Publ. RIMS, Kyoto Univ.
26 (1990), 539–584

On the Microlocal Structure of the Regular Prehomogeneous Vector Space Associated with $SL(5) \times GL(4)$

By

Ikuzo OZEKI*

Introduction

Let $\rho = \Lambda_2 \otimes \Lambda_1$ be an irreducible representation of $G = SL(5, \mathbf{C}) \times GL(4, \mathbf{C})$ on $V = \overset{2}{\Lambda}\mathbf{C}^5 \otimes \mathbf{C}^4 (= V(10) \otimes V(4))$. Then we have a Zariski-dense G-orbit, namely, the triplet (G, ρ, V) is a prehomogeneous vector space (abbrev. P.V.). All irreducible P.V.'s are completely classified (See [SK]), and the b-functions of irreducible regular P.V.'s are already calculated (See [SKKO], [Ki], [KM], [KO], [O 2]) by using microlocal analysis (See [SKKO]) except the case of our P.V. (G, ρ, V) which has the most complicated microlocal structure among all the reduced irreducible regular P.V.'s (See [O 1]). The table of G-orbits in V was first given in [O 1]. However, one orbit corresponding to a generic point of $(SL(5) \times GL(2), \Lambda_2 \otimes \Lambda_1, \overset{2}{\Lambda}\mathbf{C}^5 \otimes \mathbf{C}^2)$ is missed in [O 1] (Remark 2.3). Later, Prof. N.Kawanaka gave the orbital decomposition of (G, ρ, V) by using a classification of nilpotent orbits of the exceptional Lie algebra E_8 (See [Ka 1], [Ka 2]).

In this paper, we shall give the detailed explanation of the original method used in [O 1] for the orbital decomposition of (G, ρ, V) (Proposition 2.1) and determine all good holonomic varieties (Theorem 3.4). These results are fundamental to investigate the microlocal structure of our P.V. from which we can get some information of the b-function (See [O 1]).

We shall roughly explain our method. Let $G' = SL(5, \mathbf{C}) \times GL(3, \mathbf{C})$, $\rho' = \Lambda_2 \otimes \Lambda_1$, $V' = \overset{2}{\Lambda}\mathbf{C}^5 \otimes \mathbf{C}^3$. Then (G', ρ', V') is an irreducible P.V. and its orbital decomposition has been completed in [Ki]. By an injection $GL(3, \mathbf{C})$

Communicated by M. Kashiwara, November 6, 1989.
* University of Tsukuba School for the Blind, Tokyo 112, Japan.

$\ni A \to \begin{pmatrix} A & \\ & 1 \end{pmatrix} \in GL(4, \mathbb{C})$, we can regard G' as a subgroup of $G = SL(5, \mathbb{C}) \times GL(4, \mathbb{C})$, and $V = \overset{2}{\Lambda} \mathbb{C}^5 \otimes \mathbb{C}^4$ can be identified with $V' \oplus V''$ with $V'' = \overset{2}{\Lambda} \mathbb{C}^5$. Let $\{x_i'\}_{i \in I}$ be representative points of G'-orbits in V' given in Table 2. Then, clearly, any point of $V = V' \oplus V''$ is G-equivalent to a point of a form (x_i', x'') for some $i \in I$ and some x'' in V''. For each case $i \in I$ in Table 2, we shall determine the G-equivalence class of $\{(x_i', x''); x'' \in V'' = \overset{2}{\Lambda} \mathbb{C}^5\}$. The cases [1] [2] [3] in Table 2 are complicated compared to the other cases. In this paper, we shall describe cases [1] [2] [3] in detail while we shall give just results in other cases.

Now we explain some basic idea used in the case [1] as an example. One can see that $x_{7,27} + x''(x'' \in V'' = \overset{2}{\Lambda} \mathbb{C}^5)$ is G-equivalent to $x_{7,27} + y_i$ $(1 \leq i \leq 5)$ where y_1, y_2, y_3 contain some parameters (See Proposition 2.15). It is difficult to eliminate these parameters. However, one can see, $x = x_{7,27} + y_1$ is a point of a G-orbit such that its closure contains the G-orbit $S_{4,11}$ in Table 1. Fortunately, we can determine all G-orbits such that its closure contains $S_{4,11}$ (See Proposition 2.11), and hence, we can determine all G-equivalence classes of $x_{7,27} + y_1$ without parameters. This idea is often used in this paper.

The author would like to express his hearty thanks to Professors Mikio Sato, Masaki Kashiwara, Tatsuo Kimura and Tamaki Yano for their invaluable advice and encouragement.

§ 1. Preliminaries

We review some fundamental notions on prehomogeneous vector spaces.

Definition 1.1. Let G be a connected linear algebraic group, ρ an n-dimensional rational representation of G on a vector space V, all defined over $\mathbb{C}$. When V has a Zariski-dense G-orbit, the triplet (G, ρ, V) is called *a prehomogeneous vector space*, and abbreviated by P.V. Let $\mathfrak{g}$ (resp. V^*, $d\rho$) be the Lie algebra of G(resp. the dual vector space of V, the infinitesimal representation of ρ). For a point x_0 of V, we define the conormal vector space $V^*_{x_0}$ by $V^*_{x_0} = \{y \in V^*; \langle d\rho(A)x_0, y \rangle = 0$ for all $A \in \mathfrak{g}\} (= (d\rho(\mathfrak{g})x_0)^{\perp})$. Thus we obtain a triplet $(G_{x_0}, \rho_{x_0}, V^*_{x_0})$ which is called *the colocalization of* (G, ρ, V) *at* x_0, where G_{x_0} denotes the isotropy subgroup $\{g \in G; \rho(g)x_0 = x_0\}$ of G at x_0 and ρ_{x_0} denotes the restriction of the contragredient representation ρ^* of ρ to G_{x_0}.

Definition 1.2. We identify the cotangent bundle T^*V with $V \times V^*$. Similarly $T^*V^* \simeq V^* \times V^{**}$ will be identified with $V \times V^*$. For a G-orbit S in V, we define *the conormal bundle* $\Lambda = T^*_S V$ of S by the Zariski-closure of $\bigcup_{x \in S} (x, V^*_x)$ in $V \times V^*$. It is also called *a holonomic variety.*

We note that Λ is G-prehomogeneous (i.e., Λ has a Zariski-dense G-orbit) if and only if a colocalization (G_x, ρ_x, V_x^*) is a P.V. for $x \in S$.

Definition 1.3. A G-orbit S in V and a G-orbit S^* in V^* are called *the dual orbit* of each other if their conormal bundles $T_S^* V$ and $T_{S^*}^* V^*$ coincide in $V \times V^*$.

One can see easily that the dual orbit S^* of a given G-orbit S is unique if it exists.

Definition 1.4. For a given triplet (G, ρ, V), let $m = \max\{\dim \rho(G)x_0 ; x_0 \in V\}$. Then we define the *corank* of (G, ρ, V) by corank $(G, \rho, V) = \dim V - m$. Note that (G, ρ, V) is a P.V. if and only if its corank is equal to zero.

Definition 1.5. For a triplet (G, ρ, V), we denote by S_{ij}^k an i-codimensional G-orbit S in V with the j-codimensional dual orbit and the corank of the colocalization of (G, ρ, V) at x in S is equal to k. We denote by Λ_{ij}^k the holonomic variety $T_{S_{ij}^k}^* V$.

Definition 1.6. Let (G, ρ, V) be a triplet and G' a subgroup of G. Assume that $V = V' \oplus V''$, V' and V'' are G'-admissible. Let $\varphi ; V \to V'$ be the natural projection and, for $x' \in V'$, let $\tilde{G}_{x'}$ be the subgroup of G fixing the fibre $\varphi^{-1}(x')$; $\tilde{G}_{x'} = \{g \in G ; \rho(g)\varphi^{-1}(x') = \varphi^{-1}(x')\}$.

Then the following proposition is obvious.

Proposition 1.7. *With the notations as above, let* $\{x_i'\}_{i \in I}$ *be representative points of* G'*-orbits in* V'. *For each* x_i', *let* $\{x_{ij}\}_{j \in J_i}$ *be representative points of* $\tilde{G}_{x_i'}$*-orbits in* $\varphi^{-1}(x_i')$. *Then any point* x *in* V *is* G*-equivalent to* x_{ij} *for some* $i \in I$ *and* $j \in J_i$, *which is not necessarily unique.*

§2. The Orbital Decomposition

Throughout this paper, we denote by (G, ρ, V) the P.V. given by $G = SL(5, \mathbf{C}) \times GL(4, \mathbf{C})$, $\rho = \Lambda_2 \otimes \Lambda_1$, $V = \overset{2}{\Lambda} \mathbf{C}^5 \otimes \mathbf{C}^4$.

Proposition 2.1. *The triplet* (G, ρ, V) *has the following* 63 *orbits* S_{ij}^k *given in Table* 1.

Table 1

	i,	j,	k	representative points
1	0,	40,	0	256 − 346 + 157 − 247 − 148 + 238 − 129 + 459
2	1,	30,	0	456 + 137 + 257 + 148 + 238 + 159 + 249
3	2,	24,	0	456 + 346 + 137 + 148 + 238 + 259
4	2,	21,	0	126 + 456 + 137 + 247 + 148 + 238 + 259
5	3,	18,	0	126 + 356 − 157 + 148 + 238 + 249
6	3,	15,	0	156 + 246 + 346 + 147 + 237 + 138 + 259
7	4,	20,	0	256 − 157 + 148 − 238 + 129 + 349

8	4,	14,	0	156 + 246 + 147 + 237 + 128 + 359
9	4,	11,	0	256 + 346 + 157 + 247 + 148 + 238 + 139
10	5,	21,	1	346 + 137 + 247 + 128 + 259 + 459
11	5,	16,	0	156 + 257 + 348 + 139 + 249
12	5,	12,	0	146 − 236 + 256 − 157 + 128 + 349
13	5,	9,	0	256 − 346 + 157 − 247 − 148 + 238 − 129
14	6,	22,	2	136 + 147 + 237 + 158 + 248 + 259
15	6,	14,	0	236 + 356 − 157 + 248 + 139
16	6,	14,	1	346 − 157 + 247 − 148 + 238 + 129
17	6,	14,	2	156 + 246 + 147 + 357 + 138 + 129
18	6,	8,	0	256 − 346 + 147 − 237 − 158 + 129
19	7,	15,	1	346 − 157 − 148 + 238 + 129
20	7,	10,	1	146 − 256 − 347 − 128 − 139
21	7,	7,	0	256 − 346 − 157 + 247 − 138 + 129
22	8,	11,	0	146 + 236 − 137 + 128 + 459
23	8,	8,	0	246 + 357 + 128 + 139
24	8,	6,	0	256 − 346 + 147 − 138 + 238 + 129
25	9,	13,	3	136 − 247 + 158 − 258 − 129
26	9,	9,	1	156 + 346 + 147 − 238 − 129
27	9,	5,	0	256 − 346 + 147 + 237 − 138 + 129
28	10,	7,	0	136 − 246 − 157 + 238 + 129
29	11,	15,	3	256 + 346 − 147 + 138 + 129
30	11,	4,	0	156 − 246 + 147 + 237 − 138 + 129
31	12,	20,	4	136 − 237 − 148 + 249
32	12,	13,	2	236 − 247 − 158 + 129
33	12,	10,	0	236 − 137 + 128 + 459
34	13,	12,	2	246 − 147 − 237 + 138 + 129
35	13,	9,	3	156 − 246 + 237 − 138 + 129
36	14,	6,	1	156 + 236 − 147 + 138 + 129
37	15,	7,	1	146 + 237 − 138 + 129
38	20,	12,	4	126 + 137 + 148 + 159
39	7,	27,	3	$346 - 2\langle 256\rangle + 8\langle 157\rangle - 247 - 2\langle 148\rangle + 238$
40	8,	18,	2	146 + 256 − 157 − 347 − 238
41	9,	13,	1	256 + 346 − 147 − 247 + 138
42	10,	12,	0	136 + 157 + 257 + 248
43	10,	10,	1	256 + 346 − 147 − 237 + 138
44	11,	8,	0	146 − 236 − 127 + 358
45	12,	5,	0	156 + 246 − 147 − 237 − 138
46	13,	9,	1	236 + 456 − 137 + 128
47	14,	6,	0	456 + 137 − 128
48	14,	6,	2	156 − 246 − 147 + 237 + 128
49	14,	4,	0	256 − 346 − 137 + 128
50	15,	11,	3	346 + 137 + 247 + 128
51	15,	3,	0	156 − 246 − 137 + 128
52	16,	5,	0	246 − 137 + 128

53	18,	3,	0	146 + 236 − 137 + 128
54	21,	5,	1	126 + 137 + 148
55	22,	6,	2	236 − 137 + 128
56	16,	16,	4	156 − 246 − 147 + 237
57	18,	8,	2	156 + 236 + 147
58	20,	4,	0	126 + 347
59	21,	2,	0	146 − 236 − 127
60	24,	2,	0	136 − 127
61	27,	7,	3	136 + 246
62	30,	1,	0	126
63	40,	0,	0	0

Remark 2.2. In Table 1, we use the following abbreviation: for example, $236 - 137 + 128 + 459$ stands for $(u_2 \wedge u_3) \otimes u_6 - (u_1 \wedge u_3) \otimes u_7 + (u_1 \wedge u_2) \otimes u_8 + (u_4 \wedge u_5) \otimes u_9 (\in V)$, where the $u_i \wedge u_j (1 \leq i < j \leq 5)$ and $u_k (6 \leq k \leq 9)$ span $V(10)$ and $V(4)$ over $\mathbf{C}$ respectively. The representative points in Table 1 (or 2) are chosen so that the nilpotent parts of the isotropy subalgebras are contained in the upper triangular matrices in $\mathfrak{sl}(5) \oplus \mathfrak{gl}(4)$.

Remark 2.3. In the Table 1 in [O 1], the orbit $S^4_{16,16}$ was missed. We correct it in this paper.

Remark 2.4. By comparing the G-invariant indices i, j, k, one can see immediately that these 63 orbits in Table 1 are mutually distinct.

Now, by Proposition 1.7, we shall prove that every point in V is G-equivalent to one of the representative points in Table 1.

Let $V_k(10)$ be a 10-dimensional vector space $V(10) \otimes u_k (6 \leq k \leq 9)$. We take $V_6(10) \oplus V_7(10) \oplus V_8(10)$ as V' and $V_9(10)$ as V'' in Proposition 1.7.

We identify $G = SL(5) \times GL(4)$ with the subgroup

(2.1)
$$\left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in GL(9);\ A \in SL(5),\ B \in GL(4)\right\} \text{ of } GL(9), \text{ and } G' = SL(5) \times GL(3)$$

with the subgroup

$$\text{(2.2)} \quad \left\{\begin{pmatrix} A & 0 & 0 \\ 0 & B' & 0 \\ 0 & 0 & 1 \end{pmatrix} \in GL(9);\ A \in SL(5),\ B' \in GL(3)\right\} \text{ of } G.$$

It has been proved by T.Kimura that the triplet $(G', \rho|G', V') = (SL(5) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(3))$ has 25 orbits which are represented by the following points given in Table 2 (See [Ki]).

Therefore, for a point $x = x' + x''$ with $x' \in V'$ and $x'' \in V''$, we may assume that x' belongs to one of these 25 orbits.

Table 2
representative points

[1]	$346 - 2\langle 256\rangle + 8\langle 157\rangle - 247 - 2\langle 148\rangle + 238$
[2]	$146 + 256 - 157 - 347 - 238$
[3]	$256 + 346 - 147 - 247 + 138$
[4]	$136 + 157 + 257 + 248$
[5]	$256 + 346 - 147 - 237 + 138$
[6]	$146 - 236 - 127 + 358$
[7]	$156 + 246 - 147 - 237 - 138$
[8]	$236 + 456 - 137 + 128$
[9]	$456 + 137 - 128$
[10]	$156 - 246 - 147 + 237 + 128$
[11]	$256 - 346 - 137 + 128$
[12]	$346 + 137 + 247 + 128$
[13]	$156 - 246 - 137 + 128$
[14]	$246 - 137 + 128$
[15]	$146 + 236 - 137 + 128$
[16]	$126 + 137 + 148$
[17]	$236 - 137 + 128$
[18]	$156 - 246 - 147 + 237$
[19]	$156 + 236 + 147$
[20]	$126 + 347$
[21]	$146 - 236 - 127$
[22]	$136 - 127$
[23]	$136 + 246$
[24]	126
[25]	0

In the case that x' is one of [18] $\sim$ [25] in Table 2, by repeating the same argument, we obtain 25 orbits (39) $\sim$ (63) in Table 1. Therefore, we may assume that x' is one of [1] $\sim$ [17] in Table 2.

Definition 2.5. For a point $x' = \sum_k x_k (x_k \in V_k(10),\ k = 6, 7, 8)$, we define vector spaces $T_{x'}$ and $N_{x'}$ as follows.

The subspace $T_{x'}$ of V'' is the 3-dimensional subspace spanned by $\{\rho(s_{k9}(1))x_k - x_k \mid k = 6, 7, 8\}$ where $s_{ij}(\lambda)(i \neq j)$ denotes the element of G satisfying $s_{ij}(\lambda)u_i = u_i + \lambda u_j$ and $s_{ij}(\lambda)u_k = u_k (k \neq i)$ for $\lambda \in \mathbf{C}$. Note that $s_{ij}(\lambda)$ is an element of $SL(5)$ (resp. $SL(4)$) for $1 \leq i, j \leq 5$ (resp. $6 \leq i, j \leq 9$). The space $N_{x'}$ is the quotient space $V''/T_{x'}$. Then the following lemma is obvious.

Lemma 2.6. *Let y_1 and y_2 be elements of V'' such that $y_1 \equiv y_2$ modulo $T_{x'}$ in $N_{x'}$. Then $x' + y_1$ is reduced to $x' + y_2$ by the action of $\rho(s_{k9}(\lambda))$.*

Note that $\rho(\tilde{G}_{x'})$ naturally acts on $N_{x'}$. By this lemma, the classification of the fibre $\varphi^{-1}(x')$ is reduced to the orbital decomposition of $N_{x'}$ by $\rho(\tilde{G}_{x'})$.

We remark that the Lie algebra of $\tilde{G}_{x'}$ is given by

$$(2.3)\qquad \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & B_{21} & \beta \end{pmatrix} \in \mathfrak{gl}(9);\ \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & 0 \\ 0 & 0 & \beta \end{pmatrix} \in \mathfrak{g}_{x'},\ B_{21} = (a_{96}, a_{97}, a_{98}) \right\}$$

(*cf.* (2.1), (2.2)). Now we shall show several lemmas which will be used to prove Proposition 2.1.

Proposition 2.7. *In general, any irreducible P.V. (G, ρ, V) has at most one one-codimensional orbit. In our case, the orbit $S_{1,30}$ is the unique one-codimensional orbit.*

Proof. Assume that there exist two one-codimensional orbits S_1 and S_2 which are zeros of irreducible polynomials f_1 and f_2 respectively. Then $F(x) = f_1(x)^{\deg f_2}/f_2(x)^{\deg f_1}$ gives a non-constant absolute invariant (cf. [SK]) which contradicts the prehomogeneity of (G, ρ, V). Q.E.D.

Definition 2.8. Let S be an orbit. We define $\mathscr{H}(S) = \{S' \mid S' \text{ is an } G\text{-orbit satisfying } S \subset \overline{S'}\}$.

Remark 2.9. (1) Let $x_1 \in V$, $\{y_n\}$ a sequence in V, satisfying $y_n \to 0$ $(n \to \infty)$. Suppose $x_1 \in S$, $x_1 + y_n \in S'$ for $\forall n$. Then $S' \in \mathscr{H}(S)$.

(2) Let $S' \in \mathscr{H}(S)$. Then we have $\mathscr{H}(S') \subset \mathscr{H}(S)$ and $\operatorname{codim} S' \leq \operatorname{codim} S$.

We shall use Remark 2.9 by taking $x_1 = x_0 + y_0$ where x_0 is one of representative points [1] $\sim$ [17], and $y_0 \in V'' \setminus T_{x_0}$.

Proposition 2.10. (1) *Let x_0 be a point of $S_{12,10}$. There exists a 12-dimensional G_{x_0}-invariant linear subspace H such that $V = \mathfrak{g}x_0 \oplus H$, and the triplet $(G_{x_0}, \rho|_{G_{x_0}}, H)$ is isomorphic to the irreducible regular P.V. $(SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$ (See Definition 4.4 in* [SKKO] *and* [KM]).

(2) $\mathscr{H}(S_{12,10}) = \{S_{12,10}, S_{8,11}, S^0_{6,14}, S_{5,12}, S_{5,21}, S_{4,14}, S_{4,20}, S_{3,15}, S_{2,24}, S_{2,21}, S_{1,30}, S_{0,40}\}$.

All orbits in $\mathscr{H}(S_{12,10})$ appear in Table 1.

Proof. (1) Let $x_0 = 236 - 137 + 128 + 459$ be a point of $S_{12,10}$ (See Table 1). We take $H = \{p(y) \mid y \in \mathbf{C}^{12}\}$, where

$p(y) = y_1\langle 146\rangle + y_2(\langle 246\rangle + \langle 147\rangle) + y_3\langle 247\rangle + y_4(\langle 346\rangle + \langle 148\rangle) + y_5(\langle 347\rangle + \langle 248\rangle) + y_6\langle 348\rangle + y_7\langle 156\rangle + y_8(\langle 256\rangle + \langle 157\rangle) + y_9\langle 257\rangle + y_{10}(\langle 356\rangle + \langle 158\rangle) + y_{11}(\langle 357\rangle + \langle 258\rangle) + y_{12}\langle 358\rangle$.

Since $d\rho(\mathfrak{g})x_0 = T_{x_0}S_{12,10}$ is generated by $\{126, 136, 236, 246 - 147, 256 - 157, 346 - 148, 356 - 158, 456, 127, 137, 237, 347 - 248, 357 - 258, 457, 128,$

138, 238, 458, 129, 139, 149, 159, 239, 249, 259, 349, 359, 459}, it is clear that the space $x_0 + H$ is transverse to $S_{12,10}$ at x_0.

Then, we have

$$d\rho(\mathfrak{g}_{x_0})|_H = \left\{ \begin{pmatrix} (\beta+\varepsilon)I_6 + 2\Lambda_1(F) & a_{45}I_6 \\ a_{54}I_6 & (\beta-\varepsilon)I_6 + 2\Lambda_1(F) \end{pmatrix}; F \in \mathfrak{sl}(3), \begin{pmatrix} \beta+\varepsilon & a_{45} \\ a_{54} & \beta-\varepsilon \end{pmatrix} \in \mathfrak{gl}(2) \right\}.$$

Therefore, $(G_{x_0}, \rho|_{G_{x_0}}, H)$ is isomorphic to

$$(SL(3) \times GL(2),\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2)).$$

(2) We take $V(6) = S^2(\mathbf{C}e_1 + \mathbf{C}e_2 + \mathbf{C}e_3)$, $V(2) = \mathbf{C}e_4 + \mathbf{C}e_5$, and denote $e_i \otimes e_j \otimes e_k$ by $\langle ijk \rangle$, $1 \le i, j \le 3$, $k = 4, 5$. Then, a point $q(y) \in V(6) \otimes V(2)$ is expressed as

$$\begin{aligned} q(y) = {} & y_1\langle 114\rangle + y_2(\langle 124\rangle + \langle 214\rangle) + y_3\langle 224\rangle + y_4(\langle 134\rangle + \langle 314\rangle) \\ & + y_5(\langle 234\rangle + \langle 324\rangle) + y_6\langle 334\rangle \\ & + y_7\langle 115\rangle + y_8(\langle 125\rangle + \langle 215\rangle) + y_9\langle 225\rangle + y_{10}(\langle 135\rangle + \langle 315\rangle) \\ & + y_{11}(\langle 235\rangle + \langle 325\rangle) + y_{12}\langle 335\rangle. \end{aligned}$$

The correspondence $q(y) \leftrightarrow p(y)$ gives an isomorphism $V(6) \otimes V(2) \simeq H$ as representations of G_{x_0}.

The orbital decomposition of $(SL(3) \times GL(2), 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$ is given in [KM], as in Table 3.1. Here, the first column denotes the name of the orbit in [KM], and if $y_i = 0$, we leave it a blank.

Table 3.1

[KM]	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}
I	1					−1			1			−1
II_1				1			1		1			
II_2				1					1			
III_1			1	1				1				
III_2			1	1			1					
III_3			1	1								
IV_0				1				1				
V_1^0	1								1			
V_2	1							1				
V_3	1		1									
VI	1											
VII												

By the correspondence $q(y) \to p(y)$, Table 3.1 gives the $p(y)$ for each orbit. The second column in Table 3.2 denotes the orbits in (G, ρ, V) containing $x_0 + p(y)$.

Table 3.2

[KM]	orbit	$p(y)$	dual in [KM]
I	0, 40, 0	146 − 348 + 257 − 358	VII
II_1	1, 30, 0	346 + 148 + 156 + 257	VI
II_2	2, 24, 0	346 + 148 + 257	V_3
III_1	2, 21, 0	247 + 346 + 148 + 256 + 157	V_2
III_2	3, 15, 0	247 + 346 + 148 + 156	III_2
III_3	5, 21, 1	247 + 346 + 148	III_3
IV_0	4, 20, 0	346 + 148 + 256 + 157	V_1^0
V_1^0	4, 14, 0	146 + 257	IV_0
V_2	5, 12, 0	146 + 256 + 157	III_1
V_3	6, 14, 0	146 + 247	II_2
VI	8, 11, 0	146	II_1
VII	12, 10, 0	0	I

Let S' be an orbit of (G', ρ', H), where $G' = G_{x_0}$ and $\rho' = \rho|_{G'}$. Let S' be an orbit of (G', ρ', H). Then there exists a unique G-orbit S which contains $x_0 + S'$. Let S be an orbit such that $S_{12,10} \subset \bar{S}$. Then, $A = \{y \in H ; x_0 + y \in S\}$ is invariant by G'. Since $G(x_0 + H)$ is a neighborhood of x_0, A is not empty. Hence A contains some of the representative points in Table 3.1.

Hence S must be one of the orbits in Table 3.2. Q.E.D.

Proposition 2.11. (1) *Let x_0 be a point of $S_{4,11}$. There exists a 4 dimensional linear space $x_0 + H$, transverse to $S_{4,11}$ at x_0, and $x_0 + H$ consists of points of orbits $S_{4,11}$, $S_{3,15}$, $S_{3,18}$, $S_{2,24}$, $S_{2,21}$, $S_{1,30}$ and $S_{0,40}$.*

(2) $\mathscr{H}(S_{4,11}) = \{S_{4,11}, S_{3,15}, S_{3,18}, S_{2,24}, S_{2,21}, S_{1,30}, S_{0,40}\}$. *All orbits in $\mathscr{H}(S_{4,11})$ appear in Table* 1.

Proof. (1) Let $x_0 = 256 + 346 + 157 + 247 + 148 + 238 + 139$ be a generic point of $S_{4,11}$ (See Table 1). We take $H = \{p(y);\, y = (y_2, y_3, y_4, y_5) \in \mathbf{C}^4\}$, where

$$p(y) = y_2\langle 456\rangle - y_3\langle 457\rangle + y_4\langle 458\rangle - y_5\langle 459\rangle.$$

Then we can verify that the $x_0 + H$ is transverse to $S_{4,11}$.

Now, using a basis $(X_1, \ldots, X_{40})$ of $\mathfrak{g}$, and a basis $(e_1, \ldots, e_{40})$ of V, we define

$q(x) = \sum_{i=1}^{40} x_i e_i$, where $(x_1, \ldots, x_{40}) \in \mathbf{C}^{40}$, and

$$(e_1, \ldots, e_{40}) A(q(x)) = (d\rho(X_1)q(x), \ldots, d\rho(X_{40})q(x)).$$

The $A(q(x))$ is a 40×40 matrix with linear entries. We notice that $\operatorname{codim}_V Gq(x) = \dim \mathfrak{g}_{q(x)} = \operatorname{corank} A(q(x))$.

By applying elementary transforms on $A(x_0 + p(y))$, we can find a 40×40 matrices C with entries in $\mathbf{C}$ and $D(y)$ with polynomial entries, such that $\det C$, $\det D(y) \in \mathbf{C} - \{0\}$, and

$$C\ A(x_0 + p(y))D(y) = \begin{pmatrix} L(y) & R(y) \\ 0 & I_{36} \end{pmatrix},$$

$$L(y) = \begin{pmatrix} 2y_2 & 3y_3 & 4y_4 & 5y_5 \\ 3y_3 & 4y_4 - \frac{6}{5}y_2^2 & 5y_5 - \frac{4}{5}y_2y_3 & -\frac{2}{5}y_2y_4 \\ 4y_4 & 5y_5 - \frac{4}{5}y_2y_3 & 2y_2y_4 - \frac{6}{5}y_3^2 & 3y_2y_5 - \frac{3}{5}y_3y_4 \\ 5y_5 & -\frac{2}{5}y_2y_4 & 3y_2y_5 - \frac{3}{5}y_3y_4 & 2y_3y_5 - \frac{4}{5}y_4^2 \end{pmatrix},$$

where $R(y)$ is a 4×36 matrix with polynomial entries, and I_{36} is the identity matrix of size 36.

We will classify the points in H according to the corank $L(y) = \operatorname{corank} A(x_0 + p(y))$, and determine the connected components of $D_k = \{x_0 + p(y);\ y \in \mathbf{C}^4,\ \operatorname{corank} L(y) = k\}$ for each k, $0 \le k \le 4$. Now we state a

Claim. Let E_k be a connected component of D_k, and suppose E_k is smooth and of $\operatorname{codim}_H E_k = k$. *Then, there is a unique orbit S of (G, ρ, V) such that $E_k \subset S$, and* $\operatorname{codim}_V S = k$.

Proof of the Claim. Take a point $q \in E_k$. Then we have

$$\operatorname{codim}_V Gq = \operatorname{corank} A(q) = k.$$

Since Gq and $x_0 + H$ intersect regularly, $\operatorname{codim}_H(Gq \cap H) = \operatorname{codim}_V Gq = k$. Therefore, $q \in Gq \cap H \subset E_k$, and since E_k and $Gq \cap H$ are smooth, we must have $Gq \cap H = E_k$ near q. Now take another point $q' \in E_k$. Since $Gq' \cap E_k$ is a neighborhood of q', and E_k is irreducible, $(Gq \cap E_k) \cap (Gq' \cap E_k) \neq \phi$ and hence, $q' \in Gq$. Thus the Claim is proved.

To determine connected components, we observe that the matrix $L(y)$ is identical with the matrix which defines the discriminant $d_5(1, 0, y_2, y_3, y_4, y_5)(= \det L(y))$ of the quintic polynomial $F(t) = t^5 + y_2t^3 + y_3t^2 + y_4t + y_5$ (See Example 7 in [Y 1], and [YS]). Therefore, connected components are determined by the type of multiple roots of $F(t)$. Here, $F(t)$ is called type $(k_1, \ldots, k_m)$, when

$$1 \leq m \leq 5,\ 1 \leq k_j \leq 5 \text{ for } 1 \leq j \leq m,\ k_1 + \cdots + k_m = 5, \text{ and}$$
$$F(t) = (t - \alpha_1)^{k_1} \cdots (t - \alpha_m)^{k_m},\ \alpha_i \neq \alpha_j, \text{ for } i \neq j,$$
$$k_1\alpha_1 + \cdots + k_m\alpha_m = 0.$$

Moreover, each connected component E is smooth and

$$\operatorname{codim}_H E = \underset{y \in E}{\operatorname{corank}}\, L(y) = 5 - m.$$

A representative point of each connected component and the corresponding orbit including it is as follows. In the following table, representative points of types (5), (4, 1), (3, 2), (3, 1, 1) and (2, 2, 1) are given by t^5, $(t-1)^4(t+4)$, $(t-2)^3(t+3)^2$, $t^3(t^2-1)$, and $(t^2-1)^2 t$, respectively.

Table 3.3

corank $L(y)$	type	y_2	y_3	y_4	y_5	orbit
4	5	0	0	0	0	$S_{4,11}$
3	4, 1	-10	20	-15	4	$S_{3,15}$
3	3, 2	-15	10	60	-72	$S_{3,18}$
2	3, 1, 1	-1	0	0	0	$S_{2,21}$
2	2, 2, 1	-2	0	1	0	$S_{2,24}$
1	2, 1, 1, 1	1	1	0	0	$S_{1,30}$
0	1, 1, 1, 1, 1	0	0	0	1	$S_{0,40}$

First, we note that each connected components C in Table 3.3 satisfy the property that if $(y_2, \dots, y_5) \in C$, then $(c^2 y_2, c^3 y_3, c^4 y_4, c^5 y_5) \in C$ for any $c \in \mathbf{C} - \{0\}$. Thus each of them includes the origin in its closure.

Now, we prove that $S_{3,18} \in \mathscr{H}(S_{4,11})$. Let $x_{3,18}$ be a point of $S_{3,18}$ in Table 1. By an exchange of basis $u_i \to \pm u_{g_1(i)}$ $g_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 5 & -3 & 1 & -7 & 8 & 6 & -9 \end{pmatrix}$ (we mean $u_4 \to -u_3$, etc)

$$\rho(g_1)x_{3,18} = 247 + 157 + 148 + 346 + 256 + 239.$$

By $g_2 = (u_9 \to u_8 + u_9) = s_{98}(1)$,

$$\rho(g_2 g_1)x_{3,18} = 256 + 346 + 157 + 247 + 148 + 238 + 239.$$

Set $g_3 = (u_2 \to u_1 + u_2,\ u_3 \to \frac{3}{2}u_1 + 3u_2 + u_3,\ u_4 \to -\frac{1}{2}u_1 - \frac{3}{2}u_2 - u_3 + u_4,\ u_5 \to$

$$-\frac{1}{4}u_1 - u_2 - u_3 + 2u_4 + u_5,\ u_7 \to -u_6 + u_7,\ u_8 \to -\frac{1}{2}u_6 + u_7 + u_8),$$

$$g_4 = s_{52}(3)s_{41}(3)s_{86}\left(-\frac{3}{2}\right)s_{96}(3)s_{51}\left(\frac{9}{4}\right)s_{42}\left(\frac{3}{2}\right)s_{31}\left(\frac{3}{2}\right),$$

$$g_5(\varepsilon) = (u_1 \to \varepsilon^{-2}u_1,\ u_2 \to \varepsilon^{-1}u_2,\ u_4 \to \varepsilon u_4,\ u_5 \to \varepsilon^2 u_5,\ u_6 \to \varepsilon^{-1}u_6,\ u_8 \to \varepsilon u_8,\ u_9 \to \varepsilon^2 u_9).$$

Set $g(\varepsilon) = g_5(\varepsilon)g_4g_3g_2g_1$. Then, $\lim_{\varepsilon\to 0}\rho(g(\varepsilon))x_{3,18} = x_{4,11}$.

Next set $g_1 = s_{32}(-1)$,

$$g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & -3 & 5 & 6 & 8 & 7 & 9 \end{pmatrix},\ g_3 = s_{52}(-1)s_{97}(1),$$

$$g_4 = (u_2 \to -u_1 + u_2,\ u_3 \to \frac{3}{2}u_1 - 3u_2 + u_3,\ u_4 \to \frac{1}{2}u_1 - \frac{3}{2}u_2 + u_3 + u_4,\ u_5 \to -\frac{1}{4}u_1 + u_2 - u_3 - 2u_4 + u_5,\ u_7 \to u_6 + u_7,\ u_8 \to -\frac{1}{2}u_6 - u_7 + u_8,\ u_9 \to -u_9),$$

$g_5(\varepsilon)$ be the same as above. Then, if we set $g(\varepsilon) = g_5(\varepsilon)g_4g_3g_2g_1$, we have $\lim_{\varepsilon\to 0} g(\varepsilon)x_{3,15} = x_{4,11}$.

Since there exist only two orbits of codimension 3 containing $S_{4,11}$ in its closure, orbits of types (4, 1) and (3, 2) in Table 3.3 coincide with $S_{3,15}$ and $S_{3,18}$. We do not know the mutual correspondence. But if we consider the orders of those holonomic varieties, we know that (4, 1) corresponds $S_{3,15}$ and (3, 2) does $S_{3,18}$.

Now we show the correspondence of codimension 2 orbits.

Set $g_1 = s_{68}(1)s_{35}(-1)s_{24}(1)$,

$$g_2 = s_{51}\left(-\frac{1}{4}\right)s_{86}\left(-\frac{1}{2}\right)s_{31}\left(\frac{1}{2}\right)s_{42}\left(\frac{1}{2}\right)s_{97}(1)s_{53}(1),$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_4 = s_{96}(1)s_{21}(1),$$

$$g_5 = (u_1 \to -u_5/2,\ u_2 \to u_3,\ u_3 \to u_1,\ u_5 \to -u_2,\ u_6 \to u_8,\ u_8 \to u_6,\ u_9 \to 2u_9).$$

$$g = g_5g_4g_3g_2g_1.$$

Next set $g_6 = s_{97}(-1)s_{53}(1)s_{86}(-1)s_{51}(-2)s_{31}(-1)s_{42}(-1)$,

$$g_7 = (u_1 \to iu_2,\ u_2 \to -iu_4,\ u_3 \to u_5,\ u_5 \to u_3,\ u_4 \to u_1,\ u_6 \to u_7,$$
$$u_7 \to -iu_8,\ u_8 \to iu_6,\ u_9 \to -iu_9),\ i = \sqrt{-1}.$$

Then, we have $\rho(g)(x_{4,11} + x_{(2,2,1)}) = x_{2,24}$ and

$$\rho(g_7 g_6)(x_{4,11} + x_{(3,1,1)}) = x_{2,21},$$

where $x_{(2,2,1)}(x_{(3,1,1)}$, respectively) is the representative point of the component of type (2,2,1)((3, 1, 1), respectively) in Table 3.3.

Types (2, 1, 1, 1) and (1, 1, 1, 1, 1) correspond to $S_{1,30}$ and $S_{0,40}$ since $S_{1,30}, S_{0,40} \in \mathscr{H}(S_{4,11})$ and they are unique orbits of codimensions 1 and 0.

(2) is clear by (1), since all orbits in H are weighted homogeneous of weights (2, 3, 4, 5). Thus the Proposition is proved. Q.E.D.

In the proof of the Proposition, $f(x) = \det A(q(x))$ is nothing but the fundamental relative invariant of our P.V. We also remark that $f_{S_{4,11}}(y) = \det L(y)$, which is identified with the discriminant of the quintic equation, is called the transverse localization along $S_{4,11}$ of $f(x)$ in [Y 2], which is relatively invariant by the vector fields defined by ${}^t(X_1, \dots, X_4) = {}^tL(y)(\partial)$, $(\partial) = {}^t(\partial/\partial y_2, \dots, \partial/\partial y_5)$. The localization of $f(x)$ in the sense of the Definition 6.7 in [SKKO] is the lowest homogeneous part of $f_{S_{4,11}}$, that is, $(5y_5)^4$.

Lemma 2.12. (1) *The* $GL(2)$ *acts on* $\mathbf{C}u + \mathbf{C}v$ *by* $(u, v) \to (u, v)A$ *for* $A \in GL(2)$. *This action induces the action* $2\Lambda_1$ (*resp.* $3\Lambda_1$) *of* $GL(2)$ *on the space of binary quadratic forms*:

$$V(3) = \{x_{20}u^2 + x_{11}uv + x_{02}v^2;\ x_{20}, x_{11}, x_{02} \in \mathbf{C}\}$$

(*resp. on the space of binary cubic forms*:

$$V(4) = \{x_{30}u^3 + x_{21}u^2v + x_{12}uv^2 + x_{03}v^3;\ x_{ij} \in \mathbf{C}\}).$$

The triplet $(GL(2), 2\Lambda_1, V(3))$ (*resp.* $(GL(2), 3\Lambda_1, V(4))$ *is a prehomogeneous vector space.*

(2) $(GL(2), 2\Lambda_1, V(3))$ *has* 3 *orbits represented by* $u^2 + v^2$, u^2, 0 (*or* uv, u^2, 0).

(3) $(GL(2), 3\Lambda_1, V(4))$ *has* 4 *orbits represented by* $u^3 + v^3$, u^2v, u^3, 0.

Definition 2.13. Let $M(m, n)$ be the totality of $m \times n$ matrices. Then $SL(m) \times SL(n)$ acts on $M(m, n)$ by $X \to AX^tB$ for $A \in SL(m)$, $B \in SL(n)$ and $X \in M(m, n)$. We denote this action by $\Lambda_1 \otimes \Lambda_1$.

The following lemma is well-known in linear algebra.

Lemma 2.14. *For* $m > n \geq 1$, *the triplet* $(SL(m) \times SL(n), \Lambda_1 \otimes \Lambda_1, M(m, n))$ *has* $(n + 1)$-*orbits* S_i *where* $S_i = \{X \in M(m, n);\ \mathrm{rank}\, X = i\}$ $i = 0, 1, 2, \dots, n$.

For $m = n$, together with scalar multiplication, we have the same result.

Now, we begin to prove the proposition 2.1 starting from the case [1] in Table 2.

1° (The case [1])

We choose $x' = -2\langle 256\rangle + \langle 346\rangle + 8\langle 157\rangle - \langle 247\rangle - 2\langle 148\rangle + \langle 238\rangle$ so that $\tilde{\mathfrak{g}}_{x'}$ becomes the standard form (2.4). The point x' belongs to the orbit $S_{7,27}$ in Table 1. We denote x' by $x_{7,27}$. The Lie algebra $\tilde{\mathfrak{g}}_{x_{7,27}}$ is given as follows:

$$(2.4)\quad \tilde{\mathfrak{g}}_{x_{7,27}} = \left\{ \begin{pmatrix} 4\Lambda_1(F) & 0 & 0 \\ 0 & 2\Lambda_1(F) & 0 \\ 0 & \gamma & \beta \end{pmatrix}; F = \begin{pmatrix} \alpha & s \\ t & -\alpha \end{pmatrix} \in \mathfrak{sl}(2),\ \gamma \in \mathbf{C}^3,\ \beta \in \mathfrak{gl}(1) \right\}$$

$$= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & A_3 \end{pmatrix} \in M(5),\right.$$

$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\} \text{ where}$$

$$A_1 = \begin{pmatrix} 4\alpha & 4s \\ t & 2\alpha \end{pmatrix},\ A_{12} = \begin{pmatrix} 0 \\ 3s \end{pmatrix},\ A_{21} = (0\ 2t),\ A_{23} = (2s\ 0),\ A_{32} = \begin{pmatrix} 3t \\ 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -2\alpha & s \\ 4t & -4\alpha \end{pmatrix},\ B_1 = \begin{pmatrix} 2\alpha & s & 0 \\ 2t & 0 & 2s \\ 0 & t & -2\alpha \end{pmatrix},\ B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{7,27}}$ on the vector space $N_{x_{7,27}}$ with respect to the base $\{\langle 129\rangle,\ 2\langle 139\rangle,\ (\langle 149\rangle + 2\langle 239\rangle),\ (4/13)(\langle 159\rangle + 8\langle 249\rangle),\ (\langle 259\rangle + 2\langle 349\rangle),\ 2\langle 359\rangle,\ \langle 459\rangle\}$ modulo $T_{x_{7,27}}$ is given as follows.

$$(2.5)\quad \mathfrak{h}_{x_{7,27}} = \left\{ C = \beta I_7 \oplus 6\Lambda_1(F) \in M(7):\ F = \begin{pmatrix} \alpha & s \\ t & -\alpha \end{pmatrix} \right\}$$

$$= \left\{ C = \begin{pmatrix} C_1 & C_{12} & 0 \\ C_{21} & C_2 & C_{23} \\ 0 & C_{32} & C_3 \end{pmatrix} \in M(7):\ C_1 = \begin{pmatrix} (6\alpha+\beta) & 6s \\ t & (4\alpha+\beta) \end{pmatrix},\right.$$

$$C_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 5s & 0 & 0 \end{pmatrix},\ C_{21} = \begin{pmatrix} 0 & 2t \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ C_2 = \begin{pmatrix} (2\alpha+\beta) & 4s & 0 \\ 3t & \beta & 3s \\ 0 & 4t & (-2\alpha+\beta) \end{pmatrix},$$

$$\left. C_{23} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2s & 0 \end{pmatrix},\ C_{32} = \begin{pmatrix} 0 & 0 & 5t \\ 0 & 0 & 0 \end{pmatrix},\ C_3 = \begin{pmatrix} (-4\alpha+\beta) & s \\ 6t & (-6\alpha+\beta) \end{pmatrix} \right\}$$

where $s = a_{45}$ and $t = a_{21}$.

Let $H_{x_{7,27}}$ be the connected algebraic subgroup of $GL(7)$ corresponding to the Lie algebra $\mathfrak{h}_{x_{7,27}}$.

Let $\tilde{\rho}$ be the action of $\tilde{G}_{x_{7,27}}$ on $N_{x_{7,27}}$. Then $H_{x_{7,27}}$ is the connected component of $\tilde{\rho}(\tilde{G}_{x_{7,27}})$.

Proposition 2.15. *Any orbit of the triplet* $(\tilde{G}_{x_{7,27}}, \tilde{\rho}, N_{x_{7,27}})$ *is represented by one of the following points modulo* $T_{x_{7,27}}$:

(1) $y_1 = \lambda\langle 459\rangle + \mu\langle 359\rangle + \nu(\langle 259\rangle + 2\langle 349\rangle) + \tau(\langle 159\rangle + 8\langle 249\rangle) + \langle 139\rangle$

(2) $y_2 = \lambda\langle 459\rangle + \mu\langle 359\rangle + \nu(\langle 259\rangle + 2\langle 349\rangle) + (\langle 149\rangle + 2\langle 239\rangle)$

(3) $y_3 = \lambda\langle 459\rangle + \mu\langle 359\rangle + (\langle 159\rangle + 8\langle 249\rangle)$

(4) $y_4 = \langle 459\rangle$

(5) $y_5 = 0$

For the orbit $S = Gx$ of $x = x_{7,27} + y_1$(resp. y_2, y_3), we have $S \in \mathscr{H}(S_{4,11})$(resp. $\mathscr{H}(S_{3,18})$, $\mathscr{H}(S_{4,14})$). The point $x = x_{7,27} + y_4$ belongs to the orbit $S_{5,9}$.

Proof. Put $y = [p_{12}(u_1 \wedge u_2) + 2p_{13}(u_1 \wedge u_3) + p_{14}\{(u_1 \wedge u_4) + 2(u_2 \wedge u_3)\} + 4/13 \cdot p_{15}\{(u_1 \wedge u_5) + 8(u_2 \wedge u_4)\} + p_{25}\{(u_2 \wedge u_5) + 2(u_3 \wedge u_4)\} + 2p_{35}(u_3 \wedge u_5) + p_{45}(u_4 \wedge u_5)] \otimes u_9 \in N_{x_{7,27}}$.

Let $h(s)$(resp. $h(\beta)$, $h(t)$) $= \exp C \in H_{x_{7,27}}$ where C is a matrix in (2.5) with all entries zero except s(resp. β, t).

[1]

(i) If $p_{13}, p_{14}, p_{15}, p_{25}, p_{35}$ or $p_{45} \neq 0$, then we may assume that $p_{12} = 0$ by the action of $h(s)$ where s satisfies the condition: $p_{12} + 6p_{13}s + 15p_{14}s^2 + 20p_{15}s^3 + 15p_{25}s^4 + 6p_{35}s^5 + p_{45}s^6 = 0$.

(a) If $p_{13} \neq 0$, then we may assume that $2p_{13} = 1$ by the action of $h(\beta)$ with $\beta = -\log 2p_{13}$. Moreover, we may assume that $p_{14} = 0$ by $h(t)$ with $t = -p_{14}$, i.e., we obtain $y = y_1$.

(b) If $p_{13} = 0$ and $p_{14} \neq 0$, then we may assume that $p_{14} = 1$ by $h(\beta)$ with $\beta = -\log p_{14}$. Then we may assume that $p_{15} = 0$ by $h(t)$ with $t = -\dfrac{p_{15}}{3}$, i.e., we obtain $y = y_2$.

(c) If $p_{13} = p_{14} = 0$ and $p_{15} \neq 0$, then we have $\dfrac{4}{13}p_{15} = 1$ by $h(\beta)$ with $\beta = -\log\dfrac{4}{13}p_{15}$. Moreover, we have $p_{25} = 0$ by $h(t)$ with $t = -\dfrac{p_{25}}{13}$, i.e., we obtain $y = y_3$.

(d) If $p_{13} = p_{14} = p_{15} = 0$ and p_{25} or $p_{35} \neq 0$, then we have $p_{45} = 0$ by $h(t)$

with $15t^2p_{25}+6tp_{35}+p_{45}=0$. Then it is reduced to (a) or (b) by exchanging $u_1 \rightleftarrows u_5$, $u_2 \rightleftarrows u_4$, $u_6 \rightleftarrows -u_8$ and $u_7 \to -u_7$.

(e) If $p_{13}=p_{14}=p_{15}=p_{25}=p_{35}=0$ and $p_{45}\neq 0$, then we obtain $y=y_4$ by $h(\beta)$ with $\beta=-\log p_{45}$.

(ii) If $p_{13}=p_{14}=p_{15}=p_{25}=p_{35}=p_{45}=0$ and $p_{12}\neq 0$, then this case is equivalent to (e) in (i).

(iii) If all $p_{ij}=0$, then we obtain $y=y_5$.

[2]

The point $x^*=x_{7,27}+(u_1\wedge u_3)\otimes u_9$ coincides with the representative point $x_{4,11}$ of the orbit $S_{4,11}$ in Table 1 by the transformation $u_1\to -u_1/2$, $u_2\to\sqrt{2}iu_2$, $u_4\to\sqrt{2}iu_4$, $u_5\to -u_5/2$, $u_6\to u_6/(\sqrt{2}i)$, $u_7\to u_7/2$, $u_8\to u_8/(\sqrt{2}i)$ and $u_9\to -2u_9$ where $i=\sqrt{-1}$. Therefore when the orbit S contains the point $x=x_{7,27}+y_1$, then the closure $\bar{S}$ contains $x_{4,11}$, because $\lim_{\varepsilon\to 0}\rho(g)x=x^*$ where g is given as follows with $\xi=\varepsilon^2$.

(2.6) g is a diagonal matrix in $M(9)$ with the diagonal elements $\{\varepsilon^{-2}, \varepsilon^{-1}, 1, \varepsilon, \varepsilon^2, \varepsilon^{-1}, 1, \varepsilon, \xi\}$.

Therefore, $S\in\mathscr{H}(S_{4,11})$.

The point $x^*=x_{7,27}+\{(u_1\wedge u_4)+2(u_2\wedge u_3)\}\otimes u_9$ coincides with the representative point $x_{3,18}$ of the orbit $S_{3,18}$ in Table 1 by the action of $\rho(s_{89}(1/2))$, $\rho(s_{98}(-2/5))$ and by the transformation $u_1\to -u_5/4$, $u_3\to u_4$, $u_4\to u_1$, $u_5\to u_3/2$, $u_6\to -u_8$, $u_7\to u_6$, $u_8\to 2u_7$ and $u_9\to 2u_9/5$. Therefore, when the orbit S contains the point $x=x_{7,27}+y_2$, then the closure $\bar{S}$ contains $x_{3,18}$, because $\lim_{\varepsilon\to 0}\rho(g)x=x^*$ where g is the diagonal matrix (2.6) with $\xi=\varepsilon$.

Therefore $S\in\mathscr{H}(S_{3,18})\subset\mathscr{H}(S_{4,11})$.

The point $x^*=x_{7,27}+\{(u_1\wedge u_5)+8(u_2\wedge u_4)\}\otimes u_9$ coincides with the representative point $x_{4,14}$ of the orbit $S_{4,14}$ in Table 1 by the action of $\rho(s_{79}(-1/8))$, $\rho(s_{97}(8/65))$ and by the transformation $u_1\to u_3/2$, $u_2\to u_1$, $u_3\to u_4$, $u_4\to u_2$, $u_5\to u_5/2$, $u_6\to -u_6$, $u_7\to u_9/2$, $u_8\to u_7$, $u_9\to 8u_8/65$. Therefore, when the orbit S contains the point $x=x_{7,27}+y_3$, then the closure $\bar{S}$ contains $x_{4,14}$, because $\lim_{\varepsilon\to 0}\rho(g)x=x^*$ where g is (2.6) with $\xi=1$. Therefore $S\in\mathscr{H}(S_{4,14})\subset\mathscr{H}(S_{12,10})$.

The point $x=x_{7,27}+y_4$ coincides with the representative point $x_{5,9}$ of the orbit $S_{5,9}$ in Table 1 by the transformation $u_1\to u_5/2$, $u_2\to\sqrt{2}u_4$, $u_4\to\sqrt{2}u_2$, $u_5\to u_1/2$, $u_6\to -u_8/\sqrt{2}$, $u_7\to -u_7/2$, $u_8\to u_6/\sqrt{2}$ and $u_9\to\sqrt{2}u_9$.

Q.E.D.

By this proposition, Remark 2.9 and Propositions 2.10, 2.11, we have the following result.

Corollary 2.16. *From the case* [1], *i.e., the orbits through* $x_{7,27}+y$

$\otimes u_9 (y \in V(10),\ y \neq 0)$, *we have no other orbit except* $S_{0,40}$, $S_{1,30}$, $S_{2,24}$, $S_{2,21}$, $S_{3,18}$, $S_{3,15}$, $S_{4,14}$, $S_{4,11}$ *and* $S_{5,9}$ *in Table* 1.

Remark 2.17. In the following study, we use the notations H_x, $\tilde{\rho}$ and $h(s, t, \dots)$ similarly to them in the case [1];

H_x: the connected algebraic subgroup of $GL(7)$ corresponding to the Lie algebra $\mathfrak{h}_x$ where $\mathfrak{h}_x$ is the action of $\tilde{\mathfrak{g}}_x$ on the vector space N_x;

$\tilde{\rho}$: the action of $\tilde{G}_x$ on N_x;

$h(s, t, \dots)$: an element $\exp C \in H_x$ where C is a matrix in $M(7)$ with $s, t, \dots$ and all the remaining parts zero in $\mathfrak{h}_x$(cf. the proof of Proposition 2.15).

2° (The case [2])

Put $x' = \langle 146 \rangle + \langle 256 \rangle - \langle 157 \rangle - \langle 347 \rangle - \langle 238 \rangle$. The point x' belongs to the orbit $S_{8,18}$ in Table 1. We denote x' by $x_{8,18}$. The Lie algebra $\tilde{\mathfrak{g}}_{x_{8,18}}$ is given as follows:

$$(2.7)\quad \tilde{\mathfrak{g}}_{x_{8,18}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5),\ B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where

$$A_1 = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & -2\alpha_2 & 0 \\ 0 & 0 & 2(\alpha_1 + \alpha_2) \end{pmatrix},\ A_{12} = \begin{pmatrix} a_{14} & a_{15} \\ a_{15} & 0 \\ 0 & a_{14} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -(2\alpha_1 + \alpha_2) & 0 \\ 0 & -\alpha_1 + \alpha_2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \alpha_1 + \alpha_2 & 0 & a_{14} \\ 0 & -\alpha_2 & a_{15} \\ 0 & 0 & -2\alpha_1 \end{pmatrix},\ B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{8,18}}$ on the vector space $N_{x_{8,18}}$ with respect to the base $\{\langle 129 \rangle, \langle 139 \rangle, (\langle 149 \rangle - \langle 259 \rangle), (\langle 159 \rangle - \langle 349 \rangle), \langle 249 \rangle, \langle 359 \rangle, \langle 459 \rangle\}$ modulo $T_{x_{8,18}}$ is given as follows:

$$(2.8)\quad \mathfrak{h}_{x_{8,18}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & C_3 \end{pmatrix} \in M(7);\ C_1 = \begin{pmatrix} (\alpha_1 - 2\alpha_2 + \beta) & 0 \\ 0 & (3\alpha_1 + 2\alpha_2 + \beta) \end{pmatrix}, \right.$$

$$C_{12} = \begin{pmatrix} 2a_{15} & 0 \\ 0 & 2a_{14} \end{pmatrix},\ C_{13} = \begin{pmatrix} -a_{14} & 0 & 0 \\ 0 & -a_{15} & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} (-\alpha_1 - \alpha_2 + \beta) & 0 \\ 0 & (\alpha_2 + \beta) \end{pmatrix},$$

$$C_{23} = \begin{pmatrix} 0 & 0 & -a_{15} \\ 0 & 0 & a_{14} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} (-2\alpha_1 - 3\alpha_2 + \beta) & 0 & 0 \\ 0 & (\alpha_1 + 3\alpha_2 + \beta) & 0 \\ 0 & 0 & (-3\alpha_1 + \beta) \end{pmatrix} \Bigg\} .$$

Proposition 2.18. *Any orbit of the triplet* $(\tilde{G}_{x_{8,18}}, \tilde{\rho}, N_{x_{8,18}})$ *is represented by one of the following points modulo* $T_{x_{8,18}}$.

(1) $y_1 = \lambda\langle 129\rangle + \mu\langle 139\rangle + \nu\langle 249\rangle + \tau\langle 359\rangle + \langle 459\rangle$
(2) $y_2 = \langle 129\rangle + \lambda(\langle 149\rangle - \langle 259\rangle) + \mu(\langle 159\rangle - \langle 349\rangle) + \nu\langle 249\rangle + \langle 359\rangle$
(3) $y_3 = \lambda(\langle 149\rangle - \langle 259\rangle) + \mu(\langle 159\rangle - \langle 349\rangle) + \langle 249\rangle + \langle 359\rangle$
(4) $y_4 = (\langle 149\rangle - \langle 259\rangle) + \lambda(\langle 159\rangle - \langle 349\rangle) + \langle 359\rangle$
(5) $y_5 = (\langle 159\rangle - \langle 349\rangle) + \langle 359\rangle$
(6) $y_6 = \langle 359\rangle$
(7) $y_7 = (\langle 149\rangle - \langle 259\rangle) + (\langle 159\rangle - \langle 349\rangle)$
(8) $y_8 = \langle 129\rangle + (\langle 159\rangle - \langle 349\rangle)$
(9) $y_9 = (\langle 159\rangle - \langle 349\rangle)$
(10) $y_{10} = \langle 129\rangle + \langle 139\rangle$
(11) $y_{11} = \langle 129\rangle$
(12) $y_{12} = 0.$

For the orbit S of $x = x_{8,18} + y_1$(resp. y_2), we have $S \in \mathscr{H}(S_{4,14})$(resp. $\mathscr{H}(S_{4,11})$).

For the orbit S of $x = x_{8,18} + y_3$(resp. y_4), if $\lambda\mu \neq 1/4$(resp. $\lambda \neq 0$), we have $S \in \mathscr{H}(S_{4,11})$, and if $\lambda\mu = 1/4$ (except $\lambda = \mu = 1/2$, $\lambda = \omega/2$ and $\mu = \omega^2/2$ or $\lambda = \omega^2/2$ and $\mu = \omega/2$)(resp. $\lambda = 0$), then the orbit S is $S_{4,20}$. If $\lambda = \mu = 1/2$, $\lambda = \omega/2$ and $\mu = \omega^2/2$ or $\lambda = \omega^2/2$ and $\mu = \omega/2$, then we can see that $S = S_{5,12}$.

The point $x = x_{8,18} + y_5$(resp. y_i, $i = 6, 7, \dots, 11$) coincides with the representative point $x_{5,16}$ (resp. $x_{6,8}$, $x_{1,30}$, $x_{3,18}$, $x_{5,16}$, $x_{6,14}^2$, $x_{7,7}$) of the orbit $S_{5,16}$ (resp. $S_{6,8}$, $S_{1,30}$, $S_{3,18}$, $S_{5,16}$, $S_{6,14}^2$, $S_{7,7}$) in Table 1.

Proof. Put $y = [p_{12}(u_1 \wedge u_2) + p_{13}(u_1 \wedge u_3) + p_{14}\{(u_1 \wedge u_4) - (u_2 \wedge u_5)\} + p_{15}\{(u_1 \wedge u_5) - (u_3 \wedge u_4)\} + p_{24}(u_2 \wedge u_4) + p_{35}(u_3 \wedge u_5) + p_{45}(u_4 \wedge u_5)] \otimes u_9 \in N_{x_{8,18}}$.

[1]

(i) If $p_{45} \neq 0$, then we may assume that $p_{14} = p_{15} = 0$, and $p_{45} = 1$ by the action of $h(a_{14}, a_{15})$ and $h(\beta)$ where $a_{14} = -p_{15}/p_{45}$, $a_{15} = p_{14}/p_{45}$ and $\beta = -\log p_{45}$, i.e., $y = y_1$.

(ii) If $p_{45} = 0$ and $p_{35} \neq 0$, then we may assume that $p_{13} = 0$ and $p_{35} = 1$ by the action of $h(a_{15})$ and $h(\beta)$ where $a_{15} = p_{13}/p_{35}$ and $\beta = -\log p_{35}$.

(a) If $p_{12} \neq 0$, then we obtain y_2 by the action of $\rho(g)$ where g is as follows:

(2.9) g is a diagonal matrix in $M(9)$ and its diagonal elements

$\{\sigma\varepsilon, \sigma^2, \varepsilon^2, \sigma^{-1}\varepsilon^{-2}, \sigma^{-2}\varepsilon^{-1}, \varepsilon, \sigma, \sigma^{-2}\varepsilon^{-2}, \xi\}$ where $\xi = \sigma^2\varepsilon^{-1}$, $\sigma^5 = 1/p_{12}$, and $\varepsilon = 1$.

(b) If $p_{12} = 0$ and $p_{24} \neq 0$, then we obtain y_3 by the action of $\rho(g)$ where g is the diagonal matrix (2.9) where $\sigma^3/\varepsilon^3 = 1/p_{24}$.

(c) If $p_{12} = p_{24} = 0$ and $p_{14} \neq 0$, then we obtain y_4 by the action of $\rho(g)$ where g is the diagonal matrix (2.9) with $\sigma^2/\varepsilon^2 = 1/p_{14}$.

(d) If $p_{12} = p_{14} = p_{24} = 0$ and $p_{15} \neq 0$, then we obtain y_5 by the action of $\rho(g)$ where g is (2.9) with $\sigma/\varepsilon = 1/p_{15}$.

(e) If $p_{12} = p_{14} = p_{15} = p_{24} = 0$, we obtain y_6.

(iii) If $p_{35} = p_{45} = 0$ and $p_{24} \neq 0$, then y coincides with y_3, y_4, y_5 or y_6, by the transformation $u_2 \leftrightarrow u_3$, $u_4 \leftrightarrow u_5$, $u_6 \leftrightarrow -u_7$ and $u_8 \rightarrow -u_8$.

(iv) If $p_{24} = p_{35} = p_{45} = 0$ and $p_{14}p_{15} \neq 0$, then we obtain y_7 by the action of $h(a_{14}, a_{15})$ and $h(\beta)$ where $a_{14} = -p_{13}/(2p_{15})$ and $a_{15} = p_{12}/(2p_{14})$ and $\beta = -\log p_{15}$ and by the action of $\rho(g)$ where g is the diagonal matrix (2.9) with $\xi = \sigma$ and $\sigma/\varepsilon = 1/p_{14}$.

(v) If $p_{14} = p_{24} = p_{35} = p_{45} = 0$ and $p_{15} \neq 0$, then we may assume that $p_{13} = 0$ and $p_{15} = 1$ by the action of $h(a_{14})$ and $h(\beta)$ where $a_{14} = -p_{13}/(2p_{15})$ and $\beta = -\log p_{15}$.

(a) If $p_{12} \neq 0$, then we obtain y_8 by the action of $\rho(g)$ where g is the diagonal matrix (2.9) with $\xi = \sigma$ and $\sigma^4\varepsilon = 1/p_{12}$.

(b) If $p_{12} = 0$, then we obtain y_9.

(vi) If $p_{15} = p_{24} = p_{35} = p_{45} = 0$ and $p_{14} \neq 0$, then y coincides with y_8 or y_9 by the same way in (iii).

(vii) If $p_{14} = p_{15} = p_{24} = p_{35} = p_{45} = 0$ and $p_{12}p_{13} \neq 0$, then we obtain y_{10} by the action of $h(\beta)$ where $\beta = -\log p_{13}$ and by the action of $\rho(g)$ where g is (2.9) with $\xi = (\sigma\varepsilon^3)^{-1}$ and $\sigma^2/\varepsilon^2 = 1/p_{12}$.

(viii) If $p_{12} \neq 0$ and all the remaining $p_{ij} = 0$, then we have y_{11} by the action of $h(\beta)$ where $\beta = -\log p_{12}$.

If $p_{13} \neq 0$ and the all remaining $p_{ij} = 0$, y coincides with y_{11} by the same way (iii).

[2]

The point $x^* = x_{8,18} + (u_4 \wedge u_5) \otimes u_9$ coincides with the representative point $x_{4,14}$ of the orbit $S_{4,14}$ in Table 1 by the transformation $u_1 \leftrightarrow u_4$, $u_2 \rightarrow u_3$, $u_3 \rightarrow u_5$, $u_5 \rightarrow u_2$, $u_6 \rightarrow -u_7$, $u_7 \rightarrow u_6$, $u_8 \rightarrow -u_9$ and $u_9 \rightarrow u_8$. Therefore, when the orbit S contains the point $x = x_{8,18} + y_1$, then the closure $\bar{S}$ contains $x_{4,14}$, because $\lim_{\sigma,\varepsilon \to 0} \rho(g)x = x^*$ where g is (2.9) with $\xi = (\sigma\varepsilon)^3$.

Therefore, $S \in \mathscr{H}(S_{4,14})$ by Remark 2.9.

The point $x^* = x_{8,18} + \{(u_1 \wedge u_2) + (u_3 \wedge u_5)\} \otimes u_9$ coincides with the representative point $x_{4,11}$ of the orbit $S_{4,11}$ in Table 1 by the transformation $u_1 \rightarrow u_2$, $u_2 \rightarrow u_3$, $u_3 \rightarrow u_1$, $u_4 \leftrightarrow u_5$, $u_8 \leftrightarrow u_9$ and $u_7 \rightarrow -u_7$. Therefore, when the orbit S contains the point $x = x_{8,18} + y_2$, then the closure $\bar{S}$ contains $x_{4,11}$, because $\lim_{\varepsilon\to\infty} \rho(g)x = x^*$ where g is (2.9) with $\sigma = 1$ and $\xi = 1/\varepsilon$. Therefore, $S \in \mathscr{H}(S_{4,11})$.

When the point $x = x_{8,18} + y_3$, if $\lambda\mu \neq 1/4$, then we may assume that y_3 is y in the case of [1]-(ii)-(a) by the action of $h(a_{14}, a_{15})$ where $-a_{14} + 2\lambda a_{15} \neq 0$ and $2\mu a_{14} - a_{15} = 0$. Therefore, y_3 coincides with y_2. If $\lambda\mu = 1/4$, then we may assume as follows:

(2.10) $$x = x_{8,18} + \{(u_1 \wedge u_4) + m(u_1 \wedge u_5) + m(u_2 \wedge u_4) + (u_3 \wedge u_5)\} \otimes u_9$$
$$(\text{where } m^2 = 8\mu^3)$$

by the action of $\rho(s_{69}(\lambda))$, $\rho(s_{79}(-\mu))$ and $\rho(g)$ where g is (2.9) with $\xi = \sigma^2/\varepsilon$ and $\sigma^2 = 2\mu\varepsilon^2$. The point x coincides with the representative point $x_{4,20}$ of the orbit $S_{4,20}$ in Table 1 (except when $\lambda = \mu = 1/2$, $\lambda = \omega/2$ and $\mu = \omega^2/2$ or $\lambda = \omega^2/2$ and $\mu = \omega/2$) by the following operators (2.11) $\sim$ (2.14):

(2.11) $$g_1 = s_{67}(m)s_{54}(m)s_{79}(m)s_{69}(-1)s_{32}(1),$$

(2.12) $$g_2 = s_{45}((m+1)/(1-m^2))s_{76}((m+1)/(1-m^2))s_{12}(1)s_{97}(2m/(1-m^2))$$
$$s_{96}(1/(1-m^2))s_{13}(-1/m)s_{23}(-1/(1-m^2)),$$

(2.13) g_3: the transformation $u_1 \to u_1 - u_2$, $u_4 \to (u_4 - u_5)/(1-m^2)$,
$u_6 \to (1-m^2)u_6$, $u_7 \to -u_7$,

(2.14) g_4: the transformation $u_1 \to u_3$, $u_2 \to u_1$, $u_3 \to u_5$, $u_5 \to u_2$, $u_6 \to u_9$,
$u_7 \to u_8$, $u_8 \to u_7$ and $u_9 \to -u_6$;
$\rho(g_4 s_{76}(1) g_3 g_2 g_1)x = x_{4,20}$.

If $\lambda = \mu = 1/2$, $\lambda = \omega/2$ and $\mu = \omega^2/2$ or $\lambda = \omega^2/2$ and $\mu = \omega/2$ in (2.10), then the point x coincides with the representative point $x_{5,12}$ of the orbit $S_{5,12}$ in Table 1 by the operators (2.11), (2.15) and (2.16):

(2.15) $$g_5 = s_{96}(-1/(4m^2))s_{12}(-m)s_{45}(-1/(2m))s_{23}(-1/(2m^2))s_{76}(-1/(2m)),$$

(2.16) g_6: the transformation $u_1 \leftrightarrow u_3$, $u_2 \to u_5/2m$, $u_5 \to u_2$, $u_6 \to u_9$,
$u_7 \to -u_6$, $u_8 \to -2mu_7$, and $u_9 \to u_8$;
$\rho(g_6 g_5 g_1)x = x_{5,12}$.

When $x = x_{8,18} + y_4$, if $\lambda \neq 0$, then we may assume that y_4 is y_2 by the action $h(a_{14}, a_{15})$ where $a_{14} = 1/2\lambda$, $a_{15} = 1/2$; if $\lambda = 0$, then the point x coincides with the representative point $x_{4,20}$ of the orbit $S_{4,20}$ in Table 1 by the following operators (2.17) and (2.18):

(2.17) $$g_7 = s_{96}(1/2)s_{23}(-1/2)s_{69}(-1)s_{32}(2);$$

(2.18) g_8: the transformation $u_1 \to u_3$, $u_2 \to u_1/\sqrt{2}$, $u_3 \to u_5$, $u_4 \to u_4/\sqrt{2}$, $u_5 \to u_2$, $u_6 \to \sqrt{2}u_9$, $u_7 \to -u_8$, $u_8 \to \sqrt{2}u_7$ and $u_9 \to u_6$:

$$\rho(g_8 g_7)x = x_{4,20}.$$

The point $x = x_{8,18} + y_5$ coincides with the representative point $x_{5,16}$ of the orbit $S_{5,16}$ in Table 1 by the following operators:

(2.19) $$g_9 = s_{23}(-1/2)s_{76}(1/4)s_{96}(-1/2)s_{79}(-1/2)s_{45}(1/2) s_{76}(-1/2)s_{23}(1/2)s_{97}(1)s_{13}(-1),$$

(2.20) g_{10}: the transformation $u_1 \to u_4$, $u_2 \to u_1$ $u_3 \leftrightarrow u_5$, $u_4 \to -u_2$, $u_6 \to u_9$, $u_7 \to -u_7$, $u_8 \to -u_6$ and $u_9 \to -u_8$;

$$\rho(g_{10}g_9)x = x_{5,16}.$$

The point $x = x_{8,18} + y_6$ coincides with the representative point $x_{6,8}$ of the orbit $S_{6,8}$ in Table 1 by the transformation $u_1 \leftrightarrow u_3$, $u_2 \leftrightarrow u_5$, $u_6 \to -u_6$, $u_7 \to -u_7$, $u_8 \to -u_8$.

The point $x = x_{8,18} + y_7$ belongs to the orbit $S_{1,30}$, because $\operatorname{codim}_V S = \dim G_x = 1$ where S is the orbit of x by Lemma 2.7.

The point $x = x_{8,18} + y_8$ coincides with the representative point $x_{3,18}$ of the orbit $S_{3,18}$ in Table 1 by the action of $\rho(s_{97}(1/2)s_{52}(-1/2)s_{79}(-1))$ and the transformation $u_1 \leftrightarrow u_2$, $u_3 \to -1/2u_5$, $u_4 \to u_3$, $u_5 \to u_9$, $u_6 \to u_8$, $u_7 \to -2u_6$, $u_8 \to -2u_7$ and $u_9 \to 1/2u_9$.

The point $x = x_{8,18} + y_9$ coincides with the representative point $x_{5,16}$ of the orbit $S_{5,16}$ in Table 1 by the action of $\rho(s_{97}(-1/2)s_{79}(1))$ and the transformation $u_1 \to u_4$, $u_2 \to u_1$, $u_3 \to u_5$, $u_4 \to -u_2$, $u_6 \to u_9$, $u_7 \to u_8$, $u_8 \to -u_6$ and $u_9 \to 1/2\, u_7$.

The point $x = x_{8,18} + y_{10}$ coincides with the representative point $x^2_{6,14}$ of the orbit $S^2_{6,14}$ in Table 1 by the action of $\rho(g)$ where

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);$$

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in M(5), \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix} \in M(4),$$

$$A_1 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ -i/4 & -1/4 & 1/4 \\ -i/4 & 1/4 & -1/4 \end{pmatrix}, A_2 = \begin{pmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{pmatrix}, B_1 = \begin{pmatrix} 2 & -2i \\ -2i & 2 \end{pmatrix}$$

and by the transformation $u_2 \to -u_3$, $u_3 \to u_2$, $u_4 \to iu_4$, $u_7 \to iu_7$ and $u_8 \leftrightarrow u_9$, where $i^2 = -1$.

The point $x = x_{8,18} + y_{11}$ coincides with the representative point $x_{7,7}$ of the orbit $S_{7,7}$ in Table 1 by the transformation $u_1 \leftrightarrow u_2$, $u_5 \to -u_5$, $u_6 \leftrightarrow u_7$ and $u_9 \to -u_9$. Q.E.D.

By this proposition, Remark 2.9 and Propositions 2.10, 2.11, we have the following results.

Corollary 2.19. *From the case* [2], *i.e., the orbits through* $x_{8,18} + y \otimes u_9 (y \in V(10),\ y \neq 0)$, *we have no other orbit except orbits* $S_{0,40}$, $S_{1,30}$, $S_{2,24}$, $S_{2,21}$, $S_{3,18}$, $S_{3,15}$, $S_{4,20}$, $S_{4,14}$, $S_{4,11}$, $S_{5,16}$, $S_{5,12}$, $S^2_{6,14}$, $S_{6,8}$ and $S_{7,7}$.

3° The case [3]

Put $x' = \langle 256 \rangle + \langle 346 \rangle - \langle 147 \rangle - \langle 247 \rangle + \langle 138 \rangle$. The point x' belongs to the orbit $S^1_{9,13}$ in Table 1.

We denote x' by $x^1_{9,13}$. The Lie algebra $\tilde{\mathfrak{g}}_{x^1_{9,13}}$ is given as follows:

$$(2.21)\qquad \tilde{\mathfrak{g}}_{x^1_{9,13}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \right.$$

$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where $A_1 = 2\alpha_1 I_2$, $A_{12} = \begin{pmatrix} a_{13} & a_{14} & 0 \\ 0 & a_{14} & a_{25} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} \alpha_2 & 0 & a_{14} \\ 0 & -(\alpha_1 + \alpha_2) & a_{13} \\ 0 & 0 & -3\alpha_1 \end{pmatrix},\ B_1 = \begin{pmatrix} \alpha_1 & a_{13} & a_{14} \\ 0 & -(\alpha_1 - \alpha_2) & 0 \\ 0 & 0 & -(2\alpha_1 + \alpha_2) \end{pmatrix},$$

$B_{21} = (a_{96}\ a_{97}\ a_{98})$.

The action of $\tilde{\mathfrak{g}}_{x^1_{9,13}}$ on the vector space $N_{x^1_{9,13}}$ with respect to the base $\{\langle 129 \rangle, (\langle 149 \rangle - \langle 249 \rangle), \langle 159 \rangle, \langle 239 \rangle, (\langle 259 \rangle - \langle 349 \rangle), \langle 359 \rangle, \langle 459 \rangle\}$ modulo $T_{x^1_{9,13}}$ is given as follows:

$$(2.22)\qquad \mathfrak{h}_{x^1_{9,13}} = \left\{ C = \begin{pmatrix} (4\alpha_1 + \beta) & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & C_3 \end{pmatrix} \in M(7); \right.$$

$$C_{12} = (2a_{14}\ a_{25} - a_{13}),\ C_{13} = (0\ 0\ 0),$$

$$C_2 = \begin{pmatrix} (\alpha_1 - \alpha_2 + \beta) & a_{13}/2 & 0 \\ 0 & (-\alpha_1 + \beta) & 0 \\ 0 & 0 & (2\alpha_1 + \alpha_2 + \beta) \end{pmatrix},$$

$$C_{23} = \begin{pmatrix} -a_{13} & 0 & a_{25}/2 \\ 0 & a_{13} & a_{14} \\ 2a_{14} & -a_{25} & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} (-\alpha_1+\beta) & 0 & a_{14} \\ 0 & (-3\alpha_1+\alpha_2+\beta) & 0 \\ 0 & 0 & (-4\alpha_1-\alpha_2+\beta) \end{pmatrix}\Bigg\}.$$

Proposition 2.20. *Any orbit of the triplet* $(\tilde{G}_{x^1_{9,13}}, \tilde{\rho}, N_{x^1_{9,13}})$ *is represented by one of the following points modulo* $T_{x^1_{9,13}}$:

(1) $y_1 = \lambda\langle 129\rangle + \mu\langle 159\rangle + \nu\langle 239\rangle + \tau\langle 359\rangle + \langle 459\rangle$
(2) $y_2 = \lambda\langle 159\rangle + (\langle 259\rangle - \langle 349\rangle) \quad \lambda \neq 0$
(3) $y_3 = \langle 129\rangle + (\langle 259\rangle - \langle 349\rangle)$
(4) $y_4 = (\langle 259\rangle - \langle 349\rangle)$
(5) $y_5 = \langle 159\rangle + \langle 239\rangle$
(6) $y_6 = (\langle 149\rangle - \langle 249\rangle) + \langle 239\rangle$
(7) $y_7 = \langle 239\rangle$
(8) $y_8 = \langle 159\rangle$
(9) $y_9 = \langle 129\rangle$
(10) $y_{10} = 0.$

For the orbit S of $x = x^1_{9,13} + y_1$, *we have* $S \in \mathscr{H}(S_{5,12})$. *When* $\lambda \neq 2$ *(resp.* $\lambda = 2$*), the point* $x = x^1_{9,13} + y_2$ *belongs to the orbit* $S_{2,24}$ *(resp.* $S_{5,16}$*) in Table* 1. *The point* $x = x^1_{9,13} + y_3$ *(resp.* $x^1_{9,13} + y_i$; $i = 4,\dots,9$*) belongs to the orbit* $S_{5,21}$ *(resp.* $S^0_{6,14}$, $S_{3,18}$, $S^1_{6,14}$, $S_{7,10}$, $S_{5,16}$, $S_{8,6}$*) in Table* 1.

Proof. Put

$$y = [p_{12}(u_1\wedge u_2) + p_{14}\{(u_1\wedge u_4) - (u_2\wedge u_4)\} + p_{15}(u_1\wedge u_5) + p_{23}(u_2\wedge u_3) + p_{25}\{(u_2\wedge u_5) - (u_3\wedge u_4)\} + p_{35}(u_3\wedge u_5) + p_{45}(u_4\wedge u_5)] \otimes u_9 \in N_{x^1_{9,13}}.$$

[1]

(i) If p_{35} or $p_{45} \neq 0$, then we may assume that $p_{45} \neq 0$. In fact, if $p_{45} = 0$ and $p_{35} \neq 0$, then we can obtain that $p_{35} = 0$ and $p_{45} \neq 0$ by the following operators:

(2.23) $\tilde{\rho}(g\cdot s_{12}(-1))x$ where $x = x^1_{9,13} + y$ and g is the transformation $u_2 \to u_2$, $u_3 \leftrightarrow u_4$, $u_6 \to -u_6$ and $u_7 \leftrightarrow -u_8$.

Hence we obtain that $y = y_1$ by the action of $h(a_{14})$, $h(a_{25})$ and $h(\beta)$ where $p_{25} + p_{45}a_{14} = 0$, $a_{25} = -2p_{14}/p_{25}$ and $\beta = -\log p_{45}$.

(ii) If $p_{35} = p_{45} = 0$, $p_{15}p_{25} \neq 0$ and $p_{15} \neq 2p_{25}$, then we obtain that $y = y_2$ by the action of $h(a_{13})$, $h(a_{25})$, $h(a_{14})$ and $h(\beta)$ where $a_{13} = -2p_{14}/(p_{15} - 2p_{25})$, $a_{25} = -(p_{12} - p_{23}a_{13})/p_{15}$, $a_{14} = -p_{23}/(2p_{25})$ and $\beta = -\log p_{25}$. If $p_{35} = p_{45} = 0$ and $p_{15} = 2p_{25} \neq 0$, then this case is reduced to the case (iv) or (vi) by (2.23).

(iii) If $p_{15} = p_{35} = p_{45} = 0$ and $p_{12}p_{25} \neq 0$ (resp., $p_{12} = 0$ and $p_{25} \neq 0$), then we obtain that $y = y_3$ (resp., $y = y_4$) by the action of $h(a_{14})$, $h(a_{13})$ and $h(\alpha_1, \beta)$ (resp., $h(\beta)$) where $a_{14} = -p_{23}/(2p_{25})$, $a_{13} = p_{14}/p_{25}$ and $-\alpha_1 + \beta = -\log p_{25}$, $4\alpha_1 + \beta = -\log p_{12}$(resp., $\beta = -\log p_{25}$).

(iv) If $p_{25}=p_{35}=p_{45}=0$ and $p_{15}p_{23}\neq 0$, then we obtain that $y=y_5$ by the action of $h(a_{13})$, $h(a_{25})$ and $h(\alpha_1, \alpha_2, \beta)$ where $a_{13}=-p_{14}/p_{15}$, $a_{25}=p_{12}/p_{15}$ and $-\alpha_1+\beta=-\log p_{15}$, $2\alpha_1+\alpha_2+\beta=-\log p_{23}$.

(v) If $p_{15}=p_{25}=p_{35}=p_{45}=0$ and $p_{14}p_{23}\neq 0$(resp., $p_{14}=0, p_{23}\neq 0$), then we obtain that $y=y_6$ (resp., $y=y_7$) by the action of $h(a_{13})$, $h(\alpha_1, \alpha_2, \beta)$(resp. $h(\beta)$) where $a_{13}=p_{12}$ $2\alpha_1+\alpha_2+\beta=-\log p_{23}$, $\alpha_1-\alpha_2+\beta=-\log p_{14}$(resp., $\beta=-\log p_{23}$).

(vi) If $p_{23}=p_{25}=p_{35}=p_{45}=0$ and $p_{15}\neq 0$, then we obtain $y=y_8$ by the action of $h(a_{13})$, $h(a_{25})$ and $h(\beta)$ where $a_{13}=p_{14}/p_{15}$ and $a_{25}=-p_{12}/p_{15}$ and $\beta=-\log p_{15}$.

(vii) If $p_{15}=p_{23}=p_{25}=p_{35}=p_{45}=0$ and $p_{14}\neq 0$, then the case is reduced to the case (v) above by the action (2.23).

(viii) If $p_{12}\neq 0$ and all the remaining $p_{ij}=0$, then we obtain that $y=y_9$ by the action of $h(\beta)$ where $\beta=-\log p_{12}$.

[2]

When the point $x^*=x^1_{9,13}+(u_4\wedge u_5)\otimes u_9$, x^* coincides with the representative point $x_{5,12}$ of the orbit $S_{5,12}$ in Table 1 by the action of $\rho(g\cdot s_{21}(-1))$ where g is the transformation $u_1\to u_3$, $u_2\leftrightarrow u_5$, $u_3\to u_4$, $u_4\to u_1$, $u_6\to -u_6$, $u_7\to -u_7$ and $u_8\leftrightarrow u_9$. Therefore, for the orbit S of $x=x^1_{9,13}+y_1$, the closure $\bar{S}$ contains x^*, because $\lim_{\sigma,\varepsilon\to 0}\rho(h)x=x^*$ where h is a diagonal matrix in $M(9)$ and its elements $\{\sigma^2, \sigma^2, \varepsilon, (\sigma\varepsilon)^{-1}\sigma^{-3}, \sigma, \sigma^{-1}\varepsilon, (\sigma^2\varepsilon)^{-1}, \sigma^4\varepsilon\}$. Hence, we have $S\in\mathscr{H}(S_{5,12})$ by Remark 2.9.

When the point $x=x^1_{9,13}+y_2$, if $\lambda\neq 2$, then the point x coincides with the representative point $x_{2,24}$ of the orbit $S_{2,24}$ in Table 1 by the action of $\rho(g\cdot s_{21}(-\lambda/2)\cdot s_{96}(1/2)\cdot s_{69}(-1))$ where g is the transformation $u_1\to 2u_3/(\lambda-2)$, $u_2\to u_5$, $u_3\to u_1$, $u_5\to(\lambda-2)u_2/\lambda$, $u_6\to -\lambda u_9/(\lambda-2)$, $u_7\to u_6$, $u_8\to -(\lambda-2)u_7/2$ and $u_9\to -u_8/2$; and if $\lambda=2$, the point coincides with the representative point $x_{5,16}$ of the orbit $S_{5,16}$ in Table 1 by the action of $\rho(g\cdot s_{96}(-1/2)\cdot s_{21}(1)\cdot s_{69}(1)\cdot s_{12}(-1))$ where g is the transformation $u_1\leftrightarrow u_5$, $u_2\to u_3$, $u_3\to u_4$, $u_4\to u_2$, $u_6\to -u_9$, $u_8\to -u_8$ and $u_9\to -u_6/2$.

The point $x=x^1_{9,13}+y_3$ coincides with the representative point $x_{5,21}$ of the orbit $S_{5,21}$ in Table 1 by the action of $\rho(g\cdot s_{25}(-2)\cdot s_{96}(-1/2)\cdot s_{51}(-1/2)\cdot s_{69}(1)\cdot s_{15}(2))$ where g is the transformation $u_1\to u_5$, $u_2\to\sqrt{2}u_2$, $u_4\to u_1$, $u_5\to -u_4/\sqrt{2}$, $u_6\to -u_7$, $u_7\to -u_8/\sqrt{2}$, $u_8\to u_6/\sqrt{2}$ and $u_9\to -u_9/\sqrt{2}$.

The point $x=x^1_{9,13}+y_4$ coincides with the representative point $x^0_{6,14}$ of the orbit $S^0_{6,14}$ in Table 1 by the action of $\rho(g\cdot s_{96}(1/2)\cdot s_{69}(-1))$ where g is the transformation $u_1\to u_5$, $u_2\to -u_2$, $u_3\to u_1$, $u_4\to u_3$, $u_5\to u_4$, $u_6\to -u_8$, $u_7\to u_6$, $u_8\to u_7$ and $u_9\to -u_9/2$.

The point $x=x^1_{9,13}+y_5$ coincides with the representative point $x_{3,18}$ of the orbit $S_{3,18}$ in Table 1 by the action of $\rho(g\cdot s_{42}(1)\cdot s_{96}(1)\cdot s_{89}(1)\cdot s_{21}(-1))$ where g

is the transformation $u_1 \to u_5$, $u_3 \to u_1$, $u_5 \to u_3$, $u_6 \to u_8$, $u_7 \to -u_9$, $u_8 \to u_7$ and $u_9 \to -u_6$.

The point $x = x^1_{9,13} + y_6$ coincides with the representative point $x^1_{6,14}$ of the orbit $S^1_{6,14}$ in Table 1 by the action of $\rho(g \cdot s_{97}(-1/2) \cdot s_{87}(1/2) \cdot s_{43}(1/2) \cdot s_{79}(1))$ where g is the transformation $u_1 \to u_3/2$, $u_2 \to u_1$, $u_3 \to u_4$, $u_4 \to u_2$, $u_6 \to -u_7$, $u_7 \to 2u_8$, $u_8 \to 2u_6$ and $u_9 \to -u_9/2$.

The point $x = x^1_{9,13} + y_8$ coincides with the point $x^1_{9,13} + y_2(\lambda = 2)$ above by the action of $\rho(g_2 s_{69}(1) s_{12}(-1) g_1)$ where g_1 (resp., g_2) is the transformation $u_9 \to 2u_9$ (resp., $u_2 \to -u_2$, $u_3 \leftrightarrow u_4$, $u_6 \to -u_6$, $u_7 \to -u_8$ and $u_8 \to -u_7$).

The point $x = x^1_{9,13} + y_7$ coincides with the representative point $x_{7,10}$ of the orbit $S_{7,10}$ in Table 1 by the action of $\rho(g \cdot s_{98}(1) \cdot s_{12}(-1))$ where g is the transformation $u_1 \leftrightarrow u_3$, $u_5 \to -u_5$ and $u_8 \leftrightarrow u_9$.

The point $x = x^1_{9,13} + y_9$ coincides with the representative point $x_{8,6}$ of the orbit $S_{8,6}$ by the transformation $u_1 \to -u_1$, $u_3 \leftrightarrow u_4$, $u_7 \to -u_8$, $u_8 \to -u_7$ and $u_9 \to -u_9$. Q.E.D.

By this proposition, Remark 2.9 and Propositions 2.10, 2.11, we have the following results.

Corollary 2.21. *From the case* [3], *i.e., the orbit through* $x^1_{9,13} + y \otimes u_9 (y \in V(10),\ y \neq 0)$, *we have no other orbit except* $S_{0,40}$, $S_{1,30}$, $S_{2,24}$, $S_{2,21}$, $S_{3,15}$, $S_{4,20}$, $S_{4,14}$, $S_{5,12}$, $S_{3,18}$, $S_{5,21}$, $S_{5,16}$, $S^0_{6,14}$, $S^1_{6,14}$, $S_{7,10}$ *and* $S_{8,6}$.

In each of the following cases, we omit the proof of the proposition, because it is similar to those of the preceeding propositions.

4° (The case [4])

Put $x' = \langle 136 \rangle + \langle 157 \rangle + \langle 257 \rangle + \langle 248 \rangle$. The point x' belongs to the orbit $S_{10,12}$ in Table 1. We denote x' by $x_{10,12}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{10,12}}$ is given as follows:

$$(2.24) \quad \tilde{\mathfrak{g}}_{x_{10,12}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \quad B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where $A_1 = -(\alpha_1 + \alpha_2 + \alpha_3) I_2$, $A_{12} = \begin{pmatrix} a_{13} & 0 & a_{15} \\ 0 & a_{24} & a_{15} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} 2(\alpha_1 + \alpha_2 - \alpha_3) & 0 & 0 \\ 0 & 2(\alpha_1 - \alpha_2 + \alpha_3) & 0 \\ 0 & 0 & 2(-\alpha_1 + \alpha_2 + \alpha_3) \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -(\alpha_1 - \alpha_2 - 3\alpha_3) & 0 & 0 \\ 0 & (3\alpha_1 - \alpha_2 - \alpha_3) & 0 \\ 0 & 0 & -(\alpha_1 - 3\alpha_2 + \alpha_3) \end{pmatrix}$$

and $B_{21} = (a_{96}\ a_{97}\ a_{98})$.

The action of $\tilde{g}_{x_{10,12}}$ on the vector space $N_{x_{10,12}}$ with respect to the base $\{\langle 129\rangle, \langle 149\rangle, (\langle 159\rangle - \langle 259\rangle)/2, \langle 239\rangle, \langle 349\rangle, \langle 359\rangle, \langle 459\rangle\}$ modulo $T_{x_{10,12}}$ is given as follows:

$$(2.25)\qquad \mathfrak{h}_{x_{10,12}} = \left\{ C = \begin{pmatrix} -(2\alpha_1 + 2\alpha_2 + 2\alpha_3 - \beta) & C_{12} & 0 \\ 0 & C_2 & C_{23} \\ 0 & 0 & C_3 \end{pmatrix} \in M(7);\right.$$

$$C_{12} = (a_{24}\ a_{15}\ -a_{13}),$$

$$C_2 = \begin{pmatrix} (\alpha_1 - 3\alpha_2 + \alpha_3 + \beta) & 0 & 0 \\ 0 & (-3\alpha_1 + \alpha_2 + \alpha_3 + \beta) & 0 \\ 0 & 0 & (\alpha_1 + \alpha_2 - 3\alpha_3 + \beta) \end{pmatrix},$$

$$\left. C_{23} = \begin{pmatrix} a_{13} & 0 & -a_{15} \\ 0 & a_{13} & 0 \\ -a_{24} & -a_{15} & 0 \end{pmatrix}, C_3 = \begin{pmatrix} (4\alpha_1 + \beta) & 0 & 0 \\ 0 & (4\alpha_2 + \beta) & 0 \\ 0 & 0 & (4\alpha_3 + \beta) \end{pmatrix} \right\}.$$

Proposition 2.22. *The triplet* $(\tilde{G}_{x_{10,12}}, \tilde{\rho}, N_{x_{10,12}})$ *has* 13 *orbits represented by the following points modulo* $T_{x_{10,12}}$:

(1) $y_1 = \langle 129\rangle + \langle 349\rangle + \langle 359\rangle + \langle 459\rangle$
(2) $y_2 = \langle 349\rangle + \langle 359\rangle + \langle 459\rangle$
(3) $y_3 = \langle 239\rangle + \langle 359\rangle + \langle 459\rangle$
(4) $y_4 = \langle 129\rangle + \langle 359\rangle + \langle 459\rangle$
(5) $y_5 = \langle 359\rangle + \langle 459\rangle$
(6) $y_6 = \langle 239\rangle + \langle 459\rangle$
(7) $y_7 = \langle 129\rangle + \langle 459\rangle$
(8) $y_8 = \langle 459\rangle$
(9) $y_9 = \langle 149\rangle + (\langle 159\rangle - \langle 259\rangle)/2 + \langle 239\rangle$
(10) $y_{10} = \langle 149\rangle + \langle 239\rangle$
(11) $y_{11} = \langle 239\rangle$
(12) $y_{12} = \langle 129\rangle$
(13) $y_{13} = 0$.

The point $x = x_{10,12} + y_1$ *(resp.* $x_{10,12} + y_i$; $i = 2, 3, \dots, 12$) *belongs to the orbit* $S_{0,40}$ *(resp.* $S_{1,30}, S_{1,30}, S_{2,21}, S_{4,20}, S_{2,24}, S_{5,21}, S^0_{6,14}, S_{6,22}, S_{7,15}, S_{8,8}, S^3_{9,13}$).

5° (The case 5)

Put $x' = \langle 256 \rangle + \langle 346 \rangle - \langle 147 \rangle - \langle 237 \rangle + \langle 138 \rangle$. The point x' belongs to the orbit $S_{10,10}$ in Table 1. We denote x' by $x_{10,10}$. The Lie algebra $\tilde{\mathfrak{g}}_{x_{10,10}}$ is given as follows:

$$\text{(2.26)} \quad \tilde{\mathfrak{g}}_{x_{10,10}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \right.$$
$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where $A_1 = \begin{pmatrix} 2\alpha_1 & 0 \\ 0 & -\alpha_1 - 2\alpha_2 \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{13} & a_{14} & 0 \\ 0 & a_{13} & a_{25} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} \alpha_1 + \alpha_2 & a_{34} & 2a_{13} \\ 0 & -2\alpha_1 - \alpha_2 & 0 \\ 0 & 0 & 2\alpha_2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \alpha_1 & a_{13} & a_{14} \\ 0 & \alpha_2 & a_{34} \\ 0 & 0 & -3\alpha_1 - \alpha_2 \end{pmatrix}, \text{ and } B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{10,10}}$ on the vector space $N_{x_{10,10}}$ with respect to the base $\{\langle 129 \rangle, (\langle 149 \rangle - \langle 239 \rangle)/2, \langle 159 \rangle, \langle 249 \rangle, (\langle 259 \rangle - \langle 349 \rangle)/2, \langle 359 \rangle, \langle 459 \rangle\}$ is given as follows;

$$\text{(2.27)} \quad \mathfrak{h}_{x_{10,10}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & -2\alpha_1 + \alpha_2 + \beta \end{pmatrix} \in M(7); \right.$$

$$C_1 = \begin{pmatrix} \alpha_1 - 2\alpha_2 + \beta & a_{13} & a_{25} \\ 0 & -\alpha_2 + \beta & 0 \\ 0 & 0 & 2\alpha_1 + 2\alpha_2 + \beta \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} -a_{14} & 0 & 0 \\ a_{34} & -a_{13} & a_{25} \\ 0 & 0 & a_{13} \end{pmatrix}, C_{13} = \begin{pmatrix} 0 \\ 0 \\ a_{14} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -3\alpha_1 - 3\alpha_2 + \beta & 0 & 0 \\ 0 & -\alpha_1 + \beta & 0 \\ 0 & 0 & \mathrm{a}_1 + 3\alpha_2 + \beta \end{pmatrix},$$

$$\left. C_{23} = \begin{pmatrix} -a_{25} \\ 3a_{13} \\ a_{34} \end{pmatrix} \right\}.$$

Proposition 2.23. *Any orbit of the triplet* $(\tilde{G}_{x_{10,10}}, \tilde{\rho}, N_{x_{10,10}})$ *is represented* *by one of the following points modulo* $T_{x_{10,10}}$:

(1) $y_1 = \lambda\langle 129\rangle + \mu(\langle 149\rangle - \langle 239\rangle) + \langle 459\rangle$
(2) $y_2 = \langle 249\rangle + \langle 359\rangle$
(3) $y_3 = \langle 129\rangle + (\langle 259\rangle - \langle 349\rangle) + \langle 359\rangle$
(4) $y_4 = (\langle 259\rangle - \langle 349\rangle) + \langle 359\rangle$
(5) $y_5 = \langle 129\rangle + \langle 359\rangle$
(6) $y_6 = \langle 359\rangle$
(7) $y_7 = \langle 159\rangle + \langle 249\rangle + (\langle 259\rangle - \langle 349\rangle)$
(8) $y_8 = \langle 249\rangle + (\langle 259\rangle - \langle 349\rangle)$
(9) $y_9 = \langle 159\rangle + (\langle 259\rangle - \langle 349\rangle)$
(10) $y_{10} = \langle 129\rangle + (\langle 259\rangle - \langle 349\rangle)$
(11) $y_{11} = (\langle 259\rangle - \langle 349\rangle)$
(12) $y_{12} = \langle 159\rangle + \langle 249\rangle$
(13) $y_{13} = \langle 249\rangle$
(14) $y_{14} = (\langle 149\rangle - \langle 239\rangle) + \langle 159\rangle$
(15) $y_{15} = \langle 159\rangle$
(16) $y_{16} = (\langle 149\rangle - \langle 239\rangle)$
(17) $y_{17} = \langle 129\rangle$
(18) $y_{18} = 0.$

For the orbit S of $x = x_{10,10} + y_1$, we have $S \in \mathscr{H}(S_{3,18})$.

The point $x_{10,10} + y_2$ *(resp.* $x_{10,10} + y_i$; $i = 3, 4, \ldots, 17$) *belongs to the orbit* $S_{2,21}$ *(resp.* $S_{4,14}$, $S_{5,12}$, $S_{5,9}$, $S^2_{6,14}$ $S_{3,15}$, $S_{5,21}$, $S_{4,14}$, $S^0_{6,14}$, $S_{8,11}$, $S_{4,11}$, $S^1_{6,14}$, $S_{6,8}$, $S_{7,7}$, $S_{7,10}$, $S_{9,5}$) *in Table* 1.

By this proposition, Remark 2.9 and Propositions 2.10, 2.11, we have the following results.

Corollary 2.24. *From the case* [5], *i.e., the orbits through* $x_{10,10} + y \otimes u_9 (y \in V(10),\ y \neq 0)$, *we have no other orbit except the orbits* $S_{0,40}$, $S_{1,30}$, $S_{2,24}$, $S_{2,21}$, $S_{3,18}$, $S_{3,15}$, $S_{4,14}$, $S_{4,11}$, $S_{5,21}$, $S_{5,12}$, $S_{5,9}$, $S^0_{6,14}$, $S^1_{6,14}$, $S^2_{6,14}$, $S_{6,8}$, $S_{7,10}$, $S_{7,7}$, $S_{8,11}$ *and* $S_{9,5}$.

6° (The case [6])

Put $x' = \langle 146\rangle - \langle 236\rangle - \langle 127\rangle + \langle 358\rangle$. The point x' belongs to the orbit $S_{11,8}$ in Table 1. We denote x' by $x_{11,8}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{11,8}}$ is given as follows:

$$\tilde{\mathfrak{g}}_{x_{11,8}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5),\ B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\} \tag{2.28}$$

where $A_1 = \begin{pmatrix} \alpha_1 & a_{12} & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{14} & 0 \\ a_{24} & 0 \\ a_{12} & a_{35} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} -(\alpha_1 - \alpha_2 - \alpha_3) & 0 \\ 0 & -2(\alpha_2 + \alpha_3) \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -(\alpha_2 + \alpha_3) & a_{24} & 0 \\ 0 & -(\alpha_1 + \alpha_2) & 0 \\ 0 & 0 & (2\alpha_2 + \alpha_3) \end{pmatrix}, \ B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{11,8}}$ on the vector space $N_{x_{11,8}}$ with respect to the base $\{\langle 139\rangle, (\langle 149\rangle + \langle 239\rangle)/2, \langle 159\rangle, \langle 249\rangle, \langle 259\rangle, \langle 349\rangle, \langle 459\rangle\}$ is given as follows:

$$(2.29)\qquad \mathfrak{h}_{x_{11,8}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & -(\alpha_1 + \alpha_2 + \alpha_3 - \beta) \end{pmatrix} \in M(7); \right.$$

$$C_1 = \begin{pmatrix} (\alpha_1 + \alpha_3 + \beta) & a_{12} & 0 \\ 0 & (\alpha_2 + \alpha_3 + \beta) & 0 \\ 0 & 0 & (\alpha_1 - 2\alpha_2 - 2\alpha_3 + \beta) \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} 0 & 0 & -a_{14} \\ 2a_{12} & a_{35} & -a_{24} \\ 0 & a_{12} & 0 \end{pmatrix}, \ C_{13} = \begin{pmatrix} 0 \\ 0 \\ a_{14} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} -(\alpha_1 - 2\alpha_2 - \alpha_3 - \beta) & 0 & 0 \\ 0 & -(\alpha_2 + 2\alpha_3 - \beta) & 0 \\ 0 & 0 & -(\alpha_1 + \alpha_2 + \alpha_3 - \beta) \end{pmatrix},$$

$$\left. C_{23} = \begin{pmatrix} 0 \\ a_{24} \\ -a_{35} \end{pmatrix} \right\}.$$

Proposition 2.25. *The triplet* $(\tilde{G}_{x_{11,8}}, \tilde{\rho}, N_{x_{11,8}})$ *has* 23 *orbits represented by the following points modulo* $T_{x_{11,8}}$:

(1) $y_1 = \langle 139\rangle + \langle 249\rangle + \langle 459\rangle$
(2) $y_2 = \langle 249\rangle + \langle 459\rangle$
(3) $y_3 = \langle 149\rangle + \langle 239\rangle + \langle 459\rangle$
(4) $y_4 = \langle 139\rangle + \langle 459\rangle$
(5) $y_5 = \langle 459\rangle$
(6) $y_6 = \langle 249\rangle + \langle 259\rangle + \langle 349\rangle$
(7) $y_7 = \langle 259\rangle + \langle 349\rangle$
(8) $y_8 = \langle 159\rangle + \langle 249\rangle + \langle 349\rangle$

(9) $y_9 = \langle 249 \rangle + \langle 349 \rangle$
(10) $y_{10} = \langle 159 \rangle + \langle 349 \rangle$
(11) $y_{11} = \langle 349 \rangle$
(12) $y_{12} = \langle 139 \rangle + \langle 249 \rangle + \langle 259 \rangle$
(13) $y_{13} = \langle 249 \rangle + \langle 259 \rangle$
(14) $y_{14} = \langle 139 \rangle + \langle 259 \rangle$
(15) $y_{15} = \langle 259 \rangle$
(16) $y_{16} = \langle 159 \rangle + \langle 249 \rangle$
(17) $y_{17} = \langle 139 \rangle + \langle 249 \rangle$
(18) $y_{18} = \langle 249 \rangle$
(19) $y_{19} = \langle 149 \rangle + \langle 239 \rangle + \langle 159 \rangle$
(20) $y_{20} = \langle 159 \rangle$
(21) $y_{21} = \langle 149 \rangle + \langle 239 \rangle$
(22) $y_{22} = \langle 139 \rangle$
(23) $y_{23} = 0.$

The point $x_{11,8} + y_1$ (resp. $x_{11,8} + y_i$; $i = 2,\dots,22$) *belongs to the orbit* $S_{1,30}$ (resp. $S_{2,24}$, $S_{2,24}$, $S_{3,18}$, $S_{5,16}$, $S_{2,21}$, $S_{3,18}$, $S_{3,15}$, $S_{5,21}$, $S_{6,22}$, $S_{7,15}$, $S_{4,14}$, $S_{5,12}$, $S^1_{6,14}$, $S_{7,10}$, $S_{4,14}$, $S^0_{6,14}$, $S_{8,11}$, $S_{7,15}$, $S_{9,9}$, $S_{8,8}$, $S_{10,7}$) *in Table* 1.

7° (The case [7])

Put $x' = \langle 156 \rangle + \langle 246 \rangle - \langle 147 \rangle + \langle 237 \rangle - \langle 138 \rangle$. The point x' belongs to the orbit $S_{12,5}$ in Table 1. We denote x' by $x_{12,5}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{12,5}}$ is given as follows:

$$(2.30) \qquad \tilde{\mathfrak{g}}_{x_{12,5}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \right.$$
$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where $A_1 = \begin{pmatrix} 2\alpha_1 & a_{12} & a_{13} \\ 0 & 4\alpha_1 + 3\alpha_2 & 0 \\ 0 & 0 & -4(\alpha_1 + \alpha_2) \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{14} & a_{15} \\ -a_{13} & a_{14} \\ a_{34} & a_{35} \end{pmatrix}$,

$A_2 = \begin{pmatrix} -2\alpha_1 - \alpha_2 & -(a_{12} + a_{34}) \\ 0 & 2\alpha_2 \end{pmatrix}$, $B_1 = \begin{pmatrix} -2(\alpha_1 + \alpha_2) & -a_{34} & a_{35} \\ 0 & \alpha_2 & (a_{12} - a_{34}) \\ 0 & 0 & 2(\alpha_1 + 2\alpha_2) \end{pmatrix}$,

$B_{21} = (a_{96}\ a_{97}\ a_{98})$.

The action of $\tilde{\mathfrak{g}}_{x_{12,5}}$ on the vector space $N_{x_{12,5}}$ with respect to the base $\{\langle 129 \rangle$, $(\langle 149 \rangle + \langle 239 \rangle)/2$, $(\langle 159 \rangle - \langle 249 \rangle)/2$, $\langle 259 \rangle$, $\langle 349 \rangle$, $\langle 359 \rangle$, $\langle 459 \rangle\}$ modulo $T_{x_{12,5}}$ is given as follows:

(2.31) $$\mathfrak{h}_{x_{12,5}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} \\ 0 & C_2 \end{pmatrix} \in M(7); \right.$$

$$C_1 = \begin{pmatrix} (6\alpha_1 + 3\alpha_2 + \beta) & -a_{13} & a_{14} & -a_{15} \\ 0 & (-\alpha_2 + \beta) & -(a_{12} + a_{34}) & a_{35} \\ 0 & 0 & (2\alpha_1 + 2\alpha_2 + \beta) & (2a_{12} + a_{34}) \\ 0 & 0 & 0 & (4\alpha_1 + 5\alpha_2 + \beta) \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 2a_{13} & a_{14} & a_{15} \\ 0 & a_{13} & a_{14} \\ 0 & 0 & -a_{13} \end{pmatrix}$$

$$\left. C_2 = \begin{pmatrix} (-6\alpha_1 - 5\alpha_2 + \beta) & (-a_{12} - a_{34}) & -a_{35} \\ 0 & -(4\alpha_1 + 2\alpha_2 - \beta) & a_{34} \\ 0 & 0 & (-2\alpha_1 + \alpha_2 + \beta) \end{pmatrix} \right\}.$$

Proposition 2.26. *The triplet* $(\tilde{G}_{x_{12,5}}, \tilde{\rho}, N_{x_{12,5}})$ *has* 14 *orbits represented by the following points modulo* $T_{x_{12,5}}$:

(1) $y_1 = \langle 129 \rangle + \langle 459 \rangle$
(2) $y_2 = \langle 459 \rangle$
(3) $y_3 = \langle 259 \rangle + \langle 359 \rangle$
(4) $y_4 = \langle 129 \rangle + \langle 359 \rangle$
(5) $y_5 = \langle 359 \rangle$
(6) $y_6 = \langle 259 \rangle + \langle 349 \rangle$
(7) $y_7 = \langle 159 \rangle - \langle 249 \rangle + \langle 349 \rangle$
(8) $y_8 = \langle 129 \rangle + \langle 349 \rangle$
(9) $y_9 = \langle 349 \rangle$
(10) $y_{10} = \langle 259 \rangle$
(11) $y_{11} = \langle 159 \rangle - \langle 249 \rangle$
(12) $y_{12} = \langle 149 \rangle + \langle 239 \rangle$
(13) $y_{13} = \langle 129 \rangle$
(14) $y_{14} = 0$.

The point $x_{12,5} + y_1$ (*resp.* $x_{12,5} + y_i$; $i = 2, \dots, 13$) *belongs to the orbit* $S_{2,21}$ (*resp.* $S_{3,18}, S_{3,15}, S_{5,9}, S_{6,8}, S_{4,11}, S^1_{6,14}, S_{8,6}, S_{9,5}, S_{6,22}, S_{7,15}, S_{9,9}, S_{11,4}$) *in Table* 1.

8° (The case [8])

Put $x' = \langle 236 \rangle + \langle 456 \rangle - \langle 137 \rangle + \langle 128 \rangle$. The point x' belongs to the orbit $S^1_{13,9}$. We denote x' by $x^1_{13,9}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x^1_{13,9}}$ is given as follows:

(2.32) $$\tilde{\mathfrak{g}}_{x^1_{13,9}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} -4\delta & A_{12} & 0 \\ 0 & \delta I_2 + A_2 & 0 \\ 0 & 0 & \delta I_2 + A_3 \end{pmatrix} \in M(5),$$

$$\left. B = \begin{pmatrix} -2\delta & 0 & 0 \\ 0 & 3\delta I_2 + A_2 & 0 \\ a_{96} & B_{32} & \beta \end{pmatrix} \in M(4) \right\} \text{ where } A_{12} = (a_{12}\ a_{13}),$$

$$A_2 = \begin{pmatrix} \varepsilon_1 & a_{23} \\ a_{32} & -\varepsilon_1 \end{pmatrix},\ A_3 = \begin{pmatrix} \varepsilon_2 & a_{45} \\ a_{54} & -\varepsilon_2 \end{pmatrix},\ B_{32} = (a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x^1_{13,9}}$ on the vector space $N_{x^1_{13,9}}$ with respect to the base $\{\langle 149\rangle, \langle 159\rangle, (\langle 239\rangle - \langle 349\rangle)/2, \langle 249\rangle, \langle 259\rangle, \langle 349\rangle, \langle 359\rangle\}$ modulo $T_{x^1_{13,9}}$ is given as follows:

(2.33) $$\mathfrak{h}_{x^1_{13,9}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} \\ 0 & C_2 \end{pmatrix} \in M(7);\ C_1 = \begin{pmatrix} (-3\delta+\beta)I_2 + A_3 & 0 \\ 0 & (2\delta+\beta) \end{pmatrix}, \right.$$

$$\left. C_{12} = \begin{pmatrix} a_{12}I_2 & a_{13}I_2 \\ 0 & 0 \end{pmatrix},\ C_2 = \begin{pmatrix} (2\delta+\varepsilon_1+\beta)I_2 + A_3 & a_{23}I_2 \\ a_{32}I_2 & (2\delta-\varepsilon_1+\beta)I_2 + A_3 \end{pmatrix} \right\}.$$

Proposition 2.27. *The triplet* $(\tilde{G}_{x^1_{13,9}}, \tilde{\rho}, N_{x^1_{13,9}})$ *has 9 orbits represented by the following points modulo* $T_{x^1_{13,9}}$:

(1) $y_1 = \langle 249\rangle + \langle 359\rangle$
(2) $y_2 = \langle 149\rangle + \langle 239\rangle - \langle 459\rangle + \langle 359\rangle$
(3) $y_3 = \langle 239\rangle - \langle 459\rangle + \langle 359\rangle$
(4) $y_4 = \langle 149\rangle + \langle 359\rangle$
(5) $y_5 = \langle 359\rangle$
(6) $y_6 = \langle 149\rangle + \langle 239\rangle - \langle 459\rangle$
(7) $y_7 = \langle 239\rangle - \langle 459\rangle$
(8) $y_8 = \langle 149\rangle$
(9) $y_9 = 0.$

The point $x^1_{13,9} + y_1$ *(resp.* $x^1_{13,9} + y_i$; $i = 2,\dots,8$) *belongs to the orbit* $S_{4,20}$ *(resp.* $S_{5,12}, S^0_{6,14}, S_{6,8}, S_{7,10}, S_{8,11}, S_{12,10}, S_{11,15}$) *in Table* 1.

9° (The case [9])

Put $x' = \langle 456\rangle + \langle 137\rangle - \langle 128\rangle$. The point x' belongs to the orbit $S^0_{14,6}$ in Table 1. We denote x' by $x^0_{14,6}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x^0_{14,6}}$ is given as follows:

(2.34) $$\tilde{\mathfrak{g}}_{x^0_{14,6}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9); \right.$$

$$A=\begin{pmatrix} -2(\delta_1+\delta_2) & A_{12} & 0 \\ 0 & \delta_1 I_2 + A_2 & 0 \\ 0 & 0 & \delta_2 I_2 + A_3 \end{pmatrix} \in M(5),$$

$$B=\begin{pmatrix} -2\delta_2 & 0 & 0 \\ 0 & (\delta_1+2\delta_2)I_2 + A_2 & 0 \\ a_{96} & B_{32} & \beta \end{pmatrix} \in M(4) \Bigg\}, \text{ where } A_{12}=(a_{12}\ a_{13}),$$

$$A_2=\begin{pmatrix} \varepsilon_1 & a_{23} \\ a_{32} & -\varepsilon_1 \end{pmatrix},\ A_3=\begin{pmatrix} \varepsilon_2 & a_{45} \\ a_{54} & -\varepsilon_2 \end{pmatrix} \text{ and } B_{32}=(a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x^0_{14,6}}$ on the vector space $N_{x^0_{14,6}}$ with respect to the base $\{\langle 149\rangle, \langle 159\rangle, \langle 239\rangle, \langle 249\rangle, \langle 259\rangle, \langle 349\rangle, \langle 359\rangle\}$ modulo $T_{x^0_{14,6}}$ is given as follows:

$$(2.35)\qquad \mathfrak{h}_{x^0_{14,6}}=\left\{C=\begin{pmatrix} C_1 & C_{12} \\ 0 & C_2 \end{pmatrix}\in M(7);\right.$$

$$C_1=\begin{pmatrix} (-2\delta_1-\delta_2+\beta)I_2+A_3 & 0 \\ 0 & (2\delta_1+\beta) \end{pmatrix},\ C_{12}=\begin{pmatrix} a_{12}I_2 & a_{13}I_2 \\ 0 & 0 \end{pmatrix},$$

$$\left. C_2=\begin{pmatrix} (\delta_1+\delta_2+\varepsilon_1+\beta)I_2+A_3 & a_{23}I_2 \\ a_{32}I_2 & (\delta_1+\delta_2-\varepsilon_1+\beta)I_2+A_3 \end{pmatrix}\right\}.$$

Proposition 2.28. *The triplet* $(\tilde{G}_{x^0_{14,6}}, \tilde{\rho}, N_{x^0_{14,6}})$ *has* 10 *orbits represented by the following points modulo* $T_{x^0_{14,9}}$:

(1) $y_1=\langle 239\rangle+\langle 249\rangle+\langle 359\rangle$
(2) $y_2=\langle 249\rangle+\langle 359\rangle$
(3) $y_3=\langle 149\rangle+\langle 239\rangle+\langle 359\rangle$
(4) $y_4=\langle 239\rangle+\langle 359\rangle$
(5) $y_5=\langle 149\rangle+\langle 359\rangle$
(6) $y_6=\langle 359\rangle$
(7) $y_7=\langle 149\rangle+\langle 239\rangle$
(8) $y_8=\langle 239\rangle$
(9) $y_9=\langle 149\rangle$
(10) $y_{10}=0.$

The point $x^0_{14,6}+y_1$ *(resp.* $x^0_{14,6}+y_i$; $i=2,\dots,9$*) belongs to the orbit* $S_{4,20}$ *(resp.* $S_{5,16}, S_{5,12}, S^0_{6,14}, S_{7,15}, S_{8,8}, S_{8,11}, S_{12,10}, S_{12,13}$*).*

10° (The case [10])

Put $x'=\langle 156\rangle-\langle 246\rangle-\langle 147\rangle+\langle 237\rangle+\langle 128\rangle$. The point x' belongs to the orbit $S^2_{14,6}$ in Table 1. We denote x' by $x^2_{14,6}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x^2_{14,6}}$ is given as follows:

(2.36) $$\tilde{\mathfrak{g}}_{x^2_{14,6}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} 3\delta I_2 + A_1 & A_{12} \\ 0 & -2\delta I_3 + A_2 \end{pmatrix} \in M(5), \right.$$

$$\left. B = \begin{pmatrix} -\delta I_2 + A_1 & B_{12} \\ B_{21} & B_2 \end{pmatrix} \in M(4) \right\} \text{ where } A_1 = \begin{pmatrix} \varepsilon & a_{12} \\ a_{21} & -\varepsilon \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2\varepsilon & a_{12} & 0 \\ 2a_{21} & 0 & 2a_{12} \\ 0 & a_{21} & -2\varepsilon \end{pmatrix},\ A_{12} = \begin{pmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} -(a_{14} + a_{25}) & 0 \\ (a_{13} + a_{24}) & 0 \end{pmatrix},\ B_{21} = \begin{pmatrix} 0 & 0 \\ a_{96} & a_{97} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -6\delta & 0 \\ a_{98} & \beta \end{pmatrix}.$$

The action of $\tilde{\mathfrak{g}}_{x^2_{14,6}}$ on the vector space $N_{x^2_{14,6}}$ with respect to the base $\{\langle 139 \rangle,\ (\langle 149 \rangle + \langle 239 \rangle)/2,\ (\langle 159 \rangle + \langle 249 \rangle)/2,\ \langle 259 \rangle,\ \langle 349 \rangle,\ \langle 359 \rangle,\ \langle 459 \rangle\}$ modulo $T_{x^2_{14,6}}$ is given as follows:

(2.37) $$\mathfrak{h}_{x^2_{14,6}} = \left\{ C = \begin{pmatrix} (\delta + \beta) I_4 + C_1 & C_{12} \\ 0 & (-4\delta + \beta) I_3 + C_2 \end{pmatrix} \in M(7); \right.$$

$$C_1 = \begin{pmatrix} 3\varepsilon & a_{12} & 0 & 0 \\ 3a_{21} & \varepsilon & 2a_{12} & 0 \\ 0 & 2a_{21} & -\varepsilon & 3a_{12} \\ 0 & 0 & a_{21} & -3\varepsilon \end{pmatrix},\ C_2 = \begin{pmatrix} 2\varepsilon & 2a_{12} & 0 \\ a_{21} & 0 & a_{12} \\ 0 & 2a_{21} & -2\varepsilon \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} -a_{14} & -a_{15} & 0 \\ (a_{13} + a_{24}) & -a_{25} & -a_{15} \\ a_{23} & a_{13} & (a_{14} - a_{25}) \\ 0 & a_{23} & a_{24} \end{pmatrix}.$$

Proposition 2.29. *The triplet* $(\tilde{G}_{x^2_{14,6}}, \tilde{\rho}, N_{x^2_{14,6}})$ *has 7 orbits represented by the following points modulo* $T_{x^2_{14,6}}$:

(1) $y_1 = \langle 359 \rangle$
(2) $y_2 = \langle 259 \rangle + \langle 349 \rangle$
(3) $y_3 = \langle 349 \rangle$
(4) $y_4 = \langle 139 \rangle + \langle 259 \rangle$
(5) $y_5 = \langle 159 \rangle + \langle 249 \rangle$
(6) $y_6 = \langle 139 \rangle$
(7) $y_7 = 0.$

The point $x^2_{14,6} + y_1$ *(resp.* $x^2_{14,6} + y_i$; $i = 2, \ldots, 6$) *belongs to the orbit* $S_{4,14}$ *(resp.* $S_{5,9}$, $S^1_{6,14}$, $S^3_{9,13}$, $S_{10,7}$, $S_{11,4}$) *in Table* 1.

11° (The case [11])

Put $x' = \langle 256\rangle - \langle 346\rangle - \langle 137\rangle + \langle 128\rangle$. The point x' belongs to the orbit $S_{14,4}$ in Table 1. We denote x' by $x_{14,4}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{14,4}}$ is given as follows:

$$(2.38)\qquad \tilde{\mathfrak{g}}_{x_{14,4}} = \left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\right.$$

$$A = \begin{pmatrix} -2(\delta_1+\delta_2) & 0 & A_{13} \\ 0 & \delta_1 I_2 + A_2 & A_{23} \\ 0 & 0 & \delta_2 I_2 + A_2 \end{pmatrix} \in M(5),$$

$$\left. B = \begin{pmatrix} -(\delta_1+\delta_2) & B_{12} & 0 \\ 0 & (\delta_1+2\delta_2)I_2 + A_2 & 0 \\ a_{96} & B_{32} & \beta \end{pmatrix} \in M(4)\right\}$$

where $A_{13} = (a_{14}\ a_{15})$, $A_2 = \begin{pmatrix} \varepsilon & a_{23} \\ a_{32} & -\varepsilon \end{pmatrix}$, $A_{23} = \begin{pmatrix} a_{24} & a_{25} \\ a_{34} & -a_{24} \end{pmatrix}$,

$B_{12} = (a_{14}\ a_{15})$, $B_{32} = (a_{97}\ a_{98})$.

The action of $\tilde{\mathfrak{g}}_{x_{14,4}}$ on the vector space $N_{x_{14,4}}$ with respect to the base $\{\langle 149\rangle, \langle 159\rangle, \langle 239\rangle, \langle 249\rangle, (\langle 259\rangle + \langle 349\rangle)/2, \langle 359\rangle, \langle 459\rangle\}$ modulo $T_{x_{14,4}}$ is given as follows:

$$(2.39)\qquad \mathfrak{h}_{x_{14,4}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & (2\delta_2+\beta) \end{pmatrix} \in M(7);\right.$$

$$C_1 = \begin{pmatrix} (-2\delta_1 - \delta_2 + \beta)I_2 + A_2 & 0 \\ 0 & (2\delta_1+\beta) \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{34} & -a_{24} & -a_{25} \end{pmatrix},\ C_{13} = \begin{pmatrix} -a_{15} \\ a_{14} \\ 0 \end{pmatrix},$$

$$\left. C_2 = (\delta_1+\delta_2+\beta)I_3 + A_3,\ C_{23} = \begin{pmatrix} -a_{25} \\ 2a_{24} \\ a_{34} \end{pmatrix}\right\}$$

where $A_3 = \begin{pmatrix} 2\varepsilon & a_{23} & 0 \\ 2a_{32} & 0 & 2a_{23} \\ 0 & a_{32} & -2\varepsilon \end{pmatrix}$.

Proposition 2.30. *The triplet $(\tilde{G}_{x_{14,4}}, \tilde{\rho}, N_{x_{14,4}})$ has* 12 *orbits represented by the following points modulo $T_{x_{14,4}}$:*

(1) $y_1 = \langle 239\rangle + \langle 459\rangle$

(2) $y_2 = \langle 459 \rangle$
(3) $y_3 = \langle 149 \rangle + \langle 159 \rangle + \langle 259 \rangle + \langle 349 \rangle$
(4) $y_4 = \langle 149 \rangle + \langle 259 \rangle + \langle 349 \rangle$
(5) $y_5 = \langle 259 \rangle + \langle 349 \rangle$
(6) $y_6 = \langle 159 \rangle + \langle 249 \rangle$
(7) $y_7 = \langle 149 \rangle + \langle 249 \rangle$
(8) $y_8 = \langle 249 \rangle$
(9) $y_9 = \langle 149 \rangle + \langle 239 \rangle$
(10) $y_{10} = \langle 149 \rangle$
(11) $y_{11} = \langle 239 \rangle$
(12) $y_{12} = 0.$

The point $x_{14,4} + y_1$ (*resp.* $x_{14,4} + y_i$; $i = 2,\dots,11$) *belongs to the orbit* $S_{4,20}$ (*resp.* $S_{5,16}$, $S^2_{6,14}$, $S_{7,10}$, $S_{8,8}$, $S_{7,7}$, $S_{8,6}$, $S_{9,9}$, $S_{9,5}$, $S_{11,15}$, $S^3_{13,9}$).

12° (The case [12])

Put $x' = \langle 346 \rangle - \langle 147 \rangle + \langle 237 \rangle + \langle 128 \rangle$. The point x' belongs to the orbit $S_{15,11}$ in Table 1. We denote x' by $x_{15,11}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{15,11}}$ is given as follows:

$$\text{(2.40)} \quad \tilde{\mathfrak{g}}_{x_{15,11}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & -4\delta \end{pmatrix} \in M(5), \right.$$

$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

$$\text{where } A_1 = \begin{pmatrix} (\delta+\eta)I_2 + A_2 & a_{13}I_2 \\ a_{31}I_2 & (\delta-\eta)I_2 + A_2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \varepsilon & a_{12} \\ a_{21} & -\varepsilon \end{pmatrix},\ A_{12} = {}^t(a_{15}\ a_{25}\ a_{35}\ a_{45}),\ B_1 = (-2\delta I_3 + B_2),$$

$$B_2 = \begin{pmatrix} 2\eta & a_{13} & 0 \\ 2a_{31} & 0 & 2a_{13} \\ 0 & a_{31} & -2\eta \end{pmatrix},\ B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{15,11}}$ on the vector space $N_{x_{15,11}}$ with respect to the base $\{\langle 139 \rangle, (\langle 149 \rangle + \langle 239 \rangle)/2, \langle 249 \rangle, \langle 159 \rangle, \langle 259 \rangle, \langle 359 \rangle, \langle 459 \rangle\}$ modulo $T_{x_{15,11}}$ is given as follows:

$$\text{(2.41)} \quad \mathfrak{h}_{x_{15,11}} = \left\{ C = \begin{pmatrix} (2\delta+\beta)I_3 + B_3 & C_{12} \\ 0 & C_2 \end{pmatrix} \in M(7); \right.$$

$$B_3 = \begin{pmatrix} 2\varepsilon & a_{12} & 0 \\ 2a_{21} & 0 & 2a_{12} \\ 0 & a_{21} & -2\varepsilon \end{pmatrix},\ C_{12} = \begin{pmatrix} a_{35} & 0 & -a_{15} & 0 \\ a_{45} & a_{35} & -a_{25} & -a_{15} \\ 0 & a_{45} & 0 & -a_{25} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} (-3\delta + \eta + \beta)I_2 + A_2 & a_{13}I_2 \\ a_{31}I_2 & (-3\delta - \eta + \beta)I_2 + A_2 \end{pmatrix}\Big\}.$$

Proposition 2.31. *The triplet* $(\tilde{G}_{x_{15,11}}, \tilde{\rho}, N_{x_{15,11}})$ *has* 6 *orbits represented by the following points modulo* $T_{x_{15,11}}$:

(1) $y_1 = \langle 159 \rangle + \langle 459 \rangle$
(2) $y_2 = \langle 139 \rangle + \langle 459 \rangle$
(3) $y_3 = \langle 459 \rangle$
(4) $y_4 = \langle 149 \rangle + \langle 239 \rangle$
(5) $y_5 = \langle 139 \rangle$
(6) $y_6 = 0.$

The point $x_{15,11} + y_1$ *(resp.* $x_{15,11} + y_i$; $i = 2, \dots, 5$) *belongs to the orbit* $S_{5,21}$ *(resp.* $S^1_{6,14}$, $S_{7,15}$, $S_{12,20}$, $S_{13,12}$).

13° (The case [13])

Put $x' = \langle 156 \rangle - \langle 246 \rangle - \langle 137 \rangle + \langle 128 \rangle$. The point x' belongs to the orbit $S_{15,3}$ in Table 1. We denote x' by $x_{15,3}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{15,3}}$ is given as follows:

$$(2.42) \qquad \tilde{\mathfrak{g}}_{x_{15,3}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \right.$$

$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where $A_1 = \begin{pmatrix} \alpha_1 & a_{12} \\ 0 & \alpha_2 \end{pmatrix}$, $A_{12} = \begin{pmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{pmatrix}$,

$$A_2 = \begin{pmatrix} 2\alpha_3 & 0 & a_{35} \\ 0 & -(\alpha_2 + \alpha_3) & a_{12} \\ 0 & 0 & -(\alpha_1 + \alpha_3) \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \alpha_3 & a_{35} & -(a_{14} + a_{25}) \\ 0 & -(\alpha_1 + 2\alpha_3) & a_{23} \\ 0 & 0 & -(\alpha_1 + \alpha_2) \end{pmatrix},\ B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{15,3}}$ on the vector space $N_{x_{15,3}}$ with respect to the base $\{\langle 149 \rangle, (\langle 159 \rangle + \langle 249 \rangle)/2, \langle 239 \rangle, \langle 259 \rangle, \langle 349 \rangle, \langle 359 \rangle, \langle 459 \rangle\}$ is given as follows:

$$(2.43) \qquad \mathfrak{h}_{x_{15,3}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & -(\alpha_1 + \alpha_2 + 2\alpha_3 - \beta) \end{pmatrix} \in M(7); \right.$$

$$C_1 = \begin{pmatrix} (\alpha_1 - \alpha_2 - \alpha_3 + \beta) & a_{12} & 0 \\ 0 & -(\alpha_3 - \beta) & 0 \\ 0 & 0 & (\alpha_2 + 2\alpha_3 + \beta) \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} 0 & a_{13} & 0 \\ 2a_{12} & a_{23} & a_{13} \\ a_{35} & -a_{24} & -a_{25} \end{pmatrix}, \ C_{13} = {}^t(-a_{15} \quad (a_{14} - a_{25}) \quad 0),$$

$$C_2 = \begin{pmatrix} -(\alpha_1 - \alpha_2 + \alpha_3 - \beta) & 0 & a_{23} \\ 0 & -(\alpha_2 - \alpha_3 - \beta) & a_{12} \\ 0 & 0 & -(\alpha_1 - \alpha_3 - \beta) \end{pmatrix},$$

$$\left. C_{23} = {}^t(a_{24} \quad -a_{35} \quad 0) \right\}.$$

Proposition 2.32 *The triplet* $(\tilde{G}_{x_{15,3}}, \tilde{\rho}, N_{x_{15,3}})$ *has* 15 *orbits represented by the following points modulo* $T_{x_{15,3}}$:

(1) $y_1 = \langle 359 \rangle + \langle 459 \rangle$
(2) $y_2 = \langle 239 \rangle + \langle 459 \rangle$
(3) $y_3 = \langle 459 \rangle$
(4) $y_4 = \langle 149 \rangle + \langle 359 \rangle$
(5) $y_5 = \langle 359 \rangle$
(6) $y_6 = \langle 259 \rangle + \langle 349 \rangle$
(7) $y_7 = \langle 349 \rangle$
(8) $y_8 = \langle 149 \rangle + \langle 259 \rangle$
(9) $y_9 = \langle 259 \rangle$
(10) $y_{10} = \langle 159 \rangle + \langle 249 \rangle + \langle 239 \rangle$
(11) $y_{11} = \langle 149 \rangle + \langle 239 \rangle$
(12) $y_{12} = \langle 239 \rangle$
(13) $y_{13} = \langle 159 \rangle + \langle 249 \rangle$
(14) $y_{14} = \langle 149 \rangle$
(15) $y_{15} = 0.$

The point $x_{15,3} + y_1$ (*resp.* $x_{15,3} + y_i$; $i = 2, \dots, 14$) *belongs to the orbit* $S_{5,12}$ (*resp.* $S_{6,8}$, $S_{7,15}$, $S^2_{6,14}$, $S_{7,10}$, $S_{7,7}$, $S_{9,9}$, $S^3_{9,13}$, $S_{10,7}$, $S_{10,7}$, $S_{11,4}$, $S^3_{13,9}$, $S_{12,13}$, $S^1_{14,6}$) in Table 1.

14° (The case [14])

Put $x' = \langle 246 \rangle - \langle 137 \rangle + \langle 128 \rangle$. The point x' belongs to the orbit $S_{16,5}$ in Table 1. We denote x' by $x_{16,5}$.

The Lie algebra $\tilde{\mathfrak{g}}_{x_{16,5}}$ is given follows:

(2.44) $$\tilde{\mathfrak{g}}_{x_{16,5}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \right.$$

$$B=\begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix}\in M(4)\Big\}$$

where $A_1=\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, $A_{12}=\begin{pmatrix} a_{13} & a_{14} & a_{15} \\ a_{23} & a_{24} & a_{25} \end{pmatrix}$,

$$A_2=\begin{pmatrix} a_3 & 0 & a_{35} \\ 0 & a_4 & a_{45} \\ 0 & 0 & -(a_1+a_2+a_3+a_4) \end{pmatrix},$$

$$B_1=\begin{pmatrix} -(a_2+a_4) & 0 & a_{14} \\ 0 & -(a_1+a_3) & a_{23} \\ 0 & 0 & -(a_1+a_2) \end{pmatrix},\ B_{21}=(a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{16,5}}$ on the vector space $N_{x_{16,5}}$ with respect to the base $\{\langle 149\rangle, \langle 239\rangle, \langle 349\rangle, \langle 159\rangle, \langle 259\rangle, \langle 359\rangle, \langle 459\rangle\}$ modulo $T_{x_{16,5}}$ is given as follows:

(2.45) $$\mathfrak{h}_{x_{16,5}}=\Big\{C=\begin{pmatrix} C_1 & C_{12} \\ 0 & C_2 \end{pmatrix}\in M(7);$$

$$C_1=\begin{pmatrix} (a_1+a_4+\beta) & 0 & a_{13} \\ 0 & (a_2+a_3+\beta) & -a_{24} \\ 0 & 0 & (a_3+a_4+\beta) \end{pmatrix},$$

$$C_{12}=\begin{pmatrix} a_{45} & 0 & 0 & -a_{15} \\ 0 & a_{35} & -a_{25} & 0 \\ 0 & 0 & a_{45} & -a_{35} \end{pmatrix},$$

$$C_2=\begin{pmatrix} -(a_2+a_3+a_4-\beta) & 0 & a_{13} & a_{14} \\ 0 & -(a_1+a_3+a_4-\beta) & a_{23} & a_{24} \\ 0 & 0 & -(a_1+a_2+a_4-\beta) & 0 \\ 0 & 0 & 0 & -(a_1+a_2+a_3-\beta) \end{pmatrix}\Big\}.$$

Proposition 2.33. *The triplet* $(\tilde{G}_{x_{16,5}}, \tilde{\rho}, N_{x_{16,5}})$ *has* 12 *orbits represented by the following points modulo* $T_{x_{16,5}}$:

(1) $y_1=\langle 359\rangle+\langle 459\rangle$
(2) $y_2=\langle 239\rangle+\langle 459\rangle$
(3) $y_3=\langle 459\rangle$

(4) $y_4 = \langle 349 \rangle + \langle 159 \rangle + \langle 259 \rangle$
(5) $y_5 = \langle 159 \rangle + \langle 259 \rangle$
(6) $y_6 = \langle 349 \rangle + \langle 259 \rangle$
(7) $y_7 = \langle 149 \rangle + \langle 259 \rangle$
(8) $y_8 = \langle 259 \rangle$
(9) $y_9 = \langle 349 \rangle$
(10) $y_{10} = \langle 149 \rangle + \langle 239 \rangle$
(11) $y_{11} = \langle 239 \rangle$
(12) $y_{12} = 0.$

The point $x_{16,5} + y_1$(*resp.* $x_{16,5} + y_i$; $i = 2,\dots,11$) *belongs to the orbit* $S^0_{6,14}$ (*resp.* $S_{7,10}$, $S_{8,8}$, $S_{8,6}$, $S^3_{9,13}$, $S_{9,9}$, $S_{10,7}$, $S_{12,13}$, $S_{12,20}$, $S_{13,12}$, $S_{15,7}$) *in Table* 1.

15° (The case [15])

Put $x' = \langle 146 \rangle + \langle 236 \rangle - \langle 137 \rangle + \langle 128 \rangle$. The point x' belongs to the orbit $S_{18,3}$ in Table 1. We denote x' by $x_{18,3}$. The Lie algebra $\tilde{\mathfrak{g}}_{x_{18,3}}$ is given as follows:

$$(2.46)\qquad \tilde{\mathfrak{g}}_{x_{18,3}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} (\delta+\eta) & A_{12} & A_{13} \\ 0 & \delta I_2 + A_2 & A_{23} \\ 0 & 0 & A_3 \end{pmatrix} \in M(5), \right.$$

$$\left. B = \begin{pmatrix} -2\delta & B_{12} & 0 \\ 0 & (-2\delta-\eta)I_2 + A_2 & 0 \\ a_{96} & B_{32} & \beta \end{pmatrix} \in M(4) \right\}$$

where $A_{12} = (a_{12}\ a_{13})$, $A_{13} = (a_{14}\ a_{15})$, $A_2 = \begin{pmatrix} \varepsilon & a_{23} \\ a_{32} & -\varepsilon \end{pmatrix}$,

$$A_{23} = \begin{pmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{pmatrix},\ A_3 = \begin{pmatrix} \delta-\eta & a_{45} \\ 0 & -4\delta \end{pmatrix},$$

$B_{12} = (a_{12} + a_{34}\quad a_{13} - a_{24})$, $B_{32} = (a_{97}\ a_{98})$.

The action of $\tilde{\mathfrak{g}}_{x_{18,3}}$ on the vector space $N_{x_{18,3}}$ with respect to the base $\{(\langle 149 \rangle - \langle 239 \rangle)/2, \langle 159 \rangle, \langle 249 \rangle, \langle 349 \rangle, \langle 259 \rangle, \langle 359 \rangle, \langle 459 \rangle\}$ modulo $T_{x_{18,3}}$ is given as follows:

$$(2.47)\qquad \mathfrak{h}_{x_{18,3}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & (-3\delta-\eta+\beta) \end{pmatrix} \in M(7); \right.$$

$$C_1 = \begin{pmatrix} (2\delta+\beta) & a_{45} \\ 0 & (-3\delta+\eta+\beta) \end{pmatrix},$$

$$C_{12} = \begin{pmatrix} (a_{12}-a_{34}) & (a_{13}+a_{24}) & -a_{35} & a_{25} \\ 0 & 0 & a_{12} & a_{13} \end{pmatrix},\ C_{13} = \begin{pmatrix} -a_{15} \\ a_{14} \end{pmatrix},$$

$$C_2 = \begin{pmatrix} (2\delta - \eta + \beta)I_2 + A_2 & a_{45}I_2 \\ 0 & (-3\delta + \beta)I_2 + A_2 \end{pmatrix}, \ C_{23} = \begin{pmatrix} -a_{25} \\ -a_{35} \\ a_{24} \\ a_{34} \end{pmatrix} \Bigg\}$$

Proposition 2.34. *The triplet $(\tilde{G}_{x_{18,3}}, \tilde{\rho}, N_{x_{18,3}})$ has 8 orbits represented by the following points modulo $T_{x_{18,3}}$:*

(1) $y_1 = \langle 459 \rangle$
(2) $y_2 = \langle 259 \rangle + \langle 349 \rangle$
(3) $y_3 = \langle 259 \rangle$
(4) $y_4 = \langle 159 \rangle + \langle 249 \rangle$
(5) $y_5 = \langle 249 \rangle$
(6) $y_6 = \langle 159 \rangle$
(7) $y_7 = \langle 149 \rangle - \langle 239 \rangle$
(8) $y_8 = 0.$

The point $x_{18,3} + y_1$ (resp. $x_{18,3} + y_i$; $i = 2, \dots, 7$) belongs to the orbit $S_{8,11}$ (resp. $S_{9,5}$, $S_{10,7}$, $S_{11,4}$, $S_{13,12}$, $S^1_{14,6}$, $S_{15,7}$) in Table 1.

16° (The case [16])

Put $x' = \langle 126 \rangle + \langle 137 \rangle + \langle 148 \rangle$. The point x' belongs to the orbit $S_{21,5}$ in Table 1. We denote x' by $x_{21,5}$. The Lie algebra $\tilde{\mathfrak{g}}_{x_{21,5}}$ is given as follows:

$$\text{(2.48)} \qquad \tilde{\mathfrak{g}}_{x_{21,5}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9); \right.$$

$$A = \begin{pmatrix} a_1 & A_{12} & a_{15} \\ 0 & A_2 & A_{23} \\ 0 & 0 & -(a_1 + a_2 + a_3 + a_4) \end{pmatrix} \in M(5),$$

$$B = \left. \begin{pmatrix} -(a_1 I_3 + {}^tA_2) & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\} \text{ where } A_{12} = (a_{12}\ a_{13}\ a_{14}),$$

$$A_2 = \begin{pmatrix} a_2 & a_{23} & a_{24} \\ a_{32} & a_3 & a_{34} \\ a_{42} & a_{43} & a_4 \end{pmatrix}, \ A_{23} = \begin{pmatrix} a_{25} \\ a_{35} \\ a_{45} \end{pmatrix}, \ B_{21} = (a_{96}\ a_{97}\ a_{98}).$$

The action of $\tilde{\mathfrak{g}}_{x_{21,5}}$ on the vector space $N_{x_{21,5}}$ with respect to the base $\{\langle 159 \rangle, \langle 349 \rangle, -\langle 249 \rangle, \langle 239 \rangle, \langle 259 \rangle, \langle 359 \rangle, \langle 459 \rangle\}$ modulo $T_{x_{21,5}}$ is given as follows:

$$\text{(2.49)} \qquad \mathfrak{h}_{x_{21,5}} = \left\{ C = \begin{pmatrix} C_1 & 0 & C_{13} \\ 0 & C_2 & C_{23} \\ 0 & 0 & C_3 \end{pmatrix} \in M(7); \ C_1 = -(a_2 + a_3 + a_4 - \beta), \right.$$

$$C_{13} = (a_{12}\ a_{13}\ a_{14}),\ C_2 = (a_2 + a_3 + a_4 + \beta)I_3 - {}^tA_2,$$

$$C_{23} = \begin{pmatrix} 0 & a_{45} & -a_{35} \\ -a_{45} & 0 & a_{25} \\ a_{35} & -a_{25} & 0 \end{pmatrix},$$

$$\left. C_3 = -(a_1 + a_2 + a_3 + a_4 - \beta)I_3 + A_2 \right\}.$$

Proposition 2.35. *The triplet* $(\tilde{G}_{x_{21,5}}, \tilde{\rho}, N_{x_{21,5}})$ *has 6 orbits represented by the following points modulo* $T_{x_{21,5}}$:

(1) $y_1 = \langle 239 \rangle + \langle 459 \rangle$
(2) $y_2 = \langle 459 \rangle$
(3) $y_3 = \langle 159 \rangle + \langle 239 \rangle$
(4) $y_4 = \langle 239 \rangle$
(5) $y_5 = \langle 159 \rangle$
(6) $y_6 = 0.$

The point $x_{21,5} + y_1$ (*resp.* $x_{21,5} + y_i$; $i = 2,\dots,5$ *belongs to the orbit* $S_{11,15}$ (*resp.* $S_{12,13}, S^1_{14,6}, S_{15,7}, S_{20,12}$) *in Table* 1.

17° (The case [17])

Put $x' = \langle 236 \rangle - \langle 137 \rangle + \langle 128 \rangle$. The point x' belongs to the orbit $S_{22,6}$ in Table 1. We denote x' by $x_{22,6}$. The Lie algebra $\tilde{\mathfrak{g}}_{x_{22,6}}$ is given as follows:

(2.50)
$$\tilde{\mathfrak{g}}_{x_{22,6}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M(9);\ A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} \in M(5), \right.$$

$$\left. B = \begin{pmatrix} B_1 & 0 \\ B_{21} & \beta \end{pmatrix} \in M(4) \right\}$$

where
$$A_1 = \begin{pmatrix} a_1 & a_{12} & a_{13} \\ a_{21} & a_2 & a_{23} \\ a_{31} & a_{32} & a_3 \end{pmatrix},\ A_{12} = \begin{pmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \\ a_{34} & a_{35} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_4 & a_{45} \\ a_{54} & a_5 \end{pmatrix},\ B_1 = (a_4 + a_5)I_3 + A_1,\ B_{21} = (a_{96}\ a_{97}\ a_{98})$$

(where $a_1 + a_2 + a_3 + a_4 + a_5 = 0$).

The action of $\tilde{\mathfrak{g}}_{x_{22,6}}$ on the vector space $N_{x_{22,6}}$ with respect to the base $\{\langle 149 \rangle, \langle 249 \rangle, \langle 349 \rangle, \langle 159 \rangle, \langle 259 \rangle, \langle 359 \rangle, \langle 459 \rangle\}$ modulo $T_{x_{22,6}}$ is given as follows:

$$(2.51)\quad \mathfrak{h}_{x_{22,6}} = \left\{ C = \begin{pmatrix} C_1 & C_{12} & C_{13} \\ C_{21} & C_2 & C_{23} \\ 0 & 0 & C_3 \end{pmatrix} \in M(7);\ C_1 = (a_4 + \beta) I_3 + A_1, \right.$$
$$C_{12} = a_{45} I_3,\ C_{13} = {}^t(-a_{15}\ -a_{25}\ -a_{35}),$$
$$C_{21} = a_{54} I_3,\ C_2 = (a_5 + \beta) I_3 + A_1,\ C_{23} = {}^t(a_{14}\ a_{24}\ a_{34}),$$
$$\left. C_3 = (a_4 + a_5 + \beta) \right\}.$$

Proposition 2.36. *The triplet* $(\tilde{G}_{x_{22,6}}, \tilde{\rho}, N_{x_{22,6}})$ *has* 4 *orbits represented by the following points modulo* $T_{x_{22,6}}$:

(1) $y_1 = \langle 459 \rangle$
(2) $y_2 = \langle 149 \rangle + \langle 359 \rangle$
(3) $y_3 = \langle 359 \rangle$
(4) $y_4 = 0.$

The point $x_{22,6} + y_1$ (*resp.* $x_{22,6} + y_i$; $i = 2, 3$) *belongs to the orbit* $S_{12,10}$ (*resp.* $S^3_{13,9}$, $S_{15,7}$) *in Table* 1.

By the above studying, we have proved Proposition 2.1.

§3. Good Holonomic Varieties of Our P.V.

Let W be the Zariski-closure of $\{(x, \varepsilon \cdot \mathrm{grad} \cdot \log f(x)) | \varepsilon \in \mathbf{C}, x \in V, f(x) \neq 0\}$ in $V \times V^*$ where f is the relative invariant of our triplet (G, ρ, V).

We recall that the conormal bundle Λ_S of an orbit S is called a good holonomic variety, when

(1) G acts on Λ_S prehomogeneously, i.e., the conormal vector space of S is a P.V., and

(2) $\Lambda_S \subset W$ (cf. Definition 4.5 in [SKKO]).

It is clear that the conormal bundle $V \times \{0\}$ of the open orbit $V - S$ is a good holonomic variety.

On the other hand, the conormal bundle $\{0\} \times V^*$ of the origin $\{0\}$ is a good holonomic variety if and only if (G, ρ, V) is a regular P.V. (cf. Proposition 4.6 in [SKKO]).

We recall the following lemmas.

Lemma 3.1. (See Corollary 6.11 in [SKKO]).

If the conormal vector space of an orbit S is an irreducible regular P.V., then the conormal bundle Λ_S *is a good holonomic variety.*

Lemma 3.2 (See Proposition 6.6 in [SKKO]).

Let Λ_S *and* $\Lambda_{S'}$ *be G-prehomogeneous conormal bundles of orbits S and S' respectively. Assume that* $\dim G_0 p = \dim V - 1$ *for some* $p \in \Lambda_S \cap \Lambda_{S'}$ *where* G_0

$=\{g\in G; f(gx)=f(x)$ *for all* $x\in V\}$.

If one of Λ_S *and* $\Lambda_{S'}$ *is a good holonomic variety, then so is the other.*

Remark 3.3. Since $G=SL(5)\times GL(4)$ is reductive, we have $(G, \rho^*, V^*) \simeq (G, \rho, V)$. For a given orbit S in (G, ρ, V), we have the dual orbit S^* in (G, ρ^*, V^*)(See Definition 1.3) which corresponds to some orbit S' in (G, ρ, V). In this case, we call S' the dual orbit of S. It is clear that if Λ_S is a good holonomic variety, then $\Lambda_{S'}$ is also a good holonomic variety.

The rest of this section is devoted to prove the following theorem.

Theorem 3.4. *In our P.V.* (G, ρ, V), *all G-prehomogeneous conormal bundles of orbits S are good holonomic varieties.*

Proof. In our P.V., the orbits whose conormal vector spaces are P.V.'s, are
given as follows: $S_{0,40}$, $S_{1,30}$, $S_{2,24}$, $S_{2,21}$, $S_{3,18}$, $S_{3,15}$, $S_{4,20}$, $S_{4,14}$, $S_{4,11}$, $S_{5,16}$, $S_{5,12}$, $S_{5,9}$, $S_{6,14}$, $S_{6,8}$, $S_{7,7}$, $S_{8,11}$, $S_{8,8}$, $S_{12,10}$, and their dual orbits.

1° (For the orbits $S_{0,40}$)
The conormal bundle $\Lambda_{0,40}$ of the open orbit $S_{0,40}$ is good.

2° (For the orbits $S_{1,30}$, $S_{4,20}$ and $S_{12,10}$)
Each of their conormal vector spaces is an irreducible regular P.V. Namely, the conormal vector space of $S_{1,30}$ (resp. $S_{4,20}$, $S_{12,10}$) is $(GL(1), \Lambda_1, V(1))$ (resp. $(GL(2), 3\Lambda_1, V(4))$, $(SL(3)\times GL(2), 2\Lambda_1\otimes\Lambda_1, V(6)\otimes V(2))$. Therefore their conormal bundles $\Lambda_{1,30}$, $\Lambda_{4,20}$ and $\Lambda_{12,10}$, are good holonomic varieties by Lemma 3.1.

3° (For the orbits $S_{3,15}$, $S_{4,11}$, $S_{5,12}$, $S_{5,9}$, $S_{7,7}$ and $S_{8,8}$)
For these orbits S_{ij}, we can prove $\Lambda_{ij}\subset W$ as follows.

For the generic point (x_{ij}, y^*_{ij}) of $\Lambda_{ij}=\overline{G(x_{ij}, y^*_{ij})}$ = the Zariski-closure of $\{(\rho(g)x_{ij}, \rho^*(g)y^*_{ij}); g\in G\}$ in Table 4, let (x_0, y^*_0) and $g=g(\varepsilon)$ be as in Table 5 corresponding to Λ_{ij}. Then we have $(x_0, \varepsilon y^*_0)\in W$. Note that W is G-admissible. One can check directly that $(x_{ij}, y^*_{ij})=\lim_{\varepsilon\to 0} g\cdot(x_0, \varepsilon y^*_0)=\lim_{\varepsilon\to 0}(\rho(g)x_0, \rho^*(g)(\varepsilon y^*_0))$ $\in W$.

Hence we have $\Lambda_{ij}=\overline{G(x_{ij}, y^*_{ij})}\subset W$.

Table 4

Λ_{ij}	x_{ij}	y^*_{ij}
$\Lambda_{3,15}$	$\langle 156\rangle+\langle 246\rangle+\langle 346\rangle+\langle 147\rangle+\langle 237\rangle+\langle 138\rangle+\langle 259\rangle$	$\langle 456\rangle-\langle 357\rangle+\langle 158\rangle+\langle 348\rangle-2\langle 248\rangle$
$\Lambda_{4,11}$	$\langle 256\rangle+\langle 346\rangle+\langle 157\rangle+\langle 247\rangle+\langle 148\rangle+\langle 238\rangle+\langle 139\rangle$	$\langle 456\rangle-\langle 357\rangle-\langle 258\rangle+\langle 348\rangle+2\langle 159\rangle-2\langle 249\rangle$

$A_{5,12}$	$\langle 146\rangle-\langle 236\rangle+\langle 256\rangle-\langle 157\rangle+\langle 128\rangle+\langle 349\rangle$	$\langle 456\rangle-\langle 148\rangle-\langle 258\rangle-2\langle 238\rangle+2\langle 357\rangle+\langle 247\rangle$
$A_{5,9}$	$\langle 256\rangle-\langle 346\rangle+\langle 157\rangle-\langle 247\rangle-\langle 148\rangle+\langle 238\rangle-\langle 129\rangle$	$\langle 456\rangle-\langle 258\rangle-\langle 348\rangle+2\langle 149\rangle+2\langle 239\rangle-3\langle 357\rangle$
$A_{7,7}$	$\langle 256\rangle-\langle 346\rangle-\langle 157\rangle+\langle 247\rangle-\langle 138\rangle+\langle 129\rangle$	$\langle 456\rangle+\langle 158\rangle+\langle 248\rangle-\langle 149\rangle-\langle 239\rangle+\langle 357\rangle$
$A_{8,8}$	$\langle 246\rangle+\langle 357\rangle+\langle 128\rangle+\langle 139\rangle$	$\langle 156\rangle+\langle 147\rangle+\langle 348\rangle+\langle 259\rangle$

y_{ij}^*: a generic point of $V_{x_{ij}}^*$.

Table 5

$V\ni x_0, f(x_0)\neq 0$, $y_0^*=\mathrm{grad}\cdot\log f(x_0)$,

g: a diagonal matrix in $M(9)$.

$A_{3,15}$
$$x_0=\langle 156\rangle+\langle 246\rangle+\langle 346\rangle+\langle 147\rangle+\langle 237\rangle+\langle 138\rangle+\langle 259\rangle+2\langle 456\rangle+\langle 158\rangle+3\langle 348\rangle$$
$$y_0^*=3\langle 156\rangle+2\langle 246\rangle+3\langle 346\rangle+6\langle 147\rangle+4\langle 237\rangle+6\langle 138\rangle+10\langle 259\rangle+\langle 456\rangle-\langle 357\rangle+\langle 158\rangle+\langle 348\rangle-2\langle 248\rangle-12\langle 136\rangle+4\langle 128\rangle-12\langle 149\rangle+12\langle 239\rangle-3\langle 359\rangle$$
$$g=\{t^{-7}, t^{-2}, t^{-2}, t^3, t^8, t^{-1}, t^4, t^9, t^{-6}\} \text{ where } t^{10}=\varepsilon$$

$A_{4,11}$
$$x_0=\langle 256\rangle+\langle 346\rangle+\langle 157\rangle+\langle 247\rangle+\langle 148\rangle+\langle 238\rangle+\langle 139\rangle+\langle 357\rangle+\langle 348\rangle-10\langle 159\rangle-15\langle 249\rangle$$
$$y_0^*=2\langle 256\rangle+8\langle 346\rangle+35\langle 157\rangle-24\langle 247\rangle+\langle 148\rangle+8\langle 238\rangle+\langle 456\rangle-\langle 357\rangle-\langle 258\rangle+\langle 348\rangle+2\langle 159\rangle-2\langle 249\rangle-18\langle 146\rangle+10\langle 236\rangle-10\langle 137\rangle+18\langle 128\rangle+10\langle 126\rangle$$
$$g=\{t^{-2}, t^{-1}, 1, t, t^2, t^{-1}, 1, t, t^2\} \text{ where } t^2=\varepsilon$$

$A_{5,12}$
$$x_0=\langle 146\rangle-\langle 236\rangle+\langle 256\rangle-\langle 157\rangle+\langle 128\rangle+\langle 349\rangle+\langle 247\rangle-2\langle 148\rangle-6\langle 258\rangle$$
$$y_0^*=3\langle 146\rangle-6\langle 236\rangle+\langle 256\rangle-9\langle 157\rangle+2\langle 128\rangle+10\langle 349\rangle+\langle 247\rangle+\langle 456\rangle+2\langle 357\rangle-\langle 148\rangle-2\langle 238\rangle-\langle 258\rangle+12\langle 126\rangle+6\langle 137\rangle+96\langle 129\rangle-12\langle 149\rangle+8\langle 239\rangle+20\langle 259\rangle-\langle 459\rangle$$
$$g=\{t^{-8}, t^{-3}, t^2, t^7, t^2, t, t^6, t^{11}, t^{-9}\} \text{ where } t^{10}=\varepsilon$$

$A_{5,9}$
$$x_0=\langle 256\rangle-\langle 346\rangle+\langle 157\rangle-\langle 247\rangle-\langle 148\rangle+\langle 238\rangle-\langle 129\rangle+5\langle 456\rangle+2\langle 258\rangle-2\langle 348\rangle-\langle 149\rangle+6\langle 239\rangle$$
$$y_0^*=4\langle 256\rangle-\langle 346\rangle+9\langle 157\rangle-\langle 247\rangle-9\langle 148\rangle+\langle 238\rangle+\langle 456\rangle-3\langle 357\rangle-\langle 258\rangle-\langle 348\rangle+2\langle 149\rangle+2\langle 239\rangle-45\langle 126\rangle+3\langle 146\rangle-7\langle 236\rangle+15\langle 128\rangle$$
$$g=\{t^{-2}, t^{-1}, 1, t, t^2, t^{-1}, 1, t, t^3\} \text{ where } t^2=\varepsilon$$

$A_{7,7}$
$$x_0=\langle 256\rangle-\langle 346\rangle-\langle 157\rangle+\langle 247\rangle-\langle 138\rangle+\langle 129\rangle+3(\langle 456\rangle+\langle 158\rangle+\langle 248\rangle-\langle 149\rangle)+4(\langle 357\rangle-\langle 239\rangle)$$
$$y_0^*=3\langle 256\rangle-4\langle 346\rangle-3\langle 157\rangle+3\langle 247\rangle-4\langle 138\rangle+3\langle 129\rangle+(\langle 456\rangle+\langle 158\rangle+\langle 248\rangle-\langle 149\rangle)+(\langle 357\rangle-\langle 239\rangle)$$
$$g=\{t^{-2}, t^{-1}, 1, t, t^2, t^{-1}, 1, t^2, t^3\} \text{ where } t^2=\varepsilon$$

$A_{8,8}$
$$x_0=\langle 246\rangle+\langle 357\rangle+\langle 128\rangle+\langle 139\rangle+4\langle 156\rangle+4\langle 147\rangle+6\langle 348\rangle+6\langle 259\rangle$$
$$y_0^*=6\langle 246\rangle+6\langle 357\rangle+4\langle 128\rangle+4\langle 139\rangle+\langle 156\rangle+\langle 147\rangle+\langle 348\rangle+\langle 259\rangle$$
$$g=\{1, \varepsilon^{-1}, \varepsilon^{-1}, \varepsilon, \varepsilon, 1, 1, \varepsilon, \varepsilon\}$$

4° (For the other orbits)

The holonomic variety $\Lambda_{2,24}$ (resp. $\Lambda_{2,21}$, $\Lambda_{4,14}$, $\Lambda_{5,16}$, $\Lambda_{6,8}$,$\Lambda_{8,11}$) has a one-codimensional G_0-prehomogeneous intersection with $\Lambda_{1,30}$ (resp. $\Lambda_{4,20}$, $\Lambda_{5,12}$, $\Lambda_{4,20}$, $\Lambda_{5,9}$, $\Lambda_{12,10}$). Therefore it is a good holonomic variety by Lemma 3.2.

Similarly, $\Lambda_{3,18}$ (resp. $\Lambda_{6,14}$) has a one-codimensional G_0-prehomogeneous intersection with $\Lambda_{2,21}$ (resp. $\Lambda_{8,11}$). Hence it is good.

By Remark 3.3, the conormal bundles of dual orbits of the above are also good holonomic varieties. Thus we obtain our assertion. Q.E.D.

References

[Ka 1] Kawanaka, N., Generalized Gelfand-Graev representations of exceptional simple algebraic groups over a finite field I, *Inventiones Math.*, **84** (1986), 575–616.

[Ka 2] ———, Orbits and stabilizers of nilpotent elements of a graded semisimple Lie algebra, *to appear.*

[Ki] Kimura, T., The *b*-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces, *Nagoya Math. J.*, **85** (1982), 1–80.

[KM] Kimura, T. and Muro, M., On some series of regular irreducible prehomogeneous vector spaces, *Proc. Japan Acad.*, **55**, Ser. A, No.10 (1979), 384–389.

[KO] Kimura, T. and Ozeki, I., On the microlocal structure of a regular prehomogeneous vector space associated with Spin (10) × *GL*(3), *Proc. Japan Acad.*, **58**, Ser. A, No. 6, (1982).

[O 1] Ozeki, I., On the microlocal structure of the regular prehomogeneous vector space associated with *SL*(5) × *GL*(4). I, *Proc. Japan Acad.*, **55**, Ser. A No 2 (1979), 37–40.

[O 2] ———, On the microlocal structure of a regular prehomogeneous vector space associated with *GL*(8), *Proc. Japan Acad.*, **56**, Ser. A, No 1, (1980), 18–21.

[SK] Sato, M. and Kimura, T., A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.*, **65** (1977), 1–155.

[SKKO] Sato, M., Kashiwara, M., Kimura, T. and Oshima, T., Micro-local analysis of prehomogeneous vector spaces, *Inventiones Math.*, **62** (1980), 117–179.

[Y1] Yano, T., On the complete localization of prehomogeneous vector spaces, *Saitama Math. J.*, **5** (1987), 27–33.

[Y2] ———, *Locally prehomogeneous spaces and their transverse localizations*, *Vol* II, *Algebraic Analysis*, Academic Press, 1988.

[YS] Yano, T. and Sekiguchi, J., The microlocal structure of weighted homogeneous polynomials associated with Coxeter systems I, II, *Tokyo J. of Math.*, **2** (1979), 193–220, **4** (1981), 1–34.