Publ. RIMS, Kyoto Univ.
26 (1990), 419–449

Sharp Estimates of Lower Bounds of Polynomial Decay Order of Eigenfunctions

Dedicated to Professor Teruo Ikebe on his sixtieth birthday

By

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§ 0. Introduction

In this paper we shall study the lower bounds of polynomial decay order as $|x| \to \infty$ of the not identically vanishing solution $u(x) \in H^2_{loc}(\Omega)$ of the second order elliptic equation in $\Omega = \{x \in \mathbf{R}^n \,|\, |x| > R_0\}$

$$-\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial x_i} + \sqrt{-1}\, b_i(x)\right) a_{ij}(x) \left(\frac{\partial}{\partial x_j} + \sqrt{-1}\, b_j(x)\right) u(x) + (q_1(x) + q_2(x)) u(x) = 0,$$

where the matrix $(a_{ij}(x))$ is uniformly positive definite, $b_i(x)$ $(1 \le i \le n)$ and $q_1(x)$ are real-valued functions, and $q_2(x)$ is a complex-valued function. Our aim is to combine the results given in Uchiyama [3], Yamada [4] and Agmon [1] in one theorem. We shall state the main parts of the assumptions for the case $a_{ij}(x) = \delta_{ij}$ (Kronecker's delta) as follows: there exist some constants β, γ_1, γ_2 and real-valued bounded functions $\sigma(r)$, $\eta(r)$ such that

$$\beta > 0, \quad \gamma_1 < 2, \quad 2 - 2\beta < \gamma_2 < 2,$$

$$\sigma(r) > 0, \quad \lim_{r \to \infty} \eta(r) = 0,$$

$$\limsup_{r \to \infty} r^{2-2\beta} \sigma(r)^{-1} [r \partial_r q_1(x) + (\gamma_1 + \eta(r)) q_1(x) + \sigma(r)^{-1} |r q_2(x)|^2 + (2 - \gamma_1)^{-1} |B(x) x|^2] < 0,$$

$$\limsup_{r \to \infty} r^{2-2\beta} \sigma(r)^{-1} [r \partial_r q_1(x) + (\gamma_2 + \eta(r)) q_1(x) + (2 - \gamma_2)^{-1} |B(x) x|^2] < 0,$$

Communicated by S. Matsuura, October 22, 1987.
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$$q_2(x) = O(r^{\beta-2}\sigma(r)),\ |B(x)x| = O(r^{\beta-1}\sqrt{\sigma(r)}) \qquad \text{as } r \to \infty,$$

where $B(x) = (\partial_i b_j(x) - \partial_j b_i(x))$ is an $n \times n$ matrix. Moreover we assume that there exist some constants $0 < a < 1$, $C > 0$, $-\infty < \delta_1 < \infty$ and $\delta_2 \leq \beta - 2$ such that

$$\int_\Omega (q_1)_-(x)|w(x)|^2 dx \leq a \int_\Omega |\nabla w(x)|^2 dx + C \int_\Omega r^{\delta_1}|w(x)|^2 dx$$
$$\text{for any } w(x) \in C_0^\infty(\Omega),$$
$$(\mathrm{Re}[q_2])_-(x) \leq Cr^{\delta_2} \qquad \text{for } |x| > R_0,$$

where $(f)_-(x) = \max\{0, -f(x)\}$ for a real-valued function $f(x)$. More detailed conditions are stated in §1. Then by Theorem 1.2 given in §1 we have

$$\liminf_{R\to\infty} R^{(\gamma_1/2)+\max\{0,\delta_1,\delta_2\}} \exp\left\{\frac{1}{2}\int_{R_0}^R \frac{\eta(r)+\sigma(r)}{r} dr\right\} \int_{R<|x|<R+1} |u(x)|^2 dx > 0.$$

Moreover if

$$\int_{R_0}^\infty R^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}} \exp\left\{-\frac{1}{2}\int_{R_0}^R \frac{\eta(r)+\sigma(r)}{r} dr\right\} dR = +\infty,$$

then we have $u(x) \notin L^2(\Omega)$.

Roughly speaking the case $\sigma(r) \equiv \varepsilon_0 > 0$ (sufficiently small), $\eta(r) \equiv 0$ and $\gamma_1 = \gamma_2$ corresponds to the result in Uchiyama [3], the case $\sigma(r) = \eta(r) = (\log r)^{-1}$ corresponds to Yamada [4] (but no detailed treatment was given), the case $\sigma(r) = \eta(r) = r^{-\varepsilon_0}$ ($\varepsilon_0 > 0$ sufficiently small) corresponds to Agmon [1]. Yamada [4] and Agmon [1] assumed $q_1(x) < 0$ for $r > R_0$, but we do not assume this condition in this paper. So our results also can be applied to the atomic-type many body potential (e.g. see Remark 1.4).

We note that the smaller $\gamma_1 < 2$ we choose, the better estimate as lower bound we have. In Example 1.7 we choose $\gamma_1 = 2 - 2\beta$ and so we cannot, in general, let $\gamma_2 = \gamma_1$. But in case $q_1(x) < 0$ for $r > R_0$ and $|B(x)x| = o(r^{\beta-1}\sqrt{\sigma(r)})$ as $r \to \infty$ we have only to choose γ_2 to satisfy $2 - 2\beta < \gamma_2 < 2$ and $\gamma_1 \leq \gamma_2$, which is the reason that Yamada [4] and Agmon [1] did not assume the condition depending on γ_2.

Example 1.7 and Remark 1.8 show the following: let

$$\begin{cases} -\Delta u(x) - (r^\theta + \lambda)u(x) + q_2(x)u(x) = 0 \qquad \text{in } \Omega\ (\theta > 0,\ -\infty < \lambda < \infty), \\ u(x) \in H^2_{loc}(\Omega), \\ \mathrm{supp}[u] \text{ is not a compact set in } \bar{\Omega}. \end{cases}$$

If

$$q_2(x) = o(r^{(\theta/2)-1}) \qquad \text{as } r \longrightarrow \infty,$$

then we have

$$\lim_{R\to\infty} R^{(\theta/2)+\varepsilon} \int_{R<|x|<R+1} |u(x)|^2 dx = \infty \qquad \text{for any } \varepsilon > 0,$$

$$u(x) \notin L^2(\Omega) \qquad \text{for } 0 < \theta < 2.$$

If

$$q_2(x) = o(r^{(\theta/2)-1}(\log r)^{-1}) \qquad \text{as } r \longrightarrow \infty,$$

then we have

$$\lim_{R\to\infty} R^{(\theta/2)}(\log R)^{\varepsilon} \int_{R<|x|<R+1} |u(x)|^2 dx = +\infty \qquad \text{for any } \varepsilon > 0,$$

$$u(x) \notin L^2(\Omega) \qquad \text{for } 0 < \theta \leq 2.$$

If

$$q_2(x) = O(r^{(\theta/2)-1-\varepsilon}) \text{ as } r \longrightarrow \infty \qquad \text{for some } \varepsilon > 0,$$

then we have

$$\liminf_{R\to\infty} R^{(\theta/2)} \int_{R<|x|<R+1} |u(x)|^2 dx > 0,$$

$$u(x) \notin L^2(\Omega) \qquad \text{for } 0 < \theta \leq 2,$$

which is the best possible result. These results show that the more gently $q_2(x)$ behaves at infinity, the better estimates as lower bounds we have.

Eastham-Kalf [2] has given fruitful informations and rich references on the problem treated in this paper.

In § 1, the assumptions and main results are explained. We give the proof of Theorem 1 in § 2 and the proof of Theorems 2 and 3 in § 3. The method of proof is similar to the one used in Uchiyama [3] and Eastham-Kalf [2, Theorem 6.3.3].

§ 1. Assumptions and Main Results

We list up the notations used here, which are the same as given in Uchiyama [3].

Notations:

$\langle \xi, \eta \rangle = \xi_1\eta_1 + \cdots + \xi_n\eta_n$ for $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n) \in \mathbf{C}^n$;

$|\xi| = (\langle \xi, \bar{\xi} \rangle)^{1/2}$ for $\xi \in \mathbf{C}^n$;
$\hat{x} = x/|x|$ and $r = |x|$ for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$;
$\partial_j = \partial/\partial x_j$ and $\partial_r = \partial/\partial r$;
$D_j = \partial_j + \sqrt{-1}\, b_j(x)$ and $D = (D_1, \dots, D_n)$;
$f'(r) = (d/dr)f(r)$ and $f''(r) = (d^2/dr^2)f(r)$;
$\nabla f = (\partial_1 f, \dots, \partial_n f)$ for a scalar-valued function $f(x)$;
$\mathrm{div}\, g = \partial_1 g_1 + \cdots + \partial_n g_n$ for a vector-valued function $g(x) = (g_1(x), \dots, g_n(x))$;
$A = A(x) = (a_{ij}(x))$ is an $n \times n$ matrix;
$B = B(x) = \mathrm{curl}\, b(x) = (\partial_i b_j(x) - \partial_j b_i(x))$ is an $n \times n$ matrix;
$(f)_-(x) = \max\{0, -f(x)\} \geq 0$ for a real-valued function $f(x)$;
$\mathrm{supp}[f]$ denotes the closure of $\{x | f(x) \neq 0\}$;
$C^j(\Omega)$ denotes the class of j-times continuously differentiable functions;
$C_0^j(\Omega) = \{f(x) \in C^j(\Omega) | \mathrm{supp}[f]$ is a compact set in $\Omega\}$;
$C_0^\infty(\Omega) = \bigcap_{j=1}^{\infty} C_0^j(\Omega)$;
$L^2(\Omega) = \left\{ f(x) \middle| \int_\Omega |f(x)|^2 dx < \infty \right\}$;
$L^2_{loc}(\Omega) = \left\{ f(x) \middle| \text{ for any compact set } K \subset \Omega, \int_K |f(x)|^2 dx < \infty \right\}$;
$H^m(\Omega)$ denotes the class of L^2-functions in Ω such that all distribution derivatives up to m belong to $L^2(\Omega)$;
$H^m_{loc}(\Omega)$ denotes the class of L^2_{loc}-functions in Ω such that all distribution derivatives up to m belong to $L^2_{loc}(\Omega)$;
$\left(\int_{|x|=t} - \int_{|x|=s} \right) f(x) dS = \int_{|x|=t} f(x) dS - \int_{|x|=s} f(x) dS.$

Next we shall state the conditions required in the theorems.

Assumptions:

(A1) each $a_{ij}(x) \in \mathbf{C}^2(\Omega)$ is a real-valued function;
(A2) $a_{ij}(x) = a_{ji}(x)$;
(A3) there exists some constant $C_1 \geq 1$ such that for any $x \in \Omega$ and any $\xi \in \mathbf{C}^n$ we have

$$C_1^{-1} |\xi|^2 \leq \langle A(x)\xi, \bar{\xi} \rangle \leq C_1 |\xi|^2;$$

(B1) each $b_i(x)$ is a real-valued function;
(B2) for any $w(x) \in H^1_{loc}(\Omega)$ we have $b_i(x) w(x)$, $(\partial_i b_j(x)) w(x) \in L^2_{loc}(\Omega)$;
(C1) $q_1(x)$ is a real-valued function;
(C2) for any $w(x) \in H^1_{loc}(\Omega)$ we have $\sqrt{|q_1(x)|}\, w(x) \in L^2_{loc}(\Omega)$;
(C3) for any $w(x) \in H^1_{loc}(\Omega)$ we have $\sqrt{|\nabla q_1(x)|}\, w(x) \in L^2_{loc}(\Omega)$;
(D1) $q_2(x)$ is a complex-valued function;

(D2) for any $w(x) \in H^1_{loc}(\Omega)$ we have $\sqrt{|q_2(x)|}\, w(x) \in L^2_{loc}(\Omega)$;

(E) there exists some constant $R_0 > 1$ such that $\Omega \supset \{x \in R^n |\ |x| > R_0\}$

(F) there exist some constants α, β, γ_1, γ_2, a_1, a_2, a_3, a_4 and some real-valued functions $\eta(r)$, $\sigma(r) \in C^1(R_0, \infty)$ such that the following (F1) ~ (F9) hold.

(F1) $0 < \alpha < \beta$, $a_1 > 1$, $a_2 > 1$, $a_3 > 0$, $a_4 > 1$;

(F2) $\sigma(r) = O(1)$ as $r \to \infty$ and $\sigma(r) > 0$ for $r > R_0$,
$\eta(r) = O(1)$ as $r \to \infty$,
$\gamma_1 + \limsup_{r \to \infty} \eta(r) < 2$,
$(2 - 2\beta <) 2 - 2\alpha < \gamma_2 + \liminf_{r \to \infty} \eta(r) \le \gamma_2 + \limsup_{r \to \infty} \eta(r) < 2$;

(F3) $\lim_{r \to \infty} r^{\beta - \alpha} \sigma(r) = \infty$;

(F4) $\lim_{r \to \infty} r^{1-\beta} \sigma(r)^{-1} \sigma'(r) = 0$ and $\lim_{r \to \infty} r^{1-\beta} \sigma(r)^{-1} \eta'(r) = 0$;

(F5) $\limsup_{r \to \infty} r^{2-2\beta} \sigma(r)^{-1} [r \langle A(x)\hat{x}, \hat{x} \rangle^{-1} \langle A(x) \nabla q_1(x), \hat{x} \rangle$
$+ (\gamma_1 + \eta(r)) q_1(x) + a_1 \sigma(r)^{-1} \langle A(x)\hat{x}, \hat{x} \rangle^{-1} |r q_2(x)|^2$
$+ a_2 (2 - \gamma_1 - \eta(r))^{-1} \langle A(x)\hat{x}, \hat{x} \rangle^{-2}$
$\cdot \langle A(x)B(x)A(x)x, B(x)A(x)x \rangle] < 0$;

(F6) $\limsup_{r \to \infty} r^{2-2\beta} \sigma(r)^{-1} [r \langle A(x)\hat{x}, \hat{x} \rangle^{-1} \langle A(x) \nabla q_1(x), \hat{x} \rangle$
$+ (\gamma_2 + \eta(r)) q_1(x) + a_3 \sigma(r)^{-1} \langle A(x)\hat{x}, \hat{x} \rangle^{-1} |r q_2(x)|^2$
$+ a_4 (2 - \gamma_2 - \eta(r))^{-1} \langle A(x)\hat{x}, \hat{x} \rangle^{-2}$
$\cdot \langle A(x)B(x)A(x)x, B(x)A(x)x \rangle] < 0$;

(F7) $\lim_{r \to \infty} \sigma(r)^{-1} (a_{ij}(x) - \delta_{ij}) = 0$, where δ_{ij} is the kronecker's delta;

(F8) $\lim_{r \to \infty} r \sigma(r)^{-1} \partial_k a_{ij}(x) = 0$;

(F9) $\lim_{r \to \infty} r^{2-\beta} \sigma(r)^{-1} \partial_k \partial_l a_{ij}(x) = 0$;

(G1) there exist some constants $0 < a_5 < 1$, $-\infty < \delta_1 < \infty$ and $C_2 > 0$ such that for any $w(x) \in C_0^\infty(\Omega)$ we have

$$\int_\Omega (q_1)_-(x) |w(x)|^2 dx \le a_5 \int_\Omega |\nabla w(x)|^2 dx + C_2 \int_\Omega r^{\delta_1} |w(x)|^2 dx;$$

(G2) there exist some constants $\delta_2 \le \beta - 2$ and $C_3 > 0$ such that for any $r > R_0$ we have
$(\mathrm{Re}[q_2])_-(x) \le C_3 \min\{r^{\delta_2}, r^{\beta-2}\sigma(r)\}$,
where $\mathrm{Re}[z]$ means the real part of $z \in \mathbf{C}$.

Now we have the

Theorem 1.1. *Let $u(x)$ satisfy*

$$(*): \begin{cases} -\langle D, AD \rangle u(x) + \{q_1(x) + q_2(x)\} u(x) = 0 \ \textit{in}\ \Omega, \\ u(x) \in H^2_{loc}(\Omega), \\ \mathrm{supp}[u] \ \textit{is not a compact set in}\ \bar{\Omega}\ (\textit{closure of}\ \Omega). \end{cases}$$

Let (*A*) $\sim$ (G) *hold. Then we have the following*:

(1) $$\liminf_{R\to\infty} R^{(\gamma_1/2)}\Phi(R)\int_{|x|=R}[|\langle ADu, \hat{x}\rangle|^2+\{r^{-2}+(q_1)_-\}|u|^2]dS>0,$$

where

$$\Phi(R)=\exp\left\{\frac{1}{2}\int_{R_0}^{R}\frac{\eta(r)+\sigma(r)}{r}dr\right\};$$

(2) *for any* $\varepsilon>0$

$$\liminf_{R\to\infty} R^{(\gamma_1/2)+\max\{0,\delta_1,\delta_2\}}\Phi(R)\int_{R<|x|<R+\varepsilon}|u|^2dx>0;$$

(3) *moreover if*

$$\int_{R_0}^{\infty} R^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}}\Phi(R)^{-1}dR=+\infty,$$

then $u(x)\notin L^2(\Omega)$.

Now we shall consider the more special case $a_{ij}(x)=\delta_{ij}$ under the weaker conditions.

Theorem 1.2. *Let* $u(x)$ *satisfy*

$$(**):\begin{cases}-\langle D, D\rangle u(x)+\{q_1(x)+q_2(x)\}u(x)=0 \text{ in } \Omega,\\ u(x)\in H^2_{loc}(\Omega),\\ \operatorname{supp}[u] \text{ is not a compact set in } \bar{\Omega}.\end{cases}$$

We assume (B) $\sim$ (G) *with* $a_{ij}(x)=\delta_{ij}$ *except for* (C3). *Instead of* (C3) *we assume*
(C3)' *for any* $w(x)\in H^1_{loc}(\Omega)$ *we have* $|\partial_r q_1(x)|^{1/2}w(x)\in L^2_{loc}(\Omega)$.
Then we have the same results as given in Theorem 1(1) $\sim$ (3), *where we replace* $a_{ij}(x)$ *with* δ_{ij}.

Lastly we shall consider the most special case $a_{ij}(x)=\delta_{ij}$ and $b_i(x)=0$ under the weakest conditions.

Theorem 1.3. *Let* $u(x)$ *satisfy*

$$(***):\begin{cases}-\Delta u(x)+\{q_1(x)+q_2(x)\}u(x)=0 \text{ in } \Omega,\\ u(x)\in H^2_{loc}(\Omega),\\ \operatorname{supp}[u] \text{ is not a compact set in } \bar{\Omega},\end{cases}$$

where Δ *is a Laplacian in* $\mathbf{R}^n$. *We assume* (C) $\sim$ (G) *with* $a_{ij}(x)=\delta_{ij}$ *and* $b_i(x)=0$ *except for* (C3) *and* (F2). *Instead of* (C3) *and* (F2) *we assume* (C3)' *and*

(F2)′ $\sigma(r) = O(1)$ *as* $r \to \infty$ *and* $\sigma(r) > 0$ *for* $r > R_0$,
$\eta(r) = O(1)$ *as* $r \to \infty$,
$\gamma_i + \eta(r) \leq 2$ *for* $i = 1, 2$ *and* $r > R_0$,
$(2 - 2\beta <) 2 - 2\alpha < \gamma_2 + \liminf_{r \to \infty} \eta(r)$.

Then we have the same results as given in Theorem 1(1) ~ (3), *where we replace* $a_{ij}(x)$ *with* δ_{ij} *and* $b_i(x)$ *with* 0.

Remark 1.4. We have the following:

(1) If $\eta(r) = 0$ and $\sigma(r) = \nu > 0$ where ν is a constant, then $\lim_{R \to \infty} R^{-\nu/2} \Phi(R) < \infty$. This case is the one considered in Uchiyama [3].
(2) If $\eta(r) = \sigma(r) = (\log r)^{-1}$, then $\lim_{R \to \infty} (\log R)^{-1} \Phi(R) < \infty$. This case almost meets with Yamada [4]. However [4] did not give its complete proof and assumed more strict conditions such as $q_1(x) < 0$ for $r > R_0$.
(3) If there exists $\lim_{R \to \infty} \Phi(R)$, the results given in Agmon [1] are almost reproduced. However [1] assumed more strict conditions such as $q_1(x) < 0$ for $r > R_0$.
(4) In our assumptions that $q_1(x) < 0$ for $r > R_0$ is not assumed. Then we can apply our theorem to the atomic type many body potential

$$q_1(x) = -\sum_{i=1}^{N} \frac{z_i}{|x^{(i)}|} + \sum_{1 \leq i < j \leq N} \frac{z_{ij}}{|x^{(i)} - x^{(j)}|} - \lambda,$$

where $x = (x^{(1)}, \ldots, x^{(N)}) \in \mathbf{R}^{3N}$, $x^{(i)} = (x_{3i-2}, x_{3i-1}, x_{3i}) \in \mathbf{R}^3$, $\lambda > 0$, z_i and z_{ij} are real constants. In this case we choose $\gamma_1 = \gamma_2 = 1$ and $\eta(r) = 0$.

Remark 1.5. If we add the following conditions

$$q_2(x) = O(r^{\beta - 2} \sigma(r)) \qquad \text{as } r \longrightarrow \infty,$$
$$|BAx| = O(r^{\beta - 1} \sqrt{\sigma(r)}) \qquad \text{as } r \longrightarrow \infty$$

in (F), then (F5) and (F6) can be replaced with weaker conditions

(F5)′
$$\begin{aligned} \limsup_{r \to \infty} r^{2-2\beta} \sigma(r)^{-1} [& r \langle A\hat{x}, \hat{x} \rangle^{-1} \langle A\nabla q_1, \hat{x} \rangle + (\gamma_1 + \eta(r)) q_1(x) \\ & + \sigma(r)^{-1} \langle A\hat{x}, \hat{x} \rangle^{-1} |r q_2(x)|^2 \\ & + (2 - \gamma_1 - \eta(r))^{-1} \langle A\hat{x}, \hat{x} \rangle^{-2} \langle ABAx, BAx \rangle] < 0; \end{aligned}$$

(F6)′
$$\begin{aligned} \limsup_{r \to \infty} r^{2-2\beta} \sigma(r)^{-1} [& r \langle A\hat{x}, \hat{x} \rangle^{-1} \langle A\nabla q_1, \hat{x} \rangle + (\gamma_2 + \eta(r)) q_1(x) \\ & + (2 - \gamma_2 - \eta(r))^{-1} \langle A\hat{x}, \hat{x} \rangle^{-2} \langle ABAx, BAx \rangle] < 0. \end{aligned}$$

In fact the quantity given in (F5) depends continuously on a_1 and a_2 under our additional conditions. So (F5)′ leads to (F5). The same happens in (F6).

Remark 1.6. If in (F5)

$$2-2\alpha<\gamma_1+\liminf_{r\to\infty}\eta(r)\leq\gamma_1+\limsup_{r\to\infty}\eta(r)<2$$

holds, then (F6) is automatically satisfied. In fact we have only to choose $\gamma_2=\gamma_1$, $a_3=a_1$ and $a_4=a_2$ in (F6).

Example 1.7. In (**) let $n=3$ and

$$q_1(x)=h(x)+V(x)-\lambda,$$
$$q_2(x)=o(r^{(\theta/2)-1})\qquad\text{as } r\longrightarrow\infty,$$
$$b_1(x)=-2^{-1}b_0x_2f(r),\ b_2(x)=2^{-1}b_0x_1f(r),\ b_3(x)=0,$$

where

$-\infty<\lambda$, $b_0<\infty$ are constants,

$h(x)$ is a negative continuous homogeneous function of degree $\theta>0$,

$V(x)$ is a real-valued function satisfying

$$V(x)=o(r^{\theta}),\ \partial_rV(x)=o(r^{\theta-1})\qquad\text{as } r\longrightarrow\infty,$$

$f(r)\in C^1(R_0,\infty)$ is a real-valued function satisfying

$$rf'(r)+2f(r)=o(r^{(\theta/2)-1})\qquad\text{as } r\longrightarrow\infty.$$

In this case (B) $\sim$ (E) are satisfied, where we replace (C3) with (C3)$'$. We choose in (F) and (G)

$$\beta=1+(\theta/2)>0,\ \gamma_1=-\theta(=2-2\beta),\ \eta(r)=\sigma(r),$$
$$\delta_1=\theta,\ \delta_2=(\theta/2)-1=\beta-2<\theta=\delta_1,$$

and let α, a_1, a_2, a_3, a_4 be arbitrary constants satisfying

$$0<\alpha<1+(\theta/2)=\beta,\ a_1>1,\ a_2>1,\ a_3>0,\ a_4>1.$$

Then we have

$$(\gamma_1/2)+\max\{0,\delta_1,\delta_2\}=\theta/2.$$

Noting

$$|Bx|^2=4^{-1}b_0^2(x_1^2+x_2^2)|rf'(r)+2f(r)|^2,$$

we have

$$r^{2-2\beta}\sigma(r)^{-1}[r\partial_rq_1(x)+(\gamma_1+\eta(r))q_1(x)+a_1\sigma(r)^{-1}|rq_2(x)|^2$$
$$+a_2(2-\gamma_1-\eta(r))^{-1}|Bx|^2]$$

$$\begin{aligned}&= r^{-\theta}h(x) + r^{-\theta}\sigma(r)^{-1}[r\partial_r V(x) + (\sigma(r) - \theta)V(x)]\\&\quad - \lambda r^{-\theta}\sigma(r)^{-1}(\sigma(r) - \theta)\\&\quad + a_1 r^{-\theta}\sigma(r)^{-2}|rq_2(x)|^2 + a_2(2 + \theta - \sigma(r))^{-1}4^{-1}b_0^2\\&\quad\times r^{-\theta}(x_1^2 + x_2^2)\sigma(r)^{-1}|rf'(r) + 2f(r)|^2.\end{aligned}$$

Now we shall consider the following three cases.

Case 1. Let

$$\begin{aligned}&V(x) = o(r^{\theta}),\ \partial_r V(x) = o(r^{\theta-1}) && \text{as } r \longrightarrow \infty,\\&q_2(x) = o(r^{(\theta/2)-1}) && \text{as } r \longrightarrow \infty,\\&rf'(r) + 2f(r) = o(r^{(\theta/2)-1}) && \text{as } r \longrightarrow \infty.\end{aligned}$$

In this case we choose

$$\sigma(r) \equiv \varepsilon,$$

where ε is a constant satisfying $0 < \varepsilon < 2 + \theta$. Noting

$$r^{-\theta}h(x) \leq \max\{h(x)|\ |x| = 1\} < 0 \text{ for any } r > R_0,$$

(F) and (G) are satisfied by any $\gamma_2 \in (\min\{2 - 2\alpha - \varepsilon, -\theta\}, 2 - \varepsilon)$. Then by Theorem 1.2 and $\lim_{r\to\infty} r^{-\varepsilon}\Phi(r) < \infty$, we have for $u(x)$ satisfying (**)

$$\begin{cases}\lim_{R\to\infty} R^{(\theta/2)+\varepsilon}\displaystyle\int_{R<|x|<R+1}|u(x)|^2dx = +\infty & \text{for any } \varepsilon > 0,\\ u(x) \notin L^2(\Omega) \quad \text{for } 0 < \theta < 2.\end{cases}$$

Case 2. Let

$$\begin{aligned}&V(x) = o(r^{\theta}(\log r)^{-1}),\ \partial_r V(x) = o(r^{\theta-1}(\log r)^{-1}) && \text{as } r \longrightarrow \infty,\\&q_2(x) = o(r^{(\theta/2)-1}(\log r)^{-1}) && \text{as } r \longrightarrow \infty,\\&rf'(r) + 2f(r) = o(r^{(\theta/2)-1}(\log r)^{-1/2}) && \text{as } r \longrightarrow \infty.\end{aligned}$$

In this case we choose for any $\varepsilon > 0$

$$\sigma(r) = \varepsilon(\log r)^{-1},$$

and then (F) and (G) are satisfied by any $\gamma_2 \in (2 - 2\alpha, 2)$. So we have by Theorem 1.2 and $\lim_{r\to\infty}(\log r)^{-\varepsilon}\Phi(r) < \infty$

$$\begin{cases}\lim_{R\to\infty} R^{(\theta/2)}(\log R)^{\varepsilon}\displaystyle\int_{R<|x|<R+1}|u(x)|^2dx = +\infty, & \text{for any } \varepsilon > 0,\\ u(x) \notin L^2(\Omega) \quad \text{for } 0 < \theta \leq 2.\end{cases}$$

Case 3. Let for some $\varepsilon > 0$

$$V(x) = O(r^{\theta-\varepsilon}),\ \partial_r V(x) = O(r^{\theta-1-\varepsilon}) \qquad \text{as } r \longrightarrow \infty,$$
$$q_2(x) = O(r^{(\theta/2)-1-\varepsilon}) \qquad \text{as } r \longrightarrow \infty,$$
$$rf'(r) + 2f(r) = O(r^{(\theta/2)-1-\varepsilon}) \qquad \text{as } r \longrightarrow \infty.$$

In this case we choose

$$\sigma(r) = r^{-\varepsilon'},$$

where ε' is a constant satisfying $0 < \varepsilon' < \min\{\varepsilon, \beta - \alpha\}$. Then (F) and (G) are satisfied by any $\gamma_2 \in (2 - 2\alpha, 2)$. So we have by Theorem 1.2 and $\lim_{r\to\infty} \Phi(r) < \infty$

$$\begin{cases} \liminf_{R\to\infty} R^{(\theta/2)} \int_{R<|x|<R+1} |u(x)|^2 dx > 0, \\ u(x) \notin L^2(\Omega) \qquad \text{for } 0 < \theta \leq 2. \end{cases}$$

Remark 1.8. The result given in Example 1.7 Case 3 is best possible. In fact we shall consider the following case in (**):

$$q_1(x) = -r^\theta \qquad (\theta > 0),$$
$$q_2(x) = 0,$$
$$(b_1(x), \dots, b_n(x)) = (0, \dots, 0).$$

Then

$$u_0(x) = r^{(2-n)/2} J_{|n-2|/(2+\theta)}\left(\frac{2}{2+\theta} r^{1+(\theta/2)}\right)$$

satisfies (**) with $\Omega = \mathbf{R}^n$, where $J_\nu(r)$ denotes the Bessel function of the first kind of order ν. This solution $u_0(x)$ satisfies

$$\begin{cases} \limsup_{R\to\infty} R^{(\theta/2)} \int_{R<|x|<R+1} |u_0(x)|^2 dx < +\infty, \\ u_0(x) \in L^2(\Omega) \qquad \text{for } \theta > 2, \end{cases}$$

since $J_\nu(r) = O(r^{-1/2})$ as $r \to \infty$.

§2. Proof of Theorem 1

In this section all the conditions (A) ~ (G) are assumed. And let $u(x)$ satisfy (*), which is given in Theorem 1.

Definition 2.1. For real-valued functions $\rho(r) \in C^2(R_0, \infty)$, $f(x) \in C^1(\Omega)$ and

$g(x) \in C^1(\Omega)$, let

$$v = v(x; \rho) = e^{\rho(r)} u(x),$$

$$k_1 = k_1(x; \rho) = -\{\rho'(r)\}^2 \langle A(x)\hat{x}, \hat{x} \rangle,$$

$$k_2 = k_2(x; \rho) = \rho''(r) \langle A(x)\hat{x}, \hat{x} \rangle + \rho'(r) \operatorname{div}\{A(x)\hat{x}\},$$

$$F(t; \rho, f, g) = \int_{|x|=t} [f(x) \langle A\hat{x}, \hat{x} \rangle \{2 \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2 - \langle ADv, \overline{Dv} \rangle$$
$$- (q_1 + k_1)|v|^2\} + g(x) \operatorname{Re}[\langle ADv, \hat{x} \rangle \bar{v}]] dS.$$

Lemma 2.2. *We have for* $t > R_0$

$$F(t; \rho, f, g) = e^{2\rho(t)} \int_{|x|=t} [f(x)\{2|\langle ADu, \hat{x} \rangle|^2 - \langle A\hat{x}, \hat{x} \rangle \langle ADu, \overline{Du} \rangle\}$$
$$+ \{2\rho' f \langle A\hat{x}, \hat{x} \rangle + g\} \operatorname{Re}[\langle ADu, \hat{x} \rangle \bar{u}]$$
$$+ \{2f\rho'^2 \langle A\hat{x}, \hat{x} \rangle^2 + (g\rho' - fq_1) \langle A\hat{x}, \hat{x} \rangle\} |u|^2] dS.$$

Proof. Noting Definition 2.1, we have the assertion by straight-forward calculation. □

Lemma 2.3. *For any* $t > s > R_0$ *we have*

$$F(t; \rho, f, g) - F(s; \rho, f, g)$$
$$= \int_{s<|x|<t} [\{2 \langle A\hat{x}, \hat{x} \rangle \partial_r f + g - \langle A\nabla f, \hat{x} \rangle - f \operatorname{div}(A\hat{x})\} \langle A\hat{x}, \hat{x} \rangle^{-1}$$
$$\cdot |\langle ADv, \hat{x} \rangle|^2$$
$$+ \{2r^{-1} f \langle A\hat{x}, \hat{x} \rangle + g - \langle A\nabla f, \hat{x} \rangle - f \operatorname{div}(A\hat{x})\}$$
$$\cdot \{\langle ADv, \overline{Dv} \rangle - \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADv, \hat{x} \rangle|^2\}$$
$$+ 2\operatorname{Re}[\langle ADv, (\nabla - \hat{x}\partial_r) f \rangle \langle \overline{ADv}, \hat{x} \rangle]$$
$$+ 2r^{-1} f\{|ADv|^2 - \langle A\hat{x}, \hat{x} \rangle \langle ADv, \overline{Dv} \rangle\}$$
$$+ 2f \operatorname{Re}[\langle (\langle ADv, \nabla \rangle A) \overline{Dv}, \hat{x} \rangle] - f \operatorname{Re}[\langle (\langle \hat{x}, A\nabla \rangle A) Dv, \overline{Dv} \rangle]$$
$$- 2f \operatorname{Re}[\sqrt{-1} \langle ABA\hat{x}, (Dv - \hat{x} \langle A\hat{x}, \hat{x} \rangle^{-1} \langle ADv, \hat{x} \rangle) \rangle \bar{v}]$$
$$+ 2f \operatorname{Re}[\langle ADv, \hat{x} \rangle \overline{q_2 v}] + \operatorname{Re}[\langle ADv, \nabla g \rangle \bar{v}]$$
$$+ \{(g - \langle A\nabla f, \hat{x} \rangle - f \operatorname{div}(A\hat{x})) q_1 - f \langle A\nabla q_1, \hat{x} \rangle + g \operatorname{Re}[q_2]\} |v|^2] dx$$
$$+ \int_{s<|x|<t} [4\rho' f |\langle ADv, \hat{x} \rangle|^2 + 2(fk_2 + g\rho') \operatorname{Re}[\langle ADv, \hat{x} \rangle \bar{v}]$$
$$+ \{(g - \langle A\nabla f, \hat{x} \rangle - f \operatorname{div}(A\hat{x})) k_1 - f \langle A\nabla k_1, \hat{x} \rangle + gk_2\} |v|^2] dx.$$

Proof. See Lemmas 2.7 and 2.8 of Uchiyama [3]. In order to obtain the above relation, the conditions (A) $\sim$ (E) are fully used. $\square$

The meaning of the following Definition 2.4 can be partly clarified by Lemma 2.5.

Definition 2.4. For $i=1, 2$ and x satisfying $r=|x|>R_0$, let

$$f_i(x)=\langle A\hat{x}, \hat{x}\rangle^{-1} r^{(\gamma_i/2)}\Phi(r),$$
$$g_i(x)=h_i(x) r^{(\gamma_i/2)-1}\Phi(r),$$
$$h_i(x)=2^{-1}\{\sigma(r)-\eta(r)-\gamma_i\}+r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{div}(A\hat{x}) + r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle,$$

where

$$\Phi(r)=\exp\left\{\frac{1}{2}\int_{R_0}^{r}\frac{\eta(r)+\sigma(r)}{r}dr\right\}.$$

And for $w\in H^2_{loc}(\Omega)$, $i=1, 2$ and x satisfying $r=|x|>R_0$, let

$$\begin{aligned}
G_i(x; w) = {} & \sigma(r)\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADw, \hat{x}\rangle|^2 \\
& + (2-\gamma_i-\eta(r))\{\langle ADw, \overline{Dw}\rangle - \langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADw, \hat{x}\rangle|^2\} \\
& + 2r\operatorname{Re}[\langle ADw, \nabla(\langle A\hat{x}, \hat{x}\rangle^{-1})\rangle\langle \overline{ADw}, \hat{x}\rangle] \\
& + 2\{\langle A\hat{x}, \hat{x}\rangle^{-1}|ADw|^2 - \langle ADw, \overline{Dw}\rangle\} \\
& + 2r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{Re}[\langle(\langle ADw, \nabla\rangle A)\overline{Dw}, \hat{x}\rangle] \\
& - r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{Re}[\langle(\langle \hat{x}, A\nabla\rangle A)Dw, \overline{Dw}\rangle] \\
& - 2\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{Re}[\sqrt{-1}\langle ABAx, (Dw-\hat{x}\langle A\hat{x}, \hat{x}\rangle^{-1} \\
& \qquad \cdot\langle ADw, \hat{x}\rangle)\rangle\overline{w}] \\
& + 2r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{Re}[\langle ADw, \hat{x}\rangle\overline{q_2 w}] + \operatorname{Re}[\langle ADw, \nabla h_i\rangle\bar{w}] \\
& + (2r)^{-1}h_i\{\eta(r)+\sigma(r)+\gamma_i-2\}\operatorname{Re}[\langle ADw, \hat{x}\rangle\bar{w}] \\
& - \{r\langle A\hat{x}, \hat{x}\rangle^{-1}\langle A\nabla q_1, \hat{x}\rangle + (\gamma_i+\eta(r))q_1 - h_i\operatorname{Re}[q_2]\}|w|^2.
\end{aligned}$$

Lemma 2.5. *We have the following relations for* $i=1, 2$:

(1) *for* $r>R_0$

$$g_i - \langle A\nabla f_i, \hat{x}\rangle - f_i\operatorname{div}(A\hat{x}) = -(\gamma_i+\eta(r))r^{(\gamma_i/2)-1}\Phi(r),$$
$$\begin{aligned}
& 2\langle A\hat{x}, \hat{x}\rangle\partial_r f_i + g_i - \langle A\nabla f_i, \hat{x}\rangle - f_i\operatorname{div}(A\hat{x}) \\
& = \{\sigma(r)+2r\langle A\hat{x}, \hat{x}\rangle\partial_r(\langle A\hat{x}, \hat{x}\rangle^{-1})\}r^{(\gamma_i/2)-1}\Phi(r),
\end{aligned}$$

$$2r^{-1}f_i\langle A\hat{x}, \hat{x}\rangle + g_i - \langle A\nabla f_i, \hat{x}\rangle - f_i \operatorname{div}(A\hat{x}) = (2-\gamma_i-\eta(r))r^{(\gamma_i/2)-1}\Phi(r),$$

(2) $F(t; \rho, f_i, g_i) - F(s; \rho, f_i, g_i)$

$$= \int_{s<|x|<t} r^{(\gamma_i/2)-1}\Phi(r)[G_i(x; v) + 4r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2$$
$$+ 2\{r\rho'' + r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{div}(A\hat{x}) + h_i\rho'\}\operatorname{Re}[\langle ADv, \hat{x}\rangle\bar{v}]$$
$$+ \langle A\hat{x}, \hat{x}\rangle\{(\gamma_i+\eta(r))\rho'^2 + 2r\rho'\rho'' - r\rho'^2\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle$$
$$+ h_i\rho'' + h_i\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{div}(A\hat{x})\}|v|^2]dx$$

for $t > s > R_0$.

Proof. Noting Definition 2.4, Lemma 2.3 and $\Phi'(r) = (2r)^{-1}\{\sigma(r) + \eta(r)\}\Phi(r)$, we have the assertions by direct calculation. □

We prepare an auxiliary Lemma 2.6.

Lemma 2.6. *We have the following*:

(1) $\lim_{r\to\infty}\sigma(r)^{-1}(\langle A\hat{x}, \hat{x}\rangle - 1) = 0$ *and* $\lim_{r\to\infty}\langle A\hat{x}, \hat{x}\rangle = 1$,

(2) $\lim_{r\to\infty} r\sigma(r)^{-1}\nabla(\langle A\hat{x}, \hat{x}\rangle) = 0$ *and* $\lim_{r\to\infty} r\sigma(r)^{-1}\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}) = 0$,

(3) $\lim_{r\to\infty} r^{2-\beta}\sigma(r)^{-1}\partial_k\partial_l(\langle A\hat{x}, \hat{x}\rangle) = 0$ *and*
$\lim_{r\to\infty} r^{2-\beta}\sigma(r)^{-1}\partial_k\partial_l(\langle A\hat{x}, \hat{x}\rangle^{-1}) = 0$,

(4) $\lim_{r\to\infty} r\sigma(r)^{-1}\{\operatorname{div}(A\hat{x}) - (n-1)r^{-1}\} = 0$ *and* $\operatorname{div}(A\hat{x}) = O(r^{-1})$ *as* $r \longrightarrow \infty$,

(5) $\lim_{r\to\infty} r^{2-\beta}\sigma(r)^{-1}\nabla\{\operatorname{div}(A\hat{x})\} = 0$,

(6) $h_i(x) = O(1)$ *as* $r \longrightarrow \infty$ *for* $i = 1, 2$,

(7) $\lim_{r\to\infty} r^{1-\beta}\sigma(r)^{-1}\nabla h_i(x) = 0$ *for* $i = 1, 2$.

Proof. We have by direct calculations

$$\nabla(\langle A\hat{x}, \hat{x}\rangle) = \langle(\nabla A)\hat{x}, \hat{x}\rangle + 2r^{-1}\{(A-E)\hat{x} - \hat{x}\langle(A-E)\hat{x}, \hat{x}\rangle\},$$
$$\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}) = -\langle A\hat{x}, \hat{x}\rangle^{-2}\nabla(\langle A\hat{x}, \hat{x}\rangle),$$
$$\partial_k\partial_l(\langle A\hat{x}, \hat{x}\rangle) = \sum_{i,j=1}^{n}\{(\partial_k\partial_l a_{ij})\hat{x}_i\hat{x}_j + (\partial_k a_{ij})\partial_l(\hat{x}_i\hat{x}_j)$$
$$+ (\partial_l a_{ij})\partial_k(\hat{x}_i\hat{x}_j) + a_{ij}\partial_k\partial_l(\hat{x}_i\hat{x}_j)\},$$
$$\partial_k\partial_l(\langle A\hat{x}, \hat{x}\rangle^{-1}) = 2\langle A\hat{x}, \hat{x}\rangle^{-3}\{\partial_k(\langle A\hat{x}, \hat{x}\rangle)\}\{\partial_l(\langle A\hat{x}, \hat{x}\rangle)\}$$
$$- \langle A\hat{x}, \hat{x}\rangle^{-2}\partial_k\partial_l(\langle A\hat{x}, \hat{x}\rangle),$$
$$\operatorname{div}(A\hat{x}) = (n-1)r^{-1} + \operatorname{div}\{(A-E)\hat{x}\},$$

$$\partial_k\{\operatorname{div}(A\hat{x})\} = \sum_{i,j=1}^{n}\{(\partial_k\partial_i a_{ij})\hat{x}_j + (\partial_k a_{ij})(\partial_i\hat{x}_j) + (\partial_i a_{ij})(\partial_k\hat{x}_j) + a_{ij}(\partial_k\partial_i\hat{x}_j)\},$$

$$h_i(x) = n-1+2^{-1}\{\sigma(r)-\eta(r)-\gamma_i\} + r\langle A\hat{x},\hat{x}\rangle^{-1}\{\operatorname{div}(A\hat{x})-(n-1)r^{-1}\} + (n-1)(\langle A\hat{x},\hat{x}\rangle^{-1}-1) + r\langle A\nabla(\langle A\hat{x},\hat{x}\rangle^{-1}),\hat{x}\rangle,$$

$$\begin{aligned}\partial_j h_i(x) = {}&\hat{x}_j\{2^{-1}(\sigma'(r)-\eta'(r)) + \langle A\hat{x},\hat{x}\rangle^{-1}\operatorname{div}(A\hat{x}) + \langle A\nabla(\langle A\hat{x},\hat{x}\rangle^{-1}),\hat{x}\rangle\}\\ &+ r[\partial_j(\langle A\hat{x},\hat{x}\rangle^{-1})\operatorname{div}(A\hat{x}) + \langle A\hat{x},\hat{x}\rangle^{-1}\partial_j\{\operatorname{div}(A\hat{x})\}]\\ &+\sum_{k,l=1}^{n} r\{\hat{x}_k(\partial_j a_{kl})\partial_l(\langle A\hat{x},\hat{x}\rangle^{-1}) + \hat{x}_k a_{kl}\partial_j\partial_l(\langle A\hat{x},\hat{x}\rangle^{-1}) + a_{kl}\partial_l(\langle A\hat{x},\hat{x}\rangle^{-1})\partial_j\hat{x}_k\},\end{aligned}$$

where $E=(\delta_{ij})$ is the $n\times n$ identity matrix. So noting (F) and

$$\partial_k\hat{x}_i = O(r^{-1}) \text{ and } \partial_k\partial_l\hat{x}_i = O(r^{-2}) \qquad \text{as } r\longrightarrow\infty,$$

$$\lim_{r\to\infty} r^{\beta}\sigma(r) = +\infty \qquad \text{(by (F3))},$$

we have the assertions. □

Lemma 2.7. *There exist some constants $C_4>0$ and $R_1\ge R_0$ such that for any $r\ge R_1$ and any $w\in H^2_{loc}(\Omega)$ we have*

$$G_1(x;w)\ge C_4\sigma(r)\{|\langle ADw,\hat{x}\rangle|^2 + r^{2\beta-2}|w|^2\}.$$

Proof. In the sequel $\varepsilon_i(r)(i=1,2,\dots)$ means a positive function for $r>R_0$ which tends to 0 as $r\to\infty$. Choose a constant a_1' satisfying $1<a_1'<a_1$. Using

$$|\langle A(x)\xi,\eta\rangle|\le\langle A(x)\xi,\bar{\xi}\rangle^{1/2}\langle A(x)\eta,\bar{\eta}\rangle^{1/2}$$
$$\text{for any } x\in\Omega \text{ and any } \xi,\eta\in\mathbf{C}^n,$$

$$\langle A\{Dw-\hat{x}\langle A\hat{x},\hat{x}\rangle^{-1}\langle ADw,\hat{x}\rangle\}, \{\overline{Dw}-\hat{x}\langle A\hat{x},\hat{x}\rangle^{-1}\langle A\overline{Dw},\hat{x}\rangle\}\rangle = \langle ADw,\overline{Dw}\rangle - \langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2$$

for any $x\in\Omega$ and any $w\in H^2_{loc}(\Omega)$, and noting Lemma 2.6, conditions (A3), (F7) and (F8), we have the following:

$$2r\operatorname{Re}[\langle ADw,\nabla(\langle A\hat{x},\hat{x}\rangle^{-1})\rangle\langle A\overline{Dw},\hat{x}\rangle]\ge-\varepsilon_1(r)\sigma(r)\langle ADw,\overline{Dw}\rangle,$$

$$2\{\langle A\hat{x},\hat{x}\rangle^{-1}|ADw|^2-\langle ADw,\overline{Dw}\rangle\} = 2\langle A\hat{x},\hat{x}\rangle^{-1}\langle ADw,(A-E)\overline{Dw}\rangle + 2(\langle A\hat{x},\hat{x}\rangle^{-1}-1)\langle ADw,\overline{Dw}\rangle\ge-\varepsilon_2(r)\sigma(r)\langle ADw,\overline{Dw}\rangle,$$

$$2r\langle A\hat{x},\hat{x}\rangle^{-1}\mathrm{Re}[\langle((\langle ADw,\nabla\rangle A)\overline{Dw},\hat{x}\rangle]\geq-\varepsilon_3(r)\sigma(r)\langle ADw,\overline{Dw}\rangle,$$

$$-r\langle A\hat{x},\hat{x}\rangle^{-1}\mathrm{Re}[\langle((\langle\hat{x},A\nabla\rangle A)Dw,\overline{Dw}\rangle]\geq-\varepsilon_4(r)\sigma(r)\langle ADw,\overline{Dw}\rangle,$$

$$\begin{aligned}-2\langle A\hat{x},\hat{x}\rangle^{-1}&\mathrm{Re}[\sqrt{-1}\langle ABAx,(Dw-\hat{x}\langle A\hat{x},\hat{x}\rangle^{-1}\langle ADw,\hat{x}\rangle)\rangle\bar{w}]\\&\geq-2\langle A\hat{x},\hat{x}\rangle^{-1}\{\langle ABAx,BAx\rangle\}^{1/2}|w|\\&\qquad\cdot\{\langle ADw,\overline{Dw}\rangle-\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\}^{1/2}\\&\geq-(2-\gamma_1-\eta)a_2^{-1}\{\langle ADw,\overline{Dw}\rangle-\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\}\\&\qquad-(2-\gamma_1-\eta)^{-1}a_2\langle A\hat{x},\hat{x}\rangle^{-2}\langle ABAx,BAx\rangle|w|^2,\end{aligned}$$

$$\begin{aligned}2r\langle A\hat{x},\hat{x}\rangle^{-1}&\mathrm{Re}[\langle ADw,\hat{x}\rangle\overline{q_2w}]\\&\geq-a_1'^{-1}\sigma(r)\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\\&\qquad-a_1'\sigma(r)^{-1}\langle A\hat{x},\hat{x}\rangle^{-1}|rq_2|^2|w|^2,\end{aligned}$$

$$\mathrm{Re}[\langle ADw,\nabla h_1\rangle\bar{w}]\geq-\varepsilon_5(r)\sigma(r)\{\langle ADw,\overline{Dw}\rangle+r^{2\beta-2}|w|^2\},$$

$$\begin{aligned}(2r)^{-1}&h_1\{\eta(r)+\sigma(r)+\gamma_1-2\}\mathrm{Re}[\langle ADw,\hat{x}\rangle\bar{w}]\\&\geq-2^{-1}(1-a_1'^{-1})\sigma(r)\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\\&\qquad-8^{-1}h_1^2(1-a_1'^{-1})^{-1}\langle A\hat{x},\hat{x}\rangle r^{-2}\sigma(r)^{-1}(\eta+\sigma+\gamma_1-2)^2|w|^2,\end{aligned}$$

$$\begin{aligned}h_1\mathrm{Re}[q_2]&\geq-(a_1-a_1')\langle A\hat{x},\hat{x}\rangle^{-1}\sigma(r)^{-1}|rq_2|^2\\&\qquad-4^{-1}(a_1-a_1')^{-1}\sigma(r)h_1^2\langle A\hat{x},\hat{x}\rangle r^{-2}.\end{aligned}$$

Noting Definition 2.4 and

$$\begin{aligned}\langle ADw,\overline{Dw}\rangle&=\{\langle ADw,\overline{Dw}\rangle-\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\}\\&\qquad+\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2,\end{aligned}$$

we have

$$\begin{aligned}G_1(x;w)\geq{}&\sigma(r)\{2^{-1}(1-a_1'^{-1})-\sum_{i=1}^{5}\varepsilon_i(r)\}\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\\&+\{(2-\gamma_1-\eta(r))(1-a_2^{-1})-\sigma(r)\sum_{i=1}^{5}\varepsilon_i(r)\}\\&\qquad\cdot\{\langle ADw,\overline{Dw}\rangle-\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADw,\hat{x}\rangle|^2\}\\&-\{r\langle A\hat{x},\hat{x}\rangle^{-1}\langle A\nabla q_1,\hat{x}\rangle+(\gamma_1+\eta(r))q_1+a_1\sigma(r)^{-1}\\&\qquad\cdot\langle A\hat{x},\hat{x}\rangle^{-1}|rq_2|^2\\&\qquad+a_2(2-\gamma_1-\eta(r))^{-1}\langle A\hat{x},\hat{x}\rangle^{-2}\langle ABAx,BAx\rangle\}|w|^2\\&-\{\varepsilon_5(r)\sigma(r)r^{2\beta-2}+8^{-1}h_1^2(1-a_1'^{-1})^{-1}\\&\qquad\cdot\langle A\hat{x},\hat{x}\rangle(\eta+\sigma+\gamma_1-2)^2\sigma(r)^{-1}r^{-2}\end{aligned}$$

$$+ 4^{-1}(a_1 - a_1')^{-1} h_1^2 \langle A\hat{x}, \hat{x} \rangle \sigma(r) r^{-2}\} |w|^2.$$

We note that we have by (F1) and (F3)

$$\lim_{r \to \infty} r^{\beta} \sigma(r) = +\infty.$$

Therefore by (A3), (F1), (F2), (F5) and Lemma 2.6 there exist some constants $C_4 > 0$ and $R_1 \geq R_0$ such that for any $r \geq R_1$ we have

$$\{2^{-1}(1 - a_1'^{-1}) - \sum_{i=1}^{5} \varepsilon_i(r)\} \langle A\hat{x}, \hat{x} \rangle^{-1} \geq C_4,$$

$$(2 - \gamma_1 - \eta(r))(1 - a_2^{-1}) - \sigma(r) \sum_{i=1}^{5} \varepsilon_i(r) \geq 0,$$

$$\begin{aligned} &- \{r \langle A\hat{x}, \hat{x} \rangle^{-1} \langle A\nabla q_1, \hat{x} \rangle + (\gamma_1 + \eta(r)) q_1 + a_1 \sigma(r)^{-1} \langle A\hat{x}, \hat{x} \rangle^{-1} |rq_2|^2 \\ &\quad + a_2(2 - \gamma_1 - \eta(r))^{-1} \langle A\hat{x}, \hat{x} \rangle^{-2} \langle ABAx, BAx \rangle\} \\ &\quad \geq 2C_4 r^{2\beta-2} \sigma(r), \\ &- \{\varepsilon_5(r) r^{2\beta-2} \sigma(r) + 8^{-1} h_1^2 (1 - a_1'^{-1})^{-1} \langle A\hat{x}, \hat{x} \rangle (\eta + \sigma + \gamma_1 - 2)^2 r^{-2} \sigma(r)^{-1} \\ &\quad + 4^{-1}(a_1 - a_1')^{-1} h_1^2 \langle A\hat{x}, \hat{x} \rangle r^{-2} \sigma(r)\} \\ &\quad \geq - C_4 r^{2\beta-2} \sigma(r). \end{aligned}$$

So using

$$\langle ADw, \overline{Dw} \rangle - \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADw, \hat{x} \rangle|^2 \geq 0,$$

we have the assertion. □

Lemma 2.8. *There exist some constants $C_5 > 0$, $C_6 > 0$ and $R_2 \geq R_0$ such that for any $r \geq R_2$ and $w \in H^2_{loc}(\Omega)$ we have*

$$G_2(x; w) \geq C_5 r^{2\beta-2} \sigma(r) |w|^2 - C_6 |\langle ADw, \hat{x} \rangle|^2.$$

Proof. We use the same estimates as given in the proof of Lemma 2.7 except for the following: Choose a constant a_3' to satisfy $0 < a_3' < a_3$, and we have

$$\begin{aligned} &- 2\langle A\hat{x}, \hat{x} \rangle^{-1} \mathrm{Re}[\sqrt{-1} \langle ABAx, (Dw - \hat{x} \langle A\hat{x}, \hat{x} \rangle^{-1} \langle ADw, \hat{x} \rangle) \rangle \bar{w}] \\ &\quad \geq - a_4^{-1}(2 - \gamma_2 - \eta(r)) \{\langle ADw, \overline{Dw} \rangle - \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADw, \hat{x} \rangle|^2\} \\ &\qquad - a_4 (2 - \gamma_2 - \eta(r))^{-1} \langle A\hat{x}, \hat{x} \rangle^{-2} \langle ABAx, BAx \rangle |w|^2, \\ &2r \langle A\hat{x}, \hat{x} \rangle^{-1} \mathrm{Re}[\langle ADw, \hat{x} \rangle \overline{q_2 w}] \\ &\quad \geq - a_3'^{-1} \sigma(r) \langle A\hat{x}, \hat{x} \rangle^{-1} |\langle ADw, \hat{x} \rangle|^2 - a_3' \sigma(r)^{-1} \langle A\hat{x}, \hat{x} \rangle^{-1} |rq_2|^2 |w|^2, \\ &(2r)^{-1} h_2 (\eta + \sigma + \gamma_2 - 2) \mathrm{Re}[\langle ADw, \hat{x} \rangle \bar{w}] \end{aligned}$$

$$\geq -2^{-1}\sigma(r)\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADw, \hat{x}\rangle|^2$$
$$-8^{-1}h_2^2\langle A\hat{x}, \hat{x}\rangle(\eta+\sigma+\gamma_2-2)^2 r^{-2}\sigma(r)^{-1}|w|^2,$$
$$h_2\mathrm{Re}[q_2] \geq -(a_3-a_3')\langle A\hat{x}, \hat{x}\rangle^{-1}\sigma(r)^{-1}|rq_2|^2$$
$$-4^{-1}(a_3-a_3')^{-1}\langle A\hat{x}, \hat{x}\rangle h_2^2 r^{-2}\sigma(r).$$

Combine the above estimates with the remaining ones given in the proof of Lemma 2.7, and we have by Definition 2.4

$$\begin{aligned}G_2(x; w) &\geq \sigma(r)\{2^{-1} - \sum_{i=1}^{5}\varepsilon_i(r) - a_3'^{-1}\}\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADw, \hat{x}\rangle|^2\\
&+ \{(2-\gamma_2-\eta)(1-a_4^{-1}) - \sigma(r)\sum_{i=1}^{5}\varepsilon_i(r)\}\{\langle ADw, \overline{Dw}\rangle\\
&\qquad - \langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADw, \hat{x}\rangle|^2\}\\
&- \{r\langle A\hat{x}, \hat{x}\rangle^{-1}\langle A\nabla q_1, \hat{x}\rangle + (\gamma_2+\eta(r))q_1 + a_3\sigma(r)^{-1}\langle A\hat{x}, \hat{x}\rangle^{-1}|rq_2|^2\\
&\qquad + a_4(2-\gamma_2-\eta(r))^{-1}\langle A\hat{x}, \hat{x}\rangle^{-2}\langle ABAx, BAx\rangle\}|w|^2\\
&- \{\varepsilon_5(r)r^{2\beta-2}\sigma(r) + 8^{-1}h_2^2\langle A\hat{x}, \hat{x}\rangle(\eta+\sigma+\gamma_2-2)^2 r^{-2}\sigma(r)^{-1}\\
&\qquad + 4^{-1}(a_3-a_3')^{-1}\langle A\hat{x}, \hat{x}\rangle h_2^2 r^{-2}\sigma(r)\}|w|^2.\end{aligned}$$

By (F1), (F2), (F6), Lemma 2.6 and $\lim_{r\to\infty} r^\beta\sigma(r) = +\infty$, there exist some constants $C_5 > 0$, $C_6 > 0$ and $R_2 \geq R_0$ such that for any $r \geq R_2$ we have

$$\begin{aligned}&\sigma(r)\{2^{-1} - \sum_{i=1}^{5}\varepsilon_i(r) - a_3'^{-1}\}\langle A\hat{x}, \hat{x}\rangle^{-1} \geq -C_6,\\
&(2-\gamma_2-\eta(r))(1-a_4^{-1}) - \sigma(r)\sum_{i=1}^{5}\varepsilon_i(r) \geq 0,\\
&-\{r\langle A\hat{x}, \hat{x}\rangle^{-1}\langle A\nabla q_1, \hat{x}\rangle + (\gamma_2+\eta(r))q_1 + a_3\langle A\hat{x}, \hat{x}\rangle^{-1}\sigma(r)^{-1}|rq_2|^2\\
&\qquad + a_4(2-\gamma_2-\eta(r))^{-1}\langle A\hat{x}, \hat{x}\rangle^{-2}\langle ABAx, BAx\rangle\}\\
&\quad \geq 2C_5 r^{2\beta-2}\sigma(r),\\
&-\{\varepsilon_5(r)r^{2\beta-2}\sigma(r) + 8^{-1}h_2^2\langle A\hat{x}, \hat{x}\rangle(\eta+\sigma+\gamma_2-2)^2 r^{-2}\sigma(r)^{-1}\\
&\qquad + 4^{-1}(a_3-a_3')^{-1}\langle A\hat{x}, \hat{x}\rangle h_2^2 r^{-2}\sigma(r)\}\\
&\quad \geq -C_5 r^{2\beta-2}\sigma(r).\end{aligned}$$

So we have the assertion. □

Lemma 2.9. *There exists some constant $R_3 \geq R_2$ such that for any constant $m \geq 1$ and any t, s satisfying $t \geq s \geq R_3$ we have*

$$F(t; mr^\alpha, f_2, g_2) \geq F(s; mr^\alpha, f_2, g_2).$$

Proof. In Lemma 2.5 let $i = 2$ and $\rho(r) = mr^\alpha$. Then we have

$$4r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2 = 4m\alpha r^\alpha\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2,$$

$$\begin{aligned}&2\{r\rho'' + r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) + h_2\rho'\}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]\\ &\quad = 2m\alpha r^{\alpha-1}\{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) + h_2\}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]\\ &\quad \geq -m\alpha r^\alpha\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2\\ &\qquad - m\alpha r^{\alpha-2}\{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) + h_2\}^2\langle A\hat{x}, \hat{x}\rangle|v|^2,\end{aligned}$$

$$\begin{aligned}&(\gamma_2 + \eta(r))\rho'^2 + 2r\rho'\rho'' - r\rho'^2\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle\\ &\qquad + h_2\rho'' + h_2\rho'\langle Ax, x\rangle^{-1}\mathrm{div}(A\hat{x})\\ &\quad = m^2\alpha^2 r^{2\alpha-2}\{\gamma_2 + \eta(r) + 2\alpha - 2 - r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle\}\\ &\qquad + h_2 m\alpha r^{\alpha-2}\{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\}.\end{aligned}$$

By Lemma 2.8 we have for any $r \geq R_2$

$$\begin{aligned}&G_2(x; v) + 4r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2\\ &\quad + 2\{r\rho'' + r\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) + h_2\rho'\}\mathrm{Re}[\langle ADv, \hat{x}\rangle\bar{v}]\\ &\quad + \langle A\hat{x}, \hat{x}\rangle\{(\gamma_2 + \eta(r))\rho'^2 + 2r\rho'\rho'' - r\rho'^2\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle\\ &\qquad + h_2\rho'' + h_2\rho'\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\}|v|^2\\ &\geq C_5 r^{2\beta-2}\sigma(r)|v|^2 - C_6|\langle ADv, \hat{x}\rangle|^2 + 3m\alpha r^\alpha\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2\\ &\quad + m\alpha r^{\alpha-2}[\{\gamma_2 + \eta(r) + 2\alpha - 2 - r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle\}m\alpha r^\alpha\\ &\quad + h_2\{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\}\\ &\quad - \{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) + h_2\}^2\langle A\hat{x}, \hat{x}\rangle]|v|^2.\end{aligned}$$

Noting Lemma 2.6 and (F2), there exists some constant $R_3 \geq R_2$ such that for any $r \geq R_3$ and any constant $m \geq 1$ we have

$$3m\alpha r^\alpha\langle A\hat{x}, \hat{x}\rangle^{-1} - C_6 \geq 0,$$

$$\begin{aligned}&\{\gamma_2 + \eta(r) + 2\alpha - 2 - r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle\}m\alpha r^\alpha\\ &\quad + h_2\{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\} - \{\alpha - 1 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})\\ &\qquad + h_2\}^2\langle A\hat{x}, \hat{x}\rangle\\ &\quad \geq 0.\end{aligned}$$

So we have the assertion. □

We intend to prove Theorem 1.1(1) by reduction to a contradiction.

Lemma 2.10. *If*

$$\liminf_{R\to\infty} R^{(\gamma_1/2)}\Phi(R)\int_{|x|=R}[|\langle ADu, \hat{x}\rangle|^2 + \{r^{-2} + (q_1)_-\}|u|^2]dS = 0,$$

then we have for any $m = 0, 1, 2, \dots$

$$\int_{|x|>R_0} r^{m\alpha+(\gamma_1/2)-1}\Phi(r)\sigma(r)\{|\langle ADu, \hat{x}\rangle|^2 + r^{2\beta-2}|u|^2\}dx < \infty.$$

Proof. By Lemma 2.2 and Lemma 2.6, there exist some constants $C_7 > 0$ and $R_4 \geq R_0$ such that for any $t \geq R_4$ we have

$$\begin{aligned} &F(t; 0, f_1, g_1) \\ &\quad = t^{(\gamma_1/2)}\Phi(t)\int_{|x|=t}[2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2 - \langle ADu, \overline{Du}\rangle \\ &\qquad + h_1 r^{-1}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] - q_1|u|^2]dS \\ &\quad \leq C_7 t^{(\gamma_1/2)}\Phi(t)\int_{|x|=t}[|\langle ADu, \hat{x}\rangle|^2 + \{r^{-2} + (q_1)_-\}|u|^2]dS. \end{aligned}$$

So we have

$$\liminf_{t\to\infty} F(t; 0, f_1, g_1) \leq 0.$$

Letting $t \to \infty$ along the suitable subsequence in Lemma 2.5, we have, by Lemma 2.7, for any $s \geq R_1$

$$C_4\int_{|x|>s} r^{(\gamma_1/2)-1}\Phi(r)\sigma(r)\{|\langle ADu, \hat{x}\rangle|^2 + r^{2\beta-2}|u|^2\}dx \leq -F(s; 0, f_1, g_1),$$

which shows that the assertion holds for $m = 0$. By the above estimate, and by Definitions 2.1 and 2.4 and Lemma 2.3 we have

$$\begin{aligned} &C_4\int_{R_1}^{t} s^{(m+1)\alpha-1}ds\int_{|x|>s} r^{(\gamma_1/2)-1}\Phi(r)\sigma(r)\{|\langle ADu, \hat{x}\rangle|^2 + r^{2\beta-2}|u|^2\}dx \\ &\quad \leq -\int_{R_1}^{t} s^{(m+1)\alpha-1}F(s; 0, f_1, g_1)ds \\ &\quad = \int_{R_1<|x|<t} r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)\{\langle ADu, \overline{Du}\rangle + q_1|u|^2 \\ &\qquad\qquad - 2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2 - h_1 r^{-1}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}]\}dx \\ &\quad = F(t; 0, 0, r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)) - F(R_1; 0, 0, r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)) \\ &\qquad - \int_{R_1<|x|<t} r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)[2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2 \\ &\qquad\quad + \{(m+1)\alpha + 2^{-1}(\eta(r) + \sigma(r) + \gamma_1) - 1 + h_1\}r^{-1}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] \\ &\qquad\quad + \mathrm{Re}[q_2]|u|^2]dx. \end{aligned}$$

Now we assume that the statement is true for m. Then by (G2), (F3), $\alpha < \beta$ and Lemma 2.6 we have for any $t \geq R_1$

$$
\begin{aligned}
&-\int_{R_1<|x|<t} r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)[2\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADu, \hat{x}\rangle|^2 \\
&\qquad + r^{-1}\{(m+1)\alpha+2^{-1}(\eta(r)+\sigma(r)+\gamma_1)-1+h_1\}\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}] \\
&\qquad + \mathrm{Re}[q_2]|u|^2]dx \\
&\leq \int_{|x|>R_1} r^{m\alpha+(\gamma_1/2)+2\beta-3}\Phi(r)\sigma(r)[8^{-1}r^{\alpha-2\beta}\sigma(r)^{-1}\{(m+1)\alpha \\
&+ 2^{-1}(\eta+\sigma+\gamma_1)-1+h_1\}^2\langle A\hat{x}, \hat{x}\rangle + C_3 r^{\alpha-\beta}]|u|^2 dx \\
&< +\infty.
\end{aligned}
$$

On the other hand by Lemma 2.2 we have

$$
\begin{aligned}
&F(t; 0, 0, r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)) \\
&= \int_{|x|=t} r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)\mathrm{Re}[\langle ADu, \hat{x}\rangle\bar{u}]dS \\
&\leq 2^{-1}\int_{|x|=t} r^{m\alpha+(\gamma_1/2)}\Phi(r)\sigma(r)[|\langle ADu, \hat{x}\rangle|^2 + r^{2\beta-2}\{r^{\beta-\alpha}\sigma(r)\}^{-2}|u|^2]dS.
\end{aligned}
$$

Noting that the assertion holds for m, we have by (F3)

$$\liminf_{t\to\infty} F(t; 0, 0, r^{(m+1)\alpha+(\gamma_1/2)-1}\Phi(r)) \leq 0.$$

Therefore we have

$$
\begin{aligned}
+\infty &> \int_{R_1}^{\infty} s^{(m+1)\alpha-1}ds\int_{|x|>s} r^{(\gamma_1/2)-1}\Phi(r)\sigma(r)\{|\langle ADu, \hat{x}\rangle|^2 + r^{2\beta-2}|u|^2\}dx \\
&= (m+1)^{-1}\alpha^{-1}\int_{|x|>R_1}\{r^{(m+1)\alpha}-R_1^{(m+1)\alpha}\}r^{(\gamma_1/2)-1}\Phi(r)\sigma(r) \\
&\qquad\qquad \times\{|\langle ADu, \hat{x}\rangle|^2 + r^{2\beta-2}|u|^2\}dx,
\end{aligned}
$$

which shows that the assertion is true for $m+1$. □

Lemma 2.11. *There exist some constants $R_5 > R_0$ and $C_8 > 0$ such that for any real-valued function $\psi(r) \in C_0^1(R_5, \infty)$ we have*

$$\int_\Omega \psi(r)^2\{|Du|^2 + (q_1)_-|u|^2\}dx \leq C_8\int_\Omega\{r^{\max\{\delta_1,\delta_2\}}\psi(r)^2 + \psi'(r)^2\}|u|^2 dx.$$

Proof. See Lemma 2.3 of Uchiyama [3], where (A), (G) and $\lim_{r\to\infty}\{a_{ij}(x)$

$-\delta_{ij}\} = 0$ have been used. □

Lemma 2.12. *Let R_5 be the one given in Lemma 2.11. There exists some constant $C_9 > 0$ such that for any $R \geq R_5$ and any real-valued function $\zeta(r) \in C^1[R_5, \infty)$ satisfying*

$$\int_{|x|>R_5} [r^{\max\{\delta_1,\delta_2\}}\zeta(r)^2 + \zeta'(r)^2]|u|^2 dx < \infty,$$

$$\liminf_{t\to\infty} \int_{t<|x|<t+1} \zeta(r)^2|u|^2 dx = 0,$$

we have

$$\int_{|x|>R+1} \zeta(r)^2\{|Du|^2 + (q_1)_-|u|^2\}dx$$
$$\leq C_9 \int_{|x|>R} \{r^{\max\{\delta_1,\delta_2\}}\zeta(r)^2 + \zeta'(r)^2\}|u|^2 dx + C_9 \int_{R<|x|<R+1} \zeta(r)^2|u|^2 dx.$$

Proof. See Lemma 2.4 of Uchiyama [3], which can be obtained from Lemma 2.11. □

Lemma 2.13. *There exist some constants $\delta_3 > 0$, $C_{10} > 0$ and $C_{11} > 0$ such that for any $r \geq R_0$ we have*

$$C_{10} r^{-\delta_3} \leq \Phi(r) \leq C_{11} r^{\delta_3}.$$

Proof. By (F2) there exist some constants $\delta_3 > 0$ and $R_6 \geq R_0$ such that for any $r \geq R_6$ we have

$$-2\delta_3 \leq \eta(r) + \sigma(r) \leq 2\delta_3.$$

Then we have for any $r \geq R_6$

$$\frac{1}{2}\int_{R_0}^{R_6} \frac{\eta(r)+\sigma(r)}{r} dr - \delta_3 \log\frac{r}{R_6} \leq \frac{1}{2}\int_{R_0}^{r} \frac{\eta(r)+\sigma(r)}{r} dr$$
$$\leq \frac{1}{2}\int_{R_0}^{R_6} \frac{\eta(r)+\sigma(r)}{r} dr + \delta_3 \log\frac{r}{R_6}.$$

Note that $\Phi(r)$ is continuous in $[R_0, \infty)$, where $R_0 \geq 1$. Then letting

$$C_{10} = \min\left\{R_6^{\delta_3}\exp\left\{\frac{1}{2}\int_{R_0}^{R_6} \frac{\eta(r)+\sigma(r)}{r} dr\right\}, \min_{R_0\leq r\leq R_6} r^{\delta_3}\Phi(r)\right\} > 0,$$

$$C_{11} = \max\left\{R_6^{-\delta_3}\exp\left\{\frac{1}{2}\int_{R_0}^{R_6} \frac{\eta(r)+\sigma(r)}{r} dr\right\}, \max_{R_0\leq r\leq R_6} r^{-\delta_3}\Phi(r)\right\} > 0$$

we have the assertion. □

Lemma 2.14. *If*

$$\liminf_{R\to\infty} R^{(\gamma_1/2)}\Phi(R)\int_{|x|=R}[|\langle ADu, \hat{x}\rangle|^2+\{r^{-2}+(q_1)_-\}|u|^2]dS=0,$$

then we have for any constant $m>0$

$$\int_{|x|>R_0} r^m[|Du|^2+\{1+(q_1)_-\}|u|^2]dx<\infty.$$

Proof. By Lemmas 2.10 and 2.13 and (F3) we have for any $m>0$

$$\int_{|x|>R_0} r^m|u|^2dx<\infty.$$

Let

$$\zeta(r)=r^{m/2}.$$

Then we have

$$\int_{|x|>R_5}\{r^{\max\{\delta_1,\delta_2\}}\zeta^2+\zeta'^2\}|u|^2dx$$
$$=\int_{|x|>R_5} r^m[r^{\max\{\delta_1,\delta_2\}}+4^{-1}m^2r^{-2}]|u|^2dx<\infty,$$
$$\liminf_{t\to\infty}\int_{t<|x|<t+1}\zeta^2|u|^2dx\le\liminf_{t\to\infty}t^{-1}\int_{t<|x|<t+1}r^{m+1}|u|^2dx=0.$$

Therefore by Lemma 2.12 we have the assertion. □

Lemma 2.15. *If*

$$\liminf_{R\to\infty} R^{\gamma_1/2}\Phi(R)\int_{|x|=R}[|\langle ADu, \hat{x}\rangle|^2+\{r^{-2}+(q_1)_-\}|u|^2]dS=0,$$

then for any constant $m>0$ *there exists some constant* $R_7\ge R_2$, *where* R_2 *is the one given in Lemma* 2.8, *such that for any* $t>s>R_7$ *we have*

$$e^{mt^\alpha}\int_{|x|=t}\langle A\hat{x}, \hat{x}\rangle|u|^2dS\le e^{ms^\alpha}\int_{|x|=s}\langle A\hat{x}, \hat{x}\rangle|u|^2dS.$$

Proof. For fixed $m>0$ let

$$a(r)=2^{-1}\{m\alpha r^\alpha+n\}.$$

By Lemma 2.6 and (F2) there exists some constant $R_8\ge R_2$ such that for any $r\ge R_8$ we have

$$\langle A\hat{x}, \hat{x} \rangle^{-1} r \operatorname{div}(A\hat{x}) \leq n.$$

Then for any $t > s > R_8$ we have by integration by parts

$$\left(\int_{|x|=t} - \int_{|x|=s}\right) e^{mr^\alpha} \langle A\hat{x}, \hat{x} \rangle |u|^2 dS$$
$$= \int_{s<|x|<t} e^{mr^\alpha} [2\mathrm{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + r^{-1}\{m\alpha r^\alpha + \langle A\hat{x}, \hat{x} \rangle^{-1} r \operatorname{div}(A\hat{x})\}$$
$$\cdot \langle A\hat{x}, \hat{x} \rangle |u|^2] dx$$
$$\leq 2\int_s^t e^{m\tau^\alpha} d\tau \int_{|x|=\tau} [\mathrm{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + a(r) r^{-1} \langle A\hat{x}, \hat{x} \rangle |u|^2] dS.$$

So we have only to show that there exists some constant $R_7 \geq R_8$ such that for any $\tau \geq R_7$ we have

$$\int_{|x|=\tau} [\mathrm{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + a(r) r^{-1} \langle A\hat{x}, \hat{x} \rangle |u|^2] dS \leq 0.$$

Let

$$\rho(r) = a(\tau) \log r, \ g(r) = r^{-2a(\tau)}.$$

Then by Definition 2.1 and by direct calculation we have

$$e^{2\rho(r)} g(r) = 1, \ 2\rho'(r) g(r) + g'(r) = 0,$$
$$k_1(x) = -a(\tau)^2 r^{-2} \langle A\hat{x}, \hat{x} \rangle, \ k_2(x) = -a(\tau) r^{-2} \langle A\hat{x}, \hat{x} \rangle$$
$$+ a(\tau) r^{-1} \operatorname{div}(A\hat{x}).$$

Therefore by Lemmas 2.2 and 2.3 and by Definition 2.4 we have for any $t_1 > \tau > R_8$

$$\left(\int_{|x|=t_1} - \int_{|x|=\tau}\right) [\mathrm{Re}[\langle ADu, \hat{x} \rangle \bar{u}] + a(\tau) r^{-1} \langle A\hat{x}, \hat{x} \rangle |u|^2] dS$$
$$= F(t_1; a(\tau)\log r, 0, r^{-2a(\tau)}) - F(\tau; a(\tau)\log r, 0, r^{-2a(\tau)})$$
$$= \int_{\tau<|x|<t_1} r^{-2a(\tau)} [\langle ADv, \overline{Dv} \rangle + (q_1 + \mathrm{Re}[q_2])|v|^2$$
$$+ a(\tau) r^{-2} \{r \operatorname{div}(A\hat{x}) - \langle A\hat{x}, \hat{x} \rangle - a(\tau) \langle A\hat{x}, \hat{x} \rangle\} |v|^2] dx$$
$$= -\int_\tau^{t_1} s_1^{-\{2a(\tau)+(\gamma_2/2)\}} \Phi(s_1)^{-1} [F(s_1; a(\tau)\log r, f_2, g_2)$$
$$+ 2C_3 s_1^{(\gamma_2/2)+\beta-2} \Phi(s_1) \sigma(s_1) \int_{|x|=s_1} \langle A\hat{x}, \hat{x} \rangle |v|^2 dS] ds_1$$

$$+\int_{\tau<|x|<t_1} r^{-2a(\tau)}[2\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADv,\hat{x}\rangle|^2 + h_2 r^{-1}\mathrm{Re}[\langle ADv,\hat{x}\rangle\bar{v}]$$
$$+\{\mathrm{Re}[q_2] + a(\tau)r^{-2}(r\,\mathrm{div}(A\hat{x}) - \langle A\hat{x},\hat{x}\rangle)$$
$$+ 2C_3 r^{\beta-2}\langle A\hat{x},\hat{x}\rangle\sigma(r)\}|v|^2]dx,$$

where C_3 is the one given in (G2) and $v = v(x; a(\tau)\log r)$. By (G2), Lemma 2.6 and $0 < \alpha < \beta$, there exist some constant $C_{12} > 0$ and $R_9 \geq R_8$ such that for any $r \geq \tau \geq R_9$ we have

$$2\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADv,\hat{x}\rangle|^2 + h_2 r^{-1}\mathrm{Re}[\langle ADv,\hat{x}\rangle\bar{v}]$$
$$+\{\mathrm{Re}[q_2] + a(\tau)r^{-2}(r\,\mathrm{div}(A\hat{x}) - \langle A\hat{x},\hat{x}\rangle) + 2C_3 r^{\beta-2}\langle A\hat{x},\hat{x}\rangle\sigma(r)\}|v|^2$$
$$\geq \{-8^{-1}h_2^2 r^{-2}\langle A\hat{x},\hat{x}\rangle - (\mathrm{Re}[q_2])_-(x)$$
$$+ a(\tau)r^{-2}(r\,\mathrm{div}(A\hat{x}) - \langle A\hat{x},\hat{x}\rangle) + 2C_3 r^{\beta-2}\langle A\hat{x},\hat{x}\rangle\sigma(r)\}|v|^2$$
$$\geq (2^{-1}C_3 r^{\beta-2}\sigma(r) - C_{12}r^{\alpha-2})|v|^2$$
$$\geq 0.$$

By integration by parts we have for any $t_2 \geq s_1 \geq R_8$

$$\left(\int_{|x|=t_2} - \int_{|x|=s_1}\right) r^{(\gamma_2/2)+\beta-2}\Phi(r)\sigma(r)\langle A\hat{x},\hat{x}\rangle|v|^2 dS$$
$$= \int_{s_1<|x|<t_2} r^{(\gamma_2/2)+\beta-2}\Phi(r)\sigma(r)[2\mathrm{Re}[\langle ADv,\hat{x}\rangle\bar{v}]$$
$$+ r^{-1}\{r\,\mathrm{div}(A\hat{x}) + (2^{-1}(\eta(r) + \sigma(r) + \gamma_2)$$
$$+ \beta - 2 + r\sigma(r)^{-1}\sigma'(r))\langle A\hat{x},\hat{x}\rangle\}|v|^2]dx.$$

Using Lemma 2.5 and the above relation we have for any $t_2 \geq s_1 \geq R_8$

$$F(t_2; a(\tau)\log r, f_2, g_2) - F(s_1; a(\tau)\log r, f_2, g_2)$$
$$+ 2C_3\left(\int_{|x|=t_2} - \int_{|x|=s_1}\right) r^{(\gamma_2/2)+\beta-2}\Phi(r)\sigma(r)\langle A\hat{x},\hat{x}\rangle|v|^2 dS$$
$$= \int_{s_1<|x|<t_2} r^{(\gamma_2/2)-1}\Phi(r)[G_2(x;v) + 4a(\tau)\langle A\hat{x},\hat{x}\rangle^{-1}|\langle ADv,\hat{x}\rangle|^2$$
$$+ 2a(\tau)r^{-1}\{h_2 + r\langle A\hat{x},\hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) - 1\}\mathrm{Re}[\langle ADv,\hat{x}\rangle\bar{v}]$$
$$+ \langle A\hat{x},\hat{x}\rangle a(\tau)r^{-2}[\{\gamma_2 + \eta(r) - 2 - r\langle A\nabla(\langle A\hat{x},\hat{x}\rangle^{-1}),\hat{x}\rangle\}a(\tau)$$
$$+ h_2\{r\langle A\hat{x},\hat{x}\rangle^{-1}\mathrm{div}(A\hat{x}) - 1\}]|v|^2$$
$$+ 4C_3 r^{\beta-1}\sigma(r)\mathrm{Re}[\langle ADv,\hat{x}\rangle\bar{v}]$$
$$+ 2C_3 r^{\beta-2}\langle A\hat{x},\hat{x}\rangle\sigma(r)\{r\langle A\hat{x},\hat{x}\rangle^{-1}\mathrm{div}(A\hat{x})$$

$$+ 2^{-1}(\eta(r) + \sigma(r) + \gamma_2) + \beta - 2 + r\sigma(r)^{-1}\sigma'(r)\}|v|^2]dx.$$

By $\alpha > 0$, Lemmas 2.6 and 2.8 there exist some constants $R_{10} \geq R_9(\geq R_8 \geq R_2)$ and $C_{13} > 0$ such that for any $r \geq \tau \geq R_{10}$ we have

$$G_2(x; v) \geq C_5 r^{2\beta-2}\sigma(r)|v|^2 - C_6|\langle ADv, \hat{x}\rangle|^2,$$

$$[2a(\tau)r^{-1}\{h_2 + r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{div}(A\hat{x}) - 1\} + 4C_3 r^{\beta-1}\sigma(r)]\operatorname{Re}[\langle ADv, \hat{x}\rangle\bar{v}]$$
$$\geq -a(\tau)\langle A\hat{x}, \hat{x}\rangle^{-1}|\langle ADv, \hat{x}\rangle|^2 - C_{13}(r^{\alpha-2} + r^{2\beta-2}\sigma(r)^2\tau^{-\alpha})|v|^2,$$

$$a(\tau)r^{-2}\langle A\hat{x}, \hat{x}\rangle[\{\gamma_2 + \eta(r) - 2 - r\langle A\nabla(\langle A\hat{x}, \hat{x}\rangle^{-1}), \hat{x}\rangle\}a(\tau)$$
$$+ h_2\{r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{div}(A\hat{x}) - 1\}]$$
$$\geq -C_{13}r^{2\alpha-2},$$

$$2C_3 r^{\beta-2}\langle A\hat{x}, \hat{x}\rangle\sigma(r)\{r\langle A\hat{x}, \hat{x}\rangle^{-1}\operatorname{div}(A\hat{x})$$
$$+ 2^{-1}(\eta(r) + \sigma(r) + \gamma_2) + \beta - 2 + r\sigma(r)^{-1}\sigma'(r)\}$$
$$\geq -C_{13}(r^{\beta-2} + r^{\beta-1}|\sigma'(r)|).$$

By (F1) $\sim$ (F4) there exists some constant $R_7 \geq R_{10}$ such that for any $r \geq \tau \geq R_7$ we have

$$3a(\tau)\langle A\hat{x}, \hat{x}\rangle^{-1} - C_6 \geq 0,$$
$$C_5 r^{2\beta-2}\sigma(r) - C_{13}(r^{\alpha-2} + r^{2\beta-2}\sigma(r)^2\tau^{-\alpha} + r^{2\alpha-2} + r^{\beta-2} + r^{\beta-1}|\sigma'(r)|) \geq 0.$$

Therefore we have for any $t_2 \geq s_1 \geq \tau \geq R_7$

$$F(s_1; a(\tau)\log r, f_2, g_2) + 2C_3 s_1^{(\gamma_2/2)+\beta-2}\Phi(s_1)\sigma(s_1)\int_{|x|=s_1}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS$$
$$\leq F(t_2; a(\tau)\log r, f_2, g_2) + 2C_3 t_2^{(\gamma_2/2)+\beta-2}\Phi(t_2)\sigma(t_2)\int_{|x|=t_2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS.$$

By (F2), Lemmas 2.2, 2.6, 2.13 and Definition 2.4, for any $\tau \geq R_7$ there exist some constants $R_{11} \geq \tau \geq R_7$ and $C_{14} > 0$ such that for any $t_2 \geq R_{11}$

$$F(t_2; a(\tau)\log r, f_2, g_2) + 2C_3 t_2^{(\gamma_2/2)+\beta-2}\Phi(t_2)\sigma(t_2)\int_{|x|=t_2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS$$
$$\leq C_{14}t_2^{2a(\tau)+(\gamma_2/2)+\delta_3}\int_{|x|=t_2}[|\langle ADu, \hat{x}\rangle|^2 + \{r^{\beta-2} + (q_1)_-\}|u|^2]dS.$$

Therefore by Lemma 2.14 we have

$$\liminf_{t_2\to\infty}[F(t_2; a(\tau)\log r, f_2, g_2)$$
$$+ 2C_3 t_2^{(\gamma_2/2)+\beta-2}\Phi(t_2)\sigma(t_2)\int_{|x|=t_2}\langle A\hat{x}, \hat{x}\rangle|v|^2 dS] \leq 0,$$

and then we have for any $s_1 \geq \tau \geq R_7$

$$F(s_1; a(\tau)\log r, f_2, g_2) + 2C_3 \int_{|x|=s_1} r^{(\gamma_2/2)+\beta-2}\Phi(r)\sigma(r)\langle A\hat{x}, \hat{x}\rangle |v|^2 dS \leq 0.$$

So at last we have for any $t_1 \geq \tau \geq R_7$

$$\int_{|x|=\tau} [\mathrm{Re}[\langle ADu, \hat{x}\rangle \bar{u}] + a(\tau)r^{-1}\langle A\hat{x}, \hat{x}\rangle |u|^2] dS$$
$$\leq \int_{|x|=t_1} [\mathrm{Re}[\langle ADu, \hat{x}\rangle \bar{u}] + a(\tau)r^{-1}\langle A\hat{x}, \hat{x}\rangle |u|^2] dS.$$

Letting $t_1 \to \infty$ along a suitable subsequence, we have, by Lemma 2.14, for any $\tau \geq R_7$

$$\int_{|x|=\tau} [\mathrm{Re}[\langle ADu, \hat{x}\rangle \bar{u}] + a(\tau)r^{-1}\langle A\hat{x}, \hat{x}\rangle |u|^2] dS \leq 0,$$

which is the desired result. □

Lemma 2.16. *If*

$$\liminf_{R\to\infty} R^{\gamma_1/2}\Phi(R) \int_{|x|=R} [|\langle ADu, \hat{x}\rangle|^2 + \{r^{-2} + (q_1)_-\}|u|^2] dS = 0,$$

then for any constant $m > 0$ *we have*

$$\int_{|x|>R_0} e^{mr^\alpha}[|Du|^2 + \{1 + (q_1)_-\}|u|^2] dx < \infty.$$

Proof. Replacing m with $m+2$ in Lemma 2.15 we have for any $r \geq R_{12}$

$$\int_{|x|=r} |u|^2 dS \leq C_{15} e^{-(m+2)r^\alpha},$$

where

$$R_{12} = \max\{R_5, R_7\},$$

$$C_{15} = C_1 e^{(m+2)R_7^\alpha} \int_{|x|=R_7} \langle A\hat{x}, \hat{x}\rangle |u|^2 dS,$$

R_5 is the one given in Lemma 2.12 and C_1 is the one given in (A3). So we have

$$\int_{|x|>R_{12}} e^{(m+1)r^\alpha}|u|^2 dx \leq C_{15} \int_{R_{12}}^{\infty} e^{-r^\alpha} dr < \infty.$$

Let

$$\zeta(r) = e^{(m/2)r^\alpha}.$$

Then there exists some constant $C_{16} > 0$ such that we have

$$\int_{|x|>R_{12}} \{r^{\max\{\delta_1,\delta_2\}}\zeta(r)^2 + \zeta'(r)^2\}|u|^2 dx \leq C_{16} \int_{|x|>R_{12}} e^{(m+1)r^\alpha}|u|^2 dx < \infty,$$

$$0 \leq \liminf_{t\to\infty} \int_{t<|x|<t+1} \zeta(r)^2|u|^2 dx \leq \liminf_{t\to\infty} e^{-t^\alpha} \int_{t<|x|<t+1} e^{(m+1)r^\alpha}|u|^2 dx = 0.$$

Applying Lemma 2.12 we have the assertion. □

Now we can prove Theorem 1.1(1).

Proof of Theorem 1.1(1). By Lemma 2.9 we have for any $m \geq 1$ and any $t \geq s \geq R_3$

$$F(s; mr^\alpha, f_2, g_2) \leq F(t; mr^\alpha, f_2, g_2).$$

By Lemma 2.2 and Lemma 2.13 for any $m \geq 1$ there exists some constant $C_{17} > 0$ such that for any $t \geq R_3$ we have

$$F(t; mr^\alpha, f_2, g_2) \leq C_{17} e^{(2m+1)r^\alpha} \int_{|x|=t} [|Du|^2 + \{1 + (q_1)_-\}|u|^2] dS.$$

Now we assume that Theorem 1.1(1) is not true. By Lemma 2.16 we have

$$\liminf_{t\to\infty} F(t; mr^\alpha, f_2, g_2) \leq 0,$$

and then for any $m \geq 1$ and any $s \geq R_3$ we have

$$F(s; mr^\alpha, f_2, g_2) \leq 0.$$

On the other hand for a fixed $s \geq R_3$ we have the followings:

$e^{-2ms^\alpha} F(s; mr^\alpha, f_2, g_2)$ is a quadratic in m,

the coefficient of m^2 in $e^{-2ms^\alpha} F(s; mr^\alpha, f_2, g_2)$ is

$$2\alpha^2 s^{2\alpha-2+(\gamma_2/2)} \Phi(s) \int_{|x|=s} \langle A\hat{x}, \hat{x} \rangle |u|^2 dS.$$

Since supp$[u]$ is not a compact set in $\bar{\Omega}$, there exist some constant $R_{13} \geq R_3$ such that we have

$$\int_{|x|=R_{13}} \langle A\hat{x}, \hat{x} \rangle |u|^2 dS > 0.$$

Then there exists some constant $m_0 \geq 1$ such that we have

$$F(R_{13}; m_0 r^\alpha, f_2, g_2) > 0,$$

which is a contradiction. □

In order to prove Theorem 1.1(2) we prepare the following.

Lemma 2.17. *Let $0 \leq a < b$ be constants and $v(r)$ be a real-valued function satisfying*

$$\lim_{R\to\infty} \sup\{|v(r) - v(R)| \ |R + a \leq r \leq R + b\} = 0.$$

Then for any $\varepsilon' > 0$ there exists some constant $R_{14} \geq R_0$ such that for any $R \geq R_{14}$ we have

$$(b - a - \varepsilon') \exp\{v(R)\} \leq \int_{R+a}^{R+b} \exp\{v(r)\} dr \leq (b - a + \varepsilon')\exp\{v(R)\}.$$

Proof. For any $\varepsilon' > 0$ there exists some constant $R_{14} \geq R_0$ such that for any $R \geq R_{14}$ and any r satisfying $R + a \leq r \leq R + b$ we have

$$|\exp\{v(r) - v(R)\} - 1| < \varepsilon'(b - a)^{-1}.$$

Then we have for any $R \geq R_{14}$

$$\left|\int_{R+a}^{R+b} [\exp\{v(r) - v(R)\} - 1] dr\right| < \varepsilon',$$

which shows for any $R \geq R_{14}$

$$-\varepsilon' \leq \int_{R+a}^{R+b} \exp\{v(r) - v(R)\} dr - (b - a) \leq \varepsilon'.$$
□

Now we give the proof of Theorem 1.1(2).

Proof of Theorem 1.1(2). *Let* $\varepsilon > 0$ and let for $r \geq R_0$

$$v(r) = -\{2^{-1}\gamma_1 + \max\{0, \delta_1, \delta_2\}\}\log r - \frac{1}{2}\int_{R_0}^{r} \frac{\eta(r) + \sigma(r)}{r} dr.$$

Since for any $R \geq R_0$ we have

$$\sup\{|v(r) - v(R)| \ |R + (\varepsilon/3) \leq r \leq R + (2\varepsilon/3)\}$$
$$\leq \{|(\gamma_1/2) + \max\{0, \delta_1, \delta_2\}| + 2^{-1} \sup_{r \geq R_0} |\eta(r) + \sigma(r)|\} \log \frac{R + (2\varepsilon/3)}{R},$$

$$\lim_{R\to\infty} \sup\{|v(r) - v(R)| \ |R + (\varepsilon/3) \leq r \leq R + (2\varepsilon/3)\} = 0$$

holds. By Lemma 2.17 with $a = \varepsilon/3$, $b = 2\varepsilon/3$ and $\varepsilon' = \varepsilon/6$, we have for any $R \geq R_{14}$

$$6^{-1}\varepsilon R^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}}\Phi(R)^{-1}$$
$$\leq \int_{R+(\varepsilon/3)}^{R+(2\varepsilon/3)} r^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}}\Phi(r)^{-1}dr$$
$$\leq 2^{-1}\varepsilon R^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}}\Phi(R)^{-1}.$$

By Theorem 1.1 (1) there exist some constants $R_{15} > R_0$ and $C_{18} > 0$ such that for any $R \geq R_{15}$

$$C_{18}R^{-(\gamma_1/2)}\Phi(R)^{-1} \leq \int_{|x|=R} [|\langle ADu, \hat{x}\rangle|^2 + \{r^{-2} + (q_1)_-\}|u|^2]dS$$
$$\leq \int_{|x|=R} [\langle A\hat{x}, \hat{x}\rangle\langle ADu, \overline{Du}\rangle + \{r^{-2} + (q_1)_-\}|u|^2]dS$$
$$\leq C_1^2 \int_{|x|=R} [|Du|^2 + \{r^{-2} + (q_1)_-\}|u|^2]dS,$$

where $C_1 \geq 1$ is the one given in (A3). Let $\xi_R(r) \in C_0^1(R, R+\varepsilon)$ satisfy the following: $\xi_R(r) = 1$ for $R + (\varepsilon/3) \leq r \leq R + (2\varepsilon/3)$, $0 \leq \xi_R(r) \leq 1$ for $R \leq r \leq R + \varepsilon$ and there exists some constant $C_{19} > 0$ such that for any $R \geq R_0$ and any $r \geq R$ we have $|\xi_R'(r)| \leq C_{19}$. Applying Lemma 2.11 with $\psi(r) = \xi_R(r)r^{-\frac{1}{2}\max\{0,\delta_1,\delta_2\}}$, we have for any $R \geq \max\{R_5, R_{14}, R_{15}\}$

$$6^{-1}\varepsilon C_{18}R^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}}\Phi(R)^{-1}$$
$$\leq \int_{R+(\varepsilon/3)}^{R+(2\varepsilon/3)} C_{18}r^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}}\Phi(r)^{-1}dr$$
$$\leq C_1^2 \int_{R<|x|<R+\varepsilon} (\xi_R(r)r^{-\frac{1}{2}\max\{0,\delta_1,\delta_2\}})^2\{|Du|^2 + (q_1)_-\}|u|^2dx$$
$$+ C_1^2 \int_{R<|x|<R+\varepsilon} r^{-\max\{0,\delta_1,\delta_2\}-2}|u|^2dx$$
$$\leq C_1^2C_8 \int_{R<|x|<R+\varepsilon} [1 + 2\{\xi_R'(r)^2 + 4^{-1}(\max\{0, \delta_1, \delta_2\})^2\}]|u|^2dx$$
$$+ C_1^2 \int_{R<|x|<R+\varepsilon} |u|^2dx$$
$$\leq C_{20} \int_{R<|x|<R+\varepsilon} |u|^2dx,$$

where

$$C_{20} = C_1^2[C_8\{1 + 2C_{19}^2 + 2^{-1}(\max\{0, \delta_1, \delta_2\})^2\} + 1] > 0.$$

This shows the assertion. □

Lastly we give the proof of Theorem 1.1(3).

Proof of Theorem 1.1(3). By Theorem 1.1(2) with $\varepsilon = 1$ there exist some integer $N_0 \geq R_0$ and some constant $C_{21} > 0$ such that for any integer $N \geq N_0$ we have

$$\int_{N<|x|<N+1} |u|^2 dx \geq C_{21} N^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}} \Phi(N)^{-1}.$$

Applying Lemma 2.17 with $a = 0$, $b = 1$ and $\varepsilon' = 1$, there exists some integer $N_1 \geq R_0$ such that for any integer $N \geq N_1$ we have

$$2N^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}} \Phi(N)^{-1} \geq \int_N^{N+1} r^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}} \Phi(r)^{-1} dr.$$

Let $N_2 = \max\{N_0, N_1\}$. Then for any integer $M > 0$ we have

$$\begin{aligned}\int_{N_2}^{N_2+M} |u|^2 dx &\geq C_{21} \sum_{n=N_2}^{N_2+M-1} n^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}} \Phi(n)^{-1} \\ &\geq 2^{-1} C_{21} \int_{N_2}^{N_2+M} r^{-(\gamma_1/2)-\max\{0,\delta_1,\delta_2\}} \Phi(r)^{-1} dr,\end{aligned}$$

which shows the assertion. □

§3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. By Lemma 4.1 of Uchiyama [3], Lemma 2.3 is also true under our weak condition (C3)'. So we can follow the proof of Theorem 1.1. □

Proof of Theorem 1.3. Lemma 2.3 also holds under our weak condition (C3)'. By Definition 2.4 we have

$$\begin{aligned}G_i(x; w) = {}& \sigma(r)|\partial_r w|^2 + (2 - \gamma_i - \eta(r))\{|\nabla w|^2 - |\partial_r w|^2\} + 2r\mathrm{Re}[\overline{q_2 w}\partial_r w] \\ &+ \mathrm{Re}[\langle \nabla w, \nabla h_i \rangle \bar{w}] + (2r)^{-1} h_i\{\eta(r) + \sigma(r) - \gamma_i - 2\}\mathrm{Re}[\bar{w}\partial_r w] \\ &- \{r\partial_r q_1(x) + (\gamma_i + \eta(r))q_1(x) - h_i \mathrm{Re}[q_2]\}|w|^2,\end{aligned}$$

where

$$h_i(x) = n - 1 + 2^{-1}\{\sigma(r) - \eta(r) - \gamma_i\}.$$

Since h_i is a function depending only on r, we have

$$\mathrm{Re}[\langle \nabla w, \nabla h_i \rangle \bar{w}] = h_i'(r)\mathrm{Re}[\bar{w}\partial_r w].$$

So in the estimination of $G_i(x; w)$ given in Lemma 2.7 and Lemma 2.8, we need not use the term $(2-\gamma_i-\eta(r))\{|\nabla w|^2-|\partial_r w|^2\}$, which is non-negative by our weak condition (F2)$'$. Therefore Lemmas 2.7 and 2.8 are also true and we can follow the proof of Theorem 1.1. □

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