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On the Bifurcation Set of a Polynomial Function and Newton Boundary

By

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§1. Introduction

Let $f: \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial function. It is well known that there exists a finite set $\Gamma \subseteqq \boldsymbol{C}$, such that $f: \boldsymbol{C}^n \setminus f^{-1}(\Gamma) \to \boldsymbol{C} \setminus \Gamma$ is a locally trivial fibration (see [1], [5], [13], [15], [16]). The smallest such set Γ we call the *bifurcation set*, denoted by B_f (in [1], [2] it is called the set of *atypical values*). Since the map f is not proper, the set B_f contains besides the set Σ_f of all critical values of f perhaps some other points (the "critical values at infinity" or "critical values of second type" [12]). There are some special cases when the polynomial has no critical values at infinity (hence $B_f = \Sigma_f$): Pham [13] and Fedoryuk [4] have imposed lowerbound conditions for $\|\operatorname{grad} f(x)\|$ for large values of $\|x\|$, Kouchnirenko has proved in [6] for convenient polynomials with nondegenerate Newton principal part at infinity, Broughton [1], [2] for "tame" polynomials and the first author [8], [9] for the larger class of "quasitame" polynomials.

In this note we give an *explicit* set S_f, such that $B_f \subseteqq \Sigma_f \cup S_f$. More precisely, let $\operatorname{grad} f(z) = \left(\overline{\frac{\partial f}{\partial z_1}(z)}, \cdots, \overline{\frac{\partial f}{\partial z_n}(z)}\right)$. We denote by $\mathcal{M}(f)$ the *Milnor set* of the polynomial f, namely

$$\mathcal{M}(f) = \{z \in \boldsymbol{C}^n \,;\, \text{there exists } \lambda \in \boldsymbol{C} \text{ such that } \operatorname{grad} f(z) = \lambda z\}.$$

We define the set S_f by:

$$S_f = \left\{ \begin{array}{l} c \in \boldsymbol{C}\,;\, \text{there exists a sequence } \{z^k\}_k \subseteqq M(f) \text{ such that} \\ \lim_{k\to\infty} \|z^k\| = \infty \text{ and } \lim_{k\to\infty} f(z^k) = c \end{array} \right\}$$

In the second section we prove:

Theorem 1. *Let $f: \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial map. Then $B_f \subseteqq \Sigma_f \cup S_f$.*

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In the third section we consider the Newton nondegenerate case. In this situation, the set S_f can be determined very easy and explicitly. Namely, let $f=\sum_{\nu\in N^n} a_\nu z^\nu$ be a polynomial of n variables (where $\nu=(\nu_1, \cdots, \nu_n)$ and $z^\nu=z_1^{\nu_1}\cdot \cdots \cdot z_n^{\nu_n}$ as usual) with $f(0)=0$. As in [6], [11] we denote $\operatorname{supp}(f)=\{\nu\in N^n ; a_\nu\neq 0\}$, $\overline{\operatorname{supp}(f)}$=the convex closure in R^n of $\operatorname{supp}(f)$, $\tilde{\Gamma}_-(f)$=the convex closure of $\{0\}\cup\operatorname{supp}(f)$, $\tilde{\Gamma}(f)$=the union of the closed faces of the polyhedron $\tilde{\Gamma}_-(f)$ which do not contain the origin. If $\Delta\subseteqq\tilde{\Gamma}(f)$ is a closed face, we note $f_\Delta(z)=\sum_{\nu\in\Delta} a_\nu z^\nu$ and we say that f is nondegenerate on Δ if the system of equations $\frac{\partial f_\Delta}{\partial z_1}(z)=\cdots=\frac{\partial f_\Delta}{\partial z_n}(z)=0$ has no solutions in $(C^*)^n$. We say that f is *Newton nondegenerate* if for every compact face Δ of $\tilde{\Gamma}(f)$, f is nondegenerate on Δ. By definition, f is *convenient* if the intersection of $\operatorname{supp}(f)$ with each coordinate axis is non-empty.

A closed face $\Delta\subseteqq\overline{\operatorname{supp}(f)}$ is called *bad* if:

(i) the affine subvariety of dimension$=\dim\Delta$ spaned by Δ contains the origin, and

(ii) there exists a hyperplane $H\subseteqq R^n$ with equation $a_1x_1+\cdots+a_nx_n=0$ (where $x_1, \cdots, x_n$ are the coordinates in R^n) such that

a) there exist i and j with $a_i<0$ and $a_j>0$

b) $H\cap\overline{\operatorname{supp}(f)}=\Delta$.

We can express more geometrically the condition (iia) by saying that the hyperplane H intersects the interior of the positive octant $(R_+)^n$.

Let $\mathcal{B}$ denote the set of bad faces of $\overline{\operatorname{supp}(f)}$. If $\Delta\in\mathcal{B}$ we define:

$$\Sigma_\Delta=\{f_\Delta(z^0) ; z^0\in(C^*)^n \text{ and } \operatorname{grad} f_\Delta(z^0)=0\}.$$

It is clear that $\Sigma_\Delta\subseteqq\Sigma_{f_\Delta}$, hence Σ_Δ is a *finite set.*

We have the following:

Theorem 2. *Suppose that f is not convenient, Newton nondegenerate and* $f(0)=0$. *Then* $B_f\subseteqq\Sigma_f\cup\{0\}\cup\bigcup_{\Delta\in\mathcal{B}}\Sigma_\Delta$.

In the convenient case, $B_f=\Sigma_f$ (see [6], [2]).

In the last section of this note there are some remarks. Also, for $n=2$, we compare our Theorem 2 with the result of Hà and Lê, which gives B_f in terms of the Euler-characteristic of the fibers $f^{-1}(c)$ (see [5]). In particular, for $n=2$, our formula is the best possible (see Proposition 6).

§ 2. Proof of Theorem 1

We need the following lemma, which is a direct consequence of the definitions:

Lemma 3. *Let $D \subseteqq C \setminus S_f$ be a closed disc. Then $f^{-1}(D) \cap \mathcal{M}(f)$ is bounded.*

For $a, b \in \boldsymbol{C}^n$ we note $\langle a, b\rangle = \sum_{j=1}^{n} a_j \bar{b}_j$. For $R \in (0, \infty)$ we put $S_R = \{z \in \boldsymbol{C}^n ; \|z\| = R\}$ and $B_R = \{z \in \boldsymbol{C}^n ; \|z\| \leqq R\}$.

Now for the proof of the theorem we fix $c \in \boldsymbol{C} \setminus (\Sigma_f \cup S_f)$ and D a small open disc centered at c, with the closure $\bar{D} \subseteqq \boldsymbol{C} \setminus (\Sigma_f \cup S_f)$. Let $R \in (0, \infty)$ be sufficiently large such that $f^{-1}(D) \cap \mathcal{M}(f) \cap \{z \in \boldsymbol{C}^n ; \|z\| \geqq R\} = \phi$ (this is possible by Lemma 3) and such that S_R meets transversally $f^{-1}(c')$ for all $c' \in D$ (also possible if D is small enough). It follows that $\operatorname{grad} f(z)$ and z are $\boldsymbol{C}$-linearly independent vectors for all $z \in A$, where $A = f^{-1}(D) \cap \{z \in \boldsymbol{C}^n ; \|z\| \geqq R\}$, and therefore we can find a smooth vector field $v_1(z)$ on A such that $\langle v_1(z), z\rangle = 0$ and $\langle v_1(z), \operatorname{grad} f(z)\rangle = 1$.

Let $\varepsilon > 0$ be such that for every $R' \in [R, R+\varepsilon]$ and for every $d \in D$, we have $f^{-1}(d) \pitchfork S_{R'}$. Since $D \cap \Sigma_f = \phi$, the fibration theorem of Ehresmann gives that the restriction $f : (f^{-1}(D) \cap B_{R+\varepsilon}, f^{-1}(D) \cap S_{R+\varepsilon}) \to D$ is a locally trivial fibration with the fiber $F = f^{-1}(c) \cap B_{R+\varepsilon}$, a smooth manifold with boundary $\partial F = f^{-1}(c) \cap S_{R+\varepsilon}$. Hence there exists a diffeomorphism $\psi : (F \times D, \partial F \times D) \to (f^{-1}(D) \cap B_{R+\varepsilon}, f^{-1}(D) \cap S_{R+\varepsilon})$ such that $f \circ \psi$ is the projection onto D. Thus the vector field $w : F \times D \to T(F \times D) = TF \times TD$, $w(z, d) = ((z, 0), (d, 1))$ will give a vector field v_2 on $f^{-1}(D) \cap B_{R+\varepsilon}$ such that $\langle v_2(z), \operatorname{grad} f(z)\rangle = 1$ for every z. Glueing together v_1 and v_2, we obtain a vector field v on $f^{-1}(D)$ such that $\langle v, \operatorname{grad} f\rangle = 1$ and such that for every z with $\|z\| \geqq R+\varepsilon$, we have $\langle v(z), z\rangle = 0$. Now using the solutions of the differential equation $\frac{dz}{dt} = v(z)$ we obtain that the restriction $f : f^{-1}(D) \to D$ is a trivial fibration.

§ 3. Proof of Theorem 2

We need a version of Curve Selection Lemma from Milnor's book [7]. This seems to be well known. For a proof, we refer the reader to [10].

Lemma 4. (Curve Selection Lemma) *Let $f_1, \cdots, f_q, g_1, \cdots, g_s, h_1, \cdots, h_r \in \boldsymbol{R}[X_1, \cdots, X_m]$ be polynomial functions with real coefficients. Let $U = \{x \in \boldsymbol{R}^m ; f_i(x) = 0, i = 1, \cdots, q\}$ and $W = \{x \in \boldsymbol{R}^m ; g_i(x) > 0, i = 1, \cdots, s\}$. Suppose that there exists a sequence $\{x^k\} \subseteqq U \cap W$ such that $\lim_{k \to \infty} \|x^k\| = \infty$ and for all $j \in \{1, \cdots, r\}$, $\lim_{k \to \infty} h_j(x^k) = 0$. Then there exists a real analytic curve $p : (0, \varepsilon) \to U \cap W$ with $\lim_{t \to 0} \|p(t)\| = \infty$, $\lim_{t \to 0} h_j(p(t)) = 0$ for $1 \leqq j \leqq r$ and of the form $p(t) = a t^{\alpha} + a_1 t^{\alpha+1} + \cdots$ with $a \in \boldsymbol{R}^m \setminus \{0\}$ and $\alpha < 0$.*

Using Theorem 1 and Curve Selection Lemma, it is sufficient to prove the following:

If f is a Newton nondegenerate polynomial with $f(0)=0$ and if $p(t)\in\mathcal{M}(f)$ is an analytic curve such that

$$\lim_{t\to 0}\|p(t)\|=\infty \quad \textit{and} \quad \lim_{t\to 0} f(p(t))\in \boldsymbol{C} \tag{1}$$

then $\lim_{t\to 0} f(p(t))\in\Sigma_f\cup\{0\}\cup\bigcup_{\Delta\in\mathcal{B}}\Sigma_\Delta$.

To prove this we consider the expansions

$$\begin{cases} p(t)=at^\alpha+a_1t^{\alpha+1}+\cdots \\ f(p(t))=bt^\beta+b_1t^{\beta+1}+\cdots \\ \operatorname{grad} f(p(t))=ct^\gamma+c_1t^{\gamma+1}+\cdots \end{cases} \tag{2}$$

and the identity

$$\frac{df(p(t))}{dt}=\left\langle\frac{dp}{dt},\ \operatorname{grad} f(p(t))\right\rangle. \tag{3}$$

The condition $p(t)\in\mathcal{M}(f)$ means that there exists an analytic curve $\lambda(t)\in\boldsymbol{C}$ such that for every t, we have

$$\operatorname{grad} f(p(t))=\lambda(t)p(t). \tag{4}$$

If $\operatorname{grad} f(p(t))\equiv 0$, the identity (3) shows that $f(p(t))$ is constant with respect to t, namely $f(p(t))\in\Sigma_f$.

So we can suppose that $\operatorname{grad} f(p(t))\not\equiv 0$. Similarly, $f(p(t))\not\equiv 0$, since otherwise by derivation $\lambda(t)\cdot\left\langle\frac{dp}{dt},\ p\right\rangle=0$ which is in contradiction with $\operatorname{grad} f(p(t))\not\equiv 0$ and $\left\langle\frac{dp}{dt},\ p\right\rangle\not\equiv 0$. From (4) we get also $\lambda(t)\not\equiv 0$. Let $\lambda(t)=\lambda_0t^\delta+\lambda_1t^{\delta+1}+\cdots$ be the expansion of $\lambda(t)$, where $\lambda_0\neq 0$. From (1) we can assume that $a\neq 0$, $\alpha<0$, $b\neq 0$, $\beta\geqq 0$ and $c\neq 0$. Using (4), the scalar product $\langle a, c\rangle\neq 0$, hence from the expansions (2) and the formula (3) we get that $\gamma+\alpha-1\geqq 0$ and thus $\gamma>0$.

Renumbering the coordinates, if necessary, we may assume that $p(t)=(p_1(t),\cdots,p_n(t))=(w_1^0t^{\nu_1}+w_1^1t^{\nu_1+1}+\cdots,\cdots,w_k^0t^{\nu_k}+w_k^1t^{\nu_k+1}+\cdots,0,\cdots,0)$, where $w_1^0\neq 0,\cdots,w_k^0\neq 0$ and $\alpha=\nu_1\leqq\nu_2\leqq\cdots\leqq\nu_k$. Identifying $\boldsymbol{R}^k$ with $\{x\in\boldsymbol{R}^n;\ x_{k+1}=\cdots=x_n=0\}$ we have $\operatorname{supp}(f)\cap\boldsymbol{R}^k\neq\phi$ since $f(p(t))\not\equiv 0$.

Consider the continuous function $l_\nu(x)=\sum_{j=1}^k\nu_jx_j$ on $\boldsymbol{R}^n$. Let Δ be the unique face of $\overline{\operatorname{supp}(f)}\cap\boldsymbol{R}^k$ where the restriction $l_\nu:\overline{\operatorname{supp}(f)}\cap\boldsymbol{R}^k\to\boldsymbol{R}$ takes the minimal value, say d, and let $m\in(-\infty, 0)$ be such that

$$m<\min\{l_\nu(x);\ x\in\overline{\operatorname{supp}(f)}\}.$$

Then $f(p(t))=f_\Delta(w_0^0)t^d+\cdots$, and for $j=1,\cdots,k$, $\frac{\partial f}{\partial z_j}(p(t))=\frac{\partial f_\Delta}{\partial z_j}(w^0)t^{d-\nu_j}+\cdots$, where $w^0=(w_1^0,\cdots,w_k^0,1,\cdots,1)$.

If $d>0$, we have $\lim_{t\to 0} f(p(t))=0$.

If $d=0$ and $\nu_k \leqq 0$, then $f(z_1, \cdots, z_k, 0, \cdots, 0)$ does not depend on z_1, hence $\frac{\partial f}{\partial z_1}(p(t))\equiv 0$, in contradiction with (4) and $p_1(t)\not\equiv 0$.

If $d=0$ and $\nu_k>0$, then for the hyperplane $H\subseteq \boldsymbol{R}^n$ with equation $\nu_1 x_1+\cdots+\nu_k x_k-m(x_{k+1}+\cdots+x_n)=0$ and the face Δ, the condition (ii) is fulfilled. Thus if (i) is not fulfilled, then Δ is not a bad face of $\overline{\mathrm{supp}(f)}$ and it follows that Δ is a closed face of $\tilde{\Gamma}(f)$. By the nondegeneracy condition on Δ, there exists $l\in\{1, \cdots, k\}$ such that $\frac{\partial f_\Delta}{\partial z_l}(w^0)\neq 0$. (We recall that $f_\Delta(z)$ does not depend on the variables $z_{k+1}, \cdots, z_n$.)

But this is in contradiction with the following lemma:

Lemma 5. *Let d, Δ, w^0 be defined as above. Suppose that $d\leqq 0$ and $d\cdot f_\Delta(w^0)=0$. Then there exists no $l\in\{1, \cdots, k\}$ such that $\frac{\partial f_\Delta}{\partial z_l}(w^0)\neq 0$.*

Proof of Lemma 5. Suppose that there exists $l\in\{1, \cdots, k\}$ such that $\frac{\partial f_\Delta}{\partial z_l}(w^0)\neq 0$. By condition (4) and $\gamma>0$ we get that $\delta+\nu_l=d-\nu_l>0$, hence $\nu_l<0$. Let $I=\{j; \nu_j=\nu_l\}$. Again by (4) we have for $j\in\{1, \cdots, k\}$:

$$j\in I \Longrightarrow d-\nu_j=\delta+\nu_j \quad \text{and} \quad \frac{\partial f_\Delta}{\partial z_j}(w^0)=\bar{\lambda}_0\overline{w_j^0};$$

$$j\notin I \Longrightarrow d-\nu_j<\delta+\nu_j, \quad \text{hence} \quad \frac{\partial f_\Delta}{\partial z_j}(w^0)=0.$$

Thus, from the Euler relation for the weakly quasihomogeneous polynomial f_Δ, $\sum_{j=1}^{k}\nu_j z_j\frac{\partial f_\Delta}{\partial z_j}(z)=d\cdot f_\Delta(z)$, we obtain for $z=w^0$ the absurd equality $\nu_l\cdot\bar{\lambda}_0\sum_{j\in I}|w_j^0|^2=0$. This ends the proof of Lemma 5.

It follows that (i) is also fulfilled and $\Delta\in\mathcal{B}$. By the above lemma, $\frac{\partial f_\Delta}{\partial z_l}(w^0)=0$ for all $l\in\{1, \cdots, k\}$ and thus we have $\lim_{t\to 0} f(p(t))=f_\Delta(w^0)\in\Sigma_\Delta$.

It remains to consider the case $d<0$. With this assumption it follows that Δ is a closed face of $\tilde{\Gamma}(f)$. Since $\beta\geqq 0$ we get $f_\Delta(w^0)=0$, hence the nondegeneracy condition on Δ is in contradiction with the above lemma.

§4. Some Remarks

1. S. A. Broughton described in [1] and [2] the class $\mathcal{T}$ of tame polynomials and proved that for a tame polynomial f, we have $B_f=\Sigma_f$.

In [8] and [9], the first author considered the class $Q\mathcal{T}$ of quasitame polynomials and also proved that $B_f=\Sigma_f$ for any quasitame polynomial f. If we denote by $\mathcal{MT}$ the class of $\mathcal{M}$-tame polynomials, namely the polynomials f with $S_f=\phi$, then $\mathcal{T}\subseteqq Q\mathcal{T}\subseteqq\mathcal{MT}$, the first inclusion being strict. We don't know if the second inclusion is an equality or not. For other interesting properties of these classes of polynomials see also [10].

2. Also in [8] and [9], the first author proved that $B_f\subseteqq\Lambda_f$, where

$$\Lambda_f:=\left\{\begin{array}{l} c\in\boldsymbol{C}\,;\text{ there exists a sequence } \{z^k\}_k\in\boldsymbol{C}^n \text{ such that } \lim\limits_{k\to\infty}\operatorname{grad}f(z^k)=0 \\ \text{and } \lim\limits_{k\to\infty}(f(z^k)-\langle z^k,\operatorname{grad}f(z^k)\rangle)=c \end{array}\right\}$$

It is not hard to prove that $\Sigma_f\cup S_f\subseteqq\Lambda_f$, but in general we have no equality, hence the set $\Sigma_f\cup S_f$ is a better approximation for B_f. Such an example is the polynomial $f=x^5y^3+x^5z^2+x^{11}y^3z^2+x$ which is Newton nondegenerate (hence $\Sigma_f\cup S_f$ is a finite set) but $\Lambda_f=\boldsymbol{C}$. This follows by using an analytic curve $(x(t), y(t), z(t))\in C^3$ with $x(t)=t$, $[z(t)]^2=-\frac{5}{9}t^{-6}+\lambda t^{-5}+\mu t^{-4}+\cdots$ and $[y(t)]^3=-\frac{5}{2}t^{-6}-\frac{81\lambda}{4}t^{-5}+\frac{9(4-90\mu-891\lambda^2)}{40}t^{-4}+\cdots$.

3. Our conjecture is that for a Newton nondegenerate polynomial f with $f(0)=0$ we have $\bigcup_{\Delta\in\mathcal{B}}\Sigma_\Delta\subseteqq B_f$; for the general case of polynomials, we hope that $S_f\subseteqq B_f$. But without a good description of the (all !) fibers $f^{-1}(c)$ it seems that there exists no simple way to prove this.

However, for $n=2$ we have the following:

Proposition 6. *Let $f\in\boldsymbol{C}[x, y]$ be a not convenient and Newton nondegenerate polynomial, not depending only of one variable, and such that $f(0)=0$. Then*

$$B_f=\Sigma_f\cup\{0\}\cup\bigcup_{\Delta\in\mathcal{B}}\Sigma_\Delta$$

Proof. We prove this proposition in three steps.

Step 1. *$\Sigma_f\subseteqq B_f$. The proof is clear.*

Step 2. *If $f\in\boldsymbol{C}[x, y]$ is a not convenient polynomial with $f(0)=0$, not depending only of one variable, then $0\in B_f$.*

Proof. Since f is not convenient, f has the form $f(x, y)=xg(x, y)$ (or $f(x, y)=yg(x, y)$, which case can be analysed similarly).

If $g(0, 0)=0$, then $0\in\Sigma_f\subseteqq B_f$.

If $g(0, 0)\neq 0$, then $g(x, y)=a_0+a_1y+\cdots+a_my^m+x\phi(x, y)$ with $a_0\neq 0$.

Suppose that there exists $a_i\neq 0$, $i\in\{1, \cdots, m\}$; by an easy computation we obtain that $0\in\Sigma_f$, hence $0\in B_f$.

Suppose that $a_i=0$ for all $i=1, \cdots, m$; hence $f(x, y)=x(a_0+x\phi(x, y))$ and $\phi(x, y)$ does depend on the variable y. The fiber $f^{-1}(0)$ contains a connected component diffeomorphic to $\boldsymbol{C}$. But the generic fiber $f^{-1}(\lambda)$, $\lambda\neq 0$, doesn't contain any contractible component. To see this, suppose that T is a contractible component of $f^{-1}(\lambda)$, $\lambda\neq 0$; consider the projection π of T to the x-axis. Then the image $\pi(T)$ is equal to $\boldsymbol{C}^*\setminus\{\text{finite number of points}\}$; hence T is a contractible (branched) covering of $\pi(T)$ with finite fibers and $\pi_1(\operatorname{im}\pi)\neq 0$ and is free, contradiction. Consequently $f^{-1}(0)$ and the generic fiber are not diffeomorphic, hence $0\in B_f$.

Step 3. *Let $f\in\boldsymbol{C}[x, y]$ be a not convenient and Newton nondegenerate polynomial, not depending only of one variable, and such that $f(0)=0$. If $c\in\bigcup_{\Delta\in\mathcal{B}}\Sigma_\Delta\setminus(\Sigma_f\cup\{0\})$, then $c\in B_f$.*

Proof. Indeed, in this case the Euler-characteristic of the special fiber $f^{-1}(c)$ and of the generic fiber $f^{-1}(c_{\text{gen}})$ differ. The Euler-characteristic of a fiber $f^{-1}(\lambda)$ can be computed by the following formula:

$$\chi(f^{-1}(\lambda))+\#\overline{(f^{-1}(\lambda)}\cap H_\infty)=2-(d-1)(d-2)+\Sigma\mu_i(\lambda),$$

where $\overline{f^{-1}(\lambda)}$ is the projective closure of $f^{-1}(\lambda)$, H_∞ is the hyperplane at infinity and $\Sigma\mu_i(\lambda)$ is the sum of the Milnor numbers of the singularities of $\overline{f^{-1}(\lambda)}$ at infinity (see [5]).

We show that $\Sigma\mu_i(c)>\Sigma\mu_i(c_{\text{gen}})$. This follows from the followings two lemmas: the first one describes the singularities at infinity and the second one is applied for these singularities.

Lemma 7. *Let f be as above and let d be the degree of f and $F(x, y, z)$ the homogeneousated polynomial of f. Then only $(1:0:0)$ and $(0:1:0)$ can be singularities of $\overline{f^{-1}(\lambda)}$ at infinity. If $(1:0:0)$ is a singularity of $\overline{f^{-1}(\lambda)}$ at infinity, then his equation is $g(y, z)=F(1, y, z)-\lambda z^d=0$ and the Newton polygon of this singularity can be obtained from the Newton diagram of f in the following way: let P, R, O be the points in the diagram of f with coordinates $(d, 0)$, $(0, d)$ and respectively $(0, 0)$. Then the origin in the diagram of g is P, the positive semiaxes are PR and PO, corresponding to the y-axis and respectively to the z-axis, and $\overline{\operatorname{supp}(g)}$ corresponds to $\tilde{\Gamma}_-(f)$. The nondegeneracy of f on the faces of $\tilde{\Gamma}_-(f)$ means the nondegeneracy of g on the corresponding faces of $\overline{\operatorname{supp}(g)}$. If Δ is a bad face of $\overline{\operatorname{supp}(f)}$ giving rise to the face $\bar{\Delta}$ of the Newton polygon of the singularity in 0 of g, then the values $\lambda\in\Sigma_\Delta\setminus\{0\}$ are exactely the values of λ such that the conditions of nondegeneracy on $\bar{\Delta}$ are not fulfilled.*

Lemma 8. *Let $h\in\boldsymbol{C}[x, y]$ be a convenient polynomial with* $\operatorname{grad} h(0)=0$ *and let Γ be the Newton polygon of h in origin* (*see* [6], [11] *for the definition*).

Suppose that h is nondegenerate on all the faces of Γ, excepting the 1*-dimensional face Δ of Γ which has an endpoint on the axis Ox, where the nondegeneracy condition is not fulfilled. Then the Milnor number $\mu(h, 0)$ of h in* 0 *is strictly greater than the Newton number $\nu(\Gamma)$.* (*See* [6], [11] *for the definition of $\nu(\Gamma)$.*)

We omit the proof of Lemma 7, since it is a straightforward verification. For the proof of Lemma 8, let $(a, 0)$ and (b, c) be the coordinates of the terminal points of Δ. We consider the covering $\varphi: (\mathbf{C}^2, 0)\to(\mathbf{C}^2, 0)$, $\varphi(x, y)=(x^c, y)$ and the singularity $h'(x, y)=h(\varphi(x, y))$ with Newton polygon Γ'; the relations (15) and (12) from [3] between $\mu(h', 0)$ and $\mu(h, 0)$, respectively $\nu(\Gamma)$ and $\nu(\Gamma')$ enable us to suppose that c divides $a-b$. Let $m=\dfrac{a-b}{c}$. We have $f_\Delta=\alpha x^b(y+\beta_1 x^m)\cdot \cdots \cdot(y+\beta_c x^m)$ with $\alpha, \beta_1, \cdots, \beta_c \in \mathbf{C}^*$. The degeneracy condition on Δ means that there exist $i, j\in\{1, \cdots, c\}$, $i\neq j$, such that $\beta_i=\beta_j$. Hence we can consider that $f_\Delta=\alpha x^c(y+\beta_1 x^m)^r\cdot(y+\beta_{r+1}x^m)\cdot \cdots \cdot(y+\beta_c x^m)$ for some $r\geqq 2$. We change the variables: $\tilde{x}=x$, $\tilde{y}=y+\beta_1 x^m$. Then all the faces of Γ, excepting Δ, will be faces of the new Newton polygon Γ'' and $\Gamma''\supseteqq\Gamma$, $\Gamma''\neq\Gamma$. Now it is easy to see that $\nu(\Gamma'')>\nu(\Gamma)$ and we finish the proof of Lemma 8 using Kouchnirenko's results [6].

Note that the analogue of Lemma 8 for polynomials with $n\geqq 3$ variables in general is not true (see Remarque 1.21 from [6]).

4. Our Proposition 6 (and the inequality from Step 3) can be compared with a result of Hà and Lê (which says that, for $n=2$, a bifurcation point λ is either critical point or $\chi(f^{-1}(\lambda))\neq\chi(f^{-1}(c_{\text{gen}}))$, and a result of M. Suzuki (which says that $\chi(f^{-1}(\lambda))\geqq\chi(f^{-1}(c_{\text{gen}}))$, see [14], Théorème 1).

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