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Self-duality and Integrable Systems

By

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§ 0. Introduction

In his lectures (1984-85) at Kyoto University, Professor M. Sato presented a program for generalizing the soliton theory ([9]; cf. [10]). The Kadomtsev-Petviashvili (KP) equation is a typical example of the soliton theory. The KP equation is written in the form of deformation equations of a linear ordinary differential equation. The time evolutions of a solution are interpreted as dynamical motions on an infinite dimensional Grassmann manifold ([7], [9]). The Lie algebra of microdifferential operators of one variable acts on this manifold transitively. He conjectured that any integrable systems can be written in the form of deformation equations of a linear system, and proposed to investigate a deformation of differential equations in higher dimensions. He showed a simple example of a deformation of holonomic systems in higher dimensions ([9]), and its generalization is treated in [4]. In this paper we study a deformation of $\mathcal{D}$-modules in higher dimensions.

First we review the KP equation. We denote by $\mathcal{E}$ the ring of microdifferential operators of one variable x. We fix a microdifferential operator P, and denote by t_P a time variable with respect to P. We study the following evolution equation associated to P:

$$\frac{\partial W}{\partial t_P}+WP=(WPW^{-1})_+W, \tag{0.1}$$

where $W=W(x, D_x)=1+\sum_{j<0}w_j(x)D_x^j\in\mathcal{E}$. We denote by $\mathcal{W}$ the set of such operators W. This space $\mathcal{W}$ is a group by the composition of $\mathcal{E}$. We get the KP-hierarchy taking $P=D_x^n$ ($u=1, 2, 3, \cdots$) in (0.1). The equation (0.1) defines a dynamical motion on $\mathcal{W}$. This infinitesimal action of the Lie algebra $\mathcal{E}$ on $\mathcal{W}$ is transitive.

The purpose of this article is to give a foundation for higher dimensional generalization of the KP hierarchy. Let now $\mathcal{E}$ be the ring of microdifferential operators in several variables. Similarly to the one dimensional case, fixing an operator $P\in\mathcal{E}$, we shall study the following equation

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$$\frac{\partial W}{\partial t_P}+WP=(WPW^{-1})_+W, \tag{0.2}$$

where the operator W is a 0-th order microdifferential operator. Here we choose a decomposition $\mathcal{E}=\mathcal{D}\oplus\mathcal{E}_\phi$ and $(WPW^{-1})_+\in\mathcal{D}$ is the component of WPW^{-1} according to this decomposition. In general the equation (0.2) imposes some constraints on the initial value $W(t_P=0)$, since the vector field defined by (0.2) is not tangent to the space $\mathcal{E}(0)$. There is no operator W_0 such that the equation (0.2) has a solution $W(t)\in\mathcal{E}$ with the initial value W_0 for any $P\in\mathcal{E}$. We take generators P in (0.2) only in the Lie subalgebra V of $\mathcal{E}$

$$V=\{F_0(x', D_0)x_0+\sum_{0\leq j<r}F_j(x', D_0)D_j+E(x', D_0);$$
$$F_k(x', D_0),\ E(x', D_0)\in\mathcal{E} \text{ for } 0\leqq k<r\},$$

where $x'=(x_1, x_2, \cdots, x_{r-1})$. This Lie algebra contains the transformation groups both of the self-dual Yang-Mills equations and of the self-dual Einstein equations (see [7], [8]). In §2 we will determine the subspace $\mathcal{W}$ of $\mathcal{E}(0)$ so that the vector field defined by (0.2) for any $P\in V$ is tangent to $\mathcal{W}$. The space $\mathcal{W}$ is a subgroup in $\mathcal{E}$. The Lie algebra V acts on $\mathcal{W}$ transitively.

In the case of $r=3$, our integrable system is nothing but a composed system of the self-dual Yang-Mills equations and the equations of self-dual metrics on Riemannian manifolds of dimension four. The Lie algebra V acts transitively on the space of self-dual connections on self-dual spaces. Thus we obtain a group-theoretical description of the twistor theory ([1], [5]).

Notations. We use the following notations: $\boldsymbol{Z}$ denotes the set of integers. $\boldsymbol{N}$ denotes the set of non-negative integers. We denote by $\boldsymbol{C}$ the complex number field. We denote by 1_n the unit matrix of size $n\times n$.

§1. Deformation $\mathcal{D}$-Modules

Throughout this paper we shall work in the category of formal power series, $\mathcal{O}=\boldsymbol{C}[[x]]=\boldsymbol{C}[[x_0, x_1, \cdots, x_{r-1}]](r\geqq 2)$. Let $\mathcal{D}$ be the ring of differential operators with coefficients in $\mathcal{O}$. Then every differential operator P of order m can be written as:

$$P=\sum_{\alpha\in\boldsymbol{N}^r, |\alpha|\leq m} a_\alpha(x)D_x^\alpha,$$

where $a_\alpha(x)$ are elements of $\mathcal{O}$, $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{r-1})\in\boldsymbol{N}^r$, $|\alpha|=\alpha_0+\alpha_1+\cdots+\alpha_{r-1}$, $D_x^\alpha=D_0^{\alpha_0}D_1^{\alpha_1}\cdots D_{r-1}^{\alpha_{r-1}}$ and $D_j=\partial/\partial x_j$ $(j=0, 1, \cdots, r-1)$.

The ring $\mathcal{E}$ of formal microdifferential operators is a set of formal Laurent series in $D_0, D_1, \cdots, D_{r-1}$ with only non-negative powers of $D_1, \cdots, D_{r-1}$. The precise definition is as follows. We denote by $\mathcal{E}(m)$ the space of formal series:

$$P=\sum_{\alpha\in Z\times N^{r-1},\,|\alpha|\leq m} a_\alpha(x)D_x^\alpha$$

where a_α's are elements of $\mathcal{O}$, and the summation is taken through $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_{r-1})$, where $\alpha_0\in \boldsymbol{Z}$, $\alpha_1\in \boldsymbol{N}, \cdots, \alpha_{r-1}\in \boldsymbol{N}$. We set

$$\mathcal{E}=\bigcup_{m\in Z}\mathcal{E}(m).$$

We endow the $\mathcal{O}$-module $\mathcal{E}$ with a structure of ring by extending the Leibniz formula. For two elements $P=\sum_\alpha a_\alpha D_x^\alpha$ and $Q=\sum_\beta b_\beta D_x^\beta$ of $\mathcal{E}$, we define the composition $P\circ Q$ by

$$P\circ Q=\sum_{\alpha,\beta\in Z\times N^{r-1},\gamma\in N^r}\binom{\alpha}{\gamma}a_\alpha b_\beta^{(\gamma)}D_x^{\alpha+\beta-\gamma},$$

where $b_\beta^{(\gamma)}=D_x^\gamma(b_\beta)$. The ring $\mathcal{E}$ has an increasing filtration by subspaces $\{\mathcal{E}(m)\}_{m\in Z}$. We have

$$\mathcal{E}(m)\mathcal{E}(n)=\mathcal{E}(n+m).$$

For any $\mathcal{O}$-submodule $\mathcal{L}$ of $\mathcal{E}$ we define the induced filtration $\{\mathcal{L}(m)\}_{m\in Z}$ of $\mathcal{L}$ by $\mathcal{L}(m)=\mathcal{L}\cap\mathcal{E}(m)$.

Let $\mathcal{E}_\phi$ be the $\mathcal{O}$-module consisting of the formal microdifferential operators of the form

$$\sum_{\alpha_0<0}a_\alpha(x)D_x^\alpha.$$

The ring $\mathcal{E}$ is the direct sum of $\mathcal{D}$ and $\mathcal{E}_\phi$. For any $P\in\mathcal{E}$, we define $P_+\in\mathcal{D}$ and $P_-\in\mathcal{E}_\phi$ by the decomposition of $\mathcal{E}$:

$$\mathcal{E}=\mathcal{D}\oplus\mathcal{E}_\phi$$

$$P=P_++P_-.$$

For any $\mathcal{O}$-submodule $\mathcal{L}$ of $\mathcal{E}$ we define the $\mathcal{O}$-module $\mathcal{L}_-$ by $\mathcal{L}_-=\mathcal{L}\cap\mathcal{E}_\phi$. Remark that $\mathcal{E}(0)=\mathcal{O}\oplus\mathcal{E}_\phi(0)$.

In the following we shall study a left $\mathcal{D}$-submodule I of $\mathcal{E}$ which satisfies the following condition:

$$\mathcal{E}(m)=I(m)\oplus\mathcal{E}_\phi(m)\qquad\text{for any}\quad m\in\boldsymbol{Z}. \tag{1.1}$$

For example $I=\mathcal{D}$ satisfies (1.1). We make clear the structure of such a $\mathcal{D}$-submodule I.

Lemma 1.1. *Suppose that a $\mathcal{D}$-submodule I of $\mathcal{E}$ satisfies the condition* (1.1). *Then I is generated as $\mathcal{D}$-module by a unique operator W such that*

$$W\in\mathcal{E}(0)\quad\textit{and}\quad W_+=1. \tag{1.2}$$

Proof. The operator W is obtained by decomposing the identity operator $1\in\mathcal{E}(0)$ into the sum of an operator in $I(0)$ and an operator in $\mathcal{E}_\phi(0)$ according to the condition (1.1):

$$\mathcal{E}(0)=I(0)\oplus\mathcal{E}_\phi(0)$$

$$1=W+U.$$

It is evident that W is contained in $\mathcal{E}(0)$ and that $W_+=1$. Since we have

$$\mathcal{E}(m)W=\mathcal{E}(m), \qquad \mathcal{E}_\phi(m)W=\mathcal{E}_\phi(m)$$

for any $m\in\boldsymbol{Z}$, we obtain that

$$\mathcal{E}(m)=\mathcal{D}(m)W\oplus\mathcal{E}_\phi(m) \qquad \text{for any} \quad m\in\boldsymbol{Z}.$$

Thus the $\mathcal{D}$-module $I'=\mathcal{D}W$ also satisfies (1.1). Because I contains I' and both satisfy (1.1), I coincides with I'. The uniqueness is clear. □

Remark that W in Lemma 1.1 is invertible by (1.2).

We investigate nonlinear evolution equations according to the program of M. Sato ([9], [10]). For any $P\in\mathcal{E}$ and any $\mathcal{D}$-submodule I_0 of $\mathcal{E}$ we define the time evolution I_t of I_0 by the following differential equations:

$$\frac{\partial V(t)}{\partial t}+V(t)P\in I_t \qquad \text{for any} \quad V(t)\in I_t. \tag{1.3}$$

We call $P\in\mathcal{E}$ the generator of the evolution equation (1.3).

In general we cannot find any $\mathcal{D}$-submodule I_t which solves (1.3). In this paper we shall study the case that we can find a solution I_t of (1.3) which is a $\mathcal{D}$-module satisfying (1.1) for any t. Then I_t is generated by an operator $W(t)\in\mathcal{E}(0)$ by Lemma 1.1 and we can rewrite the equation (1.3) in terms of the generator $W(t)$.

Lemma 1.2. *We fix an operator $P\in\mathcal{E}$. We assume that the solution I_t of the evolution equation* (1.3) *is a $\mathcal{D}$-submodule which satisfies* (1.1) *for any t. Then the equation* (1.3) *reduces to the following equation*

$$\frac{\partial W(t)}{\partial t}+W(t)P=(W(t)PW(t)^{-1})_+W(t), \tag{1.4}$$

where the operator $W(t)$ is the generator of I_t in Lemma 1.1.

Proof. From the equation (1.3) there exists an operator $B(t)\in\mathcal{D}$ such that

$$\frac{\partial W(t)}{\partial t}+W(t)P=B(t)W(t).$$

Thus we have

$$B(t)=W(t)PW(t)^{-1}+\frac{\partial W(t)}{\partial t}W(t)^{-1}.$$

Since the operator $W(t)$ is contained in $1+\mathcal{E}_\phi$, the operator $(\partial W(t)/\partial t)W(t)^{-1}$ is contained in $\mathcal{E}_\phi$. Thus we obtain that $B(t)=(W(t)PW(t)^{-1})_+$. □

Remark. The equation (1.4) is rewritten as

$$\frac{\partial W}{\partial t}=(WPW^{-1})_{+}W-WP$$
$$=-(WPW^{-1})_{-}W. \tag{1.5}$$

The evolution equation (1.4) is associated with an infinitesimal action ρ of $\mathcal{E}$ on the space $\mathcal{E}$. For $P\in\mathcal{E}$ the vector field $\rho(P)$ is given as follows.

$$W \longrightarrow -(WPW^{-1})_{-}W\in T_W\mathcal{E},$$

where the tangent space $T_W\mathcal{E}$ is identified with $\mathcal{E}$ by the structure of vector space of $\mathcal{E}$.

Theorem 1.3. *For any $P, Q\in\mathcal{E}$ we have*

$$\rho([P, Q])=-[\rho(P), \rho(Q)].$$

Proof. We denote by ε_1 and ε_2 the time parameters with respect to P and Q, respectively. We set $\tilde{P}=WPW^{-1}$ and $\tilde{Q}=WQW^{-1}$. We have

$$\exp(\varepsilon_1\rho(P))W\equiv(1-\varepsilon_1(\tilde{P})_{-})W \mod \varepsilon_1^2,$$
$$\exp(\varepsilon_2\rho(Q))W\equiv(1-\varepsilon_2(\tilde{Q})_{-})W \mod \varepsilon_2^2.$$

Hence we have, modulo ε_1^2, ε_2^2

$$\exp(\varepsilon_1\rho(P))(1-\varepsilon_2(WQW^{-1})_{-})$$
$$\equiv 1-\varepsilon_2((1-\varepsilon_1\tilde{P}_{-})\tilde{Q}(1+\varepsilon_1\tilde{P}_{-}))_{-}$$
$$\equiv 1-\varepsilon_2\tilde{Q}_{-}+\varepsilon_1\varepsilon_2[\tilde{P}_{-}, \tilde{Q}]_{-}.$$

Thus we obtain, modulo ε_1^2, ε_2^2

$$\exp(\varepsilon_1\rho(P))\exp(\varepsilon_2\rho(Q))W$$
$$\equiv\exp(\varepsilon_1\rho(P))(1-\varepsilon_2(WQW^{-1})_{-})W$$
$$\equiv(1-\varepsilon_2\tilde{Q}_{-}+\varepsilon_1\varepsilon_2[\tilde{P}_{-}, \tilde{Q}]_{-})(1-\varepsilon_1\tilde{P}_{-})W$$
$$\equiv(1-\varepsilon_1\tilde{P}_{-}-\varepsilon_2\tilde{Q}_{-}+\varepsilon_1\varepsilon_2([\tilde{P}_{-}, \tilde{Q}]_{-}+\tilde{Q}_{-}\tilde{P}_{-}))W.$$

Similarly we have, modulo ε_1^2, ε_2^2

$$\exp(\varepsilon_2\rho(Q))\exp(\varepsilon_1\rho(P))W$$
$$\equiv(1-\varepsilon_2\tilde{Q}_{-}-\varepsilon_1\tilde{P}_{-}+\varepsilon_1\varepsilon_2([\tilde{Q}_{-}, \tilde{P}]_{-}+\tilde{P}_{-}\tilde{Q}_{-}))W.$$

By the formula

$$\exp(\varepsilon_1\rho(P))\exp(\varepsilon_2\rho(Q))W-\exp(\varepsilon_2\rho(Q))\exp(\varepsilon_1\rho(P))W$$
$$\equiv\varepsilon_1\varepsilon_2[\rho(P), \rho(Q)]W \mod \varepsilon_1^2, \varepsilon_2^2,$$

we obtain

$$[\rho(P), \rho(Q)]W=\{[\tilde{P}_-, \tilde{Q}]_-+\tilde{Q}_-\tilde{P}_- -[\tilde{Q}_-, \tilde{P}]_- -\tilde{P}_-\tilde{Q}_-\}W$$
$$=\{[\tilde{P}_-, \tilde{Q}_+ +\tilde{Q}_-]_- -[\tilde{Q}_-, \tilde{P}_+ +\tilde{P}_-]_- +[\tilde{Q}_-, \tilde{P}_-]\}W. \tag{1.6}$$

Since $[\tilde{P}_+, \tilde{Q}_+]_-=0$, the right hand side of (1.6) is equal to

$$[\tilde{P}, \tilde{Q}]_-W=-\rho([P, Q])W. \qquad \square$$

When $P, Q\in\mathcal{E}$ commute with each other, the following equations are compatible by Theorem 1.3:

$$\frac{\partial W}{\partial s}=(WPW^{-1})_+W-WP,$$
$$\frac{\partial W}{\partial t}=(WQW^{-1})_+W-WQ,$$

where $W=W(s, t)$.

In the case $r=1$, for any $P\in\mathcal{E}$ and for any $I_0=\mathcal{D}W(0)$ the solution I_t of the equation (1.3) satisfies the condition (1.1) for any t. With the choice $P=D_0^n$ $(n=1, 2, \cdots)$, we obtain the KP-hierarchy ([7], [8]):

$$\frac{\partial W(t)}{\partial t}+W(t)D_0^n=(W(t)D_0^nW(t)^{-1})_+W(t).$$

In higher dimensional case, we must choose a nice pair of the generator P and $I_0=\mathcal{D}W(0)$ in order that I_t satisfies the condition (1.1) for any t. We shall see the evolution equation (1.4) constrains the initial value $W(0)$ in the following example.

Example 1.1. We consider (1.4) in the case $r=2$. We take D_0^2 as the generator of the equation (1.4). We write

$$W(t)=\sum_{i+j\leqq 0,\, j\geqq 0} w_{i,j}D_0^iD_1^j, \qquad w_{0,0}\equiv 1.$$

The operator $W(t)D_0^2$ is decomposed into the sum

$$\mathcal{E}=\mathcal{D}W\oplus\mathcal{E}_\phi$$
$$W(t)D_0^2=(W(t)D_0^2W(t)^{-1})_+W(t)+U.$$

Then we have

$$(W(t)D_0^2W(t)^{-1})_+=\left(D_0^2-2\frac{\partial w_{-1,1}}{\partial x_0}D_1-2\frac{\partial w_{-1,0}}{\partial x_0}\right),$$
$$U=\sum_{i+j\leqq 1,\, i<0} u_{i,j}(x)D_0^iD_1^j$$
$$=\sum_{i+j\leqq 1,\, i<0}\left(-2\frac{\partial w_{i-1,j}}{\partial x_0}+2\frac{\partial w_{-1,1}}{\partial x_0}w_{i,j-1}-\frac{\partial^2 w_{i,j}}{\partial x_0^2}\right.$$
$$\left.2\frac{\partial w_{-1,0}}{\partial x_0}w_{i,j}+2\frac{\partial w_{-1,1}}{\partial x_0}\frac{\partial w_{i,j}}{\partial x_1}\right)D_0^iD_1^i.$$

The equation (1.4) is equivalent to the following:

$$\frac{\partial w_{i,j}}{\partial t}+u_{i,j}=0 \qquad \text{for} \quad i+j\leqq 0,$$

$$\frac{\partial w_{i-1,j}}{\partial x_0}-\frac{\partial w_{-1,1}}{\partial x_0}w_{i,j-1}=0, \qquad \text{for} \quad i+j=1.$$

The second equation constrains the initial data $W(0)$.

§ 2. Integrable Systems in Higher Dimensions

In the example 1.1 we have considered the equation (1.3) for one generator $P=D_0^2$. In this section we will introduce a space V of generators, and determine the space of $\mathcal{D}$-submodules I of $\mathcal{E}$ such that the condition (1.1) is preserved under the time evolution (1.3) for any $P\in V$.

First we review two known examples, the self-dual Yang-Mills equations and the self-dual Einstein equations. We can interpret both of the equations as integrable systems of three variables (see [14]).

Example 2.1. Self-dual Yang-Mills equations (see [5], [11]). The self-dual Yang-Mills equations are written in the following form

$$\frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t)=\frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t),$$

$$\left[\frac{\partial}{\partial s}+A_1(x_1, x_2, s, t), \frac{\partial}{\partial t}+A_2(x_1, x_2, s, t)\right]=0 \tag{2.1}$$

for gauge fields $A_1, A_2\in\mathrm{Mat}(n\times n)$ on four-dimensional manifolds.

The evolution equation (1.4) is generalized to the case that W and P have matrix coefficients. We introduce the space $\mathcal{W}_{YM}(n)$ and the Lie algebra $V_{YM}(n)$:

$$\mathcal{W}_{YM}(n)=\{W(x_1, x_2, D_0)=\sum_{i\in N} w_i(x_1, x_2)D_0^{-i};$$
$$w_i\in\mathrm{Mat}(n\times n, \boldsymbol{C}[[x_1, x_2]]), w_0\equiv 1_n\}.$$

$$V_{YM}(n)=\{F_1(x_1, x_2, D_0)D_1+F_2(x_1, x_2, D_0)D_2+E(x_1, x_2, D_0);$$
$$F_1, F_2\in\mathrm{Mat}(n\times n, \mathcal{D}), E\in\mathrm{Mat}(n\times n, \mathcal{E})\}.$$

The evolution equation (1.4) for $P\in V_{YM}(n)$ with any initial value $W\in\mathcal{W}_{YM}(n)$ has a solution in $\mathcal{W}_{YM}(n)$. We consider the equation (1.4) for $P=D_0D_1, D_0D_2\in V_{YM}(n)$:

$$\frac{\partial W}{\partial s}+WD_0D_1=(WD_0D_1W^{-1})_+W,$$

$$\frac{\partial W}{\partial t}+WD_0D_2=(WD_0D_2W^{-1})_+W. \tag{2.2}$$

In terms of the coefficients w_j of W the equation (2.2) is written in the form

$$\frac{\partial w_i}{\partial s}=\frac{\partial w_{i+1}}{\partial x_1}-\frac{\partial w_1}{\partial x_1}w_i,$$

$$\frac{\partial w_i}{\partial t}=\frac{\partial w_{i+1}}{\partial x_2}-\frac{\partial w_1}{\partial x_2}w_i, \tag{2.3}$$

for $i>0$. We set $A_j=\partial w_1/\partial x_j\ (j=1, 2)$. By eliminating w_2 from the equation (2.3) for $i=1$, we obtain the equation (2.1).

Example 2.2. Self-dual Einstein equations (see [2], [12]). The self-dual Einstein equations are written in the form (see [6])

$$\frac{\partial A_1}{\partial x_2}(x_1, x_2, s, t)=\frac{\partial A_2}{\partial x_1}(x_1, x_2, s, t),$$

$$\frac{\partial B_1}{\partial x_2}(x_1, x_2, s, t)=\frac{\partial B_2}{\partial x_1}(x_1, x_2, s, t),$$

$$\left[\frac{\partial}{\partial s}+A_1(x_1, x_2, s, t)\frac{\partial}{\partial x_1}+B_1(x_1, x_2, s, t)\frac{\partial}{\partial x_2},\right.$$

$$\left.\frac{\partial}{\partial t}+A_2(x_1, x_2, s, t)\frac{\partial}{\partial x_1}+B_2(x_1, x_2, s, t)\frac{\partial}{\partial x_2}\right]=0,$$

$$\frac{\partial B_j}{\partial x_2}(x_1, x_2, s, t)+\frac{\partial A_j}{\partial x_1}(x_1, x_2, s, t)=0 \quad (j=1, 2). \tag{2.4}$$

In the following we forget the last equation in (2.4) for simplicity.

We introduce the space $\mathcal{W}_E$ and the Lie algebra V_E:

$$\mathcal{W}_E=\left\{W=\sum_{j,k\in N}\frac{1}{j!k!}G_1^jG_2^kD_1^jD_2^k\ ;\ G_i=\sum_{j<0}g_{i,j}(x_1, x_2)D_0^j\ (i=1, 2)\right\}.$$

$$V_E=\{F_1(x_1, x_2, D_0)D_1+F_2(x_1, x_2, D_0)D_2\ ;\ F_1, F_2\in\mathcal{E}\}.$$

The evolution equation (1.4) for $P\in V_E$ with any initial value $W\in\mathcal{W}_E$ has a solution in $\mathcal{W}_E$. We consider (1.4) for $P=D_0D_1, D_0D_2\in V_E$:

$$\frac{\partial W}{\partial s}+WD_0D_1=(WD_0D_1W^{-1})_+W,$$

$$\frac{\partial W}{\partial t}+WD_0D_2=(WD_0D_2W^{-1})_+W. \tag{2.5}$$

In terms of the coefficients $g_{i,j}$ the equation (2.5) is written in the form:

$$\frac{\partial g_{i,j}}{\partial s}=\frac{\partial g_{i,j-1}}{\partial x_1}-\frac{\partial g_{1,-1}}{\partial x_1}\frac{\partial g_{i,j}}{\partial x_1}-\frac{\partial g_{2,-1}}{\partial x_1}\frac{\partial g_{i,j}}{\partial x_2},$$

$$\frac{\partial g_{i,j}}{\partial t}=\frac{\partial g_{i,j-1}}{\partial x_2}-\frac{\partial g_{1,-1}}{\partial x_2}\frac{\partial g_{i,j}}{\partial x_1}-\frac{\partial g_{2,-1}}{\partial x_2}\frac{\partial g_{i,j}}{\partial x_2} \quad \text{for } i=1, 2,\ j<0. \tag{2.6}$$

We set $A_j=-\partial g_{1,-1}/\partial x_j$, $B_j=-\partial g_{2,-1}/\partial x_j$ $(j=1, 2)$. By eliminating $g_{i,-2}$ $(i=1, 2)$ from the equation (2.6) for $i=1, 2$, $j=-1$, we obtain the equation (2.4).

We shall unify these two examples and obtain more general systems. We introduce the Lie subalgebra V of $\mathcal{E}$ which contains both the Lie algebras V_{YM} and V_E:

$$V=\{F_0(x', D_0)x_0+\sum_{0<j<r} F_j(x', D_0)D_j+E(x', D_0);$$
$$F_k(x', D_0),\ E(x', D_0)\in\mathcal{E},\ (0\leq k<r)\},$$

where $x'=(x_1, x_2, \cdots, x_{r-1})$.

Lemma 2.1. *V is isomorphic to $\mathcal{D}(1)\otimes_{\mathcal{O}}\boldsymbol{C}[[x]][x_0^{-1}]$ as a Lie algebra.*

Proof. Set $X=\boldsymbol{C}^r=\{(z_0, z_1, \cdots, z_{r-1})\in\boldsymbol{C}^r\}$, and $Y=\{z\in X;\ z_0=0\}$. We take the transformation

$$x_0=z_0^2D_{z_0},\ D_{x_0}=z_0^{-1},\quad x'=z',\quad D_{x'}=D_{z'}.$$

Then the transform of V is $\mathcal{D}(1)\otimes\boldsymbol{C}[[z]][z_0^{-1}]$. □

The Lie algebra $\mathcal{D}(1)\otimes\boldsymbol{C}[[x]][x_0^{-1}]$ is the direct sum of the Lie algebra $\Theta\otimes_{\mathcal{O}}\boldsymbol{C}[[x]][x_0^{-1}]$ of vector fields and the commutative Lie algebra $\boldsymbol{C}[[x]][x_0^{-1}]$, where the Lie algebra Θ is defined by

$$\Theta=\sum_{0\leq j<r}\mathcal{O}D_j.$$

The Lie algebra $\Theta\otimes_{\mathcal{O}}\boldsymbol{C}[[x]][x_0^{-1}]$ corresponds to the infinitesimal coordinate transformations, and the Lie algebra $\boldsymbol{C}[[x]][x_0^{-1}]$ corresponds to the infinitesimal gauge transformations of a line bundle.

Now we shall determine the set of $\mathcal{D}$-submodules in $\mathcal{E}$ such that the condition (1.1) is preserved under (1.3) with respect to any $P\in V$. We introduce the subspaces $\mathcal{W}_m$ and $\mathcal{W}_{YM}$ of $\mathcal{E}$:

$$\mathcal{W}_m=\Big\{W(x, D_x)=\sum_{\alpha\in N^r}\frac{1}{\alpha!}G^\alpha x_0^{\alpha_0}D_x^{\alpha'}\in\mathcal{E};$$
$$G^\alpha=G_0^{\alpha_0}G_1^{\alpha_1}\cdots G_{r-1}^{\alpha_{r-1}}\quad\text{where}\quad G_j=G_j(x', D_0)\in\mathcal{E}(-1)\Big\},$$
$$\mathcal{W}_{YM}=\{W(x', D_0)=\sum_{i\in N}w_i(x')D_0^{-i};\ w_i\in\boldsymbol{C}[[x_1, \cdots, x_{r-1}]],\ w_0\equiv1\}.$$

Proposition 2.2. *Let $I_0=\mathcal{D}W_0$ be a $\mathcal{D}$-submodule of $\mathcal{E}$. Assume that the time evolution of I_0 for any $P\in V$ also satisfies the condition* (1.1). *Then W_0 factorizes into the product of $W_m\in\mathcal{W}_m$ and $W_{YM}\in\mathcal{W}_{YM}$;*

$$W_0=W_mW_{YM}.$$

Remark. The operator W_m corresponds to the equation of self-dual metrics (see § 3).

Proof of Proposition 2.2.
Step 1.

Lemma 2.3. *Let $W(t)$ be the solution of* (1.4) *for $P\in V$ with $W(0)=W_0$. Then $(W(t)PW(t)^{-1})_-$ is contained in $\mathcal{E}_\phi(0)$.*

Proof. It follows from the equation (1.5) that

$$\frac{\partial W(t)}{\partial t}W(t)^{-1}=-(W(t)PW(t)^{-1})_-.$$

Since the operator $(\partial W(t)/\partial t)W(t)^{-1}$ is contained in $\mathcal{E}(0)$, we obtain Lemma 2.3. □

Step 2. For any $P\in\mathcal{E}$, let $\tilde{P}$ be $W_0PW_0^{-1}$.

Lemma 2.4. *Suppose that $P, Q\in V$ commute with each other. The operator $[\tilde{P}_+, \tilde{Q}_-]_-$ is contained in $\mathcal{E}_\phi(0)$.*

Proof. We consider the solution $W=W(s, t)$ of the equation (1.5) for the operators P, Q:

$$\frac{\partial W}{\partial s}=-(WPW^{-1})_-W,$$

$$\frac{\partial W}{\partial t}=-(WQW^{-1})_-W, \tag{2.7}$$

where $W=W(s, t)$. Since $[P, Q]=0$, the system (2.7) is compatible by Theorem 1.3. It follows from Lemma 2.3 that WPW^{-1} is contained in $\mathcal{D}+\mathcal{E}_\phi(0)$. We have

$$\begin{aligned}\frac{\partial}{\partial t}(WPW^{-1})&=\frac{\partial W}{\partial t}PW^{-1}-WPW^{-1}\frac{\partial W}{\partial t}W^{-1}\\&=-(WQW^{-1})_-WPW^{-1}+WPW^{-1}(WQW^{-1})_-\\&=[WPW^{-1}, (WQW^{-1})_-].\end{aligned}$$

Hence $[\tilde{P}, \tilde{Q}_-]$ is contained in $\mathcal{D}+\mathcal{E}_\phi(0)$. Since $[\tilde{P}_-, \tilde{Q}_-]$ is contained in $\mathcal{E}_\phi(0)$ by Lemma 2.3, $[\tilde{P}_+, \tilde{Q}_-]$ is contained in $\mathcal{D}+\mathcal{E}_\phi(0)$. □

Step. 3. Recall that $\tilde{P}$ denotes $W_0PW_0^{-1}$ for any $P\in\mathcal{E}$. We define the operators $G_i, H_i\in\mathcal{E}$ $(0\leqq i<r)$ as follows:

$$\begin{aligned}&\tilde{D}_0=D_0-G_0(x, D_x),\\&\tilde{x}_i=x_i+G_i(x, D_x) \qquad \text{for } i=1, 2, \cdots, r-1,\\&\tilde{x}_0=x_0+H_0(x, D_x),\\&\tilde{D}_i=D_i+H_i(x, D_x) \qquad \text{for } i=1, 2, \cdots, r-1.\end{aligned} \tag{2.8}$$

Lemma 2.5. *Assume that W_0 satisfies the condition in Proposition 2.2. The operators G_i, H_i are written in the following form:*

$$G_i=G_i(x', D_0)\in\mathcal{E}(-1),$$
$$H_i=\sum_{0<j<r}K_{ij}(x', D_0)D_j+L_i(x, D_0) \quad \text{for} \quad 0\leqq i<r,$$
$$\text{where} \quad K_{ij}(x', D_0)\in\mathcal{E},\ L_i(x, D_0)\in\mathcal{E}(-1),$$
$$\text{and} \quad x'=(x_1, \cdots, x_{r-1}).$$

Proof. We shall show the following statement $(2.9)_n$ by the induction on n:

$$G_i=G_i'+G_i'',\ H_i=H_i'+H_i''$$

for some G_i', H_i', G_i'' and H_i'' $(0\leqq i<r)$ such that

$$G_0'',\ H_i''\in\mathcal{E}(-n),\ G_i'',\ H_0''\in\mathcal{E}(-n-1) \quad \text{for} \quad 0<i<r,$$
$$G_i'=G_i'(x', D_0)\in\mathcal{E}(-1),$$
$$H_i'=\sum_{0<j<r}K_{ij}(x', D_0)D_j+L_i(x, D_0) \quad \text{for} \quad 0\leqq i<r$$
$$\text{with} \quad K_{ij}(x', D_0),\ L_i(x, D_0)\in\mathcal{E}(-1). \tag{2.9$_n$}$$

It is evident that the statement $(2.9)_0$ is true. By assuming $(2.9)_n$ we shall prove $(2.9)_{n+1}$.

We expand the operators:

$$G_j''=\sum_{k\leqq 0,\ \alpha'\in N^{r-1}} g_{k,\alpha'}^{(j)}D_0^kD_{x'}^{\alpha'},$$
$$H_j''=\sum_{k\leqq 0,\ \alpha'\in N^{r-1}} h_{k,\alpha'}^{(j)}D_0^kD_{x'}^{\alpha'}. \tag{2.10}$$

Since G_0'' belongs to $\mathcal{E}(-n)$, we have

$$\tilde{D}_0^{n+2}=(D_0-G_0'-G_0'')^{n+2}$$
$$\equiv(D_0-G_0')^{n+2}-(n+2)G_0''(D_0-G_0')^{n+1}$$
$$\equiv(D_0-G_0')^{n+2}-(n+2)G_0''D_0^{n+1} \quad \text{modulo } \mathcal{E}(0).$$

The operator $(D_0-G_0')^{n+2}$ belongs to $\mathcal{D}+\mathcal{E}(0)$, because G_0' does not contain D_j $(0<j<r)$. Hence we obtain

$$\tilde{D}_0^{n+2}\equiv-(n+2)G_0''D_0^{n+1} \quad \text{modulo } \mathcal{D}+\mathcal{E}(0).$$

Since $\tilde{D}_0^{n+2}$ is contained in $\mathcal{D}+\mathcal{E}(0)$ for any $n\in\boldsymbol{Z}$ by Lemma 2.3, $G_0''D_0^{n+1}$ is also contained in $\mathcal{D}+\mathcal{E}(0)$. Therefore we obtain

$$g_{-n-|\alpha'|,\alpha'}^{(0)}=0 \quad \text{for} \quad |\alpha'|\geqq 2. \tag{2.11}$$

Similarly we have, modulo $\mathcal{D}+\mathcal{E}(0)$

$$\tilde{D}_0^{n+1}\tilde{D}_j \equiv -(n+1)G_0''D_0^nD_j + H_j''D_0^{n+1},$$

$$\tilde{x}_0\tilde{D}_0^{n+2} \equiv -(n+2)x_0G_0''D_0^{n+1} + H_0''D_0^{n+2},$$

$$\tilde{x}_j\tilde{D}_0^{n+1}\tilde{D}_k \equiv G_j''D_0^{n+1}D_k + x_j(H_k''D_0^{n+1} - (n+1)G_0''D_0^nD_k).$$

By Lemma 2.3, $\tilde{D}_0^{n+1}\tilde{D}_j$, $\tilde{x}_0\tilde{D}_0^{n+2}$ and $\tilde{x}_j\tilde{D}_0^{n+1}\tilde{D}_k$ are contained in $\mathcal{D}+\mathcal{E}(0)$. Thus we obtain

$$\begin{aligned} &h^{(j)}_{-|\alpha'|-n,\alpha'}=0 \qquad \text{for} \quad |\alpha'|\geqq 3, \\ &h^{(j)}_{-2-n,\alpha'+e_j}=(n+1)g^{(0)}_{-1-n,\alpha'} \qquad \text{for} \quad |\alpha'|=1, \\ &h^{(0)}_{-|\alpha'|-n-1,\alpha'}=0 \qquad \text{for} \quad |\alpha'|\geqq 2, \\ &g^{(j)}_{-|\alpha'|-n-1,\alpha'}=0 \qquad \text{for} \quad |\alpha'|\geqq 2, \end{aligned} \tag{2.12}$$

where $j=1, 2, \cdots, r-1$ and $e_j=(0, \cdots, \overset{j}{\check{1}}, \cdots, 0)\in \boldsymbol{N}^{r-1}$.

We denote by $\mathcal{V}$ and $\mathcal{V}'$ the subspaces of $\mathcal{D}+\mathcal{E}(0)$:

$$\mathcal{V}=\{\sum_{0<k<r} F_k(x, D_0)D_k + E(x, D_0);\ F_k(x, D_0),\ E(x, D_0)\in\mathcal{E}\},$$

$$\mathcal{V}'=\{\sum_{0<k<r} F_k(x', D_0)D_k + E(x, D_0);\ F_k(x', D_0),\ E(x, D_0)\in\mathcal{E}\}.$$

For $P, Q\in\mathcal{V}'$ the commutator $[P, Q]$ is contained in $\mathcal{V}$. Since H_j' $(0\leqq j<r)$ are contained in $\mathcal{V}'$, $[H_i', H_j']$ and $[x_0, H_j']$ $(0\leqq i, j<r)$ are contained in $\mathcal{V}$.

For $j>0$ we have

$$\begin{aligned}[\tilde{x}_0, \tilde{D}_j] &= [x_0, H_j] - \frac{\partial H_0}{\partial x_j} + [H_0, H_j] \\ &= [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} + ([H_0'', H_j''] + [H_0', H_j''] + [H_0'', H_j']) \\ &\quad + \left([x_0, H_j'] - \frac{\partial H_0'}{\partial x_j} + [H_0', H_j']\right) \\ &\equiv [x_0, H_j''] - \frac{\partial H_0''}{\partial x_j} \quad \text{modulo } \mathcal{E}(-n-2)+\mathcal{V}.\end{aligned}$$

Since $[\tilde{x}_0, \tilde{D}_j]=0$, $[x_0, H_j'']-(\partial H_0''/\partial x_j)$ belongs to $\mathcal{E}(-n-2)+\mathcal{V}$. By (2.10) we have, modulo $\mathcal{E}(-n-2)+\mathcal{V}$

$$[x_0, H_j''] - \frac{\partial H_0''}{\partial x_j}$$

$$\equiv \sum_{|\alpha'|\geqq 2}\left((n+|\alpha'|)h^{(j)}_{-n-|\alpha'|,\alpha'} - \frac{\partial h^{(0)}_{-n-1-|\alpha'|,\alpha'}}{\partial x_j}\right)D_0^{-1-n-|\alpha'|}D_{x'}^{\alpha'}. \tag{2.13}$$

We obtain from (2.12) and (2.13) that

$$\begin{aligned} &h^{(j)}_{-2-n,\alpha'}=0 \qquad \text{for} \quad |\alpha'|=2, \quad 0<j<r, \\ &g^{(0)}_{-1-n,\alpha'}=0 \qquad \text{for} \quad |\alpha'|=1. \end{aligned} \tag{2.14}$$

Similarly by the equations

$$[\tilde{D}_0, \tilde{D}_j]=\frac{\partial H_j}{\partial x_0}+\frac{\partial G_0}{\partial x_j}-[G_0, H_j]=0 \quad \text{for} \quad 0<j<r,$$

$$[\tilde{D}_0, \tilde{x}_j]=\frac{\partial G_j}{\partial x_0}+[G_j, G_0]-[G_0, x_j]=0 \quad \text{for} \quad 0<j<r,$$

$$[\tilde{D}_0, \tilde{x}_0]=1+[D_0, H_0]-[G_0, x_0]+[G_0, H_0]=1,$$

we obtain

$$\frac{\partial h^{(j)}_{-1-n,e_k}}{\partial x_0}=0, \quad \frac{\partial h^{(0)}_{-2-n,e_k}}{\partial x_0}=0, \quad \frac{\partial g^{(j)}_{-1-n,0}}{\partial x_0}=0. \tag{2.15}$$

For $0<i<r$ we have modulo $\mathcal{E}(-1)$

$$\begin{aligned}\tilde{D}_0^n\tilde{D}_i&=(D_0-G_0'-G_0'')^n(D_i+H_i'+H_i'')\\&\equiv((D_0-G_0')^n-nG_0''(D_0-G_0')^{n-1})(D_i+H_i'+H_i'')\\&\equiv-nG_0''D_0^{n-1}D_i+(D_0-G_0')^n(D_i+H_i')+H_i''D_0^n.\end{aligned} \tag{2.16}$$

By (2.12) and (2.14) we have

$$H_i''D_0^n\equiv\sum_{0<j<r}h^{(i)}_{-1-n,e_j}D_0^{-1}D_j \quad \text{modulo } \mathcal{E}(-1),$$

$$-nG_0''D_0^{n-1}D_i\equiv-ng^{(0)}_{-n,0}D_0^{-1}D_i \quad \text{modulo } \mathcal{E}(-1).$$

We set K_{ij} and L_i in $(2.9)_n$ as follows:

$$K_{ij}(x', D_0)=\sum_{m<0}k_{i,j,m}(x')D_0^m,$$

$$L_i(x, D_0)=\sum_{m<0}l_{i,m}(x')D_0^m.$$

By $(2.9)_n$ we have

$$\begin{aligned}(\tilde{D}_0\tilde{D}_i)_+&=(D_0-G_0)(D_i+H_i'+H_i'')\\&=D_0D_i+\{D_0(H_i'+\sum_{1\le j<r}K_{ij}+L_i)\}_+\\&=D_0D_i+\sum_{1\le j<r}(h^{(i)}_{-1,e_j}+k_{i,j,-1})D_j+l_{i,-1}.\end{aligned} \tag{2.17}$$

Sublemma. *We have*

$$\frac{\partial g^{(0)}_{-n,0}}{\partial x_0}=0. \tag{2.18}$$

Proof. The commutator $[(\tilde{D}_0\tilde{D}_i)_+, \tilde{D}_0^n\tilde{D}_i]$ is contained in $\mathcal{D}+\mathcal{E}(0)$ by Lemma 2.4. By (2.16) and (2.17) we have, modulo $\mathcal{E}(0)$

$$\begin{aligned}&[(\tilde{D}_0\tilde{D}_i)_+, \tilde{D}_0^n\tilde{D}_i]\\&\equiv[(\tilde{D}_0\tilde{D}_i)_+, -ng^{(0)}_{-n,0}D_0^{-1}D_i+\sum_{0<j<r}h^{(i)}_{-1-n,e_j}D_0^{-1}D_j+(D_0-G_0')^n(D_i+H_i')]\end{aligned}$$

$$\equiv[D_0,\ D_i,\ -ng^{(0)}_{-n,0}D_0^{-1}D_i]$$
$$+[(\tilde{D}_0\tilde{D}_i)_+,\ \sum_{0<j<r} h^{(i)}_{-1-n,e_j}D_0^{-1}D_j+(D_0-G_0')^n(D_i+H_i')].$$

Since $(\tilde{D}_0\tilde{D}_i)_+$ and $\sum_{0<j<r}h^{(i)}_{-1-n,e_j}D_0^{-1}D_j+(D_0-G_0')^n(D_i+H_i')$ are contained in $\mathcal{V}$, the operator

$$[D_0D_i,\ -ng^{(0)}_{-n,0}D_0^{-1}D_i]$$
$$=-n\frac{\partial g^{(0)}_{-n,0}}{\partial x_0}D_0^{-1}D_i^2-n\frac{\partial g^{(0)}_{-n,0}}{\partial x_i}D_i-n\frac{\partial^2 g^{(0)}_{-n,0}}{\partial x_0\partial x_i}$$

is contained in $\mathcal{D}+\mathcal{E}(0)$. This implies (2.18). □

It follows from (2.11), (2.12) and (2.14) that we have

$$G_0'-g^{(0)}_{-n,0}D_0^{-n}\in\mathcal{E}(-n-1),$$
$$G_i'-g^{(i)}_{-n-1,0}D_0^{-n-1}\in\mathcal{E}(-n-2),$$
$$H_0'-(\sum_{0<k<r}h^{(0)}_{-n-2,e_k}D_0^{-n-2}D_k+h^{(0)}_{-n-1,0}D_0^{-n-1})\in\mathcal{E}(-n-2),$$
$$H_i'-(\sum_{0<k<r}h^{(i)}_{-n-1,e_k}D_0^{-n-1}D_k+h^{(i)}_{-n,0}D_0^{-n})\in\mathcal{E}(-n-1),$$

for $0<i<r$. Thus we obtain $(2.9)_{n+1}$ from (2.15) and (2.18). □

Step 4. Now we shall prove Proposition 2.2. We introduce a micro-differential operator

$$W_m=\sum_{\alpha\in N^r}\frac{1}{\alpha!}G^\alpha x^{\alpha_0}D_{x'}^{\alpha'},$$

where $G^\alpha=G_0^{\alpha_0}G_1^{\alpha_0}\cdots G_{r-1}^{\alpha_{r-1}}$ is given in Lemma 2.5. Then we have

$$[D_0,\ W_m]=\sum_{\alpha\in N^r}\frac{\alpha_0}{\alpha!}G^\alpha x_0^{\alpha_0-1}D_{x'}^{\alpha'}$$
$$=G_0\sum_{\alpha\in N^r}\frac{1}{\alpha!}G^\alpha x_0^{\alpha_0}D_{x'}^{\alpha'}=G_0W_m.$$

Hence we get

$$W_mD_0W_m^{-1}=D_0-G_0. \tag{2.20}$$

Similarly we obtain that

$$W_mx_jW_m^{-1}=x_j+G_j\quad(1\leqq j<r). \tag{2.21}$$

Set $W_{YM}=W_m^{-1}W$. Then it follows from (2.8), (2.20) and (2.21) that W_{YM} commutes with $D_0,\ x_1,\ \cdots,\ x_{r-1}$. Therefore W_{YM} is contained in $\mathcal{W}_{YM}$.

Thus we have completed the proof of Proposition 2.2. □

We set

$$\mathcal{W}=\{W_mW_{YM};\ W_m\in\mathcal{W}_m,\ W_{YM}\in\mathcal{W}_{YM}\}.$$

We shall investigate the structure of $\mathcal{W}$, $\mathcal{W}_m$ and $\mathcal{W}_{YM}$.

Proposition 2.6. *The spaces $\mathcal{W}$, $\mathcal{W}_m$ and $\mathcal{W}_{YM}$ are groups by the composition of microdifferential operators, and $\mathcal{W}_{YM}$ is a normal subgroup in $\mathcal{W}$.*

Proof. It is evident that $\mathcal{W}_{YM}$ is an abelian group.

Let $W_m=\sum_\alpha(1/\alpha!)G^\alpha x_0^{\alpha_0}D_{x'}^{\alpha'}$ and $W'_m=\sum_\beta(1/\beta!)F^\beta x_0^{\beta_0}D_{x'}^{\beta'}$ be operators in $\mathcal{W}_m$. For any microdifferential operator $P(x', D_0)$, we set

$$\tilde{P}:=W_mPW_m^{-1}=P(x_1+G_1, \cdots, x_{r-1}+G_{r-1}, D_0-G_0).$$

The last equality follows from (2.20) and (2.21). Noting that $\tilde{P}$ commutes with G_j, the composition

$$\begin{aligned} W_mW'_m&=\sum_\gamma\frac{1}{\beta!}\tilde{F}^\beta W_m x_0^{\beta_0}D_{x'}^{\beta'} \\ &=\sum_{\alpha,\beta}\frac{1}{\beta!}\tilde{F}^\beta\frac{1}{\alpha!}G^\alpha x_0^{\alpha_0+\beta_0}D_{x'}^{\alpha'+\beta'} \\ &=\sum_\gamma\frac{1}{\gamma!}(\tilde{F}+G)^\gamma x_0^{\gamma_0}D_{x'}^{\gamma'} \end{aligned} \tag{2.22}$$

is contained in $\mathcal{W}_m$. For $W_{YM}=\sum_i\mathcal{W}_i(x')D_0^{-i}\in W_{YM}$ the operator

$$\begin{aligned} &W_mW_{YM}W_m^{-1} \\ &=\sum_{i\geqq 0}w_i(x_1+G_1, x_2+G_2, \cdots, x_{r-1}+G_{r-1})(D_0-G_0)^{-i} \end{aligned} \tag{2.23}$$

is contained in $\mathcal{W}_{YM}$. Since $\mathcal{W}=\mathcal{W}_m\mathcal{W}_{YM}$, it follows from (2.22) and (2.23) that $\mathcal{W}$ is a group and that $\mathcal{W}_{YM}$ is a normal subgroup of $\mathcal{W}$. □

We define the Lie subalgebras V_m and V_{YM} of V

$$V_m=\{F_0(x', D_0)x_0+\sum_{0<j<r}F_j(x', D_0)D_j\,;\ F_k(x', D_0)\in\mathcal{E}, (0\leqq k<r)\},$$

$$V_{YM}=\{E(x', D_0)\,;\ E(x', D_0)\in\mathcal{E}\}.$$

We have

$$V=V_m\oplus V_{YM}.$$

Proposition 2.7. *For any $P\in V$ (resp. V_m, V_{YM}) and $W\in\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$), we have $WPW^{-1}\in V$ (resp. V_m, V_{YM}).*

Proof. For $W=\sum_\alpha(1/\alpha!)G^\alpha x_0^{\alpha_0}D_{x'}^{\alpha'}\in\mathcal{W}_m$ we have

$$\begin{aligned} [D_i, W]&=\sum_{0\leqq j<r}\sum_\alpha\frac{\alpha_j}{\alpha!}G^{\alpha-\varepsilon_j}\frac{\partial G_j}{\partial x_i}x_0^{\alpha_0}D_{x'}^{\alpha'} \\ &=\sum_{0<j<r}\frac{\partial G_j}{\partial x_i}WD_j+\frac{\partial G_0}{\partial x_i}Wx_0 \end{aligned}$$

where $i=1, 2, \cdots, r-1$, and $\varepsilon_j=(0, \cdots, \overset{j}{1}, \cdots, 0)\in N^r$. Similarly we have

$$[x_0, W]=\sum_{0<j<r}[x_0, G_j]WD_j+[x_0, G_0]Wx_0.$$

Hence we obtain

$$WD_iW^{-1}=D_i-\sum_{0<j<r}\frac{\partial G_j}{\partial x_i}WD_jW^{-1}-\frac{\partial G_0}{\partial x_i}Wx_0W^{-1} \quad \text{for} \quad 0<i<r,$$

$$Wx_0W^{-1}=x_0-\sum_{0<j<r}[x_0, G_j]WD_jW^{-1}-[x_0, G_0]Wx_0W^{-1}. \tag{2.24}$$

We set

$$G_{ij}=\begin{cases}\dfrac{\partial G_j}{\partial x_i} & \text{for} \quad 0<i<r,\\ [x_0, G_j] & i=0,\end{cases}$$

for $0\leqq j<r$. Then we have

$$x_0=(1+G_{00})Wx_0W^{-1}+\sum_{0<j<r}G_{0j}WD_jW^{-1},$$

$$D_i=G_{i0}Wx_0W^{-1}+\sum_{0<j<r}(\delta_{ij}+G_{ij})WD_jW^{-1} \quad \text{for} \quad 0<i<r. \tag{2.25}$$

Since $G_{ij}\in\mathcal{E}(-1)$, the matrix $(\delta_{ij}+G_{ij})_{0\leqq i,j<r}$ is invertible. Hence WD_iW^{-1} $(0<i<r)$ and Wx_0W^{-1} are contained in V_m, because G_{ij} is independent of x_0, independent of $x_0, D_1, \cdots D_{r-1}$. For any operator $P=P(x', D_0)\in\mathcal{E}$, WPW^{-1} is independent of $x_0, D_1, \cdots, D_{r-1}$ by (2.20) and (2.21). Hence we get Proposition 2.7 for V_m and $\mathcal{W}_m$.

The proposition is evident for V_{YM} and $\mathcal{W}_{YM}$. For $W_m\in\mathcal{W}_m$ and $E\in V_{YM}$ the operator $W_mEW_m^{-1}$ is contained in V_{YM} by (2.20) and (2.21). For $W_{YM}\in\mathcal{W}_{YM}$ we have

$$W_{YM}D_iW_{YM}^{-1}=D_i-\frac{\partial W_{YM}}{\partial x_i}W_{YM}^{-1} \quad \text{for} \quad 0<i<r,$$

$$W_{YM}x_0W_{YM}^{-1}=x_0-[x_0, W_{YM}]W_{YM}^{-1}.$$

Since W_{YM} commutes with $D_0, x_1, \cdots, x_{r-1}$, the operator $W_{YM}PW_{YM}^{-1}$ is contained in V for any $P\in V$. Since $V=V_m\oplus V_{YM}$ and $\mathcal{W}=\mathcal{W}_m\mathcal{W}_{YM}$, we get Proposition 2.7. □

Example 2.3. We shall write down the evolution equations (1.4) for $W\in\mathcal{W}$ in the case $r=3$. We take D_0D_1, D_0D_2 as generators of (1.4).

We set

$$W=\Big(\sum_{\alpha\in N^3}\frac{1}{\alpha!}G^\alpha x_0^{\alpha_0}D_{x'}^{\alpha'}\Big)\Big(\sum_i w_i(x')D_0^{-i}\Big),$$

$$\text{where} \quad G_i(x_1, x_2, D_0)=\sum_{j<0}g_{i,j}(x_1, x_2)D_0^j \quad (i=1, 2).$$

Then we have

$$(\tilde{D}_0\tilde{D}_1)_+=D_0D_1-\frac{\partial g_{0,-1}}{\partial x_1}x_0-\sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_1}D_i-\frac{\partial w_1}{\partial x_1},$$

$$(\tilde{D}_0\tilde{D}_2)_+=D_0D_2-\frac{\partial g_{0,-1}}{\partial x_2}x_0-\sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_2}D_i-\frac{\partial w_1}{\partial x_2}.$$

Taking time parameters s and t with respect to D_0D_1 and D_0D_1, respectively, we obtain the evolution equations

$$\frac{\partial W}{\partial s}+WD_0D_1=\Big(D_0D_1-\frac{\partial g_{0,-1}}{\partial x_1}x_0-\sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_1}D_i-\frac{\partial w_1}{\partial x_1}\Big)W,$$

$$\frac{\partial W}{\partial t}+WD_0D_2=\Big(D_0D_2-\frac{\partial g_{0,-1}}{\partial x_2}x_0-\sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_2}D_i-\frac{\partial w_1}{\partial x_2}\Big)W. \tag{2.26}$$

It follows from (2.26) that we obtain the Zakharov-Shabat type equation

$$\Big[\frac{\partial}{\partial s}-D_0D_1+\frac{\partial g_{0,-1}}{\partial x_1}x_0+\sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_1}D_i+\frac{\partial w_1}{\partial x_1},$$

$$\frac{\partial}{\partial t}-D_0D_2+\frac{\partial g_{0,-1}}{\partial x_2}x_0+\sum_{i=1,2}\frac{\partial g_{i,-1}}{\partial x_2}D_i+\frac{\partial w_1}{\partial x_2}\Big]=0. \tag{2.27}$$

We shall investigate the infinitesimal action ρ of the Lie subalgebra V of $\mathcal{E}$. Remark that the Lie algebra of the group $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$) is canonically isomorphic to the Lie subalgebra $V_-=V\cap\mathcal{E}_\phi$ (resp. $(V_m)_-$, $(V_{YM})_-$).

Theorem 2.8. *The action ρ of V (resp. V_m, V_{YM}) preserves the space $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$). The action of V_- (resp. $(V_m)_-$, $(V_{YM})_-$) coincides with the infinitesimal right action of on the group $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$).*

Proof. For any element $P\in V$ (resp. V_m, V_{YM}) and any operator $W\in\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$), we have

$$\rho(P)=-(WPW^{-1})_-W\in T_W\mathcal{E}.$$

The tangent space $T_W\mathcal{W}$ (resp. $T_W\mathcal{W}_m$, $T_W\mathcal{W}_{YM}$) is identified with V_-W (resp. $(V_m)_-W$, $(V_{YM})_-W$). By Proposition 2.7, $-(WPW^{-1})_-W$ is contained in $T_W\mathcal{W}$ (resp. $T_W\mathcal{W}_m$, $T_W\mathcal{W}_{YM}$). Taking $P\in V_-$ (resp. $(V_m)_-$, $(V_{YM})_-$), we have $\rho(P)=-WP$. Hence the action ρ is the right action of vector fields. □

By Theorem 2.8 the Lie algebra V (resp. V_m, V_{YM}) acts on $\mathcal{W}$ (resp. $\mathcal{W}_m$, $\mathcal{W}_{YM}$) transitively.

§ 3. Twistor Theory and Integrable Systems

On oriented Riemannian manifolds of dimension four, the Weyl curvature tensor C decomposes into two components, the self-dual part C_+ and the anti-self-dual part C_-. A manifold is called self-dual (resp. anti-self-dual) when C_- (resp. C_+) vanishes. Penrose [5] showed that the vanishing of the anti-self-dual part of the Weyl tensor is precisely the integrability condition of the existence of a curved twistor space.

In this section we prove that the equation $C_+=0$ is the compatibility condition of the deformation equations of filtered $\mathcal{D}$-submodules in $\mathcal{E}$ (See [13] in which the Frobenius integrability condition of the equations of self dual metrics is discussed). We get the equations of self-dual metrics from the equation (2.26) for $W\in\mathcal{W}_m$.

Let M be a complex four-manifold and g a holomorphic metric, i.e. a non-degenerate symmetric holomorphic covariant two-tensor on M. We shall choose a holomorphic orientation on M which is necessary to define the complex Hodge $*$-operator. Our discussion being only local, we can assume the existence of two complex vector bundles S_+ and S_-: the bundles of self-dual and anti-self-dual spinors.

Let $\{e_j\}_{j=1,2,3,4}$ denote a local coframe on M such that $g=e_1e_2+e_3e_4$. We can write them in spinor language as

$$\begin{bmatrix} e_4 & e_2 \\ -e_1 & e_3 \end{bmatrix}=\begin{bmatrix} \psi_1\phi_1 & \psi_1\phi_2 \\ \psi_2\phi_1 & \psi_2\phi_2 \end{bmatrix} \tag{3.1}$$

where ψ_1, ψ_2 (resp. ϕ_1, ϕ_2) are the bases of self-dual (resp. anti-self-dual) spinor coframes.

We take $P=P(S_-)$, the projective bundle of the rank two vector bundle S_-. We parametrize S_- locally by

$$(x,\mu_1,\mu_2)\longrightarrow \mu_1\phi_1(x)+\mu_2\phi_2(x),$$

and $\mu=\mu_1/\mu_2$ is an affine coordinate for $\mu_2\neq 0$.

Theorem 3.1. ([5]) *The Riemannian manifold (M, g) is self-dual iff the following Pfaffian system Ω on P is integrable:*

$$\begin{aligned} &\theta:=d\mu+\omega_{21}\mu^2-(\omega_{22}-\omega_{11})\mu-\omega_{12}=0,\\ \Omega:\ &\sigma_1:=\mu e_4+e_2=0,\\ &\sigma_2:=-\mu e_1+e_4=0, \end{aligned} \tag{3.2}$$

where ω_{ij} is the connection form of S_- with respect to the frame ϕ_1 and ϕ_2.

Let $A(\Omega)$ be the sheaf of vector fields orthogonal to the Pfaffian system Ω. The sheaf $A(\Omega)$ is a Lie algebra iff Ω is integrable. In this case there exists a local basis (v_1, v_2) of $A(\Omega)$ such that $[v_1, v_2]=0$.

Proposition 3.2. *Assume (M, g) is self-dual. With appropriate coordinates*

$(\lambda, x_1, x_2, s, t)$ of P, there exists a commuting basis (v_1, v_2) of $A(\Omega)$, in the following form

$$v_1=\frac{\partial}{\partial s}-\lambda\frac{\partial}{\partial x_1}-\Big(\frac{\partial R}{\partial x_1}\frac{\partial}{\partial x_1}+\frac{\partial S}{\partial x_1}\frac{\partial}{\partial x_2}+\frac{\partial T}{\partial x_1}\frac{\partial}{\partial \lambda}\Big),$$

$$v_2=\frac{\partial}{\partial t}-\lambda\frac{\partial}{\partial x_2}-\Big(\frac{\partial R}{\partial x_2}\frac{\partial}{\partial x_1}+\frac{\partial S}{\partial x_2}\frac{\partial}{\partial x_2}+\frac{\partial T}{\partial x_2}\frac{\partial}{\partial \lambda}\Big), \tag{3.3}$$

where the functions R, S and T do not depend on λ.

Proof. First we notice the following lemma.

Lemma 3.3. ([3]) *Let (M, g) be a self-dual Riemannian four-manifold. Then there exist local coordinates (p_1, p_2, q_1, q_2) of M such that*

$$g=\sum_{i,j=1,2}P_{ij}(p, q)dp_i dq_j.$$

It follows from Lemma 3.3 that we can take local frames $\{e_j\}_{j=1,2,3,4}$ as follows:

$$e_1=-dp_1, \quad e_2=-(P_{11}dq_1+P_{12}dq_2),$$

$$e_4=dp_2, \quad e_3=P_{21}dq_1+P_{22}dq_2.$$

By Theorem 3.1 the ideal $\mathcal{J}$ generated by $(\theta, \sigma_1, \sigma_2)$ is closed under the exterior derivative d. Thns we have

$$d\sigma_1\wedge\theta\wedge\sigma_1\wedge\sigma_2=0,$$

$$d\sigma_2\wedge\theta\wedge\sigma_1\wedge\sigma_2=0. \tag{3.4}$$

By direct calculations we have

$$d\sigma_1\wedge\theta\wedge\sigma_1\wedge\sigma_2=\Big(\Big(\frac{\partial P_{12}}{\partial q_1}-\frac{\partial P_{11}}{\partial q_2}\Big)\mu^2+K\mu+L\Big)d\mu\wedge dp_1\wedge dp_2\wedge dq_1\wedge dq_2,$$

$$d\sigma_2\wedge\theta\wedge\sigma_1\wedge\sigma_2=\Big(\Big(\frac{\partial P_{22}}{\partial q_1}-\frac{\partial P_{21}}{\partial q_2}\Big)\mu^2+M\mu+N\Big)d\mu\wedge dp_1\wedge dp_2\wedge dq_1\wedge dq_2,$$

for functions K, L, M and N independent of μ. Thus we have

$$\frac{\partial P_{12}}{\partial q_1}-\frac{\partial P_{11}}{\partial q_2}=0, \qquad \frac{\partial P_{22}}{\partial q_1}-\frac{\partial P_{21}}{\partial q_2}=0.$$

Hence we can define new coordinates (x_1, x_2, s, t, μ) by the following equations:

$$\frac{\partial x_1}{\partial q_i}=-P_{1i}, \qquad \frac{\partial x_2}{\partial q_i}=P_{2i}, \quad (i=1, 2)$$

$$s=p_2, \qquad t=-p_1.$$

The differential forms θ, σ_1 and σ_2 are written in these coordinates as follows:

$$\theta=d\mu+\mu^2(E_1e_2+E_2e_3)+\mu(F_1e_2+F_2e_3)+\sum_j J_je_j,$$

$$\sigma_1=\mu ds+(dx_1+A_1ds+A_2dt),$$

$$\sigma_2=\mu dt+(dx_2+B_1ds+B_2dt),$$

for functions A_j, B_j, E_j, F_j ($j=1, 2$), and J_j ($j=1, 2, 3, 4$) independent of μ. It is easily verified that the following vectors v_1, v_2 belong to $A(\Omega)$:

$$v_1=-\mu\frac{\partial}{\partial x_1}+\frac{\partial}{\partial s}-A_1\frac{\partial}{\partial x_1}-B_1\frac{\partial}{\partial x_2}+(E_1\mu^3+F_1\mu^2+J_2\mu-J_4)\frac{\partial}{\partial\mu},$$

$$v_2=-\mu\frac{\partial}{\partial x_2}+\frac{\partial}{\partial t}-A_2\frac{\partial}{\partial x_1}-B_2\frac{\partial}{\partial x_2}+(E_2\mu^3+F_2\mu^2+J_3\mu-J_1)\frac{\partial}{\partial\mu}.$$

We set the vector field

$$l_1=\frac{\partial}{\partial s}-A_1\frac{\partial}{\partial x_1}-B_1\frac{\partial}{\partial x_2},$$

$$l_2=\frac{\partial}{\partial t}-A_2\frac{\partial}{\partial x_1}-B_2\frac{\partial}{\partial x_2}.$$

The commutator $[v_1, v_2]$ is written in the following form

$$\begin{aligned}[v_1, v_2]&=(E_2\mu^3+F_2\mu^2)\frac{\partial}{\partial x_1}-(E_1\mu^3+F_1\mu^2)\frac{\partial}{\partial x_2}\\&+\Big\{(E_2F_1-E_1F_2)\mu^4+(2E_2J_1-2E_1J_2+l_1(E_2)-l_2(E_1))\mu^3\\&+\Big(F_2J_1-F_1J_2+\frac{\partial J_2}{\partial x_2}-\frac{\partial J_3}{\partial x_1}+l_1(F_2)-l_2(F_1)\Big)\mu^2\Big\}\frac{\partial}{\partial\mu}+\mu u_1+u_0,\end{aligned}\tag{3.5}$$

where the coefficients of vectors u_1 and u_0 are independent of μ.

By the integrability condition, $[v_1, v_2]$ is a linear combination of v_1 and v_2 and since $[v_1, v_2]$ does not contain $(\partial/\partial s)$ nor $(\partial/\partial t)$, v_1 and v_2 commute with each other. It follows from (3.5) that

$$E_j=0,\quad F_j=0\ (j=1, 2),\quad \frac{\partial J_2}{\partial x_2}=\frac{\partial J_3}{\partial x_1}.$$

Thus there exists a function $f=f(x_1, x_2, s, t)$ such that

$$\frac{\partial f}{\partial x_1}=J_2,\qquad \frac{\partial f}{\partial x_2}=J_3.$$

We can take new coordinates $(\lambda=\mu+f, x_1, x_2, s, t)$. With these coordinates we have

$$v_1=\frac{\partial}{\partial s}-\lambda\frac{\partial}{\partial x_1}-\Big(A_1\frac{\partial}{\partial x_1}+B_1\frac{\partial}{\partial x_2}+C_1\frac{\partial}{\partial\lambda}\Big),$$

$$v_2=\frac{\partial}{\partial t}-\lambda\frac{\partial}{\partial x_2}-\Big(A_2\frac{\partial}{\partial x_1}+B_2\frac{\partial}{\partial x_2}+C_2\frac{\partial}{\partial\lambda}\Big),$$

where $C_1=(\partial f/\partial s)-(J_4+J_2A_1+J_3B_1)$ and $C_2=(\partial f/\partial t)-(J_1+J_2A_2+J_3B_2)$. Again taking the coefficients of μ in $[v_1, v_2]$, we obtain that

$$\frac{\partial A_1}{\partial x_2}=\frac{\partial A_2}{\partial x_1}, \quad \frac{\partial B_1}{\partial x_2}=\frac{\partial B_1}{\partial x_1}, \quad \frac{\partial C_1}{\partial x_2}=\frac{\partial C_1}{\partial x_1}.$$

Thus there exist functions R, S and T which are independent of λ such that

$$\frac{\partial R}{\partial x_j}=A_j, \quad \frac{\partial S}{\partial x_j}=B_j, \quad \frac{\partial T}{\partial x_j}=C_j \quad \text{for} \quad j=1, 2.$$

This completes the proof of Proposition 3.2. □

Remark. By Theorem 3.1 and Proposition 3.2 the equations of self-dual metrics are equivalent to the compatibility condition $[v_1, v_2]=0$:

$$\frac{\partial^2 T}{\partial t\partial x_1}-\frac{\partial^2 T}{\partial s\partial x_2}-\frac{\partial R}{\partial x_2}\frac{\partial^2 T}{\partial x_1^2}+\left(\frac{\partial R}{\partial x_1}-\frac{\partial S}{\partial x_2}\right)\frac{\partial^2 T}{\partial x_1\partial x_2}+\frac{\partial S}{\partial x_1}\frac{\partial^2 T}{\partial x_2^2}=0,$$

$$\frac{\partial^2 R}{\partial t\partial x_1}-\frac{\partial^2 R}{\partial s\partial x_2}-\frac{\partial T}{\partial x_2}-\frac{\partial R}{\partial x_2}\frac{\partial^2 R}{\partial x_1^2}+\left(\frac{\partial R}{\partial x_1}-\frac{\partial S}{\partial x_2}\right)\frac{\partial^2 R}{\partial x_1\partial x_2}+\frac{\partial S}{\partial x_1}\frac{\partial^2 R}{\partial x_2^2}=0,$$

$$\frac{\partial^2 S}{\partial t\partial x_1}-\frac{\partial^2 S}{\partial s\partial x_2}+\frac{\partial T}{\partial x_1}-\frac{\partial R}{\partial x_2}\frac{\partial^2 S}{\partial x_1^2}+\left(\frac{\partial R}{\partial x_1}-\frac{\partial S}{\partial x_2}\right)\frac{\partial^2 S}{\partial x_1\partial x_2}+\frac{\partial S}{\partial x_1}\frac{\partial^2 S}{\partial x_2^2}=0. \tag{3.6}$$

In the equation (2.26) we take $W\in\mathcal{W}_m$. In this case w_1 vanishes. Replacing $g_{0,1}$, $g_{1,1}$ and $g_{2,1}$ with R, S and T, respectively. The equations (2.27) reduces to (3.6). Therefore we have

Theorem 3.4. *The Lie algebra V_m acts on the space of self-dual metrics. This action is transitive.*

Let $\mathcal{W}_0$ be the space of $W\in\mathcal{W}$ which commute with D_0. In the equation (2.26) we take $W\in\mathcal{W}_0$. In this case G_0 vanishes. The equation (2.27) reduces to the composed system of the self-dual Yang-Mills equations and the self-dual Einstein equations (see Examples 2.1 and 2.2). Let V_0 be the Lie subalgebra of V:

$$V_0=\{\sum_{0\leq j<r} F_j(x', D_0)D_j+E(x', D_0);$$
$$F_k(x', D_0)\in\mathcal{E}(0\leqq k<r),\ E(x', D_0)\in\mathcal{E}\}.$$

Then V_0 acts on $\mathcal{W}_0$. Thus the self-dual Einstein equations are a specialization of our integrable system.

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