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$\bar{\partial}$ Cohomology of Complex Lie Groups

By

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Introduction

Let H be a toroidal group of complex dimension n, that is, H is a connected complex Lie group without non constant global holomorphic functions (such a Lie group is called also an (H, C)-group ([7, 8])). Let $\mathscr{K}$ be the real Lie algebra of a maximal compact real Lie subgroup of H. Put $q := \dim_{\boldsymbol{C}} \mathscr{K} \cap \sqrt{-1}\,\mathscr{K}$. Let $\mathcal{O}_H$ be the structure sheaf of H. By the result of the previous paper [3], H satisfies either of the following statements 1 and 2.

1. $H^p(H, \mathcal{O}_H)$ is finite dimensional for any p.
2. $H^p(H, \mathcal{O}_H)$ is a non-Hausdorff locally convex space for $1 \leq p \leq q$.

We say that H is *of finite type* if H satisfies the above property 1 and *of non-Hausdorff type* if H satisfies the above property 2, respectively.

The purpose of this paper is to investigate the cohomology groups $H^p(G, \mathcal{O})$ $(p \geq 1)$ of a complex Lie group G by the theory of $\bar{\partial}$-cohomology. We shall show the cohomology groups $H^p(G, \mathcal{O})$ $(p \geq 1)$ are completely determined by the type of the maximal toroidal subgroup

$$G^0 := \{x \mid f(x) = f(e) \textit{ for every holomorphic function } f \textit{ on } G\}$$

of G, where e is the unit element of G. By the result of [7] G^0 is a connected abelian complex Lie subgroup of G.

We shall prove the following theorems.

Theorem I. *Let G be a connected complex Lie group of complex dimension $n+l$ and $G^0 = \boldsymbol{C}^n/\Gamma$ the maximal toroidal subgroup of G of complex dimension n. Then the following statements* (1), (2), (3) *and* (4) *are equivalent.*

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(1) G^0 *is of finite type.*

(2) $H^p(G, \mathcal{O}) \cong$

$$\begin{cases} H^0(G/G^0, \mathcal{O}) \otimes \mathbf{C}\{d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p} | 1 \leq i_1 < \cdots < i_p \leq q\} & \text{for } 1 \leq p \leq q \\ 0 & \text{for } p \geq q+1 \end{cases}$$

(3) $H^p(G, \mathcal{O})$ *has a Hausdorff topology for any p.*

(4) $\bar{\partial} H^0(G, \mathscr{E}^{0,p-1})$ *is a closed subspace of the Fréchet space* $H^0(G, \mathscr{E}^{0,p})$ *for* $p \geq 1$, *where* $\mathscr{E}^{r,s}$ *denotes the sheaf of germs of* C^∞ (r, s)*-forms on G.*

Theorem I gives a characterization of a complex Lie group G whose cohomology groups $H^p(G, \mathcal{O})$ have Hausdorff topology.

Theorem II. *Let G be a connected complex Lie group and* G^0 *the maximal toroidal subgroup of G. Then*

(1) $\dim H^1(G, \mathcal{O}) = 0 \Leftrightarrow G$ *is a Stein group.*

(2) $0 < \dim H^1(G, \mathcal{O}) < \infty \Leftrightarrow G = G^0$ *is a toroidal group of finite type.*

(3) $\dim H^1(G, \mathcal{O}) = \infty$ *and* $H^1(G, \mathcal{O})$ *has Hausdorff topology* $\Leftrightarrow 0 < \dim G^0 < \dim G$ *and* G^0 *is of finite type. In this case*

$$H^p(G, \mathcal{O}) \cong H^0(G/G^0, \mathcal{O}) \otimes H^p(G^0, \mathcal{O}).$$

(4) $H^1(G, \mathcal{O})$ *has non-Hausdorff topology* $\Leftrightarrow \dim G^0 > 0$ *and* G^0 *is of non-Hausdorff type.*

§1. Preliminaries

Let G be a connected complex Lie group with the Lie algebra $\mathscr{G}$, G^0 the maximal toroidal subgroup of G, K a maximal compact real Lie subgroup of G with the Lie algebra $\mathscr{K}$, $K_{\mathbf{C}}$ the complex Lie subgroup with the Lie subalgebra $\mathscr{K}_{\mathbf{C}} := \mathscr{K} + \sqrt{-1}\,\mathscr{K}$ of $\mathscr{G}$ and Z the connected center of $K_{\mathbf{C}}$. Then $K_{\mathbf{C}}$ is closed in G ([6]). From the result of [6] G is biholomorphic onto $K_{\mathbf{C}} \times \mathbf{C}^a$ and there exists a connected Stein subgroup S_0 of $K_{\mathbf{C}}$ such that, for the connected center Z of $K_{\mathbf{C}}$,

$$\rho_0 \colon Z \times S_0 \ni (x, y) \longmapsto xy \in K_{\mathbf{C}}$$

is a finite covering homomorphism. By the result of [7, 8] $G^0 \subset Z$ and $Z \cong G^0 \times \mathbf{C}^{*r} \times \mathbf{C}^u$ for some non-negative integers r and u. Then we may assume $G = K_{\mathbf{C}} \times \mathbf{C}^a$ and $Z = G^0 \times \mathbf{C}^{*r} \times \mathbf{C}^u$. Taking a Stein subgroup $S := \mathbf{C}^{*r} \times \mathbf{C}^u \times S_0 \times \mathbf{C}^a$ of $Z \times S_0 \times \mathbf{C}^a$, we get a finite covering homomorphism

$$\rho \colon G^0 \times S \ni (x_0, x_1, x_2, x_3, x_4) \longmapsto (\rho_0((x_0, x_1, x_2), x_3), x_4) \in G.$$

From this homomorphism ρ we can assume $G = G^0 \times S/\ker\rho$. Let $\pi_1: G^0 \times S \to G^0$ and $\pi_2: G^0 \times S \to S$ be the canonical projections. Since $\pi_2(\ker\rho)$ is a finite subgroup of S, $S/\pi_2(\ker\rho)$ is a Stein group. We obtain a homomorphism

$$\pi: G^0 \times S/\ker\rho \ni (a, b)\ker\rho \longmapsto b\pi_2(\ker\rho) \in S/\pi_2(\ker\rho)$$

for $a \in G^0$, $b \in S$. From this projection π $G = G^0 \times S/\ker\rho$ is regarded as a fiber bundle over the Stein group $S/\pi_2(\ker\rho)$ whose fiber is isomorphic onto G^0 and whose structure group is the finite subgroup $\pi_1(\ker\rho)$ of G^0. Then $S/\pi_2(\ker\rho)$ is isomorphic onto G/G^0. Since G^0 is abelian, we obtain $\pi_1(\ker\rho) \subset K^0$, where K^0 is a maximal compact subgroup of G^0. We put $n: = \dim_{\mathbf{C}} G^0$, $l := \dim_{\mathbf{C}} S$. Then $\dim_{\mathbf{C}} G = n + l$. Let $\{U_\alpha\}$ be a Stein covering of S such that $\tilde{U}_\alpha := U_\alpha(\pi_2(\ker\rho)$ is biholomorphic onto U_α and $\mathscr{U} := \{\tilde{U}_\alpha\}$ is a Stein covering of $S/\pi_2(\ker\rho)$ with a biholomorphic mapping

$$h_\alpha: \pi^{-1}(\tilde{U}_\alpha) \ni (a, b)\ker\rho \longmapsto (\tilde{b}_\alpha, a_\alpha) \in \tilde{U}_\alpha \times G^0 \tag{1.1}$$

for each α, where $a_\alpha \in G^0$ and $b_\alpha \in U_\alpha$ satisfying $(a, b)\ker\rho = (a_\alpha, b_\alpha)\ker\rho$ and $\tilde{b}_\alpha = b_\alpha(\pi_2(\ker\rho))$. Then

$$h_\alpha h_\beta^{-1}(\tilde{b}_\beta, \alpha_\beta) = (\tilde{b}_\alpha, a_\alpha) = (\tilde{b}_\alpha, f_{\alpha\beta} a_\beta). \tag{1.2}$$

Since $\pi_1(\ker\rho)$ is a finite subgroup, the holomorphic mapping $f_{\alpha\beta}: \tilde{U}_\alpha \cap \tilde{U}_\beta \to \pi_1(\ker\rho)$ is locally constant. Taking a refinement Stein covering of $\mathscr{U}$, we may assume $\mathscr{U}$ is locally finite.

Throughout this paper we assume $G^0 = \mathbf{C}^n/\Gamma$, where Γ is a discrete lattice of $\mathbf{C}^n$ generated by $\mathbf{R}$-linearly independent vectors $\{e_1, e_2, \ldots, e_n, v_1 = (v_{11}, \ldots, v_{1n}), v_2 = (v_{21}, \ldots, v_{2n}), \ldots, v_q = (v_{q1}, \ldots, v_{qn})\}$ over $\mathbf{Z}$ and e_i denotes the i-th unit vector of $\mathbf{C}^n$. We take $\Re v_i$, $\Im v_i \in \mathbf{R}^n$ with $v_i = \Re v_i + \sqrt{-1}\Im v_i$. Since $e_1, e_2, \ldots, e_n$, $v_1, v_2, \ldots, v_q$ are $\mathbf{R}$-linearly independent, $\Im v_1, \Im v_2, \ldots, \Im v_q$ are $\mathbf{R}$-linearly independent. Then without loss of generality we may assume $\det[\Im v_{ij}; 1 \le i, j \le q] \ne 0$ from now on. We set

$$K_{m,i} := \sum_{j=1}^{n} v_{ij} m_j - m_{n+i} \quad \text{and} \quad K_m := \max\{|K_{m,i}|; 1 \le i \le q\}$$

for $m = (m_1, m_2, \ldots, m_{n+q}) \in \mathbf{Z}^{n+q}$. Since G^0 is toroidal, $K_m > 0$ for any $m \in \mathbf{Z}^{n+q} \setminus \{0\}$ ([5], [8]). We have the following theorem ([3]).

Theorem 1.1. *Let $G^0 = \mathbf{C}^n/\Gamma$ be a toroidal group. Then the following statements* (1) *and* (2) *are equivalent.*
(1) *There exists $a > 0$ such that*

$$\sup_{m \ne 0} \exp(-a\|m^*\|)/K_m < \infty,$$

where $\|m^*\| = \max\{|m_i|; 1 \le i \le n\}$.

$$(2)\qquad \dim H^p(G^0, \mathcal{O}) = \begin{cases} \dfrac{q!}{(q-p)!p!} & \text{if } 1 \le p \le q \\ 0 & \text{if } p > q. \end{cases}$$

We denote by π_q the projection $\mathbf{C}^n \ni (z^1, \dots, z^n) \mapsto (z^1, \dots, z^q) \in \mathbf{C}^q$. Since $\pi_q(e_i)$, $\pi_q(v_i)$ $(1 \le i \le q)$ are $\mathbf{R}$-linearly independent π_q induces the $\mathbf{C}^{*n-q}$-principal bundle

$$\pi_q\colon \mathbf{C}^n/\Gamma \ni z + \Gamma \longrightarrow \pi_q(z) + \Gamma^* \in T^q_{\mathbf{C}} := \mathbf{C}^q/\Gamma^*$$

over the complex q dimensional torus $T^q_{\mathbf{C}}$, where $\Gamma^* := \pi_q(\Gamma)$ ([9]). We put

$$\alpha_{ij} := \begin{cases} \Re v_{ij} & (1 \le i \le q,\ 1 \le j \le n) \\ 0 & (q+1 \le i \le n,\ 1 \le j \le n) \end{cases}$$

$$\beta_{ij} := \begin{cases} \Im v_{ij} & (1 \le i \le q,\ 1 \le j \le n) \\ \delta_{ij} & (q+1 \le i \le n,\ 1 \le j \le n) \end{cases}$$

$[\gamma_{ij};\ 1 \le i, j \le n] := [\beta_{ij};\ 1 \le i, j \le n]^{-1}$ and $v_i := \sqrt{-1}e_i$ for $q+1 \le i \le n$. Since $\{e_1, \dots, e_n, v_1, \dots, v_n\}$ are $\mathbf{R}$-linearly independent, we have an isomorphism

$$\phi\colon \mathbf{C}^n \ni (z^1, \dots, z^n) \longrightarrow (t^1, \dots; t^{2n}) \in \mathbf{R}^{2n}$$

as a real Lie group, where $(z^1, \dots, z^n) = \sum_{i=1}^n (t^i e_i + t^{n+i} v_i)$. Then we obtain the relations

$$(1.3)\qquad t^i = x^j - \sum_{i,k=1}^n y^k \gamma_{ki} \alpha_{ij} \quad \text{and} \quad t^{n+j} = \sum_{i=1}^n y^i \gamma_{ij}$$

for $1 \le j \le n$, where $z^i = x^i + \sqrt{-1}y^i$. We put $t = (t', t'')$, $t' = (t^1, \dots, t^{n+q}) \in \mathbf{R}^{n+q}$ and $t'' = (t^{n+q+1}, \dots, t^{2n}) \in \mathbf{R}^{n+q}$. ϕ induces the isomorphism $\phi\colon \mathbf{C}^n/\Gamma \cong \mathbf{T}^{n+q} \times \mathbf{R}^{n-q}$ as a real Lie group, where $\mathbf{T}^{n+q}$ is a real $(n+q)$ dimensional real torus. It follows from (1.3) that

$$(1.4)\qquad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left[\frac{\partial}{\partial t^i} + \sqrt{-1}\left\{-\sum_{j,k=1}^n \gamma_{ik}\alpha_{kj}\frac{\partial}{\partial t^j} + \sum_{j=1}^n \gamma_{ij}\frac{\partial}{\partial t^{n+j}}\right\}\right].$$

Then for $q+1 \le i \le n$ we have

$$(1.5)\qquad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial t^i} + \sqrt{-1}\frac{\partial}{\partial t^{n+i}}\right).$$

Let $(w^1_\alpha, \dots, w^l_\alpha)$ be a coordinate system in $\tilde{U}_\alpha$. For any $g \in \pi^{-1}(\tilde{U}_\alpha)$, we put $h_\alpha(g) = (w^1_\alpha(g), \dots, w^l_\alpha(g),\ z^1_\alpha(g), \dots, z^n_\alpha(g)) \in \tilde{U}_\alpha \times \mathbf{C}^n/\Gamma$, and $K_\alpha(g) = (z^1_\alpha(g), \dots, z^n_\alpha(g)) \in \mathbf{C}^n/\Gamma$ and $\hat{K}_\alpha(g) = \phi \circ K_\alpha(g) = t_\alpha(g) = (t^1_\alpha(g), \dots, t^{2n}_\alpha(g)) \in \mathbf{T}^{n+q} \times \mathbf{R}^{n-q}$. Since $f_{\alpha\beta} \in \pi_1(\ker \rho) \subset \mathbf{C}^n/\Gamma$, $(f_{\alpha\beta})^s \in \Gamma$, where $s := \#(\ker \rho)$. We put $\phi(f_{\alpha\beta}) = (f^1_{\alpha\beta}, \dots, f^{2n}_{\alpha\beta})$. Then $\phi((f_{\alpha\beta})^s) = (sf^1_{\alpha\beta}, \dots, sf^{2n}_{\alpha\beta}) \in \mathbf{Z}^{n+q} \times \{0\}$. Thus $\phi(f_{\alpha\beta}) =$

$(f^1_{\alpha\beta},\dots,f^{n+q}_{\alpha\beta},0\dots,0)\in \boldsymbol{R}^{2n}$. From (1.2) for $g\in\pi^{-1}(\tilde U_\alpha\cap\tilde U_\beta)$, $t_\alpha(g)=t_\beta(g)+\phi(f_{\alpha\beta})$. Putting $t_\alpha=(t'_\alpha,t''_\alpha)$, $t'_\alpha=(t^1_\alpha,\dots,t^{n+q}_\alpha)$, $t''_\alpha=(t^{n+q+1}_\alpha,\dots,t^{2n}_\alpha)$, and $f'_{\alpha\beta}=(f^1_{\alpha\beta},\dots,f^{n+q}_{\alpha\beta})$

$$t'_\alpha=t'_\beta+f'_{\alpha\beta}\quad\text{and}\quad t''_\alpha=t''_\beta. \tag{1.6}$$

Since $z^i_\alpha=t^i_\alpha+\sum_{j=1}^n t^{n+j}_\alpha v_{ji}$,

$$z^i_\alpha=z^i_\beta+f^i_{\alpha\beta}+\sum_{j=1}^q f^{n+j}_{\alpha\beta}v_{ji}\quad\text{and then}\quad d\bar z^i_\alpha=d\bar z^i_\beta. \tag{1.7}$$

§2. The Cohomology Groups $H^p(U\times G^0,\mathcal{O})$

We consider the cohomology groups $H^p(U\times G^0,\mathcal{O})$ of the product manifold of an open polydisc U in $\boldsymbol{C}^l$ and $G^0=\boldsymbol{C}^n/\Gamma$. As in §1, we have the isomorphism $\phi\colon \boldsymbol{C}^n/\Gamma\ni(z^1,\dots,z^n)+\Gamma\mapsto(t^1,\dots,t^{2n})+\phi(\Gamma)\in\boldsymbol{T}^{n+q}\times\boldsymbol{R}^{n-q}$ and the projection $\pi_q\colon \boldsymbol{C}^n/\Gamma\ni(z^1,\dots,z^n)+\Gamma\mapsto(z^1,\dots,z^q)+\Gamma^*\in\boldsymbol{T}^q_{\boldsymbol{C}}$. We put $U=\{(w^1,\dots,w^l)\in\boldsymbol{C}^l \mid |w^i|<d,\ i=1,\dots,l\}$. We have a diffeomorphism $\tilde\phi\colon U\times\boldsymbol{C}^n/\Gamma\ni(w^1,\dots,w^l,z+\Gamma)\mapsto(w^1,\dots,w^l,\pi_q(z+\Gamma),\ \xi^1,\dots,\xi^{n-q})\in U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q}$, where $\xi^i=\exp(2\pi\sqrt{-1}(t^{q+i}+\sqrt{-1}t^{n+q+i}))$ and $z=(z^1,\dots,z^n)$. Let $\mathcal{E}=\mathcal{E}_X$ be the sheaf of germs of C^∞ functions on a complex manifold X and $\mathcal{E}^{r,s}=\mathcal{E}^{r,s}_X$ the sheaf of germs of C^∞ (r,s)-forms on X. We define the sheaf $\mathcal{F}$ on $U\times G^0$ and the sheaf $\mathcal{G}$ on $U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q}$ as follows:

$$\mathcal{F}:=\{f\in\mathcal{E}_{U\times G^0}\mid \frac{\partial f}{\partial\bar w^i}=0,\ \frac{\partial f}{\partial\bar z^j}=0,\ i=1,\dots,l,\ j=q+1,\dots,n\}\ \text{and}$$

$$\mathcal{G}:=\{g\in\mathcal{E}_{U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q}}\mid \frac{\partial g}{\partial\bar w^i}=0,\ \frac{\partial g}{\partial\bar\xi^k}=0,\ i=1,\dots,l,\ k=1,\dots,n-q\}.$$

$\phi^*\colon H^0(W,\mathcal{g})\ni g\mapsto g\circ\tilde\phi\in H^0(\phi^{-1}(W),\mathcal{F})$ is an isomorphism for any open subset W of $U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q}$. Then ϕ^* induces an isomorphism $H^p(U\times G^0,\mathcal{F})\cong H^p(U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q},\mathcal{g})$ for any p.

Lemma 2.1. $H^p(U\times G^0,\mathcal{F})=0$ *for* $p\geq 1$.

Proof. It is equivalent to prove $H^p(U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q},\mathcal{g})=0$. We put $X:=U\times\boldsymbol{T}^q_{\boldsymbol{C}}\times\boldsymbol{C}^{*n-q}$. Let f be a C^∞ function in a neighborhood of $x\in X$. We put

$$\bar\partial' f:=\sum_{i=1}^l\frac{\partial f}{\partial\bar w^i}d\bar w_i+\sum_{k=1}^{n-q}\frac{\partial f}{\partial\bar\xi^j}d\bar\xi_j.$$

$\bar\partial'$ is also defined for C^∞ (r,s)-forms in a neighborhood of $x\in X$. We have an exact sequence on X,

$$0\longrightarrow\mathcal{G}\longrightarrow\mathcal{E}^{0,0}\longrightarrow\mathcal{E}^{0,1}\longrightarrow\cdots\longrightarrow\mathcal{E}^{0,n+l-q}\longrightarrow 0.$$

Then

$$H^p(X, \mathscr{G}) = \frac{\{\varphi \in H^0(X, \mathscr{E}^{0,p}) | \bar{\partial}' \varphi = 0\}}{\bar{\partial}' H^0(X, \mathscr{E}^{0,p-1})}.$$

We put $X_n := \{(w, \pi_q(z + \Gamma), \xi) \in X \,|\, |w^i| < d - \frac{1}{n}, \frac{1}{n} < |\xi^j| < n\}$. In case $l = q = 1$, for any C^∞-function $f(w, \pi_q(z + \Gamma), \xi)$ in X, we put

$$g_1(w, \pi_q(z + \Gamma), \xi) := \frac{1}{2\pi\sqrt{-1}} \iint_{|u| < d - \frac{1}{n}} \frac{f(u, \pi_q(z + \Gamma), \xi)}{u - w} du \wedge d\bar{u} \text{ and}$$

$$g_2(w, \pi_q(z + \Gamma), \xi) := \frac{1}{2\pi\sqrt{-1}} \iint_{\frac{1}{n} < |\zeta| < n} \frac{f(w, \pi_q(z + \Gamma), \zeta)}{\zeta - \xi} d\zeta \wedge d\bar{\zeta}.$$

Then in X_n we have

$$\frac{\partial g_1}{\partial \bar{w}} = f \quad \text{and} \quad \frac{\partial g_2}{\partial \bar{\xi}} = f.$$

Using this fact and the standard argument for the Dolbeault lemma ([1, 4]), we can complete the proof of this lemma.

Let $\hat{\pi}_q \colon U \times G^0 \ni (w, z + \Gamma) \mapsto (w, \pi_q(z + \Gamma)) \in U \times T^q_{\mathbf{C}}$, where $w = (w^1, \dots, w^l)$ and $z = (z^1, \dots, z^n)$. We put $\mathscr{F}^{r,s} := \mathscr{F} \otimes \hat{\pi}_q^* \mathscr{E}^{r,s}_{T^q_{\mathbf{C}}}$.

As an immediate consequence of Lemma 2.1, we have the following

Lemma 2.2. *The sequence*

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{F}^{0,0} \longrightarrow \mathscr{F}^{0,1} \longrightarrow \cdots \longrightarrow \mathscr{F}^{0,q} \longrightarrow 0$$

is exact on $U \times G^0$ *and*

$$H^p(U \times G^0, \mathscr{O}) = \frac{\{\varphi \in H^0(U \times G^0, \mathscr{F}^{0,p}) | \bar{\partial}\varphi = 0\}}{\bar{\partial} H^0(U \times G^0, \mathscr{F}^{0,p-1})}$$

for $p \geq 1$.

Let $\varphi \in H^0(U \times G^0, \mathscr{F}^{0,p})$. Since $G = \mathbf{C}^n/\Gamma$ has global 1-forms $dz^1, \dots, dz^n$, $d\bar{z}^1, \dots, d\bar{z}^n$ for the coordinate system $z = (z^1, \dots, z^n) \in \mathbf{C}^n$, we can write

$$\varphi = \frac{1}{p!} \sum_{1 \leq i_1, \dots, i_p \leq q} \varphi_{i_1 \dots i_p} d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p},$$

where $\varphi_{i_1 \dots i_p} \in H^0(U \times G^0, \mathscr{F})$ and skew-symmetric in all indices. We expand $\varphi_{i_1 \dots i_p}$ on $U \times G^0$:

$$\varphi_{i_1 \dots i_p}(w, t) = \sum_{m \in \mathbf{Z}^{n+q}} a^m_{i_1 \dots i_p}(w, t'') \exp(2\pi\sqrt{-1} \langle m, t' \rangle), \tag{2.1}$$

where $\langle m, t' \rangle = \sum_{i=1}^{n+q} m_i t_i$ and $a^m_{i_1 \dots i_p}(w, t'') \in C^\infty(U \times \boldsymbol{R}^{n-q})$. We put $\varphi^m_{i_1 \dots i_p}(w, t) = a^m_{i_1 \dots i_p}(w, t'') \exp(2\pi\sqrt{-1}\langle m, t' \rangle)$ and $\varphi^m = \frac{1}{p!}\sum_{1 \le i_1, \dots, i_p \le q} \varphi^m_{i_1 \dots i_p}$ $d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p}$. Then $\varphi = \sum_{m \in \boldsymbol{Z}^{n+q}} \varphi^m$. It follows from (1.4), (1.5) and (2.1) that

$$\text{(2.2)} \qquad \frac{\partial \varphi^m_{i_1 \dots i_p}(w, t)}{\partial \bar{z}^i} =$$

$$\begin{cases} \pi \sum_{k=1}^{q} \gamma_{ik} K_{m,k} \varphi^m_{i_1 \dots i_p}(w, t) & (1 \le i \le q) \\ \sqrt{-1}\left(\pi m_i a^m_{i_1 \dots i_p}(w, t'') + \dfrac{1}{2}\dfrac{\partial a^m_{i_1 \dots i_p}(w, t'')}{\partial t^{n+i}}\right)\exp(2\pi\sqrt{-1}\langle m, t' \rangle) & (q+1 \le i \le n). \end{cases}$$

Since $\varphi_{i_1 \dots i_p} \in H^0(U \times G^0, \mathscr{F})$, $\varphi_{i_1 \dots i_p}$ are holomorphic in $w^1, \dots, w^l$ and $z^{q+1}, \dots, z^n$. Therefore from (2.1) and (2.2), we have

(2.3)

$$\varphi^m_{i_1 \dots i_p}(w, t) = \sum_{m \in \boldsymbol{Z}^{n+q}} c^m_{i_1 \dots i_p}(w) \exp(-2\pi \sum_{i=q+1}^{n} m_i t^{n+i}) \exp(2\pi\sqrt{-1}\langle m, t' \rangle),$$

where $c^m_{i_1 \dots i_p}(w)$ are holomorphic functins in U. Let $m \in \boldsymbol{Z}^{n+q} \setminus \{0\}$ and $s(m) := \min\{s \mid |K_{m,s}| = K_m,\ 1 \le s \le q\}$. We put

$$c^{m,s}_{i_1 \dots i_{p-1}} := \sum_{i=1}^{n} \beta_{si} c^m_{i i_1 \dots i_{p-1}}.$$

Since $K_m > 0$ for $m \in \boldsymbol{Z}^{n+q} \setminus \{0\}$, we can put $d^m_{i_1 \dots i_{p-1}} := \dfrac{c^{m,s(m)}_{i_1 \dots i_{p-1}}}{\pi K_{m,s(m)}}$ and

$$\text{(2.4)} \qquad \psi^m := \frac{1}{(p-1)!} \sum_{1 \le i_1, \dots, i_{p-1} \le q} d^m_{i_1 \dots i_{p-1}}(w) \exp(-2\pi \sum_{i=q+1}^{n} m_i t^{n+i})$$
$$\times \exp(2\pi\sqrt{-1}\langle m, t' \rangle) d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_{p-1}}.$$

From the similar argument to the previous paper [3,5] we have the following

Lemma 2.3. *Let* $\varphi = \sum_{m \in \boldsymbol{Z}^{n+q}} \varphi^m \in H^0(U \times G^0, \mathscr{F}^{0,p})$ *be a* $\bar{\partial}$*-closed form. Take the* $(0, p-1)$*-form* ψ^m *defined by* (2.4) *for* $m \in \boldsymbol{Z}^{n+q} \setminus \{0\}$, *then*

$$\varphi = \frac{1}{p!} \sum_{1 \le i_1, \dots, i_{p-1} \le q} c^0_{i_1 \dots i_p}(w) d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p} + \sum_{m \in \boldsymbol{Z}^{n+q} \setminus \{0\}} \bar{\partial}\psi^m,$$

where $c^0_{i_1 \dots i_p}(w)$ *are holomorphic in* U. *In particular,* $\varphi^0 = \frac{1}{p!}\sum_{1 \le i_1, \dots, i_{p-1} \le q} c^0_{i_1 \dots i_p}$ $(w) d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p} \in \bar{\partial} H^0(U \times G^0, \mathscr{F}^{0,p-1})$ *if and only if* $\varphi^0 = 0$.

Proposition 2.1. *If G^0 is of finite type, then*

$$H^p(U \times G^0, \mathcal{O}) \cong H^0(U, \mathcal{O}) \otimes C\{d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p} \mid 1 \le i_1 < \cdots < i_p \le q\}$$

for $1 \le p \le q$ *and* $H^p(U \times G^0, \mathcal{O}) = 0$ for $p \ge q+1$.

Proof. As an immediate consequence of Lemma 2.2 we have $H^p(U \times G^0, \mathcal{O}) = 0$ for $p \ge q+1$. Let $\varphi = \sum_{m \in Z^{n+q}} \varphi^m \in H^0(U \times G^0, \mathcal{F}^{0,p})$ be a $\bar{\partial}$-closed form, where $\varphi^m := \frac{1}{p!} \sum_{1 \le i_1, \dots, i_p \le q} c^m_{i_1, \dots, i_p}(w) \exp(-2\pi \sum_{i=q+1}^n m_i t^{n+i}) \exp(2\pi\sqrt{-1}\langle m, t'\rangle) d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p}$. Similarly to [4, Lemma 7], for any compact subset K of U, any $R > 0$ and any $k > 0$, we get

$$C_K(k, R) := \sup_{w \in K} \{ |c^m_{i_1 \dots i_p}(w)| \, \|m'\|^k R^{\|m''\|} \mid m \in Z^{n+q} \} < +\infty$$

where $\|m'\| = \max\{|m_i|, |m_{n+i}| \mid 1 \le i \le q\}$ and $\|m''\| = \max\{|m_j| \mid q+1 \le j \le n\}$. Since G^0 is of finite type, the statement (1) of Theorem 1.1 in § 1 holds. By Lemma 2.3 we have $(0, p-1)$-form ψ^m defined by (2.4) such that $\varphi^m = \bar{\partial}\psi^m$ for $m \in Z^{n+q} \setminus \{0\}$. Using a similar argument to [4] and the statement (1) of Theorem 1.1 we obtain

$$|d^m_{i_1 \dots i_{p-1}}(w)| \, \|m'\|^k R^{\|m''\|} < \infty.$$

This means that $\sum_{m \in Z^{n+q} \setminus \{0\}} \psi^m$ converges to a $(0, p-1)$-form $\psi \in H^0(U \times G^0, \mathcal{F}^{0,p-1})$. Then $\phi = \phi^0 + \bar{\partial}\psi$, where $\phi^0 = \frac{1}{p!} \sum_{1 \le i_1, \dots, i_p \le q} c^0_{i_1 \dots i_p}(w) d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p}$ and $c^0_{i_1 \dots i_p}$ are holomorphic in U. This completes the proof.

In Proposition 2.1 we get a kind of Künneth formula by the $\bar{\partial}$-cohomology theory. This can be also obtained by a result of Kaup [2]. Kaup [2] shows this formula for Fréchet sheaves using the Cěch cohomology theory.

We have two isomorphisms

$$I : H^p(U \times G^0, \mathcal{O}) \cong \frac{\{\varphi \in H^0(U \times G^0, \mathcal{E}^{0,p}) \mid \bar{\partial}\varphi = 0\}}{\bar{\partial} H^0(U \times G^0, \mathcal{E}^{0,p-1})}$$

$$J : H^p(U \times G^0, \mathcal{O}) \cong \frac{\{\varphi \in H^0(U \times G^0, \mathcal{F}^{0,p}) \mid \bar{\partial}\varphi = 0\}}{\bar{\partial} H^0(U \times G^0, \mathcal{F}^{0,p-1})}.$$

Combining Lemma 2.3 with the existence of the isomorphisms I and J, we get the following

Lemma 2.4. *For any $\bar{\partial}$-closed form $\varphi \in H^0(U \times G^0, \mathcal{E}^{0,p})$, there exist a holomorphic $(0, p)$-form $h = \frac{1}{p!} \sum_{1 \le i_1, \dots, i_p \le q} c_{i_1 \dots i_p}(w) d\bar{z}^{i_1} \wedge \dots \wedge d\bar{z}^{i_p}$, $\psi \in H^0(U \times G^0, \mathcal{E}^{0,p-1})$ and $\hat{\psi}^m \in H^0(U \times G^0, \mathcal{F}^{0,p-1})$ such that*

$$\varphi = h + \sum_{m \in \mathbf{Z}^{n-q} \setminus \{0\}} \bar{\partial}\hat{\psi}^m + \bar{\partial}\psi .$$

§ 3. The Cohomology Groups $H^p(G, \mathcal{O})$

In this section we shall prove Theorem I and Theorem II. As in § 1, let G be a connected complex Lie group of complex dimension $(n + l)$, $G^0 = C^n/\Gamma$ the toroidal subgroup of complex dimension n, $\mathcal{U} = \{\tilde{U}_\alpha | \alpha = 0, 1, 2, \dots\}$ a Stein covering of $S/\pi_2(\ker \rho) \cong G/G^0$ and $\pi: G \to S/\pi_2(\ker \rho)$ the projection. We put $\mathcal{E} = \mathcal{E}_G^{0,0}$ and $\mathcal{E}^{r,s} = \mathcal{E}_G^{r,s}$. We use coordinate systems $w_\alpha^1, \dots, w_\alpha^l, z_\alpha^1, \dots, z_\alpha^n, t_\alpha^1, \dots, t_\alpha^{2n}$ in $\pi^{-1}(\tilde{U}_\alpha) \cong \tilde{U}_\alpha \times G^0$ as in § 1. Put

$$\mathcal{F} := \{f \in \mathcal{E} | \frac{\partial f}{\partial \bar{w}_\alpha^i} = 0, \frac{\partial f}{\partial \bar{z}_\alpha^j} = 0 \text{ in } \pi^{-1}(\tilde{U}_\alpha),\ i = 1, \dots, l, j = q + 1, \dots, n\}.$$

From (1.7) the sheaf $\mathcal{F}$ is well-defined on G and then $\mathcal{F}^{r,s}$ is also defined on G. To calculate the cohomology groups $H^p(G, \mathcal{O})$, we use the Dolbeault isomorphism I in § 2. Let $\varphi \in H^0(G, \mathcal{E}^{0,p})$ be a $\bar{\partial}$-closed form. We put $\varphi_\alpha := \varphi | \pi^{-1}(\tilde{U}_\alpha)$. Then φ_α is a $\bar{\partial}$-closed form on $\pi^{-1}(\tilde{U}_\alpha) \cong \tilde{U}_\alpha \times G^0$. We write

$$\varphi_\alpha = \frac{1}{p!} \sum_{1 \le i_1, \dots, i_p \le l+n} \varphi_{\alpha, i_1 \dots i_p} d\bar{\zeta}_\alpha^{i_1} \wedge \dots \wedge d\bar{\zeta}_\alpha^{i_p},$$

where $\zeta_\alpha^i = w_\alpha^i\ (i = 1, \dots, l)$, $\zeta_\alpha^{l+j} = z_\alpha^j\ (j = 1, \dots, n)$ and $\varphi_{\alpha, i_1 \dots i_p} \in H^0(\pi^{-1}(\tilde{U}_\alpha), \mathcal{E})$. Put $\varphi_\alpha = \sum_{m \in \mathbf{Z}^{n+q}} \varphi_\alpha^m$,

$$\varphi_\alpha^m = \frac{1}{p!} \sum_{1 \le i_1, \dots, i_p \le l+n} \varphi_{\alpha, i_1 \dots i_p}^m d\bar{\zeta}_\alpha^{i_1} \wedge \dots \wedge d\bar{\zeta}_\alpha^{i_p}$$

and $\varphi_{\alpha, i_1 \dots i_p} = \sum_{m \in \mathbf{Z}^{n+q}} \varphi_{\alpha, i_1 \dots i_p}^m = \sum_{m \in \mathbf{Z}^{n+q}} a_{\alpha, i_1 \dots i_p}^m (w_\alpha, t''_\alpha) \times \exp(2\pi\sqrt{-1} \langle m, t'_\alpha \rangle)$. From (1.6) and (1.7), in $\pi^{-1}(\tilde{U}_\alpha) \cap \pi^{-1}(\tilde{U}_\beta)$

$$\varphi_\alpha^m = \varphi_\beta^m \tag{3.1}$$

for all $m \in \mathbf{Z}^{n+q}$. By Lemma 2.4 we have a holomorphic $(0, p)$-form $h_\alpha = \frac{1}{p!} \sum_{1 \le i_1, \dots, i_p \le q} h_{\alpha, i_1 \dots i_p}(w)\, d\bar{z}_\alpha^{i_1} \wedge \dots \wedge d\bar{z}_\alpha^{i_p}$, $\chi_\alpha = \sum_{m \in \mathbf{Z}^{n+q}} \chi_\alpha^m \in H^0(\pi^{-1}(\tilde{U}_\alpha), \mathcal{E}^{0,p-1})$ and $\hat{\psi}_\alpha^m \in H^0(\pi^{-1}(\tilde{U}_\alpha), \mathcal{F}^{0,p-1})$ for each $m \in \mathbf{Z}^{n+q} \setminus \{0\}$ such that

$$\varphi_\alpha = h_\alpha + \sum_{m \in \mathbf{Z}^{n+q} \setminus \{0\}} \bar{\partial}\hat{\psi}_\alpha^m + \bar{\partial}\chi_\alpha. \tag{3.2}$$

We put $\psi_\alpha^m := \hat{\psi}_\alpha^m + \chi_\alpha^m$ for $m \in \mathbf{Z}^{n+q} \setminus \{0\}$ and $\psi_\alpha^0 := \chi_\alpha^0$, then we have

$$\varphi_\alpha^0 = h_\alpha + \bar{\partial}\psi_\alpha^0 \text{ and } \varphi_\alpha^m = \bar{\partial}\psi_\alpha^m \tag{3.3}$$

for $m \in \mathbf{Z}^{n+q} \setminus \{0\}$. From (3.1) we obtain

$$h_\alpha + \bar\partial\psi_\alpha^0 = h_\beta + \bar\partial\psi_\beta^0 \quad\text{and}\quad \bar\partial\psi_\alpha^m = \bar\partial\psi_\beta^m. \tag{3.4}$$

Then, by Lemma 2.3 $h_\alpha - h_\beta = \bar\partial(\psi_\beta^0 - \psi_\alpha^0) = 0$. Thus

$$h_\alpha = h_\beta. \tag{3.5}$$

We put $\Phi^m := \bar\partial\psi_\alpha^m \in H^0(G, \mathscr{E}^{0,p})$ and $\Phi^{m,1} = \delta(\{\psi_\alpha^m\}) = \{\psi_\beta^m - \psi_\alpha^m\} \in Z^1(\{\pi^{-1}(\tilde U_\alpha)\}, \bar\partial\mathscr{E}^{0,p-2})$. In case $m \neq 0$, by (3.2) we have $\Psi^{m,1} \in C^1(\{\pi^{-1}(\tilde U_\alpha)\}, \mathscr{E}^{0,p-2})$ such that $\Phi^{m,1} = \bar\partial\Psi^{m,1}$. Continuing the above argument we get $\Phi^{m,p} \in Z^p(\{\pi^{-1}(\tilde U_\alpha)\}, \mathcal{O})$. Since $I([\Phi^{m,p}]) = [\Phi^m]$ and $H^p(\{\pi^{-1}(\tilde U_\alpha)\}, \mathcal{O}) \cong H^p(\{\tilde U_\alpha\}, \mathcal{O}) = 0$, we have $\Psi^m \in H^0(G, \mathscr{E}^{0,p-1})$ such that

$$\Phi^m = \bar\partial\Psi^m. \tag{3.6}$$

In case $m = 0$, by (3.2) we have $\hat\Phi^{0,1} \in C^1(\{\pi^{-1}(\tilde U_\alpha)\}, \mathcal{O}^{0,p-1})$ and $\Psi^{0,1} \in C^1(\{\pi^{-1}(\tilde U_\alpha)\}, \mathscr{E}^{0,p-2})$, satisfying $\Phi^{0,1} = \hat\Phi^{0,1} + \bar\partial\Psi^{0,1}$, where $\mathcal{O}^{0,r}$ is the sheaf of germs of holomorphic $(0, r)$-forms on G. Then $0 = \delta\Phi^{0,1} = \delta\hat\Phi^{0,1} + \bar\partial\,\delta\Psi^{0,1}$. By Lemma 2.3 and Lemma 2.4 $\delta\hat\Phi^{0,1} = 0$. Therefore $\hat\Phi^{0,1} \in Z^1(\{\pi^{-1}(\tilde U_\alpha)\}, \mathcal{O}^{0,p-1}) \cong Z^1(\{\tilde U_\alpha\}, \mathcal{O}^{0,p-1})$. Then we have $H^{0,0} \in C^0(\{\pi^{-1}(\tilde U_\alpha)\}, \mathcal{O}^{0,p-1})$ such that $\hat\Phi^{0,1} = \delta H^{0,0}$. Replace ψ_α^0 by $\psi_\alpha^0 - H^{0,0}$ and put $\Phi^{0,1} := \delta(\psi_\alpha^0 - H^{0,0})$, then $\Phi^{0,1} = \bar\partial\Psi^{0,1}$. Continuing the similar argument we get $\Phi^{0,p} \in Z^p(\{\pi^{-1}(\tilde U_\alpha)\}, \mathcal{O})$. Thus we have $\Psi^0 \in H^0(G, \mathscr{E}^{0,p-1})$ such that

$$\Phi^0 = \bar\partial\Psi^0. \tag{3.7}$$

By (3.2), (3.5), (3.6) and (3.7) we have the following

Proposition 3.1. *Let $\varphi \in H^0(G, \mathscr{E}^{0,p})$ be a $\bar\partial$-closed form. Then we have a holomorphic $(0, p)$-form $h = \frac{1}{p!}\sum_{1\le i_1,\dots,i_p\le q} h_{\alpha,i_1\dots i_p}(w_\alpha)\, d\bar z_\alpha^{i_1} \wedge \dots \wedge d\bar z_\alpha^{i_p}$ on G and C^∞ $(0, p-1)$-forms Ψ^m on G for each $m \in \mathbf{Z}^{n+q}$ satisfying $\varphi = h + \sum_{m\in\mathbf{Z}^{n+q}} \bar\partial\Psi^m$.*

Now we start proving Theorem I. Assume (1) holds. We have the isomorphism I. Let $\varphi \in H^0(G, \mathscr{E}^{0,p})$ be a $\bar\partial$-closed form. We put $\varphi_\alpha := \varphi|\pi^{-1}(\tilde U_\alpha)$. By (3.2) we can write $\varphi_\alpha = h_\alpha + \sum_{m\in\mathbf{Z}^{n+q}\setminus\{0\}} \bar\partial\hat\psi_\alpha^m + \bar\partial\chi_\alpha$, where h_α and $\hat\psi_\alpha^m$ and χ_α are the same as in (3.2). In case $1 \le p \le q$, by Proposition 2.1, $\hat\psi_\alpha = \sum_{m\in\mathbf{Z}^{n+q}\setminus\{0\}} \hat\psi_\alpha^m$ converges in $H^0(\pi^{-1}(\tilde U_\alpha), \mathscr{F}^{0,p-1})$ for each α. We have $\varphi_\alpha = h_\alpha + \bar\partial\hat\psi_\alpha + \bar\partial\chi_\alpha$. We put $\psi_\alpha^0 := \chi_\alpha^0$ and $\psi_\alpha^1 = \sum_{m\in\mathbf{Z}^{n+q}\setminus\{0\}} (\hat\psi_\alpha^m + \chi_\alpha^m)$. Then $\varphi_\alpha = h_\alpha + \bar\partial\psi_\alpha^0 + \bar\partial\psi_\alpha^1$. Put $\Phi^i := \bar\partial\psi_\alpha^i \in H^0(G, \bar\partial\mathscr{E}^{0,p-1})$ $(i = 0, 1)$. Then similarly to getting (3.6) and (3.7) we have $\Psi^i \in H^0(G, \mathscr{E}^{0,p-1})$ satisfying $\varphi = h + \bar\partial\Psi^0 + \bar\partial\Psi^1$, where $h|\pi^{-1}(\tilde U_\alpha) = h_\alpha$. In case $p > q$, by (3.2) $\varphi_\alpha = \bar\partial\chi_\alpha$. Then we can get $\chi \in H^0(G, \mathscr{E}^{0,p-1})$ satisfying $\varphi = \bar\partial\chi$ similarly to the case $1 \le p \le q$. It is obvious that (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Finally we prove (4) $\Rightarrow$ (1). Suppose G^0 is not of finite type. Then the statement (1) in Theorem 1.1 in § 1 doesn't hold. Namely there exists $\varepsilon > 0$ such that we can choose a sequence $\{m_\mu | \mu \ge 1\} \in \mathbf{Z}^{n+q}\setminus\{0\}$ satisfying $\exp(-\varepsilon\|m'_\mu\| - \|m''_\mu\|)/K_{m_\mu} > \mu$ for any $\mu \ge 1$ ([3, Lemma 4.2]). Put

$$\delta^m := \begin{cases} \exp\{-\varepsilon \|m'_\mu\| - \|m''_\mu\|)/K_{m_\mu} & m = m_\mu \text{ for some } \mu \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

For each α, we put

$$\psi^m_\alpha := \{\sum_\gamma \exp(2\pi\sqrt{-1} < m, f'_{\gamma\alpha} >)\} \, \delta^m \exp(-2\pi \sum_{i=q+1}^{n} m_i t_\alpha^{n+i})$$
$$\times \exp(2\pi\sqrt{-1} < m, t'_\alpha >)$$

in $\pi^{-1}(\tilde{U}_\alpha)$. From (1.6), in $\pi^{-1}(\tilde{U}_\alpha) \cap \pi^{-1}(\tilde{U}_\beta)\, \psi^m_\alpha = \psi^m_\beta$. Then we have $\psi^m \in H^0(G, \mathscr{E}^{0,0})$ such that $\psi^m | \pi^{-1}(\tilde{U}_\alpha) = \psi^m_\alpha$. By the same argument of [3,4] $\sum_{m \in \mathbf{Z}^{n+q} \setminus \{0\}} \bar{\partial}\psi^m$ converges to a form $\varphi \in H^0(G, \mathscr{E}^{0,1})$. By the choice of the sequence $\{m_\mu\}$, the formal series $\sum_m \psi^m$ cannot converge to any function in $H^0(G, \mathscr{E})$. Suppose $\varphi = \bar{\partial}\lambda$ for some $\lambda = \sum_m \lambda^m$. Then we can see $\lambda^m = \psi^m$ for $m \neq 0$. It is a contradiction. Then $\varphi = \lim_{N\to\infty}(\bar{\partial}\sum_{\|m\|<N}\psi^m)$ belongs not to $\bar{\partial}H^0(G, \mathscr{E}^{0,0})$, but to the closure of $\bar{\partial}H^0(G, \mathscr{E}^{0,0})$. This contradicts the statement (4).

Finally we prove Theorem II.

(1) Suppose $H^1(G, \mathcal{O}) = 0$. By Theorem I

$$H^1(G, \mathcal{O}) \cong H^0(G/G^0, \mathcal{O}) \otimes \mathbf{C}\{d\bar{z}^i \mid 1 \leq i \leq q\}.$$

If $G^0 \neq \{e\}$, then $q \geq 1$. This contradics our assumption. Then we have $G^0 = \{e\}$. This means $G = G/G^0$ is a Stein group.

(2) Since $1 \leq \dim H^1(G, \mathcal{O}) < \infty$, $H^1(G, \mathcal{O})$ has a Hausdorff topology. Then by Theorem I,

$$H^1(G, \mathcal{O}) \cong H^0(G/G^0, \mathcal{O}) \otimes \mathbf{C}\{d\bar{z}^i \mid 1 \leq i < q\}.$$

If $\dim G/G^0 \geq 1$, then G/G^0 is a Stein group and $H^0(G/G^0, \mathcal{O})$ is of infinite dimensional. It contradicts our assumption. Then $G = G^0$.

(3) Suppose $\dim H^1(G, \mathcal{O}) = \infty$ and $H^1(G, \mathcal{O})$ has a Hausdorff topology. By Theorem I, G^0 is of finite type and $H^1(G, \mathcal{O}) \cong H^0(G/G^0, \mathcal{O}) \otimes \mathbf{C}\{d\bar{z}^i \mid 1 \leq i \leq q\}$. Since $\dim H^1(G, \mathcal{O}) = \infty$, $0 < \dim G^0 < \dim G$.

(4) Suppose $H^1(G, \mathcal{O})$ has a non-Hausdorff topology, by Theorem I, G^0 is of non-Hausdorff type. The converse is clear.

References

[1] Gunning, R. C. and Rossi, H., *Analytic functions of several complex variables*, Prentice Hall, Inc., Englewood Cliffs, N. J., 1965.

[2] Kaup, L., Eine Künnethformel für Fréchetgarben, *Math. Z.*, **97** (1967), 158–168.

[3] Kazama, H., $\bar{\partial}$-Cohomology of (H, C)-groups, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 297–317.

[4] Kazama, H. and Shon, K. H., Characterizations of the $\bar{\partial}$-cohomology groups for a family of

weakly pseudoconvex manifolds, *J. Math. Soc. Japan*, **39** (1987), 686–700.

[5] Kazama, H. and Umeno, T., Complex abelian Lie groups with finite-dimensional cohomology groups, *J. Math. Soc. Japan*, **36** (1984), 91–106.

[6] Matsushima, Y., Espaces homogènes de Stein des groupes de Lie complexes, *Nagoya Math. J.*, **16** (1960), 205–218.

[7] Morimoto, A., Non-compact complex Lie groups without non-constant holomorphic functions, *Proc. Conf. on Complex Analysis, Minneapolis* 1964, 256–272, Springer 1965.

[8] ———, On the classification of non-compact complex abelian Lie groups, *Trans. Amer. Math. Soc.*, **123** (1966), 200–228.

[9] Takeuchi, S., On completeness of holomorphic principal bundles, *Nagoya Math. J.*, **57** (1974), 121–138.