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Vanishing in Highest Degree for Solutions of D-Modules and Perverse Sheaves

By

Pierre SCHAPIRA*

Abstract

Let M be a real analytic manifold of dimension n, X a complexification of M, $\mathcal{M}$ a coherent module over the sheaf of rings $\mathscr{E}_X$ of microdifferential operators. We prove the vanishing of the group $\mathscr{E}xt^n_{\mathscr{E}_X}(\mathcal{M}, \mathscr{C}_M)$, where $\mathscr{C}_M$ denotes the sheaf of Sato's microfunctions. The proof makes use of the duality of perverse sheaves.

§1. Perverse Sheaves

Let X be a complex manifold of dimension n, and let k be a commutative field. We denote by $D^b(X)$ the derived category of the category of bounded complexes of sheaves of k-vector spaces on X, and by $D^b_{\mathbf{C}-c}(X)$ the full subcategory consisting of objects with $\mathbf{C}$-constructible cohomology. In other words, F is an object of $D^b_{\mathbf{C}-c}(X)$ iff there exists a complex analytic stratification $X = \cup X_\alpha$ such that for any $j \in Z$ and any α, the sheaf $H^j(F)_{|X_\alpha}$ is locally constant of finite rank. To $F \in Ob(D^b(X))$ one associates:

$$\mathrm{D}'F = R\mathscr{H}om(F, k_X)$$
$$\mathrm{D}F = R\mathscr{H}om(F, \omega_X),$$

where $\omega_X \cong or_X\,[2n]$ is the dualizing complex (and or_X the orientation sheaf), on X. Now, let F be an object of $D^b_{\mathbf{C}-c}(X)$ and consider the conditions below.

(1.1) *For any complex submanifold Y of X of codimension d, $H^j_Y(F)|_Y$ is zero for $j < d$.*

(1.2) *For any $j \in Z$, $H^j(F)$ is supported by a complex analytic subset of codimension $\geq j$.*

Here, we shall say that F is perverse if it satisfies the conditions (1.1) and (1.2). Remark that this definition differs from that of [B-B-D] by a shift, but it will be more convenient for our purpose. As a consequence of (1.1), one

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* Université Paris Nord, Départment de Mathématiques, 93430 Villetaneuse France.

obtains, for F perverse:

(1.3) *F is concentrated in degrees* ≥ 0.

Moreover, one can easily show (eg. cf. [K-S 2]) that F satisfies (1.1) if and only if $D'F$ satisfies (1.2). Therefore, if F is perverse, one gets:

(1.4) $D'F$ is perverse.

Proposition 1. *Let S be a closed subset of X, and let $x \in S$, x being non isolated in S. Let F be a perverse object of $D^b_{C-c}(X)$.*

(a) *One has* $H^j(R\Gamma_{\{x\}}(F_S)) = 0$ *for* $j \leq 0$.

(b) *If moreover S is subanalytic, then* $H^j(R\Gamma_S(F)_x = 0$ *for* $j \geq 2n$.

Proof. (a) The sheaf $H^0(F)$ being $\mathbf{C}$-constructible, one deduces from (1.1) that it satisfies to "the principle of analytic continuation", that is, if u is a section of $H^0(F)$ on an open subset of X, the support of u is both open and closed in this open subset. Therefore $H^0(F)|_S$ also satisfies to this principle, and $H^0_{\{x\}}(F_S) \cong H^0_{\{x\}}(X; F_S) \cong \Gamma_{\{x\}}(S; H^0(F|_S)) = 0$.

(b) By the theory of duality for $\mathbf{R}$-constructible sheaves, (cf. [V], [K-S1], [K-S2]), one has:

$$DR\Gamma_S(F) \cong (DF)_S$$

$$(R\Gamma_S(F))_x \cong \Gamma_{\{x\}}((DF)_S))^*$$

where * denotes the duality functor $Hom(\cdot, \mathrm{k})$. It is then enough to apply the result of (a) to the perverse object $DF[-2n]$.

§2. Application to $\mathscr{E}_X$-Modules

Let M be a real analytic manifold of dimension n, X a complexification of M. One denotes by $\mathscr{E}_X$ the sheaf on T^*X of (finite order) microdifferential operators (cf. [S-K-K], or cf. [S] for an introduction to the theory of $\mathscr{E}_X$-modules). One denotes by $\mathscr{C}_M$ the sheaf on T^*_MX of Sato's microfunctions (cf. [S-K-K]). Recall that the restriction of the sheaf $\mathscr{E}_X$ to the zero-section of T^*X identified to X is nothing but the sheaf $\mathscr{D}_X$ of differential operators on X, and the restriction of the sheaf $\mathscr{C}_M$ to the zero-section of T^*_MX identified to M is nothing but the sheaf $\mathscr{B}_M$ of Sato's hyperfunctions. Also recall the definition of the sheaf of microfunctions:

(2.1) $$\mathscr{C}_M \cong \mu_M(O_X) \otimes or_{M/X}[n],$$

where μ_M is the functor of Sato's microlocalization (cf. [K-S1] or [K-S2]) for a detailed construction of this functor).

Theorem 2. *Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-module. Then:*

$$\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M) = 0 \qquad \textit{for } j \geq n.$$

In particular, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$*-module, then*:

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) = 0 \qquad \textit{for } j \geq n.$$

Of course the vanishing of these groups for $j > n$ is obvious, since $\mathcal{M}$ locally admits a free resolution of length n.

Proof. (a) Suppose first that $\mathcal{M} = \mathcal{E}_X \otimes_{\mathcal{D}_X} \mathcal{N}$, where $\mathcal{N}$ is a holonomic $\mathcal{D}_X$-module. (We write $\mathcal{N}$ or $\mathcal{D}_X$ instead of $\pi^{-1}\mathcal{N}$ or $\pi^{-1}\mathcal{D}_X$ in the formulas if there is no risk of confusion. Here, π denotes as usual the projection $T^*X \to X$). Set:

$$F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Then:

$$\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M) \cong \mu_M(F)[n].$$

By a theorem of Kashiwara [K1], one knows that F is perverse (over the field $\mathbb{C}$). Applying Proposition 1(b), we get that $H^j(\mu_M(F)[n]) = 0$ for $j \geq n$.
(b) Assume now that $\mathcal{M}$ is a holonomic $\mathcal{E}_X$-module, and let $p \in T^*_M X$. If p belongs to the zero-section, $\mathcal{M}$ is a $\mathcal{D}_X$-module and this is the previous situation. Otherwise we shall use three results of Kashiwara-Kawai ([K-K]). First, there exists a real contact transformation which put char($\mathcal{M}$), the characteristic variety of $\mathcal{M}$, in a generic position. Second, we may assume $\mathcal{M}$ is regular holonomic, and finally $\mathcal{M}$ is generated by a holonomic $\mathcal{D}_X$-module. Hence, the theorem is proved in the holonomic case.
(c) To treat the general case, we denote by * the functor $\mathcal{N} \to \mathcal{E}xt^n_{\mathcal{E}_X}(\mathcal{N}, \mathcal{E}_X)$. It is proved by Kashiwara in [K2] that if $\mathcal{M}$ is coherent, then $\mathcal{M}^*$ is holonomic, $\mathcal{M}^{***} \cong \mathcal{M}^*$ and $\mathcal{M}^{**}$ is a submodule of $\mathcal{M}$. Define the coherent $\mathcal{E}_X$-module $\mathcal{K}$ by the exact sequence:

$$0 \longrightarrow \mathcal{M}^{**} \longrightarrow \mathcal{M} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Since $\mathcal{K}^* = 0$, $\mathcal{K}$ locally admits a projective resolution of length $n-1$. Therefore $\mathcal{E}xt^n_{\mathcal{E}_X}(\mathcal{K}, \mathcal{C}_M) = 0$ and $\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M) \cong \mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}^{**}, \mathcal{C}_M) = 0$ for $j \geq n$.
This completes the proof.

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