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Solvability of Linear Functional Equations in Lebesgue Spaces

By

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§ 1. Introduction

Let A be a closed linear operator on a Banach space X. This paper is concerned with the solvability and approximate solutions of the equation $Ax=y$ for a given $y \in X$, especially when X is a Lebesgue space L_p, $1 \leqq p < \infty$. The domain, null space, and range will be denoted by $D(A)$, $N(A)$, and $R(A)$, respectively.

Let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets, indexed by a directed set $\mathcal{A}$ of bounded linear operators on X with the following properties:

(a) $\|A_\alpha\| \leqq M$ for all $\alpha \in \mathcal{A}$;

(b) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all $\alpha \in \mathcal{A}$;

(c) $R(A_\alpha) \subset D(A)$ for all $\alpha \in \mathcal{A}$, w-$\lim_\alpha AA_\alpha x = 0$ for all $x \in X$, and s-$\lim_\alpha A_\alpha Ax = 0$ for all $x \in D(A)$.

(d) $B_\alpha^* x^* = \phi(\alpha) x^*$ for all $x^* \in R(A)^\perp$ $(=N(A^*)$ in case $D(A)=X)$ with $\lim_\alpha |\phi(\alpha)| = \infty$.

We call $\{A_\alpha\}$ a system of *almost invariant integrals* for $A+I$ and $\{B_\alpha\}$ the system of *companion integrals*. The terminologies go back to those of Eberlein [4] and Dotson [2] for the case $A=T-I$ with T bounded. The following two theorems concerning the convergence of $\{A_\alpha x\}$ and $\{B_\alpha y\}$ have been established in [8]:

(i) *$\{A_\alpha x\}$ converges if and only if it contains a weakly convergent subnet, if and only if $x \in N(A) \oplus \overline{R(A)}$, and the mapping P: $x \to$ s-$\lim_\alpha A_\alpha x$ is a bounded projection with $R(P)=N(A)$, $N(P)=\overline{R(A)}$ and $D(P)=N(A) \oplus \overline{R(A)}$,*

(ii) *$\{B_\alpha y\}$ converges if and only if it contains a weakly convergent subnet, if and only if $y \in A(D(A) \cap \overline{R(A)})$. The limit $x=$ s-$\lim_\alpha B_\alpha y$ is the unique solution of the equation $Ax=y$ in $\overline{R(A)}$.*

In a reflexive space, the weak sequential precompactness of bounded sets

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implies that $D(P)=X$ and $R(A)=A(D(A)\cap\overline{R(A)})$. The following theorem [8, Corollary 1.8] is then easily deduced form (ii).

Theorem 1. *If X is a reflexive space, then, under the conditions* (a), (b), (c), *and* (d), *the following statements are equivalent*:

(1) $y\in R(A)$;

(2) $\{B_\alpha y\}$ *is bounded*;

(3) *There is a subnet* $\{B_\beta\}$ *of* $\{B_\alpha\}$ *such that* $x=\text{w-lim}_\beta B_\beta y$ *exists*;

(4) $x=\text{s-lim}_\alpha B_\alpha y$ *exists.*

Moreover, the x *in* (3) *and* (4) *is the unique solution of* $Ax=y$ *in* $\overline{R(A)}$.

This theorem holds in particular for any Lebesgue space $L_p(S, \Sigma, \mu)$ with $1<p<\infty$. In general, while the implications "(3)$\Leftrightarrow$(4)$\Rightarrow$(1)$\Rightarrow$(2)" always hold (due to (ii), (a), and (b)), the other two implications "(2)$\Rightarrow$(1)" and "(1)$\Rightarrow$(4)" may not hold in a nonreflexive space (cf. [9] and [8, Remark 1.7]). However, with some additional assumption, we shall prove in section 2 the following positive result for $L_1(S, \Sigma, \mu)$.

Theorem 2. *Let* $X=L_1(S, \Sigma, \mu)$ *wih* μ *a* σ*-finite measure. If* $\{A_\alpha\}$ *and* $\{B_\alpha\}$ *satisfy* (a) *with* $M=1$, (b), (c), *and* (d), *then* (1) *and* (2) *are equivalent. If, in addition to the above assumption,* μ *is a finite measure and* $\|A_\alpha f\|_\infty\leqq K\|f\|_\infty$ *for all* $f\in L_\infty(S, \Sigma, \mu)$ *and* $\alpha\in\mathcal{A}$, *then the statements* (1), (2), (3), *and* (4) *are equivalent, and the limit* x *in* (3) *and* (4) *is the unique solution of* $Ax=y$ *in* $\overline{R(A)}$.

These general theorems can be used to study the solvability and various approximate solutions of the linear functional equation $Ag=f$ in $L_p(S, \Sigma, \mu)$, $1\leqq p<\infty$. For illustration we shall display in sections 3 and 4 applications to n-times integrated semigroups and cosine operator functions, respectively. Applications to other methods of solving $(I-T)x=y$ such as those considered in [8] are also possible. In particular, theorems of Lin and Sine [7], and of Krengel and Lin [6, Theorem 3.1] can be deduced from this result. In section 3, the almost everywhere pointwise convergence of the approximate solutions of $Ag=f$ will also be observed for the case that A is the generator of a C_0-semigroup of contractions on $L_1(S, \Sigma, \mu)$ that also fulfills the condition that $\|T(t)f\|_\infty\leqq K\|f\|_\infty$ for all $f\in L_1\cap L_\infty$ and $t\geqq 0$.

§ 2. Proof of Theorem 2

Suppose (1) holds, i.e., $y=Ax$. Then (a) and (b) imply that $\|B_\alpha y\|=\|B_\alpha Ax\|=\|(I-A_\alpha)x\|\leqq(1+M)\|x\|$ for all $\alpha\in\mathcal{A}$, i.e., (2) holds.

Conversely, if $\{B_\alpha y\}$ is bounded, we first show that $A_\alpha y\to 0$. Indeed, (d) implies that for each $x^*\in R(A)^\perp$ we have

$$\|B_\alpha y\|\|x^*\| \geqq |\langle B_\alpha y, x^*\rangle| = |\langle y, \phi(\alpha)x^*\rangle| = |\phi(\alpha)||\langle y, x^*\rangle|,$$

which would be unbounded unless $\langle y, x\rangle=0$. Hence y belongs to ${}^\perp(R(A)^\perp)=\overline{R(A)}$. This fact with assumptions (a) and (c) implies that $A_\alpha y$ converges in norm to 0.

Next, let LIM_β be a Banach limit on the space of bounded functions on $\mathcal{A}$, and define a linear functional q on $L_1(\mu)^*=L_\infty(\mu)$ by $q(x^*)=\operatorname{LIM}_\beta\langle B_\beta y, x^*\rangle$, $x^*\in L_\infty(\mu)$. Then q belongs to $X^{**}=L_\infty(\mu)^*=ba(S, \Sigma, \mu)$, the space of bounded finitely additive measures (=charges)$\ll\mu$, and $\|q\|\leqq\sup_\alpha\|B_\alpha y\|$. We have for $x^*\in X^*$ and $\alpha\in\mathcal{A}$

$$\begin{aligned}[A_\alpha^{**}q](x^*)=q(A_\alpha^*x^*)&=\operatorname{LIM}_\beta\langle B_\beta y, A_\alpha^* x^*\rangle\\ &=\operatorname{LIM}_\beta\langle(I-B_\alpha A)B_\beta y, x^*\rangle\\ &=\operatorname{LIM}_\beta\langle(B_\beta y-B_\alpha(I-A_\beta)y, x^*\rangle\\ &=\operatorname{LIM}_\beta\langle(B_\beta y, x^*\rangle-\langle B_\alpha y, x^*\rangle+\lim_\beta\langle A_\beta y, B_\alpha^* x^*\rangle\\ &=q(x^*)-\langle B_\alpha y, x^*\rangle,\end{aligned}$$

(b) and the fact that $Py=0$ having been used. Hence $A_\alpha^{**}q=q-B_\alpha y$ for all $\alpha\in\mathcal{A}$.

$L_1(S, \Sigma, \mu)$ can be identified, via the Radon-Nikodym theorem, with $M(S, \Sigma, \mu)$, the subspace of $ba(S, \Sigma, \mu)$ which consists of all countably additive measures$\ll\mu$. Decomposing $q=q_1+q_2$ with $q_1\in M(S, \Sigma, \mu)$ and q_2 a pure charge (cf. [12]), and using the contraction assumption and the fact that the norm of an element of $ba(S, \Sigma, \mu)$ is the sum of the norms of its two parts, we obtain the estimate:

$$\begin{aligned}\|q_2\|\geqq\|A_\alpha^{**}q_2\|&=\|q_1-B_\alpha y-A_\alpha^{**}q_1+q_2\|\\ &=\|q_1-B_\alpha y-A_\alpha q_1\|+\|q_2\|,\end{aligned}$$

which shows that $q_1=B_\alpha y+A_\alpha q_1\in D(A)$ and $Aq_1=AB_\alpha y+AA_\alpha q_1=y-A_\alpha y+AA_\alpha q_1$ for all $\alpha\in\mathcal{A}$. Taking limits yields that $y=Aq_1\in R(A)$. Thus we have proved the equivalence of (1) and (2).

Since, as mentioned in the introduction, the conditions (3) and (4) are equivalent to that y belongs to $A(D(A)\cap\overline{R(A)})=A(D(A)\cap D(P))$, which is equal to $R(A)$ when $D(P)=X$, the second part of Theorem 2 follows from the next lemma.

Lemma 3. *Let (S, Σ, μ) be a finite measure space and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be bounded operators on $X=L_1(S, \Sigma, \mu)$ as well as on $L_\infty(S, \Sigma, \mu)$ which satisfy* (b), (c), *and* (d). *Suppose further that* $\|A_\alpha f\|_1 \leqq M\|f\|_1$ *for all* $f \in L_1(S, \Sigma, \mu)$ *and* $\|A_\alpha h\|_\infty \leqq K\|h\|_\infty$ *for all* $h \in L_\infty(S, \Sigma, \mu)$ *and for all* $\alpha \in \mathcal{A}$. *Then* $\{A_\alpha f\}$ *converges in* $L_1(S, \Sigma, \mu)$ *for all* f *in* $L_1(S, \Sigma, \mu)$.

Proof. If h is a simple function, then $\left|\int_E (A_\alpha h) d\mu\right| \leqq K\|h\|_\infty \mu(E)$ which converges to 0 uniformly for $\alpha \in \mathcal{A}$ as $\mu(E) \to 0$. Hence $\{A_\alpha h; \alpha \in \mathcal{A}\}$ is weakly sequentially precompact in $L_1(S, \Sigma, \mu)$ (see [3, Corollary IV.8.11]). It follows from (i) in the introduction that $\{A_\alpha h\}$ converges in $L_1(S, \Sigma, \mu)$. Since the set of all simple functions is dense in $L_1(S, \Sigma, \mu)$ and since $\{A_\alpha\}$ is uniformly bounded, the convergence of $\{A_\alpha f\}$ holds for all f in $L_1(S, \Sigma, \mu)$.

§ 3. Generators of *n*-times Integrated Semigroups

A strongly continuous family $\{T(t); t \geqq 0\}$ of bounded operators on X is called a *n-times integrated semigroup* if $T(0)=I$ and $T(t)T(s)=T(t+s)(t, s \geqq 0)$ in case $n=0$, and if $T(0)=0$ and

$$T(t)T(s)=\frac{1}{(n-1)!}\left\{\int_t^{t+s}(t+s-r)^{n-1}T(r)dr-\int_0^s(t+s-r)^{n-1}T(r)dr\right\} \quad (t, s \geqq 0)$$

in case $n \geqq 1$. A 0-times integrated semigroup is just the classical C^0-semigroup. $T(\cdot)$ is said to be *non-degenerate* if $T(t)x=0$ for all $t>0$ implies $x=0$, and *exponentially bounded* if $\|T(t)\| \leqq Me^{wt}$ for some $M \geqq 1$, $w>0$ and for all $t \geqq 0$. For a non-degenerate and exponentially bounded $T(\cdot)$ there exists a uniquely determined closed operator A, called the *generator* of $T(\cdot)$, such that $(w, \infty) \subset \rho(A)$ and $(\lambda-A)^{-1}x=\int_0^\infty \lambda^n e^{-\lambda t}T(t)x dt$ for all $x \in X$ and $\lambda>w$. For the definitions and basic properties we refer to Arendt [1], and Tanaka and Miyadera [11].

It is known [1, Proposition 3.3] that $\int_0^t T(s)x ds \in D(A)$ and $A\int_0^t T(s)x ds= T(t)x-(t^n/n!)x$ for all $x \in X$; $\int_0^t T(t)Ax ds=T(t)x-(t^n/n!)x$ for all $x \in D(A)$. Hence, if we put $A_t:=(n+1)!t^{-n-1}\int_0^t T(s)ds$ and $B_t:=-(n+1)!t^{-n-1}\int_0^t\int_0^s T(u)du ds$ for $t>0$, then the closedness of A implies that $B_tA \subset AB_t=I-A_t$ and $A_tA \subset AA_t=(n+1)!t^{-n-1}T(t)-(n+1)t^{-1}I$. Thus (c) holds if $t^{-n-1}T(t)$ converges strongly to 0 as $t \to \infty$. In particular, both (a) and (c) will hold in case $\|T(t)\|=O(t^n)(t \to \infty)$. To verify (d) let $x^* \in R(A)^\perp$. Then $\langle T(u)x-(u^n/n!)x, x^*\rangle=\left\langle A\int_0^u T(s)x ds, x^*\right\rangle =0$ for all $u \geqq 0$, so that

$$\langle x, B_t^* x^*\rangle=\langle B_t x, x^*\rangle=-(n+1)!t^{-n-1}\int_0^t\int_0^s T(u)x, x^*>du ds$$

$$=-(n+1)t^{-n-1}\int_0^t\int_0^s u^n du ds\langle x, x^*\rangle$$

$$=-\frac{t}{n+2}\langle x, x^*\rangle \quad \text{for all} \quad x\in X.$$

That is, the condition (d) holds with $\phi(t)=-\dfrac{t}{n+2}$.

On the other hand, if $\|A_t\|\leqq M$ for all $t\geqq 0$, then

$$\begin{aligned}\|(\lambda-A)^{-1}x\| &\leqq \left\|\int_0^\infty e^{-\lambda t}\lambda^n T(t)x\,dt\right\| \\ &\leqq \lambda^{n+1}\int_0^\infty e^{-\lambda t}\left\|\int_0^t T(s)x\,ds\right\|dt \\ &\leqq \frac{\lambda^{n+1}}{(n+1)!}M\int_0^\infty e^{-\lambda t}t^{n+1}dt\|x\| \\ &= \frac{M}{\lambda}\|x\|\end{aligned}$$

for all $x\in X$ and $\lambda>0$, so that $\{\lambda(\lambda-A)^{-1}\}_{\lambda>0}$ is a system of almost invariant integrals and $\{(\lambda-A)^{-1}\}_{\lambda>0}$ the associated system of companion integrals (see [8, Example V]).

Now Theorems 1 and 2 can be applied to the two pairs $\{\{A_t\}, \{B_t\}\}$ and $\{\{\lambda(\lambda-A)^{-1}\}, \{(\lambda-A)^{-1}\}\}$ to deliver the next theorem, which is concerned with the equivalence of the following conditions:

(S1) $y\in R(A)$;

(S2) $\sup_{\lambda>0}\|(\lambda-A)^{-1}y\|<\infty$;

(S3) $x=\text{w-}\lim_{k\to\infty}(A-\lambda_k)^{-1}y$ exists for some sequence $\{\lambda_k\}\to 0^+$;

(S4) $x=\text{s-}\lim_{\lambda\to 0^+}(A-\lambda)^{-1}y$ exists;

(S5) $\sup_{t>0}\left\|t^{-n-1}\int_0^t\int_0^s T(u)y\,du\,ds\right\|<\infty$;

(S6) $x=-\text{w-}\lim_{k\to\infty}(n+1)!\,t_k^{-n-1}\int_0^{t_k}\int_0^s T(u)y\,du\,ds$ exists for some sequence $\{t_k\}\to\infty$;

(S7) $x=-\text{s-}\lim_{t\to\infty}(n+1)!\,t^{-n-1}\int_0^t\int_0^s T(u)du\,ds$ exists;

(S8) $\sup_{t>0}\left\|t^{-n}\int_0^t T(s)y\,ds\right\|<\infty$.

Theorem 4. *Let $T(\cdot)$ be a non-degenerate, exponentially bounded, n-times integrated semigroup on X, and A be its generator. Suppose that $\left\|(n+1)!\,t^{-n-1}\int_0^t T(s)x\,ds\right\|\leqq M\|x\|$ for all $x\in X$ and $t>0$ and that $t^{-n-1}T(t)\to 0$*

strongly as $t\to\infty$.

(i) *If* X *is reflexive, then the conditions* (S1)-(S7) *are equivalent to each other.*

(ii) *If* $X=L_1(S, \Sigma, \mu)$ *with* μ *a* σ*-finite measure, and if* $M=1$, *then conditions* (S1), (S2), *and* (S5) *are equivalent; they are also equivalent to* (S3), (S4), (S6) *and* (S7) *in case* μ *is a finite measure and* $\left\|t^{-n-1}\int_0^t T(s)f\,ds\right\|_\infty \leqq K\|f\|_\infty$ *for all* $f\in L_\infty(S, \Sigma, \mu)$ *and all* $t>0$.

Remark. If $T(\cdot)$ satisfies the growth condition $\|T(t)\|\leqq Mt^n/(n+1)!$, $t\geqq 0$, then the hypothesis of Theorem 4 is satisfied and (S8) can be added as another equivalent condition. In fact, it is easy to see that (S1)$\Rightarrow$(S8)$\Rightarrow$(S5) in this case.

The following corollary for contraction C_0-semigroups on $L_1(S, \Sigma, \mu)$ is a specialization of Theorem 4; the first part of it is due to Krengel and Lin [6] (see also [9]).

Corollary 5. *Let* A *be the generator of a* C_0*-semigroup* $T(\cdot)$ *of contractions on* $L_1(S, \Sigma, \mu)$, *with* μ *a* σ*-finite measure. Then with* $n=0$ *the conditions* (S1), (S2), (S5), *and* (S8) *are equivalent. If, in addition,* μ *is finite and* $\sup_{t>0}\left\|t^{-1}\int_0^t T(s)f\,ds\right\|_\infty \leqq K\|f\|_\infty$ *for all* $f\in L_\infty(S, \Sigma, \mu)$, *then* (S1)-(S8), *with* $n=0$, *all are equivalent.*

For a given function $f\in R(A)$ in $L_p(S, \Sigma, \mu)$, $1\leqq p<\infty$, we now consider the almost everywhere convergence of $B_t f$. Suppose a pointwise ergodic theorem for the system $\{A_t\}$ holds so that $A_t x$ converges almost everywhere on S for all $x\in L_p$. Then for any solution g of the equation $Ag=f$, $B_t f= B_t Ag=(I-A_t)g$ surely converges almost everywhere. If $Ag=f$ has a solution g in $D(A)\cap\overline{R(A)}$ (This is always the case when $\{A_t\}$ is mean ergodic, i.e. $D(P)=X$), then $A_t g$ converges to $Pg=0$ almost everywhere on S and $B_t f$ converges to g almost everywhere on S. In what follows we deduce from Theorem 4, Corollary 5, and the Cesàro and Abelian pointwise ergodic theorems in [5] a pointwise convergence theorem for the approximate solutions $\{B_t f\}$ of $Ag=f$.

Let A be the generator of a C_0-semigroup $T(\cdot)$ of contractions on $L_1(S, \Sigma, \mu)$ such that, for some $K\geqq 1$, $\sup_{t>0}\|T(t)f\|_\infty\leqq K\|f\|_\infty$ for all $f\in L_1(S, \Sigma, \mu)\cap L_\infty(S, \Sigma, \mu)$. Then, given any $p\in[1, \infty)$, each $T(t)$ can be extended to a linear operator, still denoted by $T(t)$, on $L_p(S, \Sigma, \mu)$ with $\|T(t)\|_p\leqq K$ and $\{T(t); t\geqq 0\}$ is also a C_0-semigroup of operators on $L_p(S, \Sigma, \mu)$ (cf. [5, p. 96]). Let A still denote the generator of the semigroup thus obtained. Under these assumptions we can formulate the following Theorem.

Theorem 6. *Let* $1 \leqq p < \infty$. (i) $f \in L_p(S, \Sigma, \mu)$ *satisfies* $\sup_{t>0} \left\| \int_0^t T(u) f du \right\|_p < \infty$ *if and only if* $Ag = f$ *is solvable in* $L_p(S, \Sigma, \mu)$. (ii) *If* $Ag = f$ *is solvable, then the limits*

$$\lim_{t \to \infty} t^{-1} \int_0^t \int_0^u (-T(v)f)(s) dv du \quad \textit{and} \quad \lim_{\lambda \to 0^+} [(A-\lambda)^{-1} f](s)$$

exist and coincide almost everywhere on S. (iii) *If* $1 < p < \infty$, *or if* $p = 1$ *and* μ *is a finite measure, then the limits in* (ii) *converge in* $\|\cdot\|_p$ *and the limit function* g *is the unique solution of* $Ag = f$ *in the* $\|\cdot\|_p$*-closure of* $R(A | L_p)$.

We end this section with a concrete application to the equation $\Delta g = f$ in $L_p(R^n)$, $1 \leqq p < \infty$, where Δ is the Laplacian $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. It is known that Δ generates the *Gauss-Weierstrass semigroup* $T(\cdot)$, which is defined by $T(0) = I$ and

$$(T(t)f)(x) = (4\pi t)^{-n/2} \int_{R^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy, \quad f \in L_p(R^n),\ t > 0.$$

This is a C_0-semigroup of contractions on $L_p(R^n)$. Hence we can formulate the following specialization of Theorem 6.

Corollary 7. *Let* f *be a function in* $L_p(R^n)$, $1 \leqq p < \infty$. *Then the equation* $\Delta g = f$ *is solvable if and only if*

$$\sup_{\lambda > 0} \left\| \int_0^\infty e^{-\lambda t} (4\pi t)^{-n/2} \int_{R^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy dt \right\|_p < \infty,$$

if and only if

$$\sup_{t > 0} \left\| \int_0^t (4\pi s)^{-n/2} \int_{R^n} \exp\left(-\frac{|x-y|^2}{4s}\right) f(y) dy ds \right\|_p < \infty.$$

When $p > 1$, *a solution is given by*

$$g(x) = -\lim_{\lambda \to 0^+} \int_0^\infty e^{-\lambda t} (4\pi t)^{-n/2} \int_{R^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy dt,$$

$$= -\lim_{t \to \infty^+} t^{-1} \int_0^t \int_0^s (4\pi u)^{-n/2} \int_{R^n} \exp\left(-\frac{|x-y|^2}{4u}\right) f(y) dy du ds,$$

the convergence being valid in the sense of pointwise almost everywhere as well as in the sense of $\|\cdot\|_p$.

§4. Generators of Cosine Operator Functions

A strongly continuous family $\{C(t); t \in R\}$ of bounded linear operators on X is called a *cosine operator function* if $C(0) = I$ and $C(t+s) + C(t-s) = 2C(t)C(s)$,

$s, t \in R$. The associated sine function $S(\cdot)$ is defined by $S(t)x=\int_0^t C(s)x\,ds$, $x \in X$. The generator $A:=C''(0)$ is a densely defined closed operator. There exist $M \geqq 1$ and $w \geqq 0$ such that $\|C(t)\| \leqq Me^{w|t|}$, $t \in R$. The resolvent set $\rho(A)$ contains all λ^2 with $\lambda > w$, and for each such λ

$$\lambda(\lambda^2-A)^{-1}x=\int_0^\infty e^{-\lambda t}C(t)x\,dt \qquad \text{for all} \quad x \in X.$$

See, e.g., Sova [10] for these and other properties of $C(\cdot)$.

For $t>0$, let $A_t:=2t^{-2}\int_0^t S(s)ds$ and $B_t:=-2t^{-2}\int_0^t\int_0^s\int_0^u S(v)dvduds$. Then we have $B_tA \subset AB_t=I-A_t$, $A_tA \subset AA_t=2t^{-2}(C(t)-I)$, and $B_t^*x^*=\frac{t^2}{12}x^*$ for all $x^* \in N(A^*)$ (see [8], Example VII). Hence $\{A_t\}$ is a system of almost invariant integrals for $A+I$ and $\{B_t\}$ is its associated system of companion integrals if $\|A_t\| \leqq M$ and if $t^{-2}C(t) \to 0$ strongly as $t \to \infty$. Moreover, as was in the case of semigroup, the condition $\|A_t\| \leqq M$ also implies that $\{\lambda(\lambda-A)^{-1}\}_{\lambda>0}$ is a system of almost invariant integrals and $\{(\lambda-A)^{-1}\}_{\lambda>0}$ the associated system of companion integrals.

From Theorems 1 and 2 we can immediately deduce the next theorem, which is concerned with the equivalence among the following conditions:

(C1) $y \in R(A)$;

(C2) $\sup_{\lambda>0}\|(\lambda-A)^{-1}y\|<\infty$;

(C3) $x=\text{w-}\lim_{k\to\infty}(A-\lambda_k)^{-1}y$ exists for some sequence $\{\lambda_k\} \to 0^+$;

(C4) $x=\text{s-}\lim_{\lambda\to 0^+}(A-\lambda)^{-1}y$ exists;

(C5) $\sup_{t>0}\left\|t^{-2}\int_0^t\int_0^s\int_0^u S(v)y\,dvduds\right\|<\infty$;

(C6) $x=-\text{w-}\lim_{k\to\infty}2t_k^{-2}\int_0^{t_k}\int_0^s\int_0^u S(v)y\,dvduds$ exists for some sequence $\{t_k\}\to\infty$;

(C7) $x=-\text{s-}\lim_{t\to\infty}2t^{-2}\int_0^t\int_0^s\int_0^u S(v)y\,dvduds$ exists;

(C8) $\sup_{t>0}\left\|\int_0^t S(s)y\,ds\right\| \leqq \infty$.

Theorem 8. *Let $C(\cdot)$ be a cosine operator function on X. Suppose that $\left\|2t^{-2}\int_0^t S(s)ds\right\| \leqq M$ for all $t>0$ and $t^{-2}C(t)\to 0$ strongly as $t\to\infty$.*

(i) *If X is reflexive, then condition* (C1)-(C7) *are equivalent to each other.*

(ii) *If $X=L_1(S, \Sigma, \mu)$ with μ a σ-finite measure, and if $M=1$, then conditions* (C1), (C2), and (C5) *are equivalent; moreover, they are also equivalent to condi-*

tions (C3), (C4), (C6), *and* (C7) *when* μ *is finite and* $\left\|2t^{-2}\int_0^t S(s)f\,ds\right\|_\infty \leqq K\|f\|_\infty$ *for all* $f\in L_\infty(S, \Sigma, \mu)$.

Remark. If $\|C(t)\|\leqq M$ for all $t\geqq 0$, then both cases (i) and (ii), the condition (C8) can be added as an equivalent condition, because (C1)$\Rightarrow$(C8)$\Rightarrow$(C2).

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