 2)$ with smooth boundary $S = \partial\Omega$, $x = (x_1, \cdots, x_n)$, T is a positive constant, k is a positive integer, the coefficients h_j, a_{ij}, a_0, a_j and $d \in \mathscr{B}([0, T] \times \bar{\Omega})$, the coefficients α_j, β and $\gamma \in \mathscr{B}([0, T] \times S)$, $s \in S$, $v(s) = (v_1(s), \cdots, v_n(s))$ is the inner unit normal at s and $v \cdot \alpha = \sum_{j=1}^{n} v_j(s) \cdot \alpha_j(s) = 0$.

We set

$$
(2.11)\quad \begin{cases} L_0\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} - 2\sum_{j=1}^{n} h_j(t, x)\frac{\partial^2}{\partial t \partial x_j} - \sum_{i,j=1}^{n} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j} \\ B_0\left(t, s, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \left\{\sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)v_j(s)\right\}^{-1/2} \\ \qquad \cdot\left\{\sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)\frac{\partial}{\partial x_j} + \left(\sum_{j=1}^{n} h_j(t, s)v_j(s)\right)\frac{\partial}{\partial t}\right\} \\ \qquad + \sum_{j=1}^{n} \alpha_j(t, s)\frac{\partial}{\partial x_j} - \beta(t, s)\frac{\partial}{\partial t}. \end{cases}
$$

The functions $\tilde{c}(t, s)$ and $\tilde{b}(t, s, \eta)$ are determined by

$$
(2.12)\qquad B_0(t, s, \tau, \xi v(s) + \eta) = \tilde{\xi} + \tilde{b}(t, s, \eta)\cdot \tilde{d}_0(\eta) - \tilde{c}(t, s)\cdot \tilde{\tau}
$$

where

$$
(2.13)\quad \begin{cases} \tilde{\xi} = -\frac{1}{2}\left\{\sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)v_j(s)\right\}^{-1/2} \frac{\partial}{\partial \xi}L_0(t, s, \tau, \xi v(s) + \eta) \\ \quad = \sqrt{X}\xi + X^{-1/2}(A\tau + Y) \\ \tilde{\tau} = \frac{1}{2}\left\{1 + \left(\sum_{j=1}^{n} h_j(t, s)v_j(s)\right)^2 \cdot \left(\sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)v_j(s)\right)^{-1}\right\}^{1/2} \\ \qquad \cdot \frac{\partial}{\partial \tau}\{L_0(t, s, \tau, \xi v(s) + \eta) + \xi(t, s, \tau, \xi v(s) + \eta)^2\} \\ \quad = \left(1 + \frac{A^2}{X}\right)^{1/2}\{\tau + (X + A^2)^{-1}\cdot(AY - XB)\} \\ \tilde{d}_0(\eta) = \{\tilde{\tau}^2 - \tilde{\xi}^2 - L_0(t, s, \tau, \xi v(s) + \eta)\}^{1/2} \\ \qquad = \left\{\frac{(AY - XB)^2}{X(X + A^2)} - \frac{Y^2}{X} + Z\right\}^{1/2} \\ A = \sum_{j=1}^{n} h_j(t, s)v_j(s),\ B = \sum_{j=1}^{n} h_j(t, s)\eta_j, \\ X = \sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)v_j(s),\ Y = \sum_{i,j=1}^{n} a_{ij}(t, s)v_i(s)\eta_j, \\ Z = \sum_{i,j=1}^{n} a_{ij}(t, s)\eta_i\eta_j \end{cases}
$$

for all $\xi \in R^1$ and all $\eta \in (\mathbb{R}^n - \{v(s)\})$ satisfying $\sum_{j=1}^{n} v_j(s)\eta_j = 0$.

Remark 3. In (2.13), $\tilde{\tau}$ is independent of ξ, and $\tilde{d}_0(\eta)$ depends on only t, s and η.

We assume the following conditions for the problem (P.2) :

(A.III) The operator L_0 in (2.11) is regularly hyperbolic on $[0, T] \times \bar{\Omega}$ and $\sum_{i,j=1}^n a_{ij}(t, s)\nu_i(s)\nu_j(s) > 0$ on $[0, T] \times S$.

(A.IV) (i) α_j and β are real valued functions.

(ii) $\tilde{c}(t, s) \geqq 0$

(iii) $\tilde{c}(t, s)^2 \geqq \tilde{b}(t, s, \eta)^2$

where $\tilde{c}$ and $\tilde{b}$ are same functions as in (2.12).

Remark 4. By (A.III), we have that h_j and a_{ij} are real valued functions, and $\tilde{d}_0(\eta)^2 > 0$. Therefore, by (A.IV)-(i), we obtain that $\tilde{c}$ and $\tilde{b}$ are real valued functions. Also, by (A.III) and (A.IV), the problem (P.2) is L^2-well-posed for $t > 0$. (see [16]).

Now, for any $s \in S$, we have a smooth coordinate transformation Ψ: $V \to W$ such that

(i) $\Psi(s_0) = y_0 = (0, y_0') = (0, y_{02}, \cdots, y_{0n})$.

(ii) V and W are neighborhoods of s_0 and y_0 respectively.

(iii) Ψ: $V \to W$ is a bijection.

(iv) $\Psi(V \cap \Omega) = W \cap \mathbb{R}^n_+$, $\mathbb{R}^n_+ = \{y = (y_1, y_2, \cdots, y_n) | y_1 > 0\}$.

(v) $\Psi(V \cap S) = W \cap \mathbb{R}^{n-1}$.

(vi) L_2 is transformed into $\tilde{L}_2$ where

$$\tilde{L}_2 = \frac{\partial^2}{\partial t^2} - 2t^k \sum_{j=1}^n \tilde{h}_j(t, y)\frac{\partial^2}{\partial t \partial y_j} - t^{2k} \sum_{i,j=1}^n \tilde{a}_{ij}(t, y)\frac{\partial^2}{\partial y_i \partial y_j} + \tilde{a}_0(t, y)\frac{\partial}{\partial t} + t^{k-1} \sum_{j=1}^n \tilde{a}_j(t, y)\frac{\partial}{\partial y_j} + \tilde{d}(t, y) \tag{2.14}$$

for all $(t, y) \in [0, T] \times (W \cap \overline{\mathbb{R}^n_+})$.

And

(vii) B_2 is transformed into $\tilde{B}_2$ where

$$\begin{aligned} \tilde{B}_2 = \frac{1}{\sqrt{\tilde{a}_{11}(t, 0, y')}} &\left\{ t^k \left[\tilde{a}_{11}(t, 0, y')\frac{\partial}{\partial y_1} + \sum_{j=2}^n \tilde{a}_{1j}(t, 0, y')\frac{\partial}{\partial y_j} \right] \right. \\ &\left. + \tilde{h}_1(t, 0, y')\frac{\partial}{\partial t} \right\} + t^k \sum_{j=2}^n \tilde{b}_j(t, y')\frac{\partial}{\partial y_j} - \tilde{c}(t, y') \\ &\cdot \left(1 + \frac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\right)^{1/2} \left\{ \frac{\partial}{\partial t} - \left(1 + \frac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')}\right)^{-1} \right. \\ &\left. \cdot t^k \sum_{j=2}^n \left(\tilde{h}_j(t, 0, y') - \frac{\tilde{h}_1(t, 0, y')}{\tilde{a}_{11}(t, 0, y')}\tilde{a}_{1j}(t, 0, y') \right)\frac{\partial}{\partial y_j} \right\} + \tilde{\gamma}(t, y') \end{aligned} \tag{2.15}$$

for all $(t, 0, y') \in [0, T] \times (W \cap \mathbb{R}^{n-1})$ (see [14: p. 484–485]).

Then, by (A.III) and (A.IV), we have

$$\begin{cases} \tilde{c}(t, y') \geqq 0 \\ \tilde{c}(t, y')^2 \geqq \tilde{b}(t, y', \eta')^2 \end{cases} \tag{2.16}$$

for all $(t, y', \eta') \in [0, T] \times (W \cap \mathbb{R}^{n-1}) \times (\mathbb{R}^{n-1} - \{0\})$ where

$$(2.17)\quad \begin{cases} \tilde{b}(t, y', \eta') = \sum_{j=2}^{n} \tilde{b}_j(t, y')\eta_j/\tilde{d}_0(\eta') \\ \tilde{d}_0(\eta') = \left[\sum_{i,j=2}^{n} \tilde{a}_{ij}(t, 0, y')\eta_i\eta_j - \dfrac{1}{\tilde{a}_{11}(t, 0, y')}\left(\sum_{j=2}^{n} \tilde{a}_{1j}(t, 0, y')\eta_j \right)^2 \right. \\ \qquad \left. + \left(1 + \dfrac{\tilde{h}_1(t, 0, y')^2}{\tilde{a}_{11}(t, 0, y')} \right)^{-1} \left\{ \sum_{j=2}^{n} \left(\tilde{h}_j(t, 0, y') - \dfrac{\tilde{h}_1(t, 0, y')}{\tilde{a}_{11}(t, 0, y')}\tilde{a}_{1j}(t, 0, y') \right)\eta_j \right\}^2 \right]^{1/2}. \end{cases}$$

Let us define $u_{r+2}(x)$ recursively by the formulae

$$(2.18)\quad u_{r+2}(x) = -\sum_{j=0}^{r} {}_rC_j\{A_1^{(j)}(0, x: D_x)u_{r+1-j}(x) + A_2^{(j)}(0, x, D_x)u_{r-j}(x)\} + f^{(r)}(0, x) \qquad (r = 0, 1, 2, \cdots)$$

where

$$(2.19)\quad \begin{cases} L_2 = D_t^2 + A_1(t, x: D_x)D_t + A_2(t, x: D_x) \\ A_i^{(j)}(t, x: \xi) = (D_t^j A_i)(t, x, \xi) \\ f^{(r)}(t, x) = (D_t^r f)(t, x) \\ D_t = \dfrac{\partial}{\partial t}, \text{ etc.} \end{cases}$$

Definition. We say that the data $\{u_0(x), u_1(x), f(t, x), g(t, s)\}$ satisfy the compatibility conditions of infinite order provided that

$$(2.20)\quad \left(\frac{\partial}{\partial t}\right)^r (B_2[u]|_s)|_{t=0} = \left(\frac{\partial}{\partial t}\right)^r g|_{t=0} \qquad (r = 0, 1, 2, \cdots).$$

Another of the main results in this paper is the following :

Theorem. *Assume the conditions* (A.III) *and* (A.IV) *for the problem* (P.2). *Then, for any data* $u_0(x)$, $u_1(x) \in H^\infty(\Omega)$, $f(t, x) \in \mathscr{E}_t^\infty(H^\infty(\Omega))$ *and* $g(t, s) \in \mathscr{E}_t^\infty(H^\infty(S))$ *which satisfy the compatibility condition of infinite order, there is a unique solution of the problem* (P.2) *which belongs to* $\mathscr{E}_t^\infty(H^\infty(\Omega))$.

The plan of the proof of Theorem is essentially same as that of [23: §3 and §4]. By the systematization of the problem (P.1) obtained in §3, we can get our Theorem. Therefore, we omit the proof of Theorem in our paper.

Remark 5. Instead of the condition (A.IV), we assume the following condition (A.V) for the problem (P.2):

(A.V) (i) Re $\tilde{c}(t, s) \geqq 0$.

(ii) There is a positive constant $\delta(0 < \delta < \frac{1}{2})$ such that

① $$\text{Re}\,\tilde{c}(t, s)^2 \geqq \{\text{Re}\,\tilde{b}(t, s, \eta)\}^2 + \{\text{Re}\,\tilde{c}(t, s)\cdot\text{Im}\,\tilde{b}(t, s, \eta) - \text{Im}\,\tilde{c}(t, s)\cdot\text{Re}\,\tilde{b}(t, s, \eta)\}^2$$

for all $(t, s, \eta) \in \{(t, s, \eta) \in [0, T] \times S \times \{\mathbb{R}^{n-1} - \nu(s)\} \mid \text{Re}\,\tilde{c}(t, s) > \delta\}$.

② α_j and β are real valued functions and

$$\tilde{c}(t, s)^2 \geqq \tilde{b}(t, s, \eta)^2$$

for all $(t, s, \eta) \in \{(t, s, \eta) \in [0, T] \times S \times \{\mathbf{R}^{n-1} - \nu(s)\} | \delta \geqq \operatorname{Re} \tilde{c}(t, s) \geqq 0\}$.

Then, under the conditions (A.III) and (A.V), we obtain our Theorem by the results in [23: §2] and in §3.

§3. A Systematization of the L^2-well-posed Mixed Problem

In this section, we are concerned with the problem (P.1) under the conditions (A.I) and (A.II), and discuss how to reduce the problem (P.1) to the mixed problem for symmetric hyperbolic pseudo differential systems of first order with non-negative type boundary condition.

We set

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} u_t - u_x + z_1(p_1 - iq_1)u_y \\ z_1(u_t + u_x) + (p_1 + iq_1)u_y \\ u_y \end{pmatrix} \tag{3.1}$$

and

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \begin{pmatrix} u_t - u_x + z_2 u_y \\ z_2(u_t + u_x) + u_y \\ u_t - u_x + w_2 u_y \\ w_2(u_t + u_x) + u_y \end{pmatrix} \tag{3.2}$$

where $u_{tt} - u_{xx} - u_{yy} = 0$, z_1, z_2 and w_2 are complex constants, and p_1 and q_1 are real constants which satisfy an inequality $p_1^2 + q_1^2 \leqq 1$.

Lemma 3.1. *U and V satisfy the following equations respectively* :

$$\begin{cases} M_1 U_t = A_1 U_x + D_1 U_y \\ M_1 = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & (1 + |z_1|^2) r_1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & (1 - |z_1|^2) r_1 \end{pmatrix} \\ D_1 = \begin{pmatrix} 0 & p_1 - iq_1 & r_1 \\ p_1 + iq_1 & 0 & z_1 r_1 \\ r_1 & \bar{z}_1 r_1 & -2r_1\{p_1 \operatorname{Re} z_1 + q_1 \operatorname{Im} z_1\} \end{pmatrix} \end{cases} \tag{3.3}$$

and

$$\text{(3.4)}\quad \begin{cases} V_t = A_2 V_x + D_2 V_y \\ A_2 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad D_2 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \end{array}\right) \end{cases}$$

where $r_1 = 1 - p_1^2 - q_1^2$.

Proof. By simple calculations, we obtain Lemma 3.1 . Q.E.D.

Remark 6. See [7] and [19] for (3.4) and [23] for (3.3).

Remark 7. By (2.7) and (2.9), the above systematization of the wave equation $u_{tt} - u_{xx} - u_{yy} = 0$ enables us to treat the Cauchy problem and the mixed problem by choosing parameters as appropriate functions or operators.

Now, by (2.9), we obtain

$$\text{(3.5)}\qquad \sigma_0(L_1) = \tilde{\xi}^2 + \tilde{d}_1(\eta')^2 - \tilde{\tau}^2$$

for the principal symbol of the operator L_1 in (P.1) where

$$\text{(3.6)}\quad \begin{cases} \tilde{\xi} = \dfrac{1}{\sqrt{a_{11}(t, x)}}\left\{a_{11}(t, x)\xi + \sum_{j=2}^{n} a_{1j}(t, x)\eta_j + h_1(t, x)\tau\right\} \\ \tilde{\tau} = \left(1 + \dfrac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2}\left\{\tau - \left(1 + \dfrac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1} \sum_{j=2}^{n}\left(h_j(t, x) - \dfrac{h_1(t, x)}{a_{11}(t, x)}a_{1j}(t, x)\right)\eta_j\right\} \\ \tilde{d}_1(\eta') = \left[\sum_{i,j=2}^{n} a_{ij}(t, x)\eta_i\eta_j - \dfrac{1}{a_{11}(t, x)}\left(\sum_{j=2}^{n} a_{1j}(t, x)\eta_j\right)^2\right. \\ \qquad\left. + \left(1 + \dfrac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1}\left\{\sum_{j=2}^{n}\left(h_j(t, x) - \dfrac{h_1(t, x)}{a_{11}(t, x)}a_{1j}(t, x)\right)\eta_j\right\}^2\right]^{1/2}. \end{cases}$$

From now on, we treat a systematization of the L^2-well-posed mixed problem with real boundary operators (P.1)

$$\text{(P.1)}\quad \begin{cases} L_1[u] = f(t, x) \\ u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x) \\ B_1[u]|_{x_1=0} = g(t, x') \\ (t, x) = (t, x_1, x') \in [0, T] \times \overline{\mathbb{R}^1_+} \times \mathbb{R}^{n-1}. \end{cases}$$

We separate it into two cases :

(I) $\qquad \frac{1}{2} > c(t, x') \geqq 0 \qquad$ for all $(t, x') \in [0, T] \times \mathbb{R}^{n-1}$

(II) $\qquad c(t, x') > \frac{1}{4} \qquad$ for all $(t, x') \in [0, T] \times \mathbb{R}^{n-1}$.

Remark 8. The case (II) is a special one of the case which is already treated in [23].

We set

$$Q_0=\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{1/2}\left\{\frac{\partial}{\partial t}-\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{-1}\cdot\left(\sum_{j=2}^{n}h_j(t,x)\frac{\partial}{\partial x_j}-\frac{h_1(t,x)}{a_{11}(t,x)}\sum_{j=2}^{n}a_{1j}(t,x)\frac{\partial}{\partial x_j}\right)\right\} \tag{3.7}$$

and

$$Q_1=\frac{1}{\sqrt{a_{11}(t,x)}}\left\{a_{11}(t,x)\frac{\partial}{\partial x_1}+\sum_{j=2}^{n}a_{1j}(t,x)\frac{\partial}{\partial x_j}+h_1(t,x)\frac{\partial}{\partial t}\right\}+\gamma(t,x')\,. \tag{3.8}$$

Let Q_2 be a pseudo differential operator with respect to $x'=(x_2,\cdots,x_n)$ with the symbol

$$\sigma(Q_2)=i\left[\sum_{i,j=2}^{n}a_{ij}(t,x)\eta_i\eta_j-\frac{1}{a_{11}(t,x)}\left(\sum_{j=2}^{n}a_{1j}(t,x)\eta_j\right)^2+\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{-1}\left(\sum_{j=2}^{n}h_j(t,x)\eta_j-\frac{h_1(t,x)}{a_{11}(t,x)}\sum_{j=2}^{n}a_{1j}(t,x)\eta_j\right)^2+1\right]^{1/2}. \tag{3.9}$$

Then, we obtain

$$C\left(\sum_{j=2}^{n}\eta_j^2+1\right)\geqq|\sigma(Q_2)|^2\geqq C^{-1}\left(\sum_{j=2}^{n}\eta_j^2+1\right)$$

for all $(t,x,\eta')\in[0,T]\times\overline{\mathbf{R}_+^n}\times\mathbf{R}^{n-1}$ where C is a positive constant. Also, we set

$$\begin{cases}b(t,x',\eta')=\sum_{j=2}^{n}b_j(t,x')\eta_j/d(t,x',\eta')\\ b_I(t,x',\eta')=\sum_{j=2}^{n}b_j(t,x')\eta_j/(-i\sigma(Q_2)|_{x_1=0})\end{cases} \tag{3.10}$$

where $b(t,x',\eta')$ and $d(t,x',\eta')$ are the same functions as in (2.2), and $\tilde{d}_I(\eta')|_{x_1=0}=d(t,x',\eta')$.

Case (I): $\frac{1}{2}>c(t,x')\geqq 0$ for all $(t,x')\in[0,T]\times\mathbf{R}^{n-1}$.

By (2.6), (2.10) and (3.6), there is a smooth, real and symmetric $(n-1)\times(n-1)$ matrix $M(t,x)$ such that

$$\tilde{d}_I(\eta')^2={}^t\eta' M\eta' \tag{3.11}$$

and

$$M>0 \qquad \text{for all} \quad (t,x)\in[0,T]\times\overline{\mathbf{R}_+^n} \tag{3.12}$$

where ${}^t\eta'=(\eta_2,\cdots,\eta_n)$.

Lemma 3.1. *(Jacobi) There is a smooth and real $(n-1)\times(n-1)$ matrix $N(t, x)$ such that*

$$ {}^tNMN = \begin{pmatrix} \alpha_2(t,x) & & 0 \\ & \ddots & \\ 0 & & \alpha_n(t,x) \end{pmatrix} > 0 \tag{3.13} $$

and

$$ C_1 > |\det N(t,x)| > C_2 \tag{3.14} $$

for all $(t, x) \in [0, T] \times \overline{\mathbb{R}^n_+}$ where C_1 and C_2 are positive constants.

Proof. See [3 : p. 300–304]. Q.E.D.

Remark 9. $N(t, x)$ and $\alpha_j(t, x)$ are constant outside a compact set in $[0, T] \times \overline{\mathbb{R}^n_+}$. We set

$$ \begin{cases} \tilde{M} = M(t, 0, x') \\ \tilde{N} = N(t, 0, x') \\ \tilde{\alpha}_j = \alpha_j(t, 0, x') \qquad (j = 2, \cdots, n)\,. \end{cases} \tag{3.15} $$

Lemma 3.2. *For $c(t, x')$ and $b(t, x', \eta')$ in (2.2), we have*

$$ c(t,x')^2 \geqq \sum_{j=2}^{n} \tilde{b}_j(t,x')^2 \qquad \text{for all} \quad (t,x') \in [0,T] \times \mathbb{R}^{n-1} \tag{3.16} $$

where

$$ (\tilde{b}_2, \cdots, \tilde{b}_n) = (b_2, \cdots, b_n)\tilde{N} \begin{pmatrix} \frac{1}{\sqrt{\tilde{\alpha}_2}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{\tilde{\alpha}_n}} \end{pmatrix}. \tag{3.17} $$

Proof. By $d(t, x', \eta')^2 = \tilde{d}_I(\eta')^2|_{x_1=0}$, we obtain

$$ \begin{aligned} d(t,x',\eta')^2 &= {}^t\eta'\tilde{M}\eta' \\ &= {}^t\eta'({}^t\tilde{N})^{-1}\,{}^t\tilde{N}\tilde{M}\tilde{N}\tilde{N}^{-1}\eta' \\ &= {}^t(\tilde{N}^{-1}\eta')\cdot{}^t\tilde{N}\cdot\tilde{M}\cdot\tilde{N}\cdot(\tilde{N}^{-1}\eta') \\ &= {}^t(\tilde{N}^{-1}\eta') \begin{pmatrix} \sqrt{\tilde{\alpha}_2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\tilde{\alpha}_n} \end{pmatrix} \begin{pmatrix} \sqrt{\tilde{\alpha}_2} & & 0 \\ & & \\ 0 & & \sqrt{\tilde{\alpha}_n} \end{pmatrix} (\tilde{N}^{-1}\eta') \\ &= {}^t\zeta'\cdot\zeta' \end{aligned} \tag{3.18} $$

where

$$(3.19)\qquad \begin{cases} \eta' = {}^t(\eta_2, \cdots, \eta_n) \\ \zeta' = \begin{pmatrix} \sqrt{\tilde{\alpha}_2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\tilde{\alpha}_n} \end{pmatrix} \tilde{N}^{-1}\eta' = \begin{pmatrix} \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix}. \end{cases}$$

Then, we have

$$(3.20)\qquad \begin{aligned} \sum_{j=2}^{n} b_j(t, x')\eta_j &= (b_2, \cdots, b_n)\begin{pmatrix} \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} \\ &= (b_2, \cdots, b_n)\tilde{N}\cdot\tilde{N}^{-1}\begin{pmatrix} \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} \\ &= (b_2, \cdots, b_n)\tilde{N}\begin{pmatrix} \frac{1}{\sqrt{\tilde{\alpha}_2}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{\tilde{\alpha}_n}} \end{pmatrix}\begin{pmatrix} \sqrt{\tilde{\alpha}_2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\tilde{\alpha}_n} \end{pmatrix}\tilde{N}^{-1}\begin{pmatrix} \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} \\ &= (\tilde{b}_2, \cdots, \tilde{b}_n)\begin{pmatrix} \zeta_2 \\ \vdots \\ \zeta_n \end{pmatrix}. \end{aligned}$$

By (3.18), (3.19) and (3.20), we get

$$(3.21)\qquad b(t, x', \eta')^2 = \left(\frac{\sum_{j=2}^{n} \tilde{b}_j(t, x')\zeta_j}{\sqrt{\zeta_2^2 + \cdots + \zeta_n^2}} \right)^2$$

for all $(t, x', \zeta') \in [0, T] \times \mathbb{R}^{n-1} \times (\mathbb{R}^{n-1} - \{0\})$. Therefore, by (2.1) and (3.21), we have

$$c(t, x')^2 \geqq \sum_{j=2}^{n} \tilde{b}_j(t, x')^2 . \qquad \text{Q.E.D.}$$

For the problem (P.1), we set

$$(3.22)\qquad U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} Q_0 u - Q_1 u \\ z_1(Q_0 u + Q_1 u) \\ Q_2 u \\ u \end{pmatrix}$$

where Q_0, Q_1 and Q_2 are same operators as in (3.7), (3.8) and (3.9) respectively, and

$$(3.23)\qquad z_1(t, x') = \sqrt{\frac{1 - c(t, x')}{1 + c(t, x')}}\,.$$

Then, we have

Theorem 3.3. *The problem* (P.1) *is transformed into the system*:

$$
(3.24)\qquad \begin{cases} M_1 U_t = A_1 U_{x_1} + \sum_{j=2}^{n} B_{1j} U_{x_j} + D_1 Q_2 U + E_1 U + F_1(t, x) \\ U(0, x) = U_0(x) \\ P_1 U|_{x_1=0} = G_1(t, x') \\ (t, x) = (t, x_1, x') \in [0, T] \times \overline{\mathbb{R}^1_+} \times \mathbb{R}^{n-1} \end{cases}
$$

where

$$
M_1 = \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \frac{2}{1+c} & \\ & & & 1 \end{pmatrix} - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \frac{2c}{1+c} & \\ & & & 1 \end{pmatrix}
$$

$$
A_1 = \sqrt{a_{11}(t, x)} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \frac{2c}{1+c} & \\ & & & 1 \end{pmatrix}
$$

$$
B_{1j} = \frac{a_{1j}(t, x)}{a_{11}(t, x)} A_1 + \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1/2} \cdot \left(h_j(t, x) - \frac{h_1(t, x)}{a_{11}(t, x)} a_{1j}(t, x)\right) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \frac{2}{1+c} & \\ & & & 1 \end{pmatrix} \qquad (j = 2, \cdots, n)
$$

$$
D_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & z_1 & 0 \\ 1 & z_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$E_1 \cdots$ *a* 4×4 *pseudo differential system which has the property that for* $\sigma(E) = (e_{ij})$, *the following conditions holds* :

(i) $e_{ij}(t, x, \eta') \in C^\infty([0, T] \times \overline{\mathbb{R}^n_+} \times \mathbb{R}^{n-1})$

(ii) *for any* $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$, *there is a positive constant* $C_\theta^{(i,j)}$ *such that*

$$
\left| \left(\frac{\partial}{\partial t}\right)^{\theta_1} \left(\frac{\partial}{\partial x_1}\right)^{\theta_2} \left(\frac{\partial}{\partial x'}\right)^{\theta_3} \left(\frac{\partial}{\partial \eta'}\right)^{\theta_4} e_{ij} \right| \leqq C_\theta^{(i,j)} \langle \eta' \rangle^{-|\theta_4|}
$$

where $\langle \eta' \rangle = (\sum_{j=2}^{n} \eta_j^2 + 1)^{1/2}$.

$$F_1 = {}^t(f, z_1 f, 0, 0)$$

$$P_1 = \left(1, -z_1, -\frac{2\tilde{b}_I}{1 + c(t, x')}, 0\right) \qquad (\tilde{b}_I \in \mathcal{S}^0, \sigma(\tilde{b}_I) = b_I(t, x', \eta'))$$

$$G_1 = -\frac{2g(t, x')}{1 + c(t, x')} + T_0 u \qquad (T_0 \in \mathcal{S}^0)$$

$$\langle A_1 U, U\rangle \geqq -C\langle\!\langle U_3\rangle\!\rangle^2_{-1/2} \qquad \textit{for all} \quad U \in \operatorname{Ker} P_1 \tag{3.25}$$

and C is a positive constant.

Proof. By (3.3), (3.5) and the method in [11 : p. 69, Lemma 2.4], we have (3.24). By $\operatorname{Ker} P_1 \ni U$, we obtain

$$U_1 = \sqrt{\frac{1 - c}{1 + c}}\, U_2 + \frac{2}{1 + c} \sum_{j=2}^{n} \tilde{b}_j R_j U_3 \tag{3.26}$$

where $\tilde{b}_j(t, x)$ $(j = 2, \cdots, n)$ is the same as in (3.17),

$$\begin{cases} R_j \in \mathcal{S}^0 \\ \sigma(R_j) = \zeta_j/(-i\sigma(Q_2)|_{x_1=0}) \\ \sum_{j=2}^{n} \sigma(R_j)^2 \leqq 1 \end{cases} \tag{3.27}$$

and ζ_j $(j = 2, \cdots, n)$ is the same as in (3.19). Then, we have for $U \in \operatorname{Ker} P_1$

$$\begin{aligned} I &= \langle A_1 U, U\rangle \\ &\geqq \int_{\mathbf{R}^{n-1}} \sqrt{a_{11}(t, 0, x')} \cdot \left\{-|U_1|^2 + |U_2|^2 + \frac{2c}{1 + c}|U_3|^2\right\} dx' . \end{aligned} \tag{3.28}$$

Also, by (3.16), we get that

$$\{(t, x')|c(t, x') = 0\} \subseteq \{(t, x')|\tilde{b}_2(t, x') = \cdots = \tilde{b}_n(t, x') = 0\} .$$

Therefore, we obtain by (3.16)

$$\begin{aligned} &\left|\sqrt{\frac{1 - c}{1 + c}}\, U_2 + \frac{2}{1 + c} \sum_{j=2}^{n} \tilde{b}_j R_j U_3\right|^2 \\ &\leqq \frac{1 - c}{1 + c}|U_2|^2 + 4 \cdot \frac{\sqrt{c}}{\sqrt{1 + c}}|U_2| \cdot \frac{\sqrt{1 - c}}{1 + c}\sqrt{c}\left|\sum_{j=2}^{n} \frac{\tilde{b}_j}{c} R_j U_3\right| \\ &\quad + \frac{4}{(1 + c)^2}\left|\sum_{j=2}^{n} \tilde{b}_j R_j U_3\right|^2 \\ &\leqq \left(\frac{1 - c}{1 + c} + \frac{2c}{1 + c}\right)|U_2|^2 + \frac{2c(1 - c)}{(1 + c)^2}\left(\sum_{j=2}^{n} \frac{\tilde{b}_j^2}{c^2}\right) \cdot \left(\sum_{j=2}^{n} |R_j U_3|^2\right) \\ &\quad + \frac{4}{(1 + c)^2}\left(\sum_{j=2}^{n} \tilde{b}_j^2\right) \cdot \left(\sum_{j=2}^{n} |R_j U_3|^2\right) \end{aligned} \tag{3.29}$$

$$\leqq |U_2|^2 + \frac{2c(1-c)+4c^2}{(1+c)^2} \sum_{j=2}^{n} |R_j U_3|^2$$

$$= |U_2|^2 + \frac{2c}{1+c} \sum_{j=2}^{n} |R_j U_3|^2 .$$

By (3.28) and (3.29), we have

$$I \geqq \int_{\mathbf{R}^{n-1}} \sqrt{a_{11}(t, 0, x')} \cdot \frac{2c}{1+c} \left\{ |U_3|^2 - \sum_{j=2}^{n} |R_j U_3|^2 \right\} dx' . \tag{3.30}$$

Here, we set

$$\theta(t, x') = \frac{2\sqrt{a_{11}(t, 0, x')} \cdot c(t, x')}{1 + c(t, x')} . \tag{3.31}$$

Then, we obtain

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} \theta |R_j U_3|^2 \, dx' &= \langle \theta R_j U_3, R_j U_3 \rangle \\ &= \langle \theta R_j^* R_j U_3, U_3 \rangle + \langle R_{j1} U_3, U_3 \rangle \qquad (R_{j1} \in \mathcal{S}^{-1}) \\ &= \langle \theta S_j U_3, U_3 \rangle + \langle S_{j1} U_3, U_3 \rangle \qquad (S_{j1} \in \mathcal{S}^{-1}) \end{aligned} \tag{3.32}$$

where

$$\begin{cases} \sigma(S_j) = \zeta_j^2 / (-i\sigma(Q_2)|_{x_1=0})^2 = \sigma(R_j)^2 \\ \sum_{j=2}^{n} \sigma(S_j) = \sum_{j=2}^{n} \sigma(R_j)^2 \leqq 1 . \end{cases} \tag{3.33}$$

By (3.30), (3.32), (3.33) and the sharp form of the Gårding inequality (see [11]), we have (3.25). Q.E.D.

To estimate U_{x_1}, we have

Corollary 3.4. *The following equality holds for U in* (3.22) :

$$M_{11} U_t = A_{11} U_{x_1} + \sum_{j=2}^{n} B_{11j} U_{x_j} + D_{11} Q_2 U + E_{11} U + F_{11} \tag{3.34}$$

where

$$M_{11} = \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} I - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}} \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$A_{11} = \sqrt{a_{11}(t, x)} \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$B_{11j} = \frac{a_{1j}(t, x)}{a_{11}(t, x)} A_{11} + \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1/2} \cdot \left(h_j(t, x) - \frac{h_1(t, x)}{a_{11}(t, x)} a_{1j}(t, x)\right) I$$

$$(j = 2, \cdots, n)$$

$$D_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & z_1 & 0 \\ 1 & 0 & d_{33}^{(1)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (d_{33}^{(1)} \in \mathcal{S}^0)$$

$E_{11} \ldots$ a 4×4 pseudo differential system with the same property as E_1

and

$$F_{11} = F_1 .$$

Proof. Operating the pseudo differential operator Q_2 for $U_1 = Q_0 u - Q_1 u$, we have

$$(3.35) \qquad Q_2 U_1 = Q_2 \left\{\left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}}\right\} \frac{\partial u}{\partial t} - Q_2 \sqrt{a_{11}(t, x)} \frac{\partial u}{\partial x_1} + Q_2 T_1 u$$

where

$$(3.36) \qquad T_1 = \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1/2} \sum_{j=2}^{n} \left\{h_j(t, x) - \frac{h_1(t, x)}{a_{11}(t, x)} a_{1j}(t, x)\right\} \frac{\partial}{\partial x_j}$$
$$- \frac{1}{\sqrt{a_{11}(t, x)}} \sum_{j=2}^{n} a_{1j}(t, x) \frac{\partial}{\partial x_j} - \gamma(t, x') .$$

Therefore, we obtain

$$(3.37) \qquad \left\{\left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}}\right\} \frac{\partial}{\partial t}(Q_2 u)$$
$$= \sqrt{a_{11}(t, x)} \frac{\partial}{\partial x_1}(Q_2 u) + Q_2 U_1$$
$$- \left[Q_2, \left\{\left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}}\right\}\right] \frac{\partial u}{\partial t}$$
$$+ \left\{\left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}}\right\} Q_{2t} u$$
$$+ [Q_2, \sqrt{a_{11}(t, x)}] \frac{\partial u}{\partial x_1} - \sqrt{a_{11}} Q_{2x_1} u - T_1 Q_2 u - [Q_2, T_1] u .$$

Then, we have Corollary 3.4. Q.E.D.

Case (II): $c(t, x') > \frac{1}{4}$ for all $(t, x') \in [0, T] \times \mathbb{R}^{n-1}$.

We set

$$U=\begin{pmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\\U_7\end{pmatrix}=\begin{pmatrix}Q_0u-Q_1u+pQ_2u\\Q_0u+Q_1u+pQ_2u\\Q_2u\\Q_0u-Q_1u+z_2pQ_2u\\z_2(Q_0u+Q_1u)+pQ_2u\\Q_2u\\\sigma u\end{pmatrix} \tag{3.38}$$

where σ is a positive constant, p is a pseudo differential operator with respect to $x'=(x_2,\dots,x_n)$ whose symbol is equal to

$$\sigma(p)=-\frac{b_I(t,x',\eta')}{c(t,x')} \tag{3.39}$$

and

$$z_2=\frac{c(t,x')-1}{c(t,x')+1}\,. \tag{3.40}$$

By (2.1) and (3.40), we have

$$\inf_{(t,x',\eta')\in[0,T]\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}}\{1-\sigma(p)^2\}\geqq 0 \tag{3.41}$$

and

$$\sup_{(t,x')\in[0,T]\times\mathbb{R}^{n-1}}|z_2|<1\,. \tag{3.42}$$

Then, we obtain the following

Theorem 3.5. *The problem* (P.1) *is transformed into the system* :

$$\begin{cases}M_2U_t=A_2U_{x_1}+\sum_{j=2}^{n}B_{2j}U_{x_j}+D_2Q_2U+E_2U+F_2(t,x)\\U(0,x)=U_0(x)\\P_2U|_{x_1=0}=G_2(t,x')\\(t,x)\in[0,T]\times\overline{\mathbb{R}^n_+}\end{cases} \tag{3.43}$$

where

$$M_2=\left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{1/2}\begin{pmatrix}1&&&&&&\\&1&&&&0&\\&&2r&&&&\\&&&1&&&\\&&&&1&&\\&0&&&&(1+|z_2|^2)r&\\&&&&&&1\end{pmatrix}$$

$$-\frac{h_1(t,x)}{\sqrt{a_{11}(t,x)}}\begin{pmatrix} -1 & & & & & \text{\Large 0} & \\ & 1 & & & & & \\ & & 0 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ \text{\Large 0} & & & & & (1-|z_2|^2)r & \\ & & & & & & 1 \end{pmatrix}$$

$$A_2 = \sqrt{a_{11}(t,x)}\cdot\begin{pmatrix} -1 & & & & & \text{\Large 0} & \\ & 1 & & & & & \\ & & 0 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ \text{\Large 0} & & & & & (1-|z_2|^2)r & \\ & & & & & & 1 \end{pmatrix}$$

$$B_{2j} = \frac{h_1(t,x)}{a_{11}(t,x)}A_2 + \left(1+\frac{h_1(t,x)^2}{a_{11}(t,x)}\right)^{-1/2}\left(h_j(t,x) - \frac{h_i(t,x)}{a_{11}(t,x)}a_{1j}(t,x)\right).$$

$$\cdot\begin{pmatrix} 1 & & & & & \text{\Large 0} & \\ & 1 & & & & & \\ & & 2r & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ \text{\Large 0} & & & & & (1+|z_2|^2)r & \\ & & & & & & 1 \end{pmatrix} \qquad (j = 2,\cdots,n)$$

$$D_2 = \left(\begin{array}{ccc|cccc} 0 & p & r & & & & \\ p & 0 & r & & \text{\Large 0} & & \\ r & r & -2pr & & & & \\ \hline & & & 0 & p & r & 0 \\ & \text{\Large 0} & & p & 0 & z_2 r & 0 \\ & & & r & z_2 r & -2rpz_2 & 0 \\ & & & 0 & 0 & 0 & 0 \end{array}\right)$$

$r\cdots$ *a pseudo differential operator with respect to* $x' = (x_2,\cdots,x_n)$ *whose symbol is equal to* $1-\sigma(p)^2$.

$E_2\cdots$ *a* 7×7 *pseudo differential system with the same property as* E_1.

$F_2 = {}^t(f, f, 0, f, z_2 f, 0, 0)$

$$P_2 = \begin{pmatrix} 1 & z_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$G_2 = {}^t\left(-\frac{2g}{c(t,x')+1} + T_2 u, -\frac{2g}{c(t,x')+1} + T_2 u\right) \qquad (T_2 \in S^0)$$

(3.44) $$\langle A_2 U, U \rangle \geqq -C\{\langle\langle U_3 \rangle\rangle^2_{-1/2} + \langle\langle U_6 \rangle\rangle^2_{-1/2}\} \qquad \text{for all} \quad U \in \operatorname{Ker} P_2$$

and C is a positive constant.

Proof. See [23 : p. 261–263]. Q.E.D.

Corollary 3.6. *For all U in* (3.38), *there are positive constants σ_0 and C such that for all $\sigma \geqq \sigma_0$*

(3.45) $$(M_2 U, U) \geqq C(U, U)$$

where σ is the same constant in (3.38).

Proof. See [23 : p. 263–264]. Q.E.D.

Remark 10. For all U in (3.38), there are positive constants σ_1, C_1 and C_2 such that for all $\sigma \geqq \sigma_1$

(3.46) $$\begin{cases} (M_2 U_t, U_t) \geqq C_1\left(\|u_{tt}\|^2 + \sum_{j=1}^{n} \|u_{x_j t}\|^2\right) - C_2(U, U) \\ (M_2 U_{x_j}, U_{x_j}) \geqq C_1\left(\|u_{tx_j}\|^2 + \sum_{k=1}^{n} \|u_{x_k x_j}\|^2\right) - C_2(U, U) \qquad (j = 2, \cdots, n). \end{cases}$$

To estimate U_{x_1}, we have

Corollary 3.7. *The following equality holds for U in* (3.38):

(3.47) $$M_{21} U_t = A_{21} U_{x_1} + \sum_{j=2}^{n} B_{21j} U_{x_j} + D_{21} Q_2 U + E_{21} U + F_{21}$$

where

$$M_{21} = \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{1/2} I - \frac{h_1(t, x)}{\sqrt{a_{11}(t, x)}} \begin{pmatrix} -1 & & & & & & 0 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ 0 & & & & & & 1 \end{pmatrix}$$

$$A_{21} = \sqrt{a_{11}(t, x)} \begin{pmatrix} -1 & & & & & & 0 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ 0 & & & & & & 1 \end{pmatrix}$$

$$B_{21j} = \frac{a_{1j}(t, x)}{a_{11}(t, x)} A_{21} + \left(1 + \frac{h_1(t, x)^2}{a_{11}(t, x)}\right)^{-1/2} \left(h_j(t, x) - \frac{h_1(t, x)}{a_{11}(t, x)} a_{1j}(t, x)\right) I$$

$$(j = 2, \cdots, n)$$

$$D_{21} = \left(\begin{array}{ccc|cccc} 0 & p & r & & & & \\ p & 0 & r & & \mathbb{O} & & \\ 1 & 0 & d_{33}^{(2)} & & & & \\ \hline & & & 0 & p & r & 0 \\ & \mathbb{O} & & p & 0 & z_2 r & 0 \\ & & & 1 & 0 & d_{66}^{(2)} & 0 \\ & & & 0 & 0 & 0 & 0 \end{array}\right) \quad (d_{33}^{(2)}, d_{66}^{(2)} \in S^0)$$

$E_{21} \cdots$ a 7×7 pseudo differential system with the same property as E_1

and

$$F_{21} = F_2 .$$

Proof. Operating the pseudo differential operator Q_2 for $U_1 = Q_0 u - Q_1 u + p_1 Q_2 u$ and $U_4 = Q_0 u - Q_1 u + z_2 p Q_2 u$, we obtain Corollary 3.7. Q.E.D.

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