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Some Inequalities for Generalized Commutators

By

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Abstract

Let A, B be linear operators on a Banach space $\mathscr{H}$ with spectra in the set $S = \mathbf{C}\backslash(-\infty, 0)$, for which

$$\|t(A+t)^{-1}\| \leqq M, \ \|t(B+t)^{-1}\| \leqq N \qquad (t>0).$$

Then for a certain class of holomorphic functions f preserving S and $f(0) = \lim_{s\to 0+} f(s) = 0$, one has

$$\|f(A)X - Xf(B)\| \leqq af(b\|AX - XB\|) \qquad \text{for all } X \in B(\mathscr{H}), \ \|X\| \leqq 1,$$

where $a = 2(M+N)$, $b = MN/(M+N)$.

At that, if $\alpha \in \mathbf{C}$, $0 < \mathrm{Re}\alpha \leqq 1$, then for all $X \in B(\mathscr{H})$

$$\|A^\alpha X - XB^\alpha\| \leqq C(M, N, \alpha)\|X\|^{1-\mathrm{Re}\alpha}\|AX - XB\|^{\mathrm{Re}\alpha}$$

where

$$C(M, N, \alpha) = \frac{|\sin\alpha\pi|(MN)^{\mathrm{Re}\alpha}(M+N)^{1-\mathrm{Re}\alpha}}{\pi\mathrm{Re}\alpha(1-\mathrm{Re}\alpha)}.$$

Also

$$\|\exp(-tA^{1/2})X - X\exp(-tB^{1/2})\| \leqq C(M, N)t^{2/3}\|X\|^{2/3}\|AX - XB\|^{1/3},$$

where

$$C(M, N) = 3\pi^{-1}(MN)^{1/3}(M+N)^{2/3}, \ t \geqq 0, \ X \in B(\mathscr{H}).$$

§ 1. Introduction

Recently F. Kittaneh and H. Kosaki [4] obtained the interesting inequalities:

Let A, B be two positive operators on a Hilbert space $\mathscr{H}$ and f – an operator monotone function on $(0, \infty)$. Then

(1.1) If $\lim_{s\to 0+} f(s) = 0$, one has

$$\|f(A) - f(B)\| \leqq f(\|A - B\|),$$

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(1.2) If $A \geqq a \geqq 0$ and $B \geqq b \geqq 0$, one has for all $X \in B(\mathscr{H})$

$$\|f(A)X - Xf(B)\|_p \leqq C(a, b)\|AX - XB\|_p, \; 1 \leqq p \leqq \infty,$$

where

$$C(a, b) = \begin{cases} \dfrac{f(a) - f(b)}{a - b} & a \neq b \\ f'(a) & a = b \end{cases}.$$

In particular, when $f(s) = s^\alpha$, $0 < \alpha \leqq 1$, and $A \geqq c > 0$, $B \geqq c > 0$, (1.2) turns into

(1.3) $$\|A^\alpha X - XB^\alpha\|_p \leqq \alpha c^{\alpha - 1}\|AX - XB\|_p \qquad (X \in B(\mathscr{H})).$$

Unfortunately, when $c = 0$, or $a = b = 0$ and $f'(a) = \infty$ in (1.2), it is impossible to estimate $\|f(A)X - Xf(B)\|_p$ in terms of $\|AX - XB\|_p$ there even for $p = \infty$.

Such an estimate is sometimes important. For instance, in another development – for use in C^*-algebra theory, W. Arveson proved the following result (see [1], the Lemma on p. 332)

(1.4) Let $\mathscr{U}$ be a C^*-algebra, f – a continuous function on [0, 1] (if $\mathscr{U}$ has no unit, one assumes $f(0) = 0$) and let $\varepsilon > 0$. There exists $\delta > 0$ such that any time when a, x are in the unit ball of $\mathscr{U}$ and $a \geqq 0$, one has

$$\|ax - xa\| < \delta \qquad \text{implies } \|f(a)x - xf(a)\| < \varepsilon.$$

For $f(s) = s^\alpha$, $0 < \alpha < 1$, a more precise estimate was found in the paper [7] (Lemma 2.1):

(1.5) If a, x are elements in a C^*-algebra and $a \geqq 0$, then for any α, $0 < \alpha < 1$ one has

$$\|a^\alpha x - xa^\alpha\| \leqq (1 - \alpha)^{\alpha - 1}\|x\|^{1 - \alpha}\|ax - xa\|^\alpha.$$

As mentioned in the remarks on p. 4 of [7], U. Haagerup has reduced the constant $(1 - \alpha)^{\alpha - 1}$ in (1.5) to $\sin\alpha\pi(\pi\alpha(1 - \alpha))^{-1}$. We have come to this result independently and we present the refined inequality here in a general setting – see below (2.6).

The aim of these notes is to describe a method of obtaining inequalities for generalized commutators and to illustrate it by some examples, thus complementing the results of Kittaneh-Kosaki. The inequalities here are stated for Banach space operators, although Banach algebra elements could be used also. The operator framework keeps in line with the notations in Kittaneh-Kosaki's paper and makes it possible to consider unbounded operators as well.

§ 2. Inequalities

We consider operator monotone functions f on $[0, \infty)$ of the form

$$(2.1)\qquad f(z) = kz + \int_0^\infty \frac{z}{z+t}\,d\mu(t), \qquad k \geqq 0,$$

$$f(0) = \lim_{s\to 0+} f(s) = 0, \quad \int_1^\infty d\mu(t)/t < \infty, \qquad \mu(0) = 0,$$

and for the monotone increasing function μ we assume also that μ' exists and is positive in $(0, \infty)$. We also consider a pair of two bounded linear operators A, B on a complex Banach space $\mathscr{H}$ satisfying

$$(2.2)\qquad \mathrm{Sp}(A),\ \mathrm{Sp}(B) \subset \mathbf{C}\backslash(-\infty, 0),$$

$$\|t(A+t)^{-1}\| \leqq M, \ \|t(B+t)^{-1}\| \leqq N \qquad \text{for all } t > 0.$$

For such functions f and operators A, B one can define

$$f(A) = kA + \int_0^\infty A(A+t)^{-1}\,d\mu(t) \qquad \text{and similarly } f(B).$$

As $$\int_1^\infty \|A(A+t)^{-1}\|\,d\mu(t) \leqq M\|A\| \int_1^\infty \frac{d\mu(t)}{t} < \infty$$

and $$\int_0^1 \|A(A+t)^{-1}\|\,d\mu(t) = \int_0^1 \|1 - t(A+t)^{-1}\|\,d\mu(t) \leqq (M+1)\int_0^1 d\mu(t) < \infty,$$

the above integral is a uniformly convergent Bochner-Stieltjes operator valued integral and $f(A)$ is a bounded linear operator on $\mathscr{H}$.

Let X be in the unit ball of $B(\mathscr{H})$ and put $c = \|AX - XB\|$. For all $s \geqq 0$ we have

$$(2.3)\qquad \|f(A)X - Xf(B)\| = \|k(AX - XB) + \int_0^s t(X(B+t)^{-1} - (A+t)^{-1}X)\,d\mu(t)$$

$$+ \int_s^\infty t(A+t)^{-1}(AX - XB)(B+t)^{-1}\,d\mu(t)\|$$

$$\leqq kc + (M+N)\mu(s) + cMN\int_s^\infty \frac{d\mu(t)}{t}.$$

This last expression takes its minimum for $s \geqq 0$ in $s = \lambda = cMN/(M+N)$, as its derivative with respect to s for $s > 0$ is

$$\mu'(s)(M + N - cMNs^{-1}).$$

Let $a = 2(M+N)$ and let $MN \geqq 1/2$ or $k = 0$. One easily checks that

$$kc + (M + N)\int_0^{\lambda} d\mu(t) + cMN\int_{\lambda}^{\infty} d\mu(t)/t$$

$$\leqq a\left(k\lambda + \int_0^{\lambda}\frac{\lambda}{\lambda + t}d\mu(t) + \int_{\lambda}^{\infty}\frac{\lambda}{t + \lambda}d\mu(t)\right) = af(\lambda).$$

This way we have proved the theorem:

Theorem 2.1. *Let A, B be two operators on the Banach space $\mathscr{H}$ satisfying* (2.2). *Then for every function f of the form* (2.1) *and every $X \in B(\mathscr{H})$, $\|X\| \leqq 1$:*

$$\|f(A)X - Xf(B)\| \leqq af(b\|AX - XB\|) \tag{2.4}$$

where $a = 2(M + N)$, $b = MN/(M + N)$ and $MN \geqq 1/2$ or $k = 0$.

Note that when $MN \geqq 1$ and $\int_0^{\infty} d\mu(t)/t < \infty$, i.e. $f'(0)$ is finite, it follows from (2.3) for $s = 0$:

$$\|f(A)X - Xf(B)\| \leqq f'(0)MN\|AX - XB\| \qquad \text{for all } X \in B(\mathscr{H}) \tag{2.5}$$

in accordance with (1.2).

We shall present now one variety of (2.4) which is of particular interest. Starting from the representation

$$z^{\alpha} = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty}\frac{z}{z + t}t^{\alpha - 1}dt, \ z \in \mathbf{C}\backslash(-\infty, 0), \ 0 < \mathrm{Re}\,\alpha \leqq 1$$

one can define the fractional powers

$$A^{\alpha} = \frac{\sin\alpha\pi}{\pi}\int_0^{\infty}(A + t)^{-1}At^{\alpha - 1}dt \text{ and in the same way } B^{\alpha} \text{ (see [2])}.$$

Proposition 2.2. *For A, B satisfying* (2.2), $0 < \mathrm{Re}\,\alpha \leqq 1$, *and all $X \in B(\mathscr{H})$ one has*

$$\|A^{\alpha}X - XB^{\alpha}\| \leqq C(M, N, \alpha)\|X\|^{1 - \mathrm{Re}\,\alpha}\|AX - XB\|^{\mathrm{Re}\,\alpha} \tag{2.6}$$

where $$C(M, N, \alpha) = \frac{|\sin\alpha\pi|}{\pi\,\mathrm{Re}\,\alpha(1 - \mathrm{Re}\,\alpha)}(MN)^{\mathrm{Re}\,\alpha}(M + N)^{1 - \mathrm{Re}\,\alpha}$$

Proof. Proceeding as in (2.3) we find for every $s > 0$ and $X \in B(\mathscr{H})$

$$\|A^{\alpha}X - XB^{\alpha}\|$$
$$\leqq \frac{|\sin\alpha\pi|}{\pi}\left(\|X\|(M + N)\int_0^{s}|t^{\alpha - 1}|dt + \|AX - XB\|MN\int_s^{\infty}|t^{\alpha - 2}|dt\right)$$

$$\leqq \frac{|\sin\alpha\pi|}{\pi}\left(\frac{\|X\|(M+N)}{\mathrm{Re}\alpha}s^{\mathrm{Re}\alpha}+\frac{\|AX-XB\|MN}{1-\mathrm{Re}\alpha}s^{\mathrm{Re}\alpha-1}\right) \qquad (\text{as } |t^{\gamma}|\leqq t^{\mathrm{Re}\gamma}).$$

Minimizing the right hand side for $s>0$ we get (2.6).

Proposition 2.3. *Let H be a Hilbert space, $\|\cdot\|_p$ – the Schatten norm, $1\leqq p\leqq\infty$, f – a function as in* (2.1) *and $A, B\in B(\mathscr{H})$ – operators satisfying* (2.2). *Then one has*

(2.7) $\|f(A)X-Xf(B)\|_p\leqq af(b\|AX-XB\|_p)$ *for all* $X\in B(\mathscr{H})$, $\|X\|_p\leqq 1$,

where $a=2(M+N)$, $b=MN/(M+N)$ *and either* $MN\geqq 1/2$ *or* $k=0$.
Also

(2.8) $$\|A^{\alpha}X-XB^{\alpha}\|_p\leqq C(M,N,\alpha)\|X\|_p^{1-\mathrm{Re}\alpha}\|AX-XB\|_p^{\mathrm{Re}\alpha}$$

for all $X\in B(\mathscr{H})$, $\|X\|_p<\infty$, $0<\mathrm{Re}\alpha\leqq 1$ *and* $C(M,N,\alpha)$ *as in* (2.6).

Proof. We repeat the proofs of (2.4) and (2.6), using $\|\cdot\|_p$ instead of $\|\cdot\|$ and estimating on the right hand sides of the inequalities in the following way:

$$\|X(B+t)^{-1}-(A+t)^{-1}X\|_p\leqq(M+N)\|X\|_p t^{-1},$$
$$\|(A+t)^{-1}(AX-XB)(B+t)^{-1}\|_p\leqq MN\|AX-XB\|_p t^{-2}.$$

Remarks. The technique used in the above proofs – dividing the integral in two parts on $[0,s]$ and $[s,\infty)$ estimated in different ways and then minimizing for $s>0$ – is not new. It has been used, for instance, by Matsaev and Palant [6] for obtaining the inequality

(2.9) $$\|A^{\alpha}-B^{\alpha}\|\leqq C(M,N,\alpha)\|A-B\|^{\alpha}$$
$$(0<\alpha\leqq 1,\ C(M,N,\alpha) \text{ as in } (2.6))$$

for operators A, B essentially as in (2.2).

This method has been used also for proving moment type inequalities

(2.10) $$\|A_x^{\alpha}\|\leqq C(M,\alpha)\|x\|^{1-\alpha}\|Ax\|^{\alpha} \qquad (x\in\mathscr{H})$$
$$(0<\alpha\leqq 1,\ C(M,\alpha)=C(M,1,\alpha))$$

for A as in (2.2) – see [5], [8].

Note that Matsaev-Palant's inequality (2.9) follows immediately from (2.6) by putting there $X=1=$ identity operator. In the same way we obtain from (2.4) the immediate corollary:

Corollary 2.4. *Let f be as in* (2.1) *and let A, B satisfy* (2.2). *Then*

(2.11) $$\|f(A)-f(B)\|\leqq af(b\|A-B\|)$$

(a, b *as in* (2.4)), *which is a Banach space variety of* (1.1).

In this connection see also Theorem 3.4 in [4]. A natural modification of the proof turns it into a theorem for generalized commutators.

§ 3. Further Inequalities

We want to point out that many functions of the form

$$f(z) = \int_0^\infty \frac{z}{z+t} g(t)dt, \text{ or } f(z) = \int_0^\infty \frac{h(t)}{z+t} dt, \ z \in \mathbf{C} \backslash (-\infty, 0),$$

with explicitly given g, h can successfully be used for obtaining moment type inequalities for generalized commutators via the above method. To illustrate this we shall present the following example:

$$\exp(-tz^{1/2}) = \frac{1}{\pi} \int_0^\infty \frac{\sin(t\sqrt{\lambda})}{z+\lambda} d\lambda \qquad (t \geqq 0) \text{(see [2], [10])}.$$

It is well-known that if A is a closed linear operator satisfying (2.2), its square root $A^{1/2}$ (defined as in Section 2) is the generator of a (holomorphic) one-parameter operator semigroup which can be determined by the formula

$$\exp(-tA^{1/2}) = \frac{1}{\pi} \int_0^\infty (A+\lambda)^{-1} \sin(t\sqrt{\lambda}) d\lambda, \qquad t \geqq 0, \tag{3.1}$$

(see [2], Section 5).

Proposition 3.1. *Let A, B be closed (possibly unbounded) linear operators on the Banach space $\mathscr{H}$ satisfying* (2.2). *Then for any $X \in B(\mathscr{H})$ for which $AX - XB$ is bounded, we have*

$$\|\exp(-tA^{1/2})X - X\exp(-tB^{1/2})\| \leqq C(M, N) t^{2/3} \|X\|^{2/3} \|AX - XB\|^{1/3}, \tag{3.2}$$

$$C(M, N) = 3\pi^{-1}(MN)^{1/3}(M+N)^{2/3}, \qquad t \geqq 0.$$

Proof. For all $s > 0$:

$$\begin{aligned} &\pi \|\exp(-tA^{1/2})X - X\exp(-tB^{1/2})\| \\ &\leqq t \int_0^s \left| \frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}} \right| \sqrt{\lambda} \|(A+\lambda)^{-1}X - X(B+\lambda)^{-1}\| d\lambda \\ &\quad + \int_s^\infty |\sin(t\sqrt{\lambda})| \ \|(A+\lambda)^{-1}(AX - XB)(B+\lambda)^{-1}\| d\lambda \\ &\leqq 2t\|X\|(M+N)\sqrt{s} + MN\|AX - XB\| s^{-1}. \end{aligned}$$

Minimizing this for $s > 0$ we come to (3.2).

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