

Partial *-Algebras of Closable Operators

I. The Basic Theory and the Abelian Case

By

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Abstract

This paper, the first of two, is devoted to a systematic study of partial *-algebras of closable operators in a Hilbert space (partial Op*-algebras). After setting up the basic definitions, we describe canonical extensions of partial Op*-algebras by closure and introduce a new bounded commutant, called quasi-weak. We initiate a theory of abelian partial *-algebras. As an application, we analyze thoroughly the partial Op*-algebras generated by a single closed symmetric operator.

§ 1. Introduction

Ever since the pioneering days of Heisenberg's Matrix Mechanics, operator algebras have played a prominent role in quantum theories. For instance, the algebraic language is by now standard in quantum statistical mechanics (e. g. see the monograph of Bratteli and Robinson [1]). However the algebras used in this context consist invariably of bounded operators (in particular representations of abstract C*-algebras) and their bicommutants, that is, von Neumann algebras. In particular the latter play a crucial role in the Tomita-Takesaki theory [1].

Yet this framework is often too narrow for applications. The next step is to consider algebras of unbounded operators, consisting of operators with a common dense invariant domain. Take for instance a nonrelativistic one particle quantum system. In the Schrödinger representation, with Hilbert space $L^2(\mathbf{R}^3)$, the canonical variables are represented by the operators \mathbf{q} and \mathbf{p} , both unbounded and obeying the canonical commutation relations $[p_j, q_k] = i\delta_{jk}$. The natural domain associated to this system is of course Schwartz space $\mathcal{S}(\mathbf{R}^3)$, and the corresponding *-algebra $\mathcal{L}^\dagger(\mathcal{S})$ consists of all operators A such that $A\mathcal{S} \subset \mathcal{S}$ and $A^*\mathcal{S} \subset \mathcal{S}$. This algebra (slightly generalized if spin is

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considered) appears as the basic object for nonrelativistic Quantum Mechanics, and it has quite remarkable properties (see the forthcoming monograph [2] for a detailed study). Elaboration of this model eventually led to the well-developed theory of Op^* -algebras [3-5].

However, it is sometimes unnatural, even impossible, to demand a common invariant domain for all relevant operators in a given problem. To give a trivial example: in the simple case described above, \mathcal{S} is invariant under \mathfrak{q} and \mathfrak{p} , but it is of course not invariant under any of their spectral projections. Another instance is a Wightman field theory, where the natural (Gårding) domain is not always invariant under the elements of (local) field algebras [6]. Furthermore, the algebra $\mathcal{L}^\dagger(\mathcal{S})$ is not complete under the strong* topology, which is both natural and convenient, and its completion is not an algebra. This state of affairs prompted W. Karwowski and one of us (JPA) to introduce [7] the concept of *partial *-algebra of operators*, or *partial Op^* -algebra*. A systematic analysis of those was undertaken in recent papers with Mathot and Trapani [8, 9], where, in particular, various notions of commutants and bicommutants were introduced and studied.

Of course partial Op^* -algebras are only a special case of (topological) *partial *-algebra*. This concept, originally due to Borchers [10], was also developed in Ref. 7. Several other types of partial *-algebras have been discussed in the literature, for instance:

(1) The *(topological) quasi *-algebras*, introduced by Lassner [11] and obtained by completion of a topological *-algebra for which the multiplication is separately, but not jointly, continuous.

(2) The *left partial Hilbert algebras*, introduced by one of us (AI) in the context of a generalized Tomita-Takesaki theory [12], and also, independently, by Ekhaguere [13].

(3) Partial *-algebras of operators on *partial inner product spaces* [14, 15].

The objective of the present paper is to continue the systematic study of partial Op^* -algebras, and, in parallel, to develop a theory of representations of abstract partial *-algebras, following the pattern familiar for *-algebras and based on the Gel'fand-Naimark-Segal (GNS) construction. This approach has been highly successful in statistical mechanics and in quantum field theory, and so it seems desirable to extend it to *partial *-algebras* as well.

Before giving a detailed outline of the two papers, we have to emphasize a crucial change with respect to the previous work on the subject. Instead of considering *closed* operators on a Hilbert space, with a common core \mathcal{D} , we take their restriction to \mathcal{D} : our partial Op^* -algebras consist now of *closable* operators. The two approaches are fully equivalent, but the new one simplifies the picture on several counts. It facilitates the comparison with Op^* -algebras and clarifies the extension theory, which is especially useful for representations. In order to make the papers self-contained, we will reproduce

here the main definitions, indicating along the way the differences (in particular, in notation) with the earlier presentation. We refer to Ref. 8 for further details.

The present paper (Part I) is organized as follows. We begin, in Section 2, by examining the notion of *abelian* or *commutative* partial *-algebra; the structure of those is more complicated than in the usual case, because the partial multiplication need not be associative. Section 3 reviews the basic theory of partial *-algebras of *closable* operators on a Hilbert space. After recasting the main definitions in this new framework, we present a systematic theory of *extensions* of partial Op*-algebras, elaborating on the results of Refs. 7 and 8. We also define a new kind of bounded commutant, called *quasi-weak*. The decisive aspect here is to incorporate in the definition of the commutant the lack of associativity of the partial multiplication, a characteristic property of partial Op*-algebras, and a very troublesome one. All these notions will prove useful in the theory of representations described in Part II. Section 4 is devoted to a thorough analysis of the partial Op*-algebras generated by a single symmetric operator, and their bounded commutants. Here again the situation is rather tricky, but a quick look at differential operators on a finite interval shows that all possible pathologies do occur in practice. In particular, these partial Op*-algebras are often really partial, that is, they are not *-algebras, and they are rarely abelian. Of course the situation simplifies considerably, but not completely, when the generating operator is self-adjoint: even then, the operator generates often a genuine, non abelian, partial Op*-algebra. In the Appendix, finally, we relate the quasi-weak bounded commutant to the various types of commutants, bounded or unbounded, introduced in the earlier papers.

The central topic of Part II is the theory of representations abstract partial *-algebras. The definition is the natural one: representation of a partial *-algebra \mathfrak{A} is a homomorphism of \mathfrak{A} into some partial Op*-algebra (see Section 2 below). For these representations, the notions of extension and of adjoint, familiar for representations of *-algebras, are defined with help of the extension theory described here in Section 3. When it comes to the explicit realization of a representation, the GNS construction [1] is usually the answer, so we have to extend it to partial *-algebras. The problem is that, for a *partial* *-algebra, positivity of a linear functional is not well-defined in general, hence we shall consider positive *sesquilinear* forms instead. So the question we want to address in Part II reads: to characterize a class of positive sesquilinear forms on a partial *-algebra that makes possible a GNS construction.

In the case of a *-algebra \mathfrak{A} , a positive sesquilinear form ϕ on $\mathfrak{A} \times \mathfrak{A}$ is called *invariant* if $\phi(x^*y, z) = \phi(y, xz)$, for all $x, y, z \in \mathfrak{A}$ and then the GNS construction for ϕ is always possible. But if \mathfrak{A} is partial *-algebra, the products x^*y, xz are not always defined and the definition above is not acceptable. We will provide in Part II, Sec. 3, a new definition of invariance for sesquilinear forms which does the job: as we show, the GNS construction is now possible for

every invariant positive sesquilinear (i. p. s.) form and the resulting representation has all the expected properties. The new concept of i. p. s. form generalizes that of *h-form*, introduced previously by Lassner and one of us (JPA) for the same purpose [16, 17]; the latter turns out to be unduly restrictive in practice.

If \mathfrak{A} is a topological quasi $*$ -algebra [11], it is easy to construct i. p. s. forms on $\mathfrak{A} \times \mathfrak{A}$ by taking limits of suitable linear forms on the dense subalgebra \mathfrak{A}_0 . But this approach fails for general partial Op^* -algebra. For that reason, we are led to introduce a restricted class, called *partial GW^* -algebras*, characterized essentially by a large content in bounded operators. This class of partial Op^* -algebras seems interesting in itself, and we study them in detail in Part II, Section 4. In fact, they behave in many respects as a natural generalization of von Neumann (W^*) algebras, and of topological quasi $*$ -algebras as well. A related topic discussed in Part II (Section 5) is the question of standardness of GNS representation of partial $*$ -algebras. In the abelian case, for instance, we show that the well-known criterion of Powers [3] extends to abelian partial GW^* -algebras.

Of course, there are many more aspects to the theory of partial $(\text{Op})^*$ -algebras, and a number of topics will be discussed in several subsequent papers. For instance:

- the notion of cyclic or strongly cyclic vector for a representation [18] and correlatively the regularity properties of i. p. s. forms [19].
- the order structure of the space of i. p. s. forms, in particular the extension to partial $*$ -algebras of the well-known Radon-Nikodym theorem and Lebesgue decomposition theorem [20-22].
- the *extendability* of i. p. s. forms upon addition of a unit, in case the original partial $*$ -algebra does not contain one.
- the *normality* of i. p. s. forms on partial $(\text{Op})^*$ -algebras, which is crucial for applications to quantum theories.

The moral of the whole story is clear: partial $*$ -algebras are vastly more complex objects than $*$ -algebras. This shows up in a particularly vivid fashion in the study of the abelian case, outlined in the present Part I. Yet a surprisingly large number of results do extend naturally from Op^* -algebras to a class of partial Op^* -algebras, namely the partial GW^* -algebras. This gives hope that this theory will provide a viable framework for physical applications.

§2. Abelian Partial $*$ -Algebras

A *partial $*$ -algebra* is a complex vector space \mathfrak{A} with an involution $x \rightarrow x^*$ (i. e. $(x + \lambda y)^* = x^* + \bar{\lambda}y^*$, $x^{**} = x$) and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ such that:

- (i) $(x, y) \in \Gamma$ iff $(y^*, x^*) \in \Gamma$;
- (ii) if $(x, y) \in \Gamma$ and $(x, z) \in \Gamma$, then $(x, \lambda y + \mu z) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$;
- (iii) whenever $(x, y) \in \Gamma$, there exists an element $x \cdot y \in \mathfrak{A}$ with the usual

properties of the multiplication :

$$x \cdot (y + \lambda z) = x \cdot y + \lambda(x \cdot z) \text{ and } (x \cdot y)^* = y^* \cdot x^*, \text{ for } (x, y), (x, z) \in \Gamma \text{ and } \lambda \in \mathbb{C}.$$

An element e of \mathfrak{A} is said to be a *unit* if $e^* = e$, $(e, x) \in \Gamma$ and $ex = xe = x$ for every $x \in \mathfrak{A}$.

Whenever $(x, y) \in \Gamma$, we say that x is a *left multiplier* of y and y a *right multiplier* of x , and write $x \in L(y)$ and $y \in R(x)$. By (ii), $L(x)$ and $R(x)$ are vector subspaces of \mathfrak{A} . For a subset $\mathfrak{N} \subset \mathfrak{A}$, we write

$$L(\mathfrak{N}) = \bigcup_{x \in \mathfrak{N}} L(x), \quad R(\mathfrak{N}) = \bigcup_{x \in \mathfrak{N}} R(x).$$

Notice that the multiplication is *not* required to be associative, but it must be distributive with respect to the addition by (iii). This lack of associativity makes the structure of abelian partial *-algebras much trickier than usual; we shall study this in detail below. However, in some cases, a weak form of associativity is useful: a partial *-algebra \mathfrak{A} is called *semi-associative* [8] if $y \in R(x)$ implies $y \cdot z \in R(x)$ for every $z \in R(\mathfrak{A})$ and then one has $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. We shall meet examples below.

A **-homomorphism* of a partial *-algebra \mathfrak{A} into another one \mathfrak{B} is a linear map σ such that (i) $\sigma(x^*) = \sigma(x)^*$ for each $x \in \mathfrak{A}$, and (ii) whenever $x \in L(y)$ in \mathfrak{A} , then $\sigma(x) \in L(\sigma(y))$ in \mathfrak{B} and $\sigma(x) \cdot \sigma(y) = \sigma(x \cdot y)$. Notice that, even if σ is a *-homomorphism of \mathfrak{A} onto \mathfrak{B} and it is a bijection, σ^{-1} is not necessarily a *-homomorphism. A *-homomorphism σ of \mathfrak{A} onto \mathfrak{B} is said to be a **-isomorphism* if it is a bijection and σ^{-1} is a *-homomorphism.

Abelianness in the context of partial *-algebras is defined in the natural way:

Definition 2.1. *A partial *-algebra \mathfrak{A} is said to be abelian, or commutative, if the following conditions hold:*

- (i) $(x, y) \in \Gamma \Leftrightarrow (y, x) \in \Gamma, \quad x, y \in \mathfrak{A};$
- (ii) $x \cdot y = y \cdot x, (x, y) \in \Gamma,$

or, equivalently,

- (i) $L(x) = R(x), \forall x \in \mathfrak{A};$
- (ii) $x \cdot y = y \cdot x, \forall x \in \mathfrak{A}, y \in R(x).$

Examples of abelian partial *-algebras are quite familiar.

Example 1. *Partial *-algebras of functions:*

Let Ω be a Lebesgue-measurable set in \mathbb{R}^n ; as usual, we denote by $L^p(\Omega)$ the Banach space of all measurable functions $f: \Omega \rightarrow \mathbb{C}$ such that $\|f\|_p \equiv (\int_{\Omega} |f|^p dt)^{1/p} < \infty$. For $f \in L^p(\Omega)$, consider the following set of real numbers:

$$\omega(f) = \{q \in [1, \infty) \mid \|f\|_q < \infty\}.$$

We can now define a partial multiplication in $L^p(\Omega)$, taking Γ as

$$\Gamma = \{(f, g) \in L^p(\Omega) \times L^p(\Omega) \mid \exists r \in \omega(f); pr(r - p)^{-1} \in \omega(g)\}.$$

By Hölder’s inequality, $(f, g) \in \Gamma$ implies that the (ordinary) product fg belongs to $L^p(\Omega)$. It is easy to check that under this operation $L^p(\Omega)$ is a semi-associative partial $*$ -algebra. Abelianness is evident.

Example 2. *Partial $*$ -algebras of polynomials:*

Let $\mathfrak{P}(z)$ be the set of all complex polynomials of arbitrary degree in the real variable z . $\mathfrak{P}(z)$ is an abelian $*$ -algebra (where $*$ is understood to be the complex conjugation).

$\mathfrak{P}(z)$ contains plenty of abelian partial $*$ -algebras. Let us denote by $\mathfrak{P}_r(z)$ the following subset of $\mathfrak{P}(z)$ (by δp we mean the degree of the polynomial p):

$$\mathfrak{P}_r(z) = \{p(z) \in \mathfrak{P}(z) \mid \delta p \leq r\}.$$

It is readily checked that $\mathfrak{P}_r(z)$ is an abelian partial $*$ -algebra when we take as Γ the following set

$$\Gamma = \{(p, q) \in \mathfrak{P}_r(z) \times \mathfrak{P}_r(z) \mid \delta p + \delta q \leq r\}.$$

Typical examples of abelian algebras are those generated by one hermitian element of a non-abelian algebra. If \mathfrak{A} is a partial $*$ -algebra and x an element of \mathfrak{A} , we may consider the partial $*$ -algebra $\mathfrak{M}(x)$ generated by x as the intersection of all partial $*$ -subalgebras of \mathfrak{A} containing x . This is clearly a well-defined object. If \mathfrak{A} is an algebra and $x = x^*$, then $\mathfrak{M}(x)$ consists of all polynomials in x and it is abelian, but this need not be true any more in our case. Indeed, pathologies may arise from the following two facts which are peculiar to partial $*$ -algebras:

- (i) an element x is not necessarily a multiplier of itself;
- (ii) if they are defined, we may have several n^{th} powers of x , because of the failure of associativity.

So abelianness may fail. Let us examine the first few possibilities to clarify our discussion.

Case 1: x is not a multiplier of itself

$$\mathfrak{M}(x) = \{\lambda + \mu x \mid \lambda, \mu \in \mathbb{C}\} = \mathfrak{P}_1(x), \text{ and it is abelian.}$$

Case 2: $x \in R(x)$ ($\Leftrightarrow x \in L(x)$), but $x \notin R(x^2)$, $x^2 = x \cdot x$.

(i) We may have $x^2 \notin R(x^2)$; in this case:

- (i₁) If $q(x) \notin R(p(x))$, for all polynomials p and q with $1 \leq \delta p \leq 2$ and $\delta q = 2$, then

$$\mathfrak{M}(x) = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 \mid \alpha_i \in \mathbb{C}\} = \mathfrak{P}_2(x), \text{ abelian.}$$

(i₂) We may have $q(x) \in R(p(x))$ for some polynomials p and q with $1 \leq \delta p \leq 2$ and $\delta q = 2$. In this case,

$$p(x)q(x) \in \mathfrak{M}(x) \text{ and } \mathfrak{B}_2(x) \not\subseteq \mathfrak{M}(x).$$

(ii) We may have $x^2 \in R(x^2)$. In that case, neither $x^2 \cdot x$, nor $x \cdot x^2$ are defined, but $x^2 \cdot x^2$ is defined. Then the structure of $\mathfrak{M}(x)$ is more complex than in case (i).

In view of such pathologies, we have to be careful! We begin with the definition of powers, in a recursive way.

Definition 2.2. *Let \mathfrak{A} be a partial *-algebra with unit 1 and $x \in \mathfrak{A}$. Then:*

- (1) *Given an integer $n \geq 2$, we say that the n^{th} power of x is defined iff all products $x^k \cdot x^{n-k}$, $1 \leq k \leq n-1$, exist and coincide. Their common value is denoted, obviously, by x^n .*
- (2) *The element x is called well-behaved if its n^{th} power x^n is defined whenever any one product $x^k \cdot x^{n-k}$, $1 \leq k \leq n-1$, exists.*

Remark. If $x = x^*$, it is enough to consider the powers x^k with $1 \leq k \leq [n/2]$, where $[n/2]$ is the integer part of $n/2$.

Thus well-behaved elements are those for which associativity holds when one multiplies their powers.

Clearly if the n^{th} power of x is defined, so are all k^{th} powers x^k , $k < n$. Hence there must be a largest integer n for which this is true.

Definition 2.3. *Given any $x \in \mathfrak{A}$, the length of x is the largest number $l(x) \in \mathbb{N} \cup \{\infty\}$ such that all powers x^k , $1 \leq k \leq l(x)$, are defined.*

Examples 3. (1) *If \mathfrak{A} is an associative partial *-algebra, then all elements of \mathfrak{A} are well-behaved.*

(2) *If \mathfrak{A} is a semi-associative partial *-algebra, then each $x \in R(\mathfrak{A}) \cap L(\mathfrak{A})$ is well-behaved and has infinite length.*

Let \mathfrak{A} be a partial *-algebra with unit 1 and $x = x^* \in \mathfrak{A}$. As we will see in Section 4 below, the length $l(x)$ is the crucial concept for characterizing the behaviour of x . If x is well-behaved, $l(x)$ is just the largest number such that $x \in R(x^k)$ for all $k \leq l(x) - 1$ and $x \notin R(x^{l(x)})$. Thus it has powers $x, x^2, \dots, x^{l(x)}$, and no product $x^p \cdot x^q$ with $p + q > l(x)$ can exist. This *does* happen, however, if x is not well-behaved. So we define the following subsets of \mathfrak{A} :

$$\mathfrak{R}(x) = \left\{ \sum_{k=0}^{l(x)} \alpha_k x^k; \alpha_k \in \mathbb{C}, k = 0, 1, 2, \dots, l(x) \right\},$$

$$\mathfrak{S}(x) = \mathfrak{M}(x) - \mathfrak{R}(x).$$

Thus we have:

$$\mathfrak{M}(x) = \mathfrak{R}(x) + \mathfrak{S}(x),$$

and we call $\mathfrak{R}(x)$ the *regular* part of $\mathfrak{M}(x)$, $\mathfrak{S}(x)$ its *singular* part. We will study these concepts in detail in Section 4 below in the case of partial $*$ -algebras of closable operators.

§3. The Basic Theory of Partial Op $*$ -Algebras

3.A. General Definitions, Abelianness

In order to make the paper self-contained, we will rewrite here the main definitions about partial Op $*$ -algebras, with the new convention that all operators are now restricted to the common domain \mathcal{D} , hence *closable* instead of *closed* as before. Along the way we will indicate the differences with the earlier presentation.

As usual, \mathcal{H} denotes a Hilbert space, fixed once and for all, and \mathcal{D} a dense subspace of \mathcal{H} . We denote by $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all (closable) linear operators X such that $D(X) = \mathcal{D}$, $D(X^*) \supset \mathcal{D}$ [in the notation of Ref. 8, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \equiv \mathcal{C}_0(\mathcal{D}, \mathcal{H}) = \mathcal{C}(\mathcal{D}) \upharpoonright \mathcal{D}$]. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial $*$ -algebra [8] with respect to the following operations: the usual sum $X_1 + X_2$, the scalar multiplication λX , the involution $X \mapsto X^\dagger \equiv X^* \upharpoonright \mathcal{D}$ and the *weak* partial multiplication $X_1 \square X_2 = X_1^{\dagger*} X_2$, defined whenever X_1 is a *weak left multiplier* of X_2 ($X_1 \in L^w(X_2)$ or $X_2 \in R^w(X_1)$), that is, iff $X_2 \mathcal{D} \subset D(X_1^{\dagger*})$ and $X_1^{\dagger*} \mathcal{D} \subset D(X_2)$. When we regard $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ as a partial $*$ -algebra with those operations, we denote it by $\mathcal{L}_w^\dagger(\mathcal{D}, \mathcal{H})$ [thus $\mathcal{L}_w^\dagger(\mathcal{D}, \mathcal{H}) = \mathcal{C}^w(\mathcal{D}) \upharpoonright \mathcal{D}$]. Then a *weak partial Op $*$ -algebra* on \mathcal{D} is a $*$ -subalgebra \mathfrak{M} of $\mathcal{L}_w^\dagger(\mathcal{D}, \mathcal{H})$; that is \mathfrak{M} is a subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that $X^\dagger \in \mathfrak{M}$ whenever $X \in \mathfrak{M}$ and $X_1 \square X_2 \in \mathfrak{M}$ for any $X_1, X_2 \in \mathfrak{M}$ such that $X_1 \in L^w(X_2)$.

On $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ we also consider the *strong* partial multiplication: $X_1 \cdot X_2 = \overline{X_1} X_2$, defined whenever $X_1 \in L^s(X_2)$ (or $X_2 \in R^s(X_1)$), which means $X_2 \mathcal{D} \subset D(\overline{X_1})$ and $X_1^{\dagger} \mathcal{D} \subset \mathcal{D}(\overline{X_2})$ [in the earlier notation, $\overline{X_1} \equiv X_1^{\dagger*}$ and $\overline{X_2} \equiv X_2^{\dagger}$]. Equipped with this partial multiplication, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is denoted by $\mathcal{L}_s^\dagger(\mathcal{D}, \mathcal{H})$. We remind the reader that $\mathcal{L}_s^\dagger(\mathcal{D}, \mathcal{H})$ is in general *not* a partial $*$ -algebra, because the strong partial multiplication is not distributive with respect to the addition [7, 8]. Thus we have to make a distinction. A subspace of $\mathcal{L}_s^\dagger(\mathcal{D}, \mathcal{H})$ which is stable under all operations will be called a *strong pseudo-partial Op $*$ -algebra* on \mathcal{D} ; this is the case in particular of $\mathcal{L}_s^\dagger(\mathcal{D}, \mathcal{H})$ itself. If, in addition, the distributive law holds, we will speak of *strong partial Op $*$ -algebra*.

As before [7], an operator $X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is called *standard* if it verifies any of the equivalent conditions: (i) $\overline{X} = X^{\dagger*}$; (ii) $\overline{X}^\dagger = X^*$; (iii) \mathcal{D} is a core for X^* . This is the case, in particular, if \overline{X} is self-adjoint or normal. Similarly, a partial Op $*$ -algebra \mathfrak{M} is *standard* if every operator $X \in \mathfrak{M}$ is standard. Notice

that in this case we don't have to distinguish between strong and weak multiplication, since the two notions coincide on standard operators.

Let \mathfrak{N} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then there is a minimal *weak* partial Op^* -algebra on \mathcal{D} containing \mathfrak{N} , which we denote by $\mathfrak{M}_w[\mathfrak{N}]$. But there does not necessarily exist a minimal *strong* partial Op^* -algebra containing \mathfrak{N} . When it exists, we denote it by $\mathfrak{M}_s[\mathfrak{N}]$ (this object was introduced in Ref. 7 and denoted there $\mathfrak{M}[\mathfrak{N}]$). Clearly $\mathfrak{M}_s[\mathfrak{N}]$ exists if and only if there exists a strong partial Op^* -algebra containing \mathfrak{N} . Since this need not be the case, we consider the minimal strong pseudo-partial Op^* -algebra containing \mathfrak{N} , i.e. the minimal \dagger -invariant subspace $\mathfrak{B}_s[\mathfrak{N}]$ which is stable under the strong partial multiplication and contains \mathfrak{N} . It is easily shown that $\mathfrak{B}_s[\mathfrak{N}]$ always exists and $\mathfrak{B}_s[\mathfrak{N}] \subset \mathfrak{M}_w[\mathfrak{N}]$; if $\mathfrak{M}_s[\mathfrak{N}]$ does exist, then $\mathfrak{M}_s[\mathfrak{N}] \subset \mathfrak{B}_s[\mathfrak{N}] \subset \mathfrak{M}_w[\mathfrak{N}]$. As in Ref. 7, $\mathfrak{M}_w[\mathfrak{N}]$ and, when it exists, $\mathfrak{M}_s[\mathfrak{N}]$ are called the partial Op^* -algebras *generated* by \mathfrak{N} . The case where \mathfrak{N} consists of the restriction of a single closed symmetric operator will be investigated in Section 4 below.

We will need in the sequel several topologies on partial Op^* -algebras (see Ref. 3 for a detailed study). The locally convex topology on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ generated by the family of seminorms: $p_{\xi, \eta}(X) = |(X\xi|\eta)|$, $\xi, \eta \in \mathcal{D}$ (resp. $p_\xi(X) = \|X\xi\|$, $\xi \in \mathcal{D}$; $p_\xi^*(X) = \|X\xi\| + \|X^\dagger\xi\|$, $\xi \in \mathcal{D}$) is called the *weak* topology (resp. the *strong* topology; the *strong** topology), and denoted by t_w (resp. t_s ; t_{s^*}). We recall that $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is complete for t_{s^*} , but not in general for t_w or t_s [24]. For $\mathfrak{N} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, we denote by $[\mathfrak{N}]^{s^*}$ the t_{s^*} -closure of \mathfrak{N} . In particular, $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = [\mathcal{L}^\dagger(\mathcal{D})]^{s^*}$, i.e. the completion of the associated Op^* -algebra $\mathcal{L}^\dagger(\mathcal{D})$, namely

$$\mathcal{L}^\dagger(\mathcal{D}) = \{A \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}); A\mathcal{D} \subset \mathcal{D} \text{ and } A^*\mathcal{D} \subset \mathcal{D}\}.$$

Let \mathfrak{M} be a weak partial Op^* -algebra on \mathcal{D} . We put

$$\mathcal{D}^\infty(\mathfrak{M}) = \{\{\xi_n\} \subset \mathcal{D}; \sum_{n=1}^\infty \|\xi_n\|^2 < \infty \text{ and } \sum_{n=1}^\infty \|X\xi_n\|^2 < \infty \text{ for all } X \in \mathfrak{M}\}.$$

The locally convex topology on \mathfrak{M} generated by the family of seminorms on \mathfrak{M} :

$$p_{\{\xi_n\}, \{\eta_n\}}(X) = \sum_{n=1}^\infty |(X\xi_n|\eta_n)|, \{\xi_n\}, \{\eta_n\} \in \mathcal{D}^\infty(\mathfrak{M})$$

$$\text{(resp. } p_{\{\xi_n\}}(X) = (\sum_{n=1}^\infty \|X\xi_n\|^2)^{1/2};$$

$$p_{\{\xi_n\}}^*(X) = p_{\{\xi_n\}}(X) + p_{\{\xi_n\}}(X^\dagger), \{\xi_n\} \in \mathcal{D}^\infty(\mathfrak{M}))$$

is called the σ -*weak* (resp. σ -*strong*, σ^* -*strong*) topology relative to \mathfrak{M} , and denoted by $t_{\sigma w}^{\mathfrak{M}}$ (resp. $t_{\sigma s}^{\mathfrak{M}}$, $t_{\sigma s^*}^{\mathfrak{M}}$).

Finally we come to the question of abelianness. According to the abstract theory, a partial Op^* -algebra \mathfrak{A} is abelian if it satisfies the conditions of Definition 2.1. Now, for an algebra \mathfrak{A} of *bounded* operators, this may be rephrased into the global condition $\mathfrak{A} \subseteq \mathfrak{A}'$. To get a similar statement for an algebra of unbounded operators, a fortiori for a partial Op^* -algebra \mathfrak{M} , we have

to choose first the type of commutant, necessarily unbounded. Several possibilities are at our disposal [9], for instance the *weak unbounded commutant* [23, 24]:

$$\mathfrak{M}'_{\sigma} = \{ Y \in \mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H}); (X\xi | Y\eta) = (Y^{\dagger}\xi | X^{\dagger}\eta) \text{ for each } \xi, \eta \in \mathcal{D} \text{ and } X \in \mathfrak{M} \}, \quad (3.1)$$

or the weak and strong *natural commutants*, which are smaller:

$$\mathfrak{M}'_{\square} = \{ Y \in \mathfrak{M}'_{\sigma}; Y \in L^w(X) \cap R^w(X), Y \square X = X \square Y, \text{ for all } X \in \mathfrak{M} \}, \quad (3.2)$$

$$\mathfrak{M}'_{\cdot} = \{ Y \in \mathfrak{M}'_{\sigma}; Y \in L^s(X) \cap R^s(X), Y \cdot X = X \cdot Y, \text{ for all } X \in \mathfrak{M} \}. \quad (3.3)$$

However, a global condition like $\mathfrak{M} \subseteq \mathfrak{M}'_{\square}$ or $\mathfrak{M} \subseteq \mathfrak{M}'_{\cdot}$ forces \mathfrak{M} to be an algebra. As for the condition $\mathfrak{M} \subseteq \mathfrak{M}'_{\sigma}$, it implies indeed that \mathfrak{M} is abelian, as follows from Proposition 3.1(i) below, but the converse is not true in general. Notice that this stronger condition $\mathfrak{M} \subseteq \mathfrak{M}'_{\sigma}$ is verified in the particular case of the partial *-algebra of commuting normal operators studied in Ref. 25; we will recover this result in Section 4 below. Thus, for partial Op*-algebras, it seems that abelianness cannot be formulated globally, but only elementwise. The conditions may be formulated in several equivalent ways. Indeed:

Proposition 3.1. *Let $\mathfrak{M} \subset \mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$ be a weak partial Op*-algebra. Then the following conditions are equivalent.*

- (i) $\mathfrak{M} \cap R^w(X) \subset \{X\}'_{\square}, \forall X \in \mathfrak{M}.$
- (ii) $\mathfrak{M} \cap L^w(X) \subset \{X\}'_{\square}, \forall X \in \mathfrak{M}.$
- (iii) $\mathfrak{M} \cap R^w(X) \subset \{X\}'_{\sigma}, \forall X \in \mathfrak{M}.$
- (iv) $\mathfrak{M} \cap L^w(X) \subset \{X\}'_{\sigma}, \forall X \in \mathfrak{M}.$
- (v) \mathfrak{M} is abelian.

Proof. \bullet (i) \Rightarrow (ii): We show that $\mathfrak{M} \cap R^w(X) = \mathfrak{M} \cap L^w(X)$. If $\mathfrak{M} \cap R^w(X) \subset \{X\}'_{\square}$ for all $X \in \mathfrak{M}$, then clearly $\mathfrak{M} \cap R^w(X) \subset \mathfrak{M} \cap L^w(X)$; and so $\mathfrak{M} \cap L^w(X) = \mathfrak{M} \cap R^w(X^{\dagger})^{\dagger} = (\mathfrak{M} \cap R^w(X^{\dagger}))^{\dagger} \subset (\mathfrak{M} \cap L^w(X^{\dagger}))^{\dagger} = \mathfrak{M} \cap R^w(X)$. Hence $\mathfrak{M} \cap L^w(X) = \mathfrak{M} \cap R^w(X) \subset \{X\}'_{\square}$ for all $X \in \mathfrak{M}$. We can prove the implication (ii) \Rightarrow (i) in a similar way.

\bullet (i) \Rightarrow (iii) and (ii) \Rightarrow (iv): This is obvious.

\bullet (iii) \Rightarrow (v): Let $Y \in \mathfrak{M} \cap R^w(X)$; then $Y \in \{X\}'_{\sigma}$, i.e.

$$(Y\xi | X^{\dagger}\eta) = (X\xi | Y^{\dagger}\eta), \forall \xi, \eta \in \mathcal{D}. \quad (3.4)$$

Since $Y \in R^w(X)$ we have $(Y\xi | X^{\dagger}\eta) = ((X \square Y)\xi | \eta) = (X\xi | Y^{\dagger}\eta)$, which implies that $X: \mathcal{D} \rightarrow D(Y^{\dagger*})$. In an analogous way we get $Y^{\dagger}: \mathcal{D} \rightarrow D(X^*)$. Thus $X \in R^w(Y)$ and from (3.4) $X \square Y = Y \square X$. The implication (iv) \Rightarrow (v) is proven in the same way, and (v) \Rightarrow (i) is clear from the definition. \square

An analogous statement holds true if \mathfrak{M} is a strong partial *-algebra: \mathfrak{M} is abelian iff $\mathfrak{M} \cap R^s(X) \subset \{X\}'$, $\forall X \in \mathfrak{M}$, etc.

3.B. Extensions of Partial Op*-Algebras

As for Op*-algebras, a partial Op*-algebra may be extended by closure to a larger domain, and in fact two such extensions, a priori different, have been described in Refs.7,8. Several other extensions will be defined below. To make the discussion more systematic, it is handy to consider first the notion of *extension* itself, at the purely algebraic level.

Let $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{H}$ and $\mathfrak{M}_1, \mathfrak{M}_2$ two \dagger -invariant subsets of $\mathcal{L}^\dagger(\mathcal{D}_1, \mathcal{H})$ and $\mathcal{L}^\dagger(\mathcal{D}_2, \mathcal{H})$ respectively. If there exists a bijection ε of \mathfrak{M}_1 onto \mathfrak{M}_2 such that $X \subset \varepsilon(X)$ for all $X \in \mathfrak{M}_1$, then \mathfrak{M}_2 is said to be an *extension* of \mathfrak{M}_1 , and this is denoted by $\mathfrak{M}_2 \equiv \varepsilon(\mathfrak{M}_1) \succ \mathfrak{M}_1$. Then, as it is easily shown, ε is linear and \dagger -invariant, and if $\varepsilon(X_1) \square \varepsilon(X_2)$ is well-defined, then so is $X_1 \square X_2$, but the converse does not necessarily hold.

If $\varepsilon(\mathfrak{M}_1) \succ \mathfrak{M}_1$, and $X_1 \square X_2$ (resp. $X_1 \bullet X_2$) is well-defined if and only if $\varepsilon(X_1) \square \varepsilon(X_2)$ (resp. $\varepsilon(X_1) \bullet \varepsilon(X_2)$) is well-defined, then $\varepsilon(\mathfrak{M}_1)$ is said to be an *algebraic extension* of \mathfrak{M}_1 for the weak (resp. strong) partial multiplication, which is denoted by $\varepsilon(\mathfrak{M}_1) \succ_w \mathfrak{M}_1$ (resp. $\varepsilon(\mathfrak{M}_1) \succ_s \mathfrak{M}_1$). The following result is immediate:

Lemma 3.2. *Suppose \mathfrak{M}_1 is a weak partial Op*-algebra on \mathcal{D}_1 and $\mathfrak{M}_2 = \varepsilon(\mathfrak{M}_1) \succ \mathfrak{M}_1$. Then \mathfrak{M}_2 is a weak partial Op*-algebra on \mathcal{D}_2 and ε^{-1} is a *-homomorphism of \mathfrak{M}_2 onto \mathfrak{M}_1 , but ε is not necessarily a *-homomorphism. In particular, if $\varepsilon(\mathfrak{M}_1) \succ_w \mathfrak{M}_1$, then ε is a *-isomorphism of \mathfrak{M}_1 onto \mathfrak{M}_2 .*

Remark. In the strong case, the additional assumption $\mathfrak{M}_1 \prec_s \mathfrak{M}_2$ is needed; otherwise we don't know if $\varepsilon(\mathfrak{M}_1)$ is a strong partial Op*-algebra on \mathcal{D}_2 .

We describe now three canonical extensions of partial Op*-algebras; the first two have been discussed already in Ref.8. Further extensions will be introduced in Section 3.C below, under additional conditions.

Let \mathfrak{M} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. We denote by $t_{\mathfrak{M}}$ the induced topology on \mathcal{D} defined by the family of seminorms $\{\|\cdot\|_X; X \in \mathfrak{M}\}$:

$$\|\xi\|_X = \|\xi\| + \|X\xi\|, \xi \in \mathcal{D}.$$

If the locally convex space $(\mathcal{D}, t_{\mathfrak{M}})$ is complete, then \mathfrak{M} is said to be *closed*.

Let \mathfrak{M} be a \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and denote by $\tilde{\mathcal{D}}(t_{\mathfrak{M}})$ the completion of \mathcal{D} relative to the topology $t_{\mathfrak{M}}$. Put

$$\tilde{\mathfrak{M}}(X) (\equiv \tilde{X}) = \overline{X} \upharpoonright \tilde{\mathcal{D}}(t_{\mathfrak{M}}), X \in \mathfrak{M}.$$

Then $\hat{\mathfrak{M}}(\equiv \hat{\mathfrak{M}})$ is a closed \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\hat{\mathcal{D}}(t_{\mathfrak{M}}), \mathcal{H})$ which is *minimal* among closed extensions of \mathfrak{M} . In particular, if \mathfrak{M} is a weak (resp. strong) partial Op*-algebra on \mathcal{D} , then $\hat{\mathfrak{M}}$ is a weak (resp. strong) partial Op*-algebra on $\hat{\mathcal{D}}(t_{\mathfrak{M}})$ satisfying $\mathfrak{M} <_w \hat{\mathfrak{M}}$ (resp. $\mathfrak{M} <_s \hat{\mathfrak{M}}$) and $\hat{\iota}$ is a *-isomorphism of \mathfrak{M} onto $\hat{\mathfrak{M}}$.

We put

$$\hat{\mathcal{D}}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(\bar{X})$$

$$\hat{\iota}(X) (\equiv \hat{X}) = \bar{X} \upharpoonright \hat{\mathcal{D}}(\mathfrak{M}), \quad X \in \mathfrak{M}.$$

If $\hat{\mathcal{D}}(\mathfrak{M}) = \mathcal{D}$, then \mathfrak{M} is said to be *fully closed*. It is clear that

$$\mathcal{D} \subset \hat{\mathcal{D}}(t_{\mathfrak{M}}) \subset \hat{\mathcal{D}}(\mathfrak{M}),$$

and hence if \mathfrak{M} is fully closed, then it is closed. The converse is false for a general subset \mathfrak{M} , as it is already the case for Op*-algebras, but we conjecture that the two notions coincide for vector subspaces [25].

Let \mathfrak{M} be a \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then [8] $\hat{\mathfrak{M}}$ is a fully closed \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\hat{\mathcal{D}}(\mathfrak{M}), \mathcal{H})$, which is minimal among fully closed extensions of \mathfrak{M} . In particular, if \mathfrak{M} is a weak partial Op*-algebra on \mathcal{D} , then $\hat{\mathfrak{M}}$ is a fully closed weak partial Op*-algebra on $\hat{\mathcal{D}}(\mathfrak{M})$ satisfying $\mathfrak{M} <_w \hat{\mathfrak{M}}$ and $\hat{\iota}$ is a *-isomorphism of \mathfrak{M} onto $\hat{\mathfrak{M}}$.

If \mathfrak{M} is a *strong* partial Op*-algebra on \mathcal{D} , $\hat{\mathfrak{M}}$ is a fully closed, \dagger -invariant subspace of $\mathcal{L}^\dagger(\hat{\mathcal{D}}(\mathfrak{M}), \mathcal{H})$, and it is stable under the string partial multiplication. However distributivity may fail, and so $\hat{\mathfrak{M}}$ is in general only a strong *pseudo*-partial Op*-algebra on $\hat{\mathcal{D}}(\mathfrak{M})$ (this was overlooked in Ref. 7). If distributivity holds in $\hat{\mathfrak{M}}$, then $\hat{\iota}^{-1}$ is a *-homomorphism of $\hat{\mathfrak{M}}$ onto \mathfrak{M} , but not a *-isomorphism in general.

Next we define the adjoint of a \dagger -invariant subset \mathfrak{M} of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. We put

$$\mathcal{D}^*(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(X^*)$$

$$\iota^*(X) = X^{\dagger*} \upharpoonright \mathcal{D}^*(\mathfrak{M}), \quad X \in \mathfrak{M}.$$

Then $\iota^*(X)$ is a closable operator in \mathcal{H} satisfying $\iota^*(X) \supset \hat{X}$ for each $X \in \mathfrak{M}$, but $\mathcal{D}^*(\mathfrak{M})$ is not necessarily contained in $D(\iota^*(X)^*)$, and so $\iota^*(\mathfrak{M})$ is not \dagger -invariant (however, $\iota^*(\mathfrak{M})$ is invariant under another, less natural, involution, namely: $X \rightarrow (X \upharpoonright \mathcal{D})^* \upharpoonright \mathcal{D}^*(\mathfrak{M})$, as shown in Ref. 7).

We now put

$$\mathcal{D}^{**}(\mathfrak{M}) = \bigcap_{X \in \mathfrak{M}} D(\iota^*(X)^*)$$

$$\iota^{**}(X) = \iota^*(X^{\dagger*})^* \upharpoonright \mathcal{D}^{**}(\mathfrak{M}), \quad X \in \mathfrak{M}.$$

Lemma 3.3. $i^{**}(\mathfrak{M})$ is a fully closed \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}^{**}(\mathfrak{M}), \mathcal{H})$ and an extension of $\widehat{\mathfrak{M}}$. Suppose \mathfrak{M} is a weak partial Op^* -algebra on \mathcal{D} . Then $i^{**}(\mathfrak{M})$ is a fully closed weak partial Op^* -algebra on $\mathcal{D}^{**}(\mathfrak{M})$, which is an extension of $\widehat{\mathfrak{M}}$.

Proof. For each $\xi \in \mathcal{D}^*(\mathfrak{M})$, $\eta \in \widehat{\mathcal{D}}(\mathfrak{M})$ and $X \in \mathfrak{M}$, we have:

$$(i^*(X^\dagger)\xi|\eta) = (X^*\xi|\eta) = (\xi|\overline{X}\eta) = (\xi|\hat{i}(X)\eta),$$

and hence $\hat{i}(X) \subset i^{**}(X)$ for each $X \in \mathfrak{M}$. Furthermore, since $X^{\dagger*} \supset i^*(X^\dagger)^*$, we have

$$\hat{i}(X) \subset i^{**}(X) \subset i^*(X) \tag{3.5}$$

for all $X \in \mathfrak{M}$. We show that $i^{**}(X)^\dagger = i^{**}(X^\dagger)$ for each $X \in \mathfrak{M}$. This follows from the equality:

$$\begin{aligned} (i^{**}(X)\xi|\eta) &= (i^*(X^\dagger)^*\xi|\eta), \text{ by (3.5)} \\ &= (\xi|i^*(X^\dagger)\eta), \text{ by (3.5)} \\ &= (\xi|i^{**}(X^\dagger)\eta) \end{aligned}$$

for each $\xi, \eta \in \mathcal{D}^{**}(\mathfrak{M})$ and $X \in \mathfrak{M}$. Hence, $i^{**}(\mathfrak{M})$ is a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}^{**}(\mathfrak{M}), \mathcal{H})$ which is an extension of $\widehat{\mathfrak{M}}$. It is clear that $i^{**}(\mathfrak{M})$ is fully closed. The rest follows from Lemma 3.2. \square

In view of the relations (3.5), it is natural to extend to the present case the terminology used for Op^* -algebras [4,5]:

Definition 3.5. Let \mathfrak{M} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then \mathfrak{M} is said to be self-adjoint if $\mathcal{D}^*(\mathfrak{M}) = \mathcal{D}$, essentially self-adjoint if $\mathcal{D}^*(\mathfrak{M}) = \widehat{\mathcal{D}}(\mathfrak{M})$, and algebraically self-adjoint if $\mathcal{D}^*(\mathfrak{M}) = \mathcal{D}^{**}(\mathfrak{M})$.

As a final remark, we may quote the following useful results [8]:

- (i) if $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is fully closed, then $\mathcal{L}_s^\dagger(\mathcal{D}, \mathcal{H})$ is semi-associative;
- (ii) if $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is self-adjoint, then $\mathcal{L}_w^\dagger(\mathcal{D}, \mathcal{H})$ is semi-associative.

To give a simple example, let $\mathcal{D} = \mathcal{D}^\infty(T) = \bigcap_{k=1}^\infty D(T^k)$ for a self-adjoint operator T . Then $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is self-adjoint, since every power T^k is self-adjoint and belongs to $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, in fact to $\mathcal{L}^\dagger(\mathcal{D})$. Furthermore, $R^w(\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}))$ is the set of all bounded operators B such that $B\mathcal{D} \subset \mathcal{D}$.

3.C. Bounded Commutants

In the study of Op^* -algebras [4, 22, 24, 27-32], an important rôle is played by the weak bounded commutant. The same is true for partial Op^* -algebras, where the weak bounded commutant \mathfrak{M}'_w of \mathfrak{M} is defined exactly in the same way [8, 9, 23], namely:

$$\mathfrak{M}'_w = \{C \in \mathcal{B}(\mathcal{H}); (CX\xi|\eta) = (C\xi|X^\dagger\eta), \text{ for each } X \in \mathfrak{M} \text{ and } \xi, \eta \in \mathcal{D}\}. \tag{3.6}$$

However, this commutant is not sufficient in general, because no requirement of associativity was made for the partial multiplication. The same problem appears with the other notions of bounded commutant introduced in Ref.9 (the so-called natural bounded commutants). In order to remedy this difficulty, we introduce a new bounded commutant, called *quasi-weak* and denoted \mathfrak{M}'_{qw} . The idea is to incorporate the lack of associativity in the very definition of the commutant. As we will show in the second paper, this is exactly what is needed for the study of representations. The relationship between the quasi-weak bounded commutant and the other types of commutants introduced in Ref.9 is discussed in the Appendix.

Let \mathfrak{M} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Its *quasi-weak* bounded commutant is defined as follows:

$$\begin{aligned} \mathfrak{M}'_{qw} = \{C \in \mathfrak{M}'_w; (CX_1^\dagger \xi | X_2 \eta) = (C\xi | (X_1 \square X_2)\eta) \text{ and} \\ (C^* X_1^\dagger \xi | X_2 \eta) = (C^* \xi | (X_1 \square X_2)\eta) \end{aligned} \tag{3.7}$$

for all $X_1, X_2 \in \mathfrak{M}$ s.t. $X_1 \in L^w(X_2)$ and all $\xi, \eta \in \mathcal{D}$.

Exactly as \mathfrak{M}'_w , \mathfrak{M}'_{qw} is a weakly closed, $*$ -invariant subspace of $\mathcal{B}(\mathcal{H})$, but none of them is necessarily an algebra, even if \mathfrak{M} is an Op^* -algebra [4, 27, 29].

Remark 3.6. (1) *Suppose \mathfrak{M} is a weak partial Op^* -algebra on \mathcal{D} . Then*

$$\begin{aligned} \mathfrak{M}'_{qw} = \{C \in \mathfrak{M}'_w; (CX_1^\dagger \xi | X_2 \eta) = (C\xi | (X_1 \square X_2)\eta) \\ \text{for all } X_1, X_2 \in \mathfrak{M} \text{ s.t. } X_1 \in L^w(X_2) \text{ and all } \xi, \eta \in \mathcal{D}\}, \end{aligned}$$

that is, the two conditions in Eq. (3.7) are equivalent.

(2) *Comparing the new bounded commutant with the earlier ones [9], one gets readily the following inclusions (see Appendix):*

$$\mathfrak{M}'_s \subset \mathfrak{M}'_b \subset \mathfrak{M}'_{qw} \subset \mathfrak{M}'_{\square b} = \mathfrak{M}'_w. \tag{3.8}$$

Notice that the last equality is valid for any \dagger -invariant subset \mathfrak{M} . This fact was overlooked in Ref.9 but it has no consequences for that paper.

Let \mathfrak{M} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then $\mathfrak{M}'_{qw} \subsetneq \mathfrak{M}'_w$ in general, and examples are easily constructed (see Example 5 at the end of Section 4). So we may ask, when does the equality $\mathfrak{M}'_{qw} = \mathfrak{M}'_w$ hold? One obvious case is when $\mathfrak{M} \subset \mathcal{L}^\dagger(\mathcal{D})$. For the general case, the following easy result gives a sufficient condition.

Lemma 3.7. *Let \mathfrak{M} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Consider the following statements:*

- (1) \mathfrak{M} is essentially self-adjoint.
- (2) $\mathfrak{M}'_w \mathcal{D} \subset \mathcal{D}(\mathfrak{M})$.

(3) \overline{X} is affiliated with $(\mathfrak{M}'_w)'$ for each $X \in \mathfrak{M}$.

(4) \mathfrak{M}'_w is a von Neumann algebra, which equals \mathfrak{M}'_{q_w} .

Then the following implications hold:

$$(1) \implies \begin{array}{c} (2) \\ \updownarrow \\ (3) \end{array} \implies (4)$$

□

We remark that the converse implications need not hold even if \mathfrak{M} is an Op*-algebra [4, 27, 29]; also, \mathfrak{M}'_w is not necessarily a von Neumann algebra, even if it coincides with \mathfrak{M}'_{q_w} , but we don't know if the converse holds.

Lemma 3.8. (1) Let \mathfrak{M} be a †-invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then:

$$\mathfrak{M}'_w = \widehat{\mathfrak{M}}'_w = \widehat{\mathfrak{M}}'_w = i^{**}(\mathfrak{M})'_w$$

and

$$i^{**}(\mathfrak{M})'_{q_w} \subset \widehat{\mathfrak{M}}'_{q_w} \subset \widehat{\mathfrak{M}}'_w \subset \mathfrak{M}'_{q_w}$$

(2) Suppose \mathfrak{M} is a weak partial Op*-algebra. Then:

$$i^{**}(\mathfrak{M})'_{q_w} \subset \widehat{\mathfrak{M}}'_{q_w} = \widehat{\mathfrak{M}}'_w = \mathfrak{M}'_{q_w}$$

(3) Suppose \mathfrak{M} is an algebraically self-adjoint †-invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then

$$i^{**}(\mathfrak{M})'_{q_w} = \widehat{\mathfrak{M}}'_{q_w} = \widehat{\mathfrak{M}}'_w = \mathfrak{M}'_{q_w}$$

Proof. (1) It is clear that

$$i^{**}(\mathfrak{M})'_w \subset \mathfrak{M}'_w = \widehat{\mathfrak{M}}'_w = \widehat{\mathfrak{M}}'_w$$

$$i^{**}(\mathfrak{M})'_{q_w} \subset \widehat{\mathfrak{M}}'_{q_w} \subset \widehat{\mathfrak{M}}'_w \subset \mathfrak{M}'_{q_w}$$

Since

$$(X^\dagger \xi | C\eta) = (C^* X^\dagger \xi | \eta) = (i^*(X^\dagger) C^* \xi | \eta) = (C^* \xi | i^{**}(X)\eta) = (\xi | C i^{**}(X)\eta),$$

for each $X \in \mathfrak{M}$, $C \in \mathfrak{M}'_w$, $\xi \in \mathcal{D}$ and $\eta \in \mathcal{D}^{**}(\mathfrak{M})$, it follows that

$$\mathfrak{M}'_w \mathcal{D}^{**}(\mathfrak{M}) \subset \mathcal{D}^*(\mathfrak{M}),$$

$$i^*(X)C\eta = C i^{**}(X)\eta$$

for each $X \in \mathfrak{M}$, $C \in \mathfrak{M}'_w$ and $\eta \in \mathcal{D}^{**}(\mathfrak{M})$, which implies

$$(i^{**}(X)\xi | C\eta) = (C^* i^{**}(X)\xi | \eta) = (i^*(X)C^* \xi | \eta) = (C^* \xi | i^{**}(X^\dagger)\eta),$$

for each $X \in \mathfrak{M}$, $C \in \mathfrak{M}'_w$ and $\xi, \eta \in \mathcal{D}^{**}(\mathfrak{M})$. Hence, $C \in i^{**}(\mathfrak{M})'_w$ for each $C \in \mathfrak{M}'_w$.

(2) Take an arbitrary $C \in \mathfrak{M}'_{qw}$. Let $X, Y \in \mathfrak{M}$ such that $\hat{X} \in L^w(\hat{Y})$. Since $\hat{\iota}: \mathfrak{M} \rightarrow \hat{\mathfrak{M}}$ is a $*$ -isomorphism, $X \in L^w(Y)$ and $\hat{X} \square \hat{Y} = \widehat{X \square Y}$. For each $\xi, \eta \in \hat{\mathcal{D}}(\mathfrak{M})$, there exist sequences $\{\xi_n\}, \{\eta_m\}$ in \mathcal{D} such that:

$$\begin{aligned} \lim_{n \rightarrow \infty} \xi_n &= \xi \text{ and } \lim_{n \rightarrow \infty} X^\dagger \xi_n = \widehat{X^\dagger \xi} = \hat{X}^\dagger \xi, \\ \lim_{m \rightarrow \infty} \eta_m &= \eta \text{ and } \lim_{m \rightarrow \infty} Y \eta_m = \hat{Y} \eta. \end{aligned}$$

Then we have

$$\begin{aligned} (C \widehat{X^\dagger \xi} | \hat{Y} \eta) &= \lim_{m \rightarrow \infty} (C \hat{X}^\dagger \xi | Y \eta_m) \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (C X^\dagger \xi_n | Y \eta_m) \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (C \xi_n | (X \square Y) \eta_m) \\ &= \lim_{m \rightarrow \infty} (C \xi | (X \square Y) \eta_m) \\ &= \lim_{m \rightarrow \infty} ((X \square Y)^* C \xi | \eta_m) \\ &= ((X \square Y)^* C \xi | \eta) \\ &= (C \xi | \widehat{X \square Y} \eta) \\ &= (C \xi | (\hat{X} \square \hat{Y}) \eta). \end{aligned}$$

Hence, $C \in \mathfrak{M}'_{qw}$.

(3) Since $i^{**}(\mathfrak{M})$ is a self-adjoint \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}^{**}(\mathfrak{M}), \mathcal{H})$, it follows from Lemma 3.7 that $i^{**}(\mathfrak{M})'_{qw} = i^{**}(\mathfrak{M})'_w$. Hence, we have by (1):

$$i^{**}(\mathfrak{M})'_w = i^{**}(\mathfrak{M})'_{qw} \subset \mathfrak{M}'_{qw} \subset \mathfrak{M}'_w = i^{**}(\mathfrak{M})'_w. \quad \square$$

If \mathfrak{A} is an Op^* -algebra, additional extensions of \mathfrak{A} may be defined [31] under the condition that its weak bounded commutant \mathfrak{A}'_w be an algebra (see also Ref.32). Similar results hold for partial Op^* -algebras, as will be shown now. The new extensions will play a crucial rôle in the theory of the so-called partial GW^* -algebras, to be developed in Part II.

Theorem 3.9. *Let \mathfrak{M} be a \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.*

(1) *Suppose \mathfrak{M}'_w is an algebra. Put*

$$\mathcal{D}_w(\mathfrak{M}) = \{ \sum_{k=1}^n C_k \xi_k; C_k \in \mathfrak{M}'_w, \xi_k \in \mathcal{D}, k = 1, \dots, n, n \in \mathbb{N} \}$$

$$\varepsilon_w(X)(\sum_{k=1}^n C_k \xi_k) = \sum_{k=1}^n C_k X \xi_k, \quad X \in \mathfrak{M}, \quad \sum_{k=1}^n C_k \xi_k \in \mathcal{D}_w(\mathfrak{M}).$$

Then $\varepsilon_w(\mathfrak{M})$ is a \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\mathcal{D}_w(\mathfrak{M}), \mathcal{H})$ such that

- (1)₁ $\mathfrak{M} < \varepsilon_w(\mathfrak{M})$,
- (1)₂ $\varepsilon_w(\mathfrak{M})'_w = \mathfrak{M}'_w$,
- (1)₃ $\varepsilon_w(\mathfrak{M})'_w \mathcal{D}_w(\mathfrak{M}) = \mathcal{D}_w(\mathfrak{M})$.

If \mathfrak{M} is a weak partial Op^* -algebra then ε_w is a \dagger -invariant linear bijection of \mathfrak{M} onto the partial Op^* -algebra $\varepsilon_w(\mathfrak{M})$, ε_w^{-1} is a $*$ -homomorphism, but ε_w is not necessarily a $*$ -homomorphism.

(2) Suppose \mathfrak{M}'_{qw} is an algebra. Put

$$\begin{aligned} \mathcal{D}_{qw}(\mathfrak{M}) &= \{ \sum_{k=1}^n C_k \xi_k; C_k \in \mathfrak{M}'_{qw}, \xi_k \in \mathcal{D}, k = 1, \dots, n, n \in \mathbf{N} \} \\ \varepsilon_{qw}(X)(\sum_{k=1}^n C_k \xi_k) &= \sum_{k=1}^n C_k X \xi_k, \quad X \in \mathfrak{M}, \quad \sum_{k=1}^n C_k \xi_k \in \mathcal{D}_{qw}(\mathfrak{M}). \end{aligned}$$

Then $\varepsilon_{qw}(\mathfrak{M})$ is a \dagger -invariant subset (resp. subspace) of $\mathcal{L}^\dagger(\mathcal{D}_{qw}(\mathfrak{M}), \mathcal{H})$ such that

- (2)₁ $\mathfrak{M} <_w \varepsilon_{qw}(\mathfrak{M})$,
- (2)₂ $\varepsilon_{qw}(\mathfrak{M})'_w = \mathfrak{M}'_{qw}$,
- (2)₃ $\varepsilon_{qw}(\mathfrak{M})'_w \mathcal{D}_{qw}(\mathfrak{M}) = \mathcal{D}_{qw}(\mathfrak{M})$.

If \mathfrak{M} is a weak partial Op^* -algebra, then ε_{qw} is a $*$ -isomorphism of the partial Op^* -algebra \mathfrak{M} onto the partial Op^* -algebra $\varepsilon_{qw}(\mathfrak{M})$.

Proof. We prove the statement (2). A similar reasoning applies to the statement (1). Since

$$\begin{aligned} (\varepsilon_{qw}(X) \sum_{k=1}^n C_k \zeta_k | \sum_{j=1}^m C'_j \zeta'_j) &= \sum_k \sum_j (C_k X \zeta_k | C'_j \zeta'_j) \\ &= \sum_k \sum_j (C'_j * C_k X \zeta_k | \zeta'_j) \\ &= \sum_k \sum_j (C'_j * C_k \zeta_k | X^\dagger \zeta'_j) \\ &= (\sum_k C_k \zeta_k | \sum_j C'_j X^\dagger \zeta'_j) \\ &= (\sum_k C_k \zeta_k | \varepsilon_{qw}(X^\dagger) \sum_j C'_j \zeta'_j) \end{aligned}$$

for each $X \in \mathfrak{M}$ and $\sum_k C_k \zeta_k, \sum_j C'_j \zeta'_j \in \mathcal{D}_{qw}(\mathfrak{M})$, it follows that $\varepsilon_{qw}(\mathfrak{M})$ is a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}_{qw}(\mathfrak{M}), \mathcal{H})$ satisfying the statements $\mathfrak{M} < \varepsilon_{qw}(\mathfrak{M})$, (2)₂ and (2)₃.

We show $\mathfrak{M} <_w \varepsilon_{qw}(\mathfrak{M})$. Let $X_1, X_2 \in \mathfrak{M}$ such that $X_1 \in L^w(X_2)$. Then we have

$$\begin{aligned} (\varepsilon_{qw}(X_1)^\dagger \sum_{k=1}^n C_k \zeta_k | \varepsilon_{qw}(X_2) \sum_{j=1}^m C'_j \zeta'_j) &= \sum_k \sum_j (C_k X_1^\dagger \zeta_k | C'_j X_2 \zeta'_j) \\ &= \sum_k \sum_j (C'_j * C_k X_1^\dagger \zeta_k | X_2 \zeta'_j), \quad (\text{by } C'_j * C_k \in \mathfrak{M}'_{qw}) \\ &= \sum_k \sum_j (C'_j * C_k \zeta_k | (X_1 \square X_2) \zeta'_j) \end{aligned}$$

and

$$\begin{aligned}
 &= (\sum_k C_k \zeta_k | \sum_j C'_j (X_1 \square X_2) \zeta'_j) \\
 &(\varepsilon_{q_w}(X_2) \sum_{k=1}^n C_k \zeta_k | \varepsilon_{q_w}(X_1)^\dagger \sum_{j=1}^m C'_j \zeta'_j) \\
 &= (\sum_k C_k \zeta_k | \sum_j C'_j (X_1 \square X_2)^\dagger \zeta'_j)
 \end{aligned}$$

for each $\sum_k C_k \zeta_k, \sum_j C'_j \zeta'_j \in \mathcal{D}_{q_w}(\mathfrak{M})$. It follows that $\varepsilon_{q_w}(X_1) \in L^w(\varepsilon_{q_w}(X_2))$, and hence $\mathfrak{M} \prec_w \varepsilon_{q_w}(\mathfrak{M})$, which implies by Lemma 3.2 that, if \mathfrak{M} is a weak partial Op*-algebra on \mathcal{D} , then ε_{q_w} is a *-isomorphism of \mathfrak{M} onto the weak partial Op*-algebra $\varepsilon_{q_w}(\mathfrak{M})$ on $\mathcal{D}_{q_w}(\mathfrak{M})$. This completes the proof. \square

Let \mathfrak{N} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. As above, we denote by $\mathfrak{M}_w[\mathfrak{N}]$ (resp. $\mathfrak{M}_s[\mathfrak{N}]$, $\mathfrak{B}_s[\mathfrak{N}]$) the weak (resp. strong, strong pseudo-) partial Op*-algebra generated by \mathfrak{N} . We investigate the relations between their commutants \mathfrak{N}'_w , \mathfrak{N}'_{q_w} , $\mathfrak{M}_s[\mathfrak{N}]'_w$, $\mathfrak{B}_s[\mathfrak{N}]'_w$, $\mathfrak{M}_w[\mathfrak{N}]'_w$ and $\mathfrak{M}_w[\mathfrak{N}]'_{q_w}$.

Proposition 3.10. *Let \mathfrak{N} be a \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then the following statements hold.*

(1) $\mathfrak{M}_w[\mathfrak{N}]'_{q_w} \subset \mathfrak{M}_w[\mathfrak{N}]'_w \subset \mathfrak{B}_s[\mathfrak{N}]'_w = \mathfrak{N}'_w$.

Suppose $\mathfrak{M}_s[\mathfrak{N}]$ exists. Then

$$\mathfrak{M}_s[\mathfrak{N}]'_w = \mathfrak{B}_s[\mathfrak{N}]'_w = \mathfrak{N}'_w.$$

(2) Suppose \mathfrak{N}'_{q_w} is an algebra. Then

$$\mathfrak{M}_w[\mathfrak{N}]'_{q_w} \subset \mathfrak{N}'_{q_w} = \varepsilon_{q_w}(\mathfrak{N})'_w = \mathfrak{M}_w[\varepsilon_{q_w}(\mathfrak{N})]'_w = \mathfrak{M}_w[\varepsilon_{q_w}(\mathfrak{N})]'_{q_w}.$$

In particular, if $\mathfrak{N}'_{q_w} \mathcal{D} = \mathcal{D}$, then

$$\mathfrak{M}_w[\mathfrak{N}]'_{q_w} = \mathfrak{N}'_{q_w}.$$

(3) Suppose \mathfrak{N}'_w is an algebra. Then

$$\mathfrak{M}_w[\mathfrak{N}]'_w \subset \mathfrak{N}'_w = \varepsilon_w(\mathfrak{N})'_w = \mathfrak{M}_w[\varepsilon_w(\mathfrak{N})]'_w = \mathfrak{M}_w[\varepsilon_w(\mathfrak{N})]'_{q_w}.$$

In particular, if $\mathfrak{N}'_w \mathcal{D} = \mathcal{D}$, then

$$\mathfrak{M}_w[\mathfrak{N}]'_{q_w} = \mathfrak{M}_w[\mathfrak{N}]'_w = \mathfrak{N}'_w = \mathfrak{N}'_{q_w}.$$

Proof. (1) It is clear that

$$\mathfrak{M}_w[\mathfrak{N}]'_{q_w} \subset \mathfrak{M}_w[\mathfrak{N}]'_w \subset \mathfrak{B}_s[\mathfrak{N}]'_w \subset \mathfrak{N}'_w.$$

We show $\mathfrak{N}'_w \subset \mathfrak{B}_s[\mathfrak{N}]'_w$. Take an arbitrary $C \in \mathfrak{N}'_w$ and define the set:

$$\begin{aligned}
 \mathfrak{R}_C &= \{X \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}); (X \xi | C \eta) = (C^* \xi | X^\dagger \eta) \text{ and} \\
 &(X \xi | C^* \eta) = (C \xi | X^\dagger \eta), \text{ for all } \xi, \eta \in \mathcal{D}\} \\
 &= \{C, C^*\}'_\sigma,
 \end{aligned}$$

where \mathfrak{N}'_o denotes again the weak unbounded commutant (3.1) of \mathfrak{N} . Then \mathfrak{K}_C is a \dagger -invariant subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ containing \mathfrak{N} . Furthermore, \mathfrak{K}_C is stable under the strong partial multiplication, as follows from Ref.9, Prop.3.2(i), and can be checked readily. Thus $\mathfrak{B}_s[\mathfrak{N}] \subset \mathfrak{K}_C$. Hence $C \in \mathfrak{B}_s[\mathfrak{N}]'_w$ and we have $\mathfrak{N}'_w = \mathfrak{B}_s[\mathfrak{N}]'_w$.

Suppose $\mathfrak{M}_s[\mathfrak{N}]$ exists. Then, since

$$\mathfrak{N} \subset \mathfrak{M}_s[\mathfrak{N}] \subset \mathfrak{B}_s[\mathfrak{N}],$$

it follows that

$$\mathfrak{N}'_w \supset \mathfrak{M}_s[\mathfrak{N}]'_w \supset \mathfrak{B}_s[\mathfrak{N}]'_w = \mathfrak{N}'_w,$$

which completes the proof of (1).

To show the statements (2) and (3), we first show that $\mathfrak{M}_w[\mathfrak{N}]'_w = \mathfrak{N}'_w$ if $\mathfrak{N}'_w \mathcal{D} = \mathcal{D}$. It is clear that $\mathfrak{M}_w[\mathfrak{N}]'_w \subset \mathfrak{N}'_w$. Take again an arbitrary $C \in \mathfrak{N}'_w$. Since $\mathfrak{N}'_w \mathcal{D} \subset \mathcal{D}$, it follows from Ref.9, Prop.3.3, that \mathfrak{K}_C is a weak partial Op*-algebra containing \mathfrak{N} , which implies $\mathfrak{M}_w[\mathfrak{N}] \subset \mathfrak{K}_C$. Hence we have $C \in \mathfrak{M}_w[\mathfrak{N}]'_w$ for each $C \in \mathfrak{N}'_w$. Thus finally we get $\mathfrak{M}_w[\mathfrak{N}]'_w = \mathfrak{N}'_w$.

(2) By Theorem 3.9 we have $\varepsilon_{q_w}(\mathfrak{N}'_w \mathcal{D}_{q_w}(\mathfrak{N})) = \mathcal{D}_{q_w}(\mathfrak{N})$, and hence statement (2) follows from the fact just proven.

(3) This follows similarly from Theorem 3.9 and the fact proven above. This completes the proof. □

§4. Partial Op*-Algebras Generated by a Symmetric Operator

4.A. The Partial Op*-Algebras $\mathfrak{M}_s(T^{(1)})$ and $\mathfrak{M}_w(T^{(1)})$

Given a sense domain \mathcal{D} , we will study in this section the partial Op*-algebra generated by a single symmetric element X of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Of course, there are two of them, a weak one and a strong one, if the latter exists. To start with, we have to distinguish between weak and strong powers of X , and between weakly and strongly well-behaved elements, in the sense of Section 2. The main question is whether these partial Op*-algebras are in fact abelian, as one would naively expect. Also, what is the structure of their commutants?

Let T be a closed symmetric operator in \mathcal{H} and \mathcal{D} a core for T , i.e. $T = \overline{T \upharpoonright \mathcal{D}}$. When $\mathcal{D} \subset D(T^n)$ for some $n \in \mathbb{N} \cup \{\infty\}$, where $D(T^\infty) \equiv \mathcal{D}^\infty(T) = \bigcap_{k=1}^\infty D(T^k)$, we define:

$$T^{[k]} = T^k \upharpoonright \mathcal{D}, \text{ for } k = 1, 2, \dots, n.$$

Notice the relations

$$T^{[k]} = T^{[k]\dagger} \subset T^k \subset T^{k*} \subset T^{[k]*}.$$

We also define spaces of polynomials:

$$\mathfrak{P}_n(T^{11}) = \{\sum_{k=0}^n \alpha_k T^{11k}; \alpha_k \in \mathbb{C}, k = 1, 2, \dots, n\}, \text{ if } n \in \mathbb{N},$$

$$\mathfrak{P}(T^{11}) \equiv \mathfrak{P}_\infty(T^{11}) = \{\sum_{k=0}^m \alpha_k T^{11k}; \alpha_k \in \mathbb{C}, k = 1, 2, \dots, m; m \in \mathbb{N}\}, \text{ if } n = \infty.$$

Given a complex polynomial $P(t) = \sum_{k=0}^n \alpha_k t^k$, we put:

$$P(T^{11}) = \sum_{k=0}^n \alpha_k T^{11k}.$$

We first investigate the structure of the weak partial Op^* -algebra $\mathfrak{M}_w(T^{11})$ and the strong partial Op^* -algebra $\mathfrak{M}_s(T^{11})$ generated by T^{11} . It is clear that, if $T\mathcal{D} \subset \mathcal{D}$, then $\mathfrak{M}_w(T^{11}) = \mathfrak{M}_s(T^{11}) = \mathfrak{P}(T^{11})$. which is an Op^* -algebra on \mathcal{D} . For an arbitrary operator T , we will show below that, contrary to the case of a general \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, the strong partial Op^* -algebra $\mathfrak{M}_s(T^{11})$ always exists and one has, for some n :

$$\mathfrak{M}_s(T^{11}) \subset \mathfrak{P}_n(T^{11}) \subset \mathfrak{M}_w(T^{11}).$$

The following results of Schmüdgen [33] are well-known.

Lemma 4.1. *Let T be a closed symmetric operator in a Hilbert space \mathcal{H} . Then the following statements hold.*

- (1) $P(T)$ is a closed operator in \mathcal{H} for every complex polynomial P .
- (2) The norms $\|\cdot\|_{P(T)}$ and $\|\cdot\|_{T^n}$ are equivalent for every complex polynomial P of degree n .
- (3) A subspace $\mathcal{D} \subset D(T^n)$ is a core for T^n iff it is a core for every complex polynomial $P(T)$ of degree n .

As a consequence of this lemma, it suffices to consider powers of T for controlling arbitrary polynomials in T . First we define properly the weak and the strong powers of $T^{11} \equiv T \upharpoonright \mathcal{D}$. Once and for all, n denotes the largest number in $\mathbb{N} \cup \{\infty\}$ such that $\mathcal{D} \subset D(T^n)$. The following lemma is straightforward.

- Lemma 4.2.** (1) *When $n \in \mathbb{N}$, $T^{[k]} \square T^{[m-k]}$ exists and equals $T^{[m]}$ for each $m \leq n$ and each $k < m$. When $n = \infty$, $T^{[k]} \square T^{[m-k]}$ exists and equals $T^{[m]}$ for each $m \in \mathbb{N}$ and each $k < m$.*
- (2) *If $T^{[k]} \circ T^{[m]}$ exists for $k, m < n$, then $k + m \leq n$.* □

Assume $n \in \mathbb{N}$ (the case $n = \infty$ will be treated later on, in Theorem 4.6). By Lemma 4.2(1), for each $m = 1, 2, \dots, n$, $T^{[m]}$ is the weak m^{th} power $T^{11} \square \dots \square T^{11}$ (m times) of T^{11} (so the notation is consistent). Higher weak powers of T^{11} are now defined recursively. If all products $T^{[k]} \square T^{[m]}$ exist for each pair $k, m \in \mathbb{N}$ with $k + m = n + 1$ and they coincide, we say that the weak $(n + 1)^{\text{th}}$ power of T^{11} is defined and we denote it by $T^{[n+1]}$. Successive higher powers $T^{[n+2]}, T^{[n+3]}, \dots$ may be defined in the same way, if the corresponding

conditions hold.

For the strong powers, the situation changes dramatically. Let $m \leq n$. Then, if all products $T^{[k]} \cdot T^{[m-k]}$ exist for $k = 1, \dots, m$ and they coincide, we say that the strong m^{th} power of $T^{[1]}$ is defined; we denote it by $T^{(m)}$, although it coincides with $T^{[m]}$. But now the process stops at $m = n$: according to Lemma 4.2 (2), no higher strong power may be defined.

The easiest way to visualize the behavior of $T^{[1]}$ under the two partial multiplications is to use the concept of length introduced in Section 2: the *weak length* $l_w(T^{[1]})$ (resp. the *strong length* $l_s(T^{[1]})$ of $T^{[1]}$) is the largest number m in $\mathbb{N} \cup \{\infty\}$ such that the weak m^{th} power $T^{[m]}$ (resp. the strong m^{th} power $T^{(m)}$) of $T^{[1]}$ is defined. Thus the discussion above may be summarized by the inequalities $1 \leq l_w(T^{[1]}) \leq \infty$, $1 \leq l_s(T^{[1]}) \leq n$. The next result gives more information on the behavior of $T^{[1]}$.

Lemma 4.3. *Let T be a closed symmetric operator in \mathcal{H} and \mathcal{D} a core for T . Denote by n be the largest natural number such that $\mathcal{D} \subset D(T^n)$ and by m the largest number in $\mathbb{N} \cup \{0\} \cup \{\infty\}$ such that $T^n \mathcal{D} \subset D(T^{*m})$. Then the following statements hold:*

$$(1) \quad l_w(T^{[1]}) = m + n \text{ if } m \leq n, \text{ and } 2n \leq l_w(T^{[1]}) \leq m + n \text{ if } m > n,$$

$$T^{[k]} = \begin{cases} T^k \upharpoonright \mathcal{D}, & k \leq n \\ (T^*)^{k-n} T^n \upharpoonright \mathcal{D}, & n + 1 \leq k \leq l_w(T^{[1]}). \end{cases}$$

In particular, if T is self-adjoint, then $l_w(T^{[1]}) = n$.

$$(2) \quad 2 \leq l_s(T^{[1]}) \leq n;$$

$$T^{(k)} = T^{[k]} = T^k \upharpoonright \mathcal{D}, \quad k \leq l_s(T^{[1]}).$$

Proof. (1) By the definition of the weak length $l_w(T^{[1]})$, it is easily shown that $l_w(T^{[1]}) \leq m + n$ and

$$T^{[k]} = \begin{cases} T^k \upharpoonright \mathcal{D}, & k \leq n \text{ (by definition)} \\ (T^*)^{k-n} T^n \upharpoonright \mathcal{D}, & n + 1 \leq k \leq l_w(T^{[1]}). \end{cases}$$

Suppose $m \leq n$. Take an arbitrary $l \leq m + n$ and $p, q \in \mathbb{N}$ with $p + q = l$. If $q \geq n$, then $p \leq n$ and $p + q - n = l - n \leq m$, and hence we have

$$\begin{aligned} (T^{[p]} \xi \mid T^{[q]} \eta) &= (T^p \xi \mid T^{*q-n} T^n \eta) \\ &= (\xi \mid T^{*l-n} T^n \eta) \end{aligned}$$

and

$$\begin{aligned} (T^{[q]} \xi \mid T^{[p]} \eta) &= (T^{*q-n} T^n \xi \mid T^p \eta) \\ &= (T^{l-n} \xi \mid T^n \eta) \\ &= (\xi \mid T^{*l-n} T^n \eta) \end{aligned}$$

for each $\xi, \eta \in \mathcal{D}$. Hence, $T^{[p]} \square T^{[q]}$ exists and it equals $T^{*l-n} T^n \upharpoonright \mathcal{D}$. In the case $q < n$, we can show in the same way that $T^{[p]} \square T^{[q]}$ exists. Hence, we have $l_w(T^{11}) = m + n$. When $m > n$, it is similarly shown that the weak l^{th} power $T^{[l]}$ of T^{11} exists for each $l \leq 2n$, and so $2n \leq l_w(T^{11}) \leq m + n$. If T is self-adjoint, then $m = 0$, and so $l_w(T^{11}) = n$.

(2) This follows from Lemma 4.2 (2). □

We proceed to the main theorem of this section and analyze the structure of the partial Op^* -algebras $\mathfrak{M}_w(T^{11})$ and $\mathfrak{M}_s(T^{11})$. Let T, \mathcal{D} and n be as above. We begin with the weak case. When $l_w \equiv l_w(T^{11}) = \infty$, it is clear that

$$\begin{aligned} \mathfrak{M}_w(T^{11}) &= \{ \sum_{k=0}^r \alpha_k T^{[k]}; \alpha_k \in \mathbf{C}, k = 0, 1, \dots, r, r \in \mathbf{N} \cup \{0\} \} \\ &= \{ \sum_{k=0}^n \alpha_k T^k \upharpoonright \mathcal{D} + \sum_{k=1}^p \alpha_{n+k} (T^*)^k T^n \upharpoonright \mathcal{D}; \\ &\quad \alpha_k \in \mathbf{C}, k = 0, 1, \dots, n + p, p \in \mathbf{N} \}. \end{aligned}$$

Hence we have only to study the structure of $\mathfrak{M}_w(T^{11})$ when $l_w < \infty$. As usual, we denote by $\mathfrak{R}_w(T^{11})$ the *regular part* of $\mathfrak{M}_w(T^{11})$, i.e. the set $\mathfrak{P}_{l_w}(T^{11})$ of all finite linear combinations of $T^{[k]}$ ($0 \leq k \leq l_w$), and by $\mathfrak{S}_w(T^{11})$ the *singular part* $\mathfrak{M}_w(T^{11}) - \mathfrak{R}_w(T^{11})$ of $\mathfrak{M}_w(T^{11})$. The singular part is quite complicated, because T^{11} is not necessarily weakly well-behaved ($T^{[p]} \square T^{[q]}$ may exist for $1 \leq p, q \leq l_w$ and $p + q > l_w$), and, even if T^{11} is weakly well-behaved, $R_1 \square R_2$ may exist for $R_1, R_2 \in \mathfrak{R}_w(T^{11})$ and $\gamma_1 + \gamma_2 > l_w$, where γ_i is the degree of the polynomial $R_i(t)$. In fact, both $\mathfrak{M}_w(T^{11})$ and $\mathfrak{S}_w(T^{11})$ may be constructed in a recursive way. We define successively:

$\mathfrak{M}_1(T^{11})$: the set of all finite linear combinations of polynomials $G^{(1)}$, where

$$G^{(1)} = R_1 \square R_2 \text{ exists for some } R_1, R_2 \in \mathfrak{R}_w(T^{11}),$$

$\mathfrak{M}_k(T^{11})$: the set of all finite linear combinations of polynomials $G^{(k)}$, where

$$G^{(k)} = G_1^{(k-1)} \square G_2^{(k-1)} \text{ exists for some } G_1^{(k-1)}, G_2^{(k-1)} \in \mathfrak{M}_{k-1}(T^{11}), k \in \mathbf{N}.$$

$\mathfrak{S}_k(T^{11}) = \mathfrak{M}_k(T^{11}) - \mathfrak{R}_w(T^{11}), k \in \mathbf{N}$.

Then we have

$$\begin{aligned} \mathfrak{R}_w(T^{11}) &\subset \mathfrak{M}_1(T^{11}) \subset \dots \subset \bigcup_{k \in \mathbf{N}} \mathfrak{M}_k(T^{11}) = \mathfrak{M}_w(T^{11}), \\ \mathfrak{S}_1(T^{11}) &\subset \mathfrak{S}_2(T^{11}) \subset \dots \subset \bigcup_{k \in \mathbf{N}} \mathfrak{S}_k(T^{11}) = \mathfrak{S}_w(T^{11}), \\ \mathfrak{M}_w(T^{11}) &= \mathfrak{R}_w(T^{11}) + \mathfrak{S}_w(T^{11}) \\ &= \{ \sum_{k=0}^{l_w} \alpha_k T^{[k]} + S(T^{11}); \alpha_k \in \mathbf{C} (k = 0, 1, \dots, l_w) \text{ and } S \in \mathfrak{S}_w(T^{11}) \} \\ &= \{ \sum_{k=0}^n \alpha_k T^k \upharpoonright \mathcal{D} + \sum_{k=1}^{l_w-n} \alpha_{n+k} (T^*)^k T^n \upharpoonright \mathcal{D} + S(T^{11}); \\ &\quad \alpha_k \in \mathbf{C} (k = 0, 1, \dots, l_w) \text{ and } S \in \mathfrak{S}_w(T^{11}) \}. \end{aligned}$$

Let us analyze the space $\mathfrak{S}_1(T^{11})$ in more details. We put:

$$\begin{aligned} \mathfrak{S}_1^0(T^{11}) &= \{A(T^{11}); A(T^{11}) \equiv T^{[p]} \square T^{[q]} \text{ exists for some} \\ &\quad 1 \leq p, q \leq l_w \text{ and } p + q > l_w\}, \\ \mathfrak{S}_1^1(T^{11}) &= \{B(T^{11}); B(T^{11}) \equiv (\alpha_0 I + \alpha_1 T^{11} + \dots + \alpha_{l_w} T^{[l_w]}) \square \\ &\quad (\beta_0 I + \dots + \beta_{l_w} T^{[l_w]}) \\ &\quad \text{exists for some non-zero } (\alpha_0, \dots, \alpha_{l_w}), (\beta_0, \dots, \beta_{l_w}) \in \mathbf{C}^{l_w}, \\ &\quad \text{but } \alpha_{k_0} T^{[k_0]} \square \beta_{m_0} T^{[m_0]} \text{ does not exist for some } k_0, m_0\}. \end{aligned}$$

Then we have

$$\begin{aligned} \mathfrak{S}_1(T^{11}) &= \{\sum_{k=0}^{l_w} \alpha_k T^{[k]} + \sum_{i=1}^s \beta_i A_i(T^{11}) + \sum_{j=1}^t \gamma_j B_j(T^{11}); \\ &\quad A_i(T^{11}) \in \mathfrak{S}_1^0(T^{11}) (i = 1, 2, \dots, s), B_j(T^{11}) \in \mathfrak{S}_1^1(T^{11}) (j = 1, 2, \dots, t), \\ &\quad \text{and } |\beta_i| + |\gamma_j| \neq 0 \text{ for some } i, j\}. \end{aligned}$$

The structure of the spaces $\mathfrak{S}_k(T^{11}) (k \geq 2)$ is more complicated.

We turn now to the strong case, and begin by analyzing the structure of $\mathfrak{B}_s(T^{11})$, the strong pseudo-partial Op*-algebra generated by T^{11} (see Section 3. A). As before, we define the *regular* part and the *singular* part of $\mathfrak{B}_s(T^{11})$:

$$\begin{aligned} \mathfrak{R}_s(T^{11}) &= \{\sum_{k=0}^{l_s} \alpha_k T^{[k]}; \alpha_k \in \mathbf{C}, k = 0, 1, 2, \dots, l_s\}, \\ \mathfrak{S}_s(T^{11}) &= \mathfrak{B}_s(T^{11}) - \mathfrak{R}_s(T^{11}). \end{aligned}$$

Let us investigate the singular part $\mathfrak{S}_s(T^{11})$ of $\mathfrak{B}_s(T^{11})$ in more detail. First, we consider generalized powers (quasi-powers) and obtain an increasing sequence of subsets:

$$\mathfrak{R}_0 \subset \mathfrak{R}_1 \subset \mathfrak{R}_2 \subset \dots \mathfrak{R}_l \subset \mathfrak{R}_{l+1} \equiv \mathfrak{R}_s,$$

where l is the integer defined by the condition $2^{l-1} l_s < n \leq 2^l l_s$ and we have defined:

$$\begin{aligned} \mathfrak{R}_0 &= \{I, T^{(1)}, \dots, T^{(l_s)}\}, \\ \mathfrak{R}_1 &= \{T^{[r]}; l_s + 1 \leq r \leq 2l_2, T^{[p]} \cdot T^{[q]} \text{ exists for some } p, q \text{ such that} \\ &\quad 1 \leq p, q \leq l_s \text{ and } p + q = r\}, \\ \mathfrak{R}_2 &= \{T^{[r]}; l_s + 1 \leq r \leq 2^2 l_s, T^{[r]} = T^{[p]} \cdot T^{[q]} \text{ exists for some} \\ &\quad T^{[p]} \in \mathfrak{R}_0 \cup \mathfrak{R}_1 \text{ and } T^{[q]} \in \mathfrak{R}_1\}, \\ &\dots\dots \\ \mathfrak{R}_k &= \{T^{[r]}; l_s + 1 \leq r \leq 2^k l_s, T^{[r]} = T^{[p]} \cdot T^{[q]} \text{ exists for some } T^{[p]} \in \mathfrak{R}_0 \cup \mathfrak{R}_{k-1} \end{aligned}$$

$$\text{and } T^{[q]} \in \mathfrak{R}_k\}, \quad 1 \leq k \leq l + 1.$$

Notice that, for $k = l$ and $k = l + 1$, the range of the exponent r is $l_s + 1 \leq r \leq n$. Denote by $\mathfrak{R}_k^q(T^{11})$ ($1 \leq k \leq l + 1$) the set of all finite linear combinations of elements of $\mathfrak{R}_0 \cup \mathfrak{R}_k$. Then the *quasi-regular* part of $\mathfrak{B}_s(T^{11})$ is defined as

$$\begin{aligned} \mathfrak{R}_s^q(T^{11}) &\equiv \mathfrak{R}_{l+1}^q(T^{11}) \\ &= \left\{ \sum_{k=0}^{l_s} \alpha_k T^{[k]} + \sum_{r \in \mathbb{N}_s} \beta_r T^{[r]}; \alpha_k \in \mathbb{C}, k = 1, 2, \dots, l_s \text{ and } \beta_r \in \mathbb{C}, r \in \mathbb{N}_s \right\}, \end{aligned}$$

where $\mathbb{N}_s = \{r \in \mathbb{N}; T^{[r]} \in \mathfrak{R}_s\}$.

In addition, we introduce, as in the weak case, the following spaces of polynomials in T^{11} :

$\mathfrak{B}_1(T^{11})$: the set of all finite linear combinations of polynomials $P^{(1)}$, where

$$P^{(1)} = P_1^{(0)} \cdot P_2^{(0)} \text{ exists for some } P_1^{(0)}, P_2^{(0)} \in \mathfrak{R}_s(T^{11}),$$

$\mathfrak{B}_k(T^{11})$: the set of all finite linear combinations of polynomials $P^{(k)}$, where

$$\begin{aligned} P^{(k)} = P_1^{(k-1)} \cdot P_2^{(k-1)} \text{ exists for some } P_1^{(k-1)}, P_2^{(k-1)} \in \mathfrak{B}_{k-1}(T^{11}), \\ 2 \leq k \leq l + 1. \end{aligned}$$

Notice that an element $P^{(l)}$ of $\mathfrak{B}_1(T^{11})$ is a polynomial in T^{11} of degree at most n :

$$P^{(l)}(T^{11}) = \sum_{k=0}^n \gamma_k T^{[k]}, \quad \gamma_k \in \mathbb{C}.$$

Consider any $P^{(l+1)} = P_1^{(l)} \cdot P_2^{(l)} \in \mathfrak{B}_{l+1}(T^{11})$. Then we have:

$$P_2^{(l)}(T^{11}) \mathcal{D} \subset D(\overline{T^{p_1} \upharpoonright \mathcal{D}}) \subset D(T^{p_1}),$$

which implies $p_1 + p_2 \leq n$, where p_1 and p_2 are the degrees of the polynomials $P_1^{(l)}$ and $P_2^{(l)}$, respectively. Hence we have

$$\begin{aligned} \mathfrak{B}_1(T^{11}) \subset \dots \subset \mathfrak{B}_{l+1}(T^{11}) = \mathfrak{B}_s(T^{11}) \subset \mathfrak{B}_n(T^{11}) \\ \cup \qquad \qquad \cup \\ \mathfrak{R}_1^q(T^{11}) \subset \dots \subset \mathfrak{R}_{l+1}^q(T^{11}) = \mathfrak{R}_s^q(T^{11}). \end{aligned}$$

Now we define the *strongly singular* part of $\mathfrak{B}_s(T^{11})$ as follows:

$$\begin{aligned} \mathfrak{S}_k^s(T^{11}) &= \mathfrak{B}_k(T^{11}) - \mathfrak{R}_k^q(T^{11}), \quad 1 \leq k \leq l + 1, \\ \mathfrak{S}_s^s(T^{11}) &= \bigcup_{k=0}^{l+1} \mathfrak{S}_k^s(T^{11}) \\ &= \mathfrak{B}_s(T^{11}) - \mathfrak{R}_s^q(T^{11}). \end{aligned}$$

Then we have:

$$\begin{aligned} \mathfrak{B}_s(T^{11}) &= \mathfrak{R}_s(T^{11}) + \mathfrak{S}_s(T^{11}) \\ &= \mathfrak{R}_s^q(T^{11}) + \mathfrak{S}_s^s(T^{11}) \\ &= \{ \sum_{k=0}^{l_s} \alpha_k T^{(k)} + \sum_{r \in \mathbb{N}_s} \beta_r T^{[r]} + S(T^{11}); \alpha_k \in \mathbb{C} (k = 1, 2, \dots, l_s) \\ &\quad \text{and } \beta_r \in \mathbb{C} (r \in \mathbb{N}_s) \text{ and } S \in \mathfrak{S}_s^s(T^{11}) \}. \end{aligned}$$

Finally we are ready to show that $\mathfrak{M}_s(T^{11})$ exists and equals $\mathfrak{B}_s(T^{11})$. As shown above, we have $\mathfrak{B}_s(T^{11}) \subset \mathfrak{P}_n(T^{11})$. Take any $P \in \mathfrak{B}_s(T^{11})$ and any pair $Q_1, Q_2 \in L^s(P)$. It follows from Lemma 4.1 (2) that $P(T^{11})\mathcal{D} \subset D(\overline{Q(T^{11})})$ iff $P(T^{11})\mathcal{D} \subset D(\overline{T^q \upharpoonright \mathcal{D}})$, where q is the degree of the polynomial Q , so that

$$P(T^{11})\mathcal{D} \subset D(\overline{T^{q_1} \upharpoonright \mathcal{D}}) \cap D(\overline{T^{q_2} \upharpoonright \mathcal{D}}),$$

which implies

$$P(T^{11})\mathcal{D} \subset D(\overline{Q_1(T^{11}) + Q_2(T^{11})}).$$

Furthermore it is clear that

$$(\overline{Q_1(T^{11})} + \overline{Q_2(T^{11})})\mathcal{D} \subset D(\overline{P(T^{11})}).$$

Hence it follows that $Q_1 + Q_2 \in L^s(P)$, which implies that $\mathfrak{B}_s(T^{11}) = \mathfrak{M}_s(T^{11})$.

Thus $\mathfrak{R}_s(T^{11})$ (resp. $\mathfrak{R}_s^q(T^{11}), \mathfrak{S}_s(T^{11}), \mathfrak{S}_s^s(T^{11})$) will be called the *regular* (resp. *quasi-regular, singular, strongly singular*) part of $\mathfrak{M}_s(T^{11})$.

Given two polynomials P, Q in T^{11} , if $P \cdot Q$ and $Q \cdot P$ both exist, then they are equal. But $P(T^{11})\mathcal{D} \subset D(\overline{T^q \upharpoonright \mathcal{D}})$ does not necessarily imply $\overline{P(T^{11})}\mathcal{D} \subset D(\overline{T^q \upharpoonright \mathcal{D}})$, and therefore $P \in L^s(Q)$ is not equivalent to $Q \in L^s(P)$. Hence $\mathfrak{M}_s(T^{11})$ is not abelian in general. Furthermore, when $\mathfrak{R}_s^q(T^{11}) \neq \emptyset$, $\mathfrak{M}_s(T^{11})$ is not associative either.

Before casting those results in the form of a theorem, it is useful to emphasize the differences between the two cases.

(1) In the *strong* case, the powers $\{I, T^{11}, \dots, T^{[n]}\}$ always exist, and they divide into genuine powers $\{I, T^{(1)}, \dots, T^{(l_s)}\} \equiv \mathfrak{R}_0$ and the elements of the sets $\mathfrak{R}_1, \dots, \mathfrak{R}_s$, which are quasi-powers $T^q \upharpoonright \mathcal{D}$ of T^{11} : all together they generate the quasi-regular part $\mathfrak{R}_s^q(T^{11})$ of $\mathfrak{M}_s(T^{11})$. In addition, the latter may contain strongly singular elements $S \in \mathfrak{S}_s^s(T^{11})$, which are products of polynomials, but cannot be represented themselves as polynomials.

(2) In the *weak* case, if a product $T^{[p]} \square T^{[q]}$ exists for some $1 \leq p, q \leq l_w$ and $p + q > l_w$, its value is

$$T^{[p]} \square T^{[q]} = \begin{cases} (T^p \upharpoonright \mathcal{D})^* T^q \upharpoonright \mathcal{D}, & q \leq n \\ (T^p \upharpoonright \mathcal{D})^* (T^{q-l_w})^* T^n \upharpoonright \mathcal{D}, & q > n, \end{cases}$$

which means that it is not even a quasi-power of T^{11} . Hence the analysis of $\mathfrak{M}_w(T^{11})$ cannot be pushed as far as that of $\mathfrak{M}_s(T^{11})$.

Summarizing now the whole discussion, we obtain the following

Theorem 4.4. *Let T be a closed symmetric operator in \mathcal{H} , \mathcal{D} be a dense subspace in the Hilbert space $D(T)$, n be the largest natural number such that $\mathcal{D} \subset D(T^n)$ and m be the largest number in $\mathbf{N} \cup \{0\} \cup \{\infty\}$ such that $T^n \mathcal{D} \subset D(T^{*m})$. Then the following statements hold:*

- (1) $2 \leq l_s \leq n \leq \min(m, n) + n \leq l_w \leq m + n$
and

$$\mathfrak{M}_s(T^{11}) = \mathfrak{B}_s(T^{11}) \subset \mathfrak{P}_n(T^{11}) \subset \mathfrak{M}_w(T^{11}).$$

- (2) When $l_w = \infty$,

$$\begin{aligned} \mathfrak{M}_w(T^{11}) &= \left\{ \sum_{k=0}^r \alpha_k T^{[k]}; \alpha_k \in \mathbf{C}, k = 0, 1, \dots, r, r \in \mathbf{N} \cup \{0\} \right\} \\ &= \left\{ \sum_{k=0}^n \alpha_k T^k \upharpoonright \mathcal{D} + \sum_{k=1}^p \alpha_{n+k} (T^*)^k T^n \upharpoonright \mathcal{D}; \right. \\ &\quad \left. \alpha_k \in \mathbf{C}, k = 0, 1, \dots, n + p, p \in \mathbf{N} \right\}, \end{aligned}$$

and it is abelian and associative.

When $l_w < \infty$,

$$\begin{aligned} \mathfrak{M}_w(T^{11}) &= \mathfrak{R}_w(T^{11}) + \mathfrak{S}_w(T^{11}) \\ &= \left\{ \sum_{k=0}^{l_w} \alpha_k T^{[k]} + S(T^{11}); \alpha_k \in \mathbf{C} (k = 0, 1, \dots, l_w) \text{ and } S \in \mathfrak{S}_w(T^{11}) \right\}, \end{aligned}$$

where $\mathfrak{R}_w(T^{11})$ is the regular part and $\mathfrak{S}_w(T^{11})$ is the singular part of $\mathfrak{M}_w(T^{11})$. Furthermore, $\mathfrak{M}_w(T^{11})$ need be neither abelian, nor associative.

- (3) $\mathfrak{M}_s(T^{11}) = \mathfrak{R}_s(T^{11}) + \mathfrak{S}_s(T^{11})$

$$\begin{aligned} &= \mathfrak{R}_s^q(T^{11}) + \mathfrak{S}_s^s(T^{11}) \\ &= \left\{ \sum_{k=0}^{l_s} \alpha_k T^{[k]} + \sum_{r \in \mathbf{N}_s} \beta_r T^{[r]} + S(T^{11}); \alpha_k \in \mathbf{C} (k = 1, 2, \dots, l_s), \right. \\ &\quad \left. \beta_r \in \mathbf{C} (r \in \mathbf{N}_s) \text{ and } S \in \mathfrak{S}_s^s(T^{11}) \right\}, \end{aligned}$$

where $\mathfrak{R}_s(T^{11})$ (resp. $\mathfrak{R}_s^q(T^{11})$, $\mathfrak{S}_s(T^{11})$, $\mathfrak{S}_s^s(T^{11})$) is the regular (resp. quasi-regular, singular, strongly singular) part of $\mathfrak{M}_s(T^{11})$, and the latter is neither abelian, nor associative, in general. □

Clearly the situation, which is quite pathological in general, will improve if the domain \mathcal{D} is better adapted to the operator T . For instance:

Proposition 4.5. *Let T , \mathcal{D} and n be as in Theorem 4.4. Then the following*

statements hold:

(1) Suppose \mathcal{D} is a core for each T^k for $1 \leq k \leq n$. Then $l_s = n$, $\mathfrak{M}_s(T^{11}) = \mathfrak{P}_n(T^{11})$ and its full closure $\widehat{\mathfrak{M}}_s(T^{11})$ equals the fully closed strong partial Op^* -algebra $\mathfrak{M}_s(T \upharpoonright D(T^n)) = \mathfrak{P}_n(T \upharpoonright D(T^n))$ on $D(T^n)$.

(2) Suppose $T^{[k]}$ is essentially self-adjoint for each $1 \leq k \leq n$. Then $l_s = l_w = n$, $\mathfrak{M}_s(T^{11}) = \mathfrak{M}_w(T^{11}) = \mathfrak{P}_n(T^{11})$, and its full closure equals the standard partial Op^* -algebra $\mathfrak{M}_s(T \upharpoonright D(T^n)) = \mathfrak{M}_w(T \upharpoonright D(T^n)) = \mathfrak{P}_n(T \upharpoonright D(T^n))$ on $D(T^n)$.

Proof. (1) Since all strong powers up to order n are defined, we have $l_s = n$. Hence it follows from Theorem 4.4 (1), (3) that $\mathfrak{M}_s(T^{11}) = \mathfrak{P}_n(T^{11})$. From the closed graph theorem, we can conclude that $l_s(T \upharpoonright D(T^n)) = n$ and $\mathfrak{M}_s(T \upharpoonright D(T^n)) = \mathfrak{P}_n(T \upharpoonright D(T^n))$. By Lemma 4.1 we have $\widehat{\mathcal{D}}(\mathfrak{M}_s(T^{11})) = D(T^n)$ and $\overline{P(T^{11})} = P(T)$ for every polynomial P up to degree n , which implies that $\widehat{\mathfrak{M}}_s(T^{11}) = \mathfrak{M}_s(T \upharpoonright D(T^n)) = \mathfrak{P}_n(T \upharpoonright D(T^n))$.

(2) By Theorem 4.4 (1) and statement (1), we have $l_s = l_w = n$ and so:

$$\mathfrak{M}_s(T^{11}) = \mathfrak{P}_n(T^{11}) = \mathfrak{R}_w(T^{11}) \subset \mathfrak{M}_w(T^{11}).$$

Suppose $P, Q \in \mathfrak{R}_w(T^{11})$ are such that $p + q > n$ and $P \square Q$ exists. Since

$$Q(T^{11})\mathcal{D} \subset D(\overline{P}(T^{11})^*) = D(P(T)),$$

it follows from the spectral theory of self-adjoint operators that

$$\mathcal{D} \subset D(P(T)Q(T)) \subset D((PQ)(T)),$$

which implies $\mathcal{D} \subset D(T^{p+q})$. This contradicts the assumption $p + q > n$. Hence, $\mathfrak{M}_w(T^{11}) = \mathfrak{P}_n(T^{11})$. By (1) we have

$$\widehat{\mathfrak{M}}_s(T^{11}) = \widehat{\mathfrak{M}}_w(T^{11}) = \mathfrak{P}_n(T \upharpoonright D(T^n)),$$

and the standardness of $\mathfrak{P}_n(T \upharpoonright D(T^n))$ is clear. □

Conversely, some properties of $\mathfrak{M}_w(T^{11})$ (or $\mathfrak{M}_s(T^{11})$) may be used to prove that $T^{11}, T^{21}, \dots, T^{n1}$ are essentially self-adjoint. For instance, a sufficient condition is that $\widehat{\mathfrak{M}}_w(T^{11})$ be standard, as will be shown in Proposition 4.9 below.

Finally we treat the case $n = \infty$, which is of course more regular.

Theorem 4.6. *Let T and \mathcal{D} be as in Theorem 4.4. Suppose that $n = \infty$, that is $\mathcal{D} \subset \mathcal{D}^\infty(T)$. Then the following statements hold:*

- (1) $l_w = \infty$ and $\mathfrak{M}_w(T^{11}) = \mathfrak{P}(T^{11})$.
- (2) $i^*(\mathfrak{M}_w(T^{11}))$ is a closed Op -algebra on $\bigcap_{n \in \mathbb{N}} D(T^{[n]*})$ and $i^{**}(\mathfrak{M}_w(T^{11}))$ is a closed Op^* -algebra. In particular, if T is self-adjoint, then $i^{**}(\mathfrak{M}_w(T^{11}))$ is a standard Op^* -algebra on $\mathcal{D}^\infty(T)$ and equals the polynomial algebra $\mathfrak{P}(T | \mathcal{D}^\infty(T))$.
- (3) If \mathcal{D} is a core for each T^n , $n \in \mathbb{N}$, then $l_s = l_w = \infty$ and $\mathfrak{M}_s(T^{11})$

$$= \mathfrak{M}_w(T^{11}) = \mathfrak{B}(T^{11}).$$

(4) If $T^{[n]}$ is essentially self-adjoint for each $n \in \mathbb{N}$, then $l_s = l_w = \infty$, $\mathfrak{M}_s(T^{11}) = \mathfrak{M}_w(T^{11}) = \mathfrak{B}(T^{11})$, and its closure is a standard Op^* -algebra on $\mathcal{D}^\infty(T)$. All algebras in (1)-(4) are abelian and associative. The closures and full closures of each of these (partial) Op^* -algebras coincide.

Proof. (1) This follows from Theorem 4.4 (1), (3).

(2) It is clear that $\mathcal{D}^*(\mathfrak{M}_w(T^{11})) = \bigcap_{n=1}^\infty D(T^{[n]*})$. Let $\xi \in \mathcal{D}^*(\mathfrak{M}_w(T^{11}))$ and $n \in \mathbb{N}$; then one has:

$$(T^m \eta | T^{[n]*} \xi) = (T^{[n+m]} \eta | \xi) = (\eta | T^{[m+n]*} \xi)$$

for each $\eta \in \mathcal{D}$ and $m \in \mathbb{N}$, and hence $T^{[n]*} \xi \in \bigcap_{m=1}^\infty D(T^{[m]*}) = \mathcal{D}^*(\mathfrak{M}_w(T^{11}))$, which implies that $i^*(\mathfrak{M}_w(T^{11}))$ is a closed Op -algebra and $i^{**}(\mathfrak{M}_w(T^{11}))$ is a closed Op^* -algebra. Suppose T is self-adjoint. Then we show

$$\bigcap_{k=1}^n D(T^{[k]*}) = D(T^n), \quad n \in \mathbb{N}.$$

For $n = 1$, the statement is clear. Assume $\bigcap_{k=1}^n D(T^{[k]*}) = D(T^n)$, then we prove the statement for $n + 1$. Since

$$\bigcap_{k=1}^{n+1} D(T^{[k]*}) = \bigcap_{k=1}^n D(T^{[k]*}) \cap D(T^{[n+1]*}) = D(T^n) \cap D(T^{[n+1]*}),$$

we have

$$D(T^{n+1}) \subset D(T^n) \cap D(T^{[n+1]*}) = \bigcap_{k=1}^{n+1} D(T^{[k]*}).$$

Conversely, take an arbitrary vector $\xi \in \bigcap_{k=1}^{n+1} D(T^{[k]*})$. Then $\xi \in D(T^n)$ and $\xi \in D(T^{[n+1]*})$, hence we have, for every $\eta \in \mathcal{D}$,

$$(T\eta | T^n \xi) = (T^{n+1} \eta | \xi) = (\eta | T^{[n+1]*} \xi)$$

and therefore, $\xi \in D(T^{n+1})$. Hence,

$$\bigcap_{k=1}^n D(T^{[k]*}) = D(T^n), \quad n \in \mathbb{N},$$

which implies

$$\mathcal{D}^*(\mathfrak{M}_w(T^{11})) = \bigcap_{n=1}^\infty D(T^{[n]*}) = \mathcal{D}^\infty(T),$$

and thus $i^{**}(\mathfrak{M}_w(T^{11}))$ is a standard Op^* -algebra on $\mathcal{D}^\infty(T)$ and equals the polynomial algebra $\mathfrak{B}(T | \mathcal{D}^\infty(T))$.

(3) This follows from (1) and Proposition 4.5 (1).

(4) By (3) we have $\mathfrak{M}_s(T^{11}) = \mathfrak{M}_w(T^{11}) = \mathfrak{B}(T^{11})$. By Lemma 4.1, we also have

$$\begin{aligned} \widehat{\mathcal{D}}(\mathfrak{M}_s(T^{11})) &= \bigcap \{D(\overline{(P(T) \upharpoonright \mathcal{D}})); P \text{ is a polynomial}\} \\ &= \bigcap \{D(P(T)); P \text{ is a polynomial}\} \end{aligned}$$

$$= \bigcap_{n=1}^{\infty} D(T^n) = \mathcal{D}^{\infty}(T),$$

and hence it follows from [30] that the closure of $\mathfrak{M}_s(T^{11})$ is a standard Op*-algebra on $\mathcal{D}^{\infty}(T)$.

It is clear that all algebras in (1)-(4) are abelian and associative. Since $l_w = \infty$, it follows that $\overline{P} \square P$ exists for any polynomial P , which implies that $\mathfrak{M}_w(T^{11}) = \mathfrak{P}(T^{11})$ is directed; that is, for any pair $P, Q \in \mathfrak{P}(T^{11})$, there exists an element $R \in \mathfrak{P}(T^{11})$ such that

$$\|P(T^{11})\xi\|^2 + \|Q(T^{11})\xi\|^2 \leq \|R(T^{11})\xi\|^2,$$

for every $\xi \in \mathcal{D}$, so that the closure and the full closure of $\mathfrak{P}(T^{11})$ coincide. \square

Remark. When T is self-adjoint, one recovers in particular the results obtained earlier [23, 34] for a family of strongly commuting self-adjoint or normal operators, namely T generates an abelian, standard polynomial algebra on $\mathcal{D}^{\infty}(T)$.

Example 4. As can be expected, all situations described above may be realized by operators of derivation on a finite interval, with various boundary conditions. Let us consider indeed the closed operators S, T and H in the Hilbert space $L^2[0, 1]$ defined by

$$\cdot D(S) = \{f \in C[0, 1] : f(x) - f(0) = \int_0^x f_1(t)dt \text{ for some } f_1 \in L^2[0, 1]\}$$

$$Sf = -if_1, f \in D(S);$$

$$\cdot D(T) = \{f \in D(S) : f(1) = f(0) = 0\},$$

$$T = S|D(T);$$

$$\cdot D(H) = \{f \in D(S) : f(1) = f(0)\},$$

$$H = S|D(H).$$

As it is well-known, $T^* = S$ and H is self-adjoint. It is possible to describe explicitly the partial *-algebras generated by the above operators on the following domains:

$$\begin{aligned} \mathcal{D}_0^{(n)} &= \{f \in C^{(n)}[0, 1] ; f(0) = f(1) = 0\}, n \in \mathbf{N} \cup \{\infty\} \\ \mathcal{D}^{(n)} &= \{f \in C^{(n)}[0, 1] ; f^{(k)}(0) = f^{(k)}(1), k = 0, 1, 2, \dots, n\}, \\ \mathcal{D}^{(\infty)} &= \{f \in C^{\infty}[0, 1] ; f^{(n)}(0) = f^{(n)}(1), \forall n \in \mathbf{N}\}. \end{aligned}$$

Then the following statements hold:

- (1) $l_s(T \upharpoonright \mathcal{D}_0^{(1)}) = l_w(T \upharpoonright \mathcal{D}_0^{(1)}) = 1$ and $\mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(1)}) = \mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(1)}) = \mathfrak{P}_1(T \upharpoonright \mathcal{D}_0^{(1)})$.
- (2) Let $2 \leq n \leq \infty$. Then $l_s(T \upharpoonright \mathcal{D}_0^{(n)}) = 1, l_w(T \upharpoonright \mathcal{D}_0^{(n)}) = 2$ and

$$\mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(n)}) = \mathfrak{P}_1(T \upharpoonright \mathcal{D}_0^{(n)})$$

$$\subseteq \mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)}) = \{\alpha_0 I + \alpha_1 T \upharpoonright \mathcal{D}_0^{(n)} + \alpha_2 T^* T \upharpoonright \mathcal{D}_0^{(n)}; \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}\}.$$

(3) $l_s(H \upharpoonright \mathcal{D}^{(n)}) = l_w(H \upharpoonright \mathcal{D}^{(n)}) = n$, $\mathfrak{M}_s(H \upharpoonright \mathcal{D}^{(n)}) = \mathfrak{M}_w(H \upharpoonright \mathcal{D}^{(n)}) = \mathfrak{P}_n(H \upharpoonright \mathcal{D}^{(n)})$ and its full closure equals the standard partial Op^* -algebra $\mathfrak{M}_s(H \upharpoonright D(H^n)) = \mathfrak{M}_w(H \upharpoonright D(H^n)) = \mathfrak{P}_n(H \upharpoonright D(H^n))$ on $D(H^n)$.

Proof. (1) Since $D(T^*T) \not\subseteq \mathcal{D}_0^{(1)} \subset D(T)$, it follows that $\mathcal{D}_0^{(1)}$ is core for T and $(T \upharpoonright \mathcal{D}_0^{(1)}) \square (T \upharpoonright \mathcal{D}_0^{(1)})$ does not exist, which implies statements (1).

(2) Since $\mathcal{D}^\infty(T^*T) \subset \mathcal{D}_0^{(n)} \subset D(T^*T)$, it follows that $\mathcal{D}_0^{(n)}$ is a core for T and T^*T . Since $\mathcal{D}_0^{(n)} \not\subset D(T^2)$, it follows that $T^{[1]} \cdot T^{[1]}$ does not exist, where $T^{[1]} = T \upharpoonright \mathcal{D}_0^{(n)}$. Hence we have $l_s(T^{[1]}) = 1$ and $\mathfrak{M}_s(T^{[1]}) = \mathfrak{P}_1(T^{[1]})$. Next we show $l_w(T^{[1]}) = 2$ and $\mathfrak{M}_w(T^{[1]}) = \mathfrak{P}_2(T^{[1]})$. Since $\mathcal{D}_0^{(n)} \subset D(T^*T)$, it follows that $T^{[1]} \square T^{[1]}$ exists. Since $\mathcal{D}_0^{(n)}$ is a core for T^*T , it follows that $T^{[1]} \square T^{[2]}$ exists iff $T^*T\mathcal{D}_0^{(n)} \subset D(T^*)$ and $T\mathcal{D}_0^{(n)} \subset D(T^*T)$. But $\mathcal{D}_0^{(n)} \not\subset D(T^*T^2)$, so that $T^{[1]} \square T^{[2]}$ does not exist. Furthermore, if $T^{[2]} \square T^{[2]}$ exists, then $\mathcal{D}_0^{(n)} \subset D((T^*T)^2)$. This is a contradiction. Hence, $T^{[2]} \square T^{[2]}$ cannot exist. Thus we have $l_w(T^{[1]}) = 2$ and $\mathfrak{M}_w(T^{[1]}) = \mathfrak{P}_2(T^{[1]})$.

Suppose now that $T^{[2]}\mathcal{D}_0^{(n)} \subset D((\alpha_0 I + \alpha_1 T^{[1]} + \alpha_2 T^{[2]})^*)$ for some $\alpha_0, \alpha_1, \alpha_2 \neq 0 \in \mathbb{C}$. Then we have:

$$\begin{aligned} & (g | (\alpha_0 I + \alpha_1 T^{[1]} + \alpha_2 T^{[2]})^* T^{[2]} f) \\ &= (g | \bar{\alpha}_0 T^* T f) + (Tg | \bar{\alpha}_1 T^* T f) + (T^* Tg | \bar{\alpha}_2 T^* T f) \end{aligned}$$

for every $f, g \in \mathcal{D}_0^{(n)}$. Let $n \geq 3$. Since $\mathcal{D}_0^{(n)} \subset D(T^{*2}T)$, we have

$$(T^* Tg | \bar{\alpha}_2 T^* T f) = (g | (\alpha_0 I + \alpha_1 T^{[1]} + \alpha_2 T^{[2]})^* T^* T f - \bar{\alpha}_0 T^* T f - \bar{\alpha}_1 T^{*2} T f)$$

for every $f, g \in \mathcal{D}_0^{(n)}$, and therefore $\mathcal{D}_0^{(n)} \subset D((T^*T)^2)$. This is again a contradiction. Let $n = 2$. Since $D((T^*T)^2) \not\subseteq D(T^{*2}T) \subset \mathcal{D}_0^{(2)}$, it follows that

$$(T^* Tg | \bar{\alpha}_2 T^* T f) = (g | (\alpha_0 I + \alpha_1 T^{[1]} + \alpha_2 T^{[2]})^* T^* T f - \bar{\alpha}_0 T^* T f - \bar{\alpha}_1 T^{*2} T f)$$

for every $g \in \mathcal{D}_0^{(2)}$ and $f \in D(T^{*2}T)$, which implies $D(T^{*2}T) = D((T^*T)^2)$, another contradiction. Thus $P_1(T^{[1]}) \square P_2(T^{[1]})$ cannot exist for all polynomials P_1 and P_2 of degree 2. Similarly, $P_1(T^{[1]}) \square P_2(T^{[1]})$ cannot exist either for each polynomial P_1 of degree 1 and each polynomial P_2 of degree 2. Thus finally $\mathfrak{M}_w(T^{[1]}) = \mathfrak{M}_s(T^{[1]}) = \mathfrak{P}_2(T^{[1]})$.

(3) It is easily shown that $\mathcal{D}^{(n)} \subset D(H^n)$ for each $n \in \mathbb{N}$, and, in particular, $\mathcal{D}^{(\infty)} \subset \mathcal{D}^\infty(H)$. Hence, $H^n \upharpoonright \mathcal{D}^{(\infty)}$ is essentially self-adjoint for each $n \in \mathbb{N}$, and thus statement (3) follows from Proposition 4.5.

4. B. Bounded Commutants of $\mathfrak{M}_s(T^{(1)})$ and $\mathfrak{M}_w(T^{(1)})$

To conclude this section, we will now study the bounded commutants of the strong and weak-partial *-algebra generated by T , $\mathfrak{M}_s(T^{(1)})$ and $\mathfrak{M}_w(T^{(1)})$.

Proposition 4.7. *Let T and \mathcal{D} be as in Theorem 4.4. Then we have:*

- (1) $\mathfrak{M}_s(T^{(1)})'_w = \{T \upharpoonright \mathcal{D}\}'_w$, but in general $\mathfrak{M}_w(T^{(1)})'_w \subsetneq \mathfrak{M}_s(T^{(1)})'_w$.
- (2) $T \upharpoonright \mathcal{D}$ is essentially self-adjoint if and only if $\mathfrak{M}_s(T^{(1)})'_w$ is a von Neumann algebra.

Proof. (1) By Theorem 4.4 we know that, for some $n \in \mathbb{N} \cup \{\infty\}$, $\mathfrak{M}_s(T^{(1)}) \subseteq \mathfrak{P}_n(T^{(1)})$; then

$$\mathfrak{P}_n(T^{(1)})'_w \subseteq \mathfrak{M}_s(T^{(1)})'_w \subseteq \{T^{(1)}\}'_w.$$

The statement follows, since $\mathfrak{P}_n(T^{(1)})'_w = \{T^{(1)}\}'_w$, as it is easily shown. On the other hand, Example 5 below shows that $\mathfrak{M}_w(T^{(1)})'_w \neq \mathfrak{M}_s(T^{(1)})'_w$ in general.

- (2) This can be proved in the same way as in Ref. 3, Lemma 3.2. □

Corollary 4.8. *Let $n \in \mathbb{N} \cup \{\infty\}$ be the largest number such that $\mathcal{D} \subset D(T^n)$. Suppose that $\mathfrak{M}_w(T^{(1)}) = \mathfrak{P}_n(T^{(1)})$. Then*

$$\mathfrak{M}_w(T^{(1)})'_w = \mathfrak{M}_w(T^{(1)})'_{qw} = \mathfrak{M}_s(T^{(1)})'_w = \mathfrak{M}_s(T^{(1)})'_{qw} = \{T^{(1)}\}'_w.$$

Moreover the following statements are equivalent:

- (1) $T^{(1)}$ is essentially self-adjoint.
- (2) $\mathfrak{M}_w(T^{(1)})'_w$ is a von Neumann algebra.
- (3) $i^{**}(\mathfrak{M}_w(T^{(1)}))$ is a self-adjoint partial Op^* -algebra.

When this is the case, $i^{**}(\mathfrak{M}_w(T^{(1)}))$ coincides with the standard partial Op^* -algebra $\mathfrak{P}_n(T|D(T^n))$.

Proof. Let $C \in \{T^{(1)}\}'_w$; then for all l, m with $l + m \leq n$, we get, for $f, g \in \mathcal{D}$:

$$(CT^l f | T^m g) = (T^* C T^{l-1} f | T^m g) = (C T^{l-1} f | T^{m+1} g) = (C f | T^{l+m} g).$$

Therefore $C \in \mathfrak{M}_w(T^{(1)})'_{qw}$ and, finally

$$\mathfrak{M}_w(T^{(1)})'_w = \mathfrak{M}_w(T^{(1)})'_{qw} = \mathfrak{M}_s(T^{(1)})'_w = \mathfrak{M}_s(T^{(1)})'_{qw} = \{T^{(1)}\}'_w.$$

- ⊙ (1) \Rightarrow (2): This is clear.
- ⊙ (3) \Rightarrow (2): Since $i^{**}(\mathfrak{M}_w(T^{(1)}))$ is self-adjoint, $i^{**}(\mathfrak{M}_w(T^{(1)}))'_w$ is a von Neumann algebra. Moreover, as one readily checks

$$i^{**}(\mathfrak{M}_w(T^{(1)}))'_w = \mathfrak{M}_w(T^{(1)})'_w.$$

Therefore, $\mathfrak{M}_w(T^{(1)})'_w$ is a von Neumann algebra.

- ⊙ (3) \Rightarrow (1): As we saw in the proof of Theorem 4.6,

$$\mathcal{D}^*(\mathfrak{M}_w(T^{11})) = \bigcap_{k=1}^n D(T^{k1*}) = D(T^n).$$

Therefore $i^{**}(\mathfrak{M}_w(T \upharpoonright \mathcal{D})) = i^*(\mathfrak{M}_w(T^{11})) = \mathfrak{B}_n(T \upharpoonright D(T^n))$ is a self-adjoint partial Op*-algebra on $D(T^n)$. \square

The results just obtained about weak commutants may now in turn be used to get more information on the structure of the algebras themselves. In particular, we obtain criteria for standardness. Notice that all the conditions of the following proposition are satisfied in the case of the polynomial algebra studied in Ref. 29.

Proposition 4.9. *Let T and \mathcal{D} be as in Theorem 4.4, and $n \in \mathbb{N} \cup \{\infty\}$ be the largest number such that $\mathcal{D} \subset D(T^n)$. Then the following statements are equivalent:*

- (1) *The full closure $\widehat{\mathfrak{M}}_w(T^{11})$ of $\mathfrak{M}_w(T^{11})$ is standard.*
- (1a) *The full closure $\widehat{\mathfrak{M}}_s(T^{11})$ of $\mathfrak{M}_s(T^{11})$ is standard.*
- (2) *$\mathfrak{M}_w(T^{11}) = \mathfrak{B}_n(T^{11})$ and it is essentially self-adjoint.*
- (2a) *$\mathfrak{M}_s(T^{11}) = \mathfrak{B}_m(T^{11})$ for some $m \in \mathbb{N} \cup \{\infty\}$, and it is essentially self-adjoint.*
- (3) *T^{11} is essentially self-adjoint and $\widehat{\mathcal{D}}(\mathfrak{M}_w(T^{11})) = D(T^m)$ for some $m \in \mathbb{N} \cup \{\infty\}$.*
- (3a) *$T \upharpoonright \mathcal{D}$ is essentially self-adjoint and $\widehat{\mathcal{D}}(\mathfrak{M}_s(T^{11})) = D(T^m)$ for some $m \in \mathbb{N} \cup \{\infty\}$.*
- (4) *$T^{11}, T^{12}, \dots, T^{1n}$ are essentially self-adjoint.*

When this is the case,

$$\mathfrak{M}_w(T^{11}) = \mathfrak{M}_s(T^{11}) = \mathfrak{B}_n(T^{11}).$$

Furthermore, if $\mathfrak{M}_s(T^{11})$ is fully closed, then the above statements are equivalent to

- (5) $\{T^{11}\}'_w \mathcal{D} = \mathcal{D}$;
- (5a) $\mathfrak{M}_s(T^{11})'_w \mathcal{D} = \mathcal{D}$.

Proof. \bullet (1) \Rightarrow (4) and (1a) \Rightarrow (4) are straightforward.

\bullet (4) \Rightarrow (2) and (4) \Rightarrow (2a) follow from Proposition 4.5 (2).

\bullet (2) \Rightarrow (3) and (2a) \Rightarrow (3a) are easy consequences of Corollary 4.8.

\bullet (3) \Rightarrow (1): Since $\mathcal{D} \subset \widehat{\mathcal{D}}(\mathfrak{M}_w(T^{11})) = D(T^m)$, we have $m \leq n$. On the other hand, since $\mathfrak{B}_n(T^{11}) \subset \mathfrak{M}_w(T^{11})$, it follows that $D(T^m) = \widehat{\mathcal{D}}(\mathfrak{M}_w(T^{11})) \subset \widehat{\mathcal{D}}(\mathfrak{B}_n(T^{11})) \subset D(T^n)$, hence $n \leq m$. Therefore $m = n$ and $\mathfrak{M}_w(T^{11}) = \mathfrak{B}_n(T^{11})$, and it is a standard partial Op*-algebra on $D(T^n)$.

\bullet (3a) \Rightarrow (1a): Since $\mathcal{D} \subset \widehat{\mathcal{D}}(\mathfrak{M}_s(T^{11})) = D(T^m)$, we have $m \leq n$. Since $\widehat{\mathfrak{M}}_s(T^{11})$ is a strong partial Op*-algebra on $D(T^m)$ and T is self-adjoint, it follows that $T^k \upharpoonright D(T^m)$ is essentially self-adjoint for each $1 \leq k \leq m$ and they all belong to $\widehat{\mathfrak{M}}_s(T^{11})$, which implies that $m = n$; thus $\widehat{\mathfrak{M}}_s(T^{11})$ is a standard partial Op*-algebra on $D(T^n)$.

⊙ Let us now suppose that $\mathfrak{M}_s(T^{[1]})$ is fully closed; then (5) \Leftrightarrow (5a) follows from Proposition 4.7.

⊙ (1a) \Rightarrow (5a) is clear.

⊙ (5) \Rightarrow (4): If $\{T^{[1]}\}'_w \mathcal{D} = \mathcal{D}$, then $\{T^{[1]}\}'_w$ is a von Neumann algebra and $T^{[1]}$ is essentially self-adjoint. Let again $T = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ and $E_m = \int_{-m}^m dE(\lambda)$, $m \in \mathbb{N}$. Since $E_m \in \{T^{[1]}\}'_w$ we get $E_m \mathcal{D} \subset \mathcal{D}$, $\forall m \in \mathbb{N}$. Take now any k with $1 \leq k \leq n$. Since $\bigcup_{m=1}^{\infty} E_m \mathcal{D} \subset \bigcup_{m=1}^{\infty} E_m D(T^k)$ is dense in the Hilbert space $D(T^k)$ and $\bigcup_{m=1}^{\infty} E_m \mathcal{D} \subset \mathcal{D}$, it follows that $T^{[k]}$ is essentially self-adjoint. \square

Example 5. *By applying the previous results to the situation described in Example 4, we get the following statements.*

(1) $\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(1)})'_w = \mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(1)})_{qw} = \mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(1)})'_w = \mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(1)})_{qw}$.

(2) $T \upharpoonright \mathcal{D}_0^{(n)}$ is not essentially self-adjoint, but $\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})$ is a self-adjoint partial Op*-algebra on $\mathcal{D}_0^{(n)}$ and thus

$$\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})_{qw} = \mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})'_w \subseteq \{T \upharpoonright \mathcal{D}_0^{(n)}\}'_w = \mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(n)})'_w = \mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(n)})_{qw}$$

(3) $\mathfrak{M}_w(T \upharpoonright \mathcal{D}^{(n)})$ is a standard partial Op*-algebra on $D(T^n)$ and

$$\mathfrak{M}_w(T \upharpoonright \mathcal{D}^{(n)})_{qw} = \mathfrak{M}_w(T \upharpoonright \mathcal{D}^{(n)})'_w = \mathfrak{M}_s(T \upharpoonright \mathcal{D}^{(n)})'_w = \mathfrak{M}_s(T \upharpoonright \mathcal{D}^{(n)})_{qw} = \{T \upharpoonright \mathcal{D}^{(n)}\}'_w$$

Proof. (1) follows from Corollary 4.8.

(2) Since $\mathcal{D}_0^{(n)}$ is a core for T^*T and

$$\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)}) = \{\alpha_0 I + \alpha_1 T \upharpoonright \mathcal{D}_0^{(n)} + \alpha_2 T^*T \upharpoonright \mathcal{D}_0^{(n)}; \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C}\},$$

it follows that $\hat{\mathcal{D}}(\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})) = \mathcal{D}^*(\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})) = D(T^*T)$, and so $\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})$ is essentially self-adjoint. Then Lemma 3.7 implies that $\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})'_w = \mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})_{qw}$ and it is a von Neumann algebra, but since $T \upharpoonright \mathcal{D}_0^{(n)}$ is not essentially self-adjoint, $\{T \upharpoonright \mathcal{D}_0^{(n)}\}'_w$ is not a von Neumann algebra, so that

$$\mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})_{qw} = \mathfrak{M}_w(T \upharpoonright \mathcal{D}_0^{(n)})'_w \subseteq \{T \upharpoonright \mathcal{D}_0^{(n)}\}'_w = \mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(n)})_{qw} = \mathfrak{M}_s(T \upharpoonright \mathcal{D}_0^{(n)})'_w$$

(3) This follows from Corollary 4.8. \square

The results listed in Example 5 show that the situation for partial Op*-algebras is quite different from that of Op*-algebras:

(1) *The essential self-adjointness of $\mathfrak{M}_w(T \upharpoonright \mathcal{D})$ does not imply that $T \upharpoonright \mathcal{D}$ is essentially self-adjoint and vice versa.*

(2) *Even if $\mathfrak{M}_w(T \upharpoonright \mathcal{D})_w$ is a von Neumann algebra, this is in general not true for $\{T \upharpoonright \mathcal{D}\}'_w$.*

(3) *The equality $\mathfrak{M}_w(T \upharpoonright \mathcal{D})_{qw} = \mathfrak{M}_w(T \upharpoonright \mathcal{D})'_w = \{T \upharpoonright \mathcal{D}\}'_w$ does not imply that $\{T \upharpoonright \mathcal{D}\}'_w$ is a von Neumann algebra.*

As remarked above, these pathologies largely disappear when T is self-adjoint, since then $T \upharpoonright \mathcal{D} \equiv T^{[1]}$ is essentially self-adjoint by definition, and

$\{T \upharpoonright \mathcal{D}\}'_w$ is a von Neumann algebra.

Appendix

The bounded commutants \mathfrak{N}'_w and \mathfrak{N}'_{qw} introduced in Section 3 are the bounded parts of the following unbounded commutants, respectively:

$$\mathfrak{N}'_\sigma = \{Y \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}); (X\xi | Y\eta) = (Y^\dagger\xi | X^\dagger\eta) \text{ for each } \xi, \eta \in \mathcal{D} \text{ and } X \in \mathfrak{N}\} \tag{A.1}$$

$$\begin{aligned} \mathfrak{N}'_{q\sigma} &= \{Y \in \mathfrak{N}'_\square; ((Y \square X_1^\dagger)\xi | X_2\eta) = (Y\xi | (X_1 \square X_2)\eta) \text{ and } ((Y^\dagger \square X_1^\dagger)\xi | X_2\eta) \\ &= (Y^\dagger\xi | (X_1 \square X_2)\eta) \text{ for all } X_1, X_2 \in \mathfrak{N} \text{ s.t. } X_1 \in L^w(X_2) \text{ and all } \xi, \eta \in \mathcal{D}\}. \end{aligned} \tag{A.2}$$

In (A.2), \mathfrak{N}'_\square is the weak natural commutant [9] defined as follows:

$$\mathfrak{N}'_\square = \{Y \in \mathfrak{N}'_\sigma; Y \in L^w(X) \cap R^w(X), Y \square X = X \square Y, \text{ for all } X \in \mathfrak{N}\}. \tag{A.3}$$

Comparing those commutants with the other ones, also introduced in Ref. 9, one gets easily the following relations:

$$\mathfrak{N}' \subseteq \mathfrak{N}'_R \subseteq \mathfrak{N}'_{q\sigma} \subseteq \mathfrak{N}'_\square \subseteq \mathfrak{N}'_\sigma. \tag{A.4}$$

We remark that there is no natural equivalent \mathfrak{N}'_σ to $\mathfrak{N}'_{q\sigma}$ in terms of the strong partial multiplication. Putting $X_1 \circ X_2$ in the r.h.s of the defining equality in (A.2), one expects $\mathfrak{N}'_L \subseteq \mathfrak{N}'_{q\sigma} \subseteq \mathfrak{N}'_\square$. However:

(i) If one uses $(Y^{(\dagger)} \circ X_1^\dagger)$ in the l.h.s., then the two relations in (A.2) are valid for all $Y^{(\dagger)} \in \mathfrak{N}'_L$, the left mixed commutant introduced in Ref. 9.

(ii) If one uses instead $(Y^{(\dagger)} \square X_1^\dagger)$, then the relation holds true for all $Y^{(\dagger)} \in \mathfrak{N}'_\square$. Intersecting now all sets in (A.4) with $\mathcal{B}(\mathcal{H})$, one gets:

$$\mathfrak{N}'_{.b} = \mathfrak{N}'_{Rb} \subseteq \mathfrak{N}'_{qw} \subseteq \mathfrak{N}'_{\square b} = \mathfrak{N}'_w \tag{A.5}$$

(the last equality was not noticed in Ref. 8). As for the other putative bounded commutant discussed above, one gets immediately:

$$\mathfrak{N}'_{Lb} = \mathfrak{N}'_{.w} = \mathfrak{N}'_{\square b} = \mathfrak{N}'_w \tag{A.6}$$

so that indeed nothing new is obtained.

Coming back to the relations (A.4) and (A.5), there are many cases where some of the inclusions are in fact equalities. The following criterion is useful, and easy to check in terms of the definitions.

Lemma A.1. *Let \mathfrak{R} be an \dagger -invariant subset of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. Then:*

(1) *If $\mathfrak{R} \subseteq \mathcal{D}(\mathfrak{R})$, then*

$$\mathfrak{N}'_R \cap \mathfrak{R} = \mathfrak{N}'_{q\sigma} \cap \mathfrak{R} = \mathfrak{N}'_\square \cap \mathfrak{R} = \mathfrak{N}'_\sigma \cap \mathfrak{R}.$$

(2) *If $\mathfrak{R} \subseteq \mathcal{D}(\mathfrak{R})$, then:*

$$\mathfrak{N}'_L \cap \mathfrak{R} = \mathfrak{N}'_{\square} \cap \mathfrak{R} = \mathfrak{N}'_{\sigma} \cap \mathfrak{R}.$$

(3) If either $\mathfrak{R} \mathcal{D} \subseteq \mathcal{D}^*(\mathfrak{N})$ or $\mathfrak{N} \mathcal{D} \subseteq \mathcal{D}^*(\mathfrak{R})$, then:

$$\mathfrak{N}'_{\square} \cap \mathfrak{R} = \mathfrak{N}'_{\sigma} \cap \mathfrak{R}.$$

Proof. Let $C \in \mathfrak{N}'_{\sigma} \cap \mathfrak{R}$. Then statement (i) results from the following chain of equalities:

$$\begin{aligned} (\overline{X^\dagger} C \xi | \eta) &= (C \xi | X \eta) = (X^\dagger \xi | C^\dagger \eta) = (\xi | \overline{X} C^\dagger \eta) \\ &= (\xi | C^* X \eta) = (C^{\dagger*} X^\dagger \xi | \eta), \end{aligned}$$

which means that $C \in \mathfrak{N}'_R$. The other statements are proved in the same way. □

Applying those criteria to a number of particular cases, we recover several implications stated in Ref.9(Corollary 2.2 and Appendix):

- With $\mathfrak{R} = \mathcal{B}(\mathcal{H})$, (2) gives:

$$\mathfrak{N}'_{Lb} = \mathfrak{N}'_{\square b} = \mathfrak{N}'_w.$$

- If $\mathfrak{N} \subset \mathcal{L}^\dagger(\mathcal{D})$, (2) gives, with $\mathfrak{R} = \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$:

$$\mathfrak{N}'_L = \mathfrak{N}'_{\square} = \mathfrak{N}'_{\sigma}.$$

- If $\mathfrak{N} \subset \mathcal{B}(\mathcal{H})$, (i) gives, with $\mathfrak{R} = \mathfrak{N}'_{\sigma}$:

$$\mathfrak{N}'_R = \mathfrak{N}'_{q\sigma} = \mathfrak{N}'_{\square} = \mathfrak{N}'_{\sigma},$$

and therefore

$$\overline{\mathfrak{N}'_{Lb}} = \overline{\mathfrak{N}'_{q\sigma}} = \overline{\mathfrak{N}'_w} = \mathfrak{N}' \text{ (the usual bounded commutant).}$$

- With $\mathfrak{R} = \mathfrak{N}'_{\sigma}$, we get by (1):

$$\mathfrak{N}'_{\sigma} \mathcal{D} \subset \hat{\mathcal{D}}(\mathfrak{N}) \Rightarrow \mathfrak{N}'_R = \mathfrak{N}'_{q\sigma} = \mathfrak{N}'_{\square} = \mathfrak{N}'_{\sigma}.$$

- With $\mathfrak{R} = \mathfrak{N}'_w$, we get in the same way:

$$\mathfrak{N}'_w \mathcal{D} \subset \hat{\mathcal{D}}(\mathfrak{N}) \Rightarrow \mathfrak{N}'_b = \mathfrak{N}'_{Rb} = \mathfrak{N}'_{q\sigma} = \mathfrak{N}'_w.$$

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