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Second Microlocalization and the Mellin Transformation

By

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§ 1. Introduction

Let a conormal distribution u be a solution of the equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=f(t, x, y, u)$$

in an open set $\Omega\subset\boldsymbol{R}^3$ such that $(t, x, y)=(0, 0, 0)\in\Omega$.

In paper [1] J.M. Bony showed that if for $t<0$ u had conormal singularities situated on the characteristic surfaces Σ_1, Σ_2, Σ_3 then for $t>0$ u is regular outside Σ_1, Σ_2, Σ_3 and the forward light cone starting at the point $\Sigma_1\cap\Sigma_2\cap\Sigma_3$.

A basic tool used in the proof of this fact was, among other things, the notion of second microlocalization and in particular of second wave front set.

As it is well known the notion of the first wave front set WFu of a distribution u can be described with the help of the Fourier transformation. We show that in the case of the second wave front set an analogous role is played by the Mellin transformation. Moreover we give a new proof of Bony's theorem on propagation of 2-microlocal singularities.

§ 2. Second Wave Front Set in Terms of the Space $SP(s, s')$

Definition 1. (cf. Def. 2.5 [1]). Let $u\in D'(\boldsymbol{R}^n\setminus\{0\})$ and let u vanish outside a unit ball in $\boldsymbol{R}^n$. Let $s, s'\in\boldsymbol{R}$ be such that $s+s'$ is a non-negative integer i.e. $s+s'\in\boldsymbol{N}_0$. We say that $u\in SP(s, s')$ if

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$$\|x\|^{-s+|\lambda|}D^{\lambda}u\in L^2(\boldsymbol{R}^n) \qquad \text{for} \quad 0\le|\lambda|\le s+s' \tag{1}$$
$$\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\boldsymbol{N}_0^n\,,\quad |\lambda|=\lambda_1+\lambda_2+\cdots+\lambda_n\,.$$

For remaining (s, s') the spaces $SP(s, s')$ are defined by duality and interpolation.

Let for $\tau\in\boldsymbol{R}^n$, $x\in\boldsymbol{R}_+^n=\{x\in\boldsymbol{R}^n\colon x_i>0\ i=1,\cdots,n\}$ $x^{\tau}=x_1^{\tau_1}\cdot\cdots\cdot x_n^{\tau_n}$. The point $(a,\cdots,a)\in\boldsymbol{R}^n$ we denote by bold face $\boldsymbol{a}$.

In our case it will be more convenient to use, instead of the space $L^2(\boldsymbol{R}^n)$, the space $L^2(\boldsymbol{R}_+^n)$ with the weight x^{-1} i.e.

$$u\in L^2(\boldsymbol{R}_+^n, x^{-1}) \text{ if and only if } \int_{\boldsymbol{R}_+^n}|u(x)|^2x^{-1}dx<\infty\,.$$

For $u\in D'(\boldsymbol{R}_+^n)$, (1) can be written as $u\in SP(s, s')$ if and only if

$$x^{1/2}\|x\|^{-s+|\lambda|}D^{\lambda}u\in L^2(\boldsymbol{R}_+^n, x^{-1})\,,\qquad 0\le|\lambda|\le s+s'\,. \tag{2}$$

If supp $u\subset\Gamma$, where Γ is a proper cone in $\boldsymbol{R}_+^n$ (i.e. such that $\bar\Gamma\cap\overline{\partial\boldsymbol{R}_+^n}=\{0\}$, overbar denotes closure in $\boldsymbol{R}^n$) then there exist constants c_j, $d_j>0$ such that $c_jx_j<\|x\|<d_kx_k$, $j, k=1, 2,\cdots, n$ for $x\in\Gamma$ (shortly $\forall_k x_k\sim\|x\|$). Then (2) denotes that

$$x^{1/2-\rho}(xD)^{\lambda}u\in L^2(\boldsymbol{R}_+^n, x^{-1}) \tag{2'}$$

for $\lambda\in\boldsymbol{N}_0^n$, $\rho\in\boldsymbol{R}^n$, $0\le|\lambda|\le s+s'$, $|\rho|=\rho_1+\cdots+\rho_n=s$
where $(xD)^{\lambda}=(x_1D_1)^{\lambda_1}\cdots(x_nD_n)^{\lambda_n}$, i.e. that

$$x^{\gamma}(xD)^{\lambda}u\in L^2(\boldsymbol{R}_+^n, x^{-1}) \qquad \text{for} \quad \lambda\in\boldsymbol{N}_0^n\,,\ \gamma\in\boldsymbol{R}^n,\ |\lambda|\le s+s' \tag{3}$$

where $|\gamma|=\gamma_1+\cdots+\gamma_n\ge -s+\frac{n}{2}$, which is equivalent to

$$x^{\gamma_0}(xD)^{\lambda}u\in L^2(\boldsymbol{R}_+^n, x^{-1}) \qquad \text{for} \quad \lambda\in\boldsymbol{N}_0^n,\ |\lambda|\le s+s' \tag{3'}$$

and some $\gamma_0\in\boldsymbol{R}^n$ such that $|\gamma_0|=-s+\frac{n}{2}$.

Let $\Gamma\subset\boldsymbol{R}_+^n$ be a cone tangent only to the axis x_1 i.e. such that closure $\bar\Gamma\cap\{x\in\boldsymbol{R}^n\colon x_1=0\}=\{0\}$. Then we have $\|x\|\sim x_1$ on $\bar\Gamma$ thus the condition

$$x^{1/2}\|x\|^{-s+|\lambda|}D^{\lambda}u\in L^2(\Gamma, x^{-1})\,,\qquad |\lambda|\le s+s' \tag{4}$$

is equivalent to the condition

$$x^{1/2}x_1^{-s+|\lambda|}D^{\lambda}u\in L^2(\Gamma, x^{-1})\,,\qquad |\lambda|\le s+s'\,. \tag{5}$$

Fix $\delta\mathring{x}=(\delta\mathring{x}_1,\cdots,\delta\mathring{x}_n)\in\boldsymbol{R}_+^n$, $\mathring{\xi}=(\mathring{\xi}_1,\cdots,\mathring{\xi}_n)\in\boldsymbol{R}^n$ and let

1° $\rho\in C^{\infty}(\mathrm{i}(\boldsymbol{R}^n\setminus\{0\}))$ be a homogeneous function of order zero defined in a conical neighbourhood of $\mathrm{i}\mathring{\beta}=(\mathrm{i}\delta\mathring{x}_1\mathring{\xi}_1, \cdots, \mathrm{i}\delta\mathring{x}_n\mathring{\xi}_n)$, $\rho(\mathrm{i}\mathring{\beta})\neq 0$. The function ρ is extended to the set $\boldsymbol{R}^n+\mathrm{i}(\boldsymbol{R}^n\setminus\{0\})$ by putting $\rho(z)=\rho(\mathrm{Im}\, z)$.

2° $\kappa=\varphi\cdot\kappa'$ where φ is a C_0^{∞} bump function at zero and $\kappa'\in C^{\infty}(\boldsymbol{R}^n\setminus\{0\})$ is a cut-off function in a conical neighbourhood of $\delta\mathring{x}$ i.e. supp $\kappa'\subset\boldsymbol{R}^n_+$, κ' is homogeneous of order zero and $\kappa'(\delta\mathring{x})\neq 0$.

3° $\chi\in C^{\infty}(\mathrm{i}(\boldsymbol{R}^n\setminus\{0\}))$ be a function with supp $\chi\subset\{\mathrm{i}\tau\colon 1/4\leq\|\tau\|\}$ and $\chi(\mathrm{i}\tau)\equiv 1$ for $\tau\in\{\tau\colon 1/2\leq\|\tau\|\}$. The function χ is extended to $\boldsymbol{R}^n+\mathrm{i}(\boldsymbol{R}^n\setminus\{0\})$ by putting $\chi(z)=\chi(\mathrm{Im}\, z)$.

Definition 2. Let $(\mathring{x}, \mathring{\xi})\in T_0^*(\boldsymbol{R}^n)\simeq\{(0, \xi)\colon \xi\in\boldsymbol{R}^n\}$, $\delta\mathring{x}\in\boldsymbol{R}^n_+$, $\delta\mathring{\xi}\in\boldsymbol{R}^n$, $u\in SP(s, -\infty)=\bigcup_{\sigma'} SP(s, \sigma')$. We say that u belongs to $SP(s, s')$ 2-microlocally at the point $(\mathring{x}, \mathring{\xi}, \delta\mathring{x}, \delta\mathring{\xi})=(0, \mathring{\xi}, \delta\mathring{x}, 0)$ if there exist functions ρ, κ, χ satisfying conditions 1°, 2°, 3° respectively, such that

$$\tilde{P}(x, D)u = \chi(xD)\rho(xD)\kappa(x)u\in SP(s, s').$$

If $u\in SP(s, -\infty)$ we define its $SP(s, s')$—second wave front set (denoted $2\,\mathrm{WF}^{SP(s,s')}u$) as a closed subset of the space

$$(T^*(\boldsymbol{R}^n)\setminus T_0^*(\boldsymbol{R}^n))\cup N_{T_0^*(\boldsymbol{R}^n)}(T(T^*\boldsymbol{R}^n))$$

consisting of the points $(x, \xi)\notin T_0^*$ such that $u\notin SP(s+s', 0)$ microlocally at the point (x, ξ) and of the points $(0, \xi, \delta x, 0)$ such that $u\notin SP(s, s')$ 2-microlocally (note that in our case the space normal to $T_0^*(\boldsymbol{R}^n)$ at the point $(\mathring{x}, \mathring{\xi})$ can be identified with the set of vectors of the form $(\delta\mathring{x}, 0)$ with $\delta\mathring{x}\in\boldsymbol{R}^n$ and the topology in $N_{T_0^*(\boldsymbol{R}^n)}$ coincides with the topology of conical neighbourhoods of the vectors $\delta\mathring{x}$ in $\boldsymbol{R}^n$).

Definition 3 ([1]). We say that a function $a(x, \xi)\in C^{\infty}(\boldsymbol{R}^{2n})$ belongs to $\sum_0^{m,m'}$ if and only if

$$|D_\xi^\alpha D_x^\beta a(x, \xi)|\leq C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|+\beta|}(1+\|x\|\,\|\xi\|)^{m'-|\beta|}$$

where $\langle\xi\rangle=(1+\|\xi\|^2)^{1/2}$.

Definition 4 ([1]). We say that a function $a(x, \xi)\in C^{\infty}((\boldsymbol{R}^n\setminus\{0\}\times\boldsymbol{R}^n)$ belongs to $S\sum_0^{m,m'}$ if

a) $a(x, \xi)$ is flat in the following sense:

$$|D_\xi^\alpha D_x^\beta a(x, \xi)|\leq C_{\alpha\beta}\|x\|^{-m+|\alpha|-|\beta|}(1+\|x\|\,\|\xi\|)^{m+m'-|\alpha|}$$

b) $a(x, D)$ is properly supported, in a strong sense, in $\boldsymbol{R}^n\setminus\{0\}$ i.e.

$$\operatorname{supp} \hat{a}^2(x, x-y) \subset \left\{(x, y)\colon k^{-1} < \frac{\|y\|}{\|x\|} < k\right\}$$

for some $k>1$. Here, $\hat{a}^2$ means the Fourier transform with respect to ξ. As in [1] for the function $a(x, \xi) \in \Sigma_0^{m,m'} (S\Sigma_0^{m,m'})$ we define an operator $a(x, D) \in \mathrm{Op}(\Sigma_0^{m,m'})(\mathrm{Op}(S\Sigma_0^{m,m'}))$.

Definition 5 ([1]). Let Λ be a conic lagrangean submanifold of $T^*\mathbf{R}^n$ (here Λ means $T_0^*(\mathbf{R}^n)$). Let $(\mathring{x}, \mathring{\xi})$ belong to Λ and let $(\delta\mathring{x}, \delta\mathring{\xi})$ be a vector tangent to $T^*\mathbf{R}^n$ at that point $(\mathring{x}, \mathring{\xi})$ and not tangent to Λ at that point. If $A \in \mathrm{Op}(\Sigma_0^{m,m'})$, we will say that A is elliptic at $(\mathring{x}, \mathring{\xi}, \delta\mathring{x}, \delta\mathring{\xi})$ if one has

$$|a(\mathring{x}+\varepsilon\delta\mathring{x}, \lambda(\mathring{\xi}+\varepsilon\delta\mathring{\xi}))| \geq C\lambda^m(\lambda\varepsilon)^{m'}$$

with $C>0$, for $0<\varepsilon\leq\varepsilon_0$ and $\lambda\varepsilon>\mu_0>0$ $(A=a(x, D))$.

Definition 6 ([1]). Let $u \in D'(\mathbf{R}^n)$ and let u vanish outside a unit ball in $\mathbf{R}^n$. We say that $u \in H^{s,k}$ $(s \in \mathbf{R}, k \in \mathbf{N}_0)$ if

$$x^\alpha u \in H^{s+|\alpha|}(\mathbf{R}^n) \qquad \text{for} \quad \alpha \in \mathbf{N}_0^n,\ |\alpha| \leq k\,.$$

For remaining (s, s') the spaces $H^{s,s'}$ are defined by duality and interpolation.

Let

$$\Sigma_0^{m,-\infty} = \bigcap_{m'} \Sigma_0^{m,m'} \quad S\Sigma_0^{m,-\infty} = \bigcap_{m'} S\Sigma_0^{m,m'}$$

$$H^{s,+\infty} = \bigcap_{s'} H^{s,s'} \quad SP(s, +\infty) = \bigcap_{s'} SP(s, s')$$

$$SP(s, -\infty) = \bigcup_{s'} SP(s, s') \quad SP(-\infty, -\infty) = \bigcup_{s,s'} SP(s, s')$$

Lemma 1. *Let $\delta\mathring{x} \in \mathbf{R}_+^n$, $\|\delta\mathring{x}\|=1$, $\mathring{\xi} \in \mathbf{R}^n$. Let χ, ρ, κ be defined as in 1°–3° and $x\xi=(x_1\xi_1, \cdots, x_n\xi_n)$ then*

$$\tilde{P}(x, \xi) = \chi(\mathrm{i}x\xi)\rho(\mathrm{i}x\xi)\kappa(x)$$

belongs to $S\Sigma_0^{0,0}$ (actually to $\Sigma_0^{0,0}$, and the corresponding operator in $\mathrm{Op}(\Sigma_0^{0,0})$ is elliptic at the point $(0, \mathring{\xi}, \delta\mathring{x}, 0)$). Moreover the symbol $\tilde{P}(x, \xi)$ is concentrated around $(0, \mathring{\xi}, \delta\mathring{x})$ if and only if

$$P(x, z) = \chi(z)\rho(z)\kappa(x)$$

is concentrated around $(0, \mathrm{i}\mathring{\beta}, \delta\mathring{x})$ where $\mathrm{i}\mathring{\beta}=(\mathrm{i}\delta\mathring{x}_1\mathring{\xi}_1, \cdots, \mathrm{i}\delta\mathring{x}_n\mathring{\xi}_n)$.

Proof. We show the second part of the lemma. Since ρ is homogeneous of order 0 we have

$$\rho(\mathrm{i}x\xi)=\rho\left(\mathrm{i}\frac{x}{\|x\|}\xi\right).$$

κ is concentrated around $\delta\mathring{x}$ thus on the support of κ we have that $\frac{x}{\|x\|}$ is close to $\delta\mathring{x}$. On the support of ρ we have that $\mathrm{i}\frac{x}{\|x\|}\xi$ is close to $\mathrm{i}\mathring{\beta}$. Since for $\frac{x}{\|x\|}$ close to $\delta\mathring{x}$, $\frac{x}{\|x\|}\in \boldsymbol{R}^n_+$ it follows from the continuity of the operation of division that ξ is close to $\mathring{\xi}$ on supp $\tilde{P}(x,\xi)$.

Proposition 1. *$SP(s, s')$—second wave front set and $H^{s,s'}$—second wave front set defined by Bony coincide. More precisely: Let g be a smooth function with support in the ball $B(0,1)\subset \boldsymbol{R}$ equal to zero in a neighbourhood of zero. Then (see Remark* 2.10, [1]) *the mapping*

$$SP(s,s')\ni u\xrightarrow{\Pi} g(\|x\|\,\|D\|)u\in H^{s,s'}$$

where $g(\|x\|\,\|D\|)u=\frac{1}{(2\pi\mathrm{i})^n}\int e^{\mathrm{i}\langle x,\xi\rangle}g(\|x\|\|\xi\|)\hat{u}(\xi)d\xi$
defines an isomorphism Π *of the spaces*

$$SP(s,s')/SP(s,\infty)\xrightarrow{\Pi} H^{s,s'}/H^{s,\infty}.$$

Second wave front sets are well defined on the quotient space, thus we have well defined mappings

$$\begin{array}{ccc} SP(s,s')/SP(s,\infty) & \xrightarrow{\Pi} & H^{s,s'}/H^{s,\infty} \\ {\scriptstyle 2\mathrm{WF}^{SP(s,s')}}\searrow & & \swarrow{\scriptstyle 2\mathrm{WF}^{H^{s,s'}}} \\ & \mathcal{P} & \end{array} \tag{6}$$

where $\mathcal{P}$ is the set of closed subsets of the space

$$(T^*\boldsymbol{R}^n\setminus T^*_0(\boldsymbol{R}^n))\cup N_{T^*_0(\boldsymbol{R}^n)}(T(T^*\boldsymbol{R}^n)),$$

and $2\mathrm{WF}^B$ *means B-second wave front set.*
Assertion: *The diagram* (6) *commutes.*

Proof. We shall prove that if $u\in SP(s,s')$ 2-microlocally at the point $(\mathring{x},\mathring{\xi},\delta\mathring{x},0)$ where $(\mathring{x},\mathring{\xi})\in T^*_0(\boldsymbol{R}^n)$ then $\Pi u=g(\|x\|\,\|D\|)u\in H^{s,s'}$ 2-microlocally at that point. Let $Pf^*=\Pi$, then Pf induces an isomorphism Pf inverse to Π. According to Definition 2 we have $\tilde{P}u\in SP(s,s')$ hence

$$\tilde{P}(Pf\Pi u)\in SP(s, s') \tag{7}$$

because $Pf\Pi u-u\in SP(s, \infty)$ (Th. 29, [1]) and $\tilde{P}(x, D)\in \mathrm{Op}(S\sum_0^{0,0})$. It follows from (7) that

$$(\Pi\tilde{P}Pf)\Pi u\in H^{s,s'} .$$

It is clear that $\Pi\tilde{P}Pf$ is an operator in $\mathrm{Op}(\sum_0^{0,0})$ (for $\tilde{P}$ is an operator in $\mathrm{Op}(S\sum_0^{0,0})$). We still have to check that $\Pi\tilde{P}Pf$ is 2-microelliptic at $(\mathring{x}, \mathring{\xi}, \delta\mathring{x}, 0)$. But this is clear since the principal symbol

$$\sigma(\Pi\tilde{P}Pf) = \chi(\mathrm{i}x\xi)\tau(\mathrm{i}x\xi)\kappa(x)g(||x||\,||\xi||) = \tilde{P}(x, \xi)g(||x||\,||\xi||) .$$

(By Definition in [1] $\sigma(\Pi\tilde{P}Pf)$ is equal to the symbol $\sigma(\tilde{P})\in S\sum_0^{0,0}$ after passing through the isomorphism with the space $\sum_0^{0,0}$. But $\sigma(\tilde{P})=\tilde{P}(x, \xi)$ from the definition of the operator $\tilde{P}\in\mathrm{Op}(S\sum_0^{0,0})$ (Def. 3.5 [1]) since $\tilde{P}(x, \xi)\in S\sum_0^{0,0}$). Since Π is an isomorphism this ends the proof of commutativity of the diagram (6).

Remark. In Definition 1.9 in [1] of the second wave front set, it can be assumed that the operator $A\in\mathrm{Op}(\sum_0^{0,0})$ concentrated at $(0, \mathring{\xi})$ and 2-microelliptic at the point $(0, \mathring{\xi}, \delta\mathring{x}, 0)$ is of the form $\Pi\tilde{P}Pf$ where $\tilde{P}$ is defined by Definition 2 for suitable ρ, χ, κ.

Proof. Let $Au\in H^{s,s'}$ 2-microlocally. For suitable $\tilde{P}$, $\tilde{P}PfA^{-1}\in\mathrm{Op}(\sum_0^{0,0})$ and we have $\Pi\tilde{P}PfA^{-1}Au\in H^{s,s}$. But $A^{-1}A=I+R$, $R\in\mathrm{Op}(\sum_0^{0,-N})$ hence modulo an operator in $\mathrm{Op}(\sum_0^{0,-N})$ we have $\Pi\tilde{P}PfA^{-1}A=\Pi\tilde{P}Pf$.

Our aim is to define the second wave front set in a way analogous to the definition of the (first) wave front set in terms of the growth order of the Fourier transformation. In our case instead of the Fourier transformation we need the Mellin transformation. First we recall the definition of a Mellin distribution [2], [3].

Let $$B_+ = \{x\in\mathbf{R}_+^n : x_i\leq 1, \quad i = 1, \cdots, n\} .$$

Let $a\in\mathbf{R}^n$. Denote by $M_a=M_a(B_+)$ the space of functions $\varphi\in C^\infty(B_+)$ such that for every $\alpha\in\mathbf{N}_0^n$

$$p_{a,\alpha}(\varphi) = \sup_{x\in B_+}|(x^{a+1}(xD)^\alpha\varphi|<\infty$$

with topology given by the seminorms $p_{a,\alpha}$, $\alpha\in\mathbf{N}_0^n$. Let $\omega\in(\mathbf{R}\cup\{\infty\})^n$. We define

$$M_{(\omega)}(B_+) = \underset{\overrightarrow{a<\omega}}{\cup} M_a(B_+) \quad \text{—inductive limit.}$$

The dual space $M'_{(\omega)}(B_+)$ is a subspace of distributions and

$$M'(B_+) = \bigcup_{\omega} M'_{(\omega)}(B_+)$$

is called the space of Mellin (transformable) distributions.

Lemma 2. *Let* $u \in SP(s, s')$, $s+s' \in \boldsymbol{N}_0$, $\operatorname{supp} u \subset \Gamma$ *where* Γ *is a proper cone in* $\boldsymbol{R}^n_+$, *then* $u \in M'_{(\rho)}$ *for* $\rho \in \boldsymbol{R}^n$ $|\rho| \leq s - \frac{n}{2}$.

Proof. From (3) for $\lambda=0$ we have $x^\gamma u \in L^2(\boldsymbol{R}^n_+, x^{-1})$ for $|\gamma| \geq -s+\frac{n}{2}$ $\gamma \in \boldsymbol{R}^n$. Since multiplication by x^γ, $\gamma \in \boldsymbol{R}^n$ maps $M'_{(\omega)} \to M'_{(\omega+\gamma)}$ it is enough to show that if $u \in L^2(\boldsymbol{R}^n_+, x^{-1})$ then $u \in M'_{(0)}$. Let $\varphi \in M_a$, $a<0$; we have for $a<0$

$$|\int_{B_+} u\varphi\, dx| \leq \int_{B_+} |u|^2 x^{-1} dx \int_{B_+} x^1 |\varphi|^2 dx \leq C \int_{B_+} x^{-2a-1} dx < \infty$$

thus u is a continuous functional on the space $M_{(0)} = \bigcup_{a<0} M_a$. If $u \in M'_{(\omega)}$ we define the Mellin transform of u

$$(\mathcal{M}u)(z) = u[x^{-z-1}]\,, \qquad \operatorname{Re} z < \omega$$

(for $z \in \boldsymbol{C}^n$ and $\omega \in \boldsymbol{R}^n$, $\operatorname{Re} z < \omega$ means that $\operatorname{Re} z_j < \omega_j$ $j=1, \cdots, n$). The function $(\mathcal{M}u)(z)$ so defined is holomorphic for $\operatorname{Re} z < \omega$.

Lemma 3. $u \in L^2(\boldsymbol{R}^n_+, x^{-1})$ *if and only if* $(\mathcal{M}u)(z) \in L^2(\boldsymbol{R}^n)$ *as a function of* Im z *for each fixed* Re $z \leq 0$ (*i.e.* $\mathcal{M}(z)$ *defined for* Re $z<0$ *has a boundary value for* Re $z \to 0$, Re $z<0$ *and the boundary value is in* $L^2(\boldsymbol{R}^n)$).

Proof. The condition $u \in L^2(\boldsymbol{R}^n_+, x^{-1})$ is equivalent to the condition that the function $h(y) = u(e^{-y}) \in L^2(\boldsymbol{R}^n)$. Since

$$(\mathcal{M}u)(z) = \int_{\boldsymbol{R}^n_+} u(e^{-y}) e^{(z,y)} dy$$

we see that the latter condition is equivalent, in view of the Parsevale equality to the condition that $\mathcal{M}u(z) \in L^2(\boldsymbol{R}^n)$ for $\operatorname{Re} z=0$, hence also for $\operatorname{Re} z<0$ ($u(e^{-y})e^{(\operatorname{Re} z, y)}$ is in L^2 for $\operatorname{Re} z \leq 0$ if $u \in L^2$).

From Lemma 3 and the fact that

$$(\mathcal{M}(xD)^\lambda u)(z) = z^\lambda \mathcal{M}u(z)$$

we get the following characterization of the space $SP(s, s')$:

Corollary 1. *Let* $u \in D'(B_+)$, supp $u \subset \Gamma$ *—a proper cone, then the following conditions are equivalent:*

i) $u \in SP(s, s')$, $s+s' \in \boldsymbol{N}_0$,

ii) $\mathcal{M}u \in \mathcal{O}\left(\left\{z \in \boldsymbol{C}^n : \sum_{j=1}^n \operatorname{Re} z_j < s - \frac{n}{2}\right\}\right)$

and for every $z \in \left\{z \in \boldsymbol{C}^n : \sum_{j=1}^n \operatorname{Re} z_j \leq s - \frac{n}{2}\right\}$

$\mathcal{M}u(\operatorname{Re} z + \mathrm{i}\,\cdot) \in L^{2,s+s'}(\boldsymbol{R}^n) = L^2(\boldsymbol{R}^n, (1+||\beta||)^{s+s'})$,

iii) $\mathcal{M}u \in \mathcal{O}\left(\left\{z \in \boldsymbol{C}^n : \sum_{j=1}^n \operatorname{Re} z_j < s - \frac{n}{2}\right\}\right)$

and for some $\mathring{\alpha}$ *such that* $\sum_{j=1}^n \mathring{\alpha}_j = s - \frac{n}{2}$

$\mathcal{M}u(\mathring{\alpha} + \mathrm{i}\,\cdot) \in L^{2,s+s'}(\boldsymbol{R}^n)$.

Now let $\Gamma \subset \boldsymbol{R}^n_+$ be a cone tangent only to the axis x_1.

Proposition 2. *Let* $u \in D'(B_+)$, supp $u \subset \Gamma$ *—as above. Then the following conditions are equivalent:*

i) $u \in L^2(\Gamma, x^{-1})$,

ii) $\mathcal{M}u(z)$ *is holomorphic in the set* $\Omega_0 = \{z \in \boldsymbol{C}^n : \sum_{j=1}^n \operatorname{Re} z_j < 0, \alpha_2 < 0, \cdots, \alpha_n < 0\}$ *and for every* $\alpha \in \{\alpha \in \boldsymbol{R}^n : \alpha_1 + \cdots + \alpha_n \leq 0, \alpha_2 \leq 0, \cdots, \alpha_n \leq 0\}$, $\mathcal{M}u(\alpha + \mathrm{i}\beta) \in L^2(\boldsymbol{R}^n)$ *as a function of* β,

iii) $\mathcal{M}u$ *is holomorphic in* Ω_0 *and* $\mathcal{M}u(\mathring{\alpha} + \mathrm{i}\beta) \in L^2(\boldsymbol{R}^n)$ *for* $\mathring{\alpha} = 0$.

More generally we have

Proposition 3. *Let* $u \in D'(B_+)$, supp $u \subset \Gamma$ *as above,* $s, s' \in \boldsymbol{R}$, $s+s' \in \boldsymbol{N}_0$. *The following conditions are equivalent:*

i) $u \in SP(s, s')$,

ii) $\mathcal{M}u(z)$ *is holomorphic in the set*

$$\Omega_{s,s'} = \Big\{z \in \boldsymbol{C}^n : \operatorname{Re} z_1 + \cdots + \operatorname{Re} z_n < s - \frac{n}{2},$$
$$\operatorname{Re} z_2 < s + s' - \frac{1}{2}, \cdots, \operatorname{Re} z_n < s + s' - \frac{1}{2}\Big\}$$

and for every $\lambda \in \boldsymbol{N}_0^n$ *with* $|\lambda| \leq s+s'$ *and* $\alpha \in \boldsymbol{R}^n$ *such that*

$\alpha_1 + \cdots + \alpha_n \leq s - \frac{n}{2}, \quad \alpha_2 \leq \lambda_2 - \frac{1}{2}, \cdots, \alpha_n \leq \lambda_n - \frac{1}{2}$,

$\mathcal{M}u(\alpha + \mathrm{i}\beta) \in L^2(\boldsymbol{R}^n, (1+||\beta_1||)^{\lambda_1} \cdot \cdots \cdot |(1+||\beta_n||)^{\lambda_n})$,

iii) *$\mathcal{M}u$ is holomorphic in $\Omega_{s,s'}$ and*
$\mathcal{M}u(\alpha+i\beta)\in L^2(\boldsymbol{R}^n, (1+\|\beta_1\|)^{\lambda_1}\cdot\cdots\cdot(1+\|\beta_n\|)^{\lambda_n})$ for the points α of the form $\alpha_1=s-\frac{1}{2}-\lambda_2-\cdots-\lambda_n$, $\alpha_2=\lambda_2-\frac{1}{2}, \dots, \alpha_n=\lambda_n-\frac{1}{2}$ where
$0\le\lambda_2+\cdots+\lambda_n\le s+s'$.

Proof. From the operational properties of the Mellin transformation and Proposition 2 we obtain from (5) that for $|\lambda|\le s+s'$

$$\begin{aligned}&\mathcal{M}(x^{1/2}x_1^{-s+|\lambda|}D^\lambda u)(z)\\&=p_\lambda(z)\mathcal{M}u\left(z_1+s-\frac{1}{2}-\lambda_2-\cdots-\lambda_n,\ z_2+\lambda_2-\frac{1}{2},\ \dots,\ z_n+\lambda_n-\frac{1}{2}\right)\end{aligned}$$

belongs to $L^2(\boldsymbol{R}^n)$ as a function of Im z for any fixed Re z such that

$$\operatorname{Re} z_1+\operatorname{Re} z_2+\cdots+\operatorname{Re} z_n\le 0\,,\quad \operatorname{Re} z_2\le 0,\ \dots,\ \operatorname{Re} z_n\le 0$$

where

$$\begin{aligned}p_\lambda(z)=&\left(z_1+s-\frac{1}{2}-\lambda_1-\cdots-\lambda_n+1\right)\cdot\cdots\cdot\left(z_1+s-\frac{1}{2}-\lambda_2\cdots-\lambda_n\right)\\&\cdot\cdots\cdot\left(z_n+1-\frac{1}{2}\right)\cdots\left(z_n+\lambda_n-\frac{1}{2}\right).\end{aligned}$$

By changing the variable z one can see that all assertions of Proposition 3 follow from those of Proposition 2.

Proposition 4. *Let $(\mathring{x}, \mathring{\xi})\in T_0^*(\boldsymbol{R}^n)$, $\delta\mathring{x}\in\boldsymbol{R}_+^n$, $u\in SP(s, -\infty)$. $u\in SP(s, s')$ 2-microlocally at the point $(\mathring{x}, \mathring{\xi}, \delta\mathring{x}, 0)$ if and only if there exist functions χ, κ, ρ satisfying conditions 1°, 2°, 3° respectively, such that*

$$\chi(z)\rho(z)\mathcal{M}(\kappa u)(z)\,|_{z=\alpha+i\cdot}\in L^{2,s+s'}(\boldsymbol{R}^n)$$

for $\alpha_1+\cdots+\alpha_n\le s-\frac{n}{2}$.

Corollary 2. *The point $(\mathring{x}, \mathring{\xi}, \delta\mathring{x}, 0)$ does not belong to $2\mathrm{WF}^{SP(s,\infty)}$ if and only if for some χ, κ, ρ satisfying conditions 1°, 2°, 3° $\chi(z)\rho(z)\mathcal{M}(\kappa u)(z)$ as a function of* Im z *is rapidly decreasing for z:* $\operatorname{Re} z_1+\cdots+\operatorname{Re} z_n\le s-\frac{n}{2}$.

§ 3. Propagation of 2-Microlocal Singularities

Theorem 1. (*Propagation of singularities along the incoming bicharacteristic*). *Let $\delta\mathring{x}=(-1, 0)$, $\mathring{\xi}=(0, \mathring{\xi}')$ and let $v\in SP(s, \infty)$. Suppose that $v\in H^{s+\sigma}$*

microlocally at $(x, \mathring{\xi})$ *for every* $x=(x_1, 0)$ *where* $x_1<0$ *and that* $w=\frac{\partial}{\partial x_1}v\in SP(s-1, \sigma+1)$ *2-microlocally at* $(0, \mathring{\xi}, \delta\mathring{x}, 0)$. *If* $\sigma>-\frac{1}{2}$ *then* $v\in SP(s, \sigma)$ *2-microlocally at* $(0, \mathring{\xi}, \delta\mathring{x}, 0)$.

Proof. Replacing v by $q(D)v$ where $q(D)$ is a suitable pseudodifferential operator microelliptic at $\mathring{\xi}$, we may assume $\left(\text{since } \left[q(D), \frac{\partial}{\partial x_1}\right]=0\right)$ that both v and w are microlocally concentrated near $\mathring{\xi}$. Further we may multiply v by a bump function φ at zero and observe that $\frac{\partial}{\partial x_1}(\varphi v)=\left(\frac{\partial}{\partial x_1}\varphi\right)v+\varphi\frac{\partial}{\partial x_1}v$ satisfies the same assumption as w since $\left(\frac{\partial}{\partial x_1}\varphi\right)v\in H^{s+\sigma}$ $\Big($by the assumption that $v\in H^{s+\sigma}$ microlocally at $((x_1, 0), \mathring{\xi})$ with $x_1<0$ we have $\left(\frac{\partial}{\partial x_1}\varphi\right)v\in SP(s, \sigma)\subset SP(s-1, \sigma+1)$ because $0\notin \operatorname{supp}\left(\frac{\partial}{\partial x_1}\varphi\right)\Big)$. Consequently we may assume that v and w are microlocally concentrated near $(0, \mathring{\xi})$. Finally changing x_1 to $-x_1$ we assume that $\delta\mathring{x}=(1, 0)$. Let $\tilde{\chi}$ be a cut-off function at $(1, 0)$ subordinated to Γ homogeneous of order zero (supp $\tilde{\chi}\subset\Gamma$ and $\tilde{\chi}\equiv 1$ on a cone Γ_1 tangent to x, slightly smaller then Γ) such that

$$\tilde{w} = \tilde{\chi}w\in SP(s-1, \sigma+1) \text{ globally.}$$

First we show that there exists a unique Mellin distribution u such that

$$\tilde{w} = \frac{\partial}{\partial x_1}u\,. \tag{8}$$

By Proposition 3 $\mathcal{M}\tilde{w}(z)$ is holomorphic for $\sum_1^n \operatorname{Re} z_j<s-1-\frac{n}{2}$, $\operatorname{Re} z_2<s+\sigma-\frac{1}{2}, \cdots, \operatorname{Re} z_n<s+\sigma-\frac{1}{2}$. Computing formally the Mellin transform of (8) we get

$$\mathcal{M}\tilde{w}(z) = (z_1+1)\mathcal{M}u(z_1+1, z')\,, \qquad z' = (z_2, \cdots, z_n)$$

hence

$$\mathcal{M}u(z) = \frac{\mathcal{M}\tilde{w}(z_1-1, z')}{z_1} \tag{9}$$

on the set $\left\{\sum_j \operatorname{Re} z_j<s-\frac{n}{2}, \operatorname{Re} z_2<s+\sigma-\frac{1}{2}, \cdots, \operatorname{Re} z_n<s+\sigma-\frac{1}{2}, \operatorname{Re} z_1<0\right\}$.

Moreover for a fixed $\mathring{\alpha}$ such that $\sum\mathring{\alpha}_j<s-\frac{n}{2}$, $\mathring{\alpha}_2<s+\sigma-\frac{1}{2}, \cdots, \mathring{\alpha}_n<s+\sigma-\frac{1}{2}$,

$\mathring{\alpha}_1<0$ we have from Proposition 3 (ii) that $\mathcal{M}\tilde{w}(\mathring{\alpha}-1+\mathrm{i}\beta)\in L^2(\boldsymbol{R}^n)$. Since $|\mathring{\alpha}_1+\mathrm{i}\beta_1|>|\mathring{\alpha}_1|>0$ it follows that also

$$\frac{\mathcal{M}\tilde{w}(\mathring{\alpha}-1+\mathrm{i}\beta)}{\mathring{\alpha}_1+\mathrm{i}\beta_1}\in L^2(\boldsymbol{R}^n).$$

Thus the inversion theorem for the Mellin transformation (see [2]) implies the existence and uniqueness of the desired Mellin distribution u.

From (ii) in Proposition 3 we get

$$\mathcal{M}u(\alpha+\mathrm{i}\beta)\in L^2(\boldsymbol{R}^n,\ (1+||\beta_1||)^{\lambda_1}\cdot\cdots\cdot(1+||\beta_n||)^{\lambda_n})$$

for every fixed α such that $\sum\alpha_j\leq s-\frac{n}{2}$, $\alpha_2<\lambda_2-\frac{1}{2}$, $\cdots$, $\alpha_n<\lambda_n-\frac{1}{2}$, $\alpha_1<0$, since as before $|\alpha_1+\mathrm{i}\beta_1|\geq|\alpha_1|>0$.

In order to retain the information on u which will enable us to conclude that $u\in SP(s,\sigma)$ 2-microlocally at $\delta\mathring{x}=(1,0)$ we have to assume, according to (iii) in Proposition 3, that

$$\alpha_1=s-\frac{1}{2}-\lambda_2-\cdots-\lambda_n<0 \qquad \text{for} \quad 0\leq\lambda_2+\lambda_3+\cdots+\lambda_n\leq s+\sigma$$

i.e. that $\sigma>-\frac{1}{2}$ and $s<\frac{1}{2}$ (see Fig. 1)

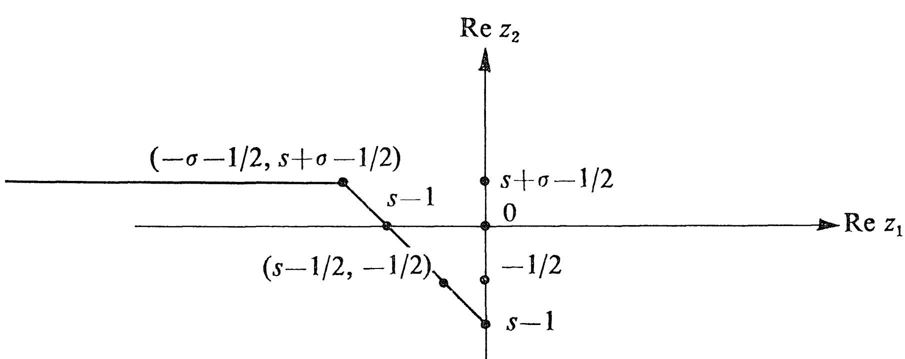

Fig. 1

Let $\Gamma_1\subset\Gamma$ be a cone tangent to the axis x_1 such that $\tilde{\chi}\equiv1$ on Γ_1. We shall prove that $u=v$ on Γ_1.

We have

$$\frac{\partial}{\partial x_1}u=\frac{\partial}{\partial x_1}v \qquad \text{on} \quad \Gamma_1.$$

Hence for $x\in\Gamma_1$ $u-v=g(x')$ —a Mellin distribution on Γ_1 in variables $x'=(x_2,\cdots,x_n)$. Since both u and v have bounded support and Γ_1 is tangent to the axis x_1 it follows that $g\equiv 0$ on Γ_1.

Let Γ_0 be the cone $\{x\in\mathbf{R}^n_+ : 0<x_1<1,\ x_2<x_1,\cdots,x_n<x_1\}$ (see Fig. 2). We want to show that $u|_{\Gamma_0}\in SP(s,\sigma)$. From Example 2 in [2] we know that if χ_0 is the characteristic function of Γ_0, we have

$$\mathcal{M}\chi_0(z)=\frac{(-1)^n}{(z_1+z_2+\cdots+z_n)z_2\cdot\cdots\cdot z_n}.$$

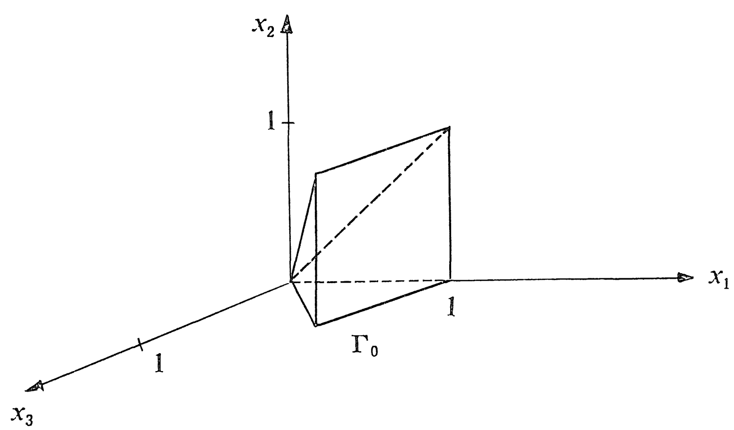

Fig. 2

Next (see [3], Proposition 5) for every $\mathring{\alpha}$ such that $\sum\mathring{\alpha}_j\leq s-\frac{n}{2}$, $\mathring{\alpha}_2\leq s+\sigma-\frac{1}{2}$, $\cdots$, $\mathring{\alpha}_n\leq s+\sigma-\frac{1}{2}$, $\mathring{\alpha}_1<0$ we have

$$\mathcal{M}(\chi_0 u)(\mathring{\alpha}+i\beta)=\lim_{\substack{\alpha\to\mathring{\alpha}\\ \alpha<\mathring{\alpha}}}(\mathcal{M}u(\mathring{\alpha}+i\gamma)*\mathcal{M}(\chi_0)(\alpha-\mathring{\alpha}+i\gamma))(\beta)$$

which, after the change of variables $\bar{\gamma}_1=\gamma_1+\cdots+\gamma_n$, $\bar{\gamma}_2=\gamma_2,\cdots,\bar{\gamma}_n=\gamma_n$ in the convolution, is the n-dimensional Hilbert transform of $\mathcal{M}u(\mathring{\alpha}+i\cdot)$. Now it is a well known fact that the Hilbert transform maps L^2-functions into L^2-functions. This proves that $\mathcal{M}(\chi_0 u)(\mathring{\alpha}+i\beta)\in L^2$, and the same is true for $\mathcal{M}(x^{1/2}x_1^{-s+|\lambda|}\chi_0 D^\lambda u)$ for $|\lambda|\leq s+\sigma$. This in view of Lemma 3 shows that $u|_{\Gamma_0}\in SP(s,\sigma)$. Since $v=u$ on Γ_1 it follows that $v|_{\Gamma_1\cap\Gamma_0}\in SP(s,\sigma)$.

Finally we note that the condition $s<\frac{1}{2}$ is not an essential restriction on w. Indeed, we may replace w by $\Delta^m w$ for m large enough where Δ is the Laplace operator in $\boldsymbol{R}^n$ and use the facts that $\left[\Delta^m, \frac{\partial}{\partial x_1}\right]=0$, $\Delta^m: SP(s, s')\to SP(s-2m, s')$ and if $\Delta^m u\in SP(s, s')$, $u\in SP(s, -\infty)$ then $u\in SP(s+2m, s')$, since an operator elliptic in the classical sense is 2-microelliptic.

Lemma 4. *Let Γ be a proper cone in $\boldsymbol{R}^n_+$, and Γ' and open subcone in Γ. Write $x=(x_1, x')$ for points $x\in\boldsymbol{R}^n$ where $x'\in\boldsymbol{R}^{n-1}$. Let $g(x')$ be a Mellin distribution on Γ, independent of x_1 and such that $g(x')\in SP(s, s')$ on Γ'. Then $g(x')\in SP(s, s')$ on Γ.*

Proof. Let χ' be a cut-off function subordinated to Γ' such that $\chi'(x)g(x')\in SP(s, s')$ globally. Since for every vector $\delta x\in\Gamma$ we can find $k>0$ such that

$$\chi(x) = \chi'(kx_1, x')$$

is a cut-off function in the direction of δx, it is enough to prove that $\chi(x)g(x')\in SP(s, s')$ globally. We have

$$\begin{aligned}\mathcal{M}(\chi(x)g(x'))(z_1, z') &= 1_{x_1}\otimes g(x')[\chi(x)x_1^{-z_1-1}x'^{-z'-1}]\\ &= g[x'^{-z'-1}\int_{\boldsymbol{R}}\chi(x_1, x')x_1^{-z_1-1}dx_1]\\ &= k^{z_1}g[x'^{-z'-1}\int_{\boldsymbol{R}}\chi'(x_1, x')x_1^{-z_1-1}dx]\\ &= k^{z_1}\mathcal{M}(\chi'(x)g(x'))(z),\end{aligned}$$

because

$$\int_{\boldsymbol{R}}\chi'(kx_1, x')x_1^{-z_1-1}dx_1 = k^{z_1}\int_{\boldsymbol{R}}\chi'(x_1, x')x_1^{-z_1-1}dx_1 .$$

Since $|k^{z_1}|=k^{\mathrm{Re}\, z_1}$ thus for a fixed Re z the multiplier k^{z_1} has no influence on the behaviour of the Mellin transform as $|\mathrm{Im}\, z|\to\infty$. This in view of Proposition 3 ends the proof.

Theorem 2. (*Propagation of singularities along second bicharacteristics*). *Let Γ be a cone in $\boldsymbol{R}^n$ not tangent to x_1 i.e. such that $\bar{\Gamma}\cap\{(x_1, 0), x_1\in\boldsymbol{R}\}=\{0\}$. Suppose $v\in SP(-\infty, -\infty)$ on Γ and let $\frac{\partial}{\partial x_1}v=w\in SP(s-1, \sigma+1)$ 2-microlocally at the points $(0, \mathring{\xi}, \delta x, 0)$ for $\delta x\in\Gamma$ where $\mathring{\xi}=(0, \mathring{\xi}')$ for some fixed $\mathring{\xi}'$. If $v\in SP(s, \sigma)$ 2-microlocally at $(0, \mathring{\xi}, \delta x, 0)$ for $\delta x\in\Gamma'$—a subcone of Γ, then $v\in SP(s,\sigma)$ 2-microlocally at $(0, \mathring{\xi}, \delta x, 0)$ for $\delta x\in\Gamma$.*

Proof. We assume that u and w are microlocally concentrated near $\mathring{\xi}$. Further (subject to a transformation of the form

$$\begin{bmatrix} \pm 1 & B \\ 0 & A \end{bmatrix}$$

where $A \in GL(n-1)$, $B \in \boldsymbol{R}^{n-1}$, which preserves $\frac{\partial}{\partial x_1}$) we may assume that Γ is a proper subcone of $\boldsymbol{R}^n_+$. Let $\tilde{\chi}$ be a cut-off function subordinated to Γ (i.e. $\tilde{\chi} \in C^\infty(\boldsymbol{R}^n_+)$, $\tilde{\chi}$ homogeneous of order zero, supp $\tilde{\chi} \subset \Gamma$ and $\tilde{\chi} \equiv 1$ on a slightly smaller proper subcone of Γ), and φ a bump function at 0 such that $\tilde{w} = \varphi \tilde{\chi} w \in SP(s-1, \sigma+1)$ globally.

Analogously to the proof of Theorem 1 we show that there exists a Mellin distribution u such that $\tilde{w} = \frac{\partial}{\partial x_1} u$. Computing the Mellin transformation we get

$$\mathcal{M}u(z) = \frac{\mathcal{M}\tilde{w}(z_1 - 1, z')}{z_1}$$

and as in the proof of Theorem 1 we conclude in view of Corollary 2 that

$$\mathcal{M}u(\alpha + i\beta) \in L^{2, s+\sigma}(\boldsymbol{R}^n)$$

for every fixed α such that $\sum \alpha_j \leq s - \frac{n}{2}$, $\alpha_1 < 0$.

We want to prove that $\mathcal{M}\chi u(\mathring{\alpha} + i\beta) \in L^2(\boldsymbol{R}^n)$ where $\mathring{\alpha}$ is a fixed point satisfying $\sum \mathring{\alpha}_j = s - \frac{n}{2}$, $\mathring{\alpha}_1 < 0$, and χ is the characteristic function of a proper cone containing Γ. To use the same technique as in the proof of Theorem 1 we suppose (again, by applying a transformation of the form $\begin{bmatrix} 1 & B \\ 0 & A \end{bmatrix}$) that $\Gamma \subset \{x \in \boldsymbol{R}^n_+ : x_2 < x_1, \cdots, x_n < x_1\}$. We also suppose that supp $\tilde{w} \subset \{x \in \boldsymbol{R}^n_+ : x_1 \leq 1\}$. For $x \in \Gamma$ we have $x_1 < C_2 x_2, \cdots, x_1 < C_n x_n$ for some positive constants $C_2, C_3, \cdots, C_n$. Since $u(x_1, x') = -\int_{x_1}^\infty \tilde{w}(t, x')\, dt$ it follows that for $x \in$ supp u $x_1 < C_2 x_2, \cdots, x_1 < C_n x_n$ and $x_1 \leq 1$. Thus if χ^0 is the characteristic function of the set $\Gamma^0_1 = \{x \in \boldsymbol{R}^n_+ : x_1 \leq 1, x_2 < x_1, \cdots, x_n < x_1\}$ it follows that supp $\chi^0 u$ is a proper cone $\tilde{\Gamma}$ in $\boldsymbol{R}^n_+$ since $x_j \sim x_k$ $j, k = 1, 2, \cdots, n$ on supp $\chi^0 u$. The proof that $\mathcal{M}\chi^0 u(\mathring{\alpha} + i\beta) \in L^2(\boldsymbol{R}^n)$ is the same as in the proof of Theorem 1. Analogously, in view of Corollary 1 we get $u \in SP(s, \sigma)$ on $\tilde{\Gamma}$. The final thing now is to compare u and v. On the set where $\varphi\tilde{\chi} \equiv 1$ we have $\frac{\partial}{\partial x_1} u = \frac{\partial}{\partial x_1} v$

thus $u-v=g(x')$ is a Mellin distribution on that set depending only on x'. Since on a smaller subcone $v\in SP(s, \sigma)$ it follows that $g(x')\in SP(s, \sigma)$ on that subcone. Now we apply Lemma 4 to get $g(x')\in SP(s, \sigma)$ on Γ. This ends the proof.

Theorem 3. (*Propagation of singularities along the outgoing bicharacteristic*). *Let* $\delta\mathring{x}=(1, 0)$, $\mathring{\xi}=(0, \mathring{\xi}')$ *and let* $v\in SP(s, -\infty)$ *be such that* $v=SP(s, \sigma)$ *2-microlocally at the points* $(0, \mathring{\xi}, \delta x, 0)$ *where* $\delta x\in \boldsymbol{R}^n_+$. *Suppose that*

$$w=\frac{\partial}{\partial x_1}v$$

and $w\in SP(s-1, \sigma+1)$ *2-microlocally at* $(0, \mathring{\xi}, \delta\mathring{x}, 0)$. *If* $\sigma<-\frac{1}{2}$ *then* $v\in SP(s, \sigma)$ *2-microlocally at* $(0, \mathring{\xi}, \delta\mathring{x}, 0)$.

Proof. By applying to v a suitable cut-off function $\tilde{\chi}$ at $(1, 0)$ subordinated to a cone $\Gamma\subset\boldsymbol{R}^n_+$ tangent only to x_1, and a cut-off function ρ in the direction $\mathring{\xi}$ we may assume that supp $v\subset\Gamma$ and $w\in SP(s-1, \sigma+1)$ locally at zero. Thus for a suitable bump function φ at zero $\tilde{w}=\varphi w\in SP(s-1, \sigma+1)$ globally.

As in the proof of Theorem 1 we prove the existence (and uniqueness) of a Mellin distribution u such that

$$\tilde{w}=\frac{\partial}{\partial x_1}u\,.$$

Moreover

$$\mathcal{M}u(z)=\frac{H(z)}{z_1}\qquad\text{for}\quad \operatorname{Re} z_1<0,\ \operatorname{Re} z\in A$$

where $H(z)=\mathcal{M}(x_1\,\tilde{w})(z)$ is holomorphic on the set $A+\mathrm{i}\boldsymbol{R}^n$ and

$$A=\left\{\alpha\in\boldsymbol{R}^n\colon \sum\alpha_j<s-\frac{n}{2},\ \alpha_2<s+\sigma-\frac{1}{2},\ \cdots,\ \alpha_n<s+\sigma-\frac{1}{2}\right\}.$$

Further for points $\alpha\in\bar{A}$, $H(\alpha+\mathrm{i}\beta)\in L^2(\boldsymbol{R}^n)$ as a function of β. We want to invert $F(z)=H(z)/z_1$ at a point $\mathring{\alpha}\in A$ such that $\mathring{\alpha}_1>0$. To this end we use the generalized Mellin transformation $M^{\mathring{\alpha}}$ introduced in [3]. We have

$$(M^{\mathring{\alpha}}f)(\beta)=F(\mathring{\alpha}+\mathrm{i}\beta)$$

where

$$f(x)=\int_{\boldsymbol{R}^n}F(\mathring{\alpha}+\mathrm{i}\gamma)x^{\mathring{\alpha}+\mathrm{i}\gamma}d\gamma\,. \tag{9}$$

Take a point $\tilde{\alpha}=(\tilde{\alpha}_1, \mathring{\alpha}')$ where $\tilde{\alpha}_1<0$, $\mathring{\alpha}=(\mathring{\alpha}_1, \mathring{\alpha}')$. Then

$$u(x) = \int_{R^n} F(\tilde{\alpha}+i\gamma)x^{\tilde{\alpha}+i\gamma}d\gamma .$$

Fixing $\mathring{\alpha}'+i\gamma'$ and applying the residuum theorem in variable z_1 we get

$$\int_R F(\tilde{\alpha}+i\gamma)x_1^{\tilde{\alpha}_1+i\gamma_1}d\gamma_1 = \int_R F(\mathring{\alpha}+i\gamma)x_1^{\mathring{\alpha}_1+i\gamma_1}d\gamma_1+H(0, \mathring{\alpha}'+i\gamma') .$$

Thus we find

$$u(x) = f(x)+ \int_{R^{n-1}} H(0, \mathring{\alpha}'+i\gamma')x'^{\mathring{\alpha}'+i\gamma'}d\gamma' .$$

Since by the definition of the Mellin transformation

$$H(0, \mathring{\alpha}+i\gamma') = x_1\tilde{w}[x_1^{-1}x'^{-\mathring{\alpha}'-i\gamma'-1}] = \tilde{w}[x'^{-\mathring{\alpha}'-i\gamma'-1}] ,$$

we have by the inversion rule (9)

$$\int_{R^{n-1}} H(0, \mathring{\alpha}'+i\gamma')x'^{\mathring{\alpha}'+i\gamma'-1}d\gamma' = \int_R \tilde{w}(x_1, x')dx_1$$

$$= \int_R \frac{\partial}{\partial x_1} u(x_1, x')dx_1 = -u(0, x')$$

because u has bounded support as a Mellin distribution. Thus

$$f(x) = u(x)-u(0, x') .$$

Since $f(x)=u(x)-u(0, x')=\int_0^{x_1} w(t, x')dt$ we see that $\operatorname{supp} f\subset\Gamma$. To prove that $f\in SP(s, \sigma)$ locally we have to ensure analogously to the proof of Theorem 1 that

$$\alpha_1 = s-1/2-\lambda_2-\cdots-\lambda_n>0 \qquad \text{for} \quad 0\leq\lambda_2+\lambda_3+\cdots+\lambda_n\leq s+\sigma$$

i.e. that $\sigma<1/2$ and $s>1/2$. Now multiplying f by the characteristic function of the unit cube $\{0<x\leq 1\}$, which under the Mellin transformation amounts to computing the Hilbert transform of $M^{\mathring{\alpha}}f$ $\left(\text{since } \mathcal{M}\chi(z)=\frac{(-1)^n}{z_1\cdot\cdots\cdot z_n}\right)$, we conclude, as in the proof of Theorem 1, that $f\in SP(s, \sigma)$ locally.

Finally we compare f and v. We have $\frac{\partial}{\partial x_1}f-\frac{\partial}{\partial x_1}v=0$ for $x\in R^n_+$, $0<x_1<\tau$ for some τ, thus $f-v=g(x')$ for $0<x_1<\tau$. Since $\operatorname{supp} f$ $\operatorname{supp} v\subset\Gamma$ it follows that $g(x')\equiv 0$. This proves that $v\in SP(s,\sigma)$ locally at zero if $s>1/2$. The case $s\leq 1/2$ is reduced to the above by a similar reasoning as at the end of the proof of Theorem 1.

Theorems 1, 2, 3 together give the following Bony's theorem on the propagation of 2-microlocal singularities:

Theorem 4. *Let* $\mathring{\xi}=(0,\mathring{\xi}')$ *and suppose* $v\in SP(s,-\infty)$ *microlocally at* $(0,\mathring{\xi})$ *and* $v\in H^{s-1/2}$ *microlocally at* $(x_1,0,\mathring{\xi})$ *where* $x_1<0$. *If* $w=\dfrac{\partial}{\partial x_1}v\in SP(s-1,-1/2)$ *microlocally at* $(0,\mathring{\xi})$ *then* $v\in SP(s-\varepsilon,-1/2)$ *microlocally at* $(0,\mathring{\xi})$ *for any* $\varepsilon>0$.

Proof. To apply Theorem 1 we observe that for $a\in\boldsymbol{R}$ $SP(a,-1/2)\subset SP(a-\varepsilon,-1/2+\varepsilon)$, while the inclusions $SP(a,-1/2)\subset SP(a,-1/2-\varepsilon)\subset SP(a-\varepsilon,-1/2)$ allow us to apply Theorem 3.

Concluding remark. The technique of the Mellin transformation presented in the paper can also be applied to get analogous results for the spaces $H^{s,s'}$.

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