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On a Theorem by Florek and Slater on Recurrence Properties of Circle Maps

By

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Abstract

An obviously little known result by Florek and Slater about the exact recurrence times of the sequence $m\beta$ mod 1 with respect to an arbitrary connected interval I in the unit interval is generalized to disconnected intervals $I_{\widehat{a,b}} = [0, a) \cup (b, 1)$ when $b = 1 - a$, $a < 1/2$. It is shown that the formula of Florek and Slater expressi..g the possible recurrence times in terms of the interval I is valid also in our case. This let us expect that this formula is valid also for general intervals of the form $I_{\widehat{a,b}}$. The relation of this result to the recurrence properties of integrable Hamiltonian systems with two degrees of freedom is obvious.

§ 1. Introduction

Some time ago when investigating ergodic properties of certain chaotic semiflows with strange attractors [1] we came about a nice but seemingly little known result by Florek [2] and Slater [3] about the distribution of the sequence $\{n\beta \bmod 1\}$ $n \in \mathcal{N}$, for irrational β on the unit interval $I = [0, 1)$. Their result concerns the so called gap problem for this sequence: If $I_{a,b}$ denotes the interval $a \leqslant x < b$ in I, denote by $N_\beta(I_{a,b})$ the set $N_\beta(I_{a,b}) = \{n \in \mathcal{N} : n\beta \bmod 1 \in I_{a,b}\}$. If we arrange the elements of $N_\beta(I_{a,b})$ according to their order $n_1 < n_2 < n_3 < \cdots$ the gap problem consists in determining the numbers $\tau_i = n_{i+1} - n_i$ for all $i \in \mathcal{N}$. The rather astonishing result of Florek and Slater then says that for a connected interval $I_{a,b}$ the gaps τ_i can take at most three different values, expressible by the continued fraction expansion of the number β. This result has a simple interpretation in the theory of dynamical systems: if $R_\beta : S_1 \to S_1$ denotes the map $R_\beta \varphi = \varphi + \beta$ of the 1-sphere $S_1 = \mathscr{R}/\mathscr{Z}$ into itself we see that the sequence $\{n\beta \bmod 1\}$ is obviously just the orbit of the point $\varphi = 0$ under the above map. Hence Florek and Slaters's result describes the recurrence behaviour of the dynamical system $R_\beta : S_1 \to S_1$ with respect to the connected interval $I_{a,b}$ of the 1-sphere. This problem, how the orbit of a system recurs to an arbitrary set of the phase space, arises for arbitrary dynamical systems and

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plays an important role in the foundation of statistical mechanics [4], [5]. In general not much is known about the exact recurrence properties of an arbitrary system, so that it is not too astonishing that even for such a trivial system as the above pure rotation R_β of S_1 the problem is not yet completely solved. The first step to such a solution has been done by Florek and Slater who proved

Theorem 0. *For any connected interval $I_{a,b}$ in I with $0 \leqslant a < b$ and any irrational β there are at most three different gaps τ_1 in the set $N_\beta(I_{a,b})$:*

$$\tau_1 = \min\{m\colon m\beta \bmod 1 < b - a = |I_{a,b}|\}$$

$$\tau_2 = \min\{m\colon m\beta \bmod 1 > 1 - |I_{a,b}|\} \tag{1}$$

$$\tau_3 = \tau_1 + \tau_2.$$

The gap τ_3 does not arise for all intervals $I_{a,b}$.

In this form the result was announced the first time by Florek in [2] without a proof, which in fact was given for the case of intervals of the form $I_{a,b}$ with $a = 0$ by Slater in [3]. He had determined the gaps τ_i in this case already before in [6], expressing them in terms of the best Diophantine approximation denominators q_n of the number β determined by this number's continued fraction expansion. It is not clear how Flored indeed proved his result (see also a remark by Slater in [3]).

To solve the general gap problem for the rotation R_β of the 1-sphere, that means the gaps for any orbit $\{R_\beta^n \varphi\}$ for arbitrary $\varphi \in S_1$ with respect to an arbitrary connected interval in S_1, one has obviously to show that the result of Florek and Slater stays true also for intervals of the form $[0, a) \cup (b, 1)$ which are disconnected in I but become connected when regarded as a subset in S_1. It was not clear to us how Slater's approach in [3] could be applied directly to intervals of the above kind which we denote by $I_{\widehat{a,b}}$. So we had to look for another proof. We found such a proof for the case of intervals of the form $I_{a,1-a}$ with $a < 1/2$. In this case we get obviously

$$R_\beta^n 0 \in I_{a,1-a} \quad \text{iff} \quad \|n\beta\| < a, \tag{2}$$

where the symbol $\|n\beta\|$ denotes the so called Diophantine norm of the irrational β. The gap problem with respect to the interval $I_{a,1-a}$ of the sequence $\{n\beta \bmod 1\}$ is then equivalent to the gap problem of the sequence $\{\|n\beta\|\}$ with respect to the interval $I_{\widehat{0,a}}$. This last problem showed up in connection with certain ergodic properties of a class of semiflows studied in [7]: a crucial role there plays the asymptotic behaviour of the function $F_n(\lambda)$ for $\operatorname{Re}\lambda > 1$ in the limit $n \to \infty$, where this function is defined as follows

$$F_n(\lambda) = \sum_{m\colon \|q_n\beta\| < \|m\beta\| < \|q_{n-1}\beta\|} m^{-\lambda}, \tag{3}$$

where the q_n's are the denominators of the rational approximants of β defined by its continued fraction expansion.

If $M_\varphi(\beta)$ denotes for $0 < \varphi \leqslant 1/2$ the set

$$M_\varphi(\beta) := \{m \in \mathcal{N} : \|m\beta\| < \varphi\} \tag{4}$$

then we can formulate our main result as

Theorem 1. *For any irrational β with continued fraction expansion $\beta = [b_0, b_1, b_2, \dots]$ and any φ with $0 < \varphi \leqslant \|q_0\beta\|$ the set $M_\varphi(\beta)$ has at most three different gaps. For $a_i\|q_i\beta\| + \|q_{i+1}\beta\| < \varphi \leqslant (a_i + 1)\|q_i\beta\| + \|q_{i+1}\beta\|$, $1 \leqslant a_i \leqslant b_{i+1}$, there are in the case*

a) $2b_{i+1} > 2a_i \geqslant b_{i+1} + 1$ *exactly the three gaps* $q_{i-1}, q_i - q_{i-1}, q_i$,

b) $2a_i \leqslant b_{i+1} - 2$ *at most the three gaps* $q_i, q_{i+1} - 2a_iq_i, q_{i+1} - (2a_i - 1)q_i$ *for* $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^1$, *the gaps* $q_i, q_{i+1} - (2a_i + 1)q_i, q_{i+1} - 2a_iq_i$ *for* $\varphi_i^1 \leqslant \varphi \leqslant \varphi_i^2$ *and finally the gaps* $q_i, q_{i+1} - 2(a_i + 1)q_i, q_{i+1} - (2a_i + 1)q_i$ *for* $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$

c) $2a_i = b_{i+1}$ *at most the three gaps* $q_{i-1}, q_i, q_i + q_{i-1}$ q_i *for* $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^1$ *and the gaps* $q_{i-1}, q_i - q_{i-1}, q_i$ *for* $\varphi_i^1 \leqslant \varphi \leqslant \varphi_i^3$

d) $2a_i = b_{i+1} - 1$ *at most the three gaps* $q_i, q_i + q_{i-1}, q_{i-1} + 2q_i$ *for* $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^1$, *the gaps* $q_{i-1}, q_i, q_i + q_{i-1}$ *for* $\varphi_i^1 \leqslant \varphi \leqslant \varphi_i^2$ *and finally the gaps* $q_i - q_{i-1}, q_{i-1}, q_i$ *for* $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$

e) $a_i = b_{i+1}$ *at most the three gaps* $q_{i-1}, q_i - q_{i-1}, q_i$ *for* $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^2$, $q_{i-2}, q_{i-1} - q_{i-2}, q_{i-1}$ *for* $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$ *if* $b_i = 1$ *respectively* q_{i-1}, $q_i - 2q_{i-1}$,
$q_i - q_{i-1}$ *for* $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$ *if* $b_i \geqslant 2$.

The quantities φ_i^k are thereby defined as follows:

$$\varphi_i^0 = a_i\|q_i\beta\| + \|q_{i+1}\beta\|,$$

$$\varphi_i^1 = a_i\|q_i\beta\| + (l + 1)\|q_{i+1}\beta\| + \|q_{i+2}\beta\| \quad \textit{if } b_{i+2} = 2l + 1$$

respectively

$$\varphi_i^1 = a_i\|q_i\beta\| + l\|q_{i+1}\beta\| + 1/2(\|q_{i+1}\beta\| + \|q_{i+2}\beta\|) \quad \textit{if } b_{i+2} = 2l$$

$$\varphi_i^2 = (a_i + 1)\|q_i\beta\| + 1/2\|q_{i+1}\beta\|$$

$$\varphi_i^3 = (a_i + 1)\|q_i\beta\| + \|q_{i+1}\beta\|.$$

As an immediate consequence of Theorem 1 we get

Corollary 1. *There exists a constant $c = c(\lambda)$ such that for any irrational β, any $\lambda > 1$ and any $i \geqslant 1$*

$$F_i(\lambda) \leqslant c(\lambda)\, q_i^{-\lambda}.$$

Corollary 2. *For any interval $I_{a,1-a}$ with $a \leqslant \|q_0\beta\|$ the recurrence times τ_i are given by*

$$\tau_1 = \min\{m: m\beta \bmod 1 < |I_{a,1-a}| = 2a\}$$
$$\tau_2 = \min\{m: m\beta \bmod 1 > 1 - 2a\}$$
$$\tau_3 = \tau_1 + \tau_2$$

where the last one is, depending on a, not always realized.

This shows that the Florek-Slater formula is also true for this class of disconnected intervals and this let us expect the formula to be valid also for general $I_{\widehat{a,b}}$.

Remark: Our result on the gaps with respect to the interval $I_{\widehat{a,b}-a}$ can be interpreted also as a result on the sequence of gaps arising in the visits of the sequence $m\beta$ mod 1 in the set $[a, 1-a]$. Besides some partial results of Slater on this problem for the interval $[0, a]$ to our knowledge nothing is known for general intervals. The method of our proof relies on the continued fraction expansion of the number β and is in its spirit analogous to Slaters approach in [6].

§2. A Representation of $\|m\beta\|$ in Terms of $\|q_n\beta\|$

Let us start with some definitions and well known properties of the continued fraction expansion of an irrational number β and its relation to the so called Diophantine norms $\|n\beta\|$. If $\beta = [b_0, b_1, \ldots]$ is the infinite continued fraction expansion of β with $b_0 \in \mathscr{Z}$ and $b_i \in \mathscr{N}$ for $i \geqslant 1$, the n-th principal convergent p_n/q_n, $n = 0, 1, \ldots$ is defined by the finite continued fraction $p_n/q_n = [b_0, b_1, \ldots, b_n]$. The numbers p_n, q_n fulfill simple recursion relations [9]:

(4)
$$p_{n+1} = b_{n+1}\ p_n + p_{n-1}$$
$$q_{n+1} = b_{n+1}\ q_n + q_{n-1}$$

with boundary conditions $p_{-1} = 1$, $p_0 = b_0$, $q_{-1} = 0$, $q_0 = 1$. The Diophantine norms $\|m\beta\|$ of β for $m \in \mathscr{N}$ are defined as [10]

(5)
$$\|m\beta\| := \min_{r \in \mathscr{Z}} |m\beta - r|.$$

Since $|q_n\beta - p_n| = (-1)^n (\beta_{n+1}\ q_n + q_{n-1})^{-1}$ where β_n is defined through $\beta = [b_0, b_1, \ldots, b_{n-1}, \beta_n]$ (see [9]) one gets for $n \geqslant 1$:

$$|q_n\beta - p_n| = \|q_n\beta\|.$$

For $n = 0$ on the other hand one finds

(6)
$$|q_0\beta - p_0| = \begin{cases} \|q_0\beta\| & \text{if } b_i \geqslant 2 \\ 1 - \|q_0\beta\| & \text{if } b_i = 1. \end{cases}$$

Furthermore the following recursion relations hold:

(7) $$\|q_{n+1}\beta\| = -b_{n+1}\|q_n\beta\| + \|q_{n-1}\beta\| \quad \text{if } n \geqslant 2$$

(8) $$\|q_2\beta\| = -b_2\|q_1\beta\| + \begin{cases} \|q_0\beta\| & \text{if } b_1 \geqslant 2 \\ 1 - \|q_0\beta\| & \text{if } b_1 = 1, \end{cases}$$

respectively

(9) $$\|q_1\beta\| = \begin{cases} 1 - b_1\|q_0\beta\| & \text{if } b_1 \geqslant 2 \\ \|q_0\beta\| & \text{if } b_1 = 1. \end{cases}$$

Our first aim is now to express the numbers $\|m\beta\|$ in terms of the best Diophantine approximations $\|q_n\beta\|$, where the q_n's are the best Diophantine approximation denominators determined by the principal convergents (4) of the irrational β.

To start with we recall the following well known fact [8]:

Lemma 1. *If q_n, $n = 0, 1,\ldots$ denote the sequence of denominators of the principal convergents of the irrational number β then any integer $m \in \mathcal{N}$ can be uniquely written as*

(10) $$m = \sum_{k=0}^{\infty} r_k q_k$$

where the integers r_k satisfy the conditions

a) $r_0 \in \{0, 1,\ldots, b_1 - 1\}$

b) $r_k \in \{0, 1,\ldots, b_{k+1} - 1\}$ *if* $r_{k-1} \neq 0$

respectively

$r_k \in \{0, 1,\ldots, b_{k+1}\}$ *if* $r_{k-1} = 0$.

Hereby the b_i, $i \geqslant 0$, are the partial quotients of β:

$$\beta = [b_0, b_1, b_2,\ldots].$$

We want next derive a similar expansion of the number $\|m\beta\|$ in terms of the $\|q_n\beta\|$. For this let p_k be the numerators of the principal convergents p_k/q_k to β, defined by its continued fraction expansion. For $m = \sum_{k=0}^{\infty} r_k q_k$ denote by $\sigma(m)$ the unique integer such that

$$\left| m\beta - \sum_{k=0}^{\infty} r_k p_k + \sigma(m) \right| < 1/2.$$

Then obviously

$$(11)\qquad \|m\beta\| = \left|m\beta - \sum_{k=0}^{\infty} r_k p_k + \sigma(m)\right| = \left|\sigma(m) + \sum_{k=0}^{\infty} (-1)^k r_k \|q_k\beta\|\right|.$$

We can then show

Lemma 2. *For all $m \in \mathcal{N}$ with $\|m\beta\| \leqslant \|q_0\beta\|$ the number $\sigma(m)$ vanishes.*

Proof. Using the properties of the coefficients r_k in expansion (10) of m and the recursion relations (7)-(9)

$$\sum_{k=0}^{\infty} (-1)^k r_k \|q_k\beta\| \leqslant \sum_{j=0}^{\infty} r_{2j}\|q_{2j}\beta\| < (b_1 - 1)\|q_0\beta\|$$
$$+ \sum_{j=1}^{\infty} b_{2j+1}\|q_{2j}\beta\| = (b_1 - 1)\|q_0\beta\| + \|q_1\beta\| \leqslant 1 - \|q_0\beta\|,$$

where the strict inequality comes from the fact that for any $m \in \mathcal{N}$ there exists a $k(m)$ such that $r_k = 0$ for $k > k(m)$. A similar argument as above shows also that

$$\sum_{k=0}^{\infty} (-1)^k r_k \|q_k\beta\| \geqslant -\sum_{j=0}^{\infty} r_{2j+1}\|q_{2j+1}\beta\| > -\sum_{j=0}^{\infty} b_{2j+2}\|q_{2j+1}\beta\|$$
$$= -b_2\|q_1\beta\| - \|q_2\beta\| \geqslant -(1 - \|q_0\beta\|).$$

This shows that $\sigma(m) \in \{0, 1, -1\}$. Assume that $\sigma(m) \neq 0$ for some $m \in \mathcal{N}$. Then the above estimates imply

$$\|m\beta\| \geqslant \left||\sigma(m)| - \left|\sum_{k=0}^{\infty} (-1)^k r_k \|q_k\beta\|\right|\right| > 1 - (1 - \|q_0\beta\|) = \|q_0\beta\|$$

contrary to the assumption $\|m\beta\| \leqslant \|q_0\beta\|$.

The expression $\sum_{k=0}^{\infty} (-1)^k r_k \|q_k\beta\|$ can be somewhat simplified. If m namely has the expansion

$$m = \sum_{k=k_0}^{\infty} r_k q_k,\ k_0 = 0,\ r_{k_0} \neq 0$$

then we fied

Lemma 3. $\left|\sum_{k=k_0}^{\infty} (-1)^k r_k \|q_k\beta\|\right| = \sum_{k=k_0}^{\infty} (-1)^{k-k_0} r_k \|q_k\beta\|.$

Proof: We only have to show that the right hand side is positive:

$$\sum_{k=k_0}^{\infty} (-1)^{k-k_0} r_k \|q_k\beta\| \geqslant r_{k_0}\|q_{k_0}\beta\| - \sum_{j=0}^{\infty} r_{k_0+1+2j}\|q_{k_0+1+2j}\beta\|$$

$$(12)\qquad > r_{k_0}\|q_{k_0}\beta\| - (b_{k_0+2} - 1)\|q_{k_0+1}\beta\| - \sum_{j=1}^{\infty} b_{k_0+2+2j}\|q_{k_0+1+2j}\beta\|$$

$$= r_{k_0}\|q_{k_0}\beta\| - (b_{k_0+2} - 1)\|q_{k_0+1}\beta\| - \|q_{k_0+2}\beta\|.$$

If $k_0 = 0$ then $r_0 \geqslant 1$ and hence $b_1 \geqslant 2$. In this case formula (8) gives $\| q_0\beta \| = b_2 \| q_1\beta \| + \| q_2\beta \|$ and expression (12) is larger than

$$(r_0 - 1)\| q_0\beta \| + \| q_1\beta \| > 0.$$

If on the other hand $k_0 \geqslant 1$ we get using formula (7)

$$(12) \geqslant (r_{k_0} - 1)\| q_{k_0}\beta \| + \| q_{k_0+1}\beta \| > 0.$$

This proves Lemma 3.

The results of Lemma 1 to Lemma 3 put together give

Proposition 2. *If $m \in \mathcal{N}$ has for irrational β the expansion $m = \sum_{k=k_0}^{\infty} r_k q_k$, $r_{k_0} \neq 0$ then the Diophantine norms of all m* with $\| m\beta \| \leqslant \| q_0\beta \|$ *have the representation*

$$\| m\beta \| = \sum_{k=k_0}^{\infty} (-1)^{k-k_0} r_k \| q_k\beta \|$$

where the q_k's are the denominators of the principal convergents of β.

Remark: The referee kindly informed us that Prop.2 is closely related to what in discrepancy problems is called a canonical form as discussed for instance in [11].

§3. Proof of Theorem 1

The proof will be similar in spirit to the procedure in [6] to solve the gap problem for mβ mod 1. We first discuss certain φ's for for which M_φ can be determined explicitly and reduce the general problem then to this case. Let us start with the well known fact (see [6], Lemma preceeding Theorem 3) that every number φ with $0 < \varphi \leqslant \| q_0\beta \|$ can be written uniquely as

$$\varphi = a_i \| q_i\beta \| + \| q_{i+1}\beta \| + \psi$$

where the integer i takes values in the set J_β with $J_\beta = \mathcal{N}$ for $b_1 \geqslant 2$ and $J_\beta = \mathcal{N} \setminus \{1\}$ for $b_1 = 1$, the integer a_i takes values in $1 \leqslant a_i \leqslant b_{i+1}$ and the real number ψ fulfills $0 \leqslant \psi < \| q_i\beta \|$.

The special role played by these numbers was seen already in [6] where the induced partition of the interval [0, 1] was used to solve the gap problem for the sequence $m\beta$ mod 1. If we denote these numbers by $\varphi_i = \varphi_i(a_i)$ that means

$$\varphi_i(a_i) = a_i \| q_i\beta \| + \| q_{i+1}\beta \|, \ i \in J_\beta \tag{13}$$

we have

Proposition 3. *The number m belongs to* M_{φ_i} *if and only if one of the following canditions holds for the expansion* $m = \sum_{k=k_0}^{\infty} r_k q_k$, $r_{k_0} \geqslant 1$:

(3_1) $k_0 = i-1 : r_{i-1} = 1,\ b_{i+1} - a_i < r_i \leqslant b_{i+1} - 1$

(3_2) $k_0 = i-1 : r_{i-1} = 1,\ r_i = b_{i+1} - a_i$, *there exists* $n \geqslant 1$ *such that* $r_k = 0$ *for* $i+1 \leqslant k < i+2n$ *and* $r_{i+2n} \geqslant 1$

(3_3) $k_0 \geqslant i$: *if* $k_0 = i$ *then* $0 \leqslant r_i \leqslant a_i$.

Unfortunately, the partition induced by these φ_i is not yet fine enough to solve the gap problem for $\|m\beta\|$. So we have to look for finer partitions. For this consider the numbers $\psi_i = \psi_i(a_i, c_{i+1})$

$$\psi_i = a_i \|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| + \|q_{i+2}\beta\|,\ i \in J_\beta \tag{14}$$

where the integers a_i and c_{i+1} can take the values

$$1 \leqslant a_i \leqslant b_{i+1} - 1 \text{ respectively } 1 \leqslant c_{i+1} \leqslant b_{i+2},$$

and therefore are defined only for $b_{i+1} \geqslant 2$. Trivially

$$\varphi_i(a_i + 1) > \psi_i(a_i, c_{i+1}) > \varphi_i(a_i).$$

In this case we find

Proposition 4. The number $m \in \mathcal{N}$ belongs to M_{ψ_i} if and only if one of the following conditions holds for the expansion $m = \sum_{k=k_0} r_k q_k$, $r_{k_0} \geqslant 1$:

(4_1) $k_0 = i-1 : r_{i-1} = 1,\ b_{i+1} - a_i < r_i \leqslant b_{i+1} - 1$

(4_2) $k_0 = i-1 : r_{i-1} = 1,\ r_i = b_{i+1} - a_i,\ 0 \leqslant r_{i+1} \leqslant c_{i+1} - 1$

(4_3) $k_0 \geqslant i$: *if* $k_0 = i$ *then* $0 \leqslant r_i \leqslant a_i + 1$. *If* $r_i = a_i + 1$ *then* $b_{i+2} - c_{i+1} \leqslant r_{i+1} \leqslant b_{i+2} - 1$. *If* $r_{i+1} = b_{i+2} - c_{i+1}$ *then there exists* $n \geqslant 1$ *such that* $r_{i+k} = 0$ *for* $2 \leqslant k \leqslant 2n$ *and* $r_{i+2n+1} \geqslant 1$.

Because the proof of Prop. 3 is very similar to the proof of Prop. 4 we give only the last one:

Proof of Prop. 4. We show first that conditions (4_1) to (4_3) are necessary. If $k_0 \leqslant i-2$ then by Prop. 2 we get

$$\begin{aligned}\|m\beta\| &> \|q_{k_0}\beta\| - (b_{k_0+2} - 1)\|q_{k_0+1}\beta\| - \sum_{j=1}^{\infty} b_{k_0+2+2j}\|q_{k_0+1+2j}\beta\| \\ &= \|q_{k_0+1}\beta\| \geqslant \|q_{i-1}\beta\| = b_{i+1}\|q_i\beta\| + \|q_{i+1}\beta\| \geqslant a_i\|q_i\beta\| + \|q_i\beta\| \\ &\quad + \|q_{i+1}\beta\| > a_i\|q_i\beta\| + b_{i+2}\|q_{i+1}\beta\| + \|q_{i+2}\beta\| \geqslant \psi_i.\end{aligned}$$

If $k_0 = i-1$ but $r_{i-1} \geqslant 2$ we find

$$\|m\beta\| > 2\|q_{i-1}\beta\| - (b_{i+1} - 1)\|q_i\beta\| - \sum_{j=1}^{\infty} b_{i+1+2j}\|q_{i2j}\beta\| = \|q_{i-1}\beta\|$$

$$+ \|q_i\beta\| = b_{i+1}\|q_i\beta\| + \|q_{i+1}\beta\| + \|q_i\beta\| = b_{i+1}\|q_i\beta\|$$

$$+ (b_{i+2} + 1)\|q_{i+1}\beta\| + \|q_{i+2}\beta\| > \psi_i.$$

We show next that for $k_0 = i - 1$ and $r_{i-1} = 1$ the condition $r_i \geqslant b_{i+1} - a_i$ is necessary. Assume $r_i \leqslant b_{i+1} - a_i - 1$. Then

$$\|m\beta\| > \|q_{i-1}\beta\| - (b_{i+1} - a_i - 1\|q_i\beta\| - \sum_{j=1}^{\infty} b_{i+1+2j}\|q_{i+2j}\beta\|$$

$$= (a_i + 1)\|q_i\beta\| \geqslant \psi_i.$$

If $k_0 = i - 1$ and $r_i = b_{i+1} - a_i$ then condition (4_2) is necessary. Otherwise $r_{i+1} \geqslant c_{i+1}$ and we find

$$\|m\beta\| > \|q_{i-1}\beta\| - (b_{i+1} - a_i)\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| - (b_{i+3} - 1)\|q_{i+2}\beta\|$$

$$- \sum_{j=1}^{\infty} b_{i+3+2j}\|q_{i+2+2j}\beta\| = \psi_i.$$

If $k_0 \geqslant i$ then $r_i \leqslant a_i + 1$ is necessary. This is clear for $a_i = b_{i+1} - 1$. If $a_i < b_{i+1} - 1$ assume $r_i \geqslant a_i + 2$. Then

$$\|m\beta\| > (a_i + 2)\|q_i\beta\| - (b_{i+2} - 1)\|q_{i+1}\beta\| - \sum_{j=1}^{\infty} b_{i+2+2j}\|q_{i+1+2j}\beta\|$$

$$= (a_i + 1)\|q_i\beta\| + \|q_{i+1}\beta\| > \psi_i.$$

If $r_i = a_i + 1$ then $b_{i+2} - c_{i+1} \leqslant r_{i+1} \leqslant b_{i+2} - 1$ is necessary. This is clear for $c_{i+1} = b_{i+2}$. For $c_{i+1} < b_{i+2}$ assume $r_{i+1} \leqslant b_{i+2} - c_{i+1} - 1$. Then we get

$$\|m\beta\| > (a_i + 1)\|q_i\beta\| - (b_{i+2} - c_{i+1} - 1)\|q_{i+1}\beta\| - \sum_{j=1}^{\infty} b_{i+2+2j}\|q_{i+1+2j}\beta\|$$

$$= (a_i + 1)\|q_i\beta\| - b_{i+2}\|q_{i+1}\beta\| + (c_{i+1} + 1)\|q_{i+1}\beta\| - \|q_{i+2}\beta\|$$

$$= a_i\|q_i\beta\| + (c_{i+1} + 1)\|q_{i+1}\beta\| > \psi_i.$$

If finally $r_i = a_i + 1$ and $r_{i+1} = b_{i+2} - c_{i+1}$ then there must exist an $n \geqslant 1$ with $r_{i+k} = 0$ for $2 \leqslant k \leqslant 2n$ and $r_{i+2n+1} \geqslant 1$. Assume on the contrary there exists a $m \geqslant 1$ such that $r_{i+k} = 0$ for $2 \leqslant k \leqslant 2m - 1$ but $r_{i+2m} \geqslant 1$. In this case

$$\|m\beta\| > (a_i + 1)\|q_i\beta\| - (b_{i+2} - c_{i+1})\|q_{i+1}\beta\| + \|q_{i+2m}\beta\|$$

$$- (b_{i+2m+2} - 1)\|q_{i+2m+1}\beta\| - \sum_{j=1}^{\infty} b_{i+2m+2+2j}\|q_{i+2m+1+2j}\beta\|$$

$$= (a_i + 1)\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| - b_{i+2}\|q_{i+1}\beta\| + \|q_{i+2m+1}\beta\|$$

$$= a_i\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| + \|q_{i+2}\beta\| + \|q_{i+2m+1}\beta\| > \psi_i.$$

This shows that conditions (4_1) to (4_3) are necessary. Let us next show that they are also sufficient. If the expansion $m = \sum_{k=k_0} r_k q_k$ fulfills condition (4_1) we get

$$\|m\beta\| < \|q_{i-1}\beta\| - (b_{i+1} - a_i + 1)\|q_i\beta\| + (b_{i+2} - 1)\|q_{i+1}\beta\| + \sum_{j=1}^{\infty} b_{i+2+2j}\|q_{i+1+2j}\beta\| = a_i\|q_i\beta\| < \psi_i.$$

If m fulfills condition (4_2) we find

$$\|m\beta\| < \|q_{i-1}\beta\| - (b_{i+1} - a_i)\|q_i\beta\| + (c_{i+1} - 1)\|q_{i+1}\beta\| + \sum_{j=1}^{\infty} b_{i+2+2j}\|q_{i+1+2j}\beta\| = \|q_{i-1}\beta\| + a_i\|q_i\beta\| - b_{i+1}\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| - \|q_{i+1}\beta\| + \|q_{i+2}\beta\| = a_i\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| + \|q_{i+2}\beta\| = \psi_i.$$

If m fulfills condition (4_3) with $k_0 \geqslant i + 1$ we find

$$\|m\beta\| < \|q_{k_0}\beta\| + \sum_{j=1}^{\infty} b_{k_0+1+2j}\|q_{k_0+2j}\beta\| = \|q_{k_0}\beta\| + \|q_{k_0+1}\beta\| \leqslant \|q_{i+1}\beta\| + \|q_{i+2}\beta\| < \psi_i.$$

If m fulfills condition (4_3) with $k_0 = i$ and $r_i \leqslant a_i$ then

$$\|m\beta\| < a_i\|q_i\beta\| + \|q_{i+1}\beta\| < \psi_i.$$

If $r_i = a_i + 1$ and $b_{i+2} - c_{i+1} + 1 \leqslant r_{r+1} \leqslant b_{i+2} - 1$ we get

$$\|m\beta\| < (a_i + 1)\|q_i\beta\| - (b_{i+2} - c_{i+1} + 1)\|q_{i+1}\beta\| + (b_{i+3} - 1)\|q_{i+2}\beta\| + \sum_{j=1}^{\infty} b_{i+3+2j}\|q_{i+2+2j}\beta\| = a_i\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| < \psi_i.$$

If finally $r_i = a_i + 1$, $r_{i+1} = b_{i+2} - c_{i+1}$ and $r_{i+k} = 0$ for $2 \leqslant k \leqslant 2n$, $r_{i+2n+1} \geqslant 1$ we find

$$\|m\beta\| < (a_i + 1)\|q_i\beta\| - (b_{i+2} - c_{i+1})\|q_{i+1}\beta\| - r_{i2n+1}\|q_{i+2n+1}\beta\| + (b_{i+2n+3} - 1)\|q_{i+2n+2}\beta\| + \sum_{j=1}^{\infty} b_{i+2n+3+2j}\|q_{i+2n+2+2j}\beta\| \leqslant a_i\|q_i\beta\| + c_{i+1}\|q_{i+1}\beta\| + \|q_{i+2}\beta\| - \|q_{i+2n+2}\beta\| < \psi_i.$$

Consider next the numbers $\chi_i = \chi_i(a_i)$ defined as

$$\chi_i(a_i) = a_i\|q_i\beta\| + 1/2\|q_{i+1}\beta\|, \quad 1 \leqslant a_i \leqslant b_{i+1}. \tag{15}$$

It is clear that $\chi_i(a_i) < \varphi_i(a_i)$.

For the corresponding set M_{χ_i} we find

Proposition 5. *$m \in \mathcal{N}$ belongs to $M_{\chi_i}(a_i)$ if and only if its expansion $m = \sum_{k=k_0} r_k q_k$, $r_{k_0} \geqslant 1$ fulfills one of the conditions:(5_1)–(5_3) which are conditions (4_1)–(4_3) of Prop. 4 with*

$$\psi_i = \psi_i(a_i - 1,\ b_{i+1}),\ \text{or}$$

(5_4) *either $k_0 = i - 1{:}r_{i-1} = 1$, $r_i = b_{i+1} - a_i$ and there exists $n \geqslant 1$ with $r_{i+k} = 0$ for $1 \leqslant k \leqslant 2n - 1$, $r_{i+2n} \geqslant 1$ or $k_0 = i{:}r_i = a_i$ and there exists $n \geqslant 1$ with $r_{i+k} = 0$ for $1 \leqslant k \leqslant 2n - 1$ and $r_{i+2n} \geqslant 1$.*

Remarks: 1) In case $a_i = 1$ only conditions (5_3) and (5_4) make sense.
2) Condition (5_4) means that exactly one of the two numbers

$$m_1 = q_{i-1} + (b_{i+1} - a_i)q_i + \sum_{j=1+2n} r_j q_j$$

$$m_2 = a_i q_i + \sum_{j=1+2n} r_j q_j$$

belongs to the set M_{χ_i}. This follows from the fact that

$$\|m_1\beta\| + \|m_2\beta\| = 2a_i\| + \|q_{i+1}\beta\| \text{ and that } \|m\beta\| = \|n\beta\|$$

for irrational β iff $m = n$.

Finally we need for irrational β with $b_{i+2} = 2l_{i+1}$ the special numbers $\zeta_i = \zeta_i(a_i,\ l_{i+1})$:

$$\zeta_i(a_i,\ l_{i+1}) = a_i\|q_i\beta\| + l_{i+1}\|q_{i+1}\beta\| + 1/2(\|q_{i+1}\beta\| + \|q_{i+2}\beta\|).$$

They fulfill $\varphi_i(a_i) < \zeta_i(a_i,\ l_{i+1}) < \chi_i(a_i + 1) < \varphi_i(a_i + 1)$.

For them we find

Proposition 6. *The number $m \in \mathcal{N}$ belongs to the set M_{ζ_i} iff one of the following conditions holds for the expansion $m = \sum_{k=k_0} r_k q_k$, $r_{k_0} \geqslant 1$: Conditions (4_1) to (4_3) of Prop.4 for $\psi_i = \psi_i(a_i,\ l_{i+1})$, or*

(6_4) *either $k_0 = i - 1{:}r_{i-1}$, $r_i = b_{i+1} - a_i$, $r_{i+1} = l_{i+1}$ and there exists $n \geqslant 1$ with $r_{i+k} = 0$ for $2 \leqslant k \leqslant 2n - 1$ and $r_{i+2n} \geqslant 1$ or $k_0 = i$, $r_i = a_i + 1$, $r_{i+1} = l_{i+1}$ and there exists $n \geqslant 1$ such that $r_{i+k} = 0$ for $2 \leqslant k \leqslant 2n - 1$ and $r_{i+2n} \geqslant 1$.*

The proofs of all these Propositions run along the lines of the proof of Prop.4 so that we can omit them here.
Knowing this way the set M_φ explicitly for certain values of φ it is not difficult

to determine next for these φ also the gaps in the sets M_φ.

Proposition 7. *For $\beta = [b_0, b_1, \ldots]$ irrational and $\varphi = \varphi_i(a_i) = a_i \| q_i\beta \| + \| q_{i+1}\beta \|$, $i \in J_\beta$, the set M_φ has for $2a_i \geqslant b_{i+1} + 1$ the gaps q_{i-1}, $q_i - q_{i-1}$, q_i and for $2a_i \leqslant b_{i+1}$ the gaps q_i, $q_{i+1} - 2a_i q_i$, $q_{i+1} - (2a_i - 1)q_i$.*

Proposition 8. *For β as in Prop.7 and $\varphi = \chi(a_i) = a_i \| q_i\beta \| + 1/2 \ \| q_{i+1}\beta \|$ the set M_φ has for $2a_i \leqslant b_{i+1}$ the gaps q_i, $q_{i+1} - (2a_i - 1)q_i$. For $b_{i+1} = 1$ the set M_φ has the gaps q_i, q_{i-1}.*

Proposition 9. *For $\beta = [b_0, b_1, \ldots]$ irrational and $\varphi = \zeta_i = a_i \| q_i\beta \| + l_{i+1} \| q_{i+1}\beta \| + 1/2(\| q_{i+1}\beta \| + \| q_{i+2}\beta \|)$ for $b_{i+2} = 2l_{i+1}$ respectively $\varphi = \bar{\zeta}_i = a_i \| q_i\beta \| + (l_{i+1} + 1) \| q_{i+1}\beta \| + \| q_{i+2}\beta \|$ for $b_{i+2} = 2l_{i+1} + 1$ the set M_φ has for $2a_i \leqslant b_{i+1}$ the gaps q_i, $q_{i+1} - 2a_i q_i$.*

The proofs of Prop.7 to 9 are straightforward. Using Prop.3 one determines the right and left nearest neighbours of any point in the set M which determine the gaps. Because this is a rather tedious but very simple task we omit the details hera and refer for more details to the appendix.
We are now ready to prove our main result, Theorem 1.

Proof of Theorem 1. Consider the interval $a_i \| q_i\beta \| + \| q_{i+1}\beta \| \leqslant \varphi < (a_i + 1) \| q_i\beta \| + \| q_{i+1}\beta \|$, $i \in J_\beta$, $1 \leqslant a_i \leqslant b_{i+1}$. Then any φ with $0 \leqslant \varphi \leqslant \| q_0\beta \|$ is contained in one of these intervals.

For $2b_{i+1} > 2a_i \geqslant b_{i+1} + 1$ by Prpos.7 there are the three gaps q_{i-1}, $q_i - q_{i-1}$, q_i. If $a_i = b_{i+1}$ the above interval becomes $\| q_{i-1}\beta \| \leqslant \varphi < \| q_{i-1}\beta \| + \| q_i\beta \|$. For $\varphi_i^0 = \| q_{i-1}\beta \|$ the gaps are by Prop.7 again q_{i-1}, $q_i - q_{i-1}$, q_i. For $\varphi_i^2 = \| q_{i-1}\beta \| + 1/2 \| q_i\beta \|$ the gaps are by Prop.8:q_{i-1}, $q_i - q_{i-1}$ if $b_i \geqslant 2$ respectively q_{i-1}, q_{i-2} for $b_i = 1$. This shows that for all φ with $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^2$ the gaps are q_{i-1}, $q_i - q_{i-1}$, q_i, where the gaps q_i disappear one after the other by being divided into two gaps of length q_{i-1} and $q_i - q_{i-1}$ at the point $\varphi = \varphi_i^2$. At the φ-value $\varphi_i^3 = \|q_{i-1}\beta\| + \| q_i\beta \|$ by Prop.7 the gaps are q_{i-2}, $q_{i-1} - q_{i-2}$, q_{i-1} if $b_i = 1$ respectively q_{i-1}, $q_i - 2q_{i-1}$, $q_i - q_{i-1}$ if $b_i \geqslant 2$. This shows that for $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$ the longest gaps present at $\varphi = \varphi_i^2$, which are $q_i - q_{i-1}$ respectively q_{i-1}, are subdivided into two gaps of length q_{i-1} and $q_i - 2q_{i-1}$ respectively q_{i-2} and $q_{i-1} - q_{i-2}$. Hence for all φ in the interval $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$ there are only these three resulting gaps present. Therefore part a) and part e) of Theorem 1 are proved.

Let us next discuss part b). For $2a_i \leqslant b_{i+1} - 2 \leqslant b_{i+1}$ we see from Prop.7 that for $\varphi = \varphi_i^0 = a_i \| q_i\beta \| + \| q_{i+1}\beta \|$ there are the gaps q_i, $q_{i+1} - 2a_i q_i$, $q_{i+1} - (2a_i - 1)q_i$. For $\varphi = \varphi_i^1 = a_i \| q_i\beta \| + l_{i+1} \| q_{i+1}\beta \| + 1/2 \ (\| q_{i+1}\beta \| + \| q_{i+2}\beta \|)$ in case $b_{i+1} = 2l_{i+1}$ respectively $\varphi_i^1 = a_i \| q_i\beta \| + (l_{i+1} + 1) \| q_{i+1}\beta \| + \| q_{i+2}\beta \|$ for $b_{i+1} = 2l_{i+1} + 1$ we see from Prop.9 that there are the gaps q_i, $q_{i+1} - 2a_i q_i$. This shows that for φ in the interval $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^1$ all gaps of length

$q_{i+1} - (2a_i - 1)q_i$ disappear step by step by being divided up in gaps of length q_i and $q_{i+1} - 2a_iq_i$. Therefore in this φ-interval only gaps of length q_i, $q_{i+1} - 2a_iq_i$, $q_{i+1} - (2a_i - 1)q_i$ are present. Because $2(a_i + 1) \leqslant b_{i+1}$ we see from Prop. 7 that for $\varphi = \varphi_i^3 = (a_i + 1)\|q_i\beta\| + \|q_{i+1}\beta\|$ there are the gaps q_i, $q_{i+1} - 2(a_i + 1)q_i$, $q_{i+1} - (2a_i + 1)q_i$. For $\varphi = \varphi_i^2 = (a_i + 1)\|q_i\beta\| + 1/2\|q_{i+1}\beta\|$ on the other hand Prop. 8 shows that M_φ has the gaps q_i, $q_{i+1} - (2a_i + 1)q_i$. This shows that for $\varphi_i^1 \leqslant \varphi \leqslant \varphi_i^2$ the gaps of length $q_{i+1} - 2a_iq_i$ are step by step subdivided into gaps of length q_i and $q_{i+1} - (2a_i + 1)q_i$ so that in this φ- interval only the three gaps q_i, $q_{i+1} - 2a_iq_i$, $q_{i+1} - (2a_i + 1)q_i$ are present. In the interval $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$ finally the gaps of length $q_{i+1} - (2a_i + 1)q_i$ are further subdivided into gaps of length q_i and $q_{i+1} - (2a_i + 2)q_i$ so that for this φ-interval only the gaps q_i, $q_{i+1} - (2a_i + 1)q_i$, $q_{i+1} - (2a_i + 2)q_i$ are present. This proves part b) of Theorem 1.

Let us discuss next part c). If $2a_i = b_{i+1}$ then Prop. 7 shows that for $\varphi_i^0 = a_i\|q_i\beta\| + \|q_{i+1}\beta\|$ there are the gaps q_i, $q_{i+1} - 2a_iq_i$, $q_{i+1} - (2a_i - 1)q_i$. For $\varphi = \varphi_i^1$ Prop. 9 gives the gaps q_i, $q_{i+1} - 2a_iq_i$ and hence for all φ in the interval $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^1$ there are the gaps q_i, $q_{i+1} - 2a_iq_i$, $q_{i+1} - (2a_i - 1)q_i$, the last one being step by step divided up into gaps of length q_i, $q_{i+1} - 2a_iq_i$. Because $2a_i = b_{i+1}$ these gaps are therefore q_i, q_{i-1}, $q_i + q_{i-1}$. For $\varphi = \varphi_i^3 = (a_i + 1)\|q_i\beta\| + \|q_{i+1}\beta\|$ we find $2(a_i + 1) = b_{i+1} + 1$. Prop. 7 shows then that for this $\varphi = \varphi_i^3$ we find the gaps q_{i-1}, $q_i - q_{i-1}$, q_i. For $\varphi_i^1 \leqslant \varphi \leqslant \varphi_i^3$ the gap q_i, longest for $\varphi = \varphi_i^1$, is step by step subdivided into gaps of length q_{i-1}, $q_i - q_{i-1}$ which shows that in this interval the gaps are just q_{i-1}, $q_i - q_{i-1}$ and q_i.

Remains to prove part d) of Theorem 1. If $2a_i = b_{i+1} - 1$ we find for $\varphi = \varphi_i^0$ the gaps q_i, $q_{i+1} - 2a_iq_i$, $q_{i+1} - (2a_i - 1)q_i$. For $\varphi = \varphi_i^1$ Prop. 9 gives the gaps q_i, $q_{i+1} - 2a_iq_i = q_{i-1} + q_i$, that means the longest gap at $\varphi = \varphi_i^0$ disappears by divided up into gaps of length q_i and $q_{i+1} - 2a_iq_i$;hence we find for $\varphi_i^0 \leqslant \varphi \leqslant \varphi_i^1$ the three gaps q_i, $q_i + q_{i-1}$, $2q_i + q_{i-1}$. Because for $\varphi = \varphi_i^2 = (a_i + 1)\|q_i\beta\| + 1/2\|q_{i+1}\beta\|$ we have $2a_i = b_{i+1} - 1 < b_{i+1}$, Prop. 8 gives for this φ-value the gaps q_{i-1}, q_i and hence for $\varphi_i^1 \leqslant \varphi \leqslant \varphi_i^2$ we have only the gaps q_{i-1}, q_i, $q_i + q_{i-1}$. For $\varphi = \varphi_i^3$ finally Prop. 7 gives again the gaps q_{i-1}, $q_i - q_{i-1}$, q_i and hence for $\varphi_i^2 \leqslant \varphi \leqslant \varphi_i^3$ the gaps q_{i-1}, $q_i - q_{i-1}$, q_i.

This concludes the proof of Theorem 1.

Unfortunately we were not able to treat with our method also those values of φ with $\|q_0\beta\| \leqslant \varphi < 1/2$. We are quite convinced that our results extend also to this case. It would be interesting to extend our results also to rational β and to study also the relative frequencies of the different gaps.

Having established the gap structure of the sets M_φ we are now prepared to prove also Corollary 1. For $i \in J_\beta$ the set $M_i = \{m: \|q_i\beta\| < \|m\beta\| < \|q_{i-1}\beta\|\}$ is obviously contained in the set M_φ with $\varphi = b_{i+1}\|q_i\beta\| + \|q_{i+1}\beta\| = \|q_{i-1}\beta\|$. Since moreover for $\varphi' = b_{i+2}\|q_{i+1}\beta\| + \|q_{i+2}\beta\| = \|q_i\beta\|$ we have

$M_i \subset \mathscr{C}M_{\varphi'} \setminus \{q_i\}$ where

$$\mathscr{C}M_{\varphi'} = \mathscr{N} \setminus M_{\varphi'} = \{m \in \mathscr{N} : \|m\beta\| \geqslant \|q_i\beta\|\}.$$

Hence $M_i = M_\varphi \cap (\mathscr{C}M_{\varphi'} \setminus \{q_i\})$. From these remarks we deduce

Lemma 4. *$m \in M_i$ if and only if its expansion $m = \sum_{k=k_0}^{\infty} r_k q_k$, $r_{k_0} \geqslant 1$, fulfills one of the following conditions:*

(1) $k_0 = i - 1 : r_{i-1} = 1$, $b_{i+1} - 1 \geqslant r_i \geqslant 1$

(2) $k_0 = i - 1 : r_{i-1} = 1$, *there exists* $n \geqslant 1$ *such that* $r_{i+k} = 0$ *for* $0 \leqslant k \leqslant 2n - 1$ *but* $r_{i+2n} \geqslant 1$

(3) $k_0 = i : r_i = 1$, *there exists* $n \geqslant 1$ *such that* $r_{i+k} = 0$ *for* $1 \leqslant k \leqslant 2n - 1$ *but* $r_{i+2n} \geqslant 1$.

This Lemma being an immediate consequence of Prop. 3 we can omit the proof. According to Lemma 4 the set M_i can be decomposed into three disjoint subsets $M_i^{(k)}$, $k = 1, 2, 3$ defined by the three conditions (1) to (3) of Lemma 4. It is rather trivial to see that every gap within one of these sets is bounded below by q_i. This implies Corollary 1 as follows: $F_i(\lambda) = \sum_{m \in M_i} m^{-\lambda} = \sum_{n=1}^{3} \sum_{m \in M_i^{(n)}} m^{-\lambda}$. If $M_i^{(n)} = \{m_0^{(n)} < m_1^{(n)} < \cdots\}$ we find $m_0^{(1)} = q_{i-1} + q_i$, $m_0^{(2)} = q_{i-1} + q_{i+2}$, $m_0^{(3)} = q_i + q_{i+2}$ and hence $m_k^{(n)} = m_0^{(n)} + \sum_{j=1}^{k} d_{j,j-1}^{(n)}$, for all $k \geqslant 1$ with $d_{j,j-1}^{(n)} = m_j^{(n)} - m_{j-1}^{(n)}$ for all $n = 1,2,3$ and all $j \geqslant 1$. Because $d_{j,j-1}^{(n)} \geqslant q_i$ and hence $m_k^{(k)} \geqslant m_0^{(n)} + k\ q_i \geqslant (k+1)q_i$ for all $k \geqslant 0$ we get

$$F_i(\lambda) \leqslant \sum_{n=1}^{3} \sum_{k \geqslant 1} k^{-\lambda} q_i^{-\lambda}.$$

If therefore $c(\lambda) = 3 \sum_{k \geqslant 1} k^{-\lambda}$ we find

$$F_i(\lambda) \leqslant c(\lambda)\ q_i^{-\lambda}$$

for all i and all irrational β.

The proof of Corollary 2 follows immediately when comparing our Theorem 1 with the following result of Slater in [6].

Theorem 2 (Slater). *For $\varphi = a_i \|q_i\beta\| + \|q_{i+1}\beta\| + \psi$, where $0 \leqslant \psi < \|q_i\beta\|$, the gaps of the sequence $\{m\beta \bmod 1\}$ within $\{m\beta \bmod 1 < \varphi\}$ are the integers q_i, $q_{i+1} - a_i q_i$ and $q_{i+1} - (a_i - 1)q_i$.*

It turns out that the gaps in Theorem 1 are exactly the gaps of the sequence $m\beta$ mod 1 with respect to the cnnected interval $[0, 2a)$. But for such intervals Slater has shown that formulas (1) are valid. But this shows that they are valid also for the kind of disconnected intervals $I_{\widehat{a,1}-\omega}$ we are looking at.

Appendix: Proof of Prop. 7

For $m \in M_{\Phi_i}$ set $m_L = \max\{m' \in M_{\Phi_i}: m' < m\}$ respectively $m_R = \min\{m', m' \in M_{\Phi_i}: m' > m\}$. In case the first set is empty we define $m_L = 0$. The gaps d_L and d_R are then defined by $d_L = m - m_L$ and $d_R = m_R - m$. Theorem 3 shows that the set M_{Φ_i} can be decomposed into disjoint subsets $M_{\Phi_i}^{(k)}$, $k = 1,2,3$, where $m \in M_{\Phi_i}^{(k)}$ if m satisfies condition (3_k) of Prop. 3. For fixed Φ_i define the variable $s_i = b_{i+1} - a_i$ with $0 \leqslant s_{i+1} - 1$. We start our discussion with the case

I. $a_i = 1$, $a_i \leqslant s_i$ $(2a_i \leqslant b_{i+1})$

In this case $M_{\Phi_i}^{(1)}$ is empty and we have $s_i \geqslant 1$ respectively $b_{i+1} \geqslant 2$.

1) Consider $m \in M_{\Phi_i}^{(2)}$ given as $m = q_{i-1} + s_i q_i + \sum_{j=i+2n} r_j q_j$.

If $n \geqslant 2$ or $n = 1$ and $r_{i+2} \neq b_{i+3}$ we find

$$m_L = q_i + \sum_{j=1+2n} r_j q_j \in M_{\Phi_i}^{(3)}: d_L = q_{i+1} - 2q_i = q_{i+1} - 2a_i q_i$$

$$m_R = q_{i+1} + \sum_{j=1+2n} r_j q_j \in M_{\Phi_i}^{(3)}: d_R = q_{i+1} - (2a_i - 1)q_i.$$

If $n = 1$ and $r_{i+2} = b_{i+3}$ we get

$$m_R = (r_{i+3} + 1)q_{i+3} + \sum_{j=i+4} r_j g_j \in M_{\Phi_i}^{(3)}: d_R = q_i.$$

2) Consider next $m \in M_{\Phi_i}^{(3)}$, $m = \sum_{j=i+n} r_j q_j$, $n \geqslant 0$ and $r_i = 1$ if $n = 0$.

If $n = 1$ we get

$$m_L = q_i + (r_{i+1} - 1)q_{i+1} + \sum_{j=i+2} r_j q_j \in M_{\Phi_i}^{(3)}: d_L = q_{i+1} - (2a_i - 1)q_i$$

$$m_R = q_i + \sum_{j=i+1} r_j q_j \in M_{\Phi_i}^{(3)}, \text{ if } r_{i+1} < b_{i+2}: d_R = q_i$$

respectively

$$m_R = (r_{i+2k-1} + 1)q_{i+2k-1} + \sum_{j=i+2k} r_j q_j \in M_{\Phi_i}^{(3)}$$

$$\text{where } k = \min\{l: r_{i+2l-1} \neq b_{i+2l}\}: d_R = q_i.$$

If $n = 0$ we have $m = q_i + \sum_{j=i+1} r_j q_j$, $l \geqslant 1$,

$$m_L = \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_L = q_i,$$

$$m_R = q_{i-1} + s_i q_i + \sum_{j=i+l} r_j q_j \in M^{(2)}_{\Phi_i}, \text{ if } l \text{ is even and } s_i = 1:$$

$$d_R = q_{i+1} - 2a_j q_j$$

respectively

$$m_R = q_{i+1} + \sum_{j=i+l} r_j q_j, \text{ if } l \text{ is odd or } s_i \neq 1\colon d_R = q_{i+1} - (2a_i - 1)q_i$$

II. $a_i = 1,\ a_i \geqslant s_i + 1\ \ (2a_i \geqslant b_{i+1} + 1)$

In this case $s_i = 0$ respectively $b_{i+1} = a_i = 1$.

1) $m \in M^{(2)}_{\Phi_i},\ m = q_{i-1} + \sum_{j=2n} r_j q_j,\ n \geqslant 1$ and we find:

$$m_L = \sum_{j=2n} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_L = q_{i-1}$$

$$m_R = q_i + \sum_{j=2n} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_R = q_i - q_{i-1}.$$

2) $m \in M^{(3)}_{\Phi_i}\colon m = \sum_{j=i+n} r_j q_j,\ n \geqslant 0$ and $r_i = 1$ if $n = 0$.

If $n = 1$ we find

$$m_L = q_i + (r_{i+1} - 1)q_{i+1} + \sum_{j=i+2} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_L = q_{i+1} - q_i$$

$$m_R = q_i + m, \text{ if } r_{i+1} < b_{i+2}\colon d_R = q_i$$

respectively for $r_{i+1} = b_{i+2}$

$$m_R = (r_{i+2} + 1)q_{i+2} + \sum_{j=i+3} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_R = q_i.$$

If $n > 1$ and even we find

$$m_L = \sum_{j=1}^{n/2} b_{i+2j} q_{i+2j-1} + (r_{i+n} - 1)q_{i+n} + \sum_{j=i+n+1} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_L = q_i,$$

$$m_R = q_{i-1} + \sum_{j=i+n} r_j q_j \in M^{(2)}_{\Phi_i}\colon d_R = q_{i-1}$$

If $n > 1$ and odd

$$m_L = q_{i-1} + \sum_{j=1} b_{i+2j+1} q_{i+2j} + (r_{i+n} - 1)q_{i+n} + \sum_{j=i+n+1} r_j q_j \in M^{(2)}_{\Phi_i}\colon d_L = q_i$$

$m_R = q_i + \sum_{j=i+n} r_j q_j \in M^{(3)}_{\Phi_i}$: $d_R = q_i$.

If $n = 0$ and therefore $m = q_i + \sum_{j=i+l} r_j q_j$, $l \geqslant 1$ we get

$m_L = q_{i-1} + \sum_{j=i+l} r_j q_j \in M^{(2)}_{\Phi_i}$ in case l is even: $d_L = q_i - q_{i-1}$

$m_L = \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i}$ in case l is odd: $d_L = q_i$

$m_R = (r_{i+1} + 1)q_{i+1} + \sum_{j=i+2} r_j q_j \in M^{(3)}_{\Phi_i}$ in case $l = 1$: $d_R = q_{i-1}$,

$m_R = q_{i+1} + \sum_{j=i+1} r_j q_j \in M^{(3)}_{\Phi_i}$ if $l \geqslant 3$, or $l = 2$ *and* $r_{i+2} < b_{i+3}$: $d_R = q_{i-1}$

$m_R = (r_{i+3} + 1)q_{i+3} + \sum_{j=i+4} r_j q_j \in M^{(3)}_{\Phi_i}$, if $l = 2$ and $r_{i+2} = b_{i+3}$: $d_R = q_{i-1}$

This concludes the discussion of the case $a_i = 1$.

III. $a_i > 1$, $a_i \leqslant s_i$ $(2a_i \leqslant b_{i+1})$

In this case $b_{i+1} \geqslant 2$ and $M^{(k)}_{\Phi_i} \neq \phi$ for $k = 1,2,3$.

1) $m \in M^{(1)}_{\Phi_i}$, $m = q_{i-1} + (s_i + k)q_i + \sum_{j=i+n} r_j q_j$, $1 \leqslant k \leqslant a_i - 1$, $n \geqslant 1$:

For $1 < k < a_i - 1$ we get

$m_L = m - g_i \in M^{(1)}_{\Phi_i}$: $d_L = q_i$

$m_R = m + q_i \in M^{(1)}_{\Phi_i}$: $d_R = q_i$.

For $k = 1$ and therefore $m = q_{i-1} + (s_i + 1)q_i + \sum_{j=i+n} r_j q_j$, $n \geqslant 1$

we find

$m_L = q_{i-1} + s_i q_i + \sum_{j=i+n} r_j q_j \in M^{(2)}_{\Phi_i}$ if n is even: $d_L = q_i$,

$m_L = a_i q_i + \sum_{j=i+n} r_j q_j \in M^{(3)}_{\Phi_i}$ if n is odd: $d_L = q_{i+1} - (2a_i - 1)q_i$

and

$m_R = m + q_i \in M^{(1)}_{\Phi_i}$: $d_R = q_i$.

For $k = a_i - 1$ and hence $m = q_{i-1} + (b_{i+1} - 1)q_i + \sum_{j=i+n} r_j q_j$ we get

$m_L = m - q_i \in M^{(1)}_{\Phi_i}$: $d_L = q_i$

$m_R = (r_{i+1} + 1)q_{i+1} + \sum_{j=i+2} r_j q_j \in M^{(3)}_{\Phi_i}$ if $n = 1$: $d_R = q_i$

$m_R = q_{i+1} + \sum_{j=i+n} r_j q_j \in M^{(3)}_{\Phi_i}$ if $n \geqslant 3$ or $n = 2$ and $r_{i+2} < b_{i+3}$:

$d_R = q_i$,

$m_R = (r_{i+2k-1} + 1)q_{i+2k-1} + \sum_{j=i+2k} r_j q_j \in M^{(3)}_{\Phi_i}$ if $r_{i+2} = b_{i+3}$ and

$k = \min\{l : r_{i+2l} < b_{i+2l-1}\}$: $d_R = q_i$.

2) $m \in M^{(2)}_{\Phi_i}$ with $m = q_{i-1} + s_i q_i + \sum_{j=i+2n} r_j q_j$, $n \geqslant 1$. In this case we find:

$m_L = a_i q_i + \sum_{j=i+2n} r_j q_j \in M^{(3)}_{\Phi_i}$: $d_L = q_{i+1} - 2a_i q_i$

$m_R = m + q_i \in M^{(2)}_{\Phi_i}$: $d_R = q_i$.

3) $m \in M^{(3)}_{\Phi_i}$, $m = \sum_{j=i+n} r_j q_j$, $n \geqslant 0$ and $1 \leqslant r_i \leqslant a_i$ in the case $n = 0$.

If $n = 0$ and $1 \leqslant r_i < a_i$ one finds

$m_L = m - q_i \in M^{(3)}_{\Phi_i}$: $d_L = q_i$

$m_R = m + q_i \in M^{(3)}_{\Phi_i}$: $d_R = q_i$.

If $n = 0$ and $r_i = a_i$ we get

$m_L = m - q_i \in M^{(3)}_{\Phi_i}$: $d_L = q_i$

$m_R = q_{i-1} + (s_i + 1)q_i + \sum_{j=i+k} r_j q_j \in M^{(1)}_{\Phi_i}$ for k odd: $d_R = q_{i+1} - (2a_i - 1)q_i$

respectively

$m_R = q_{i-1} + s_i q_i + \sum_{j=i+k} r_j q_j \in M^{(2)}_{\Phi_i}$ for k even, $k \geqslant 2$: $d_R = q_{i+1} - 2a_i q_i$.

If $n = 1$ and hence $m = r_{i+1} q_{i+1} + \sum_{j=i+2} r_j q_j$ one gets

$m_L = q_{i-1} + (b_{i+1} - 1)q_i + m - q_{i+1} \in M^{(1)}_{\Phi_i}$: $d_L = q_i$

$m_R = m + q_i \in M^{(3)}_{\Phi_i}$ if $r_{i+1} \neq b_{i+2}$: $d_R = q_i$

$m_R = (r_{i+2} + 1)q_{i+2} + \sum_{j=i+3} r_j q_j \in M^{(3)}_{\Phi_i}$ if $r_{i+1} = b_{i+2}$: $d_R = q_i$.

If $n \geqslant 2$ and hence $m = \sum_{j=i+n} r_j q_j$ we get for n even

$$m_L = \sum_{j=1}^{n/2} b_{i+2j} q_{i+2j-1} + (r_{i+n} - 1) q_{i+n} + \sum_{j=i+n+1} r_j q_j \in M_{\Phi_i}^{(3)}: d_L = q,$$

whereas for n odd one finds

$$m_L = q_{i-1} + (b_{i+1} - 1) q_i + \sum_{j=1}^{(n-1)/2} b_{i+1+2j} q_{i+2j} + (r_{i+n} - 1) q_{i+n}$$

$$+ \sum_{j=i+n+1} r_j q_j \in M_{\Phi_i}^{(1)}: d_L = q_i$$

$m_R = m + q_i \in M_{\Phi_i}^{(3)}: d_R = q_i$ both for odd and even n.

IV. $a_i > 1,\ a_i \geqslant s_i + 1\ (2a_i \geqslant b_{i+1} + 1)$

1) $m \in M_{\Phi_i}^{(1)}: m = q_{i-1} + (s_i + k) q_i + \sum_{j=i+n} r_j q_j,\ 1 \leqslant k \leqslant a_i - 1,\ n \geqslant 1.$

If $k = 1 \neq a_i - 1$ then

$$m_L = (s_i + 1) q_i + \sum_{j=i+n} r_j q_j \in M_{\Phi_i}^{(3)}: d_L = q_{i-1}$$

$$m_R = (s_i + 2) q_i + \sum_{j=i+n} r_j q_j \in M_{\Phi_i}^{(3)} \text{ if } a_i > s_i + 1: d_R = q_i - q_{i-1}$$

respectively

$m_R = m + q_i \in M_{\Phi_i}^{(1)}$ if $a_i = s_i + 1: d_R = q_i.$

If $k = 1 = a_i - 1$ and hence $a_i = 2$ we see from $a_i \geqslant s_i + 1$ that $b_{i+1} = 3$ and therefore $m = q_{i-1} + 2q_i + \sum_{j=i+n} r_j q_j,\ n \geqslant 1.$ Hence we get

$$m_L = 2q_i + \sum_{j=i+n} r_j q_j \in M_{\Phi_i}^{(3)}: d_L = q_{i-1}$$

$$m_R = q_{i+1} + \sum_{j=i+n} r_j q_j \in M_{\Phi_i}^{(3)}: d_R = q_i.$$

If $k \geqslant 2$ we get for $k = a_i - 1: m = q_{i-1} + (b_{i+1} - 1) q_i + \sum_{j=i+n} r_j q_j,\ n \geqslant 1$

and hence

$$m_L = q_{i-1} + (b_{i+1} - 2) q_i + \sum_{j=i+n} r_j q_j \in M_{\Phi_i}^{(1)}: d_L = q_i$$

$$m_R = b_{i+1} q_i + \sum_{j=i+n} r_j q_j \in M_{\Phi_i}^{(3)}, \text{ if } a_i = b_{i+1}: d_R - q_{i-1}$$

$m_R = q_{i+1} + \sum_{j=i+n} r_j q_j \in M^{(3)}_{\Phi_i}$, if $a_i < b_{i+1}$ and $n = 1$ or $n = 3$: $d_R = q_i$

$m_R = q_{i+1} + \sum_{j=i+2} r_j q_j \in M^{(3)}_{\Phi_i}$, if $a_i < b_{i+1}$, $n = 2$ and $r_{i+2} \leqslant b_{i+3} - 1$: $d_R = q_i$

$m_R = (r_{i+3} + 1) q_{i+3} + \sum_{j=i+4} r_j q_j \in M^{(3)}_{\Phi_i}$, if $a_i < b_{i+1}$, $n = 2$, $r_{i+2} = b_{i+3}$: $d_R = q_i$.

Remains for $m \in M^{(1)}_{\Phi_i}$ the case $1 < k < a_i - 1$:

if $s_i + k \geqslant a_i + 1$ then

$$m_L = q_{i-1} + (s_i + k - 1) q_i + \sum_{j=i+n} r_j q_j \in M^{(1)}_{\Phi_i}: d_L = q_i$$

where as for $s_i + k \leqslant a_i$

$$m_L = (s_i + k) q_i + \sum_{j=i+n} r_j q_j \in M^{(3)}_{\Phi_i}: d_L = q_{i-1}.$$

The corresponding right neighbours are

$$m_R = m + q_i \in M^{(1)}_{\Phi_i} \text{ if } s_i + k \geqslant a_i: d_R = q_i$$

$$m_R = (s_i + k + 1) q_i + \sum_{j=i+n} r_j q_j \in M^{(3)}_{\Phi_i} \text{ if } s_i + k \leqslant a_i - 1: d_R = q_i - q_{i-1}.$$

2) $m \in M^{(2)}_{\Phi_i}$: $m = q_{i-1} + s_i q_i + \sum_{j=i+2n} r_j q_j$, $n \geqslant 1$.

For $a_i \geqslant s_i + 1$ one finds

$$m_L = s_i q_i + \sum_{j=i+2n} r_j q_j \in M^{(3)}_{\Phi_i}: d_L = q_{i-1}$$

$$m_R = (s_i + 1) q_i + \sum_{j=i+2n} r_j q_j \in M^{(3)}_{\Phi_i}: d_R = q_i - q_{i-1}.$$

3) $m \in M^{(3)}_{\Phi_i}$: $m = \sum_{j=i+n} r_j q_j$, $n \geqslant 0$ and $1 \leqslant r_i \leqslant a_i$ if $n = 0$.

If $n = 0$, $1 \leqslant r_i \leqslant a_i$ we therefore have $m = r_i q_i + \sum_{j=i+l} r_j q_j$, $l \geqslant 1$.

For $l = 1$ we find

$$m_L = (r_i - 1) q_i + \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i} \text{ if } r_i \leqslant s_i + 1: d_L = q_i$$

$$m_L = q_{i-1} + (r_i - 1)q_i + \sum_{j=i+l} r_j q_j \in M^{(1)}_{\Phi_i} \text{ if } r_i > s_i + 1\colon d_L = q_i - q_{i-1}.$$

For $l \geqslant 2$ we find

$$m_L = (r_i - 1)q_i + \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i},\ 1 \leqslant r_i \leqslant s_i\colon\ d_L = q_i$$

$$m_L = q_{i-1} + s_i q_i + \sum_{j=i+l} r_j q_j \in M^{(2)}_{\Phi_i},\ r_i = s_i + 1,\ l \text{ even: } d_L = q_i$$

$$m_L = s_i q_i + \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i},\ r_i = s_i + 1,\ l \text{ odd: } d_L = q_i - q_{i-1}$$

$$m_L = q_{i-1} + (r_i - 1)q_i + \sum_{j=i+l} r_j q_j \in M^{(1)}_{\Phi_i}\colon r_i \geqslant s_i + 2\colon d_L = q_i - q_{q-1}.$$

For the right neighbour we find in the case $n = 0$ and $l = 1$:

$$m_R = (r_i + 1)q_i + \sum_{j=i+1} r_j q_j \in M^{(3)}_{\Phi_i} \text{ if } r_i < s_i + 1\colon d_R = q_i$$

$$m_R = q_{i-1} + m \in M^{(1)}_{\Phi_i}, \text{ if } r_i = s_i + 1\colon d_R = q_{i-1}$$

$$m_R = q_{i-1} + m \in M^{(1)}_{\Phi_i} \text{ if } r_i > s_i + 1,\ r_i \leqslant a_i < b_{i+1} \text{ or } r_i < a_i = b_{i+1}\colon$$

$$d_R = q_{i-1}$$

$$m_R = (r_{i+1} + 1)q_{i+1} + \sum_{j=i+2} r_j q_j \in M^{(3)}_{\Phi_i} \text{ if } r_i = a_i = b_{i+1}\colon d_R = q_{i-1}.$$

For $l \geqslant 2$ we find for $1 \leqslant r_i \leqslant s_i - 1$

$$m_R = (r_i + 1)q_i + \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i}\colon d_R = q_i$$

$$m_R = q_{i-1} + s_i q_i + \sum_{j=i+l} r_j q_j \in M^{(2)}_{\Phi_i} \text{ if } r_i = s_i,\ l \text{ even: } d_R = q_{i-1}$$

$$m_R = m + q_i \in M^{(3)}_{\Phi_i} \text{ if } r_i = s_i,\ l \text{ odd: } d_R = q_i.$$

For $r_i = s_i + 1$ we find

$$m_R = q_{i-1} + (s_i + 1)q_i + \sum_{j=i+l} r_j q_j \in M^{(1)}_{\Phi_i}\colon d_R = q_{i-1}.$$

If $l \geqslant 2$ and $a_i \geqslant r_i > s_i + 1$ we see that

$$m_R = q_{i-1} + m \in M^{(1)}_{\Phi_i} \text{ if } r_i \neq b_{i+1}\colon d_R = q_{i-1}$$

$$m_R = q_{i+1} + \sum_{j=i+l} r_j q_j \in M^{(3)}_{\Phi_i} \text{ if } r_i = a_i = b_{i+1},\ r_{i+2} < b_{i+3}\colon$$

$d_R = q_{i-1}$

$m_R = (r_{i+2k-1} + 1)q_{i+2k-1} + \sum_{j=i+2k} r_j q_j \in M^{(3)}_{\Phi_i}$ if $r_i = a_i = b_{i+1}$

and $k = \min\{l: r_{i+2l} < b_{i+2l+1}\}$: $d_R = q_{i-1}$.

If $n = 1$ we have $m = \sum_{j=i+1} r_j q_j$ and hence

$$m_L = q_{i-1} + (b_{i+1} - 1)q_i + (r_{i+1} - 1)q_{i+1} + \sum_{j=i+2} r_j q_j \in M^{(1)}_{\Phi_i}: d_L = q_i$$

$m_R = q_i + m \in M^{(3)}_{\Phi_i}$ if $r_{i+1} \leqslant b_{i+2} - 1$: $d_R = q_i$

$m_R = (r_{i+2k-1} + 1)q_{i+2k-1} + \sum_{j=i+2k} r_j q_j \in M^{(3)}_{\Phi_i}$, if $r_{i+1} = b_{i+2}$ and

$k = \min\{l: r_{i+2l-1} < b_{i+2l}\}$: $d_R = q_i$.

If $n \geqslant 2$ we have for even n:

$$m_L = \sum_{j=1}^{n/2} b_{i+2j} q_{i+2j-1} + (r_{i+n} - 1)q_{i+n} + \sum_{j=i+n+1} r_j q_j \in M^{(3)}_{\Phi_i}:$$

$d_L = q_i$

and for n odd

$$m_L = q_{i-1} + (b_{i+1} - 1)q_i + \sum_{j=1}^{(n-1)/2} b_{i+1+2j} q_{i+2j} + (r_{i+n} - 1)q_{i+n} + \sum_{j=i+n+1} r_j q_j \in M^{(1)}_{\Phi_i}: d_L = q_i$$

$m_R = q_i + m \in M^{(3)}_{\Phi_i}$: $d_R = q_i$.

This then proves Prop. 7. The proofs of Prop. 8 respectively 9 proceed along the same lines.

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