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On Norm-Dependent Positive Definite Functions

By

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Summary: Any norm-dependent positive definite function on an infinite dimensional normed space can be written as a superposition of $\exp(-c\|\cdot\|^2)$. Conversely, for a Hilbert space, any superposition of $\exp(-c\|\cdot\|^2)$ is positive definite. A norm-dependent positive definite function exists only if the norm is of cotype 2. If $\exp(-\|\cdot\|^\alpha)$ is positive definite for some $\alpha>0$, such α form an interval $(0, \alpha_0]$ where $\alpha_0 \leqq 2$. If $\alpha_0=2$, then $\|\cdot\|$ is a Hilbertian norm. For (l^p), $0<p\leqq 2$, we have $\alpha_0=p$. (Though $\|x\|=(\sum_n |x_n|^p)^{1/p}$ is not a norm for $0<p<1$, the last statement remains valid).

In [1], Chapter 3, it was shown that on a Hilbert space, any positive definite function dependent only on the norm can be written in the form:

$$\chi(\xi)=\int_{[0,\infty)} \exp(-c\|\xi\|^2)d\nu(c) \tag{1}$$

where ν is a finite measur on $[0, \infty)$. The proof is based on Bernstein's theorem, which claims:

Proposition 1 (Bernstein's Theorem). *Let $f(t)$ be a function on $[0, \infty)$. If and only if $f(t)$ is continuous and completely monotone, it is the Laplace transform of a positive measure on $[0, \infty)$, namely it can be written as*

$$f(t)=\int_{[0,\infty)} \exp(-st)d\nu(s). \tag{2}$$

Here, complete monotoneness is defined as:

Definition 1. A function on $[0, \infty)$ is said to be completely monotone, if for any $t, \tau>0$ and $n=0, 1, 2, \cdots$ we have

$$(-1)^n \Delta_\tau^n f(t) \geqq 0 \tag{3}$$

where

$$\Delta_\tau f(t)=f(t+\tau)-f(t). \tag{4}$$

Note that if $f(t)$ is known to be infinitely differentiable, complete mono-

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toneness is characterized by $(-1)^n(d^n/dt^n)f(t)\geqq 0$.

The proof of Bernstein's theorem, omitted here, can be found for instance in [2], Chapter 4.

For a Hilbert space, if $\chi(\xi)=\varphi(\|\xi\|^2)$ is positive definite, then φ must be completely monotone. The proof is given in [1] and also in [3] with some related discussions. But in favor of Dvoretzky's theorem, this statement is kept valid for any infinite dimensional normed space.

Proposition 2 (**Dvoretzky's theorem**). *Let X be an infinite dimensional normed space. For any $\varepsilon>0$ and positive integer n, there exist an n-dimensional subspace R and a Hilbertian norm $\|\cdot\|_H$ on R such that*

$$(1-\varepsilon)\|\xi\|_H\leqq\|\xi\|\leqq(1+\varepsilon)\|\xi\|_H \qquad for \quad \forall\xi\in R. \tag{5}$$

This theorem appeared in [4], and arose many researcher's interest which led to more detailed discussions, for instance [5].

Proposition 3. *Let X be an infinite dimensional normed space.*

If $\chi(\xi)=\varphi(\|\xi\|^2)$ is continuous and positive definite, φ must be completely monotone.

Proof. For given $t_0>0$ and $\eta>0$, there exists an $\varepsilon>0$ such that

$$|\varphi(t)-\varphi(t_0)|\leqq\eta \qquad \text{for} \quad (1-\varepsilon)^2t_0\leqq t\leqq(1+\varepsilon)^2t_0. \tag{6}$$

For this ε and any given positive integer n, there exist an n-dimensional subspace R of X and a Hilbertian norm $\|\cdot\|_H$ on R which satisfies (5).

Let $\{e_i\}_{i=1}^n$ be a CONS of R in $\|\cdot\|_H$. Since χ is positive definite, we have

$$\sum_{i=1}^n\sum_{j=1}^n\chi\left(\sqrt{\frac{t_0}{2}}(e_i-e_j)\right)\geqq 0. \tag{7}$$

For $i\neq j$, we have $\left\|\sqrt{\frac{t_0}{2}}(e_i-e_j)\right\|_H^2=t_0$, so that $\chi\left(\sqrt{\frac{t_0}{2}}(e_i-e_j)\right)\leqq\varphi(t_0)+\eta$. Thus we get

$$n\chi(0)+n(n-1)(\varphi(t_0)+\eta)\geqq 0,$$

hence $\varphi(t_0)\geqq-\dfrac{\chi(0)}{n-1}-\eta$. Since $n>0$ and $\eta>0$ are arbitrary, we must have $\varphi(t_0)\geqq 0$.

Next, for given $t_0>0, \tau>0$ and $\eta>0$, we asssume that (6) holds also for $t_0+\tau$ instead of t_0 and that R is $(n+1)$-dimensional and $\{e_i\}_{i=0}^n$ is its CONS in $\|\cdot\|_H$. Put $\xi_i=\sqrt{\frac{t_0}{2}}e_i$, $\alpha_i=1$ for $1\leqq i\leqq n$ and $\xi_i=\sqrt{\frac{t_0}{2}}e_{n-i}+\sqrt{\tau}e_0$, $\alpha_i=-1$ for $n+1\leqq i\leqq 2n$. Then, since χ is positive definite, we have

$$0 \leqq \sum_{i,j=1}^{2n} \alpha_i \alpha_j \chi(\xi_i - \xi_j) \tag{8}$$
$$= 2 \sum_{i,j=1}^{n} \left[\chi\left(\sqrt{\frac{t_0}{2}}(e_i - e_j)\right) - \chi\left(\sqrt{\frac{t_0}{2}}(e_i - e_j) \pm \sqrt{\tau} e_0\right) \right].$$

Thus we get

$$0 \leqq n\chi(0) + n(n-1)(\varphi(t_0) + \eta) - n(\varphi(\tau) - \eta) - n(n-1)(\varphi(t_0 + \tau) - \eta),$$

hence

$$\varphi(t_0) - \varphi(t_0 + \tau) \geqq -\frac{\chi(0)}{n-1} - \eta + \frac{\varphi(\tau) - \eta}{n-1} - \eta.$$

Since $n > 0$ and $\eta > 0$ are arbitrary, we must have $\varphi(t_0) - \varphi(t_0 + \tau) \geqq 0$.

In a similar way, we can prove $(-1)^m \Delta_\tau^m \varphi(t_0) \geqq 0$ for any m, hence φ is completely monotone. q.e.d.

Combining the above Proposition 3 with Proposition 1, we obtain the following result.

Proposition 4. *Let X be an infinite dimensional normed space. If a positive definite function $\chi(\xi)$ is continuous and depends only on the norm $\|\xi\|$, it is written in the form of* (1).

Remark 1. For a Hilbert space, any function $\chi(\xi)$ in the form of (1) is positive definite, but for a general infinite dimensional normed space, the converse is false. Indeed, we know:

Proposition 5. *If $\chi(\xi) = \exp(-\|\xi\|^2)$ is positive definite on a normed space X, then X must be a Hilbert space.*

Proof. By (infinite dimensional) Bochner's theorem (for instance, c.f. [6]), χ corresponds to a σ-additive measure μ on X^a, the algebraic dual space of X. The correspondence is

$$\chi(\xi) = \int \exp(ix(\xi)) d\mu(x), \ \xi \in X, \ x \in X^a. \tag{9}$$

For a fixed $\xi \neq 0$, the equality $\chi(t\xi) = \exp(-t^2 \|\xi\|^2)$ means that $x(\xi)$ follows one-dimensional Gaussian distribution of the variance $2\|\xi\|^2$. So that we have

$$\|\xi\|^2 = \frac{1}{2} \int x(\xi)^2 d\mu(x). \tag{10}$$

Thus, the function $\Phi_\xi(x) = x(\xi)$ belongs to $L^2(\mu)$, and the map $\xi \to \frac{1}{\sqrt{2}} \Phi_\xi$ becomes a norm-preserving imbedding of X into $L^2(\mu)$. Hence X is a Hilbert space as a subspace of $L^2(\mu)$. q.e.d.

Remark 2. A norm-dependent positive definite function does not always exist. Especially, if the norm is not of cotype 2, it never exists. (cf. [6] Part B Theorem 19.7 and its corollary).

Next, we shall discuss about whether $\exp(-\|\cdot\|^{\alpha})$ is positive definite or not. The following results are essentially known ([7], [8]), but we shall formulate and prove them in our way. For a preparation, we state a lemma.

Lemma. *On* $[0, \infty)$, *the function* $f_{\alpha}(t)=\exp(-t^{\alpha})$ *is completely monotone if and only if* $0\leqq\alpha\leqq 1$.

Proof. Evidently $f_{\alpha}(t)$ is not completely monotone for $\alpha<0$. We shall check the sign of $\frac{d^{n}f_{\alpha}}{dt^{n}}$.

$$\frac{d}{dt}f_{\alpha}=-\alpha t^{\alpha-1}\exp(-t^{\alpha})\leqq 0$$

is all right if $\alpha\geqq 0$.

$$\frac{d^{2}}{dt^{2}}f_{\alpha}=[-\alpha(\alpha-1)t^{\alpha-2}+\alpha^{2}t^{2\alpha-2}]\exp(-t^{\alpha})\geqq 0$$

is true if $0\leqq\alpha\leqq 1$, but false for sufficiently small t if $\alpha>1$. Suppose that

$$\frac{d^{n}}{dt^{n}}f_{\alpha}=\sum_{k=1}^{n}a_{kn}t^{k\alpha-n}\exp(-t^{\alpha}) \quad\text{and}\quad (-1)^{n}a_{kn}\geqq 0 \quad\text{for}\quad 0\leqq\alpha\leqq 1.$$

Then we have

$$\frac{d^{n+1}}{dt^{n+1}}f_{\alpha}=\sum_{k=1}^{n}[a_{kn}(k\alpha-n)t^{k\alpha-n-1}-a_{kn}\alpha t^{(k+1)\alpha-n-1}]\exp(-t^{\alpha}).$$

This means that

$$\begin{cases} a_{1,n+1}=a_{1n}(\alpha-n) \\ a_{k,n+1}=a_{kn}(k\alpha-n)-a_{k-1,n}\alpha \quad (2\leqq k\leqq n) \\ a_{n+1,n+1}=-a_{nn}\alpha. \end{cases}$$

Thus, considering $k\leqq n$ and $0\leqq\alpha\leqq 1$, we get $(-1)^{n+1}a_{k,n+1}\geqq 0$. This assures that $f_{\alpha}(t)$ is completely monotone if $0\leqq\alpha\leqq 1$. q.e.d.

Proposition 6. *If* $\exp(-\|\xi\|^{\alpha_0})$ *is positive definite on a normed space* X, *so is* $\exp(-\|\xi\|^{\alpha})$ *for* $0\leqq\alpha\leqq\alpha_0$.

Proof. Since $\exp(-t^{\alpha/\alpha_0})$ is completely monotone, from Bernstein's theorem we have

$$\exp(-\|\xi\|^{\alpha})=\int_{[0,\infty)}\exp(-s\|\xi\|^{\alpha_0})d\nu(s). \tag{11}$$

Since positive definiteness is closed under pointwise convergence and linear combination with positive coefficients, (11) assures that $\exp(-\|\xi\|^{\alpha})$ is positive definite.

Remark 3. The set $\{\alpha>0;\ \exp(-\|\xi\|^{\alpha})$ is positive definite$\}$ forms an interval, if not empty. This interval is closed at right, since positive definiteness is closed under pointwise convergence, so that it is of the form of $(0, \alpha_0]$.

We have $\alpha_0\leqq 2$, since every norm-dependent positive definite function is written in the form of (1), and $\exp(-t^{\alpha/2})$ is not completely monotone for $\alpha>2$. We have $\alpha_0=2$ if and only if X is a Hilbert space.

Proposition 7. *Let $\varphi(\xi)$ be a non-negative function on X. Suppose that for any n, m and $t>0$, $\tau>0$, there exist ξ_i, ξ'_j $(i=1, 2, \cdots, n,\ j=1, 2, \cdots, m)$ such that*

$$\varphi(\xi_i-\xi_j)=t$$

and

$$\varphi(\pm\xi'_{j_1}\pm\cdots\pm\xi'_{j_k}+\xi_i-\xi_j)=t+k\tau$$
$$\text{for}\quad 1\leqq i\neq j\leqq n,\ 1\leqq j_1<j_2<\cdots<j_k\leqq m.$$

Then, every $\varphi(\cdot)$-dependent positive definite function $\chi(\xi)=F(\varphi(\xi))$ is written in the form of

$$\chi(\xi)=\int_{[0,\infty)}\exp(-s\varphi(\xi))d\nu(s). \tag{12}$$

Proof is obtained similarly as the proof of Proposition 3. In this case $\chi(\xi)=F(\varphi(\xi))$ implies that F is completely monotone.

Corollary. *For the space (l^p), $0<p\leqq 2$, every norm-dependent positive definite function $\chi(\xi)$ is written in the form of*

$$\chi(\xi)=\int_{[0,\infty)}\exp(-s\|\xi\|^p)d\nu(s). \tag{13}$$

Remark 4. Conversely, every $\chi(\xi)$ in the form of (13) is positive definite on (l^p), because $\exp(-|t|^p)$ is positive definite on $\boldsymbol{R}$ and $\exp(-\|\xi\|^p)=\prod_{k=1}^{\infty}\exp(-|\xi_k|^p)$.

Remark 5. The criterion of this corollary shows us that $\exp(-\|\xi\|^{p'})$ is positive definite if and only if $0<p'\leqq p$. Thus we have $\alpha_0=p$ for (l^p), $0<p\leqq 2$. (α_0 is of the same meaning as in Remark 3).

The discussions in the proof of Proposition 7 do not require any norm. So, Corollary and Remarks 4 and 5 are valid also for $0<p<1$. ($\|\xi\|=(\sum_{k=1}^{\infty}|\xi_k|^p)^{1/p}$, whether it is a norm or not).

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