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Purely Inseparable Extensions of Unique Factorization Domains

By

Jeffery Lang*

§ 0. Introduction

This article attempts to set some further groundwork for the study of codimension-one cycles of purely inseparable coverings of varieties in characteristic p. Thus it represents, hopefully, a preliminary effort. The simplest type of purely inseparable cover of a variety X with coordinate ring A in characteristic $p \neq 0$ is obtained by taking $Y = \mathrm{Spec}(A[\sqrt[p]{g}])$ for some $g \in A$. Efforts to relate the codimension one cycles of X and Y ([8], [2]) led to the ring-theoretic question,

I. If A is a UFD of characteristic $p \neq 0$, for what $g \in A$ is $A[\sqrt[p]{g}]$ a UFD?

A natural place to begin to investigate (I) is with A a polynomial ring defined over a field k of characteristic $p > 0$. When k is perfect (I) can be restated,

II. For what $g \in A = k[x_1, \dots, x_n]$ is $A^p[g]$ a UFD?

This paper investigates (I) in Section 5 when A is the coordinate ring of a surface X defined as a complete intersection and extends (II) in Section 3 to study the divisor classes of rings of the form $k[x_1^p, \dots, x_n^p, g_1, \dots, g_{n-1}]$.

After a few brief preliminaries in Section 1, some tools for calculating $Cl(A)$ are developed in Section 2, that generalize ([8], (2.6)) and ([5], II(1.3)), which played an important role in showing that for a general choice of $g \in A = k[x_1, \dots, x_n]$ $A^p[g]$ is factorial (See [5], II(2.6)). In Section 4 some examples are considered. The reader is also referred to two excellent references for the subject of divisor classes of Krull rings, Samuel's 1964 Tata notes [11] and Fossum's "*The Divisor Class Group of a Krull Domain*" [4].

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* Department of Mathematics, University of Kansas, Lawrence, KS 66045, U.S.A.
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§1. Notation and Preliminaries

(0.1) k-an algebraically closed field of characteristic $p \neq 0$.
(0.2) $A = k[x_1, \dots, x_n]$-a polynomial ring in n-indeterminates over k.
(0.3) Given $f_1, \dots, f_n \in A$, $J(f_1, \dots, f_n)$ is the determinant of the matrix $[D_i(f_j)]$, $1 \leq i, j \leq n$, where $D_i = \dfrac{\partial}{\partial x_i}$.
(0.4) Given $h \in A$, let $deg_{x_i}(h)$ denote the degree of h in x_i, $deg_{x_i, x_j}(h)$ denote the degree of h in x_i and x_j, etc...., let $deg(h)$ denote the total degree of h.
(0.5) A_k^n denotes the affine n-space over k.
(0.6) $f_1, \dots, f_{n-1} \in A$ are said to satisfy condition (*) if the variety $V \subseteq A_k^n$ defined by the $n-1$ by $n-1$ minors of the matrix $\begin{bmatrix} D_1(f_1) & \cdots & D_n(f_1) \\ \vdots & & \vdots \\ D_1(f_{n-1}) & \cdots & D_n(f_{n-1}) \end{bmatrix}$ has dimension less than $n-2$.
(0.7) If R is a Krull ring denote by $Cl(R)$ the divisor class group of R ([11], page 18) (cf. (0.2)).
(0.8) If X is a noetherian integral separated scheme which is regular in codimension one, denote by $Cl(X)$ the group of Weil divisors of X([7], page 130).
(0.9) If R is a noetherian integrally closed domain, then R is a Krull ring and $X = \text{Spec}(R)$ will be regular in codimension one, and $Cl(R)$ and $Cl(X)$ defined above are isomorphic.
(0.10) Given $g, f_1, \dots, f_{n-2} \in A$, let P be the ideal in A generated by $f_1, \dots, f_{n-2}$ and $B = A/P$. For $f \in A$, denote its image in B by $\bar{f}$. Then we say that g, $f_1, \dots, f_{n-2}$ satisfies condition (**) if all three of the following conditions are satisfied.
(i) P is a height $n-2$ prime ideal in A.
(ii) $\bar{g} \notin B^p = k[\bar{x}_1^p, \dots, \bar{x}_n^p]$
(iii) The ring $A[\omega]/(\omega^p - g, f_1, \dots, f_{n-2})$ is regular in codimension one.
Note that the ring in (iii) is a domain by (i) and (ii) and it is regular in codimension one if and only if it is normal ([7], pg. 186, Proposition 8.23).
(0.11) For a prime number p, we will let $\mathbb{F}_p$ denote the finite field of order p and the set of integers $\{0, 1, \dots, p-1\}$. It will be clear from the context which is meant.

1.1. Theorem. *Let $A \subset B$ be Krull rings. Suppose that either B is integral over A or that B is a flat A-algebra. Then there is a well defined group homomorphism $\varphi: Cl(\mathrm{A}) \to Cl(B)$.*

Let B be a Krull ring of characteristic $p > 0$. Let Δ be a derivation of L, the quotient field of B, such that $\Delta(B) \subset B$. Let $K = \ker(\Delta)$ and A

$= B \cap K$. Then A is a Krull ring with B integral over A. By (1.1) we have a map $\varphi: Cl(A) \to Cl(B)$ (see [11] pp. 19–20). Set $\mathscr{L} = \{t^{-1}\Delta t \in B | t \in L\}$ and $\mathscr{L}' = \{u^{-1}\Delta u | u$ is a unit in B$\}$. Then $\mathscr{L}$ is an additive group with subgroup $\mathscr{L}'$.

1.2. Theorem. (a) *There exists a canonical monomorphism* $\bar{\varphi}: \ker \varphi \to \mathscr{L}/\mathscr{L}'$. (b) If $[L: K] = p$ *and* $\Delta(B)$ *is not contained in any height one prime of* B, *then* $\bar{\varphi}$ *is an isomorphism* ([11], p. 62).

1.3. Theorem. (a) If $[L: K] = p$, *then there exists* $a \in A$ *such that* $\Delta^p = a\Delta$, (b) $t \in L$ *is equal to* $u^{-1}\Delta u$ *for some* $u \in L$ *if and only if* $\Delta^{p-1}t - at + t^p = 0$ ([11], pp. 63–64).

§2. Computational Tools

Let k be an algebraically closed field of characteristic $p > 0$. Let $A = k[x_1, \dots, x_n]$ be a polynomial ring in n-indeterminates over k. Let $f_1, \dots, f_{n-1} \in A$. Define a derivation D on $L = k(x_1, \dots, x_n)$ by $D(h) = J(h, f_1, \dots, f_{n-1})$ where J represents the determinant of the Jacobian matrix. That is,

$$J(h, f_1, \dots, f_{n-1}) = \det \begin{bmatrix} D_1(h) & D_2(h) & \cdots & D_n(h) \\ D_1(f_1) & D_2(f_1) & \cdots & D_n(f_1) \\ \vdots & \vdots & \vdots & \vdots \\ D_1(f_{n-1}) & D_2(f_{n-1}) & \cdots & D_n(f_{n-1}) \end{bmatrix} \text{where } D_j = \frac{\partial}{\partial x_j}.$$

We have the following generalization of ([8], page 395, (2.6)). We let $A' = k[x_1^p, \dots, x_n^p, f_1, \dots, f_{n-1}]$.

2.1. Proposition. *Assume* $D \neq 0$. *Then* (i) *there exists* $a \in A$ *such that* $D(a) = 0$ *and* $D^p = aD$, (ii) a *is given by* $a = (-1)^n \sum_{j=1}^{n-1} \sum_{r_j=0}^{p-1} f_1^{r_1} \cdots f_{n-1}^{r_{n-1}} \nabla(f_1^{p-r_1-1} \cdots f_{n-1}^{p-r_{n-1}-1})$, (iii) *For all* $t \in L$, $D^{p-1}t - at = (-1)^{n-1} \sum_{j=1}^{n-1} \sum_{r_j=0}^{p-1} f_1^{r_1} \cdots f_{n-1}^{r_{n-1}} \nabla(f_1^{p-r_1-1} \cdots f_{n-1}^{p-r_{n-1}-1} t)$, *where* $\nabla = (D_1^{p-1} \cdots D_n^{p-1})$.

Proof (i) For each $i = 1, \dots, n-1$, $f_i \notin k(x_1^p, \dots, x_n^p, f_1, \dots, f_{i-1})$ (where we let $f_0 = 0$) since $D \neq 0$. It then follows that

2.1.1. $[L: L'] = p$ and $L' = D^{-1}(0)$, where L' is the quotient field of A'.

By (1.3) there exists $a \in A \cap L'$ such that $D^p = aD$.

(ii) Will follow from (iii) by letting $t = 1$.

(iii) Case(I): The f_i contain no monomials that are p-th powers and the f_i

satisfy condition (*).

For each $i = 1, \dots, n-1$, let Δ_i be the derivation on L defined by

$$\Delta_i(h) = \det \begin{bmatrix} D_1(h) & \cdots & D_{n-1}(h) \\ D_1(f_1) & \cdots & D_{n-1}(f_1) \\ \vdots & & \vdots \\ D_1(f_{i-1}) & \cdots & D_{n-1}(f_{i-1}) \\ D_1(f_{i+1}) & \cdots & D_{n-1}(f_{i+1}) \\ \vdots & & \vdots \\ D_1(f_{n-1}) & \cdots & D_{n-1}(f_{n-1}) \end{bmatrix}$$

Since $D \neq 0$, we may assume after a permutation of the x_i that $\Delta_i(f_i) \neq 0$ for each i. Now for each $1 \leq i \leq n-1$, let $E_i = (1/\Delta_i(f_i)) \cdot \Delta_i$ and $E = E_1^{p-1} \cdots E_{n-1}^{p-1}$.

2.1.2. Claim: *$E(D^{p-1}t - at) = \nabla t$, for all $t \in L$.*

Proof of (2.1.2). If $t \in A$, $deg(Dt) \leq M + deg\ t - n$, where $M = \sum_{i=1}^{n-1} deg\, f_i$. It then follows that $deg(a) \leq (p-1)(M-n)$ and

2.1.3. $deg(D^{p-1}t - at) \leq deg(t) + (p-1)(M-n)$, for all $t \in A$.

Given $h \in A'$, there is a unique $\beta_{\bar{r}} \in A$, for each $\bar{r} \in \mathbb{F}_p^{n-1}$, such that $h = \sum_{\bar{r}} \beta_{\bar{r}}^p f^{\bar{r}}$ (where for $\bar{r} = (r_1, \dots, r_{n-1}) \in \mathbb{F}_p^{n-1}$, $f^{\bar{r}} = f_1^{r_1} \cdots f_{n-1}^{r_{n-1}}$). We have that for each $i = 1, \dots, n-1$, $E_i(h) = \sum_{\bar{r}} r_i \beta_{\bar{r}}^p f^{\bar{r}'}$, where $\bar{r}' = (r_1, \dots, r_i - 1, \dots, r_{n-1})$. Then $E_i(h) \in A'$ and $deg(E_i(h)) \leq deg(h) - deg(f_i)$. Given $t \in A$, $D^{p-1}t - at \in A'$ by (3.2) below, the proof of which is independent of this section, since $D^p = aD$ and $Da = 0$. By (2.1.3) we have for all $t \in A$,

$$E(D^{p-1}t - at) = 0, \text{ or}$$

2.1.4 $$deg(E(D^{p-1}t - at)) \leq deg\ t - (p-1)n.$$

Any differential operator on $k(x_1, \dots, x_n)$ can be written uniquely as a linear combination of $D_1^{s_1} \cdots D_n^{s_n}$, $0 \leq s_i \leq p-1$, with coefficients in L. Thus there exists unique $\alpha_{\bar{r}} \in L$, for each $\bar{r} \in \mathbb{F}_p^n$ such that

2.1.5. $E(D^{p-1} - aI) = \sum_{\bar{r}} \alpha_{\bar{r}} \partial^{\bar{r}}$, where for $\bar{r} = (r_1, \dots, r_n)$, $\partial^{\bar{r}} = D_1^{r_1} D_2^{r_2} \cdots D_n^{r_n}$.

Proceed by induction on $\sum r_i$ to show $\alpha_{\bar{r}} = 0$ for $\bar{r} \neq (p-1, \dots, p-1)$. By (2.1.4), $E(D^{p-1}(1) - a(1)) = 0$. By (2.1.5) $\alpha_{(0,\dots,0)} = 0$. Assume $\alpha_{\bar{r}} = 0$ for all $\bar{r}$

with $\sum_{r_i<k} < n(p-1)$. Let $\bar{r}^* = (r_1^*, \ldots, r_n^*)$ be such that $\sum r_i^* = k$. Substitute $t = x_1^{r_1^*} \cdots x_n^{r_n^*}$ into (2.1.5) and use (2.1.4) to obtain $r_1^*! \cdots r_n^*! \alpha_{\bar{r}^*} = 0$ which implies $\alpha_{r^*} = 0$. Therefore

2.1.6. $$E(D^{p-1} - aI) = \alpha \nabla, \text{ for some } \alpha \in L.$$

Apply both sides of (2.1.6) to $(x_1 \cdots x_n)^{p-1}$ and use (2.1.4) to see that $\alpha \in k$.

To compute α, we first note that α in (2.1.6) is invariant under a linear change of variables. This may be checked by one coordinate change at a time. If x_1 say, is replaced by $\alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1}$ $(\alpha_i \in k)$ then D_1 becomes $\alpha_1 D_1$ and ∇ becomes $\alpha_1^{p-1}\nabla$. By the chain rule D becomes α_1 D, E. remains unchanged and a becomes α_1^{p-1} a so that $E(D^{p-1} - aI)$ becomes $\alpha_1^{p-1} E(D^{p-1} - aI)$.

By (*) there is a point $Q \in k^n$ where the matrix $\begin{bmatrix} D_1(f_1)(Q) & \cdots & D_n(f_1)(Q) \\ \vdots & & \vdots \\ D_1(f_{n-1})(Q) & \cdots & D_n(f_{n-1})(Q) \end{bmatrix}$ is row independent over k.

Since α is invariant under a change in coordinates we may assume $Q = (0, \ldots, 0)$. Furthermore since α is clearly unaffected by the constant terms of the f_i, we may assume that $f_1(Q) = \cdots = f_n(Q) = 0$. We then have that the degree one forms of the f_i are k-independent. Therefore, after another linear change we may assume that the lowest degree form of f_i is x_i $(1 \leq i \leq n-1)$. Again apply both sides of (2.1.6) to $(x_1 \cdots x_n)^{p-1}$ and compare the 0-degree terms. On the right we get $(-1)^n \alpha$ and on the left we get $(-1)^n$. Hence $\alpha = 1$ in (2.1.6).

Now let $t \in L$. Since $D^{p-1}t - at \in L'$, there are unique $\beta_{\bar{r}} \in L$ such that $D^{p-1}t - at = \sum_{\bar{r}} \beta_{\bar{r}}^p f^{\bar{r}}$. Fix $\bar{s} = (s_1, \ldots, s_{n-1}) \in \mathbb{F}_p^{n-1}$. Then on the one hand we know by (2.1.6) with $\alpha = 1$ that $E(D^{p-1}(f^{\bar{s}}t) - af^{\bar{s}}t) = \nabla(f^{\bar{s}}t)$. On the other hand, we have $E(D^{p-1}(f^{\bar{s}}t) - af^{\bar{s}}t) = E(f^{\bar{s}}(D^{p-1}t - at)) = (-1)^{n-1}\beta_{\bar{r}_0}^p$ where $\bar{r}_0 = (p-1-s_1, \ldots, p-1-s_{n-1})$, from which (iii) follows,

Case (II). The f_i contain no monomials that are p^{th} powers.

Assume the coefficients of the f_i are algebraically independent over k. Then (*) is satisfied and hence the formula in (iii) holds. Therefore it will hold after any specialization of the coefficients, since with respect to the differential operators D and ∇ they are constants. Finally observe that if the f_i are replaced by h_i, $1 \leq i \leq n-1$, such that $f_i - h_i \in B = k[x_1^p, \ldots, x_n^p]$, then D and hence a (such that $D^p = aD$) remain unchanged. The next lemma shows that the right side of the equality in (2.1 iii) also remains unchanged by such a substitution. Case II showed that the desired formula holds whenever the f_i contain no p-th powers. Thus the general case now follows from the above observations.

2.2. Lemma. *Assume* $f_1, \dots, f_{n-1}, h_1 \in A$ with $f_1 - h_1 \in B = k[x_1^p, \dots, x_n^p]$. *Then for all* $t \in L$,

$$\sum_{s_i} f_1^{s_1} \cdots f_{n-1}^{s_{n-1}} \nabla(f_1^{p-s_1-1} \cdots f_{n-1}^{p-s_{n-1}-1} t) =$$

$$= \sum_{s_i} h_1^{s_1} f_2^{s_2} \cdots f_{n-1}^{s_{n-1}} \nabla(h_1^{p-s_1-1} f_2^{p-s_2-1} \cdots f_{n-1}^{p-s_{n-1}-1} t)$$

Proof. $h_1 = f_1 + \alpha$, for some $\alpha \in B$. Let $t \in L$ and $t_0 = f_2^{p-s_2-1} \cdots f_{n-1}^{p-s_{n-1}-1} t$. Then $\sum_{s_i} h_1^{s_1} f_2^{s_2} \cdots f_{n-1}^{s_{n-1}} \nabla(h_1^{p-s_1-1} \cdots f_{n-1}^{p-s_{n-1}-1} t) = \sum_{(s_2,\dots,s_{n-1})} f_2^{s_2} \cdots f_{n-1}^{s_{n-1}} \sum_{s_1=0}^{p-1} h_1^{s_1} \nabla(h_1^{p-s_1-1} t_0)$. So it is enough to show that $\sum_{s=0}^{p-1} h^s \nabla(h^{p-s-1} t) = \sum_{s=0}^{p-1} f^s \nabla(f^{p-s-1} t)$, *when* $h - f = \alpha \in B$.

We have $\sum_{s=0}^{p-1} h^s \nabla(h^{p-s-1} t)$

$$= \sum_{s=0}^{p-1} \sum_{i=0}^{s} \binom{s}{i} f^i \alpha^{s-i} \sum_{j=0}^{p-1-s} \binom{p-1-s}{j} \alpha^{p-1-s-j} \nabla(f^j t)$$

$$= \sum_{s=0}^{p-1} \sum_{i=0}^{s} \sum_{j=0}^{p-1-s} \binom{s}{i} \binom{p-1-s}{j} f^i \alpha^{p-1-i-j} \nabla(f^j t)$$

$$= \sum_{j=0}^{p-1} \alpha^{-j} \nabla(f^j t) \sum_{s=0}^{p-1} \binom{p-1-s}{j} \alpha^{p-1-s} \sum_{i=0}^{s} \binom{s}{i} f^i \alpha^{s-i}$$

$$= \sum_{j=0}^{p-1} \alpha^{-j} \nabla(f^j t) \sum_{s=0}^{p-1} \binom{p-1-s}{j} \alpha^{p-1-s} (f+\alpha)^s$$

$$= \sum_{j=0}^{p-1} (-1)^j \nabla(f^j t) \sum_{s=0}^{p-1} \binom{p-1-j}{s} (-1)^s \alpha^{p-1-j-s} (f+\alpha)^s$$

$$\left(\text{Note that } \binom{p-1-s}{j}\binom{p-1}{s} = \binom{p-1-j}{s}\binom{p-1}{j} \text{ and } \binom{p-1}{j} = (-1)^j, \text{ etc.}\right)$$

$$= \sum_{j=0}^{p-1} (-1)^j (\nabla(f^j t))(\alpha - (f+\alpha))^{p-1-j}$$

$$= \sum_{j=0}^{p-1} f^{p-1-j} \nabla(f^j t) = \sum_{s=0}^{p-1} f^s \nabla(f^{p-1-s} t).$$

The next proposition generalizes ([3], page 74, Theorem (3.4)). In the two variable case it was used to prove that a generic Zariski surface has 0-divisor class group ([9]).

Let $S = S(f_1, \dots, f_{n-1}) = \{Q \in k^n: \operatorname{rank}[D_i(f_j)](Q) < n-1\}$. Let C be the matrix

$$C = \begin{bmatrix} D_1(f_1) & \cdots & D_{n-1}(f_1) & D_n(f_1) \\ \vdots & & \vdots & \vdots \\ D_1(f_{n-2}) & \cdots & D_{n-1}(f_{n-2}) & D_n(f_{n-2}) \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

and $C^* = (C^{-1})^t$ (= the transpose of C^{-1}). Let $[g_1, \dots, g_n]$ and $[h_1, \dots, h_n]$ be the $n-1-st$ and n-th rows of C^*, respectively. Let $E_1^2 = \sum_{i,i=1}^{n} g_i g_j \frac{\partial^2}{\partial x_i \partial x_j}$, $E_2^2 = \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2}{\partial x_i \partial x_j}$, and $E_1 E_2 = \sum_{i,j=1}^{n} g_i h_j \frac{\partial^2}{\partial x_i \partial x_j}$. Let M_i be the cofactor of $D_{n-1}(f_i)$ in the matrix

$$\begin{bmatrix} D_1(f_1) & \cdots & D_{n-1}(f_1) \\ \vdots & & \vdots \\ D_1(f_{n-1}) & \cdots & D_{n-1}(f_{n-1}) \end{bmatrix}, \quad 1 \le i \le n-1.$$

Let $H = [\sum_{j=1}^{n-1} M_j E_1 E_2(f_j)]^2 - [\sum_{j=1}^{n-1} M_j E_1^2(f_j)][\sum_{j=1}^{n-1} M_j E_2^2(f_j)]$.

2.3. Proposition. *For all* $Q \in S$, $a(Q) = (H(Q))^{(p-1)/2}$, *where* a *is as in* (2.1).

Proof. It is a straightforward linear algebra to check that for all $Q \in S$, $g_i(Q)$ and $h_i(Q)$ are independent of the order of $f_1, \dots, f_{n-1}$ up to a change of the same sign. It follows that $H(Q)$ is independent of the order of $f_1, \dots, f_{n-1}$.

Let $Q \in S$ be a point where the rank $[D_i(f_j)(Q)] = n-2$. After a change of coordinates, which will not alter D (hence a) or H, we may assume $Q = (0, \dots, 0)$. By the above remark, we may assume $M_{n-1}(Q) \neq 0$. Then

2.3.1 $\quad a_1 D_i(f_1)(Q) + \cdots + a_{n-2} D_i(f_{n-2})(Q) = D_i(f_{n-1})(Q)$, $1 \le i \le n$, where

$$a_r = -M_r(Q)/M_{n-1}(Q), \ 1 \le r \le n-2.$$

Replacing f_j by $f_j - f_j(Q)$, $1 \le j \le n-1$ also does not change D (and hence a) or H so that we may assume $f_j(Q) = 0$, $1 \le j \le n-1$.

Temporarily, we replace f_{n-1} by $f_{n-1} - \sum_{j=1}^{n-2} a_j f_j$. Then D and a remain unchanged, and after this substitution we have that

2.3.2 $$f_j(Q) = D_i(f_{n-1})(Q) = 0, \quad 1 \le j \le n-1,\ 1 \le i \le n.$$

Now make the change in coordinates

2.3.3 $$\bar{x}_i = \sum_{l=1}^{n} D_l(f_i)(Q)\cdot x_l,\ 1 \le i \le n-2,\ \bar{x}_{n-1} = x_{n-1},\ \bar{x}_n = x_n.$$

Then Q remains $(0, \ldots, 0)$ and (2.3.2) still holds. By the chain rule we have for all $h \in L$,

2.3.4 $$D_i(h) = \sum_{j=1}^{n-2} D_i(f_j)(Q)\cdot h_{\bar{x}_j} \qquad \text{for} \quad 1 \le i \le n-2.$$

and $$D_i(h) = \sum_{j=1}^{n-2} D_i(f_j)(Q)\cdot h_{\bar{x}_j} + h_{\bar{x}_i} \qquad \text{for} \quad n-1 \le i \le n.$$

Then $D =$

$$\det\begin{bmatrix} \sum_{j=1}^{n-2} D_1(f_j)(Q)\cdot\frac{\partial}{\partial \bar{x}_j} & \cdots\cdots & \sum_{j=1}^{n-2} D_{n-1}(f_j)(Q)\cdot\frac{\partial}{\partial \bar{x}_j} + \frac{\partial}{\partial \bar{x}_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sum_{j=1}^{n-2} D_1(f_j)(Q)\cdot(f_{n-2})_{\bar{x}_j} & \cdots\cdots & \sum_{j=1}^{n-2} D_{n-1}(f_j)(Q)\cdot(f_{n-2})_{\bar{x}_j} + (f_{n-2})_{\bar{x}_n} \\ \sum_{j=1}^{n-2} D_1(f_j)(Q)\cdot(f_{n-1})_{\bar{x}_j} & \cdots\cdots & \sum_{j=1}^{n-2} D_{n-1}(f_j)(Q)\cdot(f_{n-1})_{\bar{x}_j} + (f_{n-1})_{\bar{x}_n} \end{bmatrix}$$

$$\det\begin{bmatrix} \frac{\partial}{\partial \bar{x}_1} & \frac{\partial}{\partial \bar{x}_2} & \cdots & \frac{\partial}{\partial \bar{x}_n} \\ (f_1)_{\bar{x}_1} & (f_1)_{\bar{x}_2} & \cdots & (f_1)_{\bar{x}_n} \\ \cdots & \cdots & \cdots\ \cdots & \\ (f_{n-1})_{\bar{x}_1} & (f_{n-1})_{\bar{x}_2} & \cdots & (f_{n-1})_{\bar{x}_n} \end{bmatrix}$$

$$\begin{bmatrix} D_1(f_1)(Q) & \cdots & D_{n-2}(f_1)(Q) & D_{n-1}(f_1)(Q) & D_n(f_1)(Q) \\ D_1(f_2)(Q) & \cdots & D_{n-2}(f_2)(Q) & D_{n-1}(f_2)(Q) & D_n(f_2)(Q) \\ \vdots & & \vdots & \vdots & \vdots \\ D_1(f_{n-2})(Q) & \cdots & D_{n-2}(f_{n-2})(Q) & D_{n-1}(f_{n-2})(Q) & D_n(f_{n-2})(Q) \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

$= M_{n-1}(Q)\,[\bar{J}(\ ,f_1,\dots,f_{n-1})]$, where $\bar{J}$ is the determinant of the jacobian matrix with respect to $\bar{x},\cdots,\bar{x}_n$.

Let $\bar{D} = \bar{J}(\ ,f_1,\dots,f_{n-1})$ and $\bar{a}$ be such that $\bar{D}^p = \bar{a}\bar{D}$ (see (2.1)).

Then $a = (M_{n-1}(Q))^{p-1}\bar{a}$. Then $a(Q) = (M_{n-1}(Q))^{p-1}\bar{a}(Q)$. We have by (2.1) and (2.3.2) that $\bar{a}(\bar{Q}) = (-1)^n\bar{\nabla}((f_1\cdots f_n)^{p-1})(Q)$ where $\bar{\nabla} = \dfrac{\partial^{n(p-1)}}{\partial\bar{x}_1^{p-1}\cdots\partial\bar{x}_n^{p-1}}$. Also by the change in coordinates (2.3.3) we have

2.3.5. $f_i = \bar{x}_i + f_i^+$, $1 \le i \le n-2$, where $f_i^+ = f_i -$ (leading form of f_i) and

$$f_{n-1} = \frac{1}{2}\sum_{i,j=1}^{n}(f_{n-1})_{\bar{x}_i\bar{x}_j}(Q)\bar{x}_i\bar{x}_j + f_{n-1}^+ \quad \text{if} \quad p > 2,$$

$$f_{n-1} = \sum_{i<j}(f_{n-1})_{\bar{x}_i\bar{x}_j}(Q)\bar{x}_i\bar{x}_j + f_{n-1}^+) \quad \text{if} \quad p = 2.$$

Thus the initial form of $(f_1\cdots f_{n-1})^{p-1}$ is

2.3.6. $(\bar{x}_1\cdots\bar{x}_{n-2})^{p-1}((f_{n-1})_{\bar{x}_{n-1}\bar{x}_{n-1}}(Q)\dfrac{\bar{x}_{n-1}^2}{2} + (f_{n-1})_{\bar{x}_n\bar{x}_n}(Q)\dfrac{\bar{x}_n^2}{2} + (f_{n-1})_{\bar{x}_{n-1}\bar{x}_n}$ $(Q)\cdot\bar{x}_{n-1}\bar{x}_n)^{p-1} + g$, where g is homogeneous of degree $n(p-1)$, with

$$deg_{\bar{x}_{n-1}\bar{x}_n}(g) < 2(p-1), \text{ if } p > 2.$$

If $p = 2$, the expression $(f_{n-1})_{\bar{x}_{n-1}\bar{x}_{n-1}}(Q)\dfrac{\bar{x}_{n-1}^2}{2} + (f_{n-1})_{\bar{x}_n\bar{x}_n}(Q)\dfrac{\bar{x}_n^2}{2}$ is deleted from (2.3.6).

It then follows that if $p > 2$,

2.3.7. $\bar{a}(Q) = (-1)^n\dfrac{\partial^{2(p-1)}}{\partial\bar{x}_{n-1}^{p-1}\partial\bar{x}_n^{p-1}}\cdot$

$$\left[(f_{n-1})_{\bar{x}_{n-1}\bar{x}_{n-1}}(Q)\frac{\bar{x}_{n-1}^2}{2} + (f_{n-1})_{\bar{x}_n\bar{x}_n}(Q)\frac{\bar{x}_n^2}{2} + (f_{n-1})_{\bar{x}_{n-1}\bar{x}_n}(Q)\bar{x}_{n-1}\bar{x}_n\right]^{p-1}$$

$$= \left[(f_{n-1})^2_{x_{n-1}x_n} - (f_{n-1})_{\bar{x}^{n-1}\bar{x}_{n-1}}(f_{n-1})_{\bar{x}_n\bar{x}_n}\right]^{(p-1)/2}(Q) \text{ by}$$

(2.4) below. If $p = 2$, $\bar{a}(Q) = (f_{n-1})^{p-1}_{\bar{x}_{n-1}\bar{x}_n}(Q)$. Therefore $a(Q) = [(M_{n-1}(f_{n-1})_{\bar{x}_{n-1}\bar{x}_n})^2 - (M_{n-1}(f_{n-1})_{\bar{x}_{n-1}\bar{x}_{n-1}})(M_{n-1}(f_{n-1})_{\bar{x}_n\bar{x}_n})]^{(p-1)/2}(Q)$.

We have that

$$\begin{bmatrix}\bar{x}_1\\ \vdots\\ \bar{x}_n\end{bmatrix} = C(Q)\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$$

Therefore $$\begin{bmatrix} \dfrac{\partial}{\partial \bar{x}_1} \\ \vdots \\ \dfrac{\partial}{\partial \bar{x}_n} \end{bmatrix} = C^*(Q) \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{bmatrix}$$

We then obtain

$$a(Q) = [(M_{n-1}E_1E_2(f_{n-1}))^2 - (M_{n-1}E_1^2(f_{n-1}))(M_{n-1}E_2^2(f_{n-1}))]^{(p-1)/2}(Q)$$

Now back substitute to replace f_{n-1} by $f_{n-1} + \sum_{r=1}^{n-2} \frac{M_r(Q)}{M_{n-1}(Q)} f_r$ to obtain $a(Q) = (H(Q))^{(p-1)/2}$.

Now if Q were a point such that rank $[D_i(f_j)(Q)] < n-2$ then we could have assumed that the first $n-2$ rows of $[D_i(f_j)(Q)]$ are dependent as well. Then clearly $M_{n-1}(Q) = 0$ and by an argument similar to that used above, we obtain $\bar{a}(Q) = 0$. Thus we obtain $a(Q) = (H(Q))^{(p-1)/2}$, for all $Q \in S$.

2.4. Lemma. *Let k be a ring of characteristics $p > 0$. Let A, B and $C \in k$ and $F = Ax^2 + By^2 + Cxy \in k[x, y]$. Then* $\dfrac{\partial^{2p-2}(F^{p-1})}{\partial x^{p-1} \partial y^{p-1}} = (C^2 - 4AB)^{(p-1)/2}$.

Proof. The coefficient of $x^{p-1}y^{p-1}$ in F^{p-1} if $p > 2$ is

$$\sum_{i=0}^{(p-1)/2} \binom{p-1}{2i}\binom{2i}{i} C^{p-1-2i}(AB)^i = \sum_{i=0}^{(p-1)/2} (-1)^{2i}\binom{2i}{i} C^{p-1-2i}(AB)^i$$

$$= \sum_{i=0}^{(p-1)/2} \binom{2i}{i} C^{p-1-2i}(AB)^i = \sum_{i=0}^{(p-1)/2} \binom{2i}{i} (C^2)^{[(p-1)/2]-i}(AB)^i$$

$$= \sum_{i=0}^{(p-1)/2} (-1)^i \binom{(p-1)/2}{i} 2^{2i}(C^2)^{[(p-1)/2]-i}(AB)^i$$

$$= \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i} (C^2)^{(p-1)/2-i}(-4AB)^i$$

$= (C^2 - 4AB)^{(p-1)/2}$. If $p = 2$, the lemma is clearly true.

§ 3. The Fixed Subring of a Polynomial Ring

Let k, A, A', L, L', and $D = J(\ ,f_1, \ldots, f_{n-1})$ be as in Section 2. Let $\mathscr{L}$ be the additive group of logarithmic derivatives of D in A, $\mathscr{L} = \{h^{-1}Dh \in A : h \in L\}$. Assume (*) holds. Let $f = f_1 \cdots f_{n-1}$. For $I = (i_1, \cdots, i_{n-1}) \in \mathbb{F}_p^{n-1}$, let $f^I = f_1^{i_1} \cdots f_{n-1}^{i_{n-1}}$.

3.1. Lemma. *Let $F \subset A_k^{2n-1}$ be the variety defined by the equations* w_i^p

$= f_i(x_1, \ldots, x_n)$, $1 \leq i \leq n-1$. *Then the coordinate ring of F is isomorphic to A'.*

Proof. Let $\tilde{A} = k[x_1, \ldots, x_n, w_1, \ldots, w_{n-1}]$. Let $\phi: \tilde{A} \to A'$ be the ring homomorphism that sends x_i to x_i^p, w_j to f_j and α to α^p for all $1 \leq i \leq n$, $1 \leq j \leq n-1$, $\alpha \in k$. (Note ϕ is not a k-homomorphism.) Then $w_j^p - f_j \in \ker \phi$. Let $Q \subseteq \tilde{A}$ be the ideal generated by $w_j^p - f_j$, $1 \leq j \leq n-1$. Then Q is a prime ideal in $\tilde{A}$ of height $n-1$. Therefore $\ker \phi = Q$ by ([7] page 6, 18A).

3.2. Lemma. $D^{-1}(0) \cap A = A'$.

Proof. Let $B = D^{-1}(0) \cap A$. Then $k[x_1^p, \ldots, x_n^p] \subset A' \subseteq B \subseteq A$. B is integral over A'. For each i, $f_{i+1} \notin k(x_1^p, \ldots, x_n^p, f_1, \ldots, f_i)$ by (*). Thus $[L': k(x_1^p, \ldots, x_n^p)] = p^{n-1}$. Also by (*), $D_{x_i} \neq 0$ for some i. Therefore the quotient field of B is not L and A' and B have the same quotient field. Since A' is normal, $A' = B$.

3.3. Lemma. $Cl(A') \simeq \mathscr{L}$.

Proof. By (*) the image of D is not contained in any height one prime of A. By (1.2) and (3.2), $Cl(A') \simeq \mathscr{L}$ since the units of A are the nonzero elements of k.

3.4. Lemma. *Let $t \in \mathscr{L}$. Then $deg(t) \leq M - n$, where $M = \sum_{i=1}^{n-1} deg(f_i)$.*

Proof. $t \in \mathscr{L}$ implies there exists a $g \in L$ such that $g^{-1}Dg = t$. Multiplying g by an element of A^p, if necessary, we may assume $g \in A$. $deg(Dg) \leq (deg\, g - 1) + \sum_{i=1}^{n-1} deg(f_i - 1) = deg\, g + M - n$. Therefore $deg(t) \leq M - n$.

3.5. Lemma. *Let $\mathscr{G} = \{t \in A : \nabla(f^{p-1}t) = (-1)^n t^p\}$, where $f = f_1 \cdots f_{n-1}$ and $\nabla = D_1^{p-1} \cdots D_n^{p-1}$. Then $\mathscr{G}$ is a p-group of type $(p, \ldots, p)$ of order p^N with $N \leq \binom{M}{n}$.*

Proof. Let $t \in \mathscr{G}$. $\nabla(f^{p-1}t) = (-1)^n t^p$ implies $p\, deg\, t \leq (p-1)\, deg(f) + deg(t) - n(p-1)$. Thus $deg(t) \leq M - n$. Write $t = \sum_{|J| \leq M-n} \alpha_J x^J$ where for $J = (i_1, \ldots, i_n) \in (\mathbb{Z}^+)^n$, $x^J = x_1^{i_1} \cdots x_n^{i_n}$ and $|J| = \sum_{j=1}^{n} i_j$. Comparing coefficients on both sides of the quality $\nabla(f^{p-1}t) = (-1)^n t^p$ we obtain for each J_0 with $|J_0| \leq M - n$ an equation of the form $L_{J_0} = \alpha_{J_0}^p$, where L_{J_0} is a linear expression in the α_J with coefficients in k.

There are a total of $\binom{n + (M-n)}{n} = \binom{M}{n}$ such equations. The ring R

$= k[\alpha_J]_{|J| \le M-n}$ with the relations $L_J = \alpha_J^p$ is a finite dimensional k-vector spaced spanned by all monomials in the α_J of degree less than or equal to $(p-1)\binom{M}{n}$. Thus R has a finite number of maximal ideals ([10], p. 89).

Thus the $\binom{M}{n}$ equations $L_J = \alpha_J^p$ intersect at a finite number of points. There is no solution to these equations at infinity. By Bezout's Theorem this number is at most $p^{\binom{M}{n}}$ ([6], p. 670). Therefore $\mathscr{G}$ is of order at most $p^{\binom{M}{n}}$. $\mathscr{G}$ is a p-group of type $(p, \ldots, p)$ since $\mathscr{G} \subset A$.

3.6. Proposition. *Let $F \subset A_k^{2n-1}$ be the variety defined by $w_i^p = f_i(x_1, \ldots, x_n)$, $1 \le i \le n-1$. Then $Cl(F)$ is a finite p-group of type $(p, \ldots, p)$ of order p^N where $N \le \binom{M}{n}$.*

Proof. By (1.2), (3.1), (3.2), and (3.3), $Cl(F) \simeq \mathscr{L}$. By (1.3b) an element $t \in A$ is in $\mathscr{L}$ if and only if $D^{p-1}t - at = -t^p$. By (2.1) $t \in \mathscr{L}$ if and only if

3.6.1. $\quad V(f^{p-1}t) = (-1)^n t^p$ and $V(f^J t) = 0$

for all $J \in \mathbb{F}_p^{n-1}$ with $J \ne (p-1, \ldots, p-1)$.

Thus $\mathscr{L} \subset \mathscr{G}$. Now use (3.5).

3.7. Lemma. *Let $f \in A$ be such that $s^{-1}Ds \in A$. Assume $s = g^r h$, where $g \in A$ is irreducible, $r \ne 0 (\mathrm{mod}\ p)$ is a positive integer and $h \in A$ is relatively prime to g. Then $g^{-1}Dg \in A$.*

Proof. Let $t = s^{-1}Ds$. Then $st = Ds = rg^{r-1}h\,Dg + g^r D(h)$. Then g divides $rhDg$ and hence g divides Dg,

§ 4. Examples

4.1. Remark. From the proof of (3.6) we see that the calculation of $Cl(X)$ is equivalent to determining the number of solutions to a corresponding system of equations of the form

4.1.1. $L_J = \alpha_J^p$, $L_{J'} = 0$ where the J, $J' \in \mathbb{F}_p^{n-1}$ and the L_J and $L_{J'}$ are linear expressions in the α_J.

[1] provides an algorithm for finding the number of solutions to such a p-linear system of equations and a computer program for determining this number when the coefficients of the f_i belong to a finite field, so that the computation of $Cl(F)$ in this case is a programmable process.

4.2. Remark. Let h_i, $1 \le i \le n-1$, be homogeneous elements of A of degree s_i

with $s_i \neq 0 (\text{mod } p)$. If the h_i satisfy (*), then for each pair (i, j) with $i \neq j$, h_i and h_j have no common factors in A and each h_i has no multiple factors in A. Let $X \subset A_k^{2n-1}$ be defined by the equations $w_i^p = h_i(x_1, \dots, x_n)$ $1 \leq i \leq n-1$. The next example studies $Cl(X)$.

4.3. Example. By (4.2) each $h_i = H_{i1} \cdots H_{ir_i}$, where the H_{ij} are distinct irreducible homogeneous elements of A. Let $D = J(\ , h_1, \cdots, h_{n-1})$ and $\mathcal{L}$ the group of logarithmic derivatives of D in A. Let $h = h_1 \cdots h_{n-1}$ and $M = deg\ h$. Let $t \in \mathcal{L}$. By (3.6.1)

4.3.1. $$\nabla(h^{p-1}t) = (-1)^n t^p.$$

Assume that the lowest degree form of t is of degree s and the highest degree form of t is of degree m. Compare the lowest and highest degree forms on both sides of the equality in (4.3.1) we obtain $ps \geq (p-1)M + s - n(p-1)$ and $pm \leq (p-1)M + m - n(p-1)$. Then $m \leq M - n \leq s$ and hence t is homogeneous of degree $M - n$. Repeat the same argument used in the proof of (3.5) to obtain $|Cl(X)| = p^s$ with $s \leq \binom{M-1}{n-1}$.

Now assume that the h_i satisfy the additional condition that the variety $Y \subseteq A_k^n$ defined by $h_1 = \cdots = h_{n-1} = 0$ has a finite number of singularities. (When $n = 2$, this condition is implied by (*).) For each pair (i,j), $1 \leq i \leq n-1$, $1 \leq j \leq r_i$, let $t_{ij} = H_{ij}^{-1} D(H_{ij})$. By (3.7), $t_{ij} \in \mathcal{L}$ for each (i, j).

4.3.2. Claim. *The t_{ij} are $\mathbb{F}_p$-independent.*

Assume $d_{ij} \in \mathbb{F}_p$ and $\sum d_{ij} t_{ij} = 0$. Let $H = \prod H_{ij}^{d_{ij}}$. Then $DH = 0$. Noting that $Dh_i = 0 (1 \leq i \leq n-1)$, we may assume that $d_1 r_1 = \ldots = d_{n-1} r_{n-1} = 0$. By Euler's formula the determinant of the matrix

$$\begin{bmatrix} D_1(H) & \cdots & D_{j-1}(H) & sH & D_{j+1}(H) & \cdots & D_n(H) \\ D_1(h_1) & \cdots & D_{j-1}(h_1) & s_1 h_1 & D_{j+1}(h_1) & \cdots & D_n(h_1) \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ D_1(h_{n-1}) & \cdots & D_{j-1}(h_{n-1}) & s_{n-1} h_{n-1} & D_{j+1}(h_{n-1}) & \cdots & D_n(h_{n-1}) \end{bmatrix}$$

column j ——↑

is 0 for each $j = 1, \dots, n$, where $s = deg(H)$.

This shows that either $s \equiv 0 (\text{mod } p)$ or Y has an infinite number of singularities satisfying the equation $H_{1r_1} = \cdots = H_{n-1 r_{n-1}} = 0$. Thus $s \equiv 0$ (mod p).

If some $d_{ij} \neq 0$ we may assume without loss of generality that $d_{11} \neq 0$. Let $H_0 = h_1^{p-d_{11}} H$. Let H' be obtained from H_0 by factoring out all p-th powers. Then $deg(H') \equiv s_1(p - d_{11}) \neq 0$ (mod p) and the factors H_{11},

$H_{2r_2}, \ldots, H_{n-1r_{n-1}}$ do not appear in H'. Repeat the above argument to obtain $d_{ij} = 0$ for all $2 \leq i \leq n-1$ and that the exponent of H_{1r_1} in H' must also be 0. But this implies that $p - d_{11} = 0$. Contradiction!

Thus if we let m be the number of factors in $h = h_1 \cdots h_{n-1}$ we have that the order of $Cl(X)$ is p^r for some r with $m - n + 1 \leq r \leq \binom{M-1}{n-1}$.

4.4. Remark. When $n = 2$, (4.3) implies that $r = m - 1$, which was first proved in [8].

4.5. Example. Let $f_i(x_n), g_i(x_n) \in k[x_n]$, $1 \leq i \leq n-1$. Let $f(x_n) = f_1(x_n) \cdots f_{n-1}(x_n)$. Assume $f(x_n)$ has r distinct roots, $\theta_1, \ldots, \theta_r$. For each i, let $F_i = x_i f_i(x_n) + x_n g_i(x_n)$. Assume the F_i satisfy (*). Let $D = J(\ , F_1, \ldots, F_{n-1})$ and $\mathscr{L}$ be the group of logarithmic derivatives of D in A. We will show that $\mathscr{L}$ is of order p^r generated by the logarithmic derivatives, $D(x_n - \theta_j)/(x_n - \theta_j)$, $1 \leq j \leq r$ in A. Thus the group of Weil divisors of the variety defined by the equations, $w_i^p = F_i$, $1 \leq i \leq n-1$, will be a direct sum of r copies of $\mathbb{F}_p$.

Let $t \in \mathscr{L}$. Given $h \in A$, $deg_{x_i}(Dh) \leq deg_{x_i}(h)$, $1 \leq i \leq n-1$. Therefore $deg_{x_i}(t) = 0$ for $1 \leq i \leq n-1$. Thus $t \in k[x_n]$. Let Δ be the k-derivation on $k(x_1, \ldots, x_n)$ defined by $\Delta = t^{-1}D$. By Hochschild's formula ([11], pg. 64, (3.2)), $\Delta^p = \Delta$. Hence $(\Delta - (p-1)I) \cdots (\Delta - 2I)(\Delta - I)\Delta = \Delta^p - \Delta = 0$, where I is the identity mapping of $k(x_1, \ldots, x_n)$ into $k(x_1, \ldots, x_n)$. Clearly $\Delta(x_n) \neq 0$. Set $y_1 = \Delta(x_n)$, $y_2 = (\Delta - I)y_1, \ldots, y_p = (\Delta - (p-1)I)y_{p-1} (= 0)$. First we observe that if $x \in k(x_n)$ then $\Delta(x) \in k(x_n)$. Hence $y_1, \ldots, y_{p-1} \in k(x_n)$. Next we have that for some $l = 2, \ldots, p-1$, $y_{l-1} \neq 0$ and $y_l = (\Delta - (l-1)I)\ y_{l-1} = 0$. Therefore $\Delta(y_{l-1}) = (l-1)y_{l-1}$, which implies that $D(y_{l-1})/y_{l-1} = (l-1)t$. Let q be the inverse of $l-1$ modulo p. Let $y = y_{l-1}^q$. Then $D(y)/y = t$. Thus we've shown that there exists $y \in k(x_n)$ such that $Dy/y = t$. Multiplying y by an element of $k[x_n^p]$, if necessary, we may assume $y \in k[x_n]$.

Factor y into a product of linear factors, $y = (x_n - \alpha_1)^{s_1} \cdots (x_n - \alpha_m)^{s_m}$ where $\alpha_1, \ldots, \alpha_m \in k$ are pairwise distinct. If $s_i \geq p$ for some s_i, then $(x_n - \alpha_i)^{-p}y$ will yield the same logarithmic derivative as y, so we may assume that $1 \leq s_i \leq p-1$ for each s_i. By (3.7), $D(x_n - \alpha_i)/x_n - \alpha_i \in \mathscr{L}$ for each $i = 1, \ldots, m$. But for each i, $D(x_n - \alpha_i) = D(x_n) = (-1)^{n+1} f(x_n)$. Therefore $x_n - \alpha_i$ is a factor of $f(x_n)$ in $k[x_n]$. We conclude that $\alpha_i \in \{\theta_1, \ldots, \theta_r\}$ for each $i = 1, \ldots, m$. Thus $t = D(y)/y = \sum_{i=1}^{m} s_i(D(x_n - \alpha_i)/(x_n - \alpha_i))$ belongs to the $\mathbb{F}_p$-space spanned by $\{D(x_n - \theta_i)/(x_n - \theta_i) \colon 1 \leq i \leq r\}$. These polynomials are easily seen to be $\mathbb{F}_p$-independent. Thus $\mathscr{L}$ has order p^r.

§5. Purely Inseparable Covers of Dimension Two Factorial Domains

Let $g, f_1, \dots, f_{n-2} \in A = k[x_1, \dots, x_n]$, where k is algebraically closed of characteristic $p \neq 0$. Let $D = J(\ , g, f_1, \dots, f_{n-2})$. Assume that the ideal $P = (f_1, \dots, f_{n-2})$ is a height $n-2$ prime ideal in A. Let $B = A/P$. For $f \in A$, denote its image in B by $\bar{f}$. Then $B = k[\bar{x}_1, \dots, \bar{x}_n]$. Let $C = B^p[\bar{g}] = k[\bar{x}_1^p, \dots, \bar{x}_n^p, \bar{g}]$. Denote by $\bar{L}$ and $\bar{K}$ the quotient field of B and C, respectively. D will induce a k-derivation, $\bar{D}$, on $\bar{L}$. Throughout this section assume (**) (See (0.10).). Let $W \subseteq A_k^{n+1}$ be the variety defined by the equations $f_1 = \cdots = f_{n-2} = w^p - g = 0$.

5.1. Lemma. (i) $\bar{D}^{-1}(0) \cap B = C$, (ii) *C is isomorphic to the coordinate ring of W*, (iii) $[\bar{L}: \bar{K}] = p$, (iv) $\bar{D}(B)$ *is not contained in any height one prime of B.*

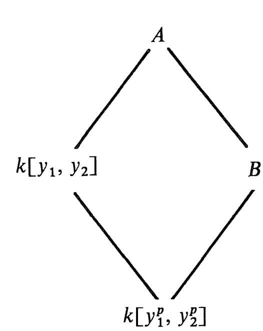
Proof. Consider the surjection $\phi: A[w] \to C$ given by $x_i \to \bar{x}_i^p$, $1 \leq i \leq n$, $w \to \bar{g}$, and $\alpha \to \alpha^p$, for all $\alpha \in k$. Then the ideal $I \subseteq A[w]$ generated by $f_1, \dots, f_{n-2}$, $w^p - g$ is contained in ker ϕ and is a prime ideal of height $n-1$ since $\bar{g} \neq \bar{h}^p$ for any $\bar{h} \in B$ by assumption. Since the dimension of C is 2, the height of ker ϕ is $n-1$. Thus $A[w]/I \cong C$, which proves (ii).

We have $B^p \subseteq C \subseteq \bar{D}^{-1}(0) \cap B \subseteq B$ and $[\bar{K}: \bar{L}^p] = p$. By lemma (5.2)(below), $[\bar{L}: \bar{L}^p] = p^2$. Therefore C and $\bar{D}^{-1}(0) \cap B$ have the same quotient field. By (ii) C is normal, which gives $C = \bar{D}^{-1}(0) \cap B$. Also $[\bar{L}: \bar{K}] = [\bar{L}: \bar{L}^p]/[\bar{K}: \bar{L}^p] = p$. Hence (iii).

(iv) is immediate from the assumption on W.

5.2. Lemma. *Let k be a perfect field of characteristic* $p \neq 0$. *Let A be a finitely generated k-integral domain of dimension* 2. *Let* $B = A^p$. *Then the degree of A over B is* p^2.

Proof. $A = k[u_1, \dots, u_n]$ for some $u_i \in A$. Then $B = k[u_1^p, \dots, u_n^p]$. By Noether's normalization theorem there exists $y_1, y_2 \in A$ such that A is separably algebraic over $k[y_1, y_2]$ and y_1, y_2 are algebraically independent over k. We then have the diagram of inclusions



Let L, L' be the quotient fields of A, B, respectively. Clearly $[L: k(y_1, y_2)] = [L': k(y_1^p, y_2^p)]$ and the result follows.

5.3. Corollary. *Let C be as in (5.1). $Cl(C) \cong \bar{\mathscr{L}}/\bar{\mathscr{L}}'$, where $\bar{\mathscr{L}} = \{\bar{f}^{-1}\bar{D}(\bar{f}) \in B\}$, $\bar{\mathscr{L}}' = \{\bar{u}^{-1}D(\bar{u})$: $\bar{u}$ is a unit in $B\}$.*

Proof. Use (5.1) and (1.2).

Throughout the remainder of this section assume that each f_i is homogeneous of degree s_i, $1 \le i \le n-2$ and g is homogeneous of degree $s \neq 0 \pmod p$. Let $M = s + \sum_{i=1}^{n-2} s_i$.

5.4. Lemma. *Let $\bar{w} \in \bar{\mathscr{L}}$. Then there exists homogeneous $t \in A$ of degree $M-n$ such that $\bar{t} = \bar{w}$.*

Proof. Let $w \in A$ be a representative of $\bar{w}$ of minimal degree. Let $deg(w) = d$. Then $w = \sum_{j=0}^{d} w_j$, where $w_j \in A$ is homogeneous of degree j. Note $\bar{w}_d \neq 0$ by minimality. Let $a \in A$ be such that $D^p = aD$. Then $\bar{D}^p = \bar{a}\bar{D}$. By (1.3) $\bar{D}^{p-1}(\bar{w}) - \bar{a}\bar{w} + \bar{w}^p = 0$. Then $\sum_{j=0}^{d} (D^{p-1}(w_j) - aw_j + w_j^p) \in P$. P being homogeneous implies that $w_d \in P$ or $D^{p-1}(w_j) - aw_j + w_d^p \in P$ for some $j = 0, 1, \ldots, d$ with $deg(D^{p-1}w_j - aw_j) = deg(w_d^p)$. (Note if h is homogeneous of degree r, then $D^{p-1}h - ah$ is homogeneous of degree $(p-1)(M-n) + r$ or $D^{p-1}h - ah = 0$.) Since $\bar{w}_d \neq 0$, it must be that $pd = (p-1)(M-n) + j$ for some $j = 0, 1, \ldots, d$. Then $pd \le (p-1)(M-n) + d$, which implies that $d \le M - n$.

5.4.1. The two sets, $\{j: \bar{D}^{p-1}\bar{w}_j - \bar{a}\bar{w}_j \neq 0\}$ and $\{j: \bar{w}_j \neq 0\}$, have the same number of elements since $\sum_{j=0}^{d} \bar{D}^{p-1}\bar{w}_j - \bar{a}\bar{w}_j + \bar{w}_j^p = 0$ and P is homogeneous.

This shows that $\bar{D}^{p-1}\bar{w}_d - \bar{a}\bar{w}_d \neq 0$. (Note $\bar{w}_{j=0} \Rightarrow \bar{D}^{p-1}\bar{w}_j - \bar{a}\bar{w}_j = 0$.) Therefore $D^{p-1}w_d - aw_d$ and w_d^p have the same degree and $\bar{D}^{p-1}\bar{w}_d - \bar{a}\bar{w}_d + \bar{w}_d^p = 0$. Thus $\bar{w}_d \in \bar{\mathscr{L}}$ by (1.3). Then $\bar{w} - \bar{w}_d = \sum_{j=1}^{d-1} \bar{w}_j \in \bar{\mathscr{L}}$. Repeat the same argument beginning with (5.4.1) to obtain $\bar{w}_j \in \bar{\mathscr{L}}$, $j = 1, \ldots, d$. If $\bar{w}_j \neq 0$ then this implies that $D^{p-1}w_j - aw_j$ and w_j^p have the same degree, but this is only possible if $j = M - n$. Thus it must be that $d = M - n$, $\bar{w}_j = 0$ for $j < M - n$ and $\bar{w} = \bar{w}_d$.

5.5. Lemma. *Let $\bar{w} \in \bar{\mathscr{L}}$. Then there exists homogeneous $y \in A$ such that $\bar{y} \neq 0$ and $\bar{y}^{-1}\bar{D}(\bar{y}) = \bar{w}$.*

Proof. By (5.4) we may assume w is homogeneous of degree $M - n$. $\bar{\Delta} = \bar{w}^{-1}\bar{D}$. By (**), $\bar{D}(\bar{x}_r) \neq 0$ for some $r = 1, \ldots, n$. Let $\bar{y}_1 = \bar{\Delta}(\bar{x}_r)$. For

$2 \le j \le p-1$, let $\bar{y}_j = \bar{\Delta}(\bar{y}_{j-1}) - (j-1)\bar{y}_{j-1}$. Then for some j, $\bar{y}_j^{-1}\bar{D}(\bar{y}_j) = (j-1)\bar{w}$ by ([11], pg 64), proof of (3.2)). Note also that $\bar{y}_j$ is of the form $\bar{v}^{-1}\bar{u}$ where $u, v \in A$ are homogeneous with $deg(u) = deg(v) + 1$. Multiply $\bar{y}_j$ by $\bar{v}^p$ to obtain a homogeneous $h \in A$ such that $\bar{h}^{-1}\bar{D}(\bar{h}) = (j-1)\bar{w}$. Choose $m \in \mathbb{F}_p$ such that $m(j-1) = 1$. Then $y = h^m$ has the desired property.

5.6. Remark. Assume that B is a unique factorization domain and that $\bar{y} \in B$ is irreducible homogeneous such that $\bar{w} = \bar{y}^{-1}\bar{D}(\bar{y}) \in \mathscr{L}$ and $\bar{w} \neq 0$. Then $\bar{x}_1\bar{w}\bar{y} =$

$$\det\begin{bmatrix} \bar{x}_1\bar{D}_1(\bar{y}) & \bar{D}_2(\bar{y}) \cdots & \bar{D}_n(\bar{y}) \\ \bar{x}_1\bar{D}_1(\bar{g}) & \bar{D}_2(\bar{g}) \cdots & \bar{D}_n(\bar{g}) \\ \vdots & \vdots & \vdots \\ \bar{x}_1\bar{D}_1(\bar{f}_{n-2}) & \bar{D}_2(\bar{f}_{n-2}) \cdots & \bar{D}_n(\bar{f}_{n-2}) \end{bmatrix}$$

$$= \det\begin{bmatrix} e\bar{y} & \bar{D}_2(\bar{y}) & \cdots & \bar{D}_n(\bar{y}) \\ s\bar{g} & \bar{D}_2(\bar{g}) & \cdots & \bar{D}_n(\bar{g}) \\ \vdots & \vdots & & \vdots \\ s_{n-2}\bar{f}_{n-2} & \bar{D}_2(\bar{f}_{n-2}) & \cdots & \bar{D}_n(\bar{f}_{n-2}) \end{bmatrix}$$

by Euler's formula, where $e = deg(y)$.

Therefore $\bar{x}_1\bar{w}\bar{y} = e\bar{y}\, M_{11} + s\bar{g}M_{21}$, where M_{11} and M_{21} are the cofactors of $e\bar{y}$ and $s\bar{g}$ in the matrix. Thus $\bar{y}$ divides $\bar{g}$ or M_{21} (recall $s \neq 0$). Similarly, if $\bar{y}$ does not divide $\bar{g}$, then $\bar{y}$ divides M_{2j}, $1 \le j \le n$.

Let $\Delta_j (1 \le j \le n)$ be the derivation on B defined by $\Delta_j =$

$$\det\begin{bmatrix} D_1 & \cdots & D_{j-1} & D_{j+1} & \cdots & D_n \\ D_1(f_1) & & D_{j-1}(f_1) & D_{j+1}(f_1) & \cdots & D_n(f_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ D_1(f_{n-2}) & & D_{j-1}(f_{n-2}) & D_{j+1}(f_{n-2}) & \cdots & D_n(f_{n-2}) \end{bmatrix}$$

Then $\bigcap_{j=1}^{n} \Delta_j^{-1}(0) \cap B = B^p$ since $\bar{g} \notin \bigcap_{j=1}^{n} \Delta_j^{-1}(0) \cap B$ by (5.1). Also $\bar{y}^{-1}\Delta_j(\bar{y}) \in B$ for $1 \le j \le n$. At this point, in order to arrive at a definitive description of $Cl(W)$ analogous to ([1], page 398, (3.2)), a condition must be added to (**) to exclude the possibility that $\bar{y}^{-1}\Delta_j(\bar{y}) \notin B$ $(1 \le j \le n)$. Hence

5.7. Theorem. *Suppose B is a unique factorization domain and that $\bar{g}$ factors in B into a product of $q+1$ distinct prime elements. Assume that either*

(i) *for each $i = 1, \ldots, n$, $\bar{x}_i \notin B^p$ and the variety defined by the equations $w^p - x_i = f_1 = \cdots = f_{n-2} = 0$ in A_k^{n+1} defines a unique factorization domain, or*

(ii) $\mathrm{End}_C(B) = B[G]$, *where $B[G]$ denotes the C-subalgebra of* $\mathrm{End}_C[B]$ *generated by B and $G = \langle \Delta_1, \ldots, \Delta_n \rangle$.*

Then the divisor class group of W is a direct sum of q copies of $\mathbb{Z}/p\mathbb{Z}$.

Some preliminary lemmas are required.

5.8. Lemma. *Assume that B is a unique factorization domain and that* $\bar{g}$ *factors in B into a product of* $q+1$ *distinct prime elements. Then there exists homogeneous polynomials* $g_1, \dots, g_{q+1} \in A$ *such that the decomposition of* $\bar{g}$ *in B into prime elements is given by* $\bar{g} = \bar{g}_1 \cdots \bar{g}_{q+1}$.

Proof. Suppose $\bar{g} = \bar{w}_1\bar{w}_2$ for some $\bar{w}_1, \bar{w}_2 \in B$. We'll show that we may choose the representatives w_1, w_2 so that they are homogeneous in A. Let $w_1 = u_0 + \cdots + u_d$, $w_2 = v_0 + \cdots + v'_d$, where u_i, v_i denote the forms of w_1, w_2 of degree i and j, respectively. Then $g - w_1w_2 \in P$. Let $r = deg(g)$. Then $\sum_{i+j=e} u_iv_j \in P$ for all $0 \leq e \leq d + d'$ with $e \neq r$. Let i_0 be minimal such that $u_{i_0} \notin P$ and j_0 be minimal such that $v_{j_0} \notin P$. Let $i_0 + j_0 = m$. Then $\sum_{i+j=m} \bar{u}_i\bar{v}_j = \bar{u}_{i_0}\bar{v}_{j_0} \neq 0$, which shows that $m = r$ and $\bar{g} = \bar{u}_{i_0}\bar{v}_{j_0}$.

5.9. Lemma. $\bar{\mathscr{L}}' = 0$.

Proof. Let $\bar{w} \in \bar{\mathscr{L}}'$. By (5.5), there exists a homogeneous element $h \in A$ such that $\bar{h} \neq 0$ and $\bar{h}^{-1}\bar{D}(\bar{h}) = \bar{w}$ in B. Also by definition of $\bar{\mathscr{L}}'$ there is a unit $\bar{u}$ in B such that $\bar{u}^{-1}\bar{D}(\bar{u}) = \bar{w}$. Let $\bar{v} = \bar{u}^{-1}$. Then $\bar{D}(\bar{v}\bar{h}) = 0$. Thus by (5.1), $\bar{v}\bar{h} = \sum_{j=0}^{p-1} \bar{\alpha}_j^p \bar{g}^j \, (\alpha_j \in A)$. Let $v \in A$ be a preimage of $\bar{v}$. Then $vh - \sum_{j=0}^{p-1} \alpha_j^p g^j \in P$. Write $v = \sum_{i=0}^{r} v_i$ with v_i the form of v of degree i. $\bar{v}$ being a unit implies $v_0 \neq 0$. Since h and g are homogeneous and P is a homogeneous ideal and $deg(g) \neq 0 (\mathrm{mod}\ p)$, we see by comparing lowest degree forms of vh and $\sum_{j=0}^{p-1} \alpha_j^p g^j$ that for some $\beta \in A$ and $j = , \dots, p-1$, $v_0h - \beta^p g^j \in P$. Therefore $\bar{v}_0\bar{h} \in C$. Since v_0(hence $\bar{v}_0$)$\in k$, $\bar{h} \in C$ and $\bar{w} = \bar{h}^{-1}D(\bar{h}) = 0$ by (5.1).

Proof of theorem (5.7): Continuing with (5.6), we have $\bar{y}^{-1}\Delta_j(\bar{y}) \in B$ for $1 \leq j \leq n$. Since $\bar{D}(\bar{y}) \neq 0$, $\Delta_j(\bar{y}) \neq 0$ for some j. If we assume (i), then either the divisor class group of the variety defined by the equations $w^p - x_j = f_1 = \cdots = f_{n-2}$ is not trivial or $\bar{y}$ is a unit in B by (1.2), which contradicts the irreducibility of $\bar{y}$ in B. If we assume (ii) then either $Cl(B^p) \neq 0$ or $\bar{y}$ is a unit in B by theorem ([4], page 93, (17.4)). Thus in either case, $\bar{y}$ is a factor of $\bar{g}$.

Let $\bar{g} = \bar{g}_1 \cdots \bar{g}_{q+1}$ be a decomposition of $\bar{g}$ in B into prime elements. Then by (3.7) and the above argument we have that the logarithmic derivatives $\bar{g}_i^{-1}\bar{D}(\bar{g}_i) \in B$ (and hence $\bar{\mathscr{L}}$) and they generate $\bar{\mathscr{L}}$. Note $\sum_{i=1}^{s+1} \frac{D(\bar{g}_i)}{\bar{g}_i} = \frac{D\bar{g}}{\bar{g}} = 0$. Therefore $\{\bar{g}_i^{-1}\bar{D}(\bar{g}_i) \colon 1 \leq i \leq q\}$ generate $\bar{\mathscr{L}}$ over $\mathbb{F}_p$. We will now show that they are $\mathbb{F}_p$-independent.

Suppose $e_i \in \mathbb{F}_p$, $1 \leq i \leq q$ are such that $\sum_{i=1}^{q} e_i \bar{g}_i^{-1} \bar{D}(\bar{g}_i) = 0$. By (5.8) we may assume that the representative $g_i \in A$ of $\bar{g}_i \in B$ is homogeneous $(1 \leq i \leq q + 1)$. Let $H = g_1^{e_1} \cdots g_q^{e_q}$. Then $\bar{D}(\bar{H}) = 0$, which implies by (5.1) that $\bar{L}^p \subset \bar{L}^p(\bar{H}) \subset \bar{L}^p(\bar{g})$. If $\bar{H} \in \bar{L}^p$ then $e_i = 0 (\text{mod } p)$, $1 \leq i \leq q$ and we're done. Otherwise $\bar{L}^p(\bar{H}) = \bar{L}^p(\bar{g})$ which implies there exists $\alpha_i \in A$ $(0 \leq i \leq q)$ such that $\bar{\alpha}_p^p \bar{H} = \sum_{i=0}^{p-1} \bar{\alpha}_i^p \bar{g}^i$. Since H, g are homogeneous elements and P a homogeneous ideal we may assume that the α_j are homogeneous polynomials as well. Since $deg(\alpha_i^p g^i) = i(deg(g))(\text{mod } p)$ and $deg(g) \neq 0 (\text{mod } p)$, it follows $\bar{\alpha}_p^p \bar{H} = \bar{\alpha}_i^p \bar{g}^i$ for some $i = 0, \ldots, p-1$. This implies that if $i \neq 0$ $\bar{g}_{q+1} \in B^p$, which contradicts (5.1). Thus $i = 0$ and $\bar{H} \in \bar{L}^p$.

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