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Remarks on the L^2-Dolbeault Cohomology Groups of Singular Algebraic Surfaces and Curves

Dedicated to Professor Koji Shiga on his sixtieth birthday

By

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Introduction

Let V be a complex algebraic variety of dimension ≤ 2 and its singularity set Sing V is assumed to be consisted of isolated singular points. Restrict the Fubini-Study metric of the ambient projective space containing V to the smooth part $V-\text{Sing } V$. Then the purpose of this paper is to investigate the relationships among various "L^2-Dolbeault cohomology groups" defined on the incomplete Kähler manifold $V-\text{Sing } V$.

As is well-known, if V is nonsingular, the so-called Dolbeault cohomology groups are defined naively, with no need of care of its metric, by $H^{p,q}(V)=\text{Ker } \bar{\partial}^{p,q}/\text{Range } \bar{\partial}^{p,q-1}$, where $\bar{\partial}^{p,q}$ is the $\bar{\partial}$-operator acting on smooth (p, q)-forms on V. However, if V is singular and one must consider those cohomology groups on $V-\text{Sing } V$, the situation changes greatly. That is, the $\bar{\partial}$-operator $\bar{\partial}^{p,q}$ is not permitted to be used so roughly as in the nonsingular case. For example, the operators $\bar{\partial}$ or the exterior derivative d acting on the following forms would define different kinds of cohomology groups:

(i) the smooth (p, q)-forms on $V-\text{Sing } V$,

(ii) (the maximal domain of $\bar{\partial}$) the square-integrable (p, q)-forms on $V-\text{Sing } V$ whose images by $\bar{\partial}$ (in the distribution sense) are also square-integrable,

(iii) (the minimal domain of $\bar{\partial}$) the (p, q)-forms which belong to the maximal

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domain and can be approximated with respect to the graph norm of $\bar{\partial}$ by smooth (p, q)-forms with compact supports on $V-\mathrm{Sing}\ V$,

(iv) the (p, q)-forms on $V-\mathrm{Sing}\ V$ which belong to the maximal domain of d and can be approximated with respect to the graph norm of d by smooth $(p+q)$-forms with compact supports on $V-\mathrm{Sing}\ V$,

and so forth. Excluding (i) which is too naive, the L^2-Dolbeault cohomology groups made from the $\bar{\partial}$-, d- operators with the domains (ii), (iii), (iv) are studied mainly by Pardon [9], Haskell [4] and the author [7] respectively and their subtle differences have been becoming clear. In the paper we will study the relationships mainly among the above cohomology groups, whose precise definitions are given at Definition 1.1.

§ 1. Some Elementary Properties of the L^2-Dolbeault Cohomology

Let us slightly modify the notations in [9] and [4], which are very useful and convenient but are somewhat ambiguous for our purpose.

From now on let X be an n-dimensional compact complex manifold. Though only the cases $n=1, 2$ are needed in the following sections, we do not dare to place a restriction on the dimension since the assertions in this section do not depend on n and it would be rather worth-while to report them for general n: *cf.* [9, § 3], in which the cases $n=1, 2$ were treated. Now the positive semi-definite Hermitian forms h_1, h_2 on X are called to be *quasi-isometric*, denoted by $h_1 \sim h_2$, if there exists a constant $C>0$ such that $C^{-1} h_1(x) \leq h_2(x) \leq Ch_1(x)$ holds for all $x \in X$. And a positive semi-definite Hermitian form γ is called *pseudo-metric* provided:

On each sufficiently small coordinate neighborhood $(U, (u_1, \cdots, u_n))$ of X, one can take holomorphic functions $\varphi_1, \cdots \varphi_n$ with $\gamma \sim \sum_{k=1}^n d\varphi_k\, d\bar{\varphi}_k$, where the right side is nonsingular outside a set of measure zero.

Then the volume element of $\sum d\varphi_k\, d\bar{\varphi}_k$ is given by $dV_\gamma = |\det(\partial\varphi_i/\partial u_j)|^2\, dU$, where dU is the volume element of the standard metric $\sum du_k\, d\bar{u}_k$. Hence the pseudo-metric γ degenerates along a divisor D_γ, defined on the neighborhood by

$$D_\gamma = \left(\det\left(\frac{\partial\varphi_i}{\partial u_j}\right) = 0\right), \tag{1.1}$$

which is called the *singular divisor* of γ.

Now we fix a pseudo-metric γ on X and take a union E of hypersurfaces containing $|D_\gamma|$ (=the support of D_γ). Let $\mathcal{A}^i$ be the sheaf of smooth i-forms on X, $\mathcal{L}^i_\gamma$ be the sheaf of locally square-integrable (with respect to γ) i-forms on X and let us define the sheaf $\mathcal{S}^i_{\gamma,E}$ by setting, for each open set $U\subset X$, $\mathcal{S}^i_{\gamma,E}(U)=\mathcal{A}^i(U-E)\cap\mathcal{L}^i_\gamma(U)$. Moreover let us define similarly the sheaves $\mathcal{A}^{p,q}$, $\mathcal{L}^{p,q}_\gamma$, $\mathcal{S}^{p,q}_{\gamma,E}$ for (p, q)-forms on X.

Definition 1.1. *Decompose the exterior derivative into $d=\partial+\bar\partial$ as usual.*

(1) *Let $\bar\partial^{p,q}_{\gamma,E}$ be the $\bar\partial$-operator with the domain $\mathcal{S}^{p,q}_{\gamma,E}(X)\cap\bar\partial^{-1}\mathcal{S}^{p,q+1}_{\gamma,E}(X)$ and its closure in $\mathcal{L}^{p,*}_\gamma(X)$ is denoted by $\hat{\bar\partial}^{p,q}_{\gamma,E}$.*

(2) *Set $\mathcal{A}^{p,q}_c(X-E)=\{\omega\in\mathcal{A}^{p,q}(X-E)\,|\,\omega\equiv 0$ near $E\}$. Then $\hat{\bar\partial}^{p,q}_{c,\gamma,E}$ denotes the closure (in $\mathcal{L}^{p,*}_\gamma(X)$) of the $\bar\partial$-operator restricted to $\mathcal{A}^{p,q}_c(X-E)$ and $\bar\partial^{p,q}_{c,\gamma,E}$ denotes its restriction to $\mathcal{A}^{p,q}(X-E)\cap\operatorname{dom}\hat{\bar\partial}^{p,q}_{c,\gamma,E}$.*

(3) *The q-th cohomology groups of the cochain complexes $\{\operatorname{dom}\bar\partial^{p,*}_{\gamma,E}\}$, $\{\operatorname{dom}\hat{\bar\partial}^{p,*}_{\gamma,E}\}$, $\{\operatorname{dom}\hat{\bar\partial}^{p,*}_{c,\gamma,E}\}$, $\{\operatorname{dom}\bar\partial^{p,*}_{c,\gamma,E}\}$ are denoted by $H^{p,q}_{\gamma,E}(X)=H^{p,q}_{\gamma,E}(X(\bar\partial))$, $\hat H^{p,q}_{\gamma,E}(X)=\hat H^{p,q}_{\gamma,E}(X(\bar\partial))$, $\hat H^{p,q}_{c,\gamma,E}(X)=\hat H^{p,q}_{c,\gamma,E}(X(\bar\partial))$, $H^{p,q}_{c,\gamma,E}(X)=H^{p,q}_{c,\gamma,E}(X(\bar\partial))$, respectively.*

(4) *Let $\bar\vartheta$ be the formal adjoint of $\bar\partial$, that is, $\bar\vartheta^{p,q}=-\bar*_\gamma\,\bar\partial^{n-p,n-1-q}\,\bar*_\gamma\colon\mathcal{A}^{p,q+1}(X)\to\mathcal{A}^{p,q}(X)$, where $\bar*_\gamma$ is the complex star operator defined by γ. Then we define the operators $\bar\vartheta^{p,q}_{\gamma,E}$, $\hat{\bar\vartheta}^{p,q}_{\gamma,E}$, $\hat{\bar\vartheta}^{p,q}_{c,\gamma,E}$, $\bar\vartheta^{p,q}_{c,\gamma,E}$ in the same way as (1)$\sim$(3).*

(5) *Let $\partial^{p,q}_{\gamma,E}$ denote the ∂-operator with the domain $\mathcal{S}^{p,q}_{\gamma,E}(X)\cap\partial^{-1}\mathcal{S}^{p+1,q}_{\gamma,E}(X)$. Then, in the same way as (1)$\sim$(4), we define the operators $\hat\partial^{p,q}_{\gamma,E}$, $\cdots$, $\partial^{p,q}_{c,\gamma,E}$, the cohomology groups $H^{p,q}_{\gamma,E}(X(\partial))$, $\cdots$, $H^{p,q}_{c,\gamma,E}(X(\partial))$, and the operators $\vartheta^{p,q}_{\gamma,E}$, $\cdots$, $\vartheta^{p,q}_{c,\gamma,E}$.*

(6) *Let $d^i_{\gamma,E}$ be the d-operator with the domain $\mathcal{S}^i_{\gamma,E}(X)\cap d^{-1}\mathcal{S}^{i+1}_{\gamma,E}(X)$. In the same way as (1)$\sim$(4), we also define $\hat d^i_{\gamma,E}$, $\cdots$, $d^i_{c,\gamma,E}$, $H^i_{\gamma,E}(X)=H^i_{\gamma,E}(X(d))$, $\cdots$, $H^i_{c,\gamma,E}(X)=H^i_{c,\gamma,E}(X(d))$, $\delta^i_{\gamma,E}$ $(=-\bar*_\gamma\,d^{2n-1-i}_{\gamma,E}\,\bar*_\gamma)$, $\cdots$, $\delta^i_{c,\gamma,E}$.*

(7) *Next we define the following "cohomology groups":*

$$H^{p,q}_{\gamma,E}(X(d))=\mathcal{L}^{p,q}_\gamma(X)\cap\operatorname{Ker}d^{p+q}_{\gamma,E}/\mathcal{L}^{p,q}_\gamma(X)\cap\operatorname{Range}d^{p+q-1}_{\gamma,E},$$

and $\hat H^{p,q}_{\gamma,E}(X(d))$, $\hat H^{p,q}_{c,\gamma,E}(X(d))$, $H^{p,q}_{c,\gamma,E}(X(d))$ are defined by replacing $d_{\gamma,E}$ with $\hat d_{\gamma,E}$, $\hat d_{c,\gamma,E}$, $d_{c,\gamma,E}$, respectively.

(8) *Moreover we define the following "harmonic spaces":*

$$\mathcal{H}^{p,q}_{\gamma,E}(X)=\mathcal{H}^{p,q}_{\gamma,E}(X(\bar\partial))=\operatorname{Ker}\hat{\bar\partial}^{p,q}_{\gamma,E}\cap\operatorname{Ker}\hat{\bar\vartheta}^{p,q-1}_{c,\gamma,E},$$
$$\mathcal{H}^{p,q}_{c,\gamma,E}(X)=\mathcal{H}^{p,q}_{c,\gamma,E}(X(\bar\partial))=\operatorname{Ker}\hat{\bar\partial}^{p,q}_{c,\gamma,E}\cap\operatorname{Ker}\hat{\bar\vartheta}^{p,q-1}_{\gamma,E},$$

and similarly $\mathcal{H}^{p,q}_{\gamma,E}(X(\partial))$, $\mathcal{H}^{p,q}_{c,\gamma,E}(X(\partial))$, $\mathcal{H}^i_{\gamma,E}(X)=\mathcal{H}^i_{\gamma,E}(X(d))$, $\mathcal{H}^i_{c,\gamma,E}(X)=$

$\mathcal{H}^{i}_{c,\gamma,E}(X(d))$ *are defined. Finally we set*

$$\mathcal{H}^{p,q}_{\gamma,E}(X(d)) = \mathcal{L}^{p,q}_{\gamma}(X) \cap \mathcal{H}^{p+q}_{\gamma,E}(X(d)) \,,$$
$$\mathcal{H}^{p,q}_{c,\gamma,E}(X(d)) = \mathcal{L}^{p,q}_{\gamma}(X) \cap \mathcal{H}^{p+q}_{c,\gamma,E}(X(d)) \,.$$

Remark. (a) At (8) all the "$\wedge$" can be removed, that is, the elements of the harmonic spaces are all smooth, because of the elliptic regularity theorem. (b) In the above notations, if $E=|D_\gamma|$, then it will be omitted: $\mathcal{S}^{p,q}_{\gamma}=\mathcal{S}^{p,q}_{\gamma,|D_\gamma|}$, $\bar\partial^{p,q}_{\gamma}=\bar\partial^{p,q}_{\gamma,|D_\gamma|}$, *etc.* Also if γ is positive definite (*i.e.*, $|D_\gamma|=\emptyset$), the subscript γ will be omitted: $\bar\partial^{p,q}$, $\bar\partial^{p,q}_{E}$, $\bar\partial^{p,q}_{c,E}$, *etc.*

Now, letting $(\hat{\bar\partial}^{p,q}_{\gamma,E})^*$, $(\hat{d}^{i}_{\gamma,E})^*$, *etc.* be the adjoint operators of $\hat{\bar\partial}^{p,q}_{\gamma,E}$, $\hat{d}^{i}_{\gamma,E}$, *etc.* with respect to the inner product $(\ ,\ )_\gamma$ of $\mathcal{L}^{*,*}_{\gamma}(X)$ or $\mathcal{L}^{*}_{\gamma}(X)$, we have, by [2],

$$(1.2)\qquad \begin{aligned} (\hat{\bar\partial}^{p,q}_{\gamma,E})^* &= \hat{\bar\vartheta}^{p,q}_{c,\gamma,E}\,, & (\hat{\bar\partial}^{p,q}_{c,\gamma,E})^* &= \hat{\bar\vartheta}^{p,q}_{\gamma,E}\,,\\ (\hat{\partial}^{p,q}_{\gamma,E})^* &= \hat{\vartheta}^{p,q}_{c,\gamma,E}\,, & (\hat{\partial}^{p,q}_{c,\gamma,E})^* &= \hat{\vartheta}^{p,q}_{\gamma,E}\,,\\ (\hat{d}^{i}_{\gamma,E})^* &= \hat{\delta}^{i}_{c,\gamma,E}\,, & (\hat{d}^{i}_{c,\gamma,E})^* &= \hat{\delta}^{i}_{\gamma,E}\,. \end{aligned}$$

Moreover there exist also the following Hodge decompositions:

$$(1.3)\qquad \begin{aligned} \mathcal{L}^{p,q}_{\gamma}(X) &= \overline{\mathrm{Range}\ \hat{\bar\partial}^{p,q-1}_{\gamma,E}} \oplus \mathcal{H}^{p,q}_{\gamma,E}(X(\bar\partial)) \oplus \overline{\mathrm{Range}\ \hat{\bar\vartheta}^{p,q}_{c,\gamma,E}}\\ &= \overline{\mathrm{Range}\ \hat{\bar\partial}^{p,q-1}_{c,\gamma,E}} \oplus \mathcal{H}^{p,q}_{c,\gamma,E}(X(\bar\partial)) \oplus \overline{\mathrm{Range}\ \hat{\bar\vartheta}^{p,q}_{\gamma,E}}\,, \end{aligned}$$

$$(1.4)\qquad \begin{aligned} \mathcal{L}^{i}_{\gamma}(X) &= \overline{\mathrm{Range}\ \hat{d}^{i-1}_{\gamma,E}} \oplus \mathcal{H}^{i}_{\gamma,E}(X(d)) \oplus \overline{\mathrm{Range}\ \hat{\delta}^{i}_{c,\gamma,E}}\\ &= \overline{\mathrm{Range}\ \hat{d}^{i-1}_{c,\gamma,E}} \oplus \mathcal{H}^{i}_{c,\gamma,E}(X(d)) \oplus \overline{\mathrm{Range}\ \hat{\delta}^{i}_{\gamma,E}}\,. \end{aligned}$$

As for (1.3), the similar decomposition holds also for the ∂-operators. Next the argument [1, §1(A)$\sim$(C)] implies

$$(1.5)\qquad \begin{aligned} &\mathrm{Range}\ \hat{d}^{i}_{\gamma,E} \cap \mathcal{A}^{i+1}(X-E) = \mathrm{Range}\ d^{i}_{\gamma,E}\,,\\ &\mathrm{Range}\ \hat{d}^{i}_{c,\gamma,E} \cap \mathcal{A}^{i+1}(X-E) = \mathrm{Range}\ d^{i}_{c,\gamma,E}\,, \end{aligned}$$

and the similar argument for $\bar\partial$ implies

$$(1.6)\qquad \begin{aligned} &\mathrm{Range}\ \hat{\bar\partial}^{p,q}_{\gamma,E} \cap \mathcal{A}^{p,q+1}(X-E) = \mathrm{Range}\ \bar\partial^{p,q}_{\gamma,E}\,,\\ &\mathrm{Range}\ \hat{\bar\partial}^{p,q}_{c,\gamma,E} \cap \mathcal{A}^{p,q+1}(X-E) = \mathrm{Range}\ \bar\partial^{p,q}_{c,\gamma,E}\,. \end{aligned}$$

The following injections can be gotten because of (1.3) and (1.6):

$$(1.7)\qquad \begin{aligned} &\mathcal{H}^{p,q}_{\gamma,E}(X(\bar\partial)) \xrightarrow{inj.} H^{p,q}_{\gamma,E}(X(\bar\partial)) \xrightarrow{inj.} \hat{H}^{p,q}_{\gamma,E}(X(\bar\partial))\,,\\ &\mathcal{H}^{p,q}_{c,\gamma,E}(X(\bar\partial)) \xrightarrow{inj.} H^{p,q}_{c,\gamma,E}(X(\bar\partial)) \xrightarrow{inj.} \hat{H}^{p,q}_{c,\gamma,E}(X(\bar\partial))\,. \end{aligned}$$

The results similar to (1.6) and (1.7) hold for ∂ and d. As a conjecture at (1.7), both of the right $\xrightarrow{inj.}$ will be isomorphic: the isomorphism $H^i_{\gamma,E}(X(d)) \xrightarrow{\cong} \hat{H}^i_{\gamma,E}(X(d))$ has already been verified by Cheeger [1, (1.5)]. Even if the conjecture is not settled, we have a useful sufficient condition for all the $\xrightarrow{inj.}$ at (1.7) to be isomorphic. That is, if $\hat{H}^{p,q}_{\gamma,E}(X(\bar{\partial}))$ and $\hat{H}^{p,q}_{c,\gamma,E}(X(\bar{\partial}))$ are finite-dimensional, then Range $\hat{\bar{\partial}}^{p,q-1}_{\gamma,E}$ and Range $\hat{\bar{\partial}}^{p,q-1}_{c,\gamma,E}$ are respectively closed in $\mathcal{L}^{p,q}_{\gamma}(X)$, which, combined with (1.3) and (1.6), implies it.

Proposition 1.2. *If E is a union of smooth hypersurfaces with normal crossings and γ is nondegenerate (i.e., positive definite), then we have*

(1) $d^i_E \subset \hat{d}^i$, $d^i \subset \hat{d}^i_{c,E}$, *hence* $\hat{d}^i_E = \hat{d}^i_{c,E}$,

(2) $\bar{\partial}^{p,q}_E \subset \hat{\bar{\partial}}^{p,q}$, $\bar{\partial}^{p,q} \subset \hat{\bar{\partial}}^{p,q}_{c,E}$, *hence* $\hat{\bar{\partial}}^{p,q}_E = \hat{\bar{\partial}}^{p,q}_{c,E}$,

(3) $\partial^{p,q}_E \subset \hat{\partial}^{p,q}$, $\partial^{p,q} \subset \hat{\partial}^{p,q}_{c,E}$, *hence* $\hat{\partial}^{p,q}_E = \hat{\partial}^{p,q}_{c,E}$.

Proof. Since the restriction of a nondegenerate metric on X to $X-E$ is conical near E, [1] implies (1). In the following let us prove (2), for which we have only to review the proof of Pardon [9, Proposition 3.2] with emphasizing a certain point different from it. First obviously [9, Lemma 3.5] holds for the general dimension case. That is, put $U=\{u=(u_1, \cdots, u_n) \in \boldsymbol{C} \mid |u_i|<1 \text{ for any } i\}$, $I \subset \{1, 2, \cdots, n\}$, $T(\varepsilon)=\cup_{i \in I}\{u \in U \mid |u_i|=\varepsilon,\ |u_j| \geq \varepsilon \text{ for any } I \ni j \neq i\}$. Then, for any nonnegative and square-integrable function f on $U-\{\prod_{i \in I} u_i=0\}$, we have

$$\liminf_{\varepsilon \to 0} \int_{T(\varepsilon)} f\, dT(\varepsilon) = 0 . \tag{1.8}$$

Now, in order to prove $\bar{\partial}^{p,q}_E \subset \hat{\bar{\partial}}^{p,q}$, it suffices to verify the following: for the open set U with $E=\{\prod_{i \in I} u_i=0\}$, if $\omega \in \operatorname{dom} \bar{\partial}^{p,q}_E$ and $\tau \in \mathcal{A}^{n-p,n-1-q}_c(U)$, we have

$$\int_U \bar{\partial}\omega \wedge \tau = (-1)^{p+q+1} \int_U \omega \wedge \bar{\partial}\tau .$$

Considering further the Stokes' theorem, we have only to prove

$$\lim_{\varepsilon \to 0} \int_{T(\varepsilon)} \omega \wedge \tau = 0 . \tag{1.9}$$

Since τ is bounded near E, expressing $\omega \wedge \tau = \sum f_i\, du_1 \wedge \cdots \wedge du_n \wedge d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_{i-1} \wedge d\bar{u}_{i+1} \wedge \cdots \wedge d\bar{u}_n$, each f_i is square-integrable with respect to a nondegenerate metric. Hence we have

$$\left|\int_{T(\varepsilon)} \omega\wedge\tau\right| \leq \int_{T(\varepsilon)} (\sum|f_i|)\, dT(\varepsilon),$$

whose right side has lim inf$=0$ by (1.8). Since the left side of (1.9) exists, certainly it must be zero. Next let us prove $\bar\partial^{p,q}\subset\hat{\bar\partial}^{p,q}_{c,E}$. Following the same idea as above, it suffices to prove (1.9) for $\omega\in\mathcal{A}^{p,q}_c(U)$ and $\tau\in\mathrm{dom}\,\bar\partial^{n-p,n-1-q}_E$. Since ω is bounded near E in this case, the above proof of (1.9) is applicable also here.

Remark. In the proof of [4, Proposition 1.17], Haskell showed $\bar\partial^{0,q}\subset\hat{\bar\partial}^{0,q}_{c,E}$ using the fact $d^i\subset\hat d^i_{c,E}$. It will be difficult, however, to verify general $\bar\partial^{p,q}\subset\hat{\bar\partial}^{p,q}_{c,E}$ using it.

Corollary 1.3. *If E is a union of smooth hypersurfaces with normal crossings and γ is nondegenerate, then we have the following natural commutative diagram:*

$$\begin{array}{ccccc}
\hat H^{p,q}(X) & \xrightarrow{\cong} & \hat H^{p,q}_E(X) & \xleftarrow{\cong} & \hat H^{p,q}_{c,E}(X)\\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong\\
H^{p,q}(X) & \xrightarrow{\cong} & H^{p,q}_E(X) & \xleftarrow{\cong} & H^{p,q}_{c,E}(X)\\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong\\
\mathcal{H}^{p,q}(X) & = & \mathcal{H}^{p,q}_E(X) & = & \mathcal{H}^{p,q}_{c,E}(X)
\end{array}$$

Also, for ∂-operators or d-operators, we have the similar commutative diagrams.

Proof. Proposition 1.2(2) implies

$$\bar\partial^{p,q}\subset\bar\partial^{p,q}_E=\bar\partial^{p,q}_{c,E}\subset\hat{\bar\partial}^{p,q}_{c,E}=\hat{\bar\partial}^{p,q}_E=\hat{\bar\partial}^{p,q}. \tag{1.10}$$

And it is well-known that we have the isomorphism $\mathcal{H}^{p,q}(X)\xrightarrow{\cong}H^{p,q}(X)\xrightarrow{\cong}\hat H^{p,q}(X)$ and they are finite-dimensional. Hence the isomorphism $\hat H^{p,q}(X)\xrightarrow{\cong}\hat H^{p,q}_E(X)$ induced from (1.10) implies the finite dimensionality of $\hat H^{p,q}_E(X)$ and the isomorphisms $\mathcal{H}^{p,q}_E(X)\xrightarrow{\cong}H^{p,q}_E(X)\xrightarrow{\cong}\hat H^{p,q}_E(X)$: see the remark following (1.7). The similar argument implies also $\mathcal{H}^{p,q}_{c,E}(X)\xrightarrow{\cong}H^{p,q}_{c,E}(X)\xrightarrow{\cong}\hat H^{p,q}_{c,E}(X)$.

Finally let $\mathcal{V}\to X$ be a holomorphic Hermitian vector bundle and let us discuss the cohomology groups induced from the $\bar\partial$-operator $\bar\partial[\mathcal{V}]$ acting on the $\mathcal{V}$-valued forms. In the same way as Definition 1.1(1)$\sim$(3), we define $\bar\partial^{p,q}_{\gamma,E}[\mathcal{V}]$, $\hat{\bar\partial}^{p,p}_{\gamma,E}[\mathcal{V}]$, $\hat{\bar\partial}^{p,q}_{c,\gamma,E}[\mathcal{V}]$, $\bar\partial^{p,q}_{c,\gamma,E}[\mathcal{V}]$, $H^{p,q}_{\gamma,E}(X;\mathcal{O}(\mathcal{V}))$, $\cdots$, $H^{p,q}_{c,\gamma,E}(X;\mathcal{O}(\mathcal{V}))$ and, using the formal adjoint $\bar\vartheta[\mathcal{V}]=-*_{\gamma,\mathcal{V}^*}\bar\partial[\mathcal{V}^*]*_{\gamma,\mathcal{V}}$, we define, as in (4), $\bar\vartheta^{p,q}_{\gamma,E}[\mathcal{V}]$, $\cdots$, $\bar\vartheta^{p,q}_{c,\gamma,E}[\mathcal{V}]$. Also $\mathcal{H}^{p,q}_{\gamma,E}(X;\mathcal{O}(\mathcal{V}))$, $\cdots$, $\mathcal{H}^{p,q}_{c,\gamma,E}(X;\mathcal{O}(\mathcal{V}))$ are defined. Then, as in (1.2), we have

$$(\hat{\bar\partial}^{p,q}_{\gamma,E}[\mathcal{V}])^* = \hat{\vartheta}^{p,q}_{c,\gamma,E}[\mathcal{V}], \quad (\hat{\bar\partial}^{p,q}_{c,\gamma,E}[\mathcal{V}])^* = \hat{\vartheta}^{p,q}_{\gamma,E}[\mathcal{V}]. \tag{1.11}$$

For the $\mathcal{V}$-valued form case, (1.4), (1.6), (1.7) and Proposition 1.2(2) are similarly verified and if E is a union of smooth hypersurfaces with normal crossings and γ is nondegenerate, then we have the natural commutative diagram similar to Corollary 1.3:

$$\begin{array}{ccccc} \hat{H}^{p,q}(X;\mathcal{O}(\mathcal{V})) & \xrightarrow{\cong} & \hat{H}^{p,q}_E(X;\mathcal{O}(\mathcal{V})) & \xleftarrow{\cong} & \hat{H}^{p,q}_{c,E}(X;\mathcal{O}(\mathcal{V})) \\ \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\ H^{p,q}(X;\mathcal{O}(\mathcal{V})) & \xrightarrow{\cong} & H^{p,q}_E(X;\mathcal{O}(\mathcal{V})) & \xleftarrow{\cong} & H^{p,q}_{c,E}(X;\mathcal{O}(\mathcal{V})) \\ \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\ \mathcal{H}^{p,q}(X;\mathcal{O}(\mathcal{V})) & = & \mathcal{H}^{p,q}_E(X;\mathcal{O}(\mathcal{V})) & = & \mathcal{H}^{p,q}_{c,E}(X;\mathcal{O}(\mathcal{V})) \end{array} \tag{1.12}$$

Pardon [9, §3] gave the isomorphism $H^{p,q}(X;\mathcal{O}(\mathcal{V})) \xrightarrow{\cong} H^{p,q}_E(X;\mathcal{O}(\mathcal{V}))$ with $\dim X=2$ by considering the resolution of $\mathcal{O}(\mathcal{V})$.

Final Remark. In the following sections, the arrows $\xrightarrow{inj.}$, $\xrightarrow{surj.}$, *etc.* among the cohomology groups always mean the existence of an injective map, a surjective map, *etc.* which are induced from natural cochain maps.

§ 2. Main Theorems for Singular Surfaces

Let S be a complex algebraic surface with isolated singular points. It does not need to be normal.

Proposition 2.1. (HSIANG-PATI [5]). *There exists a desingularization ρ: $(X, E)\to(S, \mathrm{Sing}\, S)$ such that, defining γ to be the pullback (by ρ) of the Fubini-Study metric of an ambient projective space containing S, we have,*

(0) *$E=|\rho^{-1}(\mathrm{Sing}\, S)|$ is a union of smooth curves with normal crossings,*

(−) *at each smooth point of E, there exists a local coordinate neighborhood $(U, (u, v))$ and $1\le n_1\le n_2$ (integers) such that $E=\{u=0\}$ on U and*

$$\gamma \sim du^{n_1}\, d\overline{u^{n_1}} + d(u^{n_2} v)\, d(\overline{u^{n_2} v}),$$

(+) *at each normal crossing of E, there exists a local coordinate neighborhood $(U, (u, v))$ and $1\le n_1\le n_2$, $1\le m_1\le m_2$ (integers) such that $n_1 m_2 - m_1 n_2 \neq 0$, $E=\{uv=0\}$ on U and*

$$\gamma \sim d(u^{n_1} v^{m_1})\, d(\overline{u^{n_1} v^{m_1}}) + d(u^{n_2} v^{m_2})\, d(\overline{u^{n_2} v^{m_2}}).$$

Hence γ is a pseudo-metric, which is especially called 'of Hsiang-Pati type'.

Let us fix such a desingularization ρ: $(X, E)\to(S, \mathrm{Sing}\, S)$. Set $E=\sum E_i$

(irreducible components), then for each E_i the *index* $(n_{i1}, n_{i2})=(n_1, n_2)$ is determined according to $(-)$. At $(+)$ the index of $\{u=0\}$ is (n_1, n_2) and that of $\{v=0\}$ is (m_1, m_2). Obviously it does not depend on the choice of the smooth point of E_i nor of the neighborhood. The singular divisor of the pseudo-metric γ is given by

$$D_\gamma = \sum (n_{i1}+n_{i2}-1)\, E_i \tag{2.1}$$

and its local defining function, which therefore determines a nonzero meromorphic section of the line bundle $[D_\gamma]$, can be expressed as follows:

$$a_\gamma = \begin{cases} u^{n_1+n_2-1} & :(-) \\ u^{n_1+n_2-1}\, v^{m_1+m_2-1} & :(+) \end{cases} \tag{2.2}$$

Now, since the L^2-cohomology groups and the usual L^2-Dolbeault cohomology groups of $S-\mathrm{Sing}\, S$ are naturally equal to $H^i_\gamma(X)$ and $H^{p,q}_\gamma(X)$ respectively, let us study the latter cohomology groups on X.

First, by Pardon [9], Haskell [4] and using the dual argument through $\bar{*}_\gamma$, we get

Theorem 2.2.

$$\begin{array}{ccccccc}
H^{0,0}(X) & \xrightarrow{\cong} & H^{0,0}_{c,\gamma}(X) & \xrightarrow{\cong} & H^{0,0}_\gamma(X) & \xrightarrow[a_\gamma\times]{\cong} & H^{0,0}(X;\mathcal{O}(D_\gamma)) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
\mathcal{H}^{0,0}(X) & = & \mathcal{H}^{0,0}_{c,\gamma}(X) & = & \mathcal{H}^{0,0}_\gamma(X) & \xrightarrow[a_\gamma\times]{\cong} & \mathcal{H}^{0,0}(X;\mathcal{O}(D_\gamma)) \\
\bar{*}\downarrow\cong & & \bar{*}_\gamma\downarrow\cong & & \bar{*}_\gamma\downarrow\cong & & \bar{*}_{[D_\gamma]}\downarrow\cong \\
\mathcal{H}^{2,2}(X) & & \mathcal{H}^{2,2}_\gamma(X) & = & \mathcal{H}^{2,2}_{c,\gamma}(X) & & \mathcal{H}^{2,2}(X;\mathcal{O}(-D_\gamma)) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^{2,2}(X) & \xleftarrow{\cong} & H^{2,2}_\gamma(X) & \xleftarrow{\cong} & H^{2,2}_{c,\gamma}(X) & \xleftarrow[a_\gamma\times]{\cong} & H^{2,2}(X;\mathcal{O}(-D_\gamma))
\end{array}$$

$$\begin{array}{ccccccc}
H^{0,1}(X) & \xrightarrow{\cong} & H^{0,1}_{c,\gamma}(X) & \xrightarrow{inj.} & H^{0,1}_\gamma(X) & \xrightarrow{inj.} & H^{0,1}(X;\mathcal{O}(D_\gamma)) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
\mathcal{H}^{0,1}(X) & & \mathcal{H}^{0,1}_{c,\gamma}(X) & & \mathcal{H}^{0,1}_\gamma(X) & & \mathcal{H}^{0,1}(X;\mathcal{O}(D_\gamma)) \\
\bar{*}\downarrow\cong & & \bar{*}_\gamma\downarrow\cong & & \bar{*}_\gamma\downarrow\cong & & \bar{*}_{[E_\gamma]}\downarrow\cong \\
\mathcal{H}^{2,1}(X) & & \mathcal{H}^{2,1}_\gamma(X) & & \mathcal{H}^{2,1}_{c,\gamma}(X) & & \mathcal{H}^{2,1}(X;\mathcal{O}(-D_\gamma)) \\
\downarrow\cong & & \downarrow\cong & & \downarrow inj. & & \downarrow\cong \\
H^{2,1}(X) & \xleftarrow{\cong} & H^{2,1}_\gamma(X) & \xleftarrow{surj.} & H^{2,1}_{c,\gamma}(X) & \longleftarrow & H^{2,1}(X;\mathcal{O}(-D_\gamma))
\end{array}$$

$$H^{2,1}_\gamma(X) \xleftarrow{surj.} H^{2,1}(X;\mathcal{O}(-D_\gamma))$$

$$\begin{array}{ccccccc}
H^{0,2}(X) & \xrightarrow{\cong} & H^{0,2}_{c,\gamma}(X) & \xrightarrow{surj.} & H^{0,2}_{\gamma}(X) & \xrightarrow{surj.} & H^{0,2}(X;\mathcal{O}(D_\gamma)) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
\mathcal{H}^{0,2}(X) & & \mathcal{H}^{0,2}_{c,\gamma}(X) & \hookleftarrow & \mathcal{H}^{0,2}_{\gamma}(X) & & \mathcal{H}^{0,2}(X;\mathcal{O}(D_\gamma)) \\
\bar{*}\downarrow\cong & & \bar{*}_\gamma\downarrow\cong & & \bar{*}_\gamma\downarrow\cong & & \bar{*}_{[D_\gamma]}\downarrow\cong \\
\mathcal{H}^{2,0}(X) & & \mathcal{H}^{2,0}_{\gamma}(X) & \hookleftarrow & \mathcal{H}^{2,0}_{c,\gamma}(X) & & \mathcal{H}^{2,0}(X;\mathcal{O}(-D_\gamma)) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^{2,0}(X) & \xleftarrow{\cong} & H^{2,0}_{\gamma}(X) & \xleftarrow{inj.} & H^{2,0}_{c,\gamma}(X) & \xleftarrow{inj.} & H^{2,0}(X;\mathcal{O}(-D_\gamma))\,.
\end{array}$$

Combined with the well-known isomorphisms

$$H^{p,q}(X) \xrightarrow{\cong} \hat{H}^{p,q}(X)\,, \quad H^{p,q}(X;\mathcal{O}(\pm D_\gamma)) \xrightarrow{\cong} \hat{H}^{p,q}(X;\mathcal{O}(\pm D_\gamma))\,, \tag{2.3}$$

the above theorem implies

Theorem 2.3. *As for the L^2-Dolbeault cohomology groups appeared at Theorem 2.2, excepting $H^{2,1}_{c,\gamma}(X)$, we have*

$$H^{p,q}_{\gamma}(X) \xrightarrow{\cong} \hat{H}^{p,q}_{\gamma}(X)\,, \quad H^{p,q}_{c,\gamma}(X) \xrightarrow{\cong} \hat{H}^{p,q}_{c,\gamma}(X)\,.$$

As a conjecture there must exists an isomorphism $H^{2,1}_{c,\gamma}(X) \xrightarrow{\cong} \hat{H}^{2,1}_{c,\gamma}(X)$ and they will be of finite dimension, which obviously yields $\mathcal{H}^{2,1}_{c,\gamma}(X) \xrightarrow{\cong} H^{2,1}_{c,\gamma}(X)$ at Theorem 2.2.

Let us next state the relationship among the L^2-Dolbeault cohomology groups made from the d-operators and the ones at Theorem 2.2.

Theorem 2.4.

$$\begin{array}{ccccc}
H^{0,0}(X) & \xrightarrow{\cong} & H^{0,0}_{c,\gamma}(X) & \xrightarrow{\cong} & H^{0,0}_{\gamma}(X) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
H^{0,0}(X(d)) & \xrightarrow{\cong} & H^{0,0}_{c,\gamma}(X(d)) & \xrightarrow{\cong} & H^{0,0}_{\gamma}(X(d))
\end{array}$$

$$\begin{array}{ccccc}
H^{2,2}(X) & \xleftarrow{\cong} & H^{2,2}_{\gamma}(X) & \xleftarrow{\cong} & H^{2,2}_{c,\gamma}(X) \\
\uparrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^{2,2}(X(d)) & \xleftarrow{\cong} & H^{2,2}_{\gamma}(X(d)) & \xleftarrow{\cong} & H^{2,2}_{c,\gamma}(X(d))
\end{array}$$

$$\begin{array}{ccccc}
H^{0,1}(X) & \xrightarrow{\cong} & H^{0,1}_{c,\gamma}(X) & \xrightarrow{inj.} & H^{0,1}_{\gamma}(X) \\
\uparrow\cong & & \uparrow\cong & & \uparrow inj. \\
H^{0,1}(X(d)) & \xrightarrow{\cong} & H^{0,1}_{c,\gamma}(X(d)) & \xrightarrow{\cong} & H^{0,1}_{\gamma}(X(d))
\end{array}$$

$$
\begin{array}{ccccc}
H^{2,1}(X) & \xleftarrow{\ \cong\ } & H^{2,1}_{\gamma}(X) & \xleftarrow{\ surj.\ } & H^{2,1}_{c,\gamma}(X) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
H^{2,1}(X(d)) & \overset{\cong}{\leftarrow\cdots} & H^{2,1}_{\gamma}(X(d)) & \xleftarrow{\ surj.\ } & H^{2,1}_{c,\gamma}(X(d))
\end{array}
$$

$$
\begin{array}{ccccc}
H^{1,0}(X) & \overset{inj.}{\leftarrow\cdots} & H^{1,0}_{c,\gamma}(X) & \xrightarrow{\ inj.\ } & H^{1,0}_{\gamma}(X) \\
\uparrow\cong & & \downarrow\cong & & \uparrow inj. \\
H^{1,0}(X(d)) & \xrightarrow{\ \cong\ } & H^{1,0}_{c,\gamma}(X(d)) & \xrightarrow{\ \cong\ } & H^{1,0}_{\gamma}(X(d))
\end{array}
$$

$$
\begin{array}{ccc}
H^{1,2}_{\gamma}(X) & \longleftarrow & H^{1,2}_{c,\gamma}(X) \\
\vdots\uparrow inj. & & \\
H^{1,2}_{\gamma}(X(d)) & \xleftarrow{\ surj.\ } & H^{1,2}_{c,\gamma}(X(d))\,.
\end{array}
$$

Here the broken arrows mean the existence of the maps which may not induced from the cochain complex maps.

Let us make here a brief comment on the cases $(p, q)=(1, 0), (1, 2)$. The diagrams in the cases are incomplete because both of them are related to the (1, 1)-forms. For example, though it seems at first sight that there exists a natural map from $H^{1,0}(X)=H^{1,0}_{c,E}(X)$ to $H^{1,0}_{c,\gamma}(X)$, it has a subtle problem. That is, for $[\varphi]\in H^{1,0}_{c,E}(X)$, one can take a sequence $\varphi_j\in\mathcal{A}^{1,0}_c(X-E)$ with $\varphi_j\to\varphi$ and $\bar\partial\varphi_j\to 0$ in $\mathcal{L}^{1,*}(X)$. Then certainly $\varphi_j\to\varphi$ in $\mathcal{L}^{1,0}_{\gamma}(X)$ but we do not know whether $\bar\partial\varphi_j$ tends to zero or not in $\mathcal{L}^{1,1}_{\gamma}(X)$: see Lemma 4.1. It means that φ may not define an element of $H^{1,0}_{c,\gamma}(X)$. However, in spite of such a problem, the author believes that $H^{1,0}(X)\cong H^{1,0}_{c,\gamma}(X)$ holds. If it is true, those diagrams can be improved.

In [7] we have proved

$$
H^{i}_{\gamma}(X) \cong \bigoplus_{p+q=i} H^{p,q}_{\gamma}(X(d)) \tag{2.4}
$$

if $i\neq 2$. Hence, combined with Theorem 2.4, it implies

$$
\begin{aligned}
H^{1}_{\gamma}(X) &\cong H^{0,1}_{c,\gamma}(X(\bar\partial)) \oplus H^{1,0}_{c,\gamma}(X(\partial))\,, \\
H^{3}_{\gamma}(X) &\cong H^{2,1}_{\gamma}(X(\bar\partial)) \oplus H^{1,2}_{\gamma}(X(\partial))\,.
\end{aligned} \tag{2.5}
$$

If the above conjecture is affirmative, then $H^{1,0}_{c,\gamma}(X(\partial))$ can be replaced by $H^{1,0}_{c,\gamma}(X(\bar\partial))$.

§ 3. Main Theorem for Singular Curves

Let C be a singular algebraic curve and $\rho: (X, E)\to(C, \text{Sing}\, C)$ be its normalization. At each point p of E, there exists a local coordinate neighborhood

(U, u) and an integer $m>1$ such that the pullback γ (by ρ) of the Fubini-Study metric of an ambient projective space containing C can be expressed as

$$\gamma \sim du^m d\bar{u}^m . \tag{3.1}$$

Refer to [8] *etc.* in details. The *index* $m_p=m$ (*i.e.*, the multiplicity of p) does not depend on the choice of such a neighborhood. The singular divisor of the pseudo-metric γ is given by $D_\gamma=\sum_{p\in E}(m_p-1)p$ and its local defining function can be written as $a_\gamma=u^{m-1}$.

Theorem 3.1.

$$\begin{array}{ccccccc}
H^{0,q}(X) & \xrightarrow{\cong} & H^{0,q}_{c,\gamma}(X) & \xrightarrow{\cong} & H^{0,q}_{\gamma}(X) & \xrightarrow{\cong} & H^{0,q}(X;\mathcal{O}(D_\gamma)) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \\
H^{0,q}(X(d)) & \xrightarrow{\cong} & H^{0,q}_{c,\gamma}(X(d)) & \xrightarrow{\cong} & H^{0,q}_{\gamma}(X(d)) & &
\end{array}$$

$$\begin{array}{ccccccc}
H^{1,q}(X) & \xleftarrow{\cong} & H^{1,q}_{\gamma}(X) & \xleftarrow{\cong} & H^{1,q}_{c,\gamma}(X) & \xleftarrow{\cong} & H^{1,q}(X;\mathcal{O}(-D_\gamma)) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
H^{1,q}(X(d)) & \xleftarrow{\cong} & H^{1,q}_{\gamma}(X(d)) & \xleftarrow{\cong} & H^{1,q}_{c,\gamma}(X(d)) & &
\end{array}$$

In [8] we have proved

$$H^i_\gamma(X)\cong \bigoplus_{p+q=i} H^{p,q}_\gamma(X(d)) . \tag{3.2}$$

Hence, in the curve case, the pure Hodge decomposition holds neatly:

$$H^i_\gamma(X)\cong \bigoplus_{p+q=i} H^{p,q}_\gamma(X) . \tag{3.3}$$

§ 4. Proofs of the Theorems in § 2

We use the notations in § 2. On the local coordinate neighborhood $(U, (u, v))$ given at Proposition 2.1, we get, by a straightforward computation (*cf.* [9, (2.1)~(2.3)]),

Lemma 4.1. *Let* $L^2=L^2(U, \mathrm{loc})$ *be the space of functions on* U *which are locally square-integrable with respect to a nondegenerate metric. Then we have*

$(L\colon 0, 0)$ $f\in\mathcal{L}^{0,0}_\gamma(U)$ *is equivalent to* $a_\gamma f\in L^2$,

$(L\colon 0, 1)$ $fd\bar{u}+gd\bar{v}\in\mathcal{L}^{0,1}_\gamma(U)$ *is equivalent to*

$$u^{n_1}f,\ u^{n_2-1}g\in L^2;\ (-)\,,$$
$$u^{n_1-1}v^{m_1-1}(m_1u\bar{f}-n_1v\bar{g}),\ u^{n_2-1}v^{m_2-1}(m_2u\bar{f}-n_2v\bar{g})\in L^2;\ (+)\,,$$

(L: 1, 0) *$fdu+gdv\in\mathcal{L}_\gamma^{1,0}(U)$ is equivalent to*

$$u^{n_1}f,\ u^{n_2-1}g\in L^2;\ (-)\,,$$
$$u^{n_1-1}v^{m_1-1}(m_1uf-n_1vg),\ u^{n_2-1}v^{m_2-1}(m_2uf-n_2vg)\in L^2;\ (+)\,,$$

(L: 0, 2) *$kd\bar{u}\wedge d\bar{v}\in\mathcal{L}_\gamma^{0,2}(U)$ is equivalent to $k\in L^2$,*

(L: 2, 0) *$ldu\wedge dv\in\mathcal{L}_\gamma^{2,0}(U)$ is equivalent to $l\in L^2$,*

(L: 1, 1) *$fdu\wedge d\bar{u}+gdu\wedge d\bar{v}+kdv\wedge d\bar{u}+ldv\wedge d\bar{v}\in\mathcal{L}_\gamma^{1,1}(U)$ is equivalent to*

$$u^{-n_1+n_2+1}f,\ g,\ k,\ u^{n_1-n_2-1}l\in L^2;\ (-)\,,$$
$$u^{-n_1+n_2-1}v^{-m_1+m_2-1}(m_2^2u\bar{u}f-n_2m_2u\bar{v}g-n_2m_2v\bar{u}k+n_2^2v\bar{v}l)\in L^2;\ (+)\,,$$
$$u^{-1}v^{-1}(-m_1m_2u\bar{u}f+n_1m_2u\bar{v}g+m_1n_2v\bar{u}k-n_1n_2v\bar{v}l)\in L^2;\ (+)\,,$$
$$u^{-1}v^{-1}(-m_1m_2u\bar{u}f+m_1n_2u\bar{v}g+n_1m_2v\bar{u}k-n_1n_2v\bar{v}l)\in L^2;\ (+)\,,$$
$$u^{n_1-n_2-1}v^{m_1-m_2-1}(m_1^2u\bar{u}f-n_1m_1u\bar{v}g-n_1m_1v\bar{u}k+n_1^2v\bar{v}l)\in L^2;\ (+)\,,$$

(L: 1, 2) *$kdu\wedge d\bar{u}\wedge d\bar{v}+ldv\wedge d\bar{u}\wedge d\bar{v}\in\mathcal{L}_\gamma^{1,2}(U)$ is equivalent to*

$$a_\gamma^{-1}u^{n_1}k,\ a_\gamma^{-1}u^{n_2-1}l\in L^2;\ (-)\,,$$
$$a_\gamma^{-1}u^{n_1-1}v^{m_1-1}(m_1uk-n_1vl),\ a_\gamma^{-1}u^{n_2-1}v^{m_2-1}(m_2uk-n_2vl)\in L^2;\ (+)\,,$$

(L: 2, 1) *$kdu\wedge dv\wedge d\bar{u}+ldu\wedge dv\wedge d\bar{v}\in\mathcal{L}_\gamma^{2,1}(U)$ is equivalent to*

$$a_\gamma^{-1}u^{n_1}k,\ a_\gamma^{-1}u^{n_2-1}l\in L^2;\ (-)\,,$$
$$a_\gamma^{-1}u^{m_1-1}v^{m_1-1}(m_1u\bar{k}-n_1v\bar{l}),\ a_\gamma^{-1}u^{n_2-1}v^{m_2-1}(m_2u\bar{k}-n_2v\bar{l})\in L^2;\ (+)\,,$$

(L: 2, 2) *$kdu\wedge dv\wedge d\bar{u}\wedge d\bar{v}\in\mathcal{L}_\gamma^{2,2}(U)$ is equivalent to $a_\gamma^{-1}k\in L^2$.*

The lemma obviously implies

Corollary 4.2. *Set $E=|D_\gamma|$.*

(1) $\mathcal{S}_E^{p,q}\hookrightarrow\mathcal{S}_\gamma^{p,q}\xrightarrow{a_\gamma\times}\mathcal{S}_E^{p,q}\otimes\mathcal{O}(D_\gamma);\ p+q\le 1.$

$(2)_+$ $\mathcal{S}_E^{p,q}=\mathcal{S}_\gamma^{p,q}\xrightarrow{a_\gamma\times}\mathcal{S}_E^{p,q}\otimes\mathcal{O}(D_\gamma);\ (p,q)=(0,2),(2,0).$

$(2)_-$ $\mathcal{S}_E^{p,q}=\mathcal{S}_\gamma^{p,q}\xleftarrow{a_\gamma\times}\mathcal{S}_E^{p,q}\otimes\mathcal{O}(-D_\gamma);\ (p,q)=(0,2),(2,0).$

(3) $\mathcal{S}_E^{p,q}\hookleftarrow\mathcal{S}_\gamma^{p,q}\xleftarrow{a_\gamma\times}\mathcal{S}_E^{p,q}\otimes\mathcal{O}(-D_\gamma);\ p+q\ge 3.$

As for the cohomology groups induced from the above inclusions, we get the following. Let us remark Corollary 1.3 and (1.12).

Proposition 4.3.

(P: 0, 0) $H^{0,0}(X)\xrightarrow{\cong}H_\gamma^{0,0}(X)\xrightarrow{\cong}H^{0,0}(X;\mathcal{O}(D_\gamma))$

$$(P\colon 0,1)\quad H^{0,1}(X) \xrightarrow{inj.} H^{0,1}_{\gamma}(X) \xrightarrow{inj.} H^{0,1}(X;\mathcal{O}(D_\gamma))$$

$$(P\colon 0,2)\quad H^{0,2}(X) \xrightarrow{surj.} H^{0,2}_{\gamma}(X) \xrightarrow{surj.} H^{0,2}(X;\mathcal{O}(D_\gamma))$$

$$(P\colon 1,0)\quad H^{1,0}(X) \xrightarrow{inj.} H^{1,0}_{\gamma}(X) \xrightarrow{inj.} H^{1,0}(X;\mathcal{O}(D_\gamma))$$

$$(P\colon 2,0)_+\quad H^{2,0}(X) \xrightarrow{\cong} H^{2,0}_{\gamma}(X) \xrightarrow{inj.} H^{2,0}(X;\mathcal{O}(D_\gamma))$$

$$(P\colon 2,0)_-\quad H^{2,0}(X) \xleftarrow{\cong} H^{2,0}_{\gamma}(X) \xleftarrow{inj.} H^{2,0}(X;\mathcal{O}(-D_\gamma))$$

$$(P\colon 2,1)\quad H^{2,1}(X) \xleftarrow{\cong} H^{2,1}_{\gamma}(X) \xleftarrow{surj.} H^{2,1}(X;\mathcal{O}(-D_\gamma))$$

$$(P\colon 2,2)\quad H^{2,2}(X) \xleftarrow{\cong} H^{2,2}_{\gamma}(X) \xleftarrow{\cong} H^{2,2}(X;\mathcal{O}(-D_\gamma))$$

Proof. $(P\colon 0,0)$, $(P\colon 0,1)$ and $(P\colon 0,2)$ left $\xrightarrow{surj.}$ are due to Pardon [9, Corollaries 3.8 and 4.2], which essentially come from the fact $H^0(\mathcal{O}_{D_\gamma}(D_\gamma))=0$. $(P\colon 0,2)$ right $\xrightarrow{surj.}$ is due to the injectivity of the map $H^{2,0}(X;\mathcal{O}(-D_\gamma))\to H^{2,0}(X)$ at $(P\colon 2,0)$ and the Serre duality theorem. $(P\colon 2,0)_\pm$ are obvious. As for the injectivity of $(P\colon 2,1)$ left $\xleftarrow{\cong}$: Take $\omega\in\mathcal{A}^{2,1}_{\gamma}(X)$ with $\bar\partial\omega=0$ and assume that it gives the zero element of $H^{2,1}_{E}(X)=H^{2,1}(X)$, that is, there is $\eta\in\mathcal{A}^{2,0}_{E}(X)$ such that $\bar\partial\eta=\omega$. Then ω also gives the zero element of $H^{2,1}_{\gamma}(X)$ because Lemma 4.1 $(L\colon 2,0)$ implies $\eta\in\mathcal{A}^{2,0}_{\gamma}(X)$. Next the surjectivity of $(P\colon 2,2)$ right $\xleftarrow{\cong}$ is due to $\mathcal{A}^{2,2}_{\gamma}(X)=\{\omega\in\mathcal{S}^{2,2}_{\gamma}(X-E)\,|\,a_\gamma^{-1}\omega\in L^2\}=\{\omega\in\mathcal{S}^{2,2}_{E}(X-E)\,|\,a_\gamma^{-1}\omega\in L^2\}\xleftarrow{\cong}(\mathcal{A}^{2,2}_{E}\otimes\mathcal{O}(-D_\gamma))(X)$, where $a_\gamma^{-1}\omega\in L^2$ means that, for any sufficiently small open neighborhood U of any point of E, $a_\gamma^{-1}\omega$ belongs to $L^2(U,\mathrm{loc})$: see Lemma 4.1. Further $H^{0,1}(X)\xrightarrow{inj.}H^{0,1}(X;\mathcal{O}(D_\gamma))$ and $H^{0,0}(X)\xrightarrow{\cong}H^{0,0}(X;\mathcal{O}(D_\gamma))$ imply $H^{2,1}(X)\xleftarrow{surj.}H^{2,1}(X;\mathcal{O}(-D_\gamma))$ and $H^{2,2}(X)\xleftarrow{\cong}H^{2,2}(X;\mathcal{O}(-D_\gamma))$. Now certainly $(P\colon 2,1)$ and $(P\colon 2,2)$ hold.

Proofs of Theorems 2.2 *and* 2.3. First Proposition 4.3 holds even if H is changed into $\hat{H}$. Hence the finite dimensionalities of the cohomology groups at (2.3) imply that $\hat{H}^{p,q}_{\gamma}(X)$ for $(p,q)=(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(2,0)$, $(2,1)$, $(2,2)$ are of finite dimension. Therefore, by the comment following (1.7), we have the isomorphisms $\mathcal{H}^{p,q}_{\gamma}(X)\xrightarrow{\cong}H^{p,q}_{\gamma}(X)\xrightarrow{\cong}\hat{H}^{p,q}_{\gamma}(X)$ for (p,q) above. Thus the proof of Theorem 2.3 for $H^{p,q}_{\gamma}(X)$ is complete. Next, at [4, Theorem 2.1], essentially Haskell proved the existence of the commutative diagram:

$$\begin{array}{ccccc} H^{0,q}(X) & \xrightarrow{\cong} & H^{0,q}_{c,\gamma}(X) & \nwarrow\cong & \\ \downarrow\cong & & \downarrow\cong & & \mathcal{H}^{0,q}_{c,\gamma}(X)\,. \\ \hat{H}^{0,q}(X) & \xrightarrow{\cong} & \hat{H}^{0,q}_{c,\gamma}(X) & \swarrow\cong & \end{array} \tag{4.1}$$

On the other hand, using the argument similar to the proof of Proposition 1.2, we get the implications:

$$\bar{\partial}^{2,q}_{c,\gamma} \overset{a_\gamma \times}{\longleftarrow\!\!\supset} \hat{\bar{\partial}}^{2,q}[-D_\gamma], \quad \hat{\bar{\partial}}^{2,q}_{c,\gamma} \overset{a_\gamma \times}{\longleftarrow\!\!\supset} \hat{\bar{\partial}}^{2,q}[-D_\gamma]. \tag{4.2}$$

Let us prove the left implication. Take $\varphi \in \mathrm{dom}\, \bar{\partial}^{2,q}[-D_\gamma]$. Then $a_\gamma \varphi \in \mathcal{S}^{2,q}_\gamma(X)$ because of Corollary 4.2. We use the notation in the proof of Proposition 1.2. For $\psi \in \mathrm{dom} \bar{\vartheta}^{2,q}_\gamma$ with supp $\psi \subset U$, we have, by (1.8),

$$\begin{aligned}\int_U \bar{\partial}(a_\gamma\varphi) \wedge \bar{*}_\gamma \psi - \int_U a_\gamma \varphi \wedge \bar{*}_\gamma \bar{\vartheta}_\gamma \psi &= \int \bar{\partial}(a_\gamma \varphi \wedge \bar{*}_\gamma \psi) \\ = \int d(a_\gamma \varphi \wedge \bar{*}_\gamma \psi) &= \pm \lim_{\varepsilon \to 0} \int_{T(\varepsilon)} a_\gamma \varphi \wedge \bar{*}_\gamma \psi \\ = \pm \lim_{\varepsilon \to 0} \int_{T(\varepsilon)} \varphi \wedge a_\gamma \bar{*}_\gamma \psi &= 0,\end{aligned}$$

because φ is bounded on U and the $(0, 1-q)$-form $a_\gamma \bar{*}_\gamma \psi$ has the coefficient belonging to $L^2(U, \mathrm{loc})$. Thus we get (4.2). Since it obviously implies

$$\mathrm{Ker}\, \bar{\partial}^{2,2}_{c,\gamma} \overset{\cong}{\longleftarrow} \mathrm{Ker}\, \bar{\partial}^{2,2}[-D_\gamma], \quad \mathrm{Ker}\, \hat{\bar{\partial}}^{2,2}_{c,\gamma} \overset{\cong}{\longleftarrow} \mathrm{Ker}\, \hat{\bar{\partial}}^{2,2}[-D_\gamma],$$

we have

$$H^{2,2}_{c,\gamma}(X) \overset{surj.}{\longleftarrow} H^{2,2}(X; \mathcal{O}(-D_\gamma)), \quad \hat{H}^{2,2}_{c,\gamma}(X) \overset{surj.}{\longleftarrow} \hat{H}^{2,2}(X; \mathcal{O}(-D_\gamma)). \tag{4.3}$$

Now we get Theorem 2.2 and the same one but with H replaced by $\hat{H}$ because of Proposition 4.3 and (4.1)~(4.3). The remaining part of Theorem 2.3 can be induced from Theorem 2.2 for $\hat{H}$.

Next we want to prove Theorem 2.4. First [7, § 2(c)] says

$$\begin{aligned}&\hat{d}^i_{c,\gamma} = \hat{d}^i_\gamma, \quad \hat{\delta}^i_{c,\gamma} = \hat{\delta}^i_\gamma \quad \text{for} \quad p \neq 1, 2, \\ &\mathrm{Ker}\, \hat{d}^1_{c,\gamma} = \mathrm{Ker}\, \hat{d}^1_\gamma, \quad \mathrm{Ker}\, \hat{\delta}^2_{c,\gamma} = \mathrm{Ker}\, \hat{\delta}^2_\gamma.\end{aligned} \tag{4.4}$$

Combined with [7, (5.2) and Theorems 1 and 8.6], they imply

$$\begin{aligned}&H^{p,q}(X(d)) \overset{\cong}{\longrightarrow} H^{p,q}_{c,\gamma}(X(d)) \overset{\cong}{\longrightarrow} H^{p,q}_\gamma(X(d)) \ : \ p+q \leq 1, \\ &H^{2,2}(X(d)) \overset{\cong}{\longleftarrow} H^{2,2}_\gamma(X(d)) \overset{\cong}{\longleftarrow} H^{2,2}_{c,\gamma}(X(d)), \\ &H^{p,q}_\gamma(X(d)) \overset{surj.}{\longleftarrow} H^{p,q}_{c,\gamma}(X(d)) \ : \ p+q = 3, \\ &\dim H^{p,q}(X(d)) = \dim H^{p,q}_\gamma(X(d)) \ : \ p+q = 3.\end{aligned} \tag{4.5}$$

Proof of Theorem 2.4. It is well-known that the so-called Hodge identities *etc.* imply

$$H^{p,q}(X) \cong H^{p,q}(X(d)). \tag{4.6}$$

As for the theorem for $(p, q)=(0, 0)$, $(2, 2)$, $(0, 1)$: Obviously the vertical arrows can be made and Theorem 2.2 and (4.5) guarantee the isomorphicness of the horizontal arrows. Therefore, by the commutativeness of the diagrams and (4.6), we get the isomorphicness *etc.* of the vertical arrows. As for the case $(p, q)=(2, 1)$: It suffices to show that the map $H_\gamma^{2,1}(X) \to H_\gamma^{2,1}(X(d))$ is isomorphic. It is trivially surjective and using (4.5), (4.6) and Theorem 2.2, we have $\dim H_\gamma^{2,1}(X(d))=\dim H^{2,1}(X(d))=\dim H^{2,1}(X)=\dim H_\gamma^{2,1}(X)$. Thus it is certainly isomorphic. As for the case $(p, q)=(1, 0)$: Let us first show the existence of the map $H_{c,\gamma}^{1,0}(X) \to H_{c,\gamma}^{1,0}(X(d))$. For $\varphi \in \operatorname{Ker} \bar{\partial}_{c,\gamma}^{1,0}=H_{c,\gamma}^{1,0}(X(d))$ one can take a sequence $\varphi_j \in \mathcal{A}_c^{1,0}(X-E)$ satisfying $\varphi_j \to \varphi$ and $\bar{\partial}\varphi_j \to 0$ in $\mathcal{L}_\gamma^{1,*}(X)$. Letting $(\ , \ )$ be the inner product of $\mathcal{L}_\gamma^{1,*}(X)$ and applying the Hodge identity for $\mathcal{A}_c^{1,0}$ $(X-E)$, we have

$$\begin{aligned} 2(\bar{\partial}\varphi_j, \bar{\partial}\varphi_j) &= (2\bar{\vartheta}_\gamma\bar{\partial}\varphi_j, \varphi_j) = ((\delta_\gamma d+d\delta_\gamma)\varphi_j, \varphi_j) \\ &= (d\varphi_j, d\varphi_j)+(\delta_\gamma\varphi_j, \delta_\gamma\varphi_j) \end{aligned}$$

Hence $d\varphi_j$ tends to zero in $\mathcal{L}_\gamma^1(X)$, which means that there exists the desired map. As for its injectivity: We have $\mathcal{L}_\gamma^{1,0}(X) \cap \operatorname{Range} d_{c,\gamma}^0 \subset \mathcal{L}_\gamma^{1,0}(X) \cap \operatorname{Range} d_\gamma^0$. For an element $d_\gamma^0\varphi=\partial_\gamma\varphi+\bar{\partial}_\gamma\varphi$ of the right side, since $\bar{\partial}_\gamma\varphi=0$, φ belongs to $\operatorname{Ker} \bar{\partial}_\gamma^{0,0}=H_\gamma^{0,0}(X)=\{\text{constant functions on } X\}$. That is, $d_\gamma^0\varphi=0$, which shows the injectivity of the map. The proof here guarantees further the injectivity of the map $H_\gamma^{1,0}(X(d)) \to H_\gamma^{1,0}(X)$. Finally, observing [7, Theorem 2], the case $(p, q)=(1, 2)$ is trivial.

§ 5. Proof of the Theorem in § 3

We use the notations in § 3. On the local coordinate neighborhood (U, u) given at § 3, we get, by a straightforward computation,

Lemma 5.1. *Let* $L^2=L^2(U, \text{loc})$ *be the space of locally square-integrable functions on* U *as before. Then we have*

$(L\colon 0, 0)$ $f \in \mathcal{L}_\gamma^{0,0}(U)$ *is equivalent to* $a_\gamma f \in L^2$,
$(L\colon 0, 1)$ $f d\bar{u} \in \mathcal{L}_\gamma^{0,1}(U)$ *is equivalent to* $f \in L^2$,
$(L\colon 1, 0)$ $f du \in \mathcal{L}_\gamma^{1,0}(U)$ *is equivalent to* $f \in L^2$,
$(L\colon 1, 1)$ $f du \wedge d\bar{u} \in \mathcal{L}_\gamma^{1,1}(U)$ *is equivalent to* $a_\gamma^{-1} f \in L^2$.

The lemma and an easy calculation imply

Corollary 5.2. *Set* $E=|D_\gamma|$.

(0) $\bar{\partial}^{0,q} \hookrightarrow \bar{\partial}^{0,q}_{c,\gamma} \hookrightarrow \bar{\partial}^{0,q}_{\gamma} \xrightarrow{a_\gamma \times} \bar{\partial}^{0,q}_{E}[D_\gamma]$

(1) $\bar{\partial}^{1,q}_{E} \hookleftarrow \bar{\partial}^{1,q}_{\gamma} \hookleftarrow \bar{\partial}^{1,q}_{c,\gamma} \xleftarrow{a_\gamma \times} \bar{\partial}^{1,q}_{E}[-D_\gamma)]$

Proof. All the inclusions will be obvious because of Lemma 5.1 and Proposition 1.2 (2). The inclusion $\bar{\partial}^{1,q}_{c,\gamma} \hookleftarrow \bar{\partial}^{1,q}_{E}[-D_\gamma]$ can be proved in the same way as the proof of (4.2).

Proposition 5.3.

$(P: 0, 0) \quad H^{0,0}(X) \xrightarrow{inj.} H^{0,0}_{c,\gamma}(X) \xrightarrow{\cong} H^{0,0}_{\gamma}(X) \xrightarrow{\cong} H^{0,0}(X; \mathcal{O}(D_\gamma))$

$(P: 0, 1) \quad H^{0,1}(X) \xrightarrow{\cong} H^{0,1}_{c,\gamma}(X) \xrightarrow{\cong} H^{0,1}_{\gamma}(X) \xrightarrow{\cong} H^{0,1}(X; \mathcal{O}(D_\gamma))$

$(P: 1, 0) \quad H^{1,0}(X) \xleftarrow{\cong} H^{1,0}_{\gamma}(X) \xleftarrow{\cong} H^{1,0}_{c,\gamma}(X) \xleftarrow{\cong} H^{1,0}(X; \mathcal{O}(-D_\gamma))$

$(P: 1, 1) \quad H^{1,1}(X) \xleftarrow{surj.} H^{1,1}_{\gamma}(X) \xleftarrow{\cong} H^{1,1}_{c,\gamma}(X) \xleftarrow{\cong} H^{1,1}(X; \mathcal{O}(-D_\gamma))$

Proof. Haskell [4, Theorem 3.1] and Pardon [9, Corollary 5.4] proved

$$H^{0,q}_{c,\gamma}(X) \xrightarrow{\cong} H^{0,q}_{\gamma}(X) \xrightarrow{\cong} H^{0,q}(X; \mathcal{O}(D_\gamma)) . \tag{5.1}$$

Trivially we have $H^{1,0}(X) \xleftarrow{\cong} H^{1,0}_{\gamma}(X) \xleftarrow{inj.} H^{1,0}_{c,\gamma}(X) \xleftarrow{inj.} H^{1,0}(X; \mathcal{O}(-D_\gamma))$ and, since (5.1) and the dual argument imply $\dim H^{1,0}_{\gamma}(X) = \dim H^{1,0}(X; \mathcal{O}(-D_\gamma))$, we get $(P: 1, 0)$. Similarly we have $H^{1,1}_{\gamma}(X) \xleftarrow{\cong} H^{1,1}_{c,\gamma}(X) \xleftarrow{\cong} H^{1,1}(X; \mathcal{O}(-D_\gamma))$. $(P: 0, 0)$ first $\xrightarrow{inj.}$ is trivial and we have $H^{0,0}(X) \xrightarrow{inj.} H^{0,0}(X; \mathcal{O}(D_\gamma))$. Hence, by the dual argument, we have $(P; 1, 1)$ first $\xleftarrow{surj.}$. Finally the surjection $H^{0,1}(X) \to H^{0,1}_{c,\gamma}(X)$ turns out to be isomorphic by the isomorphism $H^{1,0}(X) \xleftarrow{\cong} H^{1,0}_{\gamma}(X)$.

Proof of Theorem 3.1. We have $\hat{d}^{i}_{c,\gamma} = \hat{d}^{i}_{\gamma}$ because the pseudo-metric γ is conical near E: see [8, Theorem]. Combined with [8, Corollary 6], it implies

$$\begin{gathered} H^{0,q}(X(d)) \xrightarrow{\cong} H^{0,q}_{c,\gamma}(X(d)) \xrightarrow{\cong} H^{0,q}_{\gamma}(X(d)) , \\ H^{1,q}(X(d)) \xleftarrow{\cong} H^{1,q}_{\gamma}(X(d)) \xleftarrow{\cong} H^{1,q}_{c,\gamma}(X(d)) . \end{gathered} \tag{5.2}$$

Also in the curve case, (4.6) holds. Now the assertion in the case $(p, q) = (0, 0)$ can be obtained using [8, Corollary 5], (5.2), Proposition 5.3 for $(p, q) = (0, 0)$ and the commutativeness of the diagram. And it implies the case $(p, q) = (1, 1)$. The remaining cases $(p, q) = (0, 1)$, $(1, 0)$ are obtained using (5.2) and Proposition 5.3.

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