

On the Symmetry of b -Functions of Linear Free Divisors

by

Michel GRANGER and Mathias SCHULZE

Abstract

We introduce the concept of a prehomogeneous determinant as a possibly nonreduced version of a linear free divisor. Both are special cases of prehomogeneous vector spaces. We show that the roots of the b -function are symmetric about -1 for reductive prehomogeneous determinants and for regular special linear free divisors. For general prehomogeneous determinants, we describe conditions under which this symmetry persists.

Combined with Kashiwara's theorem on the roots of b -functions, our symmetry result shows that -1 is the only integer root of the b -function. This gives a positive answer to a problem posed by Castro-Jiménez and Ucha-Enríquez in the above cases.

We study the condition of strong Euler homogeneity in terms of the action of the stabilizers on the normal spaces.

As an application of our results, we show that the logarithmic comparison theorem holds for reductive linear Koszul free divisors exactly when they are strongly Euler homogeneous.

2010 Mathematics Subject Classification: 14M17, 20G05, 17B66, 14F40.

Keywords: linear free divisor, prehomogeneous vector space, b -function.

Contents

§1. Logarithmic comparison theorem and b -functions	480
§2. Linear free divisors and prehomogeneous determinants	482
§3. b -functions of reductive prehomogeneous determinants	487
§4. Examples	491
§5. b -functions of special linear free divisors	498
§6. Euler homogeneity and normal representations	500
§7. Koszul freeness and Euler homogeneity	502
References	504

Communicated by M. Kashiwara. Received March 4, 2009.

M. Granger: Département de Mathématiques, Université d'Angers, LAREMA, CNRS UMR n° 6093, 2 Bd Lavoisier, 49045 Angers, France;

e-mail: granger@univ-angers.fr

M. Schulze: Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA;

e-mail: mschulze@math.okstate.edu

§1. Logarithmic comparison theorem and b -functions

Let D be a reduced hypersurface in a complex n -dimensional manifold X , let $U = X \setminus D$, and let $j : U \hookrightarrow X$ be inclusion. Then the de Rham morphism

$$\Omega_X^\bullet(*D) \rightarrow \mathbf{R}j_*\mathbb{C}_U$$

is a quasi-isomorphism. This so-called *Grothendieck's comparison theorem* is the core of the proof of Grothendieck's algebraic de Rham theorem [Gro66] in which D plays the role of a divisor at infinity for a compactification. In particular, for Stein X , each cohomology class $c \in H^k(U; \mathbb{C})$ is represented as $c(\sigma) = \int_\sigma \omega$ by a differential k -form ω with finite pole order along D . The natural question of limiting this pole order dates back to Griffiths [Gri69] and has been studied later by Deligne and Dimca [DD90, Dim91], Karpishpan [Kar91], and others.

By analogy with Grothendieck's comparison theorem, one says that the *logarithmic comparison theorem (LCT)* holds for D if the inclusion

$$\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(*D)$$

is a quasi-isomorphism. The explicit characterization of hypersurfaces for which the LCT holds is an open problem (see [Tor07] for an overview). Mainly the extremal cases of isolated singularities and free divisors (as defined in [Sai80]) have been studied. For isolated singularities the LCT can be explicitly characterized in the quasihomogeneous case (see [HM98]) and there are partial results for the nonquasihomogeneous case (see [Sch07]). Also several variants of the LCT property have been introduced and studied. For instance, F. J. Calderón Moreno and L. Narváez Macarro [CN09] generalized the LCT to integrable logarithmic connections. More famous is a global algebraic version of the LCT which is conjectured to hold for all hyperplane arrangements in characteristic 0 (see [Ter78, Conj. 3.1] and [WY97, §1]).

In this article we shall be concerned with the case of free divisors. The main conjecture in this case is the following from [CMNCJ02, Conj. 1.4].

Conjecture 1.1. *For a free divisor, the LCT implies strong Euler homogeneity.*

By definition, *strong Euler homogeneity at p* for some $p \in D$ means that $f \in \mathfrak{m}_{X,p} \cdot J(f)$ for some, and hence any, local defining equation $f \in \mathcal{O}_{X,p}$ at p , where $J(f)$ denotes the ideal of partials of f . By *strong Euler homogeneity of D* without reference to the point we mean strong Euler homogeneity at all $p \in D$ (see Definition 6.1 for more details). We have proved Conjecture 1.1 for Koszul free divisors (see Section 7 for their definition) and for $n \leq 3$ in [GS06, Cor. 1.5, Thm. 1.6]. For plane curves, F. J. Calderón-Moreno et al. [CMNCJ02, Thm. 3.5]

showed that the implication in Conjecture 1.1 is even an equivalence, but for $n \geq 3$ a counter-example for the converse implication is missing. In fact the first result in this direction was a stronger sufficient condition for LCT, proved by F. J. Castro-Jiménez et al. [CNM96], which appeared to be too strong to be necessary: Any locally quasihomogeneous free divisor satisfies the LCT. By definition, *locally quasihomogeneous* hypersurfaces admit at any point a defining equation which is quasihomogeneous with respect to strictly positive weights on a local coordinate system. As observed by L. Narváez Macarro [NM08, Rem. 1.7.4], the above implication still holds true even without this strictness requirement. The resulting condition is referred to as local weak quasihomogeneity. His argument is based on the following characterization of the LCT in terms of $\mathcal{D}_X(\log D)$ -modules given in [CN05, Cor. 4.3].

Theorem 1.2. *A free divisor D satisfies the logarithmic comparison theorem if*

- (a) $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)}^L \mathcal{O}_X(D)$ is concentrated in degree 0,
- (b) $\mathcal{D}_X \otimes_{\mathcal{D}_X(\log D)} \mathcal{O}_X(D) \rightarrow \mathcal{O}_X(*D)$ is injective.

Condition (a) is fulfilled for Koszul free D and condition (b) is equivalent to $\text{Ann}_{\mathcal{D}_X}(f^{-1})$ being generated by order one operators. This latter condition is studied in [Tor04] where the following is proved in [Tor04, Cor. 1.8] for the Koszul free case.

Theorem 1.3. *Let $f \in \mathcal{O}_X$ be a Koszul free germ on $X = (\mathbb{C}^n, 0)$. Then $\text{Ann}_{\mathcal{D}_X}(f^{-1})$ is generated by order one operators if and only if*

- (a) f is Euler homogeneous,
- (b) -1 is the only integer root of the (local) b -function of f ,
- (c) f and the symbols of a basis of $\text{Der}(-\log f)$ form a $\text{gr}_F \mathcal{D}_X$ -regular sequence.

A promising class of examples to make the preceding theorems more explicit in order to better understand the LCT problem is that of *linear free divisors*. These are free divisors D in a complex vector space $X = V \cong \mathbb{C}^n$ such that $\text{Der}(-\log D)$ admits a basis of vector fields with linear coefficients. Linear free divisors have been introduced in [BM06] where they occur as discriminants in certain quiver representation spaces. Later they have been studied in [GMNS09] and [dGMS09]. We have proved a global version of the LCT [GMNS09, Thm. 1.6] for reductive linear free divisors and it is conceivable that LCT holds for any linear free divisor.

The interest in linear free divisors arises from the fact that they are special cases of prehomogeneous vector spaces, which endows them with a rich additional structure that has been studied intensively. A particular feature in view of The-

orem 1.3 is that their b -functions are much more explicit and satisfy, in certain cases, a functional equation due to M. Sato's fundamental theorem for prehomogeneous vector spaces. The present paper is motivated by the wish to understand b -functions of linear free divisors in order to apply Theorems 1.2 and 1.3. It is organized as follows:

In Section 2, we introduce *prehomogeneous determinants* as a slight generalization of linear free divisors for which all our arguments work without additional effort. We further study reductivity and a condition that is referred to as *special* in [dGMS09, Def. 2.1]. In Theorem 2.4, we give a more elementary proof of the implication “reductive implies special” from [dGMS09, Cor. 2.9] for the more general case of prehomogeneous determinants. In Section 3, we study b -functions of prehomogeneous determinants and prove our main result in Theorem 3.5 which states that the roots of the b -function of any reductive prehomogeneous determinant are symmetric about -1 , which implies, in particular, that condition (b) of Theorem 1.3 holds true.

Theorem 1.4. *The roots of the b -function of any reductive linear free divisor are symmetric about -1 .*

In Section 4, we compute the b -function for all linear free divisors in dimension up to 4, and, using the microlocal calculus from [SKKØ81], for some prehomogeneous determinants arising from quiver theory. Based on this list of examples we expect that the symmetry property holds true for any linear free divisor.

Conjecture 1.5. *The statement of Theorem 1.4 holds true also for nonreductive linear free divisors.*

Sections 5 and 6 contain additional material independent of our application to the LCT: We show in Theorem 5.5 that the symmetry statement in our main result holds true also for regular special linear free divisors, where “regular” refers to the corresponding notion for prehomogeneous vector spaces (see [Kim03, Def. 2.14]). In Proposition 6.6, we give a sufficient condition for Euler homogeneity at a point in terms of the corresponding normal representation. In Section 7, we discuss condition (c) in Theorem 1.3 in order to finally prove the following statement on the LCT for linear free divisors.

Theorem 1.6. *A reductive linear Koszul free divisor satisfies the logarithmic comparison theorem if and only if it is strongly Euler homogeneous.*

§2. Linear free divisors and prehomogeneous determinants

A reduced hypersurface D in an n -dimensional complex vector space V is called a *linear free divisor* if the module $\text{Der}(-\log D)$ of *logarithmic vector fields* has a basis of global degree 0 vector fields

$$\delta_k = A_k x = x^t A_k^t \partial_x \in \Gamma(V, \text{Der}(-\log D))_0, \quad A_k \in \mathbb{C}^{n \times n}, \quad k = 1, \dots, n.$$

Then, by Saito’s criterion [Sai80, Thm. 1.8.ii],

$$f = \det(\delta_1, \dots, \delta_n) \in \text{Sym}(V^*)$$

is a homogeneous defining equation for D of degree n . Therefore we can assume that $\delta_1 = \epsilon$ is the *Euler vector field*

$$\epsilon = \frac{1}{n} \sum_{i=1}^n x_i \partial_{x_i}$$

and then also that $\delta_2, \dots, \delta_n \in \Gamma(V, \text{Der}(-\log f))_0$ where $\text{Der}(-\log f)$ is the module of vector fields tangent to (the level sets of) f .

We denote by $G_D \subseteq \text{GL}_V$ the unit component of the stabilizer of $f \in \mathbb{P}\text{Sym}(V^*)$, and by $A_D \subseteq G_D$ that of $f \in \text{Sym}(V^*)$. The following was shown in [GMNS09, Lemmas 2.2–2.4]: G_D is a linear algebraic group with Lie algebra of infinitesimal generators $\mathfrak{g}_D = \Gamma(V, \text{Der}(-\log D))_0$, and the same holds for A_D with $\mathfrak{a}_D = \Gamma(V, \text{Der}(-\log f))_0$. By Saito’s criterion, $V \setminus D$ is a Zariski open orbit with finite isotropy groups and G_D an n -dimensional group. Conversely, any reduced discriminant of an n -dimensional linear algebraic group in GL_V with an open orbit is a linear free divisor. Thus, linear free divisors are special cases of discriminants of prehomogeneous vector spaces (G, ρ, V) characterized by the following conditions:

1. $G \subseteq \text{GL}_V$ is connected with $\dim G = \dim V$,
2. $\rho = \text{id} : G \hookrightarrow \text{GL}_V$,
3. $f(x) = \det(d\rho(A_1)x, \dots, d\rho(A_n)x) \in \text{Sym}(V^*)$, $\mathfrak{g} = \langle A_1, \dots, A_n \rangle$, is reduced.

We shall consider the more general objects defined by dropping the third condition. However, we shall frequently impose the additional condition of reductivity on G when we study b -functions in Section 3.

Definition 2.1. Let $G \subseteq \text{GL}_V$ be a connected algebraic group with $\dim G = \dim V = n$ such that (G, ρ, V) is a prehomogeneous vector space where $\rho = \text{id} : G \hookrightarrow \text{GL}_V$. Consider the relative invariant

$$(2.1) \quad f(x) = \det(d\rho(A_1)x, \dots, d\rho(A_n)x) \in \text{Sym}(V^*)$$

defined by generators A_1, \dots, A_n of the Lie algebra \mathfrak{g} of G . We call the (possibly nonreduced) discriminant divisor $D \subseteq V$ defined by f a *prehomogeneous (discriminant) determinant*, and a *linear free divisor* in case f is reduced. We call D *reductive* if G is a reductive group. We set $G_D = G$, $\mathfrak{g}_D = \mathfrak{g}$, and $\rho_D = \rho$, and we denote the character of f by $\chi_D: G_D \rightarrow \mathbb{C}^*$ and its derivative by $d\chi_D: \mathfrak{g}_D \rightarrow \mathbb{C}$. We call

$$(2.2) \quad A_D = (\ker \chi_D)^\circ, \quad \mathfrak{a}_D = \ker d\chi_D,$$

the *annihilator* of D where $^\circ$ denotes the unit component.

While G_D is determined by D if D is a linear free divisor, it is part of the data of a general prehomogeneous determinant D . Note that, in Definition 2.1, f is indeed a relative invariant by [Kim03, Thm. 2.9]. Then χ_D is not trivial by [Kim03, Prop. 2.4(2)] and we have

$$(2.3) \quad G_D/A_D \cong \mathbb{C}^*, \quad \mathfrak{g}_D/\mathfrak{a}_D \cong \mathbb{C}.$$

In particular, we may pick $A_1 \in \mathfrak{g}_D$ such that

$$(2.4) \quad d\chi_D(A_1) = 1.$$

We choose $A_1 = E/n$ if D is a linear free divisor. For $A \in \mathfrak{g}_D$, we abbreviate

$$(2.5) \quad \delta_A = \langle d\rho_D(A)x, \partial_x \rangle = x^t A^t \partial_x, \quad \delta_k = \delta_{A_k}.$$

By definition of a relative invariant [Kim03, §2.2] and [Kim03, Lem. 2.15],

$$(2.6) \quad f \circ \rho_D(g) = \chi_D(g) \cdot f, \quad \delta_A(f) = \langle d\rho_D(A)x, \text{grad}_x(f) \rangle = d\chi_D(A) \cdot f.$$

As $\deg f = n$ by (2.1) and $G_D \subseteq \text{GL}_V$ with $\dim V = n$ by Definition 2.1, we have

$$(2.7) \quad \deg \chi_D = n = \deg(\det \circ \rho_D).$$

Lemma 2.2. *For a prehomogeneous determinant D , $\chi_D = \det \circ \rho_D$ if and only if $\text{tr} \circ d\rho_D = 0$ on \mathfrak{a}_D .*

Proof. The converse being trivial, let us assume that the second condition holds. As $d\det = \text{tr}$, this gives using (2.2) that

$$\ker d(\det \circ \rho_D) \supseteq \mathfrak{a}_D = \ker d\chi_D, \quad \ker(\det \circ \rho_D) \supseteq A_D = (\ker \chi_D)^\circ.$$

Then both χ_D and $\det \circ \rho_D$ induce characters on G_D/A_D . But the latter has character group isomorphic to \mathbb{Z} by (2.3). Using (2.7), this implies the equality $\chi_D = \det \circ \rho_D$. □

Lemma 2.3. *If D is a linear free divisor or a reductive prehomogeneous determinant, one can choose $A_1 \in \mathfrak{g}_D$ with (2.4) such that*

$$(2.8) \quad \mathfrak{g}_D = \mathbb{C} \cdot A_1 \oplus \mathfrak{a}_D, \quad \mathfrak{a}_D = \langle A_2, \dots, A_n \rangle.$$

Proof. With $A_1 = E/n$, the first statement follows from (2.1) and Saito’s criterion.

By [Jac79, Ch. III, §7, Thm. 10], reductivity of \mathfrak{g}_D means that

$$(2.9) \quad \mathfrak{g}_D = \mathfrak{c}_D \oplus \mathfrak{s}_D$$

where \mathfrak{c}_D is the center of \mathfrak{g}_D , $d\rho_D(\mathfrak{c}_D)$ consists of semisimple endomorphisms, and \mathfrak{s}_D is a semisimple ideal. By semisimplicity of \mathfrak{s}_D , we have

$$(2.10) \quad \mathfrak{s}_D = [\mathfrak{s}_D, \mathfrak{s}_D].$$

Then it follows from (2.2) that

$$(2.11) \quad \mathfrak{s}_D = [\mathfrak{g}_D, \mathfrak{g}_D] \subseteq \mathfrak{a}_D$$

and hence $A_1 \notin \mathfrak{s}_D$ by (2.4). Using (2.9) and (2.11), this shows that $A_1 \in \mathfrak{c}_D$ after subtracting a suitable element of \mathfrak{s}_D . □

In order to say more, we shall focus on the reductive case for the remainder of this section. The following result is proved in [dGMS09, Cor. 2.9] in the case of linear free divisors.

Theorem 2.4. *Let D be a reductive prehomogeneous determinant. Then*

$$(2.12) \quad \chi_D = \det \circ \rho_D, \quad d\chi_D = \text{tr} \circ d\rho_D,$$

which means that $A_D \subseteq \text{SL}_V$.

Proof. We resume the notation \mathfrak{c}_D and \mathfrak{s}_D from Lemma 2.3. By (2.11), $\mathfrak{s}_D \subseteq \mathfrak{a}_D$ and hence $\mathfrak{a}_D = \mathfrak{c}'_D \oplus \mathfrak{s}_D$ where \mathfrak{c}'_D is the center of \mathfrak{a}_D .

Now let \mathfrak{h}_D be a Cartan subalgebra of \mathfrak{s}_D . Recall that $A_1 \in \mathfrak{c}_D$ by Lemma 2.3. Pick generators $S_1 = A_1, S_2, \dots, S_s$ of $\mathfrak{c}_D + \mathfrak{h}_D$ such that $d\rho_D(S_1), \dots, d\rho_D(S_s)$ are semisimple and extend them to a basis N_1, \dots, N_r , $r = n - s$, of \mathfrak{a}_D such that N_1, \dots, N_r are $(\mathfrak{c}_D + \mathfrak{h}_D)$ -homogeneous and $d\rho_D(N_1), \dots, d\rho_D(N_r)$ are nilpotent. Then, by (the proof of) [GMNS09, Thm. 6.1(4)], we have

$$(2.13) \quad d\chi_D(S_i) = \text{tr} \circ d\rho_D(S_i) + \sum_{j=1}^r [S_i, N_j]/N_j, \quad i = 1, \dots, s.$$

This implies that $d\chi_D = \text{tr} \circ d\rho_D$ on \mathfrak{c}_D . But on \mathfrak{s}_D , both $d\chi_D$ and $\text{tr} \circ d\rho_D$ are zero by (2.11) and (2.10). Now the claim follows from Lemma 2.2. □

Our motivation to study the symmetry of b -functions of prehomogeneous determinants stems from M. Sato's fundamental theorem for prehomogeneous vector spaces [Kim03, Prop. 4.18]. We shall need the following result to show that it holds for reductive prehomogeneous determinants in Theorem 3.3.

Theorem 2.5. *Let D be a reductive linear free divisor.*

- (a) *The Lie algebra representation ρ_D is defined over \mathbb{Q} with respect to some $V_{\mathbb{Q}}$.*
- (b) *With respect to any $V_{\mathbb{Q}}$ as in (a), also D is defined over \mathbb{Q} .*
- (c) *With respect to any $V_{\mathbb{Q}}$ as in (a), also G_D is defined over \mathbb{Q} .*

For the convenience of the reader we give a proof of the following general fact which is probably well-known to specialists.

Lemma 2.6. *Every finite-dimensional representation V of a complex semisimple Lie algebra \mathfrak{s} is defined over \mathbb{Q} with respect to some $V_{\mathbb{Q}}$.*

Proof. Let \mathfrak{h} be a Cartan and \mathfrak{b} be a Borel subalgebra of \mathfrak{s} . By Chevalley's Normalization [Ser01, Ch. VI, §6, Thm. 11], \mathfrak{s} has a rational form $\mathfrak{s}_{\mathbb{Q}}$. By complete reducibility we may assume that V is an irreducible \mathfrak{s} -module. Let $V_{\omega} = U(\mathfrak{s}) \otimes_{\mathfrak{b}} L_{\omega}$ where $U(\mathfrak{s})$ denotes the universal enveloping algebra of \mathfrak{s} and L_{ω} is a complex one-dimensional \mathfrak{b} -module of weight $\omega \in \mathfrak{h}^*$. Then set $E_{\omega} = V_{\omega}/N_{\omega}$ where N_{ω} is generated by all \mathfrak{s} -modules strictly contained in V_{ω} . By [Ser01, Ch. VII, §3, Thm. 2], $V = E_{\omega}$ where ω is integer on the coroots in $\mathfrak{h}_{\mathbb{Q}}$ by [Ser01, Ch. VII, §4, Prop. 3(b)]. This implies that V_{ω} is a direct sum of rational weight spaces for $\mathfrak{h}_{\mathbb{Q}}$ and is thus defined over \mathbb{Q} . Then N_{ω} is the maximal \mathfrak{s} -module in V_{ω} that meets L_{ω} in zero. Choosing a \mathbb{Q} -basis of L_{ω} , V_{ω} also decomposes as a direct sum of \mathbb{Q} -vector spaces $U(\mathfrak{s}_{\mathbb{Q}}) \otimes_{\mathfrak{b}_{\mathbb{Q}}} L_{\omega, \mathbb{Q}}$ compatible with the above direct sum decomposition. Clearly N_{ω} is homogeneous with respect to both direct sums and hence defined over \mathbb{Q} . \square

Proof of Theorem 2.5. We show that ρ_D is defined over \mathbb{Q} when restricted to \mathfrak{a}_D , which implies (a) by (2.8) and $A_1 = E/n$. Then (b) follows by (2.1), and (c) as in the proof of [GMNS09, Lem. 2.2].

We resume the notation \mathfrak{c}'_D , \mathfrak{s}_D , and \mathfrak{h}_D from the proof of Theorem 2.4. As $\mathfrak{c}'_D \oplus \mathfrak{h}_D$ is commutative, its representation on V can be diagonalized. By Lemma 2.6, the representation of \mathfrak{s}_D on each \mathfrak{c}'_D -weight space of V is defined over \mathbb{Q} . So there is a \mathbb{Q} -subspace $V_{\mathbb{Q}}$ of V with respect to which $\rho_D|_{\mathfrak{s}_D}$ is defined over \mathbb{Q} and with respect to which $\rho_D(\mathfrak{c}'_D \oplus \mathfrak{h}_D)$ consists of semisimple endomorphisms (with possibly complex weights). As $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D$ is a Cartan subalgebra of \mathfrak{g}_D , it follows from [Ser01, Ch. III, §5, Thm. 3(b)] that it equals its centralizer. Then [Sai71, Lem. 1.4] shows that ρ_D is defined over \mathbb{Q} with respect to $V_{\mathbb{Q}}$ on

$\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D$. The map $A \mapsto A - d\chi_D(A) \cdot A_1$ from $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D$ to $(\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D \oplus \mathfrak{h}_D) \cap \mathfrak{a}_D = \mathfrak{c}'_D \oplus \mathfrak{h}_D$ preserves the property of having eigenvalues in \mathbb{Q} . Thus, ρ_D is defined over \mathbb{Q} with respect to $V_{\mathbb{Q}}$ also on $\mathfrak{c}'_D \oplus \mathfrak{h}_D$ and hence on \mathfrak{a}_D . Remark that once ρ_D is defined over \mathbb{Q} on \mathfrak{g}_D , this also holds on the center $\mathbb{C} \cdot A_1 \oplus \mathfrak{c}'_D$ of \mathfrak{g}_D . \square

§3. b-functions of reductive prehomogeneous determinants

The (global) *b-function* of $f \in \mathbb{C}[x]$ is the minimal monic polynomial $B_f(s) \in \mathbb{C}[s]$ in a functional equation

$$(3.1) \quad B_f(s) \cdot f^s \in D_V[s] \cdot f^{s+1}$$

where D_V denotes the ring of algebraic differential operators on V . In general, the existence of the *b-function* has been established by I. N. Bernstein [Ber72] in the global case and by J.-E. Björk [Bjö79] in the local case. M. Kashiwara [Kas77] proved that the roots of the *b-function* are negative rational numbers.

Let (G, ρ, V) be a reductive prehomogeneous vector space. Then there is a compact Zariski dense subgroup $H \subseteq G$ and one can assume that $H \subseteq U(n)$ for some basis x of V . We refer to this basis x and the dual basis y of V^* as *unitary coordinates*. For a relative invariant f of (G, ρ, V) with character χ_f , the function $f^* \in \text{Sym}(V)$ defined by

$$(3.2) \quad f^*(y) = \overline{f(\bar{y})}$$

is then a relative invariant of (G, ρ^*, V^*) with character

$$(3.3) \quad \chi_{f^*} = \chi_f^{-1}$$

by the unitarian trick (see [Kim03, §2.3]). In [Kim03, Prop. 2.22 and 2.23], this is used to show that f^* defines functional equations

$$(3.4) \quad \begin{aligned} f^*(\partial_x) \cdot f^{s+1}(x) &= b_f(s) \cdot f^s(x), \\ f(\partial_y) \cdot (f^*)^{s+1}(y) &= b_{f^*}(s) \cdot (f^*)^s(y), \end{aligned}$$

$$(3.5) \quad b_f(s) = b_{f^*}(s),$$

as in (3.1) and that

$$(3.6) \quad \deg b_f(s) = \deg f = n.$$

From this one deduces that (G, ρ, V) is a regular prehomogeneous vector space and that any relative invariant f defining the discriminant is nondegenerate [Kim03,

Prop. 2.24]. In particular, (G, ρ^*, V^*) is again a reductive and hence a regular prehomogeneous vector space. It is shown in [Gyo91, Cor. 2.5.10] that

$$(3.7) \quad b_f(s) = B_f(s), \quad b_{f^*}(s) = B_{f^*}(s).$$

Now let D be a (not necessarily reductive) prehomogeneous determinant defined by f as in Definition 2.1. Then we abbreviate

$$B_D(s) = B_f(s), \quad b_D(s) = b_f(s),$$

where the latter is defined in the presence of the functional equations (3.4).

Definition 3.1. Let D be a prehomogeneous determinant. Consider $G_{D^*} = \rho_D^*(G_D) \subseteq \mathrm{GL}_{V^*}$ and $\rho_{D^*} = \mathrm{id}: G_{D^*} \hookrightarrow \mathrm{GL}_{V^*}$. Define D^* and f^* as D and f in Definition 2.1 with $(G, \rho, V) = (G_{D^*}, \rho_{D^*}, V^*)$. We call D^* the *dual determinant* of D . Note that D^* is a prehomogeneous determinant exactly if $f^* \neq 0$ or $D^* \neq V^*$.

For $A \in \mathfrak{g}_D$, we abbreviate

$$(3.8) \quad \delta_A^* = \langle d\rho_D^*(A)y, \partial_y \rangle = -y^t A \partial_y, \quad \delta_k = \delta_{A_k},$$

as in (2.5) using (2.4) and (2.8). Note that (2.6) also holds for f, D, δ_A replaced by f^*, D^*, δ_A^* .

The following statement is [dGMS09, Prop. 3.7] in the case of linear free divisors.

Proposition 3.2. *Together with D also D^* is a reductive prehomogeneous determinant (or a reductive linear free divisor) defined by f^* in (3.2).*

Proof. In unitary coordinates, we have $\rho_D^*(g) = (\rho_D(g)^{-1})^t = \overline{\rho_D(g)}$ for $g \in H$ and hence $d\rho_D^*(A) = -d\rho_D(A)^t = \overline{d\rho_D(A)}$ for $A \in \mathfrak{h}$ where \mathfrak{h} denotes the Lie algebra of H . As \mathfrak{h} spans \mathfrak{g} over \mathbb{C} , it follows that f^* of Definition 3.1 coincides with f^* of (3.2). This shows that $f^* \neq 0$ and that f^* is reduced if and only if f is reduced. \square

If D is a reductive prehomogeneous determinant, the two (equivalent) equalities

$$(3.9) \quad \begin{aligned} \chi_{D^*} \circ \rho_D^* &= \chi_{f^*} = \chi_f^{-1} = \chi_D^{-1}, \\ d\chi_{D^*} \circ d\rho_D^* &= d\chi_{f^*} = -d\chi_f = -d\chi_D \end{aligned}$$

follow from Proposition 3.2 and (3.3).

The following observation was the starting point for our study of the symmetry of b -functions of prehomogeneous determinants in the remainder of this section.

Theorem 3.3. *M. Sato’s fundamental theorem for homogeneous vector spaces as formulated in [Kim03, Prop. 4.18] holds true for reductive linear free divisors.*

Proof. The hypotheses for the proof of this theorem in [Kim03, §4.1] are fulfilled by Theorem 2.5(c) except for the requirement that f be irreducible. But this latter assumption is used in the proof only to ensure that f and f^* can be chosen to be defined over \mathbb{R} (see [Kim03, Prop. 4.1(1), Prop. 4.2]) and that $|\det \rho(g)| = |\chi_f(g)|^{n/d}$ for all $g \in G$ where $d = \deg f$ (see [Kim03, p. 119]). But, by Theorems 2.4 and 2.5(b), these statements hold true for any reductive linear free divisor. Indeed, it follows from Theorem 2.5(a) that the Lie algebra representation $d\rho_{D^*} : \mathfrak{g}_{D^*} = d\rho_D^*(\mathfrak{g}_D) \hookrightarrow \mathfrak{gl}_{V^*}$ is defined over \mathbb{Q} with respect to $V_{\mathbb{Q}}^*$. Then the defining equation f^* of the dual linear free divisor D^* is defined over \mathbb{Q} with respect to $V_{\mathbb{Q}}^*$ by Theorem 2.5(b). Note also that the nondegeneracy of f and f^* used in [Kim03, Cor. 4.4] follows from [Kim03, Prop. 2.24] even in the reducible case. \square

A symmetry property of the b -function as in Lemma 3.4 below is a corollary of Sato’s fundamental theorem (see [Kim03, Prop. 4.19]) for irreducible reductive prehomogeneous vector spaces. For reductive prehomogeneous determinants, we shall give an independent proof which is more elementary in this setting. Our argument also applies to regular linear free divisors (see Section 5).

Lemma 3.4. *Let D be a prehomogeneous determinant with $D^* \neq V^*$ such that (3.4), (3.5), (2.12), and (3.9) hold true. Then its b -function satisfies*

$$b_D(s) = (-1)^n \cdot b_D(-s - 2).$$

Proof. From (2.6), (2.12), and (3.9) we deduce that

$$(3.10) \quad \begin{aligned} Q_A &= \delta_A - s \cdot \text{tr}(A) = \delta_A - s \cdot \delta_A(f)/f \in \text{Ann}_{D_V[s]} f^s, \\ Q_A^* &= \delta_A^* + s \cdot \text{tr}(A) = \delta_A^* - s \cdot \delta_A^*(f^*)/f^* \in \text{Ann}_{D_{V^*}[s]} (f^*)^s, \end{aligned}$$

for $A \in \mathfrak{g}_D$ when identifying $A = d\rho_D(A)$. So by (3.4) and (3.10), we have

$$(3.11) \quad P = f^*(\partial_x) \cdot f(x) - b_D(s) \in \text{Ann}_{D_V[s]} f^s \ni Q_{A_k} = Q_k$$

for $k = 1, \dots, n$. Now let \mathcal{D}_V denote the sheaf of algebraic differential operators on V . Then, at any point $p \in U = V \setminus D$, we have

$$(3.12) \quad \text{Ann}_{\mathcal{D}_{V,p}[s]} f^s = \langle Q_1, \dots, Q_n \rangle_{\mathcal{D}_{V,p}[s]}.$$

Indeed, as $f \in \mathcal{O}_{V,p}^*$, conjugating by f^s reduces this statement to $\text{Ann}_{\mathcal{D}_{V,p}[s]} 1 = \langle \delta_1, \dots, \delta_n \rangle_{\mathcal{D}_{V,p}[s]}$, which holds true since $\langle \delta_1, \dots, \delta_n \rangle_{\mathcal{O}_{V,p}} = \langle \partial_{x_1}, \dots, \partial_{x_n} \rangle_{\mathcal{O}_{V,p}}$. The

equality (3.12) gives rise to an exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{D}_U[s]^n \xrightarrow{\langle Q_1, \dots, Q_n \rangle} \text{Ann}_{\mathcal{D}_U[s]} f^s \rightarrow 0.$$

As U is affine and the sheaf \mathcal{R} is quasicoherent, we have $H^1(U, \mathcal{R}) = 0$ (see [Meb89, Prop. 2.5.4] and [Har77, Thm. 3.5]) and this implies together with (3.11) that

$$(3.13) \quad f^m \cdot P \in \langle Q_1, \dots, Q_n \rangle_{D_V[s]}$$

for some $m \in \mathbb{N}$. Let \mathcal{F} denote the Fourier transform. Then, by (2.5), (3.8), and (3.10), we have

$$(3.14) \quad \begin{aligned} \mathcal{F}(Q_A(s)) &= \mathcal{F}(x^t A^t \partial_x - s \cdot \text{tr}(A)) = -\partial_y^t A^t y - s \cdot \text{tr}(A) \\ &= -y^t A \partial_y - (s + 1) \cdot \text{tr}(A) = \delta_A^* + (-s - 1) \cdot \text{tr}(A) \\ &= Q_A^*(-s - 1). \end{aligned}$$

Combining (3.10), (3.13), and (3.14) yields

$$\begin{aligned} \text{Ann}_{D_{V^*}[s]}(f^*)^{-s-1} \ni \mathcal{F}(f^m \cdot P) &= \mathcal{F}(f(x))^m \cdot \mathcal{F}(f^*(\partial_x) \cdot f(x) - b_D(s)) \\ &= (-1)^{nm} f(\partial_y)^m \cdot ((-1)^n \cdot f^*(y) \cdot f(\partial_y) - b_D(s)) \end{aligned}$$

and hence, using (3.4) and (3.5),

$$\begin{aligned} 0 &= f(\partial_y)^m \cdot ((-1)^n \cdot f^*(y) \cdot f(\partial_y) - b_D(s)) \cdot (f^*)^{-s-1} \\ &= f(\partial_y)^m \cdot ((-1)^n \cdot b_{D^*}(-s - 2) - b_D(s)) \cdot (f^*)^{-s-1} \\ &= ((-1)^n \cdot b_{D^*}(-s - 2) - b_D(s)) \cdot f(\partial_y)^m \cdot (f^*)^{-s-1} \\ &= ((-1)^n \cdot b_D(-s - 2) - b_D(s)) \cdot b_{D^*}(-s - 2) \cdots b_{D^*}(-s - m - 1) \cdot (f^*)^{-s-m-1}. \end{aligned}$$

Thus, $(-1)^n \cdot b_D(-s - 2) - b_D(s) = 0$ as claimed. □

We deduce the following from Lemma 3.4 using Theorem 2.4, and from (3.6), (3.7), and [Kas77].

Theorem 3.5. *For any reductive prehomogeneous determinant $D \subseteq V$ in dimension $\dim V = n$, the b -function $B_D(s) \equiv b_D(s)$ has degree n and rational negative roots symmetric about -1 . In particular, -1 is the only integer root.*

Remark 3.6. In [CU04, Prob. 3.2], Castro-Jiménez and Ucha-Enríquez ask whether -1 is the only integer root of the b -function for any linear free divisor. Theorem 3.5 gives a positive answer to this question in the case of reductive prehomogeneous determinants. In Theorem 5.5 in Section 5, we shall also settle the case of certain nonreductive linear free divisors.

§4. Examples

Up to dimension 4 all linear free divisors have been classified in [GMNS09, §6.4]. Table 1 lists their b -functions computed using Macaulay 2 [M2]. Here \mathfrak{g}_2 is the non-Abelian Lie algebra of dimension 2 and \mathfrak{g}_3 is characterized as having two-dimensional Abelian derived algebra \mathfrak{g}'_3 , on which the adjoint action of a basis vector outside \mathfrak{g}'_3 is semisimple with eigenvalues 1 and 2 (see [Jac79, Ch. I, §4]). Observe that, except for having degree greater than n , also the nonreductive divisors in Table 1 satisfy the statement of Theorem 3.5, which supports Conjecture 1.5.

Table 1: b -functions of linear free divisors in dimension $n \leq 4$

n	f	\mathfrak{g}_D	reductive?	Spec $b(s)$
1	x	\mathbb{C}	Yes	-1
2	xy	\mathbb{C}^2	Yes	$-1, -1$
3	xyz	\mathbb{C}^3	Yes	$-1, -1, -1$
3	$(y^2 + xz)z$	\mathfrak{b}_2	No	$-\frac{5}{4}, -1, -1, -\frac{3}{4}$
4	$xyzw$	\mathbb{C}^4	Yes	$-1, -1, -1, -1$
4	$(y^2 + xz)zw$	$\mathbb{C} \oplus \mathfrak{b}_2$	No	$-\frac{5}{4}, -1, -1, -1, -\frac{3}{4}$
4	$(yz + xw)zw$	$\mathbb{C}^2 \oplus \mathfrak{g}_2$	No	$-\frac{4}{3}, -1, -1, -1, -\frac{2}{3}$ $-\frac{7}{5}, -\frac{4}{3}, -\frac{6}{5}$
4	$x(y^3 - 3xyz + 3x^2w)$	$\mathbb{C} \oplus \mathfrak{g}_3$	No	$-1, -1, -1,$ $-\frac{4}{5}, -\frac{2}{3}, -\frac{3}{5}$
4	$y^2z^2 - 4xz^3 - 4y^3w + 18xyzw - 27w^2x^2$	$\mathfrak{gl}_2(\mathbb{C})$	Yes	$-\frac{7}{6}, -1, -1, -\frac{5}{6}$

Discriminants of quiver representations are at the origin of linear free divisors and form a rich source of reductive determinants. We shall briefly review the basics of the theory to understand the following three examples and refer to [BM06] for more details.

A quiver $Q = (Q_0, Q_1)$ is a directed finite graph with vertex set Q_0 and edge set $Q_1 \subseteq Q_0 \times Q_0$. With a dimension vector $\mathbf{d} = (d_i) \in \mathbb{N}^{Q_0}$ we associate the representation space

$$\text{Rep}(Q, \mathbf{d}) = \prod_{(i,j) \in Q_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

The quiver group

$$\text{GL}(Q, \mathbf{d}) = \prod_{i \in Q_0} \text{GL}_{d_i}(\mathbb{C})$$

acts on $\text{Rep}(Q, \mathbf{d})$ through the quiver representation

$$\begin{aligned} \rho_{Q, \mathbf{d}}: \text{GL}(Q, \mathbf{d}) &\rightarrow \text{GL}(\text{Rep}(Q, \mathbf{d})), \\ \rho_{Q, \mathbf{d}}((g_i)_{i \in Q_0})(\varphi_{(i,j)})_{(i,j) \in Q_1} &= (g_j \circ \varphi_{(i,j)} \circ g_i^{-1})_{(i,j) \in Q_1}. \end{aligned}$$

Note that $Z = \mathbb{C} \cdot (I_{d_i})_{i \in Q_0}$, where $I_{d_i} \in \text{GL}_{d_i}(\mathbb{C})$ denotes the unit matrix, lies in the kernel of $\rho_{Q,\mathbf{d}}$. The Tits form $q: \mathbb{C}^{Q_0} \rightarrow \mathbb{C}$ is defined by

$$q((x_i)_{i \in Q_0}) = \sum_{i \in Q_0} x_i^2 - \sum_{(i,j) \in Q_1} x_i x_j.$$

The condition $\dim G = \dim V$ in Definition 2.1 for the discriminant D of $(G, \rho, V) = (\text{GL}(Q, \mathbf{d})/Z, \rho_{Q,\mathbf{d}}, \text{Rep}(Q, \mathbf{d}))$ to define a prehomogeneous determinant is equivalent to $q(\mathbf{d}) = 1$ and reductivity is automatic if $D \neq V$ [GMNS09, §4].

The following example of a linear free divisor is the discriminant of the star quiver studied in [GMNS09, Example 5.3]. We use the microlocal calculus developed in [SKKØ81] to determine its b -function and then Macaulay 2 [M2] to verify our result.

Example 4.1. Consider the star quiver Q shown in Figure 1 with one sink, three sources, and dimension vector $\mathbf{d} = (2, 1, 1, 1)$ with $q(\mathbf{d}) = 1$. The quiver group

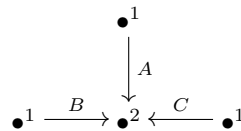


Figure 1. The star quiver in Example 4.1

$G = \text{GL}(Q, \mathbf{d}) = \text{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^3 \subseteq \text{GL}_2(\mathbb{C}) \times \text{GL}_3(\mathbb{C})$ acts on the 6-dimensional representation space $\text{Rep}(Q, \mathbf{d}) = \mathbb{C}^{2 \times 3} = \{(A \ B \ C)\}$ of 2×3 -matrices through $\rho = \rho_{Q,\mathbf{d}}$ given by

$$\rho(g)x = g_1 \cdot x \cdot g_2^{-1}, \quad g = (g_1, g_2).$$

The infinitesimal vector fields for this group action are given by

$$(4.1) \quad \xi_{i,j} = x_{i,1} \partial_{j,1} + x_{i,2} \partial_{j,2} + x_{i,3} \partial_{j,3}, \quad \xi_k = x_{1,k} \partial_{1,k} + x_{2,k} \partial_{2,k}$$

where $x_{i,j}$ are coordinates on $\mathbb{C}^{2 \times 3}$ and $\partial_{i,j} = \partial_{x_{i,j}}$. The discriminant is a reductive linear free divisor D defined by the product of maximal minors

$$f(x) = (x_{1,2}x_{2,3} - x_{2,2}x_{1,3})(x_{1,1}x_{2,3} - x_{2,1}x_{1,3})(x_{1,1}x_{2,2} - x_{2,1}x_{1,2})$$

with associated character

$$(4.2) \quad \chi(g) = \det(g_1)^3 \det(g_2)^{-2}.$$

We first enumerate the different orbits by specifying a generic element x_0 . We denote by $O_{i,j}$ (or O_i if there is only one) an orbit of dimension i of a point $x_0^{i,j}$

(or x_0^j), and by $\Lambda_{i,j}$ (or Λ_i) its conormal space. In order to calculate the dimension of the orbit of x_0 it is sufficient to determine the tangent space $T_{x_0} = d\rho(\mathfrak{g})x_0$ which is spanned by the evaluation of the vector fields (4.1) at x_0 .

1. Orbits of rank 2:

(1.a) One open orbit:

$$x_0^6 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(1.b) Three orbits of rank 2 and dimension 5 with all columns nonzero:

$$x_0^{5,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad x_0^{5,2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad x_0^{5,3} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(1.c) Three orbits of rank 2 and dimension 4 with exactly one zero column:

$$x_0^{4,1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_0^{4,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_0^{4,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

2. Orbits of rank 1:

(2.a) One orbit of rank 1 and dimension 4:

$$x_0^{4,0} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2.b) Three orbits of rank 1 and dimension 3:

$$x_0^{3,1} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_0^{3,2} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_0^{3,3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2.c) Three orbits of rank 1 and dimension 2:

$$x_0^{2,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_0^{2,2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_0^{2,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. One orbit of rank 0: $x_0^0 = 0$.

For each of the above points $x_0 = x_0^{i,j}$, we shall now compute the action ρ_{x_0} of the isotropy group G_{x_0} on the normal space $V_{x_0} = V/T_{x_0}$ to the tangent space T_{x_0} to the orbit of x_0 . We omit the trivial cases of the open orbit and of the zero orbit and describe V_{x_0} by identifying it with a representative subspace in V .

1. $x_0 = x_0^{5,1}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \times (a, 1, 1) \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{2,1}, \partial_{2,2}, \partial_{2,3}, \partial_{1,2} + \partial_{1,3} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = a \cdot x.$$

2. $x_0 = x_0^{4,3}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \times (1, a, b) \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{1,2}, \partial_{2,1}, \partial_{2,2} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = \frac{1}{b} \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & av \end{pmatrix}.$$

3. $x_0 = x_0^{4,0}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{1,2}, \partial_{1,3}, \partial_{2,1} + \partial_{2,2} + \partial_{2,3} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & u & v \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = b \cdot x.$$

This is the only non-prehomogeneous normal action.

4. $x_0 = x_0^{3,3}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \times (1, 1, c) \right\}, \quad T_{x_0} = \langle \partial_{1,1}, \partial_{1,2}, \partial_{2,1} + \partial_{2,2} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & 0 & u \\ 0 & v & w \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = \begin{pmatrix} 0 & (u + aw)/c \\ bv & bw/c \end{pmatrix}.$$

This is a prehomogeneous action with nonreduced discriminant $\{vw^2 = 0\}$.

5. $x_0 = O_{2,1}$:

$$G_{x_0} = \left\{ g = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \times (1, c, d) \right\}, \quad T_{x_0} = \langle \partial_{11}, \partial_{21} \rangle,$$

$$V_{x_0} = \left\{ x = \begin{pmatrix} 0 & r & s \\ 0 & t & u \end{pmatrix} \right\}, \quad \rho_{x_0}(g)x = \begin{pmatrix} 0 & (r + at)/c & (s + au)/d \\ 0 & bt/c & bu/d \end{pmatrix}$$

This is a prehomogeneous action with reduced discriminant $\{tu(ur - st) = 0\}$.

The holonomy diagram in Figure 2 shows the incidence relations between the good orbit conormals subject to intersections in codimension one. Recall that $\Lambda_{4,0}$ is not good as it is not prehomogeneous. We shall apply the method of [SKKÖ81] to the

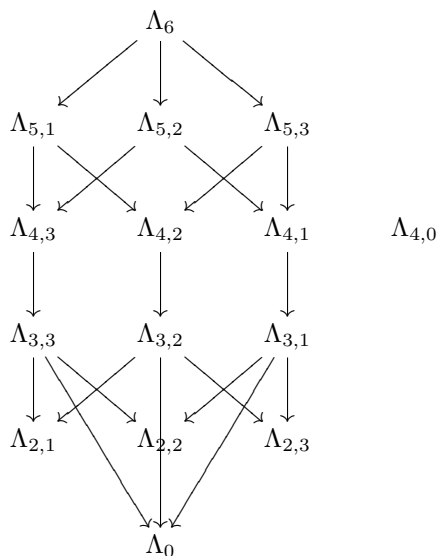


Figure 2. Holonomy diagram of Example 4.1

chain

$$(4.3) \quad \Lambda_6 \rightarrow \Lambda_{5,1} \rightarrow \Lambda_{4,3} \rightarrow \Lambda_{3,3} \rightarrow \Lambda_0.$$

An elementary local calculation shows that all the intersections in this chain are smooth and transversal at the generic point, which is the situation of [SKKØ81, Cor. 7.6]. Therefore it remains to calculate $\text{ord}_\Lambda f^s$ for all Λ in the chain (4.3) using the following formula from [SKKØ81, Prop. 4.14]:

$$(4.4) \quad \text{ord}_\Lambda f^s = -m_\Lambda s - \frac{\mu_\Lambda}{2} = d\chi(A_0) - \text{tr}_{V_{x_0}^*} d\rho_{x_0}^*(A_0) + \frac{1}{2} \dim V_{x_0}^*.$$

Here $A_0 = (A_0^1, A_0^2) \in \mathfrak{g}_{x_0}$ and $(x_0, y_0) \in \Lambda$ is generic such that $d\rho_{x_0}(A_0)y_0 = y_0$. By (4.2), we have

$$d\chi(A_0) = 3 \cdot \text{tr}(A_0^1) - 2 \cdot \text{tr}(A_0^2), \quad \text{tr}_{V_{x_0}^*} d\rho_{x_0}^*(A_0) = -\text{tr}_{V_{x_0}} d\rho_{x_0}(A_0),$$

which can be determined from the preceding computations. Note that the contra-gradient action ρ^* on the dual space $(\mathbb{C}^{2 \times 3})^*$ identified with $\mathbb{C}^{3 \times 2}$ by the trace pairing is given by

$$d\rho^*(g)\xi = g_2 \xi g_1^{-1}, \quad g = (g_1, g_2).$$

One computes

$$\begin{aligned} \text{ord}_{\Lambda_6} f^s &= 0, & \text{ord}_{\Lambda_{5,1}} f^s &= -s - \frac{1}{2}, & \text{ord}_{\Lambda_{4,3}} f^s &= -2s - 1, \\ \text{ord}_{\Lambda_{3,3}} f^s &= -5s - \frac{5}{2}, & \text{ord}_{\Lambda_0} f^s &= -6s - 3. \end{aligned}$$

Using the formula in [SKKØ81, Cor. 7.6] this yields

$$\begin{aligned} \frac{b_{\Lambda_{5,1}}}{b_{\Lambda_6}} &= s + 1, & \frac{b_{\Lambda_{4,3}}}{b_{\Lambda_{5,1}}} &= -s - \frac{1}{2} + 2s + 1 + \frac{1}{2} = s + 1, \\ \frac{b_{\Lambda_{3,3}}}{b_{\Lambda_{4,3}}} &= (3s + 2)(3s + 3)(3s + 4), & \frac{b_{\Lambda_0}}{b_{\Lambda_{3,3}}} &= -5s - \frac{5}{2} + 6s + 3 + \frac{1}{2} = s + 1. \end{aligned}$$

From this we conclude that

$$B_D(s) = b_{\Lambda_0}(s) = \left(s + \frac{2}{3}\right)(s + 1)^4 \left(s + \frac{4}{3}\right)$$

is the b -function of D , which is also confirmed by Macaulay 2 [M2].

The next two examples are reductive nonreduced prehomogeneous determinants arising from quiver representations.

Example 4.2. We modify the star quiver of Example 4.1 by adding an additional source. Consider the resulting star quiver \tilde{D}_3 in Figure 3 and the dimension vector

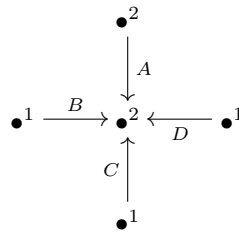


Figure 3. The star quiver in Example 4.2

$\mathbf{d} = (2, 2, 1, 1, 1)$ with $q(\mathbf{d}) = 1$. The quiver group $\text{GL}(Q, \mathbf{d}) = \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^3$ acts on a 10-dimensional representation space $\text{Rep}(Q, \mathbf{d}) = \mathbb{C}^{2 \times 2} \times \mathbb{C}^{3 \times 2} = \{(A, (B \ C \ D))\}$ and the orbit of $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\right)$ is open. The nonreduced discriminant D defined by the function

$$\begin{aligned} f(x) &= (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})^2 \cdot (x_{1,3}x_{2,4} - x_{1,4}x_{2,3}) \cdot (x_{1,3}x_{2,5} - x_{1,5}x_{2,3}) \\ &\quad \cdot (x_{1,4}x_{2,5} - x_{1,5}x_{2,4}) \end{aligned}$$

is a reductive prehomogeneous determinant. It is a union $D = D_1 \cup D_2$ where D_1 and D_2 are defined by the functions

$$(4.5) \quad \begin{aligned} f_1(x) &= (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})^2, \\ f_2(x) &= (x_{1,3}x_{2,4} - x_{1,4}x_{2,3}) \cdot (x_{1,3}x_{2,5} - x_{1,5}x_{2,3}) \cdot (x_{1,4}x_{2,5} - x_{1,5}x_{2,4}). \end{aligned}$$

Since D_1 is the square of a quadratic form and D_2 is the reductive linear free divisor from Example 4.1, we have

$$\begin{aligned} B_{D_1}(s) &= (2s + 1)(s + 1)^2(2s + 3), \\ B_{D_2}(s) &= \left(s + \frac{2}{3}\right)(s + 1)^4\left(s + \frac{4}{3}\right). \end{aligned}$$

As the functions in (4.5) depend on separate sets of variables, $B_D(s)$ divides $B_{D_1}(s) \cdot B_{D_2}(s)$ and hence

$$B_D = B_{D_1} \cdot B_{D_2} = \left(s + \frac{1}{2}\right)\left(s + \frac{2}{3}\right)(s + 1)^6\left(s + \frac{4}{3}\right)\left(s + \frac{3}{2}\right)$$

as $\deg B_D = 10 = \deg B_{D_1} + \deg B_{D_2}$ by Theorem 3.5. For the reduced discriminant D_{red} , we lose the symmetry property. Indeed, one computes

$$B_{D_{\text{red}}}(s) = \left(s + \frac{2}{3}\right)(s + 1)^5\left(s + \frac{4}{3}\right)(s + 2)$$

using the same degree argument.

Example 4.3. Consider the quiver representation associated with the quiver \tilde{A}_n in Figure 4 and the dimension vector $\mathbf{d} = (2, 1, \dots, 1)$ with $q(\mathbf{d}) = 1$. The repre-

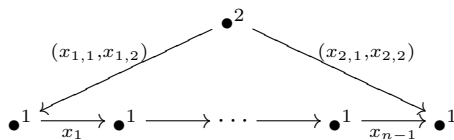


Figure 4. The quiver in Example 4.3

sentation space is $(n + 3)$ -dimensional and the orbit of

$$\left(\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}, (x_1, \dots, x_{n-1}) \right) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (1, \dots, 1) \right)$$

is open. The nonreduced discriminant D defined by the function

$$f(x) = \det \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}^2 \cdot x_1 \cdots x_{n-1} = (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})^2 \cdot x_1 \cdots x_{n-1}$$

is a reductive prehomogeneous determinant. As in Example 4.2, one computes

$$B_D(s) = \left(s + \frac{1}{2}\right)(s + 1)^{n+1} \left(s + \frac{3}{2}\right),$$

$$B_{D_{\text{red}}}(s) = (s + 1)^n (s + 2),$$

and the reduced discriminant does not have the symmetry property.

§5. *b*-functions of special linear free divisors

The symmetry of the *b*-functions of nonreductive prehomogeneous determinants in Section 4 motivates the attempt to weaken the hypotheses of Theorem 3.5. In the case of linear free divisors, this turns out to be possible.

Consider a linear free divisor *D* defined by *f* as in Definition 2.1. In [GMNS09, Thm. 6.1], we proved that \mathfrak{a}_D has a basis $A_2, \dots, A_n = S_2, \dots, S_s, N_1, \dots, N_{n-r}$ as in the proof of Theorem 2.4: the $d\rho_D(S_i)$ are semisimple and commute, the $d\rho_D(N_j)$ are nilpotent, and the N_j are homogeneous with respect to the S_i with rational weights. Then, as in (2.13), we have

$$(5.1) \quad d\chi_f(A_i) = \text{tr} \circ d\rho_D(A_i) + \sum_{j=1}^n [A_i, A_j]/A_j, \quad i = 1, \dots, s.$$

Assume now that $D^* \neq V^*$. Then the analog of (5.1) for D^* reads

$$(5.2) \quad d\chi_{f^*}(A_i) = \text{tr} \circ d\rho_D^*(A_i) + \sum_{j=1}^n [A_i, A_j]/A_j, \quad i = 1, \dots, s.$$

As $\text{tr} \circ d\rho_D^*(A_i) = -\text{tr} \circ d\rho_D(A_i)$, we obtain the equalities

$$(5.3) \quad d\chi_f(A_i) - d\chi_{f^*}(A_i) = 2\text{tr} \circ d\rho_D(A_i), \quad i = 1, \dots, n,$$

as the difference of (5.1) and (5.2) for $i = 1, \dots, s$. Note that both sides of (5.3) are zero for $i = s + 1, \dots, n$. From (5.3), we deduce the following formulas.

Lemma 5.1. *For a linear free divisor *D* with $D^* \neq V^*$,*

$$\chi_f \cdot \chi_{f^*}^{-1} = \det^2 \circ \rho_D, \quad d\chi_f - d\chi_{f^*} = 2\text{tr} \circ d\rho_D. \quad \square$$

As an immediate consequence of Lemma 5.1, (2.12) and (3.9) are equivalent.

Proposition 5.2. *For a linear free divisor *D* with $D^* \neq V^*$, we have $\chi_D = \det \circ \rho_D$ if and only if $\chi_{f^*} = \chi_f^{-1}$.* □

A relation of (3.4), (3.5), and (3.9) can be established for general prehomogeneous determinants.

Proposition 5.3. *Let D be a prehomogeneous determinant with $D^* \neq V^*$. Then (3.4) is equivalent to (3.9) and implies (3.5).*

Proof. The classical proof essentially works in our setting:

The existence of $b_f(s)$ and $b_{f^*}(s)$ satisfying (3.4) follows from (3.9) by the character argument of [Kim03, Prop. 2.22], which also gives the converse implication.

In the reductive case,

$$f(x) = \sum_{|\alpha|=n} f_\alpha \cdot x^\alpha, \quad f^*(y) = \sum_{|\alpha|=n} f_\alpha^* \cdot x^\alpha$$

have complex conjugate coefficients $f_\alpha = \overline{f_\alpha^*}$ in unitary coordinates. This is not necessary to show that

$$b_f(0) = f^*(\partial_x) \cdot f(x) = \sum_{|\alpha|=n} f_\alpha \cdot f_\alpha^* \cdot \alpha! = f(\partial_y) \cdot f^*(y) = b_{f^*}(0)$$

and similarly

$$b_f(0) \cdots b_f(m-1) = f^*(\partial_x)^m \cdot f(x)^m = f(\partial_y)^m \cdot f^*(y)^m = b_{f^*}(0) \cdots b_{f^*}(m-1)$$

for any $m \in \mathbb{N}$. The equality in (3.5) follows as in [Kim03, Prop. 2.23]. □

In [dGMS09, Def. 2.1], a linear free divisor is called *special* if (2.12) holds true. Using this terminology, we deduce from Lemma 3.4 and Propositions 5.2 and 5.3 the following result analogous to Theorem 3.5.

Theorem 5.4. *Let $D \subseteq V$ be a special linear free divisor in dimension $\dim V = n$ with $D^* \neq V^*$. Then the b-function $b_D(s)$ has degree at most n and roots symmetric about -1 .* □

Under the stronger assumption of regularity we can prove more.

Theorem 5.5. *Let $D \subseteq V$ be a regular special linear free divisor in dimension $\dim V = n$. Then the b-function $B_D(s) \equiv b_D(s)$ has degree at most n and negative rational roots symmetric about -1 . In particular, -1 is the only integer root.*

Proof. Note that the hypothesis $D^* \neq V^*$ in Theorem 5.4 becomes redundant for regular D by [Kim03, Thm. 2.16].

By [SKK081, Prop. 4.6], regularity of D is equivalent to $V^* = T_0^*V$ being a good holonomic variety. Under this latter hypothesis the arguments in [Gyo91, Lem. 2.5.7] still yield (3.7). □

Note that in Theorem 5.5 regularity of f is equivalent to $\deg b_D(s) = n$. Indeed, the degree n coefficient of $b_f(s)$ is given by $f^*(\text{grad log } f) \cdot f$ by [Kim03, Prop. 2.13(2)] and $\text{grad log } f: V \rightarrow V^*$ is a G -equivariant map by [Kim03, Prop. 2.13(1)].

§6. Euler homogeneity and normal representations

The condition of strong Euler homogeneity arose in the context of the logarithmic comparison theorem in [CMNCJ02, Conj. 1.4] (see also [CNM96] and [GS06]) and has been studied in [GMNS09, §7] in the case of linear free divisors. It is conceivable that all linear free divisors are strongly Euler homogeneous, but there is neither a proof nor a counter-example.

Definition 6.1. Let $D \subseteq V$ be a reduced hypersurface. A (local) vector field $\epsilon \in \text{Der}(-\log D)$ is called an *Euler vector field* for D if $\epsilon(f) = f$ for some (local) defining equation f of D . It is called an *Euler vector field at* $p \in D$ if in addition $\epsilon(p) = 0$. The hypersurface D is called *strongly Euler homogeneous at* $p \in D$ if there is an Euler vector field at p defined locally at p . By *strong Euler homogeneity of* D we mean that this property holds for all $p \in D$.

The following statement is implicit in the proof of [GMNS09, Lem. 7.5].

Lemma 6.2. *If D is (locally) defined by f and Euler homogeneous with Euler vector field ϵ then D is strongly Euler homogeneous if $\epsilon(p) \in \text{Der}(-\log f)(p)$ for all $p \in D$.*

Assume now that D is a linear free divisor. Then the preceding observation can be formulated as follows.

Lemma 6.3. *Strong Euler homogeneity of a linear free divisor D means that the G_D -orbits and A_D -orbits in D coincide.*

We shall now give a sufficient condition for strong Euler homogeneity of D at $x_0 \in D$. Let G_{D,x_0} be the stabilizer of x_0 in G_D with Lie algebra \mathfrak{g}_{D,x_0} and consider the representation $\rho_{D,x_0}: G_{D,x_0} \rightarrow V_{x_0}$ induced by ρ_D on the normal space $V_{x_0} = V/d\rho_D(\mathfrak{g}_D)x_0$ to the tangent space of the G_D -orbit of x_0 at x_0 .

The localization $f_{x_0} \in \text{Sym}(V^*)$ of f at x_0 [SKKÖ81, Def. 6.7] is the homogeneous nonzero polynomial defined by

$$f(x_0 + \varepsilon x') \equiv \varepsilon^k \cdot f_{x_0}(x') \pmod{\langle \varepsilon^{k+1} \rangle}.$$

By [SKKÖ81, Def. 6.8], $f_{x_0}(x')$ induces a relative invariant $f_{x_0}(y)$ of the representation $(G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$ with character $\chi_{f_{x_0}} = \chi_D$ where $y = y_1, \dots, y_k$ are

coordinates on V_{x_0} . Note however that the latter might not be a prehomogeneous vector space.

In the context of linear free divisors, Definition 2.1 applied to $(G, \rho, V) = (G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$, as in Definition 3.1, gives another natural candidate D_{x_0} for such an invariant.

Definition 6.4. Assume that A_1, \dots, A_k form a basis of \mathfrak{g}_{D,x_0} and define $D_{x_0} \subseteq V_{x_0}$ by

$$f'_{x_0}(y) = \det(d\rho_{D,x_0}(A_1)y, \dots, d\rho_{D,x_0}(A_k)y) \in \text{Sym}(V_{x_0}^*).$$

Then $f'_{x_0} \neq 0$ or $D_{x_0} \neq V_{x_0}$ is equivalent to $(G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$ being a prehomogeneous vector space.

Lemma 6.5. Let D be a linear free divisor and assume that $x_0 \in D$ and $D_{x_0} \neq V_{x_0}$. Then $f'_{x_0}(y) = f_{x_0}(y)$.

Proof. Note that $A_{k+1}x_0, \dots, A_nx_0$ form a basis of the tangent space $d\rho_D(\mathfrak{g}_D)x_0$ to the G_D -orbit of x_0 at x_0 . We may assume that $d\rho_D(\mathfrak{g}_D)x_0 = 0 \times \mathbb{C}^{n-k}$ and then that

$$(6.1) \quad (A_{k+1}x_0, \dots, A_nx_0) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

We may then choose $y = x'_1, \dots, x'_k$. Using (2.1) and (6.1), we compute

$$(6.2) \quad \begin{aligned} f(x_0 + \varepsilon x') &= \det(A_1(x_0 + \varepsilon x'), \dots, A_n(x_0 + \varepsilon x')) \\ &\equiv \varepsilon^k \cdot \det(A_1x', \dots, A_kx', A_{k+1}x_0, \dots, A_nx_0) \pmod{\langle \varepsilon^{k+1} \rangle}. \end{aligned}$$

If we denote by A'_i the matrix consisting of the upper k rows of A_i , (6.1) implies that

$$(6.3) \quad \begin{aligned} \det(A_1x', \dots, A_kx', A_{k+1}x_0, \dots, A_nx_0) &= \det(A'_1x', \dots, A'_kx') \\ &\equiv \det(d\rho_{D,x_0}(A_1)y, \dots, d\rho_{D,x_0}(A_k)y) = f'_{x_0}(y) \pmod{\langle x_{k+1}, \dots, x_n \rangle}. \end{aligned}$$

By hypothesis, $f'_{x_0} \neq 0$ and the claim follows from (6.2) and (6.3). □

Proposition 6.6. Let D be a linear free divisor and assume that $x_0 \in D$ and $D_{x_0} \neq V_{x_0}$. Then D is strongly Euler homogeneous at x_0 .

Proof. By assumption $(G_{D,x_0}, \rho_{D,x_0}, V_{x_0})$ is a prehomogeneous vector space and $f_{x_0}(y)$ is a nonzero relative invariant with character χ_D . Thus, there must be an infinitesimal vector field as in (2.5) that is not tangent to the level sets of $f_{x_0}(y)$. By (2.6), this means that $\chi_D(B) = 1$ for some $B \in \mathfrak{g}_{x_0}$. Using (2.2) and (2.4) we conclude that $B = A_1 - A$ with $A \in \mathfrak{a}_D$ and hence $\rho_D(A_1)x_0 = \rho_D(A)x_0$ by definition of \mathfrak{g}_{x_0} . Then the claim follows from Lemma 6.2. \square

§7. Koszul freeness and Euler homogeneity

Koszul freeness is a natural finiteness condition on logarithmic vector fields and is also closely related to the logarithmic comparison theorem (see [Tor04] and [CN05]). In geometric terms, *Koszul freeness* of a free divisor $D \subseteq V$ can be defined as the local finiteness of the *logarithmic stratification* by maximal integral submanifolds along logarithmic vector fields as defined in [Sai80, §3]. As shown in [GMNS09, Thm. 7.4], this condition can be formulated algebraically as the symbols of a basis of $\text{Der}(-\log D) \subseteq \mathcal{D}_V$ forming a regular sequence in $\text{gr}_F \mathcal{D}_V = \mathcal{O}_V \otimes \text{Sym}(V)$ where F denotes the order filtration on \mathcal{D}_V . Without the technical details, this equivalence can be understood as follows: At each point $p \in V$, $\text{Der}(-\log D)(p)$ spans the tangent space of the logarithmic stratum through p . Thus, the *logarithmic variety* in T^*V defined by the symbols of local bases of $\text{Der}(-\log D)$ is the union of conormals to the logarithmic strata (see [Sai80, (3.16)]). It is of (co)dimension n exactly where the symbols of n local basis elements of $\text{Der}(-\log D)$ form a regular sequence in the Cohen–Macaulay coordinate ring $\mathcal{O}_V \otimes \text{Sym}(V)$ of T^*V (see [CN02, 1.8]).

It is tempting to introduce a stronger version of Koszul freeness by imposing local finiteness of the logarithmic stratification of $\text{Der}(-\log f)$ on D . But unless D is strongly Euler homogeneous, which reduces this condition to ordinary Koszul freeness by Lemma 6.2, it is not clear that the condition in question depends only on D .

For a linear free divisor D , the logarithmic stratification is the stratification by orbits of G_D consisting of smooth locally closed algebraic varieties (see [Hum75, Prop. 8.3]). Using the action of $\mathbb{C}^* \subseteq G_D$, one can see that local finiteness of this stratification is equivalent to finiteness. Moreover, there is a natural choice of defining equation which we use to introduce the desired strong version of Koszul freeness for this class of divisors.

Definition 7.1. We call a linear free divisor D *Koszul free* if the stratification of D by orbits of G_D is finite, and *strongly Koszul free* if the same holds for the stratification by orbits of A_D .

Note that the G_D -orbit $V \setminus D$ is a union of infinitely many A_D -orbits by reasons of dimension. The algebraic definition of Koszul freeness above translates to strong Koszul freeness as follows.

Proposition 7.2. *A linear free divisor D is Koszul free if and only if the symbols of a basis of \mathfrak{g}_D form a regular sequence in $\text{gr}_F D_V$. It is strongly Koszul free if and only if the symbols of a basis of \mathfrak{a}_D and f form a regular sequence in $\text{gr}_F D_V$.*

Proof. Let $\delta_2, \dots, \delta_n$ be a basis of \mathfrak{a}_D and denote by $\sigma_F: D_V \rightarrow \text{gr}_F D_V$ the symbol map for the order filtration F on D_V . As $\text{gr}_F D_V = \text{Sym}(V^* \oplus V)$ is a polynomial ring hence Cohen–Macaulay, we have to show that the variety $\text{Var}(f, \sigma_F(\delta_2), \dots, \sigma_F(\delta_n))$ has (co)dimension n exactly if the stratification of D by orbits of A_D is finite. As \mathfrak{a}_D spans the tangent spaces of the orbits, $\text{Var}(\sigma_F(\delta_2), \dots, \sigma_F(\delta_n))$ is the union of conormals to the orbits and $\text{Var}(f) = T^*V|_D$. The second claim follows and the proof of the first one is similar. \square

Remark 7.3. The following generalization of strong Koszul freeness to arbitrary free divisors has been suggested to us by L. Narváez Macarro: Let $D \subseteq X$ be a free divisor in a complex manifold. Then we call D *strongly Koszul free* if, locally at any point $p \in D$, and for some (and hence any) reduced local defining equation f of D at p and for some (and hence any) local basis $\delta_1, \dots, \delta_n$ of $\text{Der}(-\log D)$, the symbols of f and $\delta_i - \delta_i(f)/f \cdot s$, $i = 1, \dots, n$, with respect to the (total) order filtration F form a regular sequence in $\text{gr}_F \mathcal{D}_{X,p}[s] = \text{Sym}_{\mathcal{O}_{X,p}} F^1 \mathcal{D}_{X,p}$. By [NM08, Prop. 2.2.16] and Proposition 7.2, this reduces to Definition 7.1 in the special case. Using ideas of A. Simis and T. Torrelli [CN09, Prop. 1.27], Narváez Macarro can prove that strong Koszul freeness is equivalent to being free with Jacobian ideal of linear type.

Let us now relate the properties of strong Euler homogeneity, Koszul freeness, and strong Koszul freeness for linear free divisors.

Proposition 7.4. *The following implications hold for any linear free divisor D .*

- (a) *Strong Koszul freeness implies Koszul freeness.*
- (b) *Strong Koszul freeness implies strong Euler homogeneity.*
- (c) *For Koszul free D , strong Euler homogeneity and strong Koszul freeness are equivalent.*

Proof. (a) is clear by Definition 7.1 and (c) follows from (b) and Lemma 6.3.

For $0 \leq k \leq n - 1$, let $T_k = \{p \in D \mid \text{rk } \mathfrak{a}_D(p) \leq k\}$ (see [GMNS09, §7] and [Sai80, Def. 3.12]). Then T_k is a homogeneous algebraic variety defined by f and the $(k + 1) \times (k + 1)$ -minors of the matrix $(\delta_i)_{i=2, \dots, n}$, and equals the union of the

A_D -orbits of dimension at most k . As the Euler vector field ϵ is the infinitesimal generator of the \mathbb{C}^* -action that defines homogeneity, ϵ is tangent to T_k and to all G_D -orbits. Therefore, T_k is also a union of G_D -orbits. So if D is strongly Koszul free, $T_k \setminus T_{k-1}$ is the finite union of all k -dimensional A_D -orbits each of which is also a G_D -orbit. This proves (b). \square

Finally, we are ready to prove our main application to the LCT problem.

Proof of Theorem 1.6. Let D be a reductive linear Koszul free divisor defined by f . Because of the \mathbb{C}^* -action, the (global) b -function of D coincides with the local b -function of the germ f_0 of f at the origin (see [Gyo91, Lemmas 2.5.3–2.5.4]). By Proposition 7.2 and flatness of \mathcal{O}_V over $\text{Sym}(V^*)$, Koszul freeness of a linear free divisor D is equivalent to Koszul freeness of $(f, 0)$. By the same argument, strong Koszul freeness of D is equivalent to condition (c) in Theorem 1.3 for f_0 . Therefore, the claim follows by combining Theorem 3.5 and Proposition 7.4(c) with Theorems 1.2 and 1.3. \square

Acknowledgments

The second author wishes to thank David Mond for his comments on the first draft version of this article, and Michel and Jenifer Granger and the Université d'Angers for their hospitality in Summer 2008. We are grateful to Luis Narváez Macarro for pointing out Remark 7.3.

The first author was partially supported by the ANR Sediga, Programme blanc N° ANR-08-BLAN-0317-01/02. The second author was supported by the College of Arts & Sciences at Oklahoma State University through a FY 2009 Dean's Incentive Grant.

References

- [Ber72] I. N. Bernšteĭn, Analytic continuation of generalized functions with respect to a parameter, *Funkcional. Anal. i Priložen.* **6** (1972), no. 4, 26–40 (in Russian). Zbl 0282.46038 MR 0320735
- [Bjö79] J.-E. Björk, *Rings of differential operators*, North-Holland Math. Library 21, North-Holland, Amsterdam, 1979. Zbl 0499.13009 MR 0549189
- [BM06] R.-O. Buchweitz and D. Mond, Linear free divisors and quiver representations, in *Singularities and computer algebra*, London Math. Soc. Lecture Note Ser. 324, Cambridge Univ. Press, Cambridge, 2006, 41–77. Zbl 1101.14013 MR 2228227
- [CNM96] F. J. Castro-Jiménez, L. Narváez-Macarro, and D. Mond, Cohomology of the complement of a free divisor, *Trans. Amer. Math. Soc.* **348** (1996), 3037–3049. Zbl 0862.32021 MR 1363009
- [CU04] F. J. Castro-Jiménez and J. M. Ucha-Enríquez, Testing the logarithmic comparison theorem for free divisors, *Experiment. Math.* **13** (2004), 441–449. Zbl 1071.14024 MR 2118269

- [CMNCJ02] F. J. Calderón Moreno, D. Mond, L. Narváez Macarro, and F. J. Castro Jiménez, Logarithmic cohomology of the complement of a plane curve, *Comment. Math. Helv.* **77** (2002), 24–38. Zbl 1010.32016 MR 1898392
- [CN02] F. Calderón-Moreno and L. Narváez-Macarro, The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors, *Compos. Math.* **134** (2002), 59–74. Zbl 1017.32023 MR 1931962
- [CN05] ———, Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres, *Ann. Inst. Fourier (Grenoble)* **55** (2005), 47–75. Zbl 1089.32003 MR 2141288
- [CN09] ———, On the logarithmic comparison theorem for integrable logarithmic connections, *Proc. London Math. Soc.* (3) **98** (2009), 585–606. MR 2500865
- [DD90] P. Deligne and A. Dimca, Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières, *Ann. Sci. École Norm. Sup.* (4) **23** (1990), 645–656. Zbl 0743.14028 MR 1072821
- [dGMS09] I. de Gregorio, D. Mond, and Ch. Sevenheck, Linear free divisors and Frobenius manifolds, *Compos. Math.* **145** (2009), 1305–1350. Zbl pre05625831 MR 2551998
- [Dim91] A. Dimca, On the de Rham cohomology of a hypersurface complement, *Amer. J. Math.* **113** (1991), 763–771. Zbl 0743.14029 MR 1118460
- [GMNS09] M. Granger, D. Mond, A. Nieto-Reyes, and M. Schulze, Linear free divisors and the global logarithmic comparison theorem, *Ann. Inst. Fourier (Grenoble)* **59** (2009), 811–850. Zbl 1163.32014 MR 2521436
- [GS06] M. Granger and M. Schulze, On the formal structure of logarithmic vector fields, *Compos. Math.* **142** (2006), 765–778. Zbl 1096.32016 MR 2231201
- [Gri69] Ph. A. Griffiths, On the periods of certain rational integrals. I, II, *Ann. of Math.* (2) **90** (1969), 460–495, 496–541. Zbl 0215.08103 MR 0260733
- [Gro66] A. Grothendieck, On the de Rham cohomology of algebraic varieties, *Inst. Hautes Études Sci. Publ. Math.* 1966, no. 29, 95–103. Zbl 0145.17602 MR 0199194
- [Gyo91] A. Gyoja, Theory of prehomogeneous vector spaces without regularity condition, *Publ. Res. Inst. Math. Sci.* **27** (1991), 861–922. Zbl 0773.14025 MR 1145669
- [Har77] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. 52, Springer, New York, 1977. Zbl 0531.14001 MR 0463157
- [HM98] M. P. Holland and D. Mond, Logarithmic differential forms and the cohomology of the complement of a divisor, *Math. Scand.* **83** (1998), 235–254. Zbl 0941.32027 MR 1673922
- [Hum75] J. E. Humphreys, *Linear algebraic groups*, Grad. Texts in Math. 21, Springer, New York, 1975. Zbl 0325.20039 MR 0396773
- [Jac79] N. Jacobson, *Lie algebras*, Dover Publ., New York, 1979. Republication of the 1962 original. Zbl 0121.27504 MR 0143793
- [Kar91] Y. Karpishpan, Pole order filtration on the cohomology of algebraic links, *Compos. Math.* **78** (1991), 213–226. Zbl 0743.32028 MR 1104788
- [Kas77] M. Kashiwara, B -functions and holonomic systems. Rationality of roots of B -functions, *Invent. Math.* **38** (1976/77), 33–53. Zbl 0354.35082 MR 0430304
- [Kim03] T. Kimura, *Introduction to prehomogeneous vector spaces*, Transl. Math. Monogr. 215, Amer. Math. Soc., Providence, RI, 2003. Zbl 1035.11060 MR 1944442
- [M2] D. R. Grayson and M. E. Stillman, *Macaulay 2*, a software system for research in algebraic geometry, <http://www.math.uiuc.edu/Macaulay2/>.
- [Meb89] Z. Mebkhout, *Le formalisme des six opérations de Grothendieck pour les \mathcal{D}_X -modules cohérents*, Travaux en Cours 35, Hermann, Paris, 1989. Zbl 0686.14020 MR 1008245

- [NM08] L. Narváez Macarro, Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors, in *Singularities I*, Contemp. Math. 474, Amer. Math. Soc., Providence, RI, 2008, 245–269. Zbl 1166.32006 MR 2454351
- [Sai71] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen, *Invent. Math.* **14** (1971), 123–142. Zbl 0224.32011 MR 0294699
- [Sai80] ———, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291. Zbl 0496.32007 MR 0586450
- [SKKŌ81] M. Sato, M. Kashiwara, T. Kimura, and T. Ōshima, Microlocal analysis of prehomogeneous vector spaces, *Invent. Math.* **62** (1980/81), 117–179. Zbl 0456.58034 MR 0595585
- [Sch07] M. Schulze, Logarithmic comparison theorem versus Gauss–Manin system for isolated singularities, *Adv. Geom.*, to appear.
- [Ser01] J.-P. Serre, *Complex semisimple Lie algebras*, Springer Monogr. Math., Springer, Berlin, 2001. Zbl 1058.17005 MR 1808366
- [Ter78] H. Terao, Forms with logarithmic pole and the filtration by the order of the pole, in *Proceedings of the International Symposium on Algebraic Geometry (Kyoto, 1977)*, Kinokuniya Book Store, Tokyo, 1978, 673–685. Zbl 0429.32015 MR 0578880
- [Tor04] T. Torrelli, On meromorphic functions defined by a differential system of order 1, *Bull. Soc. Math. France* **132** (2004), 591–612. Zbl 1080.32011 MR 2131905
- [Tor07] ———, Logarithmic comparison theorem and \mathcal{D} -modules: an overview, in *Singularity theory*, World Sci., Hackensack, NJ, 2007, 995–1009. Zbl 1121.32005 MR 2342946
- [WY97] J. Wiens and S. Yuzvinsky, De Rham cohomology of logarithmic forms on arrangements of hyperplanes, *Trans. Amer. Math. Soc.* **349** (1997), 1653–1662. Zbl 0948.52014 MR 1407505