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Solvability in Distributions for a Class of Singular Differential Operators, III

By

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We say that a linear partial differential equation $Pu = f$ is locally solvable in $\mathscr{D}'$ at p, if for any $f \in \mathscr{D}'$ there exists a $u \in \mathscr{D}'$ such that $Pu = f$ holds near p. The following is one of the most fundamental problems: under what condition is $Pu = f$ locally solvable in $\mathscr{D}'$?

When P is non-singular, this problem has been studied by many authors (for example, see the survey in [6, Chapter 26]). When P is singular, in [14, 15, 16] the author has established the local solvability in $\mathscr{D}'$ for singular differential operators of various types: in [14] for operators of Fuchsian type, in [15] for operators of non-Fuchsian hyperbolic type, and in [16] for operators of non-Fuchsian elliptic type.

In this paper, the author will establish the local solvability in $\mathscr{D}'$ for a class of non-Fuchsian singular partial differential operators under much more general condition.

It should be noted that the following cases were already treated as to the local solvability for singular differential operators P. When P is a Fuchsian operator of hyperbolic type, the solvability in C^∞, $\mathscr{D}'$ or $\mathscr{B}$ (where $\mathscr{B}$ means the set of all hyperfunctions) was discussed in [1, 2, 3, 4, 5, 11, 12]. When P is a Fuchsian operator of elliptic type, the solvability in $\mathscr{B}$ was discussed in [9]. When P is a non-Fuchsian operator of hyperbolic type, the solvability in C^∞ was discussed in [10, 13]. See also [7].

By the author's results (in [14, 15, 16] and this paper), we can conclude that the class of operators for which the local solvability in $\mathscr{D}'$ is valid is much wider than the class of Fuchsian operators.

§ 1. Main Theorem

Let $(t, x) = (t, x_1, \ldots, x_n) \in \mathbb{R}_t \times \mathbb{R}^n_x$ and let us consider

$$P = (t\partial_t)^m + \sum_{\substack{j+|\alpha| \leq m \\ j<m}} a_{j,\alpha}(t, x)(t\partial_t)^j \partial_x^\alpha , \tag{1.1}$$

where $m \in \{1, 2, \ldots\}$, $\partial_t = \partial/\partial t$, $\partial_x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, 2, \ldots\}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$, and the coefficients $a_{j,\alpha}(t, x)$ ($j + |\alpha| \leq m$ and $j < m$) are C^∞ functions defined in an open neighborhood U of $(0, 0)$ in $\mathbb{R}_t \times \mathbb{R}^n_x$.

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Denote by $\mathscr{D}'(U)$ the set of all distributions in (t, x) defined on U. Put $p(\tau, \xi)$ and Σ as follows:

$$p(\tau, \xi) = \tau^m + \sum_{\substack{j+|\alpha|=m \\ j<m}} a_{j,\alpha}(0, 0)\tau^j \xi^\alpha , \tag{1.2}$$

$$\Sigma = \{(\tau, \xi) \in \mathbf{R}_\tau \times \mathbf{R}^n_\xi; p(\tau, \xi) = 0\} . \tag{1.3}$$

Assume the following three conditions:

(A-1) When $j + |\alpha| = m$, $a_{j,\alpha}(t, x)$ is real-valued on U.

(A-2) $\Sigma \cap \{(0, \xi) \in \mathbf{R}_\tau \times \mathbf{R}^n_\xi; \xi \neq 0\} = \emptyset$.

(A-3) $\left(\dfrac{\partial p}{\partial \tau}\right)(\tau, \xi) \neq 0$, when $(0, 0) \neq (\tau, \xi) \in \Sigma$.

Note that P is not of Fuchsian type in t (by (A-2) and (A-3)).

Then, we can state our main theorem as follows.

Theorem 1. *Let P be the operator in* (1.1). *Assume* (A-1), (A-2) *and* (A-3). *Then, for any $f(t, x)(=f) \in \mathscr{D}'(U)$ there exists a $u(t, x)(=u) \in \mathscr{D}'(U)$ such that $Pu = f$ holds near the origin* $(0, 0)$ *in $\mathbf{R}_t \times \mathbf{R}^n_x$; that is, $Pu = f$ is locally solvable in $\mathscr{D}'$ at* $(0, 0)$.

As a special case, we have

Corollary. *Let P be the operator in* (1.1). *Assume* (A-1) *and the following: for any $\xi \in \mathbf{R}^n_\xi \setminus \{0\}$ the equation $p(\lambda, \xi) = 0$ (in $\lambda \in \mathbf{C}$) has only simple and non-zero roots. Then, $Pu = f$ is locally solvable in $\mathscr{D}'$ at* $(0, 0)$.

Remark. More precisely, we can see the following result. For any $k \in \mathbf{Z}_+$ $(=\{0, 1, 2, \dots\})$ there are $j_k \in \mathbf{Z}_+$ and an open neighborhood U_k of $(0, 0)$ in $\mathbf{R}_t \times \mathbf{R}^n_x$ which satisfy the following: for any $f \in H^{-k}(U_k)$ there exists a $u \in H^{-j_k}(U_k)$ such that $Pu = f$ holds on U_k. Here, $H^{-p}(U_k)$ denotes the usual Sobolev space on U_k.

Example. Our result can be applied to the following operators:

$$P = (t\partial_t)^2 \pm \Delta_x + a(t, x)(t\partial_t) + \langle b(t, x), \partial_x \rangle + c(t, x) ,$$

where Δ_x is the Laplacian in x.

Let us compare the above result with the result for Fuchsian operators in [14], and let us make clear the difference between them. Let

$$L = (t\partial_t)^m + \sum_{\substack{j+|\alpha|\leqq m \\ j<m}} a_{j,\alpha}(t, x)(t\partial_t)^j (t^k \partial_x)^\alpha , \tag{1.4}$$

where $(t^k\partial_x)^\alpha = (t^k\partial/\partial x_1)^{\alpha_1} \dots (t^k\partial/\partial x_n)^{\alpha_n}$ $(=t^{k|\alpha|}\partial_x^\alpha)$. Denote by $\rho_i(x)$ $(1 \leqq i \leqq m)$ the roots of $\rho^m + \sum_{j<m} a_{j,0}(0, x)\rho^j = 0$. Define $p(\tau, \xi)$ as in (1.2) (where $a_{j,\alpha}(0, 0)$ are the ones in (1.4)). Then, we already know the following result.

Theorem 2 (Fuchsian case: Tahara [14]). *Let L be the operator in* (1.4). *Assume $k \in \{1, 2, \dots\}$, $\rho_i(0) \notin \{-1, -2, \dots\}$ $(1 \leqq i \leqq m)$,* (A-1) *and the following: for any $\xi \in \mathbf{R}^n_\xi \setminus \{0\}$ the equation $p(\lambda, \xi) = 0$ (in $\lambda \in \mathbf{C}$) has only simple roots. Then, $Lu = f$ is locally solvable in $\mathscr{D}'$ at* $(0, 0)$.

Note that P in (1.1) corresponds to L with $k=0$ and that the case $k=0$ is excluded from the consideration in Theorem 2.

§2. A Priori Estimates

Before giving a proof of Theorem 1, let us establish here the following proposition.

Proposition 1. *Let P be the operator in* (1.1), *put*

$$P_{-s}=(t\partial_t-s)^m+\sum_{\substack{j+|\alpha|\leq m\\ j<m}} a_{j,\alpha}(t,x)(t\partial_t-s)^j\partial_x^\alpha \tag{2.1}$$

for $s\in\mathbf{R}$, and let $(P_{-s})^*$ be the formal adjoint operator of P_{-s}. Assume (A-1) *and* (A-3). *Then, there are $s_k>0$ ($k\in\mathbf{Z}_+$) which satisfy the following: for any $k\in\mathbf{Z}_+$ and any $s>s_k$ there are $\delta_{k,s}>0$ and an open neighborhood $V_{k,s}$ of $(0,0)$ in $\mathbf{R}_t\times\mathbf{R}_x^n$ such that the estimate*

$$\|(P_{-s})^*\varphi\|_k \geqq \delta_{k,s}\|t^{m+k-1}\varphi\|_{m+k-1} \tag{2.2}$$

holds for any $\varphi\in C_0^\infty(V_{k,s}\cap\{t>0\})$ (or $\varphi\in C_0^\infty(V_{k,s}\cap\{t<0\})$), where $\|w\|_p$ denotes the norm of w in the Sobolev space $H^p(V_{k,s}\cap\{t>0\})$ (or $H^p(V_{k,s}\cap\{t<0\})$).

First, we remark a fact on the decomposition of the following polynomial (in τ)

$$\mathring{p}(t,x,\tau,\xi)=\tau^m+\sum_{\substack{j+|\alpha|=m\\ j<m}} a_{j,\alpha}(t,x)\tau^j\xi^\alpha\,.$$

Let W be a sufficiently small neighborhood of $(0,0)$ in $\mathbf{R}_t\times\mathbf{R}_x^n$. Then, by (A-1) and (A-3) we can see that all the real roots of the equation

$$\mathring{p}(t,x,\lambda,\xi)=0 \qquad (\text{in } \lambda\in\mathbf{C}) \tag{2.3}$$

are simple for any $(t,x,\xi)\in W\times(\mathbf{R}_\xi^n\setminus\{0\})$, and that no roots of (2.3) change continuously from "real" to "non-real" when (t,x,ξ) moves in $W\times(\mathbf{R}_\xi^n\setminus\{0\})$. Therefore, denoting by $\lambda_i(t,x,\xi)$ ($1\leqq i\leqq p$) the real roots of (2.3) we have the following: (i) the number p of the real roots of (2.3) is independent of $(t,x,\xi)\in W\times(\mathbf{R}_\xi^n\setminus\{0\})$, (ii) $\lambda_i(t,x,\xi)\in C^\infty(W\times(\mathbf{R}_\xi^n\setminus\{0\}))$ ($1\leqq i\leqq p$), and (iii) $\lambda_i(t,x,\xi)\neq\lambda_j(t,x,\xi)$ for $1\leqq i\neq j\leqq p$. Hence, by putting

$$h(t,x,\tau,\xi)=\prod_{i=1}^{p}(\tau-\lambda_i(t,x,\xi))$$

we obtain a decomposition of $\mathring{p}(t,x,\tau,\xi)$ as follows:

$$\mathring{p}(t,x,\tau,\xi)=h(t,x,\tau,\xi)e(t,x,\tau,\xi)\,, \tag{2.4}$$

where $e(t,x,\tau,\xi)$ has the form

$$e(t,x,\tau,\xi)=\tau^{m-p}+\sum_{i=1}^{m-p} e_i(t,x,\xi)\tau^{m-p-i}$$

and satisfies the following: (iv) $e_i(t,x,\xi)\in C^\infty(W\times(\mathbf{R}_\xi^n\setminus\{0\}))$ ($1\leqq i\leqq m-p$), (v)

$e_i(t, x, \xi)$ is positively homogeneous of degree i in ξ, and (vi) $e(t, x, \tau, \xi) \neq 0$ for any $(t, x, \tau, \xi) \in W \times \mathbb{R}_\tau \times (\mathbb{R}^n_\xi \setminus \{0\})$.

Next, let us show two preparatory lemmas. In the discussion below, we use the following notation: $(t, x) \in [0, T] \times \mathbb{R}^n$ $(T > 0)$, $D_x = -\sqrt{-1}\partial_x$, (w, v) denotes the inner product of w and v in $L^2((0, T) \times \mathbb{R}^n)$, $\|w\|$ denotes the norm of w in $L^2((0, T) \times \mathbb{R}^n)$, $\|w\|_k$ denotes the norm of w in $H^k((0, T) \times \mathbb{R}^n)$, and

$$\text{(2.5)} \qquad |||\varphi|||_{k,s} = \sum_{i+|\alpha|\leqq k} \|(t\partial_t + s)^i \partial_x^\alpha \varphi\| .$$

Obviously we can see the following: for any $k \in \mathbb{Z}_+$ and $s \in \mathbb{R}$ there are $A_{k,s} > 0$ and $B_{k,s} > 0$ such that

$$\text{(2.6)} \qquad A_{k,s}\|t^k \varphi\|_k \leqq |||\varphi|||_{k,s} \leqq B_{k,s}\|\varphi\|_k$$

holds for any $\varphi \in H^k((0, T) \times \mathbb{R}^n)$.

Lemma 1. *Let*

$$H_s = (t\partial_t + s)^p + \sum_{i=1}^{p} a_i(t, x, D_x)(t\partial_t + s)^{p-i} ,$$

where $a_i(t, x, D_x)$ $(1 \leqq i \leqq p)$ are pseudo-differential operators with symbols $a_i(t, x, \xi)$ satisfying the following: (i) $a_i(t, x, \xi) \in C^\infty([0, T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi)$, (ii) *$a_i(t, x, \xi)$ is positively homogeneous of degree i in ξ (for $|\xi| \geqq 1$), and* (iii) *$a_i(t, x, \xi)$ is independent of x for sufficiently large $|x|$. Assume that all the roots of the equation*

$$(\sqrt{-1}\lambda)^p + \sum_{i=1}^{p} a_i(t, x, \xi)(\sqrt{-1}\lambda)^{p-i} = 0$$

(in λ) are real and simple for any $(t, x, \xi) \in [0, T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ satisfying $|\xi| \geqq 1$. Then, for any $k \in \mathbb{Z}_+$ there are $b_k > 0$ and $c_k > 0$ such that the estimate

$$|||H_s \varphi|||_{k,s} \geqq c_k s |||\varphi|||_{p+k-1,s}$$

holds for any $\varphi \in C_0^\infty((0, T), H^\infty(\mathbb{R}^n))$ and $s > b_k$.

Proof. We will prove this by reducing the problem to the one for a first-order system of pseudo-differential operators.

Denote by Λ the pseudo-differential operator on $\mathbb{R}^n_x$ corresponding to the symbol $(1 + |\xi|^2)^{1/2}$, by $S^k([0, T])$ the set of all pseudo-differential operators of order k on $\mathbb{R}^n_x$ depending smoothly on $t \in [0, T]$, and by $S^k([0, T], p \times p)$ the set of all $p \times p$ matrices with components in $S^k([0, T])$.

For $\varphi \in C_0^\infty((0, T), H^\infty(\mathbb{R}^n))$ we put $v_j \in C_0^\infty((0, T), H^\infty(\mathbb{R}^n))$ $(j = 0, 1, \ldots, p-1)$ as follows:

$$v_j = (\sqrt{-1})^{p-j-1}(t\partial_t + s)^j \Lambda^{p-j-1}\varphi .$$

Then, under the notations

$$h_i = (\sqrt{-1})^i a_i(t, x, D_x)\Lambda^{-i+1}, \qquad i = 1, \ldots, p,$$

$$A = \begin{pmatrix} 0 & -\Lambda & & & \\ & 0 & -\Lambda & & \\ & & \ddots & \ddots & \\ & & & 0 & -\Lambda \\ h_p & h_{p-1} & & \cdots & h_1 \end{pmatrix}, \qquad v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{p-1} \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ H_s\varphi \end{pmatrix}$$

we have the following relation:

$$(t\partial_t + s + \sqrt{-1}A)v = g .$$

Moreover, by the standard argument for regularly hyperbolic systems (for example, see [8, Proposition 6.4]) we can see that there are $D \in \mathcal{S}^1([0, T], p \times p)$ and M, $N \in \mathcal{S}^0([0, T], p \times p)$ satisfying $D - D^* \in \mathcal{S}^0([0, T], p \times p)$, $NA - DN \in \mathcal{S}^0([0, T], p \times p)$ and $MN - I \in \mathcal{S}^{-1}([0, T], p \times p)$.

Thus, to have Lemma 1 it is sufficient to show the following result: for any $k \in \mathbf{Z}_+$ there are $b_k > 0$ and $c_k > 0$ such that

$$\sum_{i+j\leqq k} \|(t\partial_t + s)^i \Lambda^j (t\partial_t + s + \sqrt{-1}A)v\| \geqq c_k s \sum_{i+j\leqq k} \|(t\partial_t + s)^i \Lambda^j v\| \tag{2.7}$$

holds for any $v \in C_0^\infty((0, T), H^\infty(\mathbf{R}^n))^p$ and $s > b_k$.

Let us prove (2.7) from now. Take any $v \in C_0^\infty((0, T), H^\infty(\mathbf{R}^n))^p$. Then we have

$$N(t\partial_t + s + \sqrt{-1}A)v = (t\partial_t + s + \sqrt{-1}D)Nv - tN_t'v + \sqrt{-1}(NA - DN)v .$$

Since the operators tN_t', $NA - DN$ and $D - D^*$ are bounded in $L^2((0, T) \times \mathbf{R}^n)^p$, we have

$$\begin{aligned} &\|N(t\partial_t + s + \sqrt{-1}A)v\|^2 \\ &\geqq \frac{1}{2}\|(t\partial_t + s + \sqrt{-1}D)Nv\|^2 - C_1\|v\|^2 \\ &= \frac{1}{2}\|(t\partial_t + \sqrt{-1}D)Nv\|^2 + \frac{s^2}{2}\|Nv\|^2 \\ &\quad + s\,\mathrm{Re}\,((t\partial_t + \sqrt{-1}D)Nv, Nv) - C_1\|v\|^2 \\ &\geqq \frac{1}{2}\|(t\partial_t + \sqrt{-1}D)Nv\|^2 + \frac{s^2}{2}\|Nv\|^2 - sC_2\|Nv\|^2 - C_1\|v\|^2 \end{aligned}$$

for some $C_1 > 0$ and $C_2 > 0$. Therefore, if $s > 4C_2$, we obtain

$$\|N(t\partial_t + s + \sqrt{-1}A)v\|^2 \geqq \frac{s^2}{4}\|Nv\|^2 - C_1\|v\|^2 . \tag{2.8}$$

On the other hand, since

$$\Lambda^{-1}(t\partial_t + s + \sqrt{-1}A)v = (t\partial_t + s)\Lambda^{-1}v + \sqrt{-1}\Lambda^{-1}Av$$

holds and since $\Lambda^{-1}A$ is bounded in $L^2((0, T) \times \mathbf{R}^n)^p$, for $s > 1/2$ we have

$$
\begin{aligned}
(2.9)\qquad & \|A^{-1}(t\partial_t + s + \sqrt{-1}A)v\|^2 \\
& \geqq \frac{1}{2}\|(t\partial_t + s)A^{-1}v\|^2 - C_3\|v\|^2 \\
& = \frac{1}{2}\|t\partial_t A^{-1}v\|^2 + \frac{s^2 - s}{2}\|A^{-1}v\|^2 - C_3\|v\|^2 \\
& \geqq \frac{s^2}{4}\|A^{-1}v\|^2 - C_3\|v\|^2
\end{aligned}
$$

for some $C_3 > 0$. Hence, by (2.8) and (2.9) we obtain

$$
\begin{aligned}
(2.10)\qquad & \|N(t\partial_t + s + \sqrt{-1}A)v\|^2 + \|A^{-1}(t\partial_t + s + \sqrt{-1}A)v\|^2 \\
& \geqq \frac{s^2}{4}(\|Nv\|^2 + \|A^{-1}v\|^2) - (C_1 + C_3)\|v\|^2 .
\end{aligned}
$$

Here, we note the following: there are $C_4 > 0$ and $C_5 > 0$ such that

$$
(2.11)\qquad C_4\|w\|^2 \leqq \|Nw\|^2 + \|A^{-1}w\|^2 \leqq C_5\|w\|^2
$$

holds for any $w \in C_0^\infty((0, T), H^\infty(\mathbb{R}^n))^p$. In fact, this is verified by $\|Nw\| + \|A^{-1}w\| \leqq (\|N\| + \|A^{-1}\|)\|w\|$ and $\|w\| \leqq \|MNw\| + \|(I - MN)w\| \leqq \|M\|\|Nw\| + \|(I - MN)A\| \cdot \|A^{-1}w\| \leqq (\|M\| + \|(I - MN)A\|)(\|Nw\| + \|A^{-1}w\|)$.

Therefore, by (2.10) and (2.11) we have

$$
C_5\|(t\partial_t + s + \sqrt{-1}A)v\|^2 \geqq \left(\frac{s^2}{4}C_4 - C_1 - C_3\right)\|v\|^2 .
$$

Thus, by choosing $b_0 = \max\{4C_2, 1/2, 8(C_1 + C_3)/C_4\}$ and $c_0 = (C_4/8C_5)^{1/2}$ we obtain

$$
\|(t\partial_t + s + \sqrt{-1}A)v\| \geqq c_0 s\|v\|
$$

for $s > b_0$. Thus, we have proved (2.7) for $k = 0$.

Note that

$$
(2.12)\qquad \begin{cases} A(t\partial_t + s + \sqrt{-1}A)v = (t\partial_t + s + \sqrt{-1}A + \sqrt{-1}[A, A]A^{-1})Av , \\ (t\partial_t + s)(t\partial_t + s + \sqrt{-1}A)v = (t\partial_t + s + \sqrt{-1}A)(t\partial_t + s)v + \sqrt{-1}(tA_t'A^{-1})Av \end{cases}
$$

hold and that $[A, A]A^{-1}$, $tA_t'A^{-1} \in \mathcal{S}^0([0, T], p \times p)$ are bounded in $L^2((0, T) \times \mathbb{R}^n)^p$. Therefore, by using (2.12) and by induction on k we can prove (2.7) for $k \geqq 1$ in the same way as above. Q.E.D.

Lemma 2. *Let*

$$
E_s = (t\partial_t + s)^q + \sum_{i=1}^{q} a_i(t, x, D_x)(t\partial_t + s)^{q-i} ,
$$

where $a_i(t, x, D_x)$ $(1 \leqq i \leqq q)$ are pseudo-differential operators with symbols $a_i(t, x, \xi)$ satisfying the following: (i) $a_i(t, x, \xi) \in C^\infty([0, T] \times \mathbb{R}_x^n \times \mathbb{R}_\xi^n)$, (ii) *$a_i(t, x, \xi)$ is positively homogeneous of degree i in ξ (for $|\xi| \geqq 1$), and* (iii) *$a_i(t, x, \xi)$ is independent of x for sufficiently large $|x|$. Assume that*

$$(\sqrt{-1}\tau)^q + \sum_{i=1}^{q} a_i(0, 0, \xi)(\sqrt{-1}\tau)^{q-i} \neq 0 \tag{2.13}$$

holds for any $(\tau, \xi) \in \mathbf{R}_\tau \times \mathbf{R}_\xi^n$ *satisfying* $|\xi| \geqq 1$. *Then, there are an open neighborhood* W *of* $(0, 0)$ *in* $\mathbf{R}_t \times \mathbf{R}_x^n$, $d_k > 0$ $(k \in \mathbf{Z}_+)$ *and* $C_{k,s} > 0$ $(k \in \mathbf{Z}_+$ *and* $s \in \mathbf{R})$ *such that the estimate*

$$|||E_s\varphi|||_{k,s} \geqq d_k |||\varphi|||_{q+k,s} - C_{k,s}|||\varphi|||_{q+k-1,s} \tag{2.14}$$

holds for any $\varphi \in C_0^\infty(W \cap \{t > 0\})$, $k \in \mathbf{Z}_+$ *and* $s \in \mathbf{R}$.

Proof. Put $B(r) = \{x \in \mathbf{R}^n; |x| < r\}$, and assume that $\varphi(t, x)(=\varphi) \in C_0^\infty((0, \varepsilon) \times B(r))$. Let $\mu_r(x) \in C_0^\infty(B(2r))$ be such that $\mu_r = 1$ holds on $B(r)$. Obviously we have $\varphi = \mu_r\varphi$.

Put

$$E_s^{(0,0)} = (t\partial_t + s)^q + \sum_{i=1}^{q} a_i(0, 0, D_x)(t\partial_t + s)^{q-i},$$

and choose $b_i(t, x, D_x)$, $c_{ij}(t, x, D_x) \in \mathcal{S}^i([0, T])$ so that the following relation holds:

$$a_i(t, x, D_x) = a_i(0, 0, D_x) + tb_i(t, x, D_x) + \sum_{j=1}^{n} x_j c_{ij}(t, x, D_x).$$

Then, for $\varphi \in C_0^\infty((0, \varepsilon) \times B(r))$ we have

$$\begin{aligned} E_s\varphi = E_s^{(0,0)}\varphi &+ \sum_{i=1}^{q} tb_i(t\partial_t + s)^{q-i}\varphi \\ &+ \sum_{i=1}^{q}\sum_{j=1}^{n} (x_j\mu_r)c_{ij}(t\partial_t + s)^{q-i}\varphi \\ &+ \sum_{i=1}^{q}\sum_{j=1}^{n} x_j[c_{ij}, \mu_r](t\partial_t + s)^{q-i}\varphi. \end{aligned}$$

Therefore, by the conditions $|t| \leqq \varepsilon$, $|x_j\mu_r(x)| \leqq 2r$ and $[c_{ij}, \mu_r]$ $(=c_{ij}\mu_r - \mu_r c_{ij}) \in \mathcal{S}^{i-1}([0, T])$ we obtain

$$\|E_s\varphi\| \geqq \|E_s^{(0,0)}\varphi\| - (\varepsilon + r)A_1|||\varphi|||_{q,s} - C_r|||\varphi|||_{q-1,s} \tag{2.15}$$

for some $A_1 > 0$ (independent of ε and r) and $C_r > 0$ (depending on r). Thus, in order to estimate $\|E_s\varphi\|$ from below we need to estimate $\|E_s^{(0,0)}\varphi\|$ from below.

Note the following fact. Put

$$R = \partial_z^q + \sum_{i=1}^{q} a_i(0, 0, D_x)\partial_z^{q-i}.$$

Then, by (2.13) and by using the Fourier transformation we can see the following: there are $\mu > 0$ and $A_2 > 0$ such that

$$\|R\psi\|_{(z,x)} \geqq \mu \sum_{i+|\alpha| \leqq q} \|\partial_z^i\partial_x^\alpha\psi\|_{(z,x)} - A_2 \sum_{i+|\alpha| \leqq q-1} \|\partial_z^i\partial_x^\alpha\psi\|_{(z,x)} \tag{2.16}$$

holds for any $\psi = \psi(z, x) \in C_0^\infty(\mathbf{R}_z \times \mathbf{R}_x^n)$, where $\|w\|_{(z,x)}$ is the norm of w in $L^2(\mathbf{R}_z \times \mathbf{R}_x^n)$.

By using (2.16), let us estimate $\|E_s^{(0,0)}\varphi\|$ from below. Note the following: by the change of variables $(0, T) \times \mathbf{R}_x^n \ni (t, x) \to (z, x) = (\log t, x) \in \mathbf{R}_z \times \mathbf{R}_x^n$, $t\partial_t$ is transformed into ∂_z, $E_0^{(0,0)}$ is transformed into R, $\phi(t,x)\in C_0^\infty((0,T)\times\mathbb{R}_x^n)$ is transformed into $\psi(z,x)=\phi(e^z,x)\in C_0^\infty(\mathbb{R}_z\times\mathbb{R}_x^n)$, and dt/t is transformed into dz. Therefore, by (2.16) we have

$$\left\|\frac{1}{\sqrt{t}}E_0^{(0,0)}\phi\right\| \geqq \mu\sum_{i+|\alpha|\leqq q}\left\|\frac{1}{\sqrt{t}}(t\partial_t)^i\partial_x^\alpha\phi\right\| - A_2\sum_{i+|\alpha|\leqq q-1}\left\|\frac{1}{\sqrt{t}}(t\partial_t)^i\partial_x^\alpha\phi\right\|$$

for any $\phi=\phi(t,x)\in C_0^\infty((0,T)\times\mathbb{R}_x^n)$. Moreover, by putting $\phi(t,x)=\sqrt{t}\varphi(t,x)$ we obtain

$$\|E_{1/2}^{(0,0)}\varphi\| \geqq \mu|||\varphi|||_{q,1/2} - A_2|||\varphi|||_{q-1,1/2} \tag{2.17}$$

for any $\varphi\in C_0^\infty((0,T)\times\mathbb{R}_x^n)$. Since

$$\big||||\varphi|||_{l,s} - |||\varphi|||_{l,1/2}\big| \leqq C_{l,s}|||\varphi|||_{l-1,s} \tag{2.18}$$

holds for some $C_{l,s}>0$, by (2.17) and (2.18) we obtain

$$\begin{aligned}\|E_s^{(0,0)}\varphi\| &\geqq \|E_{1/2}^{(0,0)}\varphi\| - \|(E_s^{(0,0)}-E_{1/2}^{(0,0)})\varphi\| \\ &\geqq \mu|||\varphi|||_{q,1/2} - A_2|||\varphi|||_{q-1,1/2} - A_{3,s}|||\varphi|||_{q-1,s} \\ &\geqq \mu|||\varphi|||_{q,s} - B_s|||\varphi|||_{q-1,s}\end{aligned} \tag{2.19}$$

for some $A_{3,s}>0$ and $B_s>0$ (depending on s).

Hence, by (2.15) and (2.19) we have

$$\|E_s\varphi\| \geqq (\mu-(\varepsilon+r)A_1)|||\varphi|||_{q,s} - (B_s+C_r)|||\varphi|||_{q-1,s}$$

for any $\varphi\in C_0^\infty((0,\varepsilon)\times B(r))$. Thus, by putting $\varepsilon=1/4A_1$, $r=1/4A_1$, $W=(-\varepsilon,\varepsilon)\times B(r)$, $d_0=\mu/2$ and $C_{0,s}=B_s+C_r$ we can obtain

$$\|E_s\varphi\| \geqq d_0|||\varphi|||_{q,s} - C_{0,s}|||\varphi|||_{q-1,s}\,.$$

Thus, we have proved (2.14) for $k=0$. (2.14) for $k\geqq 1$ may be proved by induction on k. Q.E.D.

Now, by using Lemmas 1 and 2 let us give a proof of Proposition 1.

Proof of Proposition 1. Let P_{-s} be as in (2.1). Then, for any $\varphi\in C^\infty(U)$ we have

$$\begin{aligned}(P_{-s})^*\varphi &= (-t\partial_t-1-s)^m + \sum_{\substack{j+|\alpha|\leqq m\\ j<m}}(-t\partial_t-1-s)^j(-\partial_x)^\alpha a_{j,\alpha}(t,x)\varphi \\ &= (-1)^m\left[(t\partial_t+s)^m + \sum_{\substack{j+|\alpha|\leqq m\\ j<m}} b_{j,\alpha}(t,x)(t\partial_t+s)^j\partial_x^\alpha\right]\varphi\end{aligned}$$

for some $b_{j,\alpha}(t,x)\in C^\infty(U)$ such that $b_{j,\alpha}(t,x)=a_{j,\alpha}(t,x)$ for $j+|\alpha|=m$ and therefore

$$\tau^m + \sum_{\substack{j+|\alpha|=m\\ j<m}} b_{j,\alpha}(0,0)\tau^j\xi^\alpha = p(\tau,\xi)\,.$$

Since we are discussing $(P_{-s})^*$ only in a small neighborhood of $(0,0)$ in $\mathbb{R}_t\times\mathbb{R}_x^n$, we may assume that $b_{j,\alpha}(t,x)$ is constant outside a small neighborhood of $(0,0)$ in $\mathbb{R}_t\times\mathbb{R}_x^n$.

Then, by (2.4), (A-1) and (A-3) we can see that $(P_{-s})^*$ is decomposed into the following form:

$$(2.20)\qquad (P_{-s})^* = (-1)^m H_s E_s + \sum_{i=1}^{m-1} Q_i(t, x, D_x)(t\partial_t + s)^{m-1-i},$$

where H_s is an operator of order p satisfying the conditions in Lemma 1, E_s is an operator of order $q(=m-p)$ satisfying the conditions in Lemma 2, and $Q_i(t, x, D_x) \in \mathcal{S}^i([0, T])$ $(1 \leqq i \leqq m-1)$.

Choose b_k, c_k, W, d_k and $C_{k,s}$ so that the conditions in Lemmas 1 and 2 hold for the operators H_s and E_s in (2.20). Let $k \in \mathbb{Z}_+$. Then, by choosing a constant $M_k > 0$ suitably we have

$$(2.21)\qquad \begin{aligned} &|||(P_{-s})^*\varphi|||_{k,s} \\ &\geqq |||H_s(E_s\varphi)|||_{k,s} - M_k|||\varphi|||_{m+k-1,s} \\ &\geqq c_k s|||E_s\varphi|||_{p+k-1,s} - M_k|||\varphi|||_{m+k-1,s} \\ &\geqq c_k s(d_{p+k-1}|||\varphi|||_{m+k-1,s} - C_{p+k-1,s}|||\varphi|||_{m+k-2,s}) - M_k|||\varphi|||_{m+k-1,s} \end{aligned}$$

for any $\varphi \in C_0^\infty(W \cap \{t > 0\})$ and $s > b_k$.

Here, we put $W(r) = \{(t, x) \in W; |x_i| < (r/\sqrt{2})(i = 1, \ldots, n)\}$ and note the following: if $\varphi \in C_0^\infty(W(r) \cap \{t > 0\})$, we have

$$(2.22)\qquad |||\varphi|||_{m+k-2,s} \leqq r|||\varphi|||_{m+k-1,s}$$

by using Poincare's inequality with respect to the x-variable.

Therefore, by (2.21) and (2.22) we have

$$\begin{aligned} |||(P_{-s})^*\varphi|||_{k,s} \geqq &\left(\frac{c_k d_{p+k-1}}{2}s - M_k\right)|||\varphi|||_{m+k-1,s} \\ &+ c_k s\left(\frac{d_{p+k-1}}{2} - rC_{p+k-1,s}\right)|||\varphi|||_{m+k-1,s}. \end{aligned}$$

Hence, by putting $s_k = (4M_k)/(c_k d_{p+k-1})$, $r_{k,s} = d_{p+k-1}/(4C_{p+k-1,s})$, $V_{k,s} = W(r_{k,s})$ and by taking $s > s_k$ we have

$$(2.23)\qquad |||(P_{-s})^*\varphi|||_{k,s} \geqq \frac{c_k d_{p+k-1}}{2}s|||\varphi|||_{m+k-1,s}$$

for any $\varphi \in C_0^\infty(V_{k,s} \cap \{t > 0\})$. Thus, by (2.6) and (2.23) we can obtain (2.2). Q.E.D.

§ 3. Proof of Theorem 1

As in [14], we put $\mathcal{D}'_0$, $\mathcal{D}'(+)$, $\mathcal{D}'(-)$, $\mathcal{D}'_{\{t=0\}}$, $\mathcal{D}'_{ext}(+)$ and $\mathcal{D}'_{ext}(-)$ as follows:

$$\mathcal{D}'_0 = \underset{W \ni (0,0)}{\text{ind}-\varinjlim}\, \mathcal{D}'(W),$$

$$\mathcal{D}'(\pm) = \underset{W \ni (0,0)}{\text{ind}-\varinjlim}\, \mathcal{D}'(W \cap \{\pm t > 0\}),$$

$$\mathscr{D}'_{\{t=0\}} = \{u \in \mathscr{D}'_0;\ \mathrm{supp}\,(u) \subset \{t=0\}\}\,,$$

$$\mathscr{D}'_{ext}(\pm) = \{u \in \mathscr{D}'(\pm);\ \text{there exists a } v \in \mathscr{D}'_0 \text{ such that } u = v \text{ on } \{\pm t > 0\}\}\,,$$

where W is an open neighborhood of $(0, 0)$ in $\mathbf{R}_t \times \mathbf{R}^n_x$. Note that $\mathscr{D}'_{ext}(\pm)$ is the set of all distributions $u \in \mathscr{D}'(\pm)$ which is extendable to a full neighborhood of $t = 0$ as a distribution.

Then, we can see that Theorem 1 is obtained by the following two facts:

(S-1) $Pu = f$ is solvable in $\mathscr{D}'_{\{t=0\}}$.

(S-2) $Pu = f$ is solvable in $\mathscr{D}'_{ext}(\pm)$.

In fact, if we know (S-1) and (S-2), the solvability of $Pu = f$ in $\mathscr{D}'_0$ is obtained by the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{D}'_{\{t=0\}} & \longrightarrow & \mathscr{D}'_0 & \longrightarrow & \mathscr{D}'_{ext}(+) \oplus \mathscr{D}'_{ext}(-) & \longrightarrow & 0 \\ & & \downarrow P & & \downarrow P & & \downarrow P & & \\ 0 & \longrightarrow & \mathscr{D}'_{\{t=0\}} & \longrightarrow & \mathscr{D}'_0 & \longrightarrow & \mathscr{D}'_{ext}(+) \oplus \mathscr{D}'_{ext}(-) & \longrightarrow & 0\,. \end{array}$$

Note that the horizontal line is exact, since for any $u \in \mathscr{D}'_{ext}(\pm)$ we can find a $v \in \mathscr{D}'_0$ such that $u = v$ on $\{\pm t > 0\}$ and $\mathrm{supp}\,(v) \subset \{\pm t \geqq 0\}$.

Hence to have Theorem 1 it is sufficient to prove (S-1) and (S-2).

Proof of (*S*-1). Put

$$C(\rho, x, \partial_x) = \rho^m + \sum_{\substack{j+|\alpha| \leqq m \\ j<m}} a_{j,\alpha}(0, x)\rho^j \partial_x^\alpha\,.$$

Let u and f be of the form

$$u = \sum_{i=0}^{N} \delta^{(i)}(t) \otimes \psi_i(x), \qquad f = \sum_{i=0}^{N} \delta^{(i)}(t) \otimes \mu_i(x)\,, \tag{3.1}$$

where $N \in \mathbf{Z}_+$, $\delta^{(i)}(t) = \partial_t^i \delta(t)$, $\delta(t) = \delta^{(0)}(t)$ is Dirac's delta-function, and $\psi_i(x)$, $\mu_i(x)$ are germs of distributions in x at the origin in $\mathbf{R}^n_x$. Then, by using the relations

$$(t\partial_t)^k \delta^{(i)}(t) = (-i-1)^k \delta^{(i)}(t) \qquad (k, i \in \mathbf{Z}_+)$$

we can see that $Pu = f$ is equivalent to the following recursive system:

$$\begin{cases} C(-N-1; x, \partial_x)\psi_N = \mu_N\,, \\ C(-N; x, \partial_x)\psi_{N-1} = \mu_{N-1} + L_{N-1,N}(x, \partial_x)\psi_N\,, \\ \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ C(-1, x, \partial_x)\psi_0 = \mu_0 + \sum_{l=1}^{N} L_{0,l}(x, \partial_x)\psi_l\,, \end{cases} \tag{3.2}$$

where $L_{i,l}(x, \partial_x)$ ($0 \leqq i \leqq N-1$ and $i+1 \leqq l \leqq N$) are differential operators of order m determined by P. Since $C(\rho, x, \partial_x)$ is assumed to be elliptic near $x = 0$ (by (A-2)), we know that the equation $C(\rho, x, \partial_x)\psi = \mu$ (where $\psi = \psi(x)$, $\mu = \mu(x)$ are distributions in x near $x = 0$) is solvable in the germ sense. Therefore, by solving (3.2) successively we can determine $\{\psi_i\}_{i=0}^N$ from the given $\{\mu_i\}_{i=0}^N$ so that $Pu = f$ holds under (3.1). This proves (S-1), because any $u, f \in \mathscr{D}'_{\{t=0\}}$ are expressed in the form (3.1). Q.E.D.

Proof of (S-2). Let $f \in \mathscr{D}'_{ext}(+)$. Then we have $f \in H^{-m-k+1}(V \cap \{t > 0\})$ for some $k \in \mathbf{Z}_+$ and some open neighborhood V of $(0, 0)$ in $\mathbf{R}_t \times \mathbf{R}_x^n$. Let s_k be the one in Proposition 1. Choose $s \in \mathbf{Z}$ satisfying $s > \max\{s_k, m + k - 1\}$ and fix it. Let $\delta_{k,s}$ and $V_{k,s}$ be the ones in Proposition 1 corresponding to these k and s. Put $W = V \cap V_{k,s}$. Then, we can see the following two facts:

$$t^{s-m-k+1}f \in H^{-m-k+1}(W \cap \{t > 0\}), \tag{3.3}$$

$$\|(P_{-s})^*\varphi\|_k \geqq \delta_{k,s}\|t^{m+k-1}\varphi\|_{m+k-1} \quad \text{for any} \quad \varphi \in C_0^\infty(W \cap \{t > 0\}). \tag{3.4}$$

Let $H_0^k(W \cap \{t > 0\})$ be the closure of $C_0^\infty(W \cap \{t > 0\})$ in the Sobolev space $H^k(W \cap \{t > 0\})$, define a linear subspace Z of $H_0^k(W \cap \{t > 0\})$ by $Z = \{(P_{-s})^*\varphi;\ \varphi \in C_0^\infty(W \cap \{t > 0\})\}$, and define a linear functional T on Z by $T((P_{-s})^*\varphi) = \langle \varphi, t^s f \rangle$. Then, by (3.3) and (3.4) we have

$$\begin{aligned} |T((P_{-s})^*\varphi)| &= |\langle t^{m+k-1}\varphi, t^{s-m-k+1}f \rangle| \\ &\leqq \|t^{m+k-1}\varphi\|_{m+k-1}\|t^{s-m-k+1}f\|_{-m-k+1} \\ &\leqq \frac{1}{\delta_{k,s}}\|(P_{-s})^*\varphi\|_k\|t^{s-m-k+1}f\|_{-m-k+1}, \end{aligned}$$

and therefore T is continuous on Z with respect to the topology induced from $H_0^k(W \cap \{t > 0\})$. Since $H^{-k}(W \cap \{t > 0\})$ is the dual space of $H_0^k(W \cap \{t > 0\})$, we can find a $v \in H^{-k}(W \cap \{t > 0\})$ such that $T(z) = \langle z, v \rangle$ for any $z \in Z$. This means that $\langle \varphi, t^s f \rangle = \langle (P_{-s})^*\varphi, v \rangle$ holds for any $\varphi \in C_0^\infty(W \cap \{t > 0\})$. Hence, we have $P_{-s}v = t^s f$ on $W \cap \{t > 0\}$; this is equivalent to $P(t^{-s}v) = f$ on $W \cap \{t > 0\}$. Thus, by putting $u = t^{-s}v$ we obtain a solution $u \in \mathscr{D}'_{ext}(+)$ of $Pu = f$. Q.E.D.

References

[1] Alinhac, S., Systèmes hyperboliques singuliers. *Asterisque*, **19** (1974), 3–24.

[2] Bove, A., Lewis, J. E. and Parenti, C., Cauchy problem for Fuchsian hyperbolic operators, *Hokkaido Math. J.*, **14** (1985), 175–248.

[3] ———, Structure properties of solutions of some Fuchsian hyperbolic equations, *Math. Ann.*, **273** (1986), 553–571.

[4] Bove, A., Lewis, J. E., Parenti, C. and Tahara, H., Cauchy problem for Fuchsian hyperbolic operators, II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **34** (1987), 127–157.

[5] Hanges, N., Parametrices and local solvability for a class of singular hyperbolic operators, *Comm. in PDE*, **3** (1978), 105–152.

[6] Hörmander, L., *The analysis of linear partial differential operators, IV.* Springer, 1985.

[7] Ivrii, V. Ja., Wave fronts of solutions of certain pseudodifferential equations, *Trudy Moscow Mat.*, **39** (1979), 49–85.

[8] Mizohata, S., *The theory of partial differential equations*, Cambridge Univ. Press, 1973.

[9] Ōaku, T., Local solvability of Fuchsian elliptic equations for hyperfunctions, *Yokohama Math. J.*, **36** (1988), 109–114.

[10] Serra, E., Local solvability for a class of totally characteristic operators, *Boll. U.M.I.*, Ser. VI, Vol. **III-C** (1984), 131–141.

[11] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, *Japan. J. Math. New Ser.*, **5** (1979), 245–347.

[12] ———, Singular hyperbolic systems, III. On the Cauchy problem for Fuchsian hyperbolic partial differential equations, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **27** (1980), 465–507.

[13] ———, Singular hyperbolic systems, IV. Remarks on the Cauchy problem for singular hyperbolic partial differential equations, *Japan. J. Math. New Ser.*, **8** (1982), 297–308.

[14] ———, On the local solvability of Fuchsian type partial differential equations, *Algebraic analysis*, Volume II (edited by Kashiwara, M. and Kawai, T.), 837–848. Academic Press, Boston, 1988.

[15] ———, Solvability in distributions for a class of singular differential operators, I. *Proc. Japan Acad.*, **64** (1988), 219–222.

[16] ———, Solvability in distributions for a class of singular differential operators, II. *Proc. Japan Acad.*, **64** (1988), 318–321.