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Remark to the Ergodic Decomposition of Measures

By

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§ 1. Introduction

Let $(X, \mathfrak{B}, \mu)$ be a measure space and $\mathfrak{A}$ be a sub-σ-field of $\mathfrak{B}$. A family $\{\mu^x\}_{x\in X}$ of probability measures on $\mathfrak{B}$, indexed by x is called a system of conditional probabilities with respect to $\mathfrak{A}$ or a disintegration of μ with respect to $\mathfrak{A}$ if it has the following properties, namely

(a) $\forall B\in\mathfrak{B}$, the function $x\mapsto\mu^x(B)$ is $\mathfrak{A}$-measurable and

(b) $\forall B\in\mathfrak{B},\quad \forall A\in\mathfrak{A},\quad \mu(B\cap A)=\int_A \mu^x(B)d\mu(x)\,.$

In general, disintegrations of μ with respect to $\mathfrak{A}$ do not exist. (See an example in the later discussions.) However, if $(X, \mathfrak{B})$ is standard (that is, the measurable space $(X, \mathfrak{B})$ is isomorphic to $(Y, \mathfrak{B}_Y)$, where Y is a Polish space and $\mathfrak{B}_Y$ is the Borel σ-field of Y), then a disintegration of any probability measures on $\mathfrak{B}$ exists for all $\mathfrak{A}$ $(\subset\mathfrak{B})$. (For example, see [1].) If a disintegration of μ with respect to $\mathfrak{A}$ exists, then for any fixed $A\in\mathfrak{A}$, $\mu^x(A)=\chi_A(x)$ holds for μ-a.e.x, where χ_A is the indicator function of A. Especially for any fixed $A\in\mathfrak{A}$, $\mu^x(A)$ takes only the values 0 or 1 for μ-a.e.x. A strengthening form of this result is as follows.

(c) For μ-a.e.x, μ^x takes only the values 0 or 1 on $\mathfrak{A}$.

If a disintegration $\{\mu^x\}_{x\in X}$ of μ with respect to $\mathfrak{A}$ satisfies (c), then it is called an ergodic decomposition. The following fact is known for the ergodic decomposition.

Theorem. *Let $(X, \mathfrak{B})$ be a standard space, $\{\mathfrak{A}_n\}$ $(n=1, 2, \cdots)$ be a decreasing*

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sequence of countably generated sub-σ-fields of $\mathfrak{B}$ *and* $\mathfrak{A}=\bigcap_{n=1}^{\infty}\mathfrak{A}_n$. *Then for any probability measure* μ *on* $\mathfrak{B}$, *the disintegration of* μ *with respect to* $\mathfrak{A}$ *is ergodic.*

For the proof, see [2] or [3].

However even in a standard space, taking a suitable sub-σ-field $\mathfrak{A}$ there does exist a probability measure whose disintegration with respect to $\mathfrak{A}$ is non ergodic. The purpose of this note is to give such an example.

§ 2. Examples

Let $\mathbb{R}^\infty$ be the countable direct product of $\mathbb{R}$, $\mathfrak{B}(\mathbb{R}^\infty)$ be the Borel σ-field on $\mathbb{R}^\infty$ and λ be the standard Lebesgue measure on (0, 1]. Take $0<s<1/2$, and using indicator function $\chi_{n,k}(\tau)$ of the intervals $((k-1)/n, k/n]$ ($n=1, 2, \cdots$, $k=1, 2, \cdots, n$) define a map $\phi(\tau)=(\phi_h(\tau))_h$ from (0, 1] to $\mathbb{R}^\infty$ such that $\phi_h(\tau)=(n^s\chi_{n,k}(\tau)+1)\sqrt{\tau}$, if $h=2^{-1}n(n-1)+k$ $(1\leqq k\leqq n)$. Then,

$$\int_0^1 \phi_h(\tau)^2\, d\lambda(\tau)\leqq 2(n^{2s}/n+1)\leqq 4\,. \tag{1}$$

Hence for all $a=(a_h)_h\in l^2$, we have

$$\textstyle\sum_{h=1}^{\infty} a_h^2\,\phi_h^2(\tau)<\infty \qquad \text{for} \quad \lambda\text{-a.e.}\,\tau. \tag{2}$$

However $\{\phi_h(\tau)\}_h$ is not bounded for each $\tau\in(0, 1]$, so

$$\forall\,\tau\in(0, 1],\ \exists\, b=(b_h)_h\in l^2,\ s.t.,\ \textstyle\sum_{h=1}^{\infty} b_h^2\,\phi_h^2(\tau)=\infty\,. \tag{3}$$

Now let g be the standard Gaussian measure with mean 0 and variance 1 on the usual Borel field $\mathfrak{B}(\mathbb{R})$, $dg(t)=(2\pi t)^{-1/2}\exp(-2^{-1}t^2)\,dt$ and G be the product measure of g, $G=\prod_{h=1}^{\infty} g$. Using transformations T_τ, S_τ on $\mathbb{R}^\infty$, T_τ: $x=(x_h)\in\mathbb{R}^\infty\mapsto(\phi_h(\tau)\,x_h)\in\mathbb{R}^\infty$, S_τ: $x=(x_h)\in\mathbb{R}^\infty\mapsto\sqrt{\tau}\,(x_h)\in\mathbb{R}^\infty$, we put $T_\tau G=G^\tau$, $S_\tau G=G_\tau$ and $\mu^\tau=2^{-1}(G^\tau+G_\tau)$. Then since

$$\textstyle\sum_{h=1}^{\infty} a_h^2\,x_h^2<\infty \quad \text{holds for } G\text{-a.e.x}=(x_h)_h \quad \text{if and only if} \quad a=(a_h)_h\in l^2\,, \tag{4}$$

we have for the spaces $H_a=\{x=(x_h)_h\in\mathbb{R}^\infty\,|\,\sum_{h=1}^{\infty} a_h^2\,x_h^2<\infty\}$ indexed by $a=(a_h)_h\in l^2$,

$$\mu^\tau(H_a)=1 \quad \text{if} \quad \textstyle\sum_{h=1}^{\infty} a_h^2\,\phi_h^2(\tau)<\infty\,, \quad \text{and} \quad \mu^\tau(H_a)=1/2 \quad \text{if} \quad \sum_{h=1}^{\infty} a_h^2\,\phi_h^2(\tau)=\infty\,. \tag{5}$$

Next take $\tau\in(0, 1]$ and fix it. Then for each n there exists unique $1\leqq k_n\leqq n$

which satisfies $\chi_{n,k_n}(\tau)=1$. Put $h_n=2^{-1}n(n-1)+k_n$. Then in virtue of the law of large numbers,

(6) $$\lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n-1)}^2}{2^{-1}n(n-1)}=1 \qquad \text{for } G\text{-a.e.x and}$$

(7) $$\lim_{n\to\infty}\frac{x_{h_1}^2+\cdots+x_{h_n}^2}{n-1}=1 \qquad \text{for } G\text{-a.e.x.}$$

Consequently, it follows from $2s<1$,

(8) $$\lim_{n\to\infty}\frac{x_{h_1}^2+\cdots+x_{h_n}^2}{2^{-1}n(n-1)}=0 \qquad \text{for } G^\tau\text{-a.e.x.}$$

Hence,

(9) $$\lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n-1)}^2}{2^{-1}n(n-1)}=\tau \qquad \text{for } G^\tau\text{-a.e.x.}$$

On the other hand, it is easy to see that

(10) $$\lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n-1)}^2}{2^{-1}n(n-1)}=\tau \qquad \text{for } G_\tau\text{-a.e.x.}$$

Thus we have,

(11) $$\lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n-1)}^2}{2^{-1}n(n-1)}=\tau \qquad \text{for } \mu^\tau\text{-a.e.x.}$$

Define $p(x)=\lim_{n\to\infty}\frac{x_1^2+\cdots+x_{2^{-1}n(n-1)}^2}{2^{-1}n(n-1)}$, if the limit exists and $p(x)=0$, otherwise. Then it follows from (11) that $p(x)=\tau$ for μ^τ-a.e.x and

(12) $$\mu^\tau(p^{-1}(E))=\chi_E(\tau) \quad \text{holds for all} \quad E\in\mathfrak{B}(\boldsymbol{R}).$$

Now put $\mu(B)=\int_0^1\mu^\tau(B)\,d\lambda(\tau)$ for $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$. Then for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$ and for all $E\in\mathfrak{B}(\boldsymbol{R})$ we have $\mu(B\cap p^{-1}(E))=\int_E\mu^\tau(B)\,d\lambda(\tau)$. Especially,

(13) $$p\mu=\lambda$$

and

(14) $$\mu(B\cap p^{-1}(E))=\int_{p^{-1}(E)}\mu^{p(x)}(B)\,d\mu(x).$$

Further from (2) and (5) we have $\mu^\tau(H_a)=1$ for λ-a.e.τ and therefore $\mu(H_a)=1$. Thus,

(15) $$\mu(B\cap H_a)=\int_{H_a}\mu^{p(x)}(B)\,d\mu(x) \qquad \text{for all} \quad B\in\mathfrak{B}(\boldsymbol{R}^\infty) \quad \text{and for all} \quad a=(a_h)_h\in l^2.$$

Let $\mathfrak{A}$ be a σ-field generated by $p^{-1}(\mathfrak{B}(\boldsymbol{R}))$ and H_a $(a\in l^2)$. Then it is easy to see that

(16) for a fixed $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$, $\mu^{p(x)}(B)$ is an $\mathfrak{A}$-measurable function of x and

(17) $\mu(B\cap A)=\int_A \mu^{p(x)}(B)\,d\mu(x)$ for all $B\in\mathfrak{B}(\boldsymbol{R}^\infty)$ and for all $A\in\mathfrak{A}$.

From (16) and (17) it follows that $\{\mu^{p(x)}\}_{x\in\boldsymbol{R}^\infty}$ is the disintegration of μ with respect to $\mathfrak{A}$. However for any τ there exists $b=(b_h)_h\in l^2$ which has property stated in (3). Consequently, $\mu^\tau(H_b)=1/2$ and therefore $\{\mu^{p(x)}\}_{x\in\boldsymbol{R}^\infty}$ is non ergodic decomposition.

Finally we will give a simple example of $(X, \mathfrak{B})$ on which a probability measure μ does not admit any disintegration with respect to a sub-σ-field $\mathfrak{A}$.

Let $X=[0, 1]$, and consider a probability measure μ on $\mathfrak{B}([0, 1])$ without atomic part. Let $\mathfrak{A}=\mathfrak{B}([0, 1])$ and let $\mathfrak{B}$ be the σ-field of all μ-measurable sets. Suppose that there would exist some disintegration $\{\mu^x\}_{x\in X}$ of μ. Then for each $A\in\mathfrak{A}$, $\mu^x(A)=\chi_A(x)$ holds for μ-a.e.x. Since $\mathfrak{A}$ is countably generated, there exists $\Omega\in\mathfrak{A}$ with $\mu(\Omega)=1$ such that $x\in\Omega$ implies $\mu^x=\delta_x$ on $\mathfrak{A}$, where δ_x is the Dirac measure at x. Especially we have $\mu^x(\{x\})=1$ for all $x\in\Omega$. Hence it holds $\mu^x=\delta_x$ on $\mathfrak{B}$ for all $x\in\Omega$. Take any $B\in\mathfrak{B}$ and put $C=\{x\in\boldsymbol{R}\mid\mu^x(B)=1\}$. Then $C\in\mathfrak{A}$ and $C\cap\Omega=B\cap\Omega$. Thus we have $B\cap\Omega\in\mathfrak{A}$ for all $B\in\mathfrak{B}$. By the way the following lemma shows that there exists $N\in\mathfrak{A}$ such that $N\subset\Omega$, ${}^\sharp N=\aleph$ and $\mu(N)=0$. It follows from these facts that ${}^\sharp\mathfrak{A}=2^\aleph$. But it contradicts to ${}^\sharp\mathfrak{A}=\aleph$, since $\mathfrak{A}$ is countably generated.

Lemma. *Let μ be a probability measure on $\mathfrak{B}([0, 1])$ without atomic part and Ω be a μ-measurable set with $\mu(\Omega)>0$. Then there exists Borel subset N of Ω such that ${}^\sharp N=\aleph$ and $\mu(N)=0$.*

Proof. Without loss of generality we may assume that Ω is a compact subset of $[0, 1]$. Put $f(t)=\mu(\Omega)^{-1}\mu(\Omega\cap[0, t])$ for $0\leqq t\leqq 1$. By the assumption f is continuous and it is easily checked that

(18) $$\mu(\Omega\cap f^{-1}([\alpha, \beta]))=(\beta-\alpha)\,\mu(\Omega) \qquad \text{for} \quad 0\leqq^\forall \alpha\leqq^\forall \beta\leqq 1\,.$$

Hence we have

(19) $$\mu(\Omega\cap f^{-1}(E))=\mu(\Omega)\,\lambda(E) \qquad \text{for all} \quad E\in\mathfrak{B}([0, 1])\,.$$

It follows from (18) that $\Omega\cap f^{-1}([\alpha, \beta])\neq\emptyset$ for $0\leqq^\forall \alpha\leqq^\forall \beta\leqq 1$. So using the complete intersection property of compact sets, $\Omega\cap f^{-1}(\alpha)\neq\emptyset$ holds for all

$\alpha \in [0, 1]$. Now take Cantor's ternary set C and put $N = \Omega \cap f^{-1}(C)$. Then $\mu(N) = 0$ holds by (19) and $^{\sharp}N = \aleph$ holds by the above arguments. Q.E.D.

References

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