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A Basis of Symmetric Tensor Representations for the Quantum Analogue of the Lie Algebras B_n, C_n and D_n

By

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Abstract

We give a basis of the finite dimensional irreducible representation of $U_q(X_n)$ ($X=B, C, D$) with highest weight $N\Lambda_1$ ($N\in\boldsymbol{Z}_{\geq 0}$), which we call "symmetric tensor representation". This basis is orthonormal and consists of weight vectors. The action of $U_q(X_n)$ is given explicitly.

§1. Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. One can associate the quantized universal enveloping algebra $U_q(\mathfrak{g})$ with each $\mathfrak{g}$([Dri], [J1]).

In [Lus], [Ro], it was shown that the usual theory of highest weight representations for $U(\mathfrak{g})$ carries over to $U_q(\mathfrak{g})$ if the parameter q is not a root of unity. In particular, finite dimensional irreducible representations of $U_q(\mathfrak{g})$ are characterized by highest weights.

For an arbitrary finite dimensional irreducible representation of $\mathfrak{gl}(n, \boldsymbol{C})$, the so-called "Gelfand-Tsetlin basis" is constructed in [GT1]. This basis is orthonormal and consists of weight vectors (with respect to the diagonal matrices). A similar construction is known for the case of $U_q(\mathfrak{gl}(n, \boldsymbol{C}))$ ([J2]). Further this basis is used to obtain the Wigner coefficients for the tensor product of an arbitrary finite dimensional irreducible representation and a vector representation ([Pas]). Therefore such a basis is useful for explicit calculations in the representation theory, mathematical physics, combinatorics, *etc.*

Assume that V is a finite dimensional irreducible representation of $\mathfrak{gl}(n, \boldsymbol{C})$. As a representation of $\mathfrak{gl}(n-1, \boldsymbol{C})$, V decomposes into irreducible components with multiplicity free. Then the Gelfand-Tsetlin basis of V is the union of the Gelfand-Tsetlin basis of the irreducible components. "Gelfand-Tsetlin basis" for $\mathfrak{o}(n)$ is also constructed similarly ([GT2]). But it is not clear whether $U_q(\mathfrak{o}(n-1))$

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can be embeded in $U_q(\mathfrak{o}(n))$ or not. Therefore, such a method cannot be applied to construct the basis for $U_q(\mathfrak{o}(n))$.

The purpose of this paper is to give a basis of the irreducible representation of $U_q(X_n)$ ($X=B, C, D$) with highest weight $N\Lambda_1$ ($N\in\boldsymbol{Z}_{\geq 0}$, where Λ_1 is the highest weight of the vector representation). We call such a representation "symmetric tensor representation of $U_q(X_n)$". This basis is orthonormal and consists of weight vectors (with respect to the Cartan subalgebra). In the construction of these bases, following two points are important. One point is to find a labelling of the bases. The other is to suppose "support property", which means the following: if a generator of $U_q(X_n)$ acts on a base, only "neighboring" (in terms of the labelling) bases appear. In order to find a labelling, we realize the symmetric tensor representations in $V_{\Lambda_1}^{\otimes N}$ (V_{Λ_1} is the vector representation). Then we give a base in the form of a linear combination of the indecomposable vectors (*i.e.*, the tensor products of bases of V_{Λ_1}) with coefficients depending on q. Taking the $q\to 0$ limit of the coefficients, the surviving indecomposable vector gives a labelling. The reason for considering such a procedure is the following. In [DJM], the one-to-one correspondence between bases of finite dimensional irreducible representations of $U_q(\mathfrak{gl}(n, \boldsymbol{C}))$ and "semi-standard tableaux" is described by the "Robinson-Schensted correspondence" or the "bump procedure". This correspondence is obtained by taking the $q\to 0$ limit of Pasquier's Wigner coefficients. In this case "semi-standard tableaux" are the labelling of bases. With such a labelling, the appropriate support property determines the coefficients of the actions of $U_q(X_n)$.

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§ 2. Preliminaries and Notations

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n. Let $A=(a_{ij})_{1\leq i,j\leq n}$ be the Cartan matrix of $\mathfrak{g}$ and let $(\alpha_i)_{1\leq i\leq n}$, $(h_i)_{1\leq i\leq n}$ be the simple roots and the simple co-roots such that $\langle h_i, \alpha_j\rangle=a_{ij}$. Let $(\ |\ )$ is the Weyl group invariant inner product on $\oplus_i \boldsymbol{C}\alpha_i$ such that $(\alpha_i|\alpha_i)=2$ if α_i is a long root. For non-zero parameter q ($q^{(\alpha_i|\alpha_i)}\neq 1$), $U_q(\mathfrak{g})$ is the $\boldsymbol{C}$-algebra generated by $\{k_i^{\pm}, X_i^+, X_i^-\}_{1\leq i\leq n}$, with relations;

$$k_i k_i^{-1}=k_i^{-1}k_i=1; \qquad [k_i, k_j]=0, \tag{2.1}$$

$$k_i X_j^+ k_i^{-1}=q_i^{a_{ij}}X_j^+; \qquad k_i X_j^- k_i^{-1}=q_i^{-a_{ij}}X_j^-, \tag{2.2}$$

$$[X_i^+, X_j^-]=\delta_{i,j}\frac{k_i^2-k_i^{-2}}{q_i^2-q_i^{-2}} \tag{2.3}$$

$$\sum_{\nu=0}^{1-a_{ij}}(-)^{\nu}\begin{bmatrix}1-a_{ij}\\ \nu\end{bmatrix}_{q_i^2}(X_i^{\pm})^{\nu}X_j^{\pm}(X_i^{\pm})^{1-a_{ij}-\nu}=0 \qquad (i\neq j). \tag{2.4}$$

Set $q_i \stackrel{\text{def}}{=} q^{(\alpha_i|\alpha_i)/4}$

$$\begin{bmatrix} m \\ n \end{bmatrix}_t \stackrel{\text{def}}{=} \frac{(t^m-t^{-m})(t^{m-1}-t^{-m+1})\cdots(t^{m-n+1}-t^{-m+n-1})}{(t-t^{-1})(t^2-t^{-2})\cdots(t^n-t^{-n})} \qquad (m>n>0),$$

$$= 1 \ (n=0, m),$$

$$= 0 \text{ otherwise.}$$

In the rest of this paper, we employ the following notation,

$$[\nu]_k \stackrel{\text{def}}{=} \frac{q^{k\nu}-q^{-k\nu}}{q^k-q^{-k}}, \qquad [\nu]=[\nu]_1.$$

Setting formally $k_i=q_i^{h_i}$, (2.3) can be rewritten as

$$[X_i^+, X_j^-]=\delta_{i,j}[h_i]_{\nu_i}, \qquad \text{where} \quad \nu_i=(\alpha_i|\alpha_i)/2.$$

We denote by $L_{\mathfrak{g}}(\Lambda)$ the finite dimensional irreducible representation of $U_q(\mathfrak{g})$ with highest weight Λ.

We shall construct the bases for the symmetric tensor representation over $U_q(X_n)$ $(X=B, C, D)$ as follows. We prepare an index set $W_X^{(N)}$ and define actions of generators of $U_q(X_n)$ on $V_X^{(N)} \stackrel{\text{def}}{=} \bigoplus_{l\in W_X^{(N)}} \boldsymbol{C}v(\ell)$ in the following form;

$$k_i^{\pm}v(\ell)=c_i^{\pm}(\ell)v(\ell), \tag{2.5}$$

$$X_i^+v(\ell)=\sum_{l'}c_i(\ell', \ell)v(\ell'), \qquad X_i^-v(\ell)=\sum_{l''}c_i(\ell, \ell'')v(\ell''). \tag{2.6}$$

(for some constants $c_i^{\pm}(\ell)$, $c_i(\ell, \ell')$ depending on q).

We equip $V_X^{(N)}$ with the $\boldsymbol{C}$-bilinear form (,) such that the basis $\{v(\ell)\}$ is an orthonormal basis. We have the following identities by (2.6),

$$(X_i^+v(\ell), v(\ell'))=(v(\ell), X_i^-v(\ell')). \tag{2.7}$$

When $c_i(\ell, \ell')\neq 0$, we write $v(\ell)\xrightarrow{i}v(\ell')$. The coefficients $c_i(\ell, \ell')$ must satisfy the following properties (2.8) and (2.9),

$$\begin{array}{ccc} v(\ell) & \xrightarrow{j} & v(\ell'') \\ \downarrow i & & \downarrow i \\ v(\ell') & \xrightarrow{j} & v(\ell^*) \end{array} \implies c_i(\ell, \ell')c_j(\ell, \ell'')=c_j(\ell', \ell^*)c_i(\ell'', \ell^*) \quad (\text{where } \ell'\neq\ell'') \tag{2.8}$$

$$\sum_{\ell''}c_i(\ell, \ell'')^2=\sum_{\ell'}c_i(\ell', \ell)^2+([h_i]_{\nu_i}v(\ell), v(\ell)). \tag{2.9}$$

In fact these conditions are equivalent to the following,

$$\ell'\neq\ell'' \implies (X_i^-v(\ell''), X_j^-v(\ell'))=(X_j^+v(\ell''), X_i^+v(\ell')), \tag{2.10}$$

$$(X_i^-v(\ell), X_i^-v(\ell))=(X_i^+v(\ell), X_i^+v(\ell))+([h_i]_{\nu_i}v(\ell), v(\ell)). \tag{2.11}$$

Remark that in our case there is a unique ℓ and a unique ℓ^* for any given ℓ' and ℓ'' $(\ell'\neq\ell'')$.

§ 3. Symmetric Tensor Representations for $U_q(B_n)$

In this section, we treat the case of B_n. Let us take the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leqq i \leqq n-1)$, $\alpha_n = \varepsilon_n$, where $(\varepsilon_1, \cdots, \varepsilon_n)$ is an orthonormal basis of the dual space of the Cartan subalgebra. Fix $N \in \boldsymbol{Z}_{\geqq 0}$ and a parameter q. We assume that q is not a root of unity. Now define $W_B^{(N)}$ and $V_B^{(N)}$,

$$W_B^{(N)} \overset{\text{def}}{=} \{\ell = (l_1, \cdots, l_n, l_{-n-1}, l_{-n}, \cdots, l_{-1}) \in \boldsymbol{Z}_{\geqq 0}^{2n+1} \mid l_1 \leqq \cdots \leqq l_{-1} = N, \ |l_{-n-1} - l_n| \leqq 1\}$$

$$V_B^{(N)} \overset{\text{def}}{=} \bigoplus_{\ell \in W_B^{(N)}} \boldsymbol{C} v(\ell).$$

We equip $V_B^{(N)}$ with the $\boldsymbol{C}$-bilinear form (,) such that $(v(\ell), v(\ell')) = \delta_{\ell, \ell'}$. We define the weight of $v(\ell)$ as follows;

$$wt(\ell) \overset{\text{def}}{=} \sum_{i=1}^{n} (l_i - l_{i-1}) \varepsilon_i - \sum_{i=1}^{n} (l_{-i} - l_{-i-1}) \varepsilon_i \qquad (l_0 = 0).$$

We set

$$\omega_i(\ell) \overset{\text{def}}{=} \frac{2(wt(\ell) \mid \alpha_i)}{(\alpha_i \mid \alpha_i)}.$$

Let $e_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0) \in \boldsymbol{Z}_{\geqq 0}^{2n+1}$.

We define the actions of generators of $U_q(B_n)$ on $V_B^{(N)}$ as follows,

$$k_i^{\pm} v(\ell) = q_i^{\pm \omega_i(\ell)} v(\ell) \tag{3.1}$$

$$X_i^{\pm} v(\ell) = a_i(\ell, \ell \pm e_i) v(\ell \pm e_i) + a_i(\ell, \ell \pm e_{-i-1}) v(\ell \pm e_{-i-1}) \quad (1 \leqq i \leqq n-1) \tag{3.2}$$

$$X_n^{+} v(\ell) = a_n(\ell, \ell') v(\ell'), \qquad X_n^{-} v(\ell) = a_n(\ell, \ell'') v(\ell'') \tag{3.3}$$

where $\ell' = \ell + e_{-n-1}$, $\ell'' = \ell - e_n$ if $l_{-n-1} = l_n$

and $\ell' = \ell + e_n$, $\ell'' = \ell - e_{-n-1}$ if $l_{-n-1} > l_n$.

The coefficients $a_i(\ell, \ell')$ satisfy $a_i(\ell, \ell') = a_i(\ell', \ell)$, and are given as follows,

$$a_i(\ell, \ell - e_i) = \left(\frac{[l_{i+1} - l_i + 1][l_i - l_{i-1}][2(l_{-i-2} - l_i + n - i) - 1]_{1/2} [2(l_{-i} - l_i + n - i) + 1]_{1/2}}{[2(l_{-i-1} - l_i + n - i) - 1]_{1/2} [2(l_{-i-1} - l_i + n - i) + 1]_{1/2}} \right)^{1/2} \quad (1 \leqq i \leqq n-2) \tag{3.4}$$

$$a_{n-1}(\ell, \ell - e_{n-1}) = \left(\frac{[l_{n-1} - l_{n-2}][2(l_{-n+1} - l_{n-1}) + 3]_{1/2} [l_n + l_{-n-1} - 2l_{n-1} + 1]_{1/2} [l_n + l_{-n-1} - 2l_{n-1} + 2]_{1/2}}{[2]_{1/2} [2(l_{-n} - l_{n-1}) + 1]_{1/2} [2(l_{-n} - l_{n-1}) + 3]_{1/2}} \right)^{1/2} \tag{3.5}$$

$$a_i(\ell, \ell - e_{-i-1}) = \left(\frac{[l_{-i} - l_{-i-1} + 1][l_{-i-1} - l_{-i-2}][2(l_{-i-1} - l_{i+1} + n - i) - 3]_{1/2} [2(l_{-i-1} - l_{i-1} + n - i) - 1]_{1/2}}{[2(l_{-i-1} - l_i + n - i) - 3]_{1/2} [2(l_{-i-1} - l_i + n - i) - 1]_{1/2}} \right)^{1/2} \quad (1 \leqq i \leqq n-2) \tag{3.6}$$

$$a_{n-1}(\ell,\ \ell-e_{-n})=$$
$$\left(\frac{[l_{-n+1}-l_{-n}+1][2(l_{-n}-l_{n-2})+1]_{1/2}[2l_{-n}-(l_n+l_{-n-1})]_{1/2}[2l_{-n}-(l_n+l_{-n-1})-1]_{1/2}}{[2]_{1/2}[2(l_{-n}-l_{n-1})-1]_{1/2}[2(l_{-n}-l_{n-1})+1]_{1/2}}\right)^{1/2} \tag{3.7}$$

$$a_n(\ell,\ \ell'')=([2l_{-n}-(l_n+l_{-n-1})+1]_{1/2}[(l_n+l_{-n-1})-2l_{n-1}]_{1/2})^{1/2} \tag{3.8}$$

($\ell''=\ell-e_n$ or $\ell''=\ell-e_{-n-1}$ according to $l_n=l_{-n-1}$ or $l_n<l_{-n-1}$).

Theorem 1. *If we define the actions of the generators by* (3.1)-(3.8), $V_B^{(N)}$ *becomes a* $U_q(B_n)$*-module isomorphic to* $L_{B_n}(N\Lambda_1)$.

Proof. We shall check (2.1)-(2.4). First of all by the definition of $a_i(\ell,\ \ell')$, we have (2.1) and (2.2). In our situation, (2.8) and (2.9) can be rewritten as in (3.9) and (3.10).

$$\begin{array}{ccc} v(\ell) & \xrightarrow{\ j\ } & v(\ell'') \\ \downarrow i & & \downarrow i \\ v(\ell') & \xrightarrow{\ j\ } & v(\ell^*) \end{array} \implies a_i(\ell,\ \ell')a_j(\ell,\ \ell'')=a_j(\ell',\ \ell^*)a_i(\ell'',\ \ell^*). \tag{3.9}$$

$X_i^+v(\ell)=a_i(\ell,\ \ell')v(\ell')+a_i(\ell,\ \ell^*)v(\ell^*)$ and $X_i^-v(\ell)=a_i(\ell,\ \ell'')v(\ell'')+a_i(\ell,\ \ell^{**})v(^{**})$ *i.e.*,

$$\begin{array}{ccccc} & & v(\ell') & & \\ & i & \uparrow i & i & \\ v(\ell^*) & \leftarrow & v(\ell) & \rightarrow & v(\ell'') \\ & & \downarrow i & & \\ & & v(\ell^{**}) & & \end{array} \implies a_i(\ell,\ \ell'')^2+a_i(\ell,\ \ell^{**})^2=a_i(\ell,\ \ell')^2+a_i(\ell,\ \ell^*)^2+[\omega_i(\ell)]_{\nu_i}. \tag{3.10}$$

We can describe $X_i^+X_j^-v(\ell)$ and $X_j^-X_i^+v(\ell)$ by the following diagrams ($i\neq j$, $i,\ j\neq n$),

$$\begin{array}{ccccc} v(\ell+e_i-e_{-j-1}) & \xleftarrow{\ j\ } & v(\ell+e_i) & \xrightarrow{\ j\ } & v(\ell+e_i-e_j) \\ \uparrow i & \text{(A)} & \uparrow i & \text{(B)} & \uparrow i \\ v(\ell-e_{-j-1}) & \xleftarrow{\ j\ } & v(\ell) & \xrightarrow{\ j\ } & v(\ell-e_j) \\ \downarrow i & \text{(C)} & \downarrow i & \text{(D)} & \downarrow i \\ v(\ell+e_{-i-1}-e_{-j-1}) & \xleftarrow{\ j\ } & v(\ell+e_{-i-1}) & \xrightarrow{\ j\ } & v(\ell+e_{-i-1}-e_j) \end{array} \tag{3.11}$$

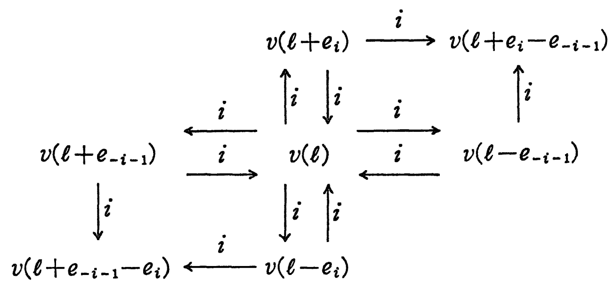
Applying (3.9) to the squares (A), (B), (C) and (D) in this diagram, we obtain

$$i\neq j \quad i,\ j\neq n \implies X_i^+X_j^-v(\ell)=X_j^-X_i^+v(\ell).$$

If $i\neq j$ and if $i=n$ or $j=n$, we may neglect two squares in the diagram (3.11). Therefore,

$$i \neq j \implies X_i^+ X_j^- v(\ell) = X_j^- X_i^+ v(\ell). \tag{3.12}$$

We can describe $X_i^+ X_i^- v(\ell)$ and $X_i^- X_i^+ v(\ell)$ by the following diagrams,



(3.13)

Note that some arrows and vertices disappear in this diagram when $i=n$. Applying (3.9) and (3.10), we obtain

$$X_i^+ X_i^- v(\ell) = X_i^- X_i^+ v(\ell) + [\omega_i(\ell)]_{\nu_i} v(\ell). \tag{3.14}$$

Here note that $\dfrac{k_i^2 - k_i^{-2}}{q_i^2 - q_i^{-2}} v(\ell) = [h_i]_{\nu_i} v(\ell) = [\omega_i(\ell)]_{\nu_i} v(\ell)$ by (3.1). We completed the proof of (2.3).

Setting $\ell_0 = (\overbrace{N, \cdots, N}^{2n+1})$ and $v_0 = v(\ell_0)$, we obtain

$$wt(\ell_0) = N\varepsilon_1 \qquad X_i^+ v_0 = 0 \qquad (1 \leqq i \leqq n). \tag{3.15}$$

To verify (2.4), let us prepare the following lemma.

Lemma 1. *If $v \in V_B^{(N)}$ satisfies $X_i^+ v = 0\,(1 \leqq i \leqq n)$, we have $v = cv_0$ for some constant c.*

Proof of Lemma 1. v can be written as $v = \sum_{\ell \in W_B^{(N)}} c(\ell) v(\ell)$ with some constants $c(\ell)$. From $X_n^+ v = 0$, we have

$$\sum_{\ell' \in W_B^{(N)}} c(\ell) a_n(\ell, \ell') v(\ell') = 0.$$

Because of the linear independence of $\{v(\ell')\}$, all the coefficients $c(\ell) a_n(\ell, \ell')$ must vanish. If $\ell' - \ell = e_{-n-1}$ or e_n, then $a_n(\ell, \ell') = a_n(\ell', \ell) \neq 0$ by (3.8). Therefore we have $c(\ell) = 0$ if ℓ does not satisfy $l_n = l_{-n-1} = l_{-n}$. From $X_{n-1}^+ v = 0$, we obtain

$$\sum_{\ell} c(\ell) w(\ell) = 0$$

where $w(\ell) = a_{n-1}(\ell, \ell+e_{n-1}) v(\ell+e_{n-1}) + a_{n-1}(\ell, \ell+e_{-n}) v(\ell+e_{-n})$ and the summation is taken over $\ell \in W_B^{(N)}$ such that $\ell+e_{n-1}$ or $\ell+e_{-n} \in W_B^{(N)}$ and $l_n = l_{-n-1} = l_{-n}$. If $\ell, \ell+e_{n-1}, \ell+e_{-n} \in W_B^{(N)}$, then $a_{n-1}(\ell, \ell+e_{n-1}), a_{n-1}(\ell, \ell+e_{-n}) \neq 0$ by (3.5) and (3.7). Therefore, $l_n = l_{-n-1} = l_{-n}$ implies that $\{w(\ell)\}$ are linearly independent.

Therefore all the coefficients must vanish. Thus, if ℓ does not satisfy $l_{n-1}=l_n=l_{-n-1}=l_{-n}=l_{-n+1}$, we obtain that $c(\ell)=0$. Arguing similarly we obtain $c(\ell)=0$, if ℓ does not satisfy $l_1=l_2=\cdots=l_{-2}=l_{-1}(=N)$. Thus we conclude $v=cv_0$ for some constant c. *q. e. d.*

Set $\xi_{i,j}^{\pm}=L.H.S.$ of (2.4). Remark that $\xi_{i,j}^{\pm}$ commute with $X_k^{\mp}(1\leqq k\leqq n)$. From that, we obtain

$$X_k^+\xi_{i,j}^-v_0=\xi_{i,j}^-X_k^+v_0=0.$$

Therefore by Lemma 1, $\xi_{i,j}^-v_0=cv_0$. Since the weight of $\xi_{i,j}^-v_0$ is different from that of v_0, we can conclude $\xi_{i,j}^-v_0=0$. Arguing similarly we find that $\xi_{i,j}^-v(\ell)=0$ for any $v(\ell)$ by the induction on the weight of $v(\ell)$. The case of $\xi_{i,j}^+$ follows from the case of $\xi_{i,j}^-$ by applying the Weyl involution. Thus, it has been shown that $V_B^{(N)}$ is a finite dimensional $U_q(B_n)$-module and has only one highest weight vector (up to constant) with highest weight $N\Lambda_1$ by Lemma 1 and (3.15) (remark $\Lambda_1=\varepsilon_1$). Therefore we conclude that $V_B^{(N)}$ is isomorphic to $L_{B_n}(N\Lambda_1)$. *q. e. d.*

§ 4. Symmetric Tensor Representations for $U_q(C_n)$

In this section, we shall treat the case of C_n. Let us take the simple roots $\alpha_i=\varepsilon_i-\varepsilon_{i+1}(1\leqq i\leqq n-1)$, $\alpha_n=2\varepsilon_n$ where $(\varepsilon_1,\cdots,\varepsilon_n)$ is an orthonormal basis of the dual space of the Cartan subalgebra. Fix N, and q as in Section 3. Now define $W_C^{(N)}$ and $V_C^{(N)}$,

$$W_C^{(N)}\stackrel{\text{def}}{=}\{\ell=(l_1,l_2,\cdots,l_n,l_{-n},\cdots,l_{-1})\in\boldsymbol{Z}_{\geqq 0}^{2n}\mid l_1\leqq\cdots\leqq l_{-1}=N\}$$

$$V_C^{(N)}\stackrel{\text{def}}{=}\bigoplus_{\ell\in W_C^{(N)}}\boldsymbol{C}v(\ell).$$

We equip $V_C^{(N)}$ with the $\boldsymbol{C}$-bilinear form (,) such that $(v(\ell),v(\ell'))=\delta_{\ell,\ell'}$ and we define the weight of $v(\ell)$ as follows;

$$wt(\ell)\stackrel{\text{def}}{=}\sum_{i=1}^{n}(l_i-l_{i-1})\varepsilon_i-\sum_{i=1}^{n-1}(l_{-i}-l_{-i-1})\varepsilon_i-(l_{-n}-l_n)\varepsilon_n\quad(l_0=0).$$

We set

$$\omega_i(\ell)\stackrel{\text{def}}{=}\frac{2(wt(\ell)\mid\alpha_i)}{(\alpha_i\mid\alpha_i)}.$$

Let $e_i=(0,\cdots,0,\overset{i}{1},0,\cdots,0)\in\boldsymbol{Z}_{\geqq 0}^{2n}$. We define the actions of generators of $U_q(C_n)$ to $V_C^{(N)}$ as follows,

$$k_i^{\pm}v(\ell)=q_i^{\pm\omega_i(\ell)}v(\ell)\tag{4.1}$$

$$X_i^{\pm}v(\ell)=a_i(\ell,\ell\pm e_i)v(\ell\pm e_i)+a_i(\ell,\ell\pm e_{-i-1})v(\ell\pm e_{-i-1})\quad(1\leqq i\leqq n-1)\tag{4.2}$$

$$X_n^{\pm}v(\ell)=a_n(\ell,\ell\pm e_n)v(\ell\pm e_n).\tag{4.3}$$

The coefficients $a_i(\ell, \ell')$ satisfy $a_i(\ell, \ell')=a_i(\ell', \ell)$ and are given as follows,

$$a_i(\ell, \ell-e_i)\overset{\text{def}}{=}\Big(\frac{[l_{i+1}-l_i+1]_{1/2}[l_i-l_{i-1}]_{1/2}[l_{-i-1}-l_i+n-i]_{1/2}[l_{-i-1}-l_i+n-i+1]_{1/2}}{[l_{-i-1}-l_i+n-i][l_{-i-1}-l_i+n-i+1]}$$
$$\frac{[l_{-i}-l_i+n-i+1][l_{-i-2}-l_i+n-i]}{[l_{-i}-l_i+n-i+1]_{1/2}[l_{-i-2}-l_i+n-i]_{1/2}}\Big)^{1/2} \quad (1\leqq i\leqq n-2) \tag{4.4}$$

$$a_{n-1}(\ell, \ell-e_{n-1})$$
$$\overset{\text{def}}{=}\Big(\frac{[l_{n-1}-l_{n-2}]_{1/2}[l_{-n}-l_{n-1}+1]_{1/2}[l_{-n}-l_{n-1}+2]_{1/2}[l_{-n+1}-l_{n-1}+2][l_n-l_{n-1}+1]}{[l_{-n}-l_{n-1}+1][l_{-n}-l_{n-1}+2][l_{-n+1}-l_{n-1}+2]_{1/2}}\Big)^{1/2} \tag{4.5}$$

$$a_i(\ell, \ell-e_{-i-1})\overset{\text{def}}{=}\Big(\frac{[l_{-i-1}-l_{-i-2}]_{1/2}[l_{-i}-l_{-i-1}+1]_{1/2}[l_{-i-1}-l_i+n-i-1]_{1/2}}{[l_{-i-1}-l_i+n-i-1]}$$
$$\frac{[l_{-i-1}-l_i+n-i]_{1/2}[l_{-i-1}-l_{i-1}+n-i][l_{-i-1}-l_{i+1}+n-i-1]}{[l_{-i-1}-l_i+n-i][l_{-i-1}-l_{i-1}+n-i]_{1/2}[l_{-i-1}-l_{i+1}+n-i-1]_{1/2}}\Big)^{1/2}$$
$$(1\leqq i\leqq n-2) \tag{4.6}$$

$$a_{n-1}(\ell, \ell-e_{-n})$$
$$\overset{\text{def}}{=}\Big(\frac{[l_{-n+1}-l_{-n}+1]_{1/2}[l_{-n}-l_{n-1}]_{1/2}[l_{-n}-l_{n-1}+1]_{1/2}[l_{-n}-l_{n-2}+1][l_{-n}-l_n]}{[l_{-n}-l_{n-1}][l_{-n}-l_{n-1}+1][l_{-n}-l_{n-2}+1]_{1/2}}\Big)^{1/2} \tag{4.7}$$

$$a_n(\ell, \ell-e_n)\overset{\text{def}}{=}([l_{-n}-l_n+1][l_n-l_{n-1}])^{1/2}. \tag{4.8}$$

Theorem 2. *If we define the actions of generators by* (4.1)-(4.8), $V_C^{(N)}$ *becomes a* $U_q(C_n)$*-module isomorphic to* $L_{C_n}(N\Lambda_1)$.

The proof being similar to that for B_n, we omit it. We only note that $v((\overbrace{N, \cdots, N}^{2n}))$ is a highest weight vector with the highest weight $N\Lambda_1$.

§ 5. Symmetric Tensor Representations for $U_q(D_n)$

In this section, we treat the case of D_n. Let us take the simple roots $\alpha_i=\varepsilon_i-\varepsilon_{i+1}\,(1\leqq i\leqq n-1)$, $\alpha_n=\varepsilon_{n-1}+\varepsilon_n$ where $(\varepsilon_1, \cdots, \varepsilon_n)$ is an orthonormal basis of the dual space of the Cartan subalgebra. Fix N, q as in Section 3. Now define $W_D^{(N)}$, $W_D^{*(N)}$ and $V_D^{(N)}$

$$W_D^{(N)}\overset{\text{def}}{=}\{\ell=(l_1, \cdots, l_{n-1}, l_{-n}, \cdots, l_{-1})\in \boldsymbol{Z}_{\geqq 0}^{2n-1}\,|\,l_1\leqq\cdots\leqq l_{n-1}<l_{-n}\leqq\cdots\leqq l_{-1}=N\}$$

$$W_D^{*(N)}\overset{\text{def}}{=}\{\ell=(l_1, \cdots, l_{n-1}, l_{-n}, \cdots, l_{-1})\in \boldsymbol{Z}_{\geqq 0}^{2n-1}\,|\,l_1\leqq\cdots\leqq l_{n-1}=l_{-n}\leqq\cdots\leqq l_{-1}=N\}$$

$$V_D^{(N)}\overset{\text{def}}{=}\Big(\bigoplus_{\ell\in W_D^{(N)}}\boldsymbol{C}v_+(\ell)\Big)\oplus\Big(\bigoplus_{\ell\in W_D^{(N)}}\boldsymbol{C}v_-(\ell)\Big)\oplus\Big(\bigoplus_{\ell\in W^*{}_D^{(N)}}\boldsymbol{C}v(\ell)\Big).$$

We shall set $v_\pm(\ell)=v(\ell)$ for $\ell\in W_D^{*(N)}$. We equip $V_D^{(N)}$ with the $\boldsymbol{C}$-bilinear form (,) such that these vectors form an orthonormal basis. We define the weight $wt_{\pm}(\ell)$ of $v_{\pm}(\ell)$ as follows,

$$wt_{\pm}(\ell) \stackrel{\text{def}}{=} \sum_{i=1}^{n-1}(l_i-l_{i-1})\varepsilon_i \pm (l_{-n}-l_{n-1})\varepsilon_n - \sum_{i=1}^{n-1}(l_{-i}-l_{-i-1})\varepsilon_i, \qquad (l_0=0).$$

We set

$$\omega_i^{\pm}(\ell) \stackrel{\text{def}}{=} \frac{2(wt_{\pm}(\ell)|\alpha_i)}{(\alpha_i|\alpha_i)}.$$

Let $\epsilon=\pm$ and $e_i=(0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0) \in \boldsymbol{Z}_{\geqq 0}^{2n-1}$.

We define the actions of the generators of $U_q(D_n)$ on $V_D^{(N)}$ as follows,

$$k_i^{\pm} v_{\epsilon}(\ell) \stackrel{\text{def}}{=} q_i^{\pm\omega_i^{\epsilon}(\ell)} v_{\epsilon}(\ell) \tag{5.1}$$

$$X_i^{\pm} v_{\epsilon}(\ell) \stackrel{\text{def}}{=} a_i(\ell, \ell \pm e_i) v_{\epsilon}(\ell \pm e_i) + a_i(\ell, \ell \pm e_{-i-1}) v_{\epsilon}(\ell \pm e_{-i-1}) \qquad (1 \leqq i \leqq n-2) \tag{5.2}$$

When $l_{n-1} < l_{-n}$,

$$X_{n-1}^{\pm} v_+(\ell) = a_{n-1}^{(+)}(\ell, \ell \pm e_{n-1}) v_+(\ell \pm e_{n-1}) \tag{5.3}$$

$$X_{n-1}^{\pm} v_-(\ell) = a_{n-1}^{(-)}(\ell, \ell \pm e_{-n}) v_-(\ell \pm e_{-n}) \tag{5.4}$$

$$X_n^{\pm} v_+(\ell) = a_n^{(+)}(\ell, \ell \pm e_{-n}) v_+(\ell \pm e_{-n}) \tag{5.5}$$

$$X_n^{\pm} v_-(\ell) = a_n^{(-)}(\ell, \ell \pm e_{n-1}) v_-(\ell \pm e_{n-1}). \tag{5.6}$$

When $l_{n-1} = l_{-n}$,

$$X_{n-1}^{+} v(\ell) = a_{n-1}^{(-)}(\ell, \ell + e_{-n}) v_-(\ell + e_{-n}) \tag{5.7}$$

$$X_{n-1}^{-} v(\ell) = a_{n-1}^{(+)}(\ell, \ell - e_{n-1}) v_+(\ell - e_{n-1}) \tag{5.8}$$

$$X_n^{+} v(\ell) = a_n^{(+)}(\ell, \ell + e_{-n}) v_+(\ell + e_{-n}) \tag{5.9}$$

$$X_n^{-} v(\ell) = a_n^{(-)}(\ell, \ell - e_{n-1}) v_-(\ell - e_{n-1}). \tag{5.10}$$

The coefficients $a_i(\ell, \ell')$ satisfy $a_i(\ell, \ell') = a_i(\ell', \ell)$ and are given as follows,

$$a_i(\ell, \ell - e_i) = \left(\frac{[l_i - l_{i-1}][l_{i+1} - l_i + 1][l_{-i-2} - l_i + n - i - 1][l_{-i} - l_i + n - i]}{[l_{-i-1} - l_i + n - i - 1][l_{-i-1} - l_i + n - i]}\right)^{1/2} \tag{5.11}$$

$$a_i(\ell, \ell - e_{-i-1}) = \left(\frac{[l_{-i-1} - l_{-i-2}][l_{-i} - l_{-i-1} + 1][l_{-i-1} - l_{i+1} + n - i - 2][l_{-i-1} - l_{i-1} + n - i - 1]}{[l_{-i-1} - l_i + n - i - 2][l_{-i-1} - l_i + n - i - 1]}\right)^{1/2} \qquad (1 \leqq i \leqq n-2) \tag{5.12}$$

$$a_{n-1}^{(+)}(\ell, \ell - e_{n-1}) = a_n^{(-)}(\ell, \ell - e_{n-1}) = ([l_{n-1} - l_{n-2}][l_{-n+1} - l_{n-1} + 1])^{1/2} \tag{5.13}$$

$$a_{n-1}^{(-)}(\ell, \ell - e_{-n}) = a_n^{(+)}(\ell, \ell - e_{-n}) = ([l_{-n+1} - l_{-n} + 1][l_{-n} - l_{n-2}])^{1/2}. \tag{5.14}$$

Theorem 3. *If we define the actions by* (5.1)-(5.14), $V_D^{(N)}$ *becomes a* $U_q(D_n)$-*module isomorphic to* $L_{D_n}(N\Lambda_1)$.

Proof. The proof is similar to B_n and C_n. We only give the proof of the following lemma.

Lemma 2. *Let* $v_0=v((\overbrace{N,\cdots,N}^{2n-1}))$. *If* $v\in V_D^{(N)}$ *satisfies* $X_i^+v=0\ (1\leqq i\leqq n)$, *then we have* $v=cv_0$ *for some constant c.*

Note that the weight of v_0 is $N\Lambda_1$ similarly to the B_n and C_n case.

Proof of Lemma 2. We can write

$$v=\sum_{\ell\in W_D^{(N)}}c_+(\ell)v_+(\ell)+\sum_{\ell\in W_D^{(N)}}c_-(\ell)v_-(\ell)+\sum_{\ell\in W^*{}_D^{(N)}}c_0(\ell)v(\ell).$$

From $X_n^+v=0$, we have

$$\sum_{\substack{\ell,\ell+e_{-n}\\ \in W_D^{(N)}}}c_+(\ell)a_n^{(+)}(\ell,\ell+e_{-n})v_+(\ell+e_{-n})+\sum_{\ell\in W_D^{(N)}}c_-(\ell)a_n^{(-)}(\ell,\ell+e_{n-1})v_-(\ell+e_{n-1})$$
$$+\sum_{\ell\in W^*{}_D^{(N)},\,\ell+e_{-n}\in W_D^{(N)}}c_0(\ell)a_n^{(+)}(\ell,\ell+e_{-n})v_+(\ell+e_{-n})=0.$$

By (5.13) and (5.14),

$$\ell,\ \ell+e_{-n}\in W_D^{(N)}\Longrightarrow a_n^{(+)}(\ell,\ell+e_{-n})\neq 0$$
$$\ell\in W_D^{(N)}\Longrightarrow \ell+e_{n-1}\in W_D^{(N)}\cup W_D^{*(N)}\quad\text{and}\quad a_n^{(-)}(\ell,\ell+e_{n-1})\neq 0$$
$$\ell\in W_D^{*(N)},\ \ell+e_{-n}\in W_D^{(N)}\Longrightarrow a_n^{(+)}(\ell,\ell+e_{-n})\neq 0,$$

and $\{v_+(\ell+e_{-n})\}_{\ell,\ell+e_{-n}\in W_D^{(N)}}$, $\{v_-(\ell+e_{n-1})\}_{\ell\in W_D^{(N)}}$ and $\{v_+(\ell+e_{-n})\}_{\ell\in W^*{}_D^{(N)},\ell+e_{-n}\in W_D^{(N)}}$ are linearly independent. Therefore all the coefficients must vanish. Hence $c_+(\ell)$, $c_-(\ell)$ and $c_0(\ell)$ vanish for such an ℓ. Thus v can be written as follows,

$$v=\sum_{\substack{\ell\in W_D^{(N)},\\ l_{n-1}<l_{-n}=l_{-n+1}}}c_+(\ell)v_+(\ell)+\sum_{\substack{\ell\in W^*{}_D^{(N)},\\ l_{n-1}=l_{-n}=l_{-n+1}}}c_0(\ell)v(\ell).$$

From $X_{n-1}^+v=0$, we have

$$\sum_{\substack{l_{n-1}<l_{-n}\\ =l_{-n+1}}}c_+(\ell)a_{n-1}^{(+)}(\ell,\ell+e_{n-1})v_+(\ell+e_{n-1})+\sum_{\substack{l_{n-1}=l_{-n}\\ =l_{-n+1}}}c_0(\ell)a_{n-1}^{(-)}(\ell,\ell+e_{-n})v_+(\ell+e_{-n})=0.$$

If $l_{n-1}<l_{-n}=l_{-n+1}$, then $a_{n-1}^{(+)}(\ell,\ell+e_{n-1})\neq 0$ by (5.13). If $l_{n-1}=l_{-n}=l_{-n+1}$, then $a_{n-1}^{(-)}(\ell,\ell+e_{-n})=0$ by (5.14). Therefore if $l_{n-1}<l_{-n}=l_{-n+1}$, $c_+(\ell)$ must vanish by the linear independence of $\{v(\ell+e_{n-1})\}_{l_{n-1}<l_{-n}=l_{-n+1}}$. Thus v can be written as follows,

$$v=\sum_{l_{n-1}=l_{-n}=l_{-n+1}}c_0(\ell)v(\ell).$$

The rest of the proof proceeds similarly to the B_n case. *q.e.d.*

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