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Construction of Gelfand-Tsetlin Basis for $\mathcal{U}_q(gl(N+1))$-modules

By

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Abstract

The Gelfand-Tsetlin basis for irreducible $\mathcal{U}_q(gl(N+1))$-modules of finite dimensions is constructed by means of the lowering operators.

§ 0. Introduction

In the previous paper [1], we constructed the Gelfand-Tsetlin basis for irreducible $\mathcal{U}_q(gl(N+1))$-modules of finite dimensions, in terms of the lowering operators. In this paper, we will give detailed accounts of our construction of this basis.

We give the result when q is a non-zero complex variable and not a root of unity, however from the view of technical arguments in its proof we also have to deal with the case of q to be a transcendental element over $\mathbf{C}$.

Let $\mathbf{K}=\mathbf{C}(q^{1/2})$ be the field of rational functions in one variable $q^{1/2}$ over the complex number field $\mathbf{C}$ if q is a transcendental element and let $\mathbf{K}=\mathbf{C}$ if q is a complex variable. The quantum universal enveloping algebra $\mathcal{U}_q=\mathcal{U}_q(gl(N+1))$ associated with $gl(N+1)$ is the associative algebra over $\mathbf{K}$([2]).

In [6] Rosso has shown the representation theory of $\mathcal{U}_q(g)$ associated with a complex simple Lie algebra g of finite dimensions. His results are easily extended for $\mathcal{U}_q(gl(N+1))$. In [2] and [5] when q is transcendental, Jimbo has shown that an irreducible finite dimensional left $\mathcal{U}(gl(N+1))$-module $\tilde{V}(\Lambda)$ with highest weight $\Lambda=(\lambda_0, \cdots, \lambda_N)(\lambda_j\in\mathbf{Z}, \lambda_0\geqq\cdots\geqq\lambda_N)$ is liftable to a $\mathcal{U}_q(gl(N+1))$-module. Namely, he has shown the existence of an irreducible left $\mathcal{U}_q(gl(N+1))$-module $V(q^{(1/2)\Lambda})$ of the same dimensions as $\tilde{V}(\Lambda)$ with highest weight $q^{(1/2)\Lambda}=(q^{(1/2)\lambda_0}, \cdots, q^{(1/2)\lambda_N})$ for a transcendental element q. Throughout this paper we set $q^{(1/2)\Lambda}=(q^{(1/2)\lambda_0}, \cdots, q^{(1/2)\lambda_N})$ for $\Lambda=(\lambda_0, \cdots, \lambda_N)$. Also we have an irreducible finite dimensional right $\mathcal{U}_q$-module $V(q^{(1/2)\Lambda})^*$ with the same properties as

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$V(q^{(1/2)\Lambda})$. This module is regarded as the dual of $V(q^{(1/2)\Lambda})$. Using these results we construct Gelfand-Tsetlin basis for $\mathcal{U}_q$-module $V(q^{(1/2)\Lambda})$. Jimbo [5] has constructed the Gelfand-Tsetlin basis for the module $V(b^{(1/2)\Lambda})$, and has written down the action of $\mathcal{U}_q$ on this basis. We will reconstruct the Gelfand-Tsetlin basis in terms of the lowering operators. (As for discussions on the classical case, see Zhelobenko [4].)

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§1. The Definition and the Representation Theory of $\mathcal{U}_q(gl(N+1))$

Let q be a transcendental element over $\mathbf{C}$ (resp. a non-zero complex variable and not a root of unity) and $\mathbf{K}=\mathbf{C}(q^{1/2})$(resp. $\mathbf{K}=\mathbf{C}$). The quantum universal enveloping algebra $\mathcal{U}_q(sl(N+1))$ [2] is the associative algebra over $\mathbf{K}$ generated by the elements $k_i^{\pm 1}$, e_i, f_i $(1\leqq i\leqq N)$ with the following relations;

$$k_i\cdot k_i^{-1}=k_i^{-1}\cdot k_i=1, \qquad k_i\cdot k_j=k_j\cdot k_i,$$

$$k_ie_jk_i^{-1}=\begin{cases} q^{-1/2}e_j \\ qe_j, \\ e_j \end{cases} \qquad k_if_jk_i^{-1}=\begin{cases} q^{1/2}f_j & (i=j\pm 1) \\ q^{-1}f_j & (i=j), \\ f_j & (i\neq j\pm 1,\ j) \end{cases}$$

$$[e_i, f_j]=\delta_{ij}\frac{k_i^2-k_i^2}{q-q^{-1}},$$

$$[e_i, e_j]=[f_i, f_j]=0 \qquad \text{for} \quad |i-j|\geqq 2,$$

$$e_i^2e_{i\pm 1}-[2]e_ie_{i\pm 1}e_i+e_{i\pm 1}e_i^2=0,$$

$$f_i^2f_{i\pm 1}-[2]f_if_{i\pm 1}f_i+f_{i\pm 1}f_i^2=0,$$

where $[m]=\dfrac{q^m-q^{-m}}{q-q^{-1}}$ is called a q-integer.

Furthermore $\mathcal{U}_q=\mathcal{U}_q(gl(N+1))$ [2] is defined by adjointing to $\mathcal{U}_q(sl(N+1))$ the elements $q^{\pm(1/2)\varepsilon_i}$ $(0\leqq i\leqq N)$ so that $k_i=q^{1/2(\varepsilon_{i-1}-\varepsilon_i)}$ and that $q^{\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_N}$ belongs to the center. We give $\mathcal{U}_q$ a structure of a non-cocommutative Hopf algebra: The coproduct Δ: $\mathcal{U}_q\to\mathcal{U}_q\otimes\mathcal{U}_q$ is defined on the generators by $\Delta(q^{\pm(1/2)\varepsilon_i})=q^{\pm(1/2)\varepsilon_i}\otimes q^{\pm(1/2)\varepsilon_i}(0\leqq i\leqq N)\Delta(e_j)=e_j\otimes k_j+k_j^{-1}\otimes e_j, \Delta(f_j)=f_j\otimes k_j+k_j^{-1}\otimes f_j(1\leqq j\leqq N)$. The counit ε: $\mathcal{U}_q\to\mathbf{K}$ and the antipode S: $\mathcal{U}_q\to\mathcal{U}_q$ are respectively defined by

$$\varepsilon(q^{\pm(1/2)\varepsilon_i})=1, \qquad \varepsilon(e_j)=\varepsilon(f_j)=0,$$

$$S(q^{\pm(1/2)\varepsilon_i})=q^{\mp(1/2)\varepsilon_i}, \quad S(e_j)=-qe_j, \quad S(f_j)=-q^{-1}f_j \ (0\leqq i\leqq N,\ 1\leqq j\leqq N).$$

Let g be a complex simple Lie algebra of finite dimensions and $\mathcal{U}_q(g)$ be a quantum universal enveloping algebra associated with g. Rosso [6] has established the representation theory of $\mathcal{U}_q(g)$. We state his results in the case of $g=sl(N+1)$. Let V be a left $\mathcal{U}_q(sl(N+1))$-module. For any $\mu=(\mu_1, \cdots, \mu_N)\in(\mathbf{K}^*)^N$, we set $V_\mu=\{v\in V \mid k_i v=\mu_i v\ (1\leqq i\leqq N)\}$. Whenever $V_\mu\neq 0$, we call it a weight space and call μ a weight of V. A left $\mathcal{U}_q(sl(N+1))$-module V is said to be a highest weight left module with highest weight $\lambda=(\lambda_1, \cdots, \lambda_N)\in(\mathbf{K}^*)^N$, if there exists a non-zero vector v such that

$$k_i v=\lambda_i v\ (1\leqq i\leqq N),\qquad e_i v=0\ (1\leqq i\leqq N),$$
$$V=\mathcal{U}_q\ (sl(N+1))\cdot v.$$

The vector v is called a highest weight vector.

Proposition 1.1 (Rosso [6]).

(i) *Let V be a highest weight left $\mathcal{U}_q(sl(N+1))$-module of highest weight λ. Then:*

(a) *V is the direct sum of its weight spaces.*

(b) *For each weight μ,* $\dim V_\mu<\infty$ *and* $\dim V_\lambda=1$.

(ii) *Any finite dimensional irreducible module is a highest weight module.*

(iii) *Let V, W be highest weight left modules of the same highest weight. If V and W are irreducible, then they are isomorphic.*

(iv) *If V is a finite dimensional irreducible left module of highest weight λ, then $\lambda=(\omega_1 q^{(1/2)\lambda_1}, \cdots, \omega_N q^{(1/2)\lambda_N})$, where for any i, $\omega_i^4=1$ and $\lambda_i\in\mathbf{Z}_+$. Conversely any weight of this form is the highest weight of a finite dimensional irreducible left module.*

(v) *Let V be a finite dimensional left module. Then the module V is completely reducible.*

(vi) *Any finite dimensional highest weight left module is irreducible.*

Further we consider a left $\mathcal{U}_q(gl(N+1))$-module V. For any $\mu=(\mu_0, \cdots, \mu_N)\in(\mathbf{K}^*)^{N+1}$, we define its weight space:

$$V_\mu=\{v\in V \mid q^{(1/2)\varepsilon_i}v=\mu_i v\ (0\leqq i\leqq N)\}.$$

A highest weight left $\mathcal{U}_q(gl(N+1))$-module V with highest weight $\lambda=(\lambda_0, \cdots, \lambda_N)$ is similarly defined: There exists a non-zero vector $v\in V$ such that

$$q^{(1/2)\varepsilon_i}v=\lambda_i v\ (0\leqq i\leqq N),\qquad e_j v=0\ (1\leqq i\leqq N),$$
$$V=\mathcal{U}_q(gl(N+1))\cdot v.$$

Using analogous arguments in [6], we can show the following proposition.

Proposition 1.2.

(i) *Let V be a highest weight left $\mathcal{U}_q(gl(N+1))$-module with highest weight λ. Then:*

(a) *V is the direct sum of its weight spaces.*

(b) *For each weight μ, $\dim V_\mu < \infty$ and $\dim V_\lambda = 1$.*

(ii) *Any finite dimensional irreducible left $\mathcal{U}_q(gl(N+1))$-module is a highest weight module.*

(iii) *If V is a finite dimensional irreducible left $\mathcal{U}_q(gl(N+1))$-module of highest weight λ, then $\lambda = (\alpha\omega_0 q^{(1/2)\lambda_0}, \cdots, \alpha\omega_N q^{(1/2)\lambda_N})$, where $\lambda_i \in \mathbf{Z}$, $\lambda_0 \geqq \cdots \geqq \lambda_N$, $\omega_i^4 = 1$, $\alpha \in \mathbf{K}^*$. Conversely any weight of this form is the highest weight of a finite dimensional irreducible left module.*

(iv) *Let V, W be highest left $\mathcal{U}_q(gl(N+1))$-modules of the same highest weight. If V and W are irreducible, then they are isomorphic.*

(v) *Any finite dimensional highest weight left $\mathcal{U}_q(gl(N+1))$-module is irreducible.*

Let $\mathbf{K}_{\alpha\cdot\omega} = \mathbf{K}\cdot w$ be 1-dimensional left $\mathcal{U}_q$-module with highest weight $\alpha\cdot\omega = (\alpha\omega_0, \cdots, \alpha\omega_N)$, $(q^{(1/2)\varepsilon_i} w = \alpha\omega_i w, e_j w = f_j w = 0, 0 \leqq i \leqq N, 1 \leqq j \leqq N)$ and let $V(q^{(1/2)\Lambda})$ be an irreducible finite dimensional left $\mathcal{U}_q$-module with highest weight $q^{(1/2)\Lambda} = (q^{(1/2)\lambda_0}, \cdots, q^{(1/2)\lambda_N})$. Then $V(q^{(1/2)\Lambda}) \otimes \mathbf{K}_{\alpha\cdot\omega}$ is an irreducible finite dimensional left $\mathcal{U}_q$-module with highest weight $\alpha\cdot\omega\cdot q^{(1/2)\Lambda} = (\alpha\omega_0 q^{(1/2)\lambda_0}, \cdots, \alpha\omega_N q^{(1/2)\lambda_N})$.

Let $\tilde{V}(\Lambda)$ be an irreducible finite dimensional left $\mathcal{U}(gl(N+1))$-module with highest weight $\Lambda = (\lambda_0, \cdots, \lambda_N)$, $(\lambda_i \in \mathbf{Z}, \lambda_0 \geqq \cdots \geqq \lambda_N)$. Then Jimbo has shown the following.

Proposition 1.3 (Jimbo [2]). *When q is transcendental, the modules $V(q^{(1/2)\Lambda})$ and $V(\tilde{\Lambda})$ are of the same dimensions, and moreover in the classical limit $(q \to 1)$ the action of $\mathcal{U}_q(gl(N+1))$ on $V(q^{(1/2)\Lambda})$ tends to that of $\mathcal{U}(gl(N+1))$ on $\tilde{V}(\Lambda)$.*

§2. Gelfand-Tsetlin Basis for Finite Dimensional $\mathcal{U}_q(gl(N+1))$-modules

In this section, we construct the Gelfand-Tsetlin basis for finite dimensional irreducible left $\mathcal{U}_q$-modules $V(q^{(1/2)\Lambda})$ with highest weight $q^{(1/2)\Lambda} = (q^{(1/2)\lambda_0}, \cdots, q^{(1/2)\lambda_N})$, $(\lambda_j \in \mathbf{Z}, \lambda_0 \geqq \cdots \geqq \lambda_N)$. Noting the argument in §1, we consider only such modules with highest weight $q^{(1/2)\Lambda}$.

Let $V = V(q^{(1/2)\Lambda})$ be the above module with the highest weight vector $|vac\rangle$:

$$V = \mathcal{U}_q |vac\rangle, \qquad e_j |vac\rangle = 0 \ (1 \leqq j \leqq N),$$

$$q^{(1/2)\varepsilon_i} |vac\rangle = q^{(1/2)\lambda_i} |vac\rangle \ (0 \leqq i \leqq N).$$

We also let $V^* = V(q^{(1/2)\Lambda})^*$ be a finite dimensional irreducible right $\mathcal{U}_q$-module generated by the highest weight vector $\langle vac|$:

$$V^* = \langle vac| \mathcal{U}_q, \qquad \langle vac| f_j = 0 \ (1 \leqq j \leqq N),$$

$$\langle vac| q^{(1/2)\varepsilon_i} = q^{(1/2)\lambda_i} \langle vac| \ (0 \leqq i \leqq N).$$

V^* is considered as the dual of V.

Let $\mathcal{U}_q(n_+)$ (resp. $\mathcal{U}_q(n_-)$) be the subalgebra of $\mathcal{U}_q(gl(N+1))$ generated by e_j's (resp. f_j's), and $\mathcal{T}$ be the subalgebra generated by $q^{\pm(1/2)\varepsilon_i}$'s. In [6], Rosso has shown the following proposition (see also [7]).

Proposition 2.1. *We have the triangular decomposition*

$$\mathcal{U}_q(gl(N+1))\cong\mathcal{U}_q(n_-)\otimes\mathcal{T}\otimes\mathcal{U}_q(n_+)$$

as vector spaces.

Through the triangular decomposition, we have a natural pairing $V^*\otimes V\to\mathbf{K}$ defined by

$$\langle vac|b\otimes a|vac\rangle\longmapsto\langle vac|ba|vac\rangle\quad(a,\,b\in\mathcal{U}_q),\tag{2.1}$$

with normalization $\langle vac|vac\rangle=1$.

We introduce elements d_{ni} and c_{in} $(0\leqq i\leqq N)$ in the subalgebra of $\mathcal{U}_q$ generated by $q^{(1/2)\varepsilon_i}$ $(0\leqq i\leqq n)$ and e_j, f_j $(1\leqq j\leqq n)$. These elements are called lowering operators and raising operators respectively.

Definition 2.2 (Lowering and raising operators).

For $0\leqq i\leqq n\leqq N$, we define the elements d_{ni}, c_{in} in the following inductive manner;

$$d_{nn}=1,\ d_{n,n-1}=f_n,$$

$$d_{ni}=\langle\varepsilon_i-\varepsilon_{n-1}+n-i\rangle f_nd_{n-1,i}-\langle\varepsilon_i-\varepsilon_{n-1}+n-i-1\rangle d_{n-1,i}f_n,$$

$$c_{nn}=1,\ c_{n-1,n}=e_n,$$

$$c_{in}=c_{i,n-1}e_n\langle\varepsilon_i-\varepsilon_{n-1}+n-i\rangle-e_nc_{i,n-1}\langle\varepsilon_i-\varepsilon_{n-1}+n-i-1\rangle,$$

where $\langle\varepsilon_i-\varepsilon_j+m\rangle=\dfrac{q^{\varepsilon_i-\varepsilon_j+m}-q^{-\varepsilon_i+\varepsilon_j-m}}{q-q^{-1}}$.

The following theorem is fundamental for the construction of the Gelfand-Tsetlin basis.

Theorem 2.3. *For fixed n, the operators d_{ni} (resp. c_{in}) $(0\leqq i\leqq n)$ mutually commute.*

We will prove this only for d_{ni}. For this end, we first note the following commutation relations:

$$f_j\langle\varepsilon_k-\varepsilon_m\rangle=\langle\varepsilon_k-\varepsilon_m-\delta_{jk}-\delta_{j-1,m}+\delta_{jm}+\delta_{j-1,k}\rangle f_j,\tag{2.2}$$

$$e_j\langle\varepsilon_k-\varepsilon_m\rangle=\langle\varepsilon_k-\varepsilon_m+\delta_{jk}+\delta_{j-1,m}-\delta_{jm}-\delta_{j-1,k}\rangle e_j.\tag{2.3}$$

Furthermore we obtain the following lemma by means of the "adjoint relations"

$$f_n^2f_{n-1}-[2]f_nf_{n-1}f_n+f_{n-1}f_n^2=0,\quad f_{n-1}^2f_n-[2]f_{n-1}f_nf_{n-1}+f_nf_{n-1}^2=0$$

and an obvious commutation relation $f_n d_{n-2,i}=d_{n-2,i}f_n$.

Lemma 2.4. *We have*

(i) $f_n^2 d_{n-1,i}-[2]f_n d_{n-1,i}f_n+d_{n-1,i}f_n^2=0 \quad (0\leqq i\leqq n-2)$,

(ii) $d_{n-1,i}^2 f_n-[2]d_{n-1,i}f_n d_{n-1,i}+f_n d_{n-1,i}^2=0 \quad (0\leqq i\leqq n-2)$,

(iii) $f_n d_{ni}=d_{ni}f_n \quad (0\leqq i\leqq n)$.

Now we prove Theorem 2.3. We simultaneously establish the following $(2.4)_n$ and $(2.5)_n$ by induction on $n\,(2\leqq n)$:

$(2.4)_n$ $\qquad d_{ni}d_{nj}=d_{nj}d_{ni} \quad (0\leqq i<j\leqq n-2)$.

$(2.5)_n$ $\qquad \langle\varepsilon_j-\varepsilon_i+i-j+1\rangle d_{n-1,i}f_n d_{n-1,j}+\langle\varepsilon_j-\varepsilon_i+i-j-1\rangle d_{n-1,j}f_n d_{n-1,i}$

$$=\langle\varepsilon_j-\varepsilon_i+i-j\rangle\{f_n d_{n-1,i}d_{n-1,j}+d_{n-1,i}d_{n-1,j}f_n\} \quad (0\leqq i<j\leqq n-2).$$

Here we should note that $(2.4)_n$ with $j=n-1$ has already been shown in Lemma 2.4 (iii), and that $(2.5)_n$ is a generalization of Lemma 2.4 (ii).

The identities $(2.4)_2$ and $(2.5)_2$ are trivial. Suppose that (2.4) and (2.5) be true for the cases of less than $n-1$. Then we first show $(2.5)_n$. Let $j\leqq n-3$. Put

$$\alpha=\varepsilon_j-\varepsilon_i+i-j, \quad \beta=\varepsilon_j-\varepsilon_{n-2}+n-j, \quad \gamma=\varepsilon_i-\varepsilon_{n-2}+n-i.$$

Note that $\alpha+\gamma=\beta$ and

$$\langle\alpha\rangle\langle\gamma-1\rangle-\langle\alpha-1\rangle\langle\gamma-2\rangle=\langle\beta-2\rangle. \tag{2.6}$$

The induction hypothesis and (2.6) lead us to

$$\begin{aligned}\text{L.H.S. of } (2.5)_n=&\langle\alpha\rangle\langle\beta-1\rangle\langle\gamma-1\rangle f_{n-1}f_n f_{n-1}d_{n-2,i}d_{n-2,j}\\ &+\langle\alpha\rangle\langle\beta-2\rangle\langle\gamma-2\rangle d_{n-2,i}d_{n-2,j}f_{n-1}f_n f_{n-1}\\ &-\langle\alpha+1\rangle\langle\beta-1\rangle\langle\gamma-2\rangle d_{n-2,i}f_{n-1}f_n f_{n-1}d_{n-2,j}\\ &-\langle\alpha-1\rangle\langle\beta-2\rangle\langle\gamma-1\rangle d_{n-2,j}f_{n-1}f_n f_{n-1}d_{n-2,i}.\end{aligned}$$

Using the adjoint relation $f_{n-1}^2 f_n-[2]f_{n-1}f_n f_{n-1}+f_n f_{n-1}^2=0$, we find that, for the proof of $(2.5)_n$, it suffices to show the following;

$$\begin{aligned}&\langle\alpha\rangle\langle\beta-1\rangle\langle\gamma-1\rangle f_{n-1}^2 d_{n-2,i}d_{n-2,j}+\langle\alpha\rangle\langle\beta-2\rangle\langle\gamma-2\rangle d_{n-2,i}d_{n-2,j}f_{n-1}^2\\ &\quad-\langle\alpha+1\rangle\langle\beta-1\rangle\langle\gamma-2\rangle d_{n-2,i}f_{n-1}^2 d_{n-2,j}\\ &\quad-\langle\alpha-1\rangle\langle\beta-2\rangle\langle\gamma-1\rangle d_{n-2,j}f_{n-1}^2 d_{n-2,i}\\ &=[2]\{\langle\alpha\rangle\langle\beta-1\rangle\langle\gamma-1\rangle f_{n-1}d_{n-2,i}f_{n-1}d_{n-2,j}\\ &\quad-\langle\alpha\rangle\langle\beta-2\rangle\langle\gamma-1\rangle f_{n-1}d_{n-2,i}d_{n-2,j}f_{n-1}\end{aligned}$$

$$-\langle\alpha\rangle\langle\beta-1\rangle\langle\gamma-2\rangle d_{n-2,i}f_{n-1}^{2}d_{n-2,j}$$
$$+\langle\alpha\rangle\langle\beta-2\rangle\langle\gamma-2\rangle d_{n-2,i}f_{n-1}d_{n-2,j}f_{n-1}\}.$$

By virtue of the induction hypothesis and the adjoint relation, we see that this identity holds. Thus we get $(2.5)_n$ for $j\leqq n-3$. Further we observe that this identity remains true whenever α, β and γ are commutative elements satisfying $\alpha+\gamma=\beta$. Hence $(2.5)_n$ with $j=n-2$ is also proved by means of Lemma 2.4(i) (replacing n with $n-1$). Now we show $(2.4)_n$. Making use of the following identities

$$f_n d_{n-1,i} f_n d_{n-1,j} - d_{n-1,i} f_n d_{n-1,j} f_n = f_n d_{n-1,j} f_n d_{n-1,i} - d_{n-1,j} f_n d_{n-1,i} f_n, \tag{2.7}$$

$$[2](f_n d_{n-1,i} f_n d_{n-1,j} - f_n d_{n-1,j} f_n d_{n-1,i}) = d_{n-1,i} f_n^2 d_{n-1,j} - d_{n-1,j} f_n^2 d_{n-1,i}, \tag{2.8}$$

we get,

$$\begin{aligned}&[2](d_{ni}d_{nj}-d_{nj}d_{ni})\\ &=\langle\varepsilon_j-\varepsilon_i+i-j+1\rangle d_{n-1,i}f_n^2 d_{n-1,j}+\langle\varepsilon_j-\varepsilon_i+i-j-1\rangle d_{n-1,j}f_n^2 d_{n-1,i}\\ &\quad-[2]\langle\varepsilon_j-\varepsilon_i+i-j\rangle f_n d_{n-1,i} d_{n-1,j} f_n\\ &=[2]f_n\times\{\text{L.H.S. of } (2.5)_n-\text{R.H.S. of } (2.5)_n\}\\ &=0.\end{aligned}$$

Thus Theorem 2.3 is proved.

For multi-indices $\alpha=(\alpha_0,\cdots\alpha_{n-1})$, $\beta=(\beta_0,\cdots,\beta_{n-1})$, $\alpha\geqq\beta$ stands for the lexicographic order. Set $d_n^{\alpha}=d_{n,0}^{\alpha_0}\cdots d_{n,n-1}^{\alpha_{n-1}}$, and $c_n{}^{\alpha}=c_{n-1,n}^{\alpha_{n-1}}\cdots c_{0,n}^{\alpha_0}$. We should note that the order of multiples in $d_n{}^{\alpha}$ and $c_n{}^{\alpha}$ is not essential thanks to Theorem 2.3. Let $\mathcal{J}_n$ be the left ideal of $\mathcal{U}_q$ generated by e_j $(1\leqq j\leqq n)$. The following proposition is the key to the construction of the Gelfand-Tsetlin basis for the module V.

Proposition 2.5. *We have, for* $\alpha\geqq\beta$,

$$c_n^{\alpha}d_n^{\beta}\equiv\delta_{\alpha\beta}[\alpha]!\prod_{i=0}^{n-1}\left\{\prod_{l=1}^{\alpha_i}\prod_{k=1}^{n-i}\langle\varepsilon_i-\varepsilon_{i+k}+k-l\rangle\times\prod_{l=1}^{\alpha_i}\prod_{K=1}^{n-i-1}\langle\varepsilon_i-\varepsilon_{i+k}+k-l+1+\alpha_{i+k}\rangle\right\}\quad mod\ \mathcal{J}_n, \tag{2.9}$$

where $[\alpha]!=[\alpha_0]!\cdots[\alpha_{n-1}]!\,([m]!=[m][m-1]\cdots[2][1])$.

To prove this proposition, we need to establish several formulas.

Lemma 2.6. *We have*

$$e_j d_{ni} \equiv 0 \ mod\ \mathcal{J}_{n-1} \ \text{for} \ 1 \leqq j \leqq n-1; \tag{2.10}$$

$$e_n d_{ni} = d_{ni}(1) e_n + d_{n-1,i} \langle \varepsilon_i - \varepsilon_n + n - i - 1 \rangle \quad \text{for} \quad 0 \leqq i \leqq n-2, \tag{2.11}$$

where $d_{ni}(1) = \langle \varepsilon_i - \varepsilon_{n-1} + n - i + 1 \rangle f_n d_{n-1,i} - \langle \varepsilon_i - \varepsilon_{n-1} + n - i \rangle d_{n-1,i} f_n$;

$$d_{ni}(1) d_{n-1,i} = d_{n-1,i} d_{ni} \quad \text{for} \quad 0 \leqq i \leqq n-2. \tag{2.12}$$

Proof. We can show (2.10) by induction on n, and can check (2.11) and (2.12) by simple computation. □

The following identity is also checked by simple calculation:

$$c_{in}^m d_{ni}^m \equiv \prod_{l=1}^{m} \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - l + 1 \rangle (e_{i+1} \cdots e_n)^m d_{ni}^m \ mod\ \mathcal{J}_{n-1}. \tag{2.13}$$

Hence the proof of Proposition 2.5 reduces to the following lemma.

Lemma 2.7. *We have*

$$\begin{aligned}(e_{i+1} \cdots e_n)^m d_{ni}^m \equiv & (e_{i+1} \cdots e_n)^{m-1} e_{i+1} \cdots e_{n-1} d_{ni}(1)^m e_n \\ & + \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - m \rangle \sum_{l=0}^{m-1} \langle \varepsilon_i - \varepsilon_n + n - i - 1 - 2l \rangle (e_{i+1} \cdots e_n)^{m-1} d_{ni}^{m-1} \\ & \qquad mod\ \mathcal{J}_{n-1};\end{aligned} \tag{2.14}$$

$$(e_{i+1} \cdots e_n)^m d_{ni}^m \equiv [m]! \prod_{l-1}^{m} \prod_{k=1}^{n-i} \langle \varepsilon_i - \varepsilon_{i+k} + k - l \rangle \ mod\ \mathcal{J}_n. \tag{2.15}$$

Proof. We only show (2.14). Suppose the m-th step be true. Then we see that

$$\begin{aligned}(e_{i+1} \cdots e_n)^{m+1} d_{ni}^{m+1} \equiv & (e_{i+1} \cdots e_n)^m e_{i+1} \cdots e_{n-1} d_{ni}(1)^m e_n d_{ni} \\ & + \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - m - 1 \rangle \sum_{l=0}^{m-1} \langle \varepsilon_i - \varepsilon_n + n - i - 2l - 3 \rangle (e_{i+1} \cdots e_n)^m d_{ni}^m \\ & \quad \text{(by the induction hypothesis and Lemma 2.6 (2.10))} \\ \equiv & (e_{i+1} \cdots e_n)^m e_{i+1} \cdots e_{n-1} d_{ni}(1)^{m+1} e_n \\ & + (e_{i+1} \cdots e_n)^m e_{i+1} \cdots e_{n-1} d_{n-1,i} d_{ni}^m \langle \varepsilon_i - \varepsilon_n + n - i - 1 \rangle \\ & + \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - m - 1 \rangle \sum_{l=1}^{m} \langle \varepsilon_i - \varepsilon_n + n - i - 1 - 2l \rangle (e_{i+1} \cdots e_n)^m d_{ni}^m \\ & \quad \text{(by Lemma 2.6 (2.11), (2.12))} \\ \equiv & (e_{i+1} \cdots e_n)^m e_{i+1} \cdots e_{n-1} d_{ni}(1)^{m+1} e_n \\ & + \prod_{k=1}^{n-i-1} \langle \varepsilon_i - \varepsilon_{i+k} + k - m - 1 \rangle \sum_{l=0}^{m} \langle \varepsilon_i - \varepsilon_n + n - i - 1 - 2l \rangle (e_{i+1} \cdots e_n)^m d_{ni}^m. \\ & \quad \text{(by Lemma 2.6 (2.11))}\end{aligned}$$

Here all the congruences mean "mod $\mathcal{I}_{n-1}$". We can easily deduce (2.15) from (2.14). □

Proof of Proposition 2.5. Since

$$(e_1\cdots e_n)^{\alpha_0} d_{n1}^{\beta_1}\cdots d_{n,n-1}^{\beta_{n-1}} \equiv d_{n1}^{\beta_1}\cdots d_{n,n-1}^{\beta_{n-1}}(e_1\cdots e_n)^{\alpha_0} \mod \mathcal{I}_{n-1},$$

we get

$$\begin{aligned} c_n^\alpha d_n^\beta &\equiv \delta_{\alpha_0\beta_0} c_{n-1,n}^{\alpha_{n-1}}\cdots c_{1n}^{\alpha_1} \sum_{l=1}^{\alpha_0}\prod_{k=1}^{n-1}\langle \varepsilon_0-\varepsilon_k+k-l+1\rangle \\ &\quad \times d_{n1}^{\beta_1}\cdots d_{n,n-1}^{\beta_{n-1}}(e_1\cdots e_n)^{\alpha_0} d_{n0}^{\alpha_0} \mod \mathcal{I}_{n-1} \\ &\equiv \delta_{\alpha_0\beta_0}[\alpha_0]!\ c_{n-1,n}^{\alpha_{n-1}}\cdots c_{1n}^{\alpha_1}\prod_{l=1}^{\alpha_0}\prod_{k=1}^{n-1}\langle \varepsilon_0-\varepsilon_k+k-l+1\rangle \\ &\quad \times d_{n1}^{\beta_1}\cdots d_{n,n-1}^{\beta_{n-1}}\prod_{l=1}^{\alpha_0}\prod_{k=1}^{n}\langle \varepsilon_0-\varepsilon_k+k-l\rangle \mod \mathcal{I}_n \\ &\equiv \delta_{\alpha_0\beta_0}[\alpha_0]!\ c_{n-1,n}^{\alpha_{n-1}}\cdots c_{1n}^{\alpha_1} d_{n1}^{\beta_1}\cdots d_{n,n-1}^{\beta_{n-1}} \\ &\quad \times \prod_{l=1}^{\alpha_0}\prod_{k=1}^{n-1}\langle \varepsilon_0-\varepsilon_k+k-l+1+\beta_k\rangle \prod_{l=1}^{\alpha_0}\prod_{k=1}^{n}\langle \varepsilon_0-\varepsilon_k+k-l\rangle \mod \mathcal{I}_n. \end{aligned}$$

Repeating this procedure, we obtain the desired formula. □

Now we are in the position to construct the Gelfand-Tsetlin basis for the module $V(q^{(1/2)\Lambda})$.

Let $\mu_{iN}=\lambda_i$. The sequence of integer vectors

$$(2.16)\qquad \mu=\begin{pmatrix}\mu_N\\ \mu_{N-1}\\ \vdots\\ \mu_1\\ \mu_0\end{pmatrix}=\begin{pmatrix}\mu_{0N} & & \mu_{1N} & \cdots\cdots & & \mu_{NN}\\ & \mu_{0,N-1} & & \mu_{1,N-1} & \mu_{N-1,N-1} & \\ & & \ddots & & \cdot^{\cdot^{\cdot}} & \\ & & \mu_{01} & & \mu_{11} & \\ & & & \mu_{00} & & \end{pmatrix}$$

is called a Gelfand-Tsetlin scheme attached to the module $V(q^{(1/2)\Lambda})$ if each pair of vectors μ_{n-1}, μ_n satisfies the condition that $\mu_{i,n}\geqq\mu_{i,n-1}\geqq\mu_{i+1,n}$ for all i, n. For each scheme, we put

$$d^\mu=d_1^{\mu_1-\mu_0}d_2^{\mu_2-\mu_1}\cdots d_N^{\mu_N-\mu_{N-1}},$$

and

$$c^\mu=c_N^{\mu_N-\mu_{N-1}}\cdots c_2^{\mu_2-\mu_1}c_1^{\mu_1-\mu_0},$$

where $\mu_n-\mu_{n-1}=(\mu_{0n}-\mu_{0,n-1},\ \cdots,\ \mu_{n-1,n}-\mu_{n-1,n-1})$.

Proposition 2.8.

(i) *The weight of the vector* $d^\mu|vac\rangle$ *is* $(q^{1/2(\sum_{i=0}^{n}\mu_{in}-\sum_{i=0}^{n-1}\mu_{i,n-1})})_{0\leq n\leq N}$.

(ii) *For Gelfand-Tsetlin schemes* μ *and* ν, *we have*

$$(2.17)\qquad \langle vac|c^\nu d^\mu|vac\rangle=\delta_{\mu\nu}N_\mu^2,\quad \textit{where}\ N_\mu^2=\prod_{n=1}^{N}\tau_n(\mu_{n-1},\ \mu_n),$$

(2.18) $\tau_n(\mu_{n-1}, \mu_n)$

$$=\prod_{0\leq i\leq j\leq n-1}\frac{[\mu_{i,n}-\mu_{j,n-1}+j-i]!}{[\mu_{i,n-1}-\mu_{j,n-1}+j-i]!}\prod_{0\leq i<j\leq n}\frac{[\mu_{in}-\mu_{jn}+j-i-1]!}{[\mu_{i,n-1}-\mu_{j,n}+j-i-1]!}.$$

Proof. The proof of (i) is straightforward, and (ii) is a direct consequence of Proprsition 2.5. □

Note that $N_\mu^2\neq 0$. Hence the vectors $\{d^\mu|vac>\}$ are linearly independent over $\mathbf{K}$.

From here to Theorem 2.11, we discuss only for a transcendental element q. By Proposition 1.3, $\dim V(q^{(1/2)\Lambda})=\dim \tilde{V}(\Lambda)$, and from the classical results [8], we know that $\dim \tilde{V}(\Lambda)$ equals to the number of the Gelfand-Tsetlin schemes. Therefore we obtain the following theorem.

Theorem 2.9. *The vectors* $|\mu>=d^\mu|vac>$*(resp.* $<\mu|=<vac|c^\mu$*), where* μ *ranges over the set of Gelfand-Tsetlin schemes, form a basis of the module* $V(q^{(1/2)\Lambda})$ *(resp.* $V(q^{(1/2)\Lambda})^*$*).*

We refer to $\{|\mu>\}$ as the Gelfand-Tsetlin basis of the module $V(q^{(1/2)\Lambda})$.

Next we consider the action of the generators of $\mathcal{U}_q$ on the Gelfand-Tsetlin basis. For a Gelfand-Tsetlin scheme μ, we set $\xi=d_{n+1}^{\mu_{n+1}-\mu_n}\cdots d_N^{\mu_N-\mu_{N-1}}|vac>$.

Proposition 2.10. *We have*

(2.19) $e_n d_n^{\mu_n-\mu_{n-1}}\xi$

$$=\sum_{j=0}^{n-1}\frac{N_{\mu'}^2}{[\mu_{j,n-1}-\mu_{n-1,n-1}+n-j]N^2_{\mu'+\delta_j(n-1)}}d_{n-1,j}d_n^{\mu_n-\mu_{n-1}-\delta_j}\xi,$$

where
$$\mu'=\begin{pmatrix} \mu_{0n} & \mu_{1n} & \cdots\cdots & \mu_{nn} \\ & \mu_{0,n-1} & \mu_{1,n-1} \cdots & \mu_{n-1,n-1} & \\ & & \mu_{0,n-1} \quad \mu_{1,n-1} & & \\ & & \ddots \quad \ddots & & \\ & & \mu_{0,n-1} & & \end{pmatrix},$$

$\mu_n-\mu_{n-1}-\delta_j=(\mu_{kn}-\mu_{k,n-1}-\delta_{jk})_{0\leq k\leq n-1}$, *and* $\mu'+\delta_j(n-1)$ *indicates to replace the* $\mu_{j,n-1}$ *in the* $(n-1)$*-th row by* $\mu_{j,n-1}+1$ *in* μ'.

Proof. From consideration on the weight of the vector $e_n d_n^{\mu_n-\mu_{n-1}}\xi$, we see that it can be expressed as follows;

(2.20) $$e_n d_n^{\mu_n-\mu_{n-1}}\xi=\sum_\alpha c_\alpha d_{n-1}^\alpha d_n^{\mu_n-\mu_{n-1}-(\alpha,1-|\alpha|)}\xi,$$

where $\alpha=(\alpha_0, \cdots, \alpha_{n-2})$ is a multi-index, $|\alpha|=\alpha_0+\cdots+\alpha_{n-2}$, and

$$\mu_n-\mu_{n-1}-(\alpha, 1-|\alpha|)$$
$$=(\mu_{0n}-\mu_{0,n-1}-\alpha_0, \cdots, \mu_{n-2,n}-\mu_{n-2,n-1}-\alpha_{n-2}, \mu_{n-1,n}-\mu_{n-1,n-1}+|\alpha|-1).$$

The coefficient c_α is given by

$$c_\alpha=\frac{(d_{n-1}^{\alpha}d_n^{\mu_n-\mu_{n-1}-(\alpha,1-|\alpha|)}\xi,\ e_n d_n^{\mu_n-\mu_{n-1}}\xi)}{\|d_{n-1}^{\alpha}d_n^{\mu_n-\mu_{n-1}-(\alpha,1-|\alpha|)}\xi\|^2}. \tag{2.21}$$

Formally rewriting the definition of d_{nj} to

$$f_n d_{n-1,j}=\langle\varepsilon_j-\varepsilon_{n-1}+n-j\rangle^{-1}\{d_{nj}+\langle\varepsilon_j-\varepsilon_{n-1}+n-j-1\rangle d_{n-1,j}f_n\},$$

we see that, for $d_{n-1}^{\alpha}=d_{n-1,j}$ $(0\leqq j\leqq n-2)$ (i.e. $|\alpha|=1$), (2.22) the numerator of (2.21)

$$=[\mu_{j,n-1}-\mu_{n-1,n-1}+n-j]^{-1}\|d_n^{\mu_n-\mu_{n-1}}\xi\|^2,$$

and that the numerator of (2.21) vanishes for $|\alpha|\geqq 2$. From (2.20), (2.21), (2.22), we get the desired result. □

We can show the following theorem which has been established by Jimbo [5].

Theorem 2.11. *The action of the generators of $\mathcal{U}_q(gl(N+1))$ on the Gelfand-Tsetlin basis $\{|\mu\rangle\}$ is expressed as follows:*

$$q^{\pm(1/2)\varepsilon_n}|\mu\rangle=q^{\pm 1/2\{\sum_{k=0}^{n}\mu_{kn}-\sum_{k=0}^{n-1}\mu_{k,n-1}\}}|\mu\rangle \quad (0\leqq n\leqq N), \tag{2.23}$$

$$e_n|\mu\rangle=\sum_{j=0}^{n-1}a_{\mu+\delta_j(n-1),\mu}|\mu+\delta_j(n-1)\rangle \quad (1\leqq n\leqq N), \tag{2.24}$$

$$f_n|\mu\rangle=\sum_{j=0}^{n-1}b_{\mu-\delta_j(n-1),\mu}|\mu-\delta_j(n-1)\rangle \quad (1\leqq n\leqq N), \tag{2.25}$$

where

$$a_{\mu+\delta_j(n-1),\mu}=-\frac{\prod_{k=0}^{n}[\mu_{kn}-\mu_{j,n-1}+j-k]}{\prod_{\substack{k=0\\(k\neq j)}}^{n-1}[\mu_{k,n-1}-\mu_{j,n-1}+j-k]}, \tag{2.26}$$

$$b_{\mu-\delta_j(n-1),\mu}=\frac{\prod_{k=0}^{n-2}[\mu_{k,n-2}-\mu_{j,n-1}+j-k]}{\prod_{\substack{k=0\\(k\neq j)}}^{n-1}[\mu_{k,n-1}-\mu_{j,n-1}+j-k]},$$

and $\mu\pm\delta_j(n-1)$ means to replace only $\mu_{j,n-1}$ with $\mu_{j,n-1}\pm 1$ in μ.

Proof. We show (2.24). Since e_n and d_{ji} $(1\leqq i\leqq j\leqq n-1)$ commute,

$$e_n d^{\mu}|vac\rangle=(d_1^{\mu_1-\mu_0}\cdots d_{n-1}^{\mu_{n-1}-\mu_{n-2}})e_n d_n^{\mu_n-\mu_{n-1}}(d_{n+1}^{\mu_{n+1}-\mu_n}\cdots d_N^{\mu_N-1})|vac\rangle.$$

Substituting (2.19) into the above identity, we get

$$e_n d^{\mu}|vac\rangle=\sum_{j=0}^{n-1}\frac{N_{\mu'}^2}{[\mu_{j,n-1}-\mu_{n-1,n-1}+n-j]N_{\mu'+\delta_j(n-1)}^2}$$
$$\times d_1^{\mu_1-\mu_0}\cdots d_{n-1}^{(\mu_{n-1}+\delta_j)-\mu_{n-2}}d_n^{\mu_n-(\mu_{n-1}+\delta_j)}\cdots d_N^{\mu_N-\mu_{N-1}}|vac\rangle.$$

Hence

$$e_n|\mu\rangle=\sum_{j=0}^{n-1}\frac{N_{\mu'}^2}{[\mu_{j,n-1}-\mu_{n-1,n-1}+n-j]N^2_{\mu'+\delta_j(n-1)}}|\mu+\delta_j(n-1)\rangle .$$

Making substitution of the formula (2.18) into the right-hand side, we obtain (2.24). Using the duality of $V(q^{(1/2)\Lambda})^*$ and $V(q^{(1/2)\Lambda})$ one can easily derive (2.25) from (2.24). □

Next we discuss for a complex variable q. The construction of the Gelfand-Tsetlin basis for $V(q^{(1/2)\Lambda})$ is as follows. Let $V=\bigoplus_{\mu}\mathbf{C}\cdot v(\mu)$, where μ ranges over the Gelfand-Tsetlin scheme attached to highest weight $q^{(1/2)\Lambda}$. We define the action of $\mathcal{U}_q$ on V:

$$q^{\pm(1/2)\varepsilon_n}v(\mu)=q^{\pm 1/2\{\sum_{k=0}^{n}\mu_{kn}-\sum_{k=0}^{n-1}\mu_{k,n-1}\}}v(\mu)\quad(0\leqq n\leqq N),$$

$$e_nv(\mu)=\sum_{j=0}^{n-1}a_{\mu+\delta_j(n-1),n}v(\mu+\delta_j(n-1))\quad(1\leqq n\leqq N),$$

$$f_nv(\mu)=\sum_{j=0}^{n-1}b_{\mu-\delta_j(n-1),\mu}v(\mu-\delta_j(n-1))\quad(1\leqq n\leqq N),$$

where $a_{\mu+\delta_j(n-1),\mu}$ and $b_{\mu-\delta_j(n-1),\mu}$ are given by (2.26). With this action, V becomes a left $\mathcal{U}_q$-module. Let $\tilde{V}=\mathcal{U}_q\cdot v(\mu_{vac})\subset V$, where μ_{vac} is the Gelfand-Tsetlin scheme such that $(\mu_{vac})_{i,n}=\lambda_i$ $(0\leqq i\leqq n\leqq N)$. Then $\tilde{V}$ is an irreducible left $\mathcal{U}_q$-module of finite dimensions with highest weight $q^{(1/2)\Lambda}$, so $\tilde{V}\cong V(q^{(1/2)\Lambda})$. Thus the vectors $|\tilde{\mu}\rangle=d^{\mu}v(\mu_{vac})$ are linearly independent over $\mathbf{C}$. This means that $\{|\tilde{\mu}\rangle\}$ is a basis of V, and further we see that $c^{\mu}|\tilde{\mu}\rangle=c^{\mu}d^{\mu}v(\mu_{vac})=N_{\mu}^2v(\mu_{vac})$. Hence V is irreducible and isomorphic to $V(q^{(1/2)\Lambda})$ as left $\mathcal{U}_q$-modules. Therefore we see that for a complex variable q, the Gelfand-Tsetlin basis of $V(q^{(1/2)\Lambda})$ and the action of $\mathcal{U}_q$ on it are the same as those for a transcendental element q.

Remark 1. Let $\mathcal{U}_n$ be a subalgebra of $\mathcal{U}_q$ generated by $q^{\pm(1/2)\varepsilon_i}$ $(0\leqq i\leqq n)$, e_j, f_j $(1\leqq j\leqq n)$. From Lemma 2.6 (2.10) and Proposition 2.8 (i), it follows that

$$e_jd_N^{\mu_N-\mu_{N-1}}|vac\rangle=0\quad(1\leqq j\leqq N-1),$$

$$q^{(1/2)\varepsilon_i}d_N^{\mu_N-\mu_{N-1}}|vac\rangle=q^{(1/2)\mu_{i,N-1}}d_N^{\mu_N-\mu_{N-1}}|vac\rangle\quad(0\leqq i\leqq N-1).$$

From Proposition 1.2 (iv), (v), we see that the vector $d^{\mu_N-\mu_{N-1}}|vac\rangle$ is the highest weight vector of the finite dimensional irreducible left $\mathcal{U}_{N-1}$-module $V(q^{(1/2)\mu_{0,N-1}},\cdots,q^{(1/2)\mu_{N-1,N-1}})$. Hence we get a weak form of the branching law for $\mathcal{U}_N\downarrow\mathcal{U}_{N-1}$.

$$(2.27)\qquad V(q^{(1/2)\Lambda})|_{\mathcal{U}_{N-1}}\supset\bigoplus_{\substack{0\leq\alpha_i\leq\lambda_i-\lambda_{i+1}\\0\leq i\leq N-1}}V(q^{1/2(\lambda_1+\alpha_0)},\cdots,q^{1/2(\lambda_N+\alpha_{N-1})}).$$

Using the branching law in the classical case ([4]), and the considerations about the dimension in Proposition 1.3 and the previous results, we get

Proposition 2.12 (*Branching law for $\mathcal{U}_N \downarrow \mathcal{U}_{N-1}$*). *We have*

$$V(q^{(1/2)\Lambda})|_{\mathcal{U}_{N-1}} = \bigoplus_{\substack{0 \leq \alpha_i \leq \lambda_i - \lambda_{i+1} \\ 0 \leq i \leq N-1}} V(q^{1/2(\lambda_1+\alpha_0)}, \cdots, q^{1/2(\lambda_N+\alpha_{N-1})}). \tag{2.28}$$

Remark 2. If q is real and positive then $\mathcal{U}_q(gl(N+1))$ has a $*$ structure. With this $*$ structure, the module $V(q^{(1/2)\Lambda})$ turns out to be unitary [1]. Further $\{|\bar{\mu}\rangle = 1/N_\mu |\mu\rangle\}$ is an orthonormal basis, and the action of $\mathcal{U}_q$ on this basis is the same as the result of Jimbo [5].

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