Publ. RIMS, Kyoto Univ.
26 (1990), 735–783

An Integral Representation of Singular Solutions and Removable Singularities of Solutions to Linear Partial Differential Equations

By

Sunao ŌUCHI*

§ 0. Introduction

Let $L(z, \partial_z)$ be a linear partial differential operator with the order $m \geqq 1$, whose coefficients are holomorphic in $\Omega = \{z \in C^{n+1};\ |z| \leqq R\}$, and K be a connected nonsingular complex hypersurface in Ω through the origin $z=0$. In the present paper we treat the equation

$$L(z, \partial_z)\, u(z) = f(z)\,, \tag{0.1}$$

where $u(z)$ may be singular on K, and $f(z)$ is holomorphic in Ω.

There are two main purposes in this paper. The one is to give an integral representation of solutions to (0.1) singular on K (Theorem 2.5). The other is to show that if $u(z)$ has some growth property near K under some conditions on $L(z, \partial_z)$, then $u(z)$ is holomorphic at K (Theorem 1.3), that is, the singularity on K is removable. The conditions on $L(z, \partial_z)$ are given by means of the characteristic indices of K and the localization on K defined in [9] and [10]. The author does not know such a theorem about removable singularities of solutions to linear partial differential equations as that in this paper. In order to show Theorem 1.3 we need the detailed analysis of the obtained integral representation and use theorems about the Laplace transform of functions with asymptotic expansions with bounds (Theorems 1.7 and 1.9), which are also the results of this paper.

We make reference to singular solutions to (0.1) in short. As for existence,

Communicated by M. Kashiwara, October 1, 1987.
* Department of Mathematics, Sophia University, Tokyo 102, Japan.

it was studied in [1], [2], [3], [13], [16] and others, where they constructed singular solutions to noncharacteristic Cauchy problem with singular initial data. The existence of solutions singular on K was also considered in [4], [11] and [14] apart from singular Cauchy problems. The condition in [11], given by means of the principal localization, is less restrictive than those in others.

The integral representation was obtained for operators with decomposable principal symbol in [6] and [7]. In the present paper we give it for a wider class of operators and its form is slightly different from that given in [6] and [7].

In §1 we give notations and definitions and state the theorem concerning removable singularities. We also give results about the relations between the Laplace transform and functions with asymptotic expansion with bounds. We don't give here the integral representation. Because it requires further preliminaries. In §2 we give them and the integral representation. Roughly speaking its kernel function takes the form

$$K(z, \lambda, t'') = \int \exp(-\lambda^{\alpha} \zeta)\, w(z, t'', \lambda, \zeta)\, d\zeta\,. \tag{0.2}$$

So construction of representation is that of $w(z, t'', \lambda, \zeta)$, which we call also the kernel function. In §3–§5 we construct $w(z, t'', \lambda, \zeta)$ and get the integral representation of solutions singular on K. In §6 we show lemmas and propositions used in the previous sections or required in the following sections. In §7 we investigate the kernel function $w(z, t'', \lambda, \zeta)$. We try to analyze its singularities with respect to ζ. The estimates of functions appearing in §3–§7 are given in §9. In §8 we complete the proof of the theorem of removable singularities (Theorem 1.3), combining the results obtained in the preceding sections with Theorems 1.7 and 1.9. In §9 we show what are left unproved, in which the estimates and some lemmas of holomorphic functions needed in construction of the integral representation are contained. In §10 we discuss about functions with asymptotic expansions and give the proofs of Theorems 1.7 and 1.9.

In this paper many constants will appear. So for simplicity we denote various constants by the same A, B, C, etc.. There will be no confusions.

§1. Notations and Definitions

First we give notations. $z=(z_0, z_1, \cdots, z_n)=(z_0, z')=(z_0, z_1, z'')$ is the coordinate of the $(n+1)$-dimensional complex space $\mathbf{C}^{n+1}$ with the norm $|z|=\max\{|z_i|;\ 0\leqq i\leqq n\}$, while $\xi=(\xi_0, \xi_1, \cdots, \xi_n)=(\xi_0, \xi_1, \xi'')=(\xi_0, \xi')$ is the dual variable. $\partial_z=(\partial_0, \partial_1, \cdots, \partial_n)=(\partial_0, \partial_1, \partial'')=(\partial_0, \partial')$, $\partial_i=\partial/\partial z_i$ is the differentiation. $\boldsymbol{N}$ is the set of natural numbers and $\boldsymbol{Q}$ is the set of rational numbers. Now let K be a nonsingular complex hypersurface through the origin $z=0$. We may choose the coordinate so that $K=\{z_0=0\}$. Then we can write $L(z, \partial_z)$ in (0.1) in the form

$$(1.1)\qquad \begin{cases} L(z, \partial_z) = \sum_{k=0}^{m} L_k(z, \partial_z)\,, \\ L_k(z, \partial_z) = \sum_{i=s_k}^{k} A_{k,i}(z, \partial')\,(\partial_0)^{k-i}\,. \end{cases}$$

Here $L_k(z, \partial_z)$ is the homogeneous part of degree k. The integers s_k $(0\leqq k\leqq m)$ are chosen so that $A_{k,s_k}(z, \xi')\not\equiv 0$ if $L_k(z, \xi)\not\equiv 0$, and we put $s_k=+\infty$ if $L_k(z, \xi)\equiv 0$.

Now let us give several definitions and notions derived from $L(z, \partial_z)$. Put $A=\{(k, s_k)\in \boldsymbol{R}^2;\ 0\leqq k\leqq m,\ s_k\neq +\infty\}$. We denote the convex hull of A by $\hat{A}$. Let Σ be the lower convex part of the boundary of $\hat{A}$, and Δ be the vertices of Σ. We set $\Delta=\{(k_i, s_{k_i});\ 0\leqq i\leqq l\}$, $m=k_0>k_1>k_2>\cdots>k_l\geqq 0$. If $l=0$, $\Sigma=\Delta=\{(m, s_m)\}$. Assume $l\geqq 1$. Then Σ consists of segments $\Sigma(\mathrm{i})$ $(1\leqq i\leqq l)$. The end points of $\Sigma(\mathrm{i})$ are $(k_{i-1}, s_{k_{i-1}})$ and (k_i, s_{k_i}) (see Fig. 1.1).

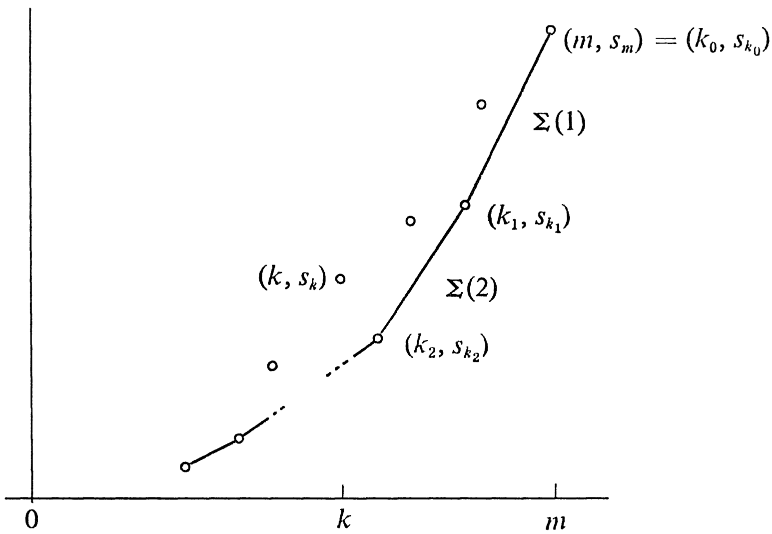

Fig. 1.1.

Set

$$(1.2)\qquad \begin{cases} \sigma_0 = +\infty\,, \\ \sigma_i = \max\,\{1, (s_{k_{i-1}}-s_{k_i})/(k_{i-1}-k_i)\} \quad (1\leqq i\leqq l)\,. \end{cases}$$

Then there is a $p\in\boldsymbol{N}$ such that

$$+\infty = \sigma_0 > \sigma_1 > \sigma_2 > \cdots > \sigma_{p-1} > \sigma_p = 1 . \tag{1.3}$$

Define for $0 \leqq i \leqq p$

$$\begin{cases} \alpha_i = (\sigma_i - 1)/\sigma_i , \\ \gamma_i = \sigma_i - 1 . \end{cases} \tag{1.4}$$

Here we mean $\alpha_0 = 1$ and $\gamma_0 = +\infty$. From the definition of σ_i, we have

Lemma 1.1. *Suppose $s_k \neq +\infty$. Then there are nonnegative $\beta_k^i \in \boldsymbol{Q}$ $(1 \leqq i \leqq p)$ such that*

$$(s_{k_{i-1}} - s_k)(1 - \alpha_i) + \beta_k^i = k_{i-1} - k , \tag{1.5}$$

and $(k, s_k) \in \Sigma(i)$ if and only if $\beta_k^i = 0$.

The proof of Lemma 1.1 will be given in §6 and we set in the sequel $\beta_k = \beta_k^1$ and $\alpha = \alpha_1$ for simplicity.

Remark 1.2. In [9] and [10] (see also [12]) characteristic indices were defined and denoted also by $\{\sigma_i\}$. In general they are different from those defined by (1.2). But if we assume some conditions on $L(z, \partial_z)$, they are coincident with each other (see Remark 1.4).

For an open set W in $\boldsymbol{C}^N$, $\tilde{W}$ means the universal covering space of W. We denote by $\mathcal{O}(W)$ the set of all holomorphic functions on W and by $\mathcal{O}(\tilde{W})$ the set of all holomorphic functions on $\tilde{W}$. Let U be a polydisk in $\boldsymbol{C}^{n+1}$ with center $z=0$. Then we set $\tilde{U}(a, b) = \{z \in \widetilde{(U - \{z_0 = 0\})};\ a < \arg z_0 < b\}$ and $\tilde{U}(a) = \tilde{U}(-a, a)$ $(a > 0)$. Hence $\mathcal{O}(U)$ is the set of all holomorphic functions in U and $\mathcal{O}(\tilde{U}(a, b))$ is the set of all holomorphic functions on the sector $\tilde{U}(a, b)$. Obviously if $b - a > 2\pi$, then $\mathcal{O}(\tilde{U}(a, b))$ contains multi-valued functions.

Now, by using these definitions, we can give a theorem about the removablity of singularity K, which is one of the main results in this paper.

Theorem 1.3. *Assume*

$$\begin{cases} \text{(a)} & \sigma_1 > 1 , \\ \text{(b)} & s_{k_{p-1}} = 0 , \\ \text{(c)} & \prod_{i=0}^{p-1} A_{k_i, s_{k_i}}(0, z', \xi') \not\equiv 0 . \end{cases} \tag{1.6}$$

Let $u(z) \in \mathcal{O}(\tilde{\Omega}(\theta_0))$ $(\theta_0 > \pi(1/2\gamma_{p-1} + 1))$ be a solution to

$$L(z, \partial_z)\, u(z) = f(z) \in \mathcal{O}(\Omega) . \tag{1.7}$$

Suppose that for any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that

(1.8) $$|u(z)| \leqq C_\varepsilon \exp(\varepsilon |z_0|^{-\gamma_{p-1}}) \quad \textit{for} \quad z \in \tilde{\Omega}(\theta_0)\,.$$

Then $u(z) \in \mathcal{O}(\Omega)$.

The proof of Theorem 1.3 is long and completed in §8. As we said in §0, Theorem 1.3 follows from the integral representation in §2.

Remark 1.4. Let $L(z, \partial_z)$ be an operator satisfying (1.6)–(c). Then $\{\sigma_i\}$ $(1 \leqq i \leqq p)$ are coincident with the characteristic indices in [9] and [10], and $A_{k_i}(0, z', \xi')$ $(0 \leqq i \leqq p-1)$ are the localizations defined there. So we can state (1.6) (a)–(c) in other conditions which are invariant under the coordinates transformations.

We give simple examples.

Corollary 1.5. *Let*

(1.9) $$L(z, \partial_z) = (\partial_0)^k + A(z, \partial')\,,$$

where $A(z, \partial')$ *is an operator with* ord. $A(z, \partial') = m > k$ *and the principal symbol* $A_m(z, \xi')$. *Assume* $A_m(0, z', \xi') \not\equiv 0$. *Then if a solution* $u(z) \in \mathcal{O}(\tilde{\Omega}(\theta_0))$ *to* (1.7), $(\theta_0 > (\pi/2)(m/k+1))$, *satisfies for any* $\varepsilon > 0$

(1.10) $$|u(z)| \leqq C_\varepsilon \exp(\varepsilon |z_0|^{-k/(m-k)})\,.$$

Then $u(z) \in \mathcal{O}(\Omega)$.

We have $\sigma_1 = m/(m-k)$ and $\sigma_2 = 1$. Hence $\gamma_1 = \sigma_1 - 1 = k/(m-k)$ for $L(z, \partial_z)$ in (1.9). More concretely let $L(z, \partial_z) = (\partial_0)^k - (-1)^{m+k}(\partial_1)^m$. Set

(1.11) $$u_1(z) = \int_0^{+\infty} \exp(-\lambda z_0 - \lambda^{k/m}(z_1 + d))\, d\lambda,\ (d > 0)\,.$$

$u_1(z)$ satisfies $L(z, \partial_z)\, u_1(z) = 0$. It holds for $u_1(z)$ that for any $\varepsilon > 0$ if $z \in \{z;\ |\arg z_0| < \frac{\pi}{2}(\frac{m}{k}+1) - \varepsilon,\ |z_1| < r_\varepsilon\}$, $|u(z)| \leqq C_\varepsilon$. So the condition $\theta_0 > \frac{\pi}{2}(\frac{m}{k}+1)$ is essential. $u_1(z)$ has the bound $|u_1(z)| \leqq A \exp(B |z_0|^{-k/(m-k)})$ on $\{z;\ |z| < r,\ -\infty < \arg z_0 < +\infty\}$. The condition (1.10) is also essential.

Now let us proceed to give the theorems about functions with asymptotic expansions. As we said in §0, they will be used to show Theorem 1.3. The proofs are in §10. Let $u(t)$ be a continuous function on $[A, +\infty)$ $(A > 0)$ such that $|u(t)| \leqq C \exp(B |t|^\gamma)$ $(\gamma > 0)$. We define the γ-Laplace transform $\hat{u}(\xi)$ of $u(t)$ by

(1.12) $$\hat{u}(\xi) = \int_a^{+\infty} \exp(\xi t)\, u(t^{1/\gamma})\, t^{-1}\, dt \quad (a > A^\gamma)\,,$$

which is holomorphic in $\{\xi;\ \mathrm{Re}\,\xi<-B\}$. The inversion formula is given by

$$u(t)=\frac{t^{\gamma}}{2\pi i}\int_{d-i\infty}^{d+i\infty}\exp(-\xi t^{\gamma})\,\hat{u}(\xi)\,d\xi\quad(d<-B)\,. \tag{1.13}$$

Definition 1.6. *We say that $u(t)$ has the γ-asymptotic expansion on $[A,+\infty)$, if for any $N\geqq 1$*

$$|u(t)-\textstyle\sum_{k=0}^{N-1}c_k t^{-k}|\leqq A_1 R^{-N}\Gamma(N/\gamma+1)|t|^{-N} \tag{1.14}$$

holds on $[A,+\infty)$.

From the definition $|c_N|\leqq A_1 R^{-N}\Gamma(N/\gamma+1)$, that is, the coefficients of the asymptotic expansion have the estimates of Gevrey type. Suppose that $u(t)$ has the asymptotic expansion (1.14). Then, by using the sequence $\{c_k\}$ $(k=0, 1,\cdots)$, define

$$g(z)=\textstyle\sum_{k=0}^{+\infty}\dfrac{c_k z^k}{\Gamma(k/\gamma+1)}\,, \tag{1.15}$$

which is holomorphic in $\{z\in \boldsymbol{C}^1;\ |z|<R\}$. We have for $\hat{u}(\xi)$

Theorem 1.7. *Assume $u(t)$ has the γ-asymptotic expansion (1.14) on $[A,+\infty)$. Then the γ-Laplace transform $\hat{u}(\xi)$ is holomorphic in $(\{\xi;\ \mathrm{Re}\,\xi<R^{\gamma},\ \xi\notin[0,R^{\gamma})\}$ and it can be holomorphically extended into $\{\xi;\ 0<|\xi|<R^{\gamma}\}$ such that $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ 0<|\xi|<R^{\gamma}\}})$, for any $\Theta>0$ and $0<c<R^{\gamma}$*

$$|\hat{u}(\xi)|\leqq M_{c,\Theta}|\log\xi|\quad in\ \{\xi;\ |\arg\xi|<\Theta,\ 0<|\xi|<c\}\,, \tag{1.16}$$

and $\{\hat{u}(\xi)-\hat{u}(\xi e^{2\pi i})\}/2\pi i=g(\xi^{1/\gamma})$, where $\xi^{1/\gamma}=|\xi|^{1/\gamma}e^{i(\arg\xi)/\gamma}$.

Now let us consider functions on a sector with asymptotic expansions with bounds. We set $\tilde{S}(a,b)=\{t\in\widetilde{(\boldsymbol{C}^1-\{0\}}\,;\ |t|\geqq A,\ a<\arg t<b\}$ $(A>0)$ and $\tilde{S}(a)=\tilde{S}(-a,a)$ $(a>0)$.

Definition 1.8. *We say that $u(t)\in\mathcal{O}(\tilde{S}(a,b))$ has the γ-asymptotic expansion*

$$u(t)\sim\textstyle\sum_{k=0}^{+\infty}c_k t^{-k}\quad at\ t=\infty\ in\ \tilde{S}(a,b)\,, \tag{1.17}$$

if for any $N>0$

$$|u(t)-\textstyle\sum_{k=0}^{N-1}c_k t^{-k}|\leqq A_1 B_1^N\Gamma(N/\gamma+1)|t|^{-N} \tag{1.18}$$

holds on any closed subsectors S_1 in $\tilde{S}(a,b)$.

If $u(t)\in\mathcal{O}(\tilde{S}(\theta_0))$ satisfies, for any $\varepsilon>0$

(1.19) $$|u(t)|\leqq C_\varepsilon \exp(\varepsilon|t|^\gamma)\ (\gamma>0) \quad \text{in } \tilde{S}(\theta_0),$$

then $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2\}})$. We have

Theorem 1.9. *Assume $u(t)\in\mathcal{O}(\tilde{S}(\theta_0))$ satisfies* (1.19). *Then $u(t)$ has the γ-asymptotic expansion*

(1.20) $$u(t)\sim\sum_{k=0}^{+\infty} c_k t^{-k} \quad at\ t=\infty\ in\ \tilde{S}(\theta_0),$$

if and only if the γ-Laplace transform $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2\}})$ satisfies the following conditions:

$\hat{u}(\xi)$ is holomorphically extensible into $\{\xi;\ 0<|\xi|<c\}$ for some $c>0$ so that $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ 0<|\xi|<c\}})$, for any $\Theta>0$

(1.21) $$|\hat{u}(\xi)|\leqq M_\Theta|\log\xi| \quad in \quad \widetilde{\{\xi;\ |\arg\xi|<\Theta,\ 0<|\xi|<c\}},$$

and $F(\xi)=\{\hat{u}(\xi)-\hat{u}(\xi e^{2\pi i})\}/2\pi i$ has the convergent power series of $\xi^{1/\gamma}$ at $\xi=0$,

(1.22) $$F(\xi)=\sum_{k=0}^{+\infty} c_k\,\xi^{k/\gamma}/\Gamma(k/\gamma+1)\ (|\xi|^{1/\gamma}<R) \quad (c\leqq R^\gamma).$$

Moreover if $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2\}})$ satisfies all above conditions and $\theta_0>\pi/2\gamma+\pi$, then $u(t)$ is holomorphic at $t=\infty$.

For functions with asymptotic expansions with Gevrey type we refer to [15], where ordinary differential equations were treated, and the papers in its references.

§ 2. Integral Representation

In §2 we show an integral representation of a singular solution $u(z)$ satisfying (0.1). From now on, we always assume $u(z)\in\mathcal{O}(\tilde{\Omega}(\theta_0))$ $(\theta_0>\pi)$, $s_m\geqq 1$ and

(2.1) $$A_{m,s_m}(0,\xi')\not\equiv 0.$$

So K is characteristic. We may assume that for $\xi'=\hat{\xi}'=(1,0,\cdots,0)$

(2.1)′ $$A_{m,s_m}(0,\hat{\xi}')\neq 0 \quad (s_m\geqq 1).$$

We try to obtain an integral representation of $u(z)$ as the sum of functions of the form

(2.2) $$\begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\Lambda(\psi)}\exp(\lambda z_0)(\log\lambda)\,d\lambda\int_{T''}K_\theta^h(z,\lambda,t'')\,\hat{u}_\theta^h(\lambda,t'')\,dt'', \\ K_\theta^h(z,\lambda,t'')=\displaystyle\int_{C(\theta)}\exp(-\lambda^\alpha\zeta)\,w^h(z,t'',\lambda,\zeta)\,d\zeta,\ (0\leqq h\leqq s_m-1). \end{cases}$$

The path $\Lambda(\psi)$, T'' and $C(\theta)$, and the functions $\hat{u}^h_\theta(\lambda, t'')$ and $w^h(z, t'', \lambda, \zeta)$ are determined in the following. In order to do so we need some preliminaries. All the proofs are given in the later sections.

Now let us explain the functions in (2.2) and the paths of integration. The explanations are divided into 3 parts.

(I) The definitions of $\hat{u}^h_\theta(\lambda, t'')$ and the path $\Lambda(\psi)$. Consider the traces of $u(z)\in\mathcal{O}(\tilde{\Omega}(\theta_0))$, $\Omega=\{z\in C^{n+1};\ |z|\leqq R\}$, to $z_1=0$,

$$u^h(z_0, z'') = (\partial/\partial z_1)^h\, u(z_0, 0, z''), \quad (0\leqq h\leqq s_m-1)\,, \tag{2.3}$$

and define

$$\hat{u}^h_\theta(\lambda, t'') = \frac{1}{2\pi i}\int_{T(\theta)} \exp(-\lambda\, t_0)\, u^h(t_0, t'')\, dt_0\,. \tag{2.4}$$

$T(\theta)$ $(-\theta_0<\theta<\theta_0-2\pi)$ is a path starting at $Re^{i(\theta+2\pi)}$, going to $\varepsilon e^{i(\theta+2\pi)}(0<\varepsilon<R)$, rounding the origin once on $|t_0|=\varepsilon$ and ending at $Re^{i\theta}$ (see Fig 2.1). For $\hat{u}^h_\theta(\lambda, t'')$ we have

Lemma 2.1. (a) *$\hat{u}^h_\theta(\lambda, t'')$ is an entire function of λ.*
(b) *For any $\varepsilon>0$, there is a $C_\varepsilon>0$ such that*

$$\sup_{|t''|\leqq R} |\hat{u}^h_\theta(\lambda, t'')|\leqq C_\varepsilon \exp(\varepsilon|\lambda|) \quad \textit{for } \lambda \textit{ with } |\arg\lambda+\theta|<\pi/2\,. \tag{2.5}$$

(c) *If* $\sup_{|t''|\leqq R} |u^h(t_0, t'')|\leqq A\exp(\delta|t_0|^{-\gamma})$ $(\gamma>0)$, *then*

$$\sup_{|t''|\leqq R} |\hat{u}^h_\theta(\lambda, t'')|\leqq B\exp(2(\delta|\lambda|^\gamma)^{1/(1+\gamma)}) \quad \textit{for } \lambda \textit{ with } |\arg\lambda+\theta|<\pi/2\,. \tag{2.6}$$

For the inversion formula we set

$$u^h_\theta(z, z'') = \frac{1}{2\pi\, i}\int_{\Lambda(\psi)} \exp(\lambda z_0)\, \hat{u}^h_\theta(\lambda, t'')\, (\log\lambda)\, d\lambda\,, \tag{2.7}$$

where $|\theta+\psi|<\pi/2$ and $\Lambda(\psi)$ is an infinite path starting at $\infty e^{i\psi}$, going around the origin once and ending at $\infty e^{i(\psi+2\pi)}$ (see Fig 2.2).

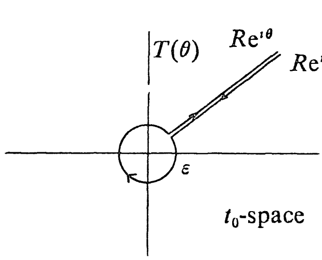

Fig. 2.1.

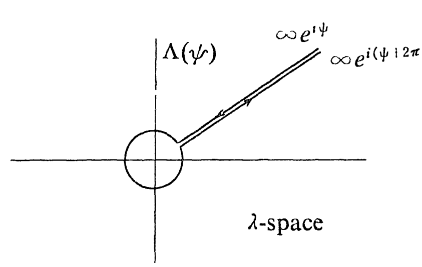

Fig. 2.2.

Lemma 2.2. *It holds that*

$$u_\theta^h(z_0, z'') = u^h(z_0, z'') + v_\theta^h(z_0, z'') \tag{2.8}$$

for z_0 *with* $\theta < \arg z_0 < \theta + 2\pi$, *where* $v_\theta^h(z_0, z'') \in \mathcal{O}\{|z_0| < R, |z''| \leqq R\}$.

The proofs of Lemma 2.1 and 2.2 are in §6.

(II) Kernel functions $w^h(z, t'', \lambda, \zeta)$ $(0 \leqq h \leqq s_m - 1)$. Each $w^h(z, t'', \lambda, \zeta)$ is determined so that it satisfies an equation. Let us derive the equation. In order to do so we derive operators $L(z, \lambda, \partial_z)$ and $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)$ from $L(z, \partial_z)$. Firstly $L(z, \lambda, \partial_z)$ is defined as follows:

$$\begin{cases} L(z, \partial_z)\{\exp(\lambda z_0) K(z, \lambda)\} = \exp(\lambda z_0) L(z, \lambda, \partial_z) K(z, \lambda), \\ L(z, \lambda, \partial_z) = L(z, \partial_0 + \lambda, \partial') = \sum_{k=0}^m \{\sum_{i=s_k}^k \lambda^{k-i} L_{k,i}(z, \partial_z)\}, \\ L_{k,i}(z, \partial_z) = \sum_{l \geqq s_k, l+j=i} A_{k,l}(z, \partial') \binom{k-l}{j} (\partial_0)^j. \end{cases} \tag{2.9}$$

Secondly we define $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)$ from $L(z, \lambda, \partial_z)$. From Lemma 1.1 we have $k - i = (1-\alpha)(m - s_m) + \alpha(m-i) - (1-\alpha)(i - s_k) - \beta_k$. Hence by omitting $\lambda^{(1-\alpha)(m-s_m)}$ and replacing λ^α by ∂_ζ in (2.9), we set

$$\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta) = \sum_{k=0}^m \{\sum_{i=s_k}^k \lambda^{-(1-\alpha)(i-s_k)-\beta_k} (\partial_\zeta)^{m-i} L_{k,i}(z, \partial_z)\}. \tag{2.10}$$

Thus we attain to the equation $w^h(z, t'', \lambda, \zeta)$ satisfies,

$$\begin{cases} \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta) w^h(z, t'', \lambda, \zeta) = 0 \\ (\partial_1)^l w(z, t'', \lambda, \zeta)|_{z_1=0} = \delta_{l,h} (2\pi i)^{-n} \zeta^{-1} \prod_{i=2}^n (t_i - z_i)^{-1}, \end{cases} \tag{2.11}$$

for $0 \leqq l \leqq s_m - 1$, where $|z| \leqq R'$ and $R' < R_1 \leqq |t_i| \leqq R$ $(i \geqq 2)$. We note that the initial values are singular at $\zeta = 0$. For the existence of $w^h(z, \lambda, t'', \zeta)$ we have

Proposition 2.3. *There is a solution* $w^h(z, t'', \lambda, \zeta)$ *of* (2.11) *which is multi-valued holomorphic in*

$$Z = \{(z, t'', \lambda, \zeta);\ |z| \leqq r,\ R_1 \leqq |t_i| \leqq R\ (i \geqq 2),\ |\lambda| \geqq A_0,\\ A^*|z_1| < |\zeta| < B^*|\lambda|^{1-\alpha}\},$$

where r $(r < R)$, A_0, A^* *and* B^* *are some positive constants, and* $|w^h(z, t'', \lambda, \zeta)| \leqq C(1 + |\zeta|^{-1} + |\log \zeta|)$ *holds.*

The proof is given in §5.

(III) Integral representation. We can define by Proposition 2.3

$$K_\theta^h(z, t'', \lambda) = \int_{C(\theta)} \exp(-\lambda^\alpha \zeta)\, w^h(z, t'', \lambda, \zeta)\, d\zeta\,, \tag{2.12}$$

where $C(\theta)=C(de^{i\theta}\lambda^{1-\alpha})$ $(0<d<B^*$, where B^* is in Proposition 2.3) is a path on the circle $|\zeta|=d|\lambda|^{1-\alpha}$ whose starting point is $de^{i\theta}\lambda^{1-\alpha}$ and goes around once on it. For $K_\theta^h(z, t'', \lambda)$, we have

Proposition 2.4. *$K_\theta^h(z, t'', \lambda)$ $(0\leqq h\leqq s_m-1)$ are single valued holomorphic functions with respect to λ in Ξ,*

$$\Xi = \{(z, t'', \lambda);\ |z|\leqq r,\ R_1\leqq |t_i|\leqq R\ (i\geqq 2),\ |\lambda|\geqq A_0\}\,, \tag{2.13}$$

and satisfy

$$\begin{cases} L(z, \lambda, \partial_z)\, K_\theta^h(z, t'', \lambda) = \exp(-de^{i\theta}\lambda)\, K'^h(z, t'', \lambda)\,, \\ (\partial/\partial z_1)^l\, K_\theta^h(z, t'', \lambda) = \delta_{l,h}(2\pi\, i)^{-n+1} \prod_{i=2}^n (t_i-z_i)^{-1} \quad (0\leqq l\leqq s_m-1)\,, \end{cases} \tag{2.14}$$

where

$$|K_\theta^h(z, t'', \lambda)| \leqq C|\lambda|^{2(1-\alpha)} \exp(c|z_1|\,|\lambda|^\alpha) \tag{2.15}$$

for λ with $|\arg\lambda+\theta|<\pi/2$, and $K'^h(z, t'', \lambda)$ is holomorphic in Ξ and

$$|K'^h(z, t'', \lambda)| \leqq A(1+|\lambda|)^N \qquad \textit{for some } N>0\,. \tag{2.16}$$

The proof is given in §5 except that $K_\theta^h(z, t'', \lambda)$ is single valued, which is proved in §9.

Finally $\int_{T''} \cdots dt''$ means $\int_{|t_2|=R} \cdots dt_2 \int_{|t_3|=R} \cdots dt_3 \cdots \int_{|t_n|=R} \cdots dt_n$. So we have $(2\pi i)^{-n+1} \int_{T''} f(t'')/\prod_{i=2}^n (t_i-z_i)\, dt''=f(z'')$ for a holomorphic function $f(z'')$.

Thus we attain to

Theorem 2.5. *Assume* (2.1)'. *Then $u(z)\in\mathcal{O}(\tilde{\Omega}(\theta_0))$ $(\theta_0>\pi)$ satisfying $L(z, \partial_z)\, u(z)=f(z)\in\mathcal{O}(\Omega)$ has an integral representation in $\tilde{U}(\theta, \theta+2\pi)$ $(-\theta_0<\theta<\theta_0-2\pi)$, $U=\{z\in \boldsymbol{C}^{n+1};\ |z|\leqq r\}$ $(r<R)$, of the form*

$$u(z) = \sum_{h=0}^{s_m-1} u_\theta^h(z)+v_\theta(z)\,, \tag{2.17}$$

where $v_\theta(z)\in\mathcal{O}(U)$ and

$$u_\theta^h(z) = \frac{1}{2\pi i}\int_{\Lambda(\psi)} \exp(\lambda z_0) \log\lambda\, d\lambda \int_{T''} K_\theta^h(z, t'', \lambda)\, \hat{u}_\theta^h(\lambda, t'')\, dt'', \quad (|\theta+\psi|<\pi/2)\,. \tag{2.18}$$

§ 3. Construction of Kernel Function $w(z, t'', \lambda, \zeta)$-(I)

Now we proceed to find a solution to the equation (2.11). We denote $w^h(z, t'', \lambda, \zeta)$ by $w(z, t'', \lambda, \zeta)$, omitting h. Let us write it again:

$$(3.1)\qquad \begin{cases} \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)\, w(z, t'', \lambda, \zeta) = 0\,, \\ (\partial_1)^l\, w(z, t'', \lambda, \zeta)\,|_{z_1=0} = \delta_{l,h}(2\pi i)^{-n}\, \zeta^{-1} \prod_{i=2}^n (t_i - z_i)^{-1}\,, \\ \quad \text{for} \quad 0 \leqq l \leqq s_m - 1\,, \end{cases}$$

where

$$(3.2)\qquad \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta) = \sum_{k=0}^m \{\sum_{i=s_k}^k \lambda^{-(1-\alpha)(i-s_k)-\beta_k} (\partial_\zeta)^{m-i}\, L_{k,i}(z, \partial_z)\}\,.$$

We construct $w(z, t'', \lambda, \zeta)$ under the condition (2.1)$'$, that is,

$$(3.3)\qquad A_{m,s_m}(0, z', \hat{\xi}') \neq 0 \qquad \text{for} \quad |z'| \leqq R_0\ (R_0 < R)\,.$$

Firstly let us introduce auxilliary functions $\{f_j(\zeta)\}$ $(j \in \boldsymbol{Z})$ used in [2],

$$(3.4)\qquad \begin{cases} f_j(\zeta) = \dfrac{\zeta^j}{(2\pi i)\, j!}\, \{\log \zeta - (1 + 1/2 + \cdots + 1/j) \quad (j \geqq 1)\,, \\ f_0(\zeta) = \dfrac{1}{2\pi i} \log \zeta\,, \\ f_j(\zeta) = (-1)^j \dfrac{(-j-1)!}{2\pi i}\, \zeta^j \quad (j \leqq -1)\,. \end{cases}$$

We note an important relation

$$(3.5)\qquad (d/d\zeta) f_j(\zeta) = f_{j-1}(\zeta)\,.$$

We try to find $w(z, t'', \lambda, \zeta)$ of the form

$$(3.6)\qquad \begin{cases} w(z, t'', \lambda, \zeta) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma V(z, t'', \lambda, \zeta, \tau)\, d\tau\,, \\ V(z, t'', \lambda, \zeta, \tau) = \sum_{p=h-1}^{+\infty} v_p(z, t'', \lambda, \tau) f_p(\zeta + \tau z_1)\,, \end{cases}$$

where γ is a closed path in τ-space which will be determined later. Thus it becomes the main purpose to obtain equations which determine $v_p(z, t'', \lambda, \tau)$ $(p \geqq h-1)$. Let us give a lemma for calculations.

Lemma 3.1. *There are operators $L_{k,i}^j(z, \partial_z)$ $(0 \leqq j \leqq i)$ with* ord. $L_{k,i}^j(z, \partial_z) \leqq j$ *and $L_{k,i}^0(z, \partial_z) = A_{k,i}(z, \hat{\xi}')$ such that*

$$(3.7)\qquad \begin{aligned} &(\partial_\zeta)^{m-i}\, L_{k,i}(z, \partial_z)\, \{v(z) f_p(\zeta + \tau z_1)\} \\ &\quad = \sum_{j=0}^i \{\tau^{i-j}\, L_{k,i}^j(z, \partial_z)\, v(z)\}\, f_{p-m+j}(\zeta + \tau z_1)\,. \end{aligned}$$

The proof is easy, so we omit it. Now we have from Lemma 3.1

(3.8) $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta) V(z, t'', \lambda, \zeta)$
$$= \sum_{p=h-1-m}^{+\infty} \{\sum_{j=0}^{m} G_j(z, \lambda, \tau, \partial_z) v_{p+m-j}(z, t'', \lambda, \tau)\} f_p(\zeta+\tau z_1),$$

where

$$(3.9) \quad \begin{cases} G_0(z, \lambda, \tau, \partial_z) = \sum_{k=0}^{m} \{\sum_{i=s_k}^{k} \lambda^{-(1-\alpha)(i-s_k)-\beta_k} \tau^i A_{k,i}(z, \hat{\xi}')\}, \\ G_j(z, \lambda, \tau, \partial_z) = \sum_{k=j}^{m} \{\sum_{i=\max(s_k, j)}^{k} \lambda^{-(1-\alpha)(i-s_k)-\beta_k} \tau^{i-j} L_{k,i}^j(z, \partial_z)\}. \end{cases}$$

We have ord. $G_j(z, \lambda, \tau, \partial_z) \leqq j$. Hence $G_0(z, \lambda, \tau, \partial_z)$ is a polynomial of τ, so we denote it by $G_0(z, \lambda, \tau)$. Set

$$(3.10) \qquad g_p(z, t'', \lambda, \tau) = -\sum_{j=1}^{m} G_j(z, \lambda, \tau, \partial_z) v_{p-j}(z, t'', \lambda, \tau).$$

We have from (3.8)

(3.11) $\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta) V(z, t'', \lambda, \zeta)$
$$= \sum_{p=h-1-m}^{+\infty} \{G_0(z, \lambda, \tau) v_{p+m}(z, t'', \lambda, \tau) - g_{p+m}(z, t'', \lambda, \tau)\} f_p(\zeta+\tau z_1).$$

Hence we'll try to determine $v_p(z, t'', \lambda, \tau)$ $(p \geqq h-1)$ by the following equations containing other unknown functions $h_p(z, t'', \lambda, \tau)$ $(p \geqq h-1)$:

$$(3.12)_{h-1} \qquad G_0(z, \lambda, \tau) v_{h-1}(z, t'', \lambda, \tau) = h_{h-1}(z, t'', \lambda, \tau),$$

$$(3.12)_p \qquad G_0(z, \lambda, \tau) v_p(z, t'', \lambda, \tau) = g_p(z, t'', \lambda, \tau) + h_p(z, t'', \lambda, \tau).$$

We'll define the path γ in (3.6) and solve the equations $(3.12)_p$ in the next section. $\{h_p(z, t'', \lambda, \tau)\}$ are polynomials of τ with degree $\leqq s_m - 1$ and are chosen so that $w(z, t'', \lambda, \zeta)$ satisfies the initial conditions in (3.1).

§ 4. Construction of Kernel Function $w(z, t'', \lambda, \zeta)$—(II)

In §4 we define the path γ, $\{v_p(z, t'', \lambda, \tau)\}$ and $\{h_p(z, t'', \lambda, \tau)\}$. Firstly we define the path γ in (3.6). In order to do so, we need a lemma concerning the roots of $G_0(z, \lambda, \tau)=0$ (see (3.9)), which is an algebraic equation of τ:

Lemma 4.1. *Assume* (3.3). *Then there are positive constants a, b and A_0 such that if $|\lambda| \geqq A_0$, then $G_0(z, \lambda, \tau) \neq 0$ on $\{a \leqq |\tau| \leqq b|\lambda|^{1-\alpha}\}$ and there exist exactly s_m roots of $G_0(z, \lambda, \tau)=0$ in $\{|\tau| < a\}$.*

The proof of Lemma 4.1 is given in §6. Lemma 4.1 means that there are exactly s_m-roots which are bounded as $|\lambda| \to +\infty$. The closed path γ in τ-space is chosen so that it encloses all the bounded roots of $G_0(z, \lambda, \tau)=0$.

Now let us proceed to the determination of $V(z, t'', \lambda, \zeta, \tau)$ in (3.6), (4.1) $$V(z, t'', \lambda, \zeta, \tau)=\sum_{p=h-1}^{+\infty} v_p(z, t'', \lambda, \tau) f_p(\zeta+\tau z_1) .$$

We put $v_p(z, t'', \lambda, \tau)=h_p(z, t'', \lambda, \tau)=0$ for $p\leqq h-2$. Consider the initial conditions of $w(z, t'', \lambda, \tau)$ in (3.1). We have

(4.2) $$(\partial_1)^l V(z, t'', \lambda, \zeta, \tau)=\sum_{p=h-1}^{+\infty}\{\sum_{i=0}^{l}\tbinom{l}{i} v_{p-i}^{(i)}(z, t'', \lambda, \tau)\tau^{l-i}\} f_{p-l}(\zeta+\tau z_1) ,$$

where $v^{(i)}(z, t'', \lambda, \tau)=(\partial_1)^i v(z, t'', \lambda, \tau)$. We define $C_{p,l}(z_0, z'', t'', \lambda)$ $(0\leqq l\leqq s_m-1)$ by

(4.3) $$\begin{aligned} C_{p,l}(z_0, z'', t'', \lambda) &= \delta_{p,l-1}\delta_{l,h}(2\pi i)^{-(n-1)}\prod_{i=2}^{n}(t_i-z_i)^{-1} \\ &-\frac{1}{2\pi i}\{\int_\gamma (\sum_{i=1}^{l}\tbinom{l}{i} v_{p-i}^{(i)}(z, t'', \lambda, \tau)\tau^{l-i})\,d\tau \\ &+\int_\gamma \frac{g_p(z, t'', \lambda, \tau)}{G_0(z, \lambda, \tau)}\tau^l\,d\tau\}\Big|_{z_1=0} . \end{aligned}$$

We note that $C_{p,l}(z_0, z'', t'', \lambda)=0$ for $p\leqq h-2$. If $v_{p-i}(z, t'', \lambda, \tau)$ $(i\geqq 1)$ are determined, since $g_p(z, t'', \lambda, \tau)$ (see (3.10)) contains only $v_{p-i}(z, t'', \lambda, \tau)$ $(i\geqq 1)$, then $C_{p,l}(z_0, z'', t'', \lambda)$ are also done. By making use of $C_{p,l}(z_0, z'', t'', \lambda)$, we determine $h_p(z, t'', \lambda, \tau)$, which is a polynomial of τ with degree$\leqq s_m-1$ so that it satisfies

(4.4) $$\frac{1}{2\pi i}\int_\gamma \frac{h_p(z, t'', \lambda, \tau)}{G_0(z, \lambda, \tau)}\tau^l\,d\tau = C_{p,l}(z_0, z'', t'', \lambda) \qquad \text{for}\quad 0\leqq l\leqq s_m-1 .$$

It follows from Lemma 9.8 in §9–II that $h_p(z, t'', \lambda, \tau)$ satisfying (4.4) uniquely exists. So we set

(4.5) $$v_p(z, t'', \lambda, \tau) = \{g_p(z, t'', \lambda, \tau)+h_p(z, t'', \lambda, \tau)\}/G_0(z, \lambda, \tau) .$$

Thus $v_p(z, t'', \lambda, \tau)$ and $h_p(z, t'', \lambda, \tau)$ are successively determined. We notice that $v_p(z, t'', \lambda, \tau)$ $(p\geqq h-1)$ have poles as functions of τ and the poles are the zeros of $G_0(z, \lambda, \tau)$.

Let us check that

(4.6) $$\begin{aligned} w(z, t'', \lambda, \zeta) &= \frac{1}{2\pi i}\int_\gamma V(z, t'', \lambda, \zeta, \tau)\,d\tau \\ &= \frac{1}{2\pi i}\int_\gamma \sum_{p=h-1}^{+\infty} v_p(z, t'', \lambda, \tau) f_p(\zeta+\tau z_1)\,d\tau \end{aligned}$$

formally satisfies (3.1). The calculation which we perform below are justified after obtaining the estimates of $v_p(z, t'', \lambda, \tau)$ and the convergence of $V(z, t'', \lambda, \zeta, \tau)$. Assuming $|\zeta|>|\tau z_1|$ for $|z|<r$ and $\tau\in\gamma$, r being a small positive constant, we have

$$(4.7)\quad \mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)\, w(z, t'', \lambda, \zeta)$$
$$= \frac{1}{2\pi i}\int_\gamma \sum_{p=h-1-m}^{+\infty} \{\sum_{j=0}^{m} G_j(z, \lambda, \tau, \partial_z)\, v_{p+m-j}(z, t'', \lambda, \tau)\} f_p(\zeta+\tau z_1)\, d\tau$$
$$= \frac{1}{2\pi i}\int_\gamma \sum_{p=h-1-m}^{+\infty} \{G_0(z, \lambda, \tau)\, v_{p+m}(z, t'', \lambda, \tau) - g_{p+m}(z, t'', \lambda, \tau)\}$$
$$f_p(\zeta+\tau z_1)\, d\tau$$
$$= \frac{1}{2\pi i}\int_\gamma \sum_{p=h-1-m}^{+\infty} h_{p+m}(z, t'', \lambda, \tau) f_p(\zeta+\tau z_1)\, d\tau = 0\,.$$

Here we note that $\tau=-\zeta/z_1$, the singular point of $\log(\zeta+\tau z_1)$, is not in the inside of γ. As for the initial values, we have from (4.2)

$$(4.8)\quad (\partial_1)^l\, w(z, t'', \lambda, \zeta)$$
$$= \frac{1}{2\pi i}\int_\gamma \sum_{p=h-1}^{+\infty} \{\sum_{i=0}^{l} \binom{l}{i} v_{p-i}^{(i)}(z, t'', \lambda, \tau)\, \tau^{l-i}\} f_{p-l}(\zeta+\tau z_1)\, d\tau\,.$$

It follows from (4.3) and (4.4) that

$$(4.9)\quad (\partial_1)^l\, w(z, t'', \lambda, \zeta)\,|_{z_1=0} = \frac{1}{2\pi i} \sum_{p=h-1}^{+\infty} \{\int_\gamma v_p(z_0, 0, z'', \lambda, \zeta)\, \tau^l\, d\tau$$
$$-\int_\gamma \frac{g_p(z, t'', \lambda, \tau)+h_p(z, t'', \lambda, \tau)}{G_0(z, \lambda, \tau)}\, \tau^l\, d\tau + \delta_{p,l-1}\, \delta_{l,h} (2\pi i)^{-n+1}$$
$$\prod_{i=2}^{n} (t_i-z_i)^{-1}\} f_{p-l}(\zeta)$$
$$= \delta_{l,h} (2\pi i)^{-n}\, \zeta^{-1} \prod_{i=2}^{n} (t_i-z_i)^{-1}\,.$$

Thus we conclude that $w(z, t'', \lambda, \zeta)$ satisfies formally the equation (3.1).

§ 5. Construction of the Integral Representation

In §5 we show the convergence of $V(z, t'', \lambda, \zeta, \tau)$, construct $w(z, t'', \lambda, \zeta)$ and $K_\theta(z, t'', \lambda)$, by integrating in τ and ζ, and attain to the integral formula. In order to do so we need the estimates of $\{v_p(z, t'', \lambda, \tau); p \geqq h-1\}$. Before we give them, let us write again the set Ξ ((2.13)), which will often appear in the sequel:

$$(5.1)\qquad \Xi = \{(z, t'', \lambda);\ |z| \leqq r,\ R_1 \leqq |t_i| \leqq R\ (i \geqq 2),\ |\lambda| \geqq \Lambda_0\}\ ,$$

where we'll make r small and Λ_0 large if necessary.

(I) The convergence of $V(z, t'', \lambda, \zeta, \tau)$. We have

Proposition 5.1. *For $v_p(z, t'', \lambda, \tau)$ $(p \geqq h-1)$ the following estimates hold: there are A and B such that*

(5.2) $\quad |v_p(z, t'', \lambda, \tau)| \leqq AB^p |\tau|^{-p-2}(p+1)! \quad$ *for* $(z, t'', \lambda, \tau) \in X$

where $X = \{(z, t'', \lambda, \tau);\ (z, t'', \lambda) \in \Xi,\ a \leqq |\tau| \leqq b|\lambda|^{1-\alpha}\}$ (*see Lemma* 4.1).

The proof of Proposition 5.1 is given with other estimates in §9. Now let us show the convergence of $V(z, t'', \lambda, \zeta, \tau)$. Set

(5.3) $$V_1(z, t'', \lambda, \zeta, \tau) = \textstyle\sum_{p=0}^{+\infty} \{(\zeta+\tau z_1)^p/p!\}\ v_p(z, t'', \lambda, \tau)$$

and

(5.4) $$V_2(z, t'', \lambda, \zeta, \tau) = \textstyle\sum_{p=1}^{+\infty} \{(1+1/2+\cdots+1/p)/p!\}(\zeta+\tau z_1)^p\ v_p(z, t'', \lambda, \tau)\,.$$

Then by noting $h-1 \geqq -1$, we have

$$V(z, t'', \lambda, \zeta, \tau) = \frac{1}{2\pi i}\{V_1(z, t'', \lambda, \zeta, \tau)\log(\zeta+\tau z_1) + V_2(z, t'', \lambda, \zeta, \tau) + (\zeta+\tau z_1)^{-1}\, v_{-1}(z, t'', \lambda, \tau)\}\,.$$

We obatin

Lemma 5.2. *$V_i(z, t'', \lambda, \zeta, \tau)$ $(i=1, 2)$ converge and $|V_i(z, t'', \lambda, \zeta, \tau)| \leqq B|\tau|^{-2}$ in $\{(z, t'', \lambda, \zeta, \tau);\ (z, t'', \lambda, \tau) \in X,\ |\zeta+\tau z_1| < A|\tau|\}$.*

Proof. From (5.2) we have
$|\zeta+\tau z_1|^p |v_p(z, t'', \lambda, \tau)|/p! \leqq AB^{p+1}|\zeta+\tau z_1|^p/|\tau|^{p+2}$. Hence if $B|\zeta+\tau z_1|/|\tau| < 1/2$, $V_i(z, t'', \lambda, \zeta, \tau)$ $(i=1, 2)$ are convergent and estimates hold.

Consequently

Proposition 5.3. *$V(z, t'', \lambda, \zeta, \tau)$ is holomorphic in*

$$Y = \{(z, t'', \lambda, \zeta, \tau);\ (z, t'', \lambda, \tau) \in X,\ 0 < |\zeta+\tau z_1| < A|\tau|\}\,.$$

Corollary 5.4. *$V(z, t'', \lambda, \zeta, \tau)$ is holomorphic and*

(5.5) $$|V(z, t'', \lambda, \zeta, \tau)| \leqq A|\tau|^{-2}(1+|\log\zeta|) + B|\zeta|^{-1}$$

in $\{(z, t'', \lambda, \zeta, \tau);\ (z, t'', \lambda, \tau) \in X,\ (A-|z_1|)|\tau| > |\zeta| > 2|\tau z_1|\}$.

Proof. We have $A|\tau| > |\zeta| + |\tau z_1| > |\zeta+\tau z_1| > |\zeta| - |\tau z_1| > 0$ and $|\tau z_1/\zeta| < 1/2$ in the domain. So $|\zeta+\tau z_1|^{-1} \leqq 2/|\zeta|$ and $|\log(\zeta+\tau z_1)| \leqq C + |\log\zeta|$. The assertion follows easily.

(II) The construction of $w(z, t'', \lambda, \zeta)$ and $K_\theta(z, t'', \lambda)$. We perform integrating in τ. We denote by $\gamma(c)$ the path in τ-space which starts at $\tau=c$, goes around once on $|\tau|=|c|$ and ends at $\tau=c\exp(2\pi i)$. Set $\gamma_1=\gamma(c)$, $a\leqq$

$|c| \leqq b|\lambda|^{1-\alpha}$ (see Proposition 5.1), and define

$$w(z, t'', \lambda, \zeta) = \int_{\gamma_1} V(z, t'', \lambda, \zeta, \tau)\, d\tau . \tag{5.6}$$

We can give the proof of Proposition 2.3 by Corollary 5.4.

Proof of Proposition 2.3. Suppose $|\tau|(A-|z_1|)>|\zeta|>2|\tau z_1|$ on $|\tau|=|c|$. Then we have $|\tau z_1/\zeta|<1/2$, $|\zeta+\tau z_1|\leqq 2/|\zeta|$ and $|\log(\zeta+\tau z_1)|\leqq C+|\log\zeta|$ on $|\tau|=|c|$, and $w(z_1, \zeta)=w(z, t'', \lambda, \zeta)$ is holomorphic in $\{(z_1, \zeta);\ |z_1|\leqq r,\ |c|(A-|z_1|)>|\zeta|>2|cz_1|\}$. Changing c in $\gamma_1=\gamma(c)$ $(a\leqq|c|\leqq b|\lambda|^{1-\alpha})$, we conclude that $w(z_1, \zeta)$ is holomorphic in $\{(z_1, \zeta);\ 2a|z_1|<|\zeta|<b|\lambda|^{1-\alpha}(A-|z_1|),\ |z_1|\leqq\min(r, A/3)\}$ and $|w(z_1, \zeta)|\leqq A(C+|\zeta|^{-1}+|\log\zeta|)$. This implies the assertion of Proposition 2.3.

Next we integrate $w(z, t'', \lambda, \zeta)$ in ζ and construct $K_\theta(z, t'', \lambda)$. Let us recall the path $C(\theta)$ in ζ-space defined in §2–III. It is a path whose stating point is $de^{i\theta}\lambda^{1-\alpha}$ $(0<d<B^*)$ and goes around once on $|\zeta|=d|\lambda|^{1-\alpha}$.

Let us show some part of Proposition 2.4 about $K_\theta(z, t'', \lambda)$.

Proof of Proposition 2.4-(I). From Proposition 2.3 we can define

$$K_\theta(z, t'', \lambda) = \int_{C(\theta)} \exp(-\lambda^\alpha\zeta) w(z, t'', \lambda, \zeta)\, d\zeta . \tag{5.7}$$

Let us deform $C(\theta)$ to the path C which starts at $de^{i\theta}\lambda^{1-\alpha}$, goes to $c_1e^{i\theta}(c_1>A^*|z_1|)$, goes around once on $|\zeta|=c_1$ and goes from $c_1e^{i(\theta+2\pi)}$ to $de^{i(\theta+2\pi)}\lambda^{1-\alpha}$, A^* being the same as in Proposition 2.3.

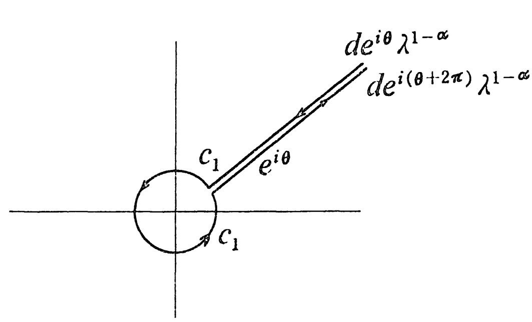

Fig. 5.1.

Thus we get, if $|\arg\lambda+\theta|<\pi/2$,

$$|K_\theta(z, t'', \lambda)|\leqq A|\lambda|^{2(1-\alpha)}\exp(c|z_1\lambda^\alpha|) . \tag{5.8}$$

It follows from the method of construction of $K_\theta(z, t'', \lambda)$ and integration by parts that

$$(5.9)\quad L(z, \lambda, \partial_z)K_\theta(z, t'', \lambda)$$
$$= \int_{C(\theta)} \exp(-\lambda^\alpha \zeta)\mathcal{L}(z, \lambda, \partial_z, \partial_\zeta)w(z, t'', \lambda, \zeta)d\zeta + \exp(-de^{i\theta}\lambda)K'(z, t'', \lambda)$$
$$= \exp(-de^{i\theta}\lambda)K'(z, t'', \lambda),$$

where $|K'(z, t'', \lambda)| \leqq A(1+|\lambda|)^N$ for some $N>0$, and

$$(\partial_1)^l K_\theta(z, t'', \lambda)|_{z_1=0} = \int_{C(\theta)} \exp(-\lambda^\alpha \zeta)(\partial_1)^l w(z, t'', \lambda, \zeta)d\zeta|_{z_1=0}$$
$$= \delta_{l,h}(2\pi i)^{-n} \prod_{i=2}^n (t_i - z_i)^{-1} \int_{C(\theta)} \exp(-\lambda^\alpha \zeta)\zeta^{-1}d\zeta$$
$$= \delta_{l,h}(2\pi i)^{-(n-1)} \prod_{i=2}^n (t_i - z_i)^{-1}.$$

The proof of Proposition 2.4 is not yet completed. The rest of it is in §9-III.

(III) The integral representation. $K_\theta(z, t'', \lambda)$ is determined in (II). Hereafter we write suffix h again, for example, $K_\theta^h(z, t'', \lambda)$, $\hat{u}_\theta^h(z)$ etc.. We set

$$(5.10)\qquad u_\theta^h(z) = \frac{1}{2\pi i}\int_{A(\psi)} \exp(\lambda z_0)\log\lambda\, d\lambda \int_{T''} K_\theta^h(z, t'', \lambda)\hat{u}_\theta^h(\lambda, t'')dt'',$$

where $|\psi+\theta|<\pi/2$ and $0\leqq h\leqq s_m-1$. Set $u_\theta(z)=\sum_{h=0}^{s_m-1} u_\theta^h(z)$. Then we have to show that $u_\theta(z)$ is a desired formula of the solution $u(z)$. We have

$$(5.11)\quad L(z, \partial_z)u_\theta^h(z)$$
$$= \frac{1}{2\pi i}\int_{A(\psi)} \exp(\lambda z_0)\log\lambda\, d\lambda \int_{T''} L(z, \lambda, \partial_z)K_\theta^h(z, t'', \lambda)\hat{u}_\theta^h(\lambda, t'')dt''$$
$$= \frac{1}{2\pi i}\int_{A(\psi)} \exp(\lambda(z_0 - de^{i\theta}))\log\lambda\, d\lambda \int_{T''} K'^h(z, t'', \lambda)\hat{u}_\theta^h(\lambda, t'')dt''$$
$$= f_\theta^h(z).$$

It is obvious that $f_\theta^h(z)$ is holomorphic in $\{|z_0|<d\} \cap \{|z|<r\}$. We have for the initial values, by Proposition 2.4,

$$(5.12)\ (\partial_1)^l u_\theta^h(z)|_{z_1=0}$$
$$= (2\pi i)^{-(n-1)}\int_{A(\psi)} \exp(\lambda z_0)\log\lambda\, d\lambda \int_{T''} (\partial_1)^l K_\theta^h(z, t'', \lambda)|_{z_1=0}\hat{u}_\theta^h(\lambda, t'')dt''$$
$$= \delta_{l,h}(2\pi i)^{-n+1}\int_{A(\psi)} \exp(\lambda z_0)\log\lambda\, d\lambda \int_{T''} \{\textstyle\prod_{i=2}^n (t_i - z_i)^{-1}\}\hat{u}_\theta^h(\lambda, t'')dt''$$
$$= \frac{\delta_{l,h}}{2\pi i}\int_{A(\psi)} \exp(\lambda z_0)\hat{u}_\theta^h(\lambda, z'')\log\lambda\, d\lambda.$$

By Lemma 2.2 we have

$$(\partial_1)^l u^h_\theta(z_0, 0, z'') = \delta_{l,h}\{(\partial_1)^h u(z_0, 0, z'')+v^h_\theta(z_0,z'')\}\ , \tag{5.13}$$

where $v^h_\theta(z_0, z'')$ is holomorphic at $z_0=z''=0$. Therefore we have

$$\begin{cases} L(z, \partial_z)(u(z)-u_\theta(z)) = f(z)-f_\theta(z), f_\theta(z) = \sum_{h=0}^{s_m-1} f^h_\theta(z)\,, \\ (\partial_1)^l(u(z)-u_\theta(z))\,|_{z_1=0} = v^l_\theta(z_0, z'') \qquad \text{for} \quad 0\leqq l\leqq s_m-1\,, \end{cases} \tag{5.14}$$

where the functions in the right hand side of (5.14) are holomorphic at $z=0$ (or $z_0=z''=0$). Therefore by the uniqueness of the Goursat's problem means that $u(z)-u_\theta(z)$ is holomorphic at $z=0$ (see §8 in [6]). So $u_\theta(z)$ is a desired integral representation of $u(z)$.

§ 6. Miscellaneous Results-(I)

In §6 we summerize what we need. Some of it was used in the previous sections and others will be used in the later sections to show estimates and to deform integration paths. This section is divided into 4 parts. They are properties of $\{\beta^i_k\}$, zeros of $G_0(z, \lambda, \tau)$, sectors S_i $(1\leqq i\leqq p-1)$ and proofs of Lemmas 2.1 and 2.2.

(I) Properties of $\{\beta^i_k\}$. We investigate $\{\beta^i_k\}$ defined in Lemma 1.1 in §1. We have set $\alpha=\alpha_1$ and $\beta_k=\beta^1_k$. Firstly we prove Lemma 1.1.

Proof of Lemma 1.1. If $k=k_{i-1}$, then we have $\beta^i_k=0$. Suppose $k\neq k_{i-1}$. Since $\sum$ is the lower convex part of the boundary of $\hat{A}$, there are $h^i_k\in \boldsymbol{Q}$, such that

$$(s_{k_{i-1}}-s_k)/(k_{i-1}-k)+h^i_k = \sigma_i = (1-\alpha_i)^{-1}\,, \tag{6.1}$$

where $h^i_k\geqq 0$ if $k<k_{i-1}$, and $h^i_k<0$ if $k>k_{i-1}$. We have (1.5), by putting $\beta^i_k= h^i_k(1-\alpha_i)\,(k_{i-1}-k)$, and $\beta^i_k=0$ if and only if $(k, s_k)\in\sum(i)$.

We further have

Proposition 6.1. *The followings hold*:

(1) $\beta_k+(\alpha-\alpha_i)s_k=\beta_{k_{i-1}}+(\alpha-\alpha_i)s_{k_{i-1}}+\beta^i_k$.

(2) *For* $(k, s_k)\in\sum(i)$, $\beta_k+(\alpha-\alpha_i)s_k=\beta_{k_{i-1}}+(\alpha-\alpha_i)s_{k_{i-1}}$, *in particular* $\beta_{k_i}+(\alpha-\alpha_i)s_{k_i}=\beta_{k_{i-1}}+(\alpha-\alpha_i)s_{k_{i-1}}$.

Proof. (1) It follows from Lemma 1.1 that $(s_m-s_k)(1-\alpha)+\beta_k=m-k$, $(s_m-s_{k_{i-1}})(1-\alpha)+\beta_{k_{i-1}}=m-k_{i-1}$ and $(s_{k_{i-1}}-s_k)(1-\alpha_i)+\beta^i_k=k_{i-1}-k$. We obtain (1) from these equalities. (2) For $(k, s_k)\in\sum(i)$, $\beta^i_k=0$. So from (1) we have the first equality and, by putting $k=k_i$, we get the second.

(II) Roots of $G_0(z, \lambda, \tau)=0$. We study the roots of $G_0(z, \lambda, \tau)=0$ (see (3.9)), which is an algebraic equation of τ. Set

$$(6.2)\qquad \begin{cases} F_0(z, \lambda, \tau) = A_{m,s_m}(z, \hat{\xi}')\tau^{s_m} \\ F_i(z, \lambda, \tau) = \sum_{(k,s_k)\in\Sigma(i)} \lambda^{-\beta_k} A_{k,s_k}(z, \hat{\xi}')\tau^{s_k} \qquad (1\leqq i\leqq p-1)\,, \end{cases}$$

where $\hat{\xi}'=(1, 0, \cdots, 0)$. We give the condition on $G_0(z, \lambda, \tau)$

$$(6.3)\text{-i}\qquad |A_{k_i,s_{k_i}}(z, \hat{\xi}')|\geqq c_0>0 \qquad \text{for} \quad |z|\leqq R_0\,.$$

Some of (6.3)-i ($0\leqq i\leqq p-1$) will be assumed in the following lemmas and propositions.

Lemma 6.2. *Assume* (6.3)-*i. Then there are positive constants* a_{i+1}, b_i *and* A_0 *such that if* $|\lambda|\geqq A_0$, *then*

$$(6.4)\qquad |G_0(z, \lambda, \tau)-\lambda^{-\beta_{k_i}}A_{k_i,s_{k_i}}(z, \hat{\xi}')\tau^{s_{k_i}}|<|\lambda^{-\beta_{k_i}}A_{k_i,s_{k_i}}(z, \hat{\xi}')\tau^{s_{k_i}}|/2$$

on $\{\tau;\ |\tau|=b_i|\lambda|^{\alpha_i-\alpha}\}\cup\{\tau;\ |\tau|=a_{i+1}|\lambda|^{\alpha_{i+1}-\alpha}\}$.

Proof. Let $|\tau|=c|\lambda|^{\alpha_i-\alpha}$. Then from Proposition 6.1 for each term in $G_0(z, \lambda, \tau)$

$$(6.5)\qquad \begin{aligned} &|\lambda^{-\beta_k-(1-\alpha)(l-s_k)}\tau^l A_{k,l}(z, \hat{\xi}')|=|c^l\lambda^{-p}A_{k,l}(z, \hat{\xi}')|\,,\\ &p = \beta_{k_i}+(\alpha-\alpha_i)s_{k_i}+\beta^i_k+(1-\alpha_i)(l-s_k)\,. \end{aligned}$$

If $l=s_k$ and $(k, s_k)\in\sum(i)$, then $p=\beta_{k_i}+(\alpha-\alpha_i)s_{k_i}$. Hence there is a small $c>0$ such that

$$\sum_{(k,s_k)\in\Sigma(i),k\neq k_i}|c^{s_k}A_{k,s_k}(z, \hat{\xi}')|<|c^{s_{k_i}}A_{k_i,s_{k_i}}(z, \hat{\xi}')|/4\,.$$

Fix $c>0$. For each term in $\{G_0(z, \lambda, \tau)-F_i(z, \lambda, \tau)\}$, $\beta^i_k>0$ or $l>s_k$. So this means $p>(\alpha-\alpha_i)s_{k_i}+\beta_{k_i}$. Therefore there is a large A_0 such that for $|\lambda|\geqq A_0$ and on $\{|\tau|=c|\lambda|^{\alpha_i-\alpha}\}$,

$$|G_0(z, \lambda, \tau)-F_i(z, \lambda, \tau)|<|\lambda^{-\beta_{k_i}}\tau^{s_{k_i}}A_{k_i,s_{k_i}}(z, \xi')/4\,.$$

Thus we have (6.4) on $\{|\tau|=b_i|\lambda||^{\alpha_i-\alpha}\}$ ($b_i=c$). Next let $|\tau|=c|\lambda|^{\alpha_{i+1}-\alpha}$, we have

$$|\lambda^{(1-\alpha)(l-s_k)-\beta_k}\tau^l A_{k,l}(z, \hat{\xi}')| = c^l|\lambda^{-q}A_{k,l}(z, \hat{\xi}')|\,,$$

where $q=\beta_{k_i}+(\alpha-\alpha_{i+1})s_{k_i}+\beta^i_k+(1-\alpha_{i+1})(l-s_k)$. Hence for a large $c>0$ we have

$$\sum_{(k,s_k)\in\Sigma(i+1),k\neq k_i} c^{s_k}|A_{k,s_k}(z,\hat{\xi}')|<c^{s_{k_i}}|A_{k_i,s_{k_i}}(z,\hat{\xi}')|/4\,.$$

Fix $c>0$. Then in the same way as above there exists *a* A_0 such that for $|\lambda|\geqq A_0$

$$|G_0(z,\lambda,\tau)-F_{i+1}(z,\lambda,\tau)|<|\tau^{s_{k_i}}A_{k_i,s_{k_i}}(z,\hat{\xi}')|/4\,.$$

Thus on $\{|\tau|=a_{i+1}|\lambda|^{\alpha_{i+1}-\alpha}\}$ ($|\lambda|\geqq A_0$, $a_{i+1}=c$), we have (6.4). This completes the proof.

Now we can show Lemma 4.1.

Proof of Lemma 4.1. We have from Lemma 6.2

$$|G_0(z,\lambda,\tau)-A_{m,s_m}(z,\hat{\xi}')\tau^{s_m}|<|A_{m,s_m}(z,\hat{\xi}')\tau^{s_m}|$$

on $\{|\tau|=b_0|\lambda|^{1-\alpha}\}\cup\{|\tau|=a_1\}$. Hence it follows from the Rouche's Theorem in the theory of functions of one complex variable that there are s_m roots of $G_0(z,\lambda,\tau)=0$ in $\{\tau;\ |\tau|<a_1\}$ and no zeros in $\{a_1\leqq|\tau|\leqq b_0|\lambda|^{1-\alpha}\}$.

Secondly we study the roots of $G_0(z,\lambda,\tau)=0$ more precisely. We have

Lemma 6.3. *Assume* (6.3)-($i-1$) *and* (6.3)-i ($i\neq 0$). *The equation* $F_i(z,\lambda,\tau)=0$ *has* $(s_{k_{i-1}}-s_{k_i})$ *non zero roots* $\{\mathring{\tau}_{i,j}(z)\lambda^{\alpha_i-\alpha};\ 1\leqq j\leqq s_{k_{i-1}}-s_{k_i}\}$ *and other roots are zero.*

Proof. Set $\tau=\eta\lambda^{\alpha_i-\alpha}$. Then, by Proposition 6.1-(2),

$$F_i(\eta\lambda^{\alpha_i-\alpha})=\lambda^{-p}\sum_{(k,s_k)\in\Sigma(i)}(\eta^{s_k}A_{k,s_k}(z,\hat{\xi}'))\,,$$

where $p=\beta_{k_i}+(\alpha-\alpha_i)s_{k_i}$. Thus $F_i(z,\lambda,\tau)=0$ has $(s_{k_{i-1}}-s_{k_i})$ non zero roots $\{\mathring{\tau}_{i,j}(z)\lambda^{\alpha_i-\alpha};\ 1\leqq j\leqq s_{k_{i-1}}-s_{k_i}\}$ and other roots are zero.

We set

$$\mathring{N}_i=\{\mathring{\tau}_{i,j}(z);\ |z|\leqq R_0,\ 1\leqq j\leqq s_{k_{i-1}}-s_{k_i}\}\,, \tag{6.6}$$

Proposition 6.4. *Assume* (6.3)-($i-1$) *and* (6.3)-i ($i\neq 0$). *The equation* $G_0(z,\lambda,\tau)=0$ *has* $(s_{k_{i-1}}-s_{k_i})$ *non zero roots* $\{\tau_{i,j}(z,\lambda);\ 1\leqq j\leqq s_{k_{i-1}}-s_{k_i}\}$ *such that for* $|\lambda|\geqq A_0$

$$|\tau_{i,j}(z,\lambda)-\mathring{\tau}_{i,j}(z)\lambda^{\alpha_i-\alpha}|\leqq A|\lambda|^{-\rho+\alpha_i-\alpha}\,, \tag{6.7}$$

A_0, A *and* ρ *being positive constants.*

Proof. We choose $\rho>0$ so that $0<\rho(s_{k_{i-1}}-s_{k_i})<d$, $d=\min\{1-\alpha_i,\ \beta_k^i\ (k\notin\Sigma(i))\}$. Put $D_\eta(i)=\{\eta;\ \mathrm{dis}(\eta,\mathring{N}_i)=|\lambda|^{-\rho}\}$, $\mathrm{dis}(\eta,K)$ being the distance

from the point η to the set K. For $\eta \in D_\eta(i)$, there is a $C>0$ such that

$$|F_i(\lambda^{\alpha_i-\alpha}\eta)| \geqq C\,|\lambda|^{-\beta_{k_i}-(\alpha-\alpha_i)s_{k_i}-\rho(s_{k_{i-1}}-s_{k_i})}\,.$$

We have from (6.5), for $\eta \in D_\eta(i)$

$$\begin{aligned}&|G_0(z, \lambda, \eta\lambda^{\alpha_i-\alpha})-F_i(z, \lambda, \eta\lambda^{\alpha_i-\alpha})|\\ &\quad\leqq C\,|\lambda|^{-\beta_{k_i}+(\alpha_i-\alpha)s_{k_i}}\Big(\textstyle\sum_{l>s_k \text{ or } (k,s_k)\notin\Sigma(i)}|\lambda|^{-(1-\alpha_i)(l-s_k)-\beta_k^i}\Big)\\ &\quad\leqq C\,|\lambda|^{-\beta_{k_i}+(\alpha_i-\alpha)s_{k_i}-d}\,.\end{aligned}$$

Thus it holds for a large Λ_0 that if $\eta \in D_\eta(i)$ and $|\lambda| \geqq \Lambda_0$, $|F_i(z, \lambda, \tau)|/2 > |G_0(z, \lambda, \tau)-F_i(z, \lambda, \tau)|$. Therefore $G_0(z, \lambda, \tau)=0$ has $(s_{k_{i-1}}-s_{k_i})$ roots in the inside of $\lambda^{(\alpha_i-\alpha)}D_\eta(i)$ by the Rouche's Theorem.

Hereafter assume $p>1$, (6.3)–i for all $0 \leqq i \leqq p-1$ and $s_{k_{p-1}}=0$. Set

$$N_i(z, \lambda) = \{\tau_{i,j}(z, \lambda)\lambda^{\alpha-\alpha_i};\ 1 \leqq j \leqq s_{k_{i-1}}-s_{k_i}\} \tag{6.8}$$

and

$$K_i(\delta) = \cup_{j=1}^{s_{k_{i-1}}-s_{k_i}}\{\tau;\ |\tau-\mathring{\tau}_{i,j}(0)| \leqq \delta\}\,. \tag{6.9}$$

We choose small δ, $R_0>0$ and a large Λ_0 so that if $|z| \leqq R_0$ and $|\lambda| \geqq \Lambda_0$,

$$K_i(\delta) \supset K_i(\delta/2) \supset N_i(z, \lambda),\ K_i(\delta) \subset \{\tau;\ b_i < |\tau| < a_i\}\ (1 \leqq i \leqq p-1)\,, \tag{6.10}$$

a_i and b_i being those in Lemma 6.2. Define the sets for $1 \leqq i \leqq p-1$

$$\tau(i) = \{\tau;\ b_i|\lambda|^{\alpha_i-\alpha} \leqq |\tau| \leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha},\ \tau \notin \lambda^{\alpha_i-\alpha}K_i(\delta)\}\,. \tag{6.11}$$

Then we have

Proposition 6.5. *Let $N(z, \lambda)$ be the set of all bounded roots of $G_0(z, \lambda, \tau)=0$ as $\lambda \to \infty$. Then $N(z, \lambda) \subset \cup_{i=1}^{p-1}\lambda^{\alpha_i-\alpha}K_i(\delta/2)$.*

We have

Proposition 6.6. *For $\tau \in \tau(i)$*

$$|G_0(z, \lambda, \tau)| \geqq C\,|\lambda|^{-\beta_{k_{i-1}}}|\tau|^{s_{k_{i-1}}}\,. \tag{6.12}$$

Proof. We have on $\{|\tau| = b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\}$ by Lemma 6.2

$$|G_0(z, \lambda, \tau)| \geqq |\lambda^{-\beta_{k_{i-1}}}A_{k_{i-1},s_{k_{i-1}}}(z, \hat{\xi}')\tau^{s_{k_{i-1}}}|/2\,.$$

Similarly on $\{|\tau| = b_i|\lambda|^{\alpha_i-\alpha}\}$ or $\tau \in \lambda^{\alpha_i-\alpha}\partial K_i(\delta)$ we have

$$|G_0(z, \lambda, \tau)| \geqq C\,|\lambda|^{-p} \geqq C\,|\lambda|^{-\beta_{k_{i-1}}}|\tau|^{s_{k_{i-1}}}\,,$$

where $p=\beta_{k_i}+(\alpha-\alpha_i)s_{k_i}=\beta_{k_{i-1}}+(\alpha-\alpha_i)s_{k_{i-1}}$ by Lemma 6.1-(2). Thus applying the maximal principle of holomorphic functions to $\tau^{s_{k_{i-1}}}G_0(z, \lambda, \tau)^{-1}$, we have $|\tau^{s_{k_{i-1}}}G_0(z, \lambda, \tau)^{-1}|\leqq C|\lambda^{\beta_{k_{i-1}}}|$. This means (6.12).

Proposition 6.7. *For $\tau\in\tau(i)$ and $l\geqq s_k$ there is an A such that*

$$\frac{|\lambda^{-(1-\alpha)(l-s_k)-\beta_k}\tau^l|}{|G_0(z, \lambda, \tau)|}\leqq A. \tag{6.13}$$

Proof. We have, on $\{|\tau|=b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\}$

$$|\lambda^{-(1-\alpha)(l-s_k)-\beta_k}\tau^l|\leqq C|\lambda|^{-(1-\alpha)(l-s_k)-\beta_k-l(\alpha-\alpha_{i-1})}.$$

Since $(1-\alpha)(l-s_k)+\beta_k+l(\alpha-\alpha_{i-1})$

$$=(1-\alpha_{i-1})(l-s_k)+(\alpha-\alpha_{i-1})s_k+\beta_k\geqq\beta_{k_{i-1}}+(\alpha-\alpha_{i-1})s_{k_{i-1}},$$

we have $|\lambda^{-(1-\alpha)(l-s_k)-\beta_k}\tau^l|\leqq C|\lambda^{-\beta_{k_{i-1}}}\tau^{s_{k_{i-1}}}|$.
On the other hand we have

$$(1-\alpha_i)(l-s_k)+(\alpha-\alpha_i)s_k+\beta_k\geqq(\alpha-\alpha_i)s_k+\beta_k\geqq\beta_{k_{i-1}}+(\alpha-\alpha_i)s_{k_{i-1}}.$$

Hence we have on $\{|\tau|=b_i|\lambda|^{\alpha_i-\alpha}\}\cup\{\lambda^{\alpha_i-\alpha}\partial K_i(\delta)\}$

$$|\lambda^{-(1-\alpha)(l-s_k)-\beta_k}\tau^l|\leqq C|\lambda^{-\beta_{k_{i-1}}}\tau^{s_{k_{i-1}}}|.$$

It follows from Proposition 6.6 that on the boundary of $\tau(i)$

$$|\lambda^{-(1-\alpha)(l-s_k)-\beta_k}\tau^l/G_0(z, \lambda, \tau)|\leqq A.$$

By the maximal principle of holomorphic functions implies (6.13) holds on $\tau(i)$.

(III) Sectors S_i $(1\leqq i\leqq p-1)$. In (III) we define sectors $\{S_i; 1\leqq i\leqq p-1)$ whose vertex is the origin in τ-space. We make use of the sectors to prove Theorem 1.3 in § 8. To define S_i we give two lemmas.

Lemma 6.8. *There is an ω_0 $(|\omega_0|=1)$ such that* $\arg(\mathring{\tau}_{i,j}(0)\omega_0)\neq\pi-\pi\alpha_i$ (mod 2π) *for all $1\leqq i\leqq p-1$ and $1\leqq j\leqq s_{k_{i-1}}-s_{k_i}$.*

Proof. Set $B=\{\mathring{\tau}_{i,j}(0); 1\leqq i\leqq p-1, 1\leqq j\leqq s_{k_{i-1}}-s_{k_i}\}$, $L_i=\{re^{i(\pi-\pi\alpha_i)}; r\geqq0\}$ and $L=\cup_{i=1}^{p-1}L_i$. B is a finite set of nonzero points and L is a finite set of half lines. So we can find an ω_0 $(|\omega_0|=1)$ such that $\omega_0B\cap L=\phi$. This implies the assertion.

It follows from Lemma 6.8 that

Lemma 6.9. *There are ω_1 $(|\omega_1|=1)$ and positive numbers r and ε_1 such that*

$\arg \tau_{i,j}(z, \lambda)\omega \neq \pi - \pi\alpha_i \pmod{2\pi}$ *for all* $|\lambda| \geqq \Lambda_0$, $|z| \leqq r$ *and* $|\omega - \omega_1| < \varepsilon_1$.

Thus we conclude:

Proposition 6.10. *There are* δ, $\varepsilon_1 > 0$, $\hat{z}_1 \neq 0$ *and for each* i $(1 \leqq i \leqq p-1)$ *an open sector* S_i *with the vertex* 0 *in* C^1 *such that*

$$S_i \ni e_i = e^{i(\pi - \pi\alpha_i)} \quad \text{and} \quad \bar{S}_i \cap (-z_1 K(\delta)) = \phi \qquad \text{for} \quad |z_1 - \hat{z}_1| < \varepsilon_1 .$$

(IV) Representation of functions in $\mathcal{O}(\tilde{\Omega}(\theta_0))$ (Proofs of Lemmas 2.1 and 2.2). Let $w(z) \in \mathcal{O}(\tilde{\Omega}(\theta_0))$ $(\theta_0 > \pi)$, $\Omega = \{|z| \leqq R\}$. Define for $-\theta_0 < \theta < \theta_0 - 2\pi$

$$\hat{w}_\theta(\lambda, z') = \frac{1}{2\pi i} \int_{T(\theta)} \exp(-\lambda t_0) w(t_0, z') dt_0 . \tag{6.14}$$

Then $\hat{w}_\theta(\lambda, z')$ is an entire function of λ and

$$|\hat{w}_\theta(\lambda, z')| \leqq C_\varepsilon \exp(\varepsilon |\lambda|) \qquad \text{for} \quad |\arg \lambda + \theta| < \pi/2 . \tag{6.15}$$

Lemma 6.11. *If* $\sup_{|t'| \leqq R} |w(t_0, t')| \leqq A \exp(\delta |t_0|^{-\gamma})$ $(r > 0)$, *then*

$$\sup_{|t|' \leqq R} |\hat{w}_\theta(\lambda, t')| \leqq B \exp(2(\delta |\lambda|^\gamma)^{1/(1+\gamma)}) \text{ for } \lambda \text{ with } |\lambda| \geqq 1 \text{ and } |\arg \lambda + \theta| < \pi/2. \tag{6.16}$$

Proof. Choose $\varepsilon = (\delta |\lambda|^{-1})^{1/(1+\gamma)}$ in the path $T(\theta)$. Then on $|t_0| = \varepsilon$, $|t_0 \lambda| + \delta |t_0|^{-\gamma} = \varepsilon |\lambda| + \delta \varepsilon^{-\gamma} = 2(\delta |\lambda|^\gamma)^{1/(1+\gamma)}$. So we have (6.16).

Set for $|\psi + \theta| < \pi/2$

$$w_\theta(z) = \frac{1}{2\pi i} \int_{\Lambda(\psi)} \exp(\lambda z_0) \hat{w}_\theta(\lambda, z') (\log \lambda) d\lambda . \tag{6.17}$$

Then $w_\theta(z) \in \mathcal{O}(\tilde{\Omega}(\theta, \theta + 2\pi))$ by (6.15). We have the relation between $w(z)$ and $w_\theta(z)$.

Lemma 6.12. $w(z) - w_\theta(z)$ *is holomorphic in* $\{z;\ |z_0| < R,\ |z'| \leqq R\}$.

Proof. From (6.17) for z_0 with $|\arg z_0 + \psi - \pi| < \pi/2$

$$\begin{aligned} w_\theta(z) &= \frac{1}{2\pi i} \int_{\Lambda(\psi)} \exp(\lambda z_0) (\log \lambda)\, d\lambda \, \frac{1}{2\pi i} \int_{T(\theta)} \exp(-\lambda t_0) w(t_0, z') dt_0 \\ &= \frac{1}{2\pi i} \int_0^{\infty e^{i\psi}} \exp(\lambda z_0) d\lambda \int_{T(\theta)} \exp(-\lambda t_0) w(t_0, z') dt_0 . \end{aligned} \tag{6.18}$$

Set $\psi = -\theta$ and let $\arg z_0 = \theta + \pi$ and $|z_0| > 2\varepsilon$. Then by the definition of $T(\theta)$, we have

$$w_\theta(z)=\frac{1}{2\pi i}\int_{T(\theta)}\{w(t_0, z')/(t_0-z_0)\}\,dt_0=w(z)-v(z)\,,$$

where $v(z)=-\frac{1}{2\pi i}\int_{T^*(\theta)}\{w(t_0, z')/(t_0-z_0)\}\,dt_0$ and $T^*(\theta)$ is a path starting at $Re^{i\theta}$ and going around on the circle $|t|=R$ once. So $v(z)$ is holomorphic in $\{z;\ |z_0|<R,\ |z'|\leqq R\}$. This completes the proof.

We have Lemma 2.1 from Lemma 6.11 and Lemma 2.2 from Lemma 6.12.

§ 7. Holomorphic Extension of the Kernel Function $w(z, t'', \lambda, \zeta)$

We try to analyze the integral representation in detail. To do so we study the singularties of the solution $w(z, t'', \lambda, \zeta)=w^h(z, t'', \lambda, \zeta)$ of (3.1). We will obtain more precise informations of the integral representation from them, which yield the results of removable singularities (Theorem 1.3).

Now we always assume

$$\text{(7.1)}\qquad \begin{cases} \prod_{i=0}^{p-1} A_{k_i,s_{k_i}}(z, \xi')\,|_{\xi'=\hat{\xi}'=(1,0,\cdots,0)}\neq 0 & \text{for}\quad |z|\leqq R_0\,,\\ \sigma_1>1 \ \text{and}\ s_{k_{p-1}}=0 \end{cases}$$

through § 7 and § 8. This means $p>1$ and (6.3)–i hold for all $0\leqq i\leqq p-1$. Let us recall the definitions of the path $\gamma(c)$ in τ-space (see § 5-II) and positive constants a_i and b_i (see Lemma 6.2 and (6.10)). Let us write the sets appearing often in the sequels:

$$\text{(7.2)}\qquad \begin{cases} \Xi=\{(z, t'', \lambda);\ |z|\leqq r,\ R_1\leqq|t_i|\leqq R\ (i\geqq 2),\ |\lambda|\geqq A_0\}\,,\\ \tau(i)=\{\tau;\ b_i|\lambda|^{\alpha_i-\alpha}\leqq|\tau|\leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha},\ \tau\notin\lambda^{\alpha_i-\alpha}K_i(\delta)\}\\ X(i)=\{(z, t'', \lambda, \tau);\ (z, t'', \lambda)\in\Xi,\ \tau\in\tau(i)\},\\ Y(i)=\{(z, t'', \lambda, \zeta, \tau);\ (z, t'', \lambda, \tau)\in X(i),\ 0<|\zeta+\tau z_1|<A|\tau|\}\\ Z(i)=\{(z, t'', \lambda, \zeta);\ (z, t'', \lambda)\in\Xi,\\ \qquad 2a_i|\lambda^{\alpha_i-\alpha}z_1|<|\zeta|<b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}(A-|z_1|)\}\,.\\ Z_{i,i+1}=\{(z, t'', \lambda, \zeta);\ (z, t'', \lambda)\in\Xi,\\ \qquad |\zeta|<b_i|\lambda|^{\alpha_i-\alpha}(A-|z_1|),\ \zeta\notin -z_1\lambda^{\alpha_i-\alpha}K_i(\delta)\}\,, \end{cases}$$

$1\leqq i\leqq p-1$, for $K_i(\delta)$ see (6.8)–(6.10).

By Proposition 5.3 $V(z, t'', \lambda, \zeta, \tau)$ is holomorphic in $Y\subset Y(1)$ and we may assume $w(z, t'', \lambda, \zeta)$ is holomorphic in $Z(1)$ (see Proof of Proposition 2.3 in § 5). Now that we assume (7.1), we have better results than Proposition 5.3.

Proposition 7.1. (1). *The following estimate holds in* $X(1)$:

(7.3) $$|v_p(z, t'', \lambda, \tau)| \leqq AB^{p+1}|\tau|^{-p-2}p!\,.$$

(2). *$V(z, t'', \lambda, \zeta, \tau)$ is holomorphic in $Y(1)$.*

If we assume (7.3) in $X(1)$, we can show (2) by the same method as in the proof of Lemma 5.2. The estimate (7.3) is shown § 9. So from Proposition 7.1 we can define, by setting $\gamma_2 = \gamma(b_1)$,

(7.4) $$w_2(z, t'', \lambda, \zeta) = \int_{\gamma_2} V(z, t'', \lambda, \zeta, \tau)d\tau\,.$$

By repeating the same argument as in the proof of Proposition 2.3 (see § 5–II) we have

Proposition 7.2. *$w_2(z, t'', \lambda, \zeta)$ is holomorphic in $\tilde{Z}_2'$, $Z_2' = \{(z, t'', \lambda, \zeta);$ $(z, t'', \lambda) \in \Xi,\ 2b_1|z_1| < |\zeta| < b_1(A-|z_1|)\}$.*

Set $w_1(z, t'', \lambda, \zeta) = w(z, t'', \lambda, \zeta)$ and

(7.5) $$w_{1,2}(z, t'', \lambda, \zeta) = \int_{\gamma_1-\gamma_2} V(z, t'', \lambda, \zeta, \tau)d\tau\,.$$

Then we have

(7.6) $$w_1(z, t'', \lambda, \zeta) = w_{1,2}(z, t'', \lambda, \zeta) + w_2(z, t'', \lambda, \zeta) \quad \text{in} \quad \widetilde{Z_1 \cap Z_2'}\,.$$

Since the path $(\gamma_1 - \gamma_2)$ can be deformed to the path $\partial K_1(\delta)$, we get

Proposition 7.3. *$w_{1,2}(z, t'', \lambda, \zeta) \in \mathcal{O}(\tilde{Z}_{1,2})$.*

We have defined $w_i(z, t'', \lambda, \zeta)$ $(i=1, 2)$ and $w_{1,2}(z, t'', \lambda, \zeta)$. Let us construct inductively $w_i(z, t'', \lambda, \zeta)$ and $w_{i,i+1}(z, t'', \lambda, \zeta)$ $(i=1, 2, \cdots, p-1)$ such that

(7.7) $$w_{i-1}(z, t'', \lambda, \zeta) = w_{i-1,i}(z, t'', \lambda, \zeta) + w_i(z, t'', \lambda, \zeta)\,.$$

By the relation (7.7) we shall get the holomorphic extension of $w(z, t'', \lambda, \zeta)$ as a function of ζ onto a covering space of

$$Z_0 = \{\zeta;\ |\zeta| < b_0|\lambda|^{1-\alpha}(A-|z_1|),\ \zeta \notin -z_1 \cup_{q=1}^{p-1} \lambda^{\alpha_q-\alpha} K_q(\delta)\}\,.$$

Define $\gamma_i = \gamma(c_i)$, $2a_i|\lambda|^{\alpha_i-\alpha} \leqq |c_i| \leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}$. Assume that $w_{i-1}(z, t'',$ $\lambda, \zeta)$ is defined and has the form

(7.8) $$w_{i-1}(z, t'', \lambda, \zeta) = \int_{\gamma_{i-1}} V^{i-1}(z, t'', \lambda, \zeta, \tau)d\tau\,,$$

where $V^{i-1}(z, t'', \lambda, \zeta, \tau) = \sum_{p=h-1}^{+\infty} v_p^{i-1}(z, t'', \lambda, \tau)f_p(\zeta + \tau z_1)$ converging in $Y(i-1)$ and $V^1(z, t'', \lambda, \zeta, \tau) = V(z, t'', \lambda, \zeta, \tau)$. Let us define $V^i(z, t'', \lambda, \zeta, \tau)$.

Set $v_p^i(z, t'', \lambda, \tau)=0$ for $p \leqq h-2$ and suppose that $v_{p-j}^i(z, t'', \lambda, \tau)$ $(j \geqq 1)$ are defined so that

$$v_{p-j}^i(z, t'', \lambda, \tau)-v_{p-j}^{i-1}(z, t'', \lambda, \tau) \in \mathcal{O}(\{|\tau| \leqq b_{i-1}|\lambda|^{\alpha_{i-1}\alpha}\}) .$$

Set $g_p^i(z, t'', \lambda, \tau)=\sum_{j=1}^m G_j(z, \lambda, \tau, \partial_z) v_{p-j}^i(z, t'', \lambda, \tau)$. Then $g_p^i(z, t'', \lambda, \tau)-g_p^{i-1}(z, t'', \lambda, \tau) \in \mathcal{O}(\{|\tau| \leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\})$. We have

Proposition 7.4. *There exists uniquely a polynomial $h_p^i(z, t'', \lambda, \tau)$ with degree $\leqq s_{k_{i-1}}-1$ such that*

$$(7.9)\qquad \int_{\gamma_i} \frac{\{g_p^{i-1}(z, t'', \lambda, \tau)-g_p^i(z, t'', \lambda, \tau)+h_p^{i-1}(z, t'', \lambda, \tau)\} \tau^l}{G_0(z, \lambda, \tau)} d\tau$$
$$= \int_{\gamma_i} \frac{h_p^i(z, t'', \lambda, \tau) \tau^l}{G_0(z, \lambda, \tau)} d\tau \qquad \textit{for} \quad 0 \leqq l \leqq s_{k_{i-1}}-1 .$$

Proof. There exist $s_{k_{i-1}}$ roots of $G_0(z, \lambda, \tau)=0$ in the inside of γ_i if $|\lambda| \geqq A_0$. Proposition 7.4 follows from Lemma 9.8 in § 9.

So set

$$v_p^i(z, t'', \lambda, \tau) = \{g_p^i(z, t'', \lambda, \tau)+h_p^i(z, t'', \lambda, \tau)\}/G_0(z, \lambda, \tau) .$$

Since $g_p^i(z, t'', \lambda, \tau)-g_p^{i-1}(z, t'', \lambda, \tau)+h_p^i(z, t'', \lambda, \tau)-h_p^{i-1}(z, t'', \lambda, \tau) \in \mathcal{O}(\{|\tau| \leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\})$, it follows from Lemma 9.7 in § 9 that (7.9) means that $v_p^i(z, t'', \lambda, \tau)-v_p^{i-1}(z, t'', \lambda, \tau) \in \mathcal{O}(\{|\tau| \leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\})$. For the estimate of $v_p^i(z, t'', \lambda, \tau)$ we have

Lemma 7.5. *It holds that*

$$(7.10)\qquad |v_p^i(z, t'', \lambda, \tau)| \leqq AB^{p+1}|\tau|^{-p-2} p! \qquad \textit{in} \quad X(i) .$$

We refer the proof of Lemma 7.5 to § 9. By Lemma 7.5 we can show the convergence of

$$(7.11)\qquad V^i(z, t'', \lambda, \zeta, \tau) = \sum_{p=h-1}^{+\infty} v_p^i(z, t'', \lambda, \tau) f_p(\zeta+\tau z_1)$$

and get

Proposition 7.6. *$V^i(z, t'', \lambda, \zeta, \tau)$ is holomorphic in $Y(i)$.*

We define

$$(7.12)\qquad w_i(z, t'', \lambda, \zeta) = \frac{1}{2\pi i} \int_{\gamma_i} V^i(z, t'', \lambda, \zeta, \tau) d\tau ,$$

and

$$(7.13)\qquad w_{i-1,i}(z, t'', \lambda, \zeta) = \frac{1}{2\pi i}\int_{\gamma_{i-1}-\gamma_i} V^{i-1}(z, t'', \lambda, \zeta, \tau)d\tau .$$

If $|\zeta|>|\tau z_1|$, since $v_p^i(z, t'', \lambda, \tau)-v_p^{i-1}(z, t'', \lambda, \tau)\in\mathcal{O}(\{|\tau|\leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\})$, $V^{i-1}(z, t'', \lambda, \zeta, \tau)-V^i(z, t'', \lambda, \zeta, \tau)\in\mathcal{O}(\{\tau; |\tau|\leqq b_{i-1}|\lambda|^{\alpha_{i-1}-\alpha}\})$, So we can replace the integrand in (7.12) by $V^{i-1}(z, t'', \lambda, \zeta, \tau)$. Then we have in $(z, t'', \lambda, \zeta)\in Z(i-1)\cap Z(i)$,

$$(7.14)\qquad w_i(z, t'', \lambda, \zeta) = \frac{1}{2\pi i}\int_{\gamma_i} V^{i-1}(z, t'', \lambda, \zeta, \tau)d\tau .$$

Thus

$$\begin{aligned}(7.15)\quad w_{i-1}(z, t'', \lambda, \zeta) &= w_{i-1,i}(z, t'', \lambda, \zeta)+\frac{1}{2\pi i}\int_{\gamma_i} V^{i-1}(z, t'', \lambda, \zeta, \tau)d\tau \\ &= w_{i-1,i}(z, t'', \lambda, \zeta)+w_i(z, t'', \lambda, \zeta) \quad \text{in} \quad \widetilde{Z(i-1)\cap Z(i)} .\end{aligned}$$

We have

Proposition 7.7. $w_{i-1,i}(z, t'', \lambda, \zeta)\in\mathcal{O}(\tilde{Z}_{i-1,i})$ *and* $w_i(z, t'', \lambda, \zeta)\in\mathcal{O}(Z(i))$.

Thus by using $w_{i-1,i}(z, t'', \lambda, \zeta)$ and $w_i(z, t'', \lambda, \zeta)$, we can extend $w(z, t'', \lambda, \zeta)=w_1(z, t'', \lambda, \zeta)\in\mathcal{O}(\tilde{Z}(1))$ holomorphically into Z_0,

$$\begin{aligned}Z_0 &= \{(z, t'', \lambda, \zeta); (z, t'', \lambda)\in\Xi,\ |\zeta|<b_0|\lambda|^{1-\alpha}(A-|z_1|),\ \zeta\notin -z_1K\} ,\\ K &= \cup_{q=1}^{p} \lambda^{\alpha_q-\alpha} K_q(\delta) ,\end{aligned}$$

in the following way:

$$\begin{aligned}(7.16)\quad w_1(z, t'', \lambda, \zeta) &= w_{1,2}(z, t'', \lambda, \zeta)+w_2(z, t'', \lambda, \zeta) \quad \text{in } \widetilde{Z_{1,2}\cap Z(2)} \\ &= w_{1,2}(z, t'', \lambda, \zeta)+w_{2,3}(z, t'', \lambda, \zeta)+w_3(z, t'', \lambda, \zeta) \quad \text{in } \widetilde{Z_{2,3}\cap Z(3)} \\ &= w_{1,2}(z, t'', \lambda, \zeta)+w_{2,3}(z, t'', \lambda, \zeta)+\cdots+w_{p-1,p}(z, t'', \lambda, \zeta) \\ &\qquad \text{in } \tilde{Z}_{p-1,p} .\end{aligned}$$

Summing up the above extension, we have

Theorem 7.8. $w(z, t'', \lambda, \zeta)$ *has a holomorphic prolongation by* (7.16) *as a function of* ζ *to some covering space* $\hat{Z}_0$ *of* Z_0.

We denote this prolonged function also by $w(z, t'', \lambda, \zeta)$. So in (2.2), the definition of $K_\theta^h(z, \lambda, t'')$, we can deform $C(\theta)$ homotopically to a path in $\hat{Z}_0$. We'll perform it in § 8 to show Theorem 1.3.

§ 8. Removability of Singularities

In this section we assume (7.1) and complete the proof of Theorem 1.3.

So Propositions in § 6 are available. Sectors $\{S_i\}$ $(1 \leqq i \leqq p-1)$ appearing in this section are those in Proposition 6.10. Set

$$\Xi' = \{(z, t'', \lambda); (z, t'', \lambda) \in \Xi, \ |z_1 - \hat{z}_1| < \varepsilon_1\}, \tag{8.1}$$

where $\varepsilon_1 > 0$ and $\hat{z}_1 \neq 0$ are also those in Proposition 6.10. We always assume $(z, t'', \lambda) \in \Xi'$ in this section. In (I)–(II), by using Propositions 6.5 and 6.10 and Theorem 7.8, we decompose $K^h_{-\pi}(z, t'', \lambda)$. In (III) we decompose $u(z)$ with the aid of the decomposition of $K^h_{-\pi}(z, t'', \lambda)$. In (IV) we complete the proof of Theorem 1.3, by using Theorems 1.7 and 1.9.

(I) Deformation of path $C(\theta)$. We have constructed $K^h_\theta(z, t'', \lambda)$ in § 5. In view of Theorem 7.8 we can deform the path $C(\theta)$ in the definition of $K^h_\theta(z, t'', \lambda)$ ((5.7)). Firstly let us define some paths in ζ-space. For a path $C = \{\zeta(t); 0 \leqq t \leqq 1\}$ and $a \in \mathbf{C}$, $aC = \{a\zeta(t); 0 \leqq t \leqq 1\}$. A_i is a straight line which starts at $d_{i-1} e^{-i\pi\alpha_{i-1}} \lambda^{\alpha_{i-1}-\alpha}$ and ends at $c_i e^{-i\pi\alpha_i} \lambda^{\alpha_i - \alpha}$ $(c_i > 2a_i|z_1| > b_i|z_1| > d_i > 0$, $i = 1, 2, \cdots, p-1$, $d_0 = d$, d being in (2.12)). B_i is a circle starting at $c_i e^{-i\pi\alpha_i}$ and enclosing $\zeta = 0$ once.

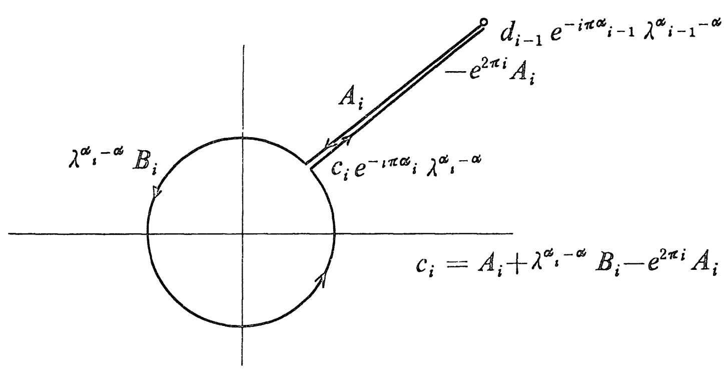

Fig. 8.1.

Set $C_i = A_i + \lambda^{\alpha_i - \alpha} B_i - e^{2\pi i} A_i$ (see Fig. 8.1.). The singularities of $w^h(z, t'', \lambda, \zeta)$ are in the inside of B_1. So we have

$$\begin{aligned} K^h_{-\pi}(z, t'', \lambda) &= \int_{C(-\pi)} \exp(-\lambda^\alpha \zeta) w^h(z, t'', \lambda, \zeta) d\zeta \\ &= \int_{A_1} \cdots d\zeta + \int_{B_1} \cdots d\zeta + \int_{-e^{2\pi i} A_1} \cdots d\zeta . \end{aligned} \tag{8.2}$$

We note that in general $w^h(z, t'', \lambda, \zeta)$ is multi-valued, so in (8.2) $\int_{A_1} \cdots d\zeta + \int_{-e^{2\pi i} A_1} \cdots d\zeta \neq 0$. Let us try to deform the path B_1 to another path. We have

Proposition 8.1. *The path B_1 in* (8.2) *can be deformed homotopically to B_1' containing the path C_2 as a subpath, $B_1'=B_1'(+)+C_2+B_1'(-)$, with the following properties*:

The paths $B_1'(+)$ and $B_1'(-)$ are in $\{d_1\leqq|\zeta|\leqq c_1\}$, $(B_1'(+)\cup B_1'(-))\cap\bar{S}_1=\phi$, $B_1'(+)\cup B_1'(-)$ encloses $-z_1K_1(\delta)$ and

$$(8.3)\quad \int_{B_1}\exp(-\lambda^{\alpha}\zeta)w^h(z,t'',\lambda,\zeta)d\zeta=\int_{B_1'(+)}\cdots d\zeta+\int_{C_2}\cdots d\zeta+\int_{B_1'(-)}\cdots d\zeta\,.$$

Proof. The singularities of $w^h(z,t'',\lambda,\zeta)$ lie in $-z_1K_1(\delta)$ or in $\bigcup_{q=2}^{p-1}\{-z_1\lambda^{\alpha_q-\alpha}K_q(\delta)\}$ by Theorem 7.8. The latter singularities are in $\{\zeta;\ |\zeta|<a_2|\lambda|^{\alpha_2-\alpha}|z_1|\}$, that is, in the inside of C_2. So we can deform B_1 so that $B_1'(+)$ and $B_1'(-)$ enclose $-z_1K_1(\delta)$ and from $(-z_1K_1(\delta))\cap\bar{S}_1=\phi$ (Proposition 6.10), $(B_1'(+)\cup B_1'(-))\cap\bar{S}_1=\phi$ and (8.3) hold.

The singularities of $w^h(z,t'',\lambda,\zeta)$ in the inside of C_2 are enclosed by $\lambda^{\alpha_2-\alpha}B_2$, more precisely, in $-z_1\lambda^{\alpha_2-\alpha}K_2(\delta)$ or in $\bigcup_{q=3}^{p-1}\{-z_1\lambda^{\alpha_q-\alpha}K_q(\delta)\}\subset\{\zeta;\ |\zeta|<a_3|\lambda|^{\alpha_3-\alpha}|z_1|\}$. We can again deform the path $C_2=A_2+\lambda^{\alpha_2-\alpha}B_2+(-e^{2\pi i}A_2)$, not changing A_2 and $e^{2\pi i}A_2$, to a path $A_2+B_2'+(-e^{2\pi i}A_2)$, where B_2' contains C_3 as a subpath and the similar results to Proposition 8.1 hold (see Fig. 8.2.).

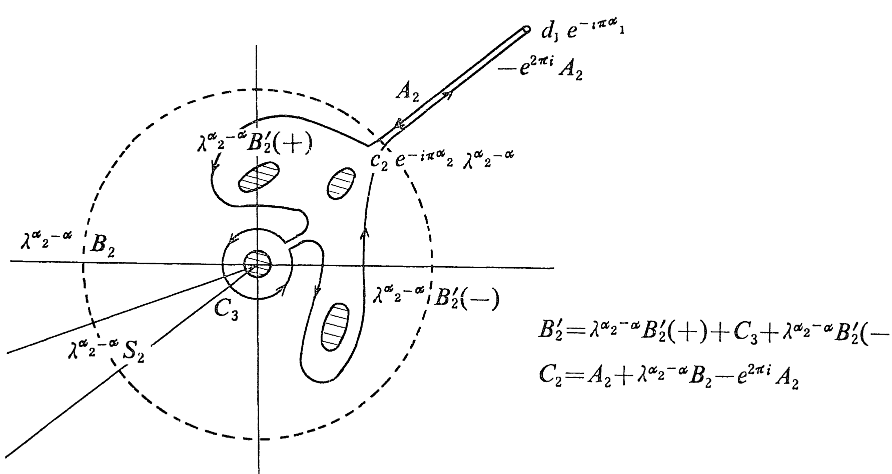

Fig. 8.2.

The singularities of $w^h(z,t'',\lambda,\zeta)$ inside of C_2 are in the parts of oblique lines in Fig. 8.2.

By the repetition of these processes of deformations we have

Proposition 8.2. *The paths $B_i (1 \leqq i \leqq p-1)$ can be deformed homotopically in Z_0 to B_i' such that*

(1) $B_i' = \lambda^{\alpha_i - \alpha} B_i'(+) + C_{i+1} + \lambda^{\alpha_i - \alpha} B_i'(-)$ $(C_p = \phi)$,

(2) *$B_i'(+)$ and $B_i'(-)$ are independent of λ and contained in $\{d_i \leqq |\zeta| \leqq c_i\}$ and $\{B_i'(+) \cup B_i'(-)\} \cap \bar{S}_i = \phi$, and it holds that*

$$(8.4) \quad \int_{\lambda^{\alpha_i - \alpha} B_i} \exp(-\lambda^{\alpha} \zeta) w^h(z, t'', \lambda, \zeta) d\zeta = \int_{\lambda^{\alpha_i - \alpha} B_i'(-)} \cdots d\zeta + \int_{C_{i+1}} \cdots d\zeta + \int_{\lambda^{\alpha_i - \alpha} B_i'(-)} \cdots d\zeta .$$

Thus Proposition 8.2 gives

Proposition 8.3. *The kernel function $K_{-\pi}^h(z, t'', \lambda)$ is represented in the following form:*

$$(8.5) \quad K_{-\pi}^h(z, t'', \lambda) = \sum_{i=1}^{p-1} \left\{ \int_{A_i} \exp(-\lambda^{\alpha} \zeta) w^h(z, t'', \lambda, \zeta) d\zeta + \int_{-e^{2\pi i} A_i} \cdots d\zeta \right\} + \sum_{i=1}^{p-1} \left\{ \int_{\lambda^{\alpha_i - \alpha} B_i'(+)} \cdots d\zeta + \int_{\lambda^{\alpha_i - \alpha} B_i'(-)} \cdots d\zeta \right\} .$$

(II) Decomposition of $K_{-\pi}^h(z, t'', \lambda)$. In order to show Theorem 1.3 we further decompose the paths. For this purpose we need lemmas and propositions about the paths A_i, $B_i'(+)$ and $B_i'(-)$.

Proposition 8.4. *Let $\zeta \in A_i$ and* $\arg \lambda = \pi$. *Then there is a $c > 0$ such that* $\operatorname{Re} \lambda^{\alpha} \zeta \geqq c |\lambda|^{\alpha_i}$.

Proof. Since A_i is $\zeta(t) = (1-t) d_i e^{-i\pi\alpha_{i-1}} \lambda^{\alpha_{i-1} - \alpha} + t c_i e^{-i\pi\alpha_i} \lambda^{\alpha_i - \alpha}$, $(0 \leqq t \leqq 1)$, $\operatorname{Re} \lambda^{\alpha} \zeta = (1-t) d_i |\lambda|^{\alpha_{i-1}} + t c_i |\lambda|^{\alpha_i}$. So there is a $c > 0$ with $\operatorname{Re} \lambda^{\alpha} \zeta \geqq c |\lambda|^{\alpha_i}$ for $0 \leqq t \leqq 1$.

Lemma 8.5. *Let $\zeta = \lambda^{\alpha_i - \alpha} \eta$, $\eta \notin \bar{S}_i$. Then there exist $c_\eta > 0$ and ψ_η with $|\psi_\eta - \pi| < \pi / 2\alpha_i$ such that* $\operatorname{Re} \lambda^{\alpha} \zeta \geqq c_\eta |\lambda|^{\alpha_i}$ *for λ with* $\arg \lambda = \psi_\eta$.

Proof. Set $\lambda = |\lambda| e^{i\psi}$ and $\eta = |\eta| e^{i\rho}$. Then $\operatorname{Re} \lambda^{\alpha} \zeta = |\lambda^{\alpha_i} \eta| \cos(\alpha_i \psi + \rho)$. Since $\eta \notin S_i$, $|\rho + \pi\alpha_i| < \pi - \varepsilon$ for some $\varepsilon > 0$. Hence there is a ψ_η such that $|\psi_\eta - \pi| < \pi / 2\alpha_i$ and $\cos(\alpha_i \psi_\eta + \rho) > 0$. Thus if $\arg \lambda = \psi_\eta$, $\operatorname{Re} \lambda^{\alpha} \zeta \geqq c_\eta |\lambda|^{\alpha_i}$ for a $c_\eta > 0$.

The proof of Lemma 8.5 also shows

Lemma 8.6. *Suppose that K is a compact set in $\boldsymbol{C}^1$ and $K \cap \bar{S}_i = \phi$. If the diameter of K is sufficiently small, then there are $C_K > 0$ and ψ_K with $|\psi_K - \pi| < \pi / 2\alpha_i$ such that* $\operatorname{Re} \lambda^{\alpha} \zeta \geqq C_K |\lambda|^{\alpha_i}$ *holds for $\eta \in K$ and λ with* $\arg \lambda = \psi_K$.

By Lemma 8.6, we can decompose the path $B_i'(\pm)$.

Proposition 8.7. *There are paths $B_{i,s}$ and constants $\psi_{i,s}$ with $|\psi_{i,s}-\pi|<\pi/2\alpha_i$ and $c_{i,s}>0$ $(1\leqq s\leqq l_i)$, which do not depend on λ such that*

(1) $B_i'(+)=\sum_{s=1}^{l_i'} B_{i,s}$ *and* $B_i'(-)=\sum_{s=l_i'+1}^{l_i} B_{i,s}$,

(2) $\mathrm{Re}\, \lambda^{\alpha}\zeta\geqq c_{i,s}|\lambda|^{\alpha_i}$ *for* $\zeta\in\lambda^{\alpha_i-\alpha}B_{i,s}$ *and* λ *with* $\arg\lambda=\psi_{i,s}$.

Let us decompose $K^h_{-\pi}(z, t'', \lambda)$ by using the paths A_i and $B_{i,s}$. Set

$$(8.6)\qquad K^h_{-\pi,i,0}(z, t'', \lambda)=\Big(\int_{A_i}+\int_{-e^{2\pi i}A_i}\Big)\exp(-\lambda^{\alpha}\zeta)w^h(z, t'', \lambda, \zeta)d\zeta,$$

$$(8.7)\qquad K^h_{-\pi,i,s}(z, t'', \lambda)=\int_{B_{i,s}(\lambda)}\exp(-\lambda^{\alpha}\zeta)w^h(z, t'', \lambda, \zeta)d\zeta,$$

$$B_{i,s}(\lambda)=\lambda^{\alpha_i-\alpha}B_{i,s}\qquad(1\leqq s\leqq l_i).$$

Then we have

$$(8.8)\qquad K^h_{-\pi}(z, t'', \lambda)=\sum_{i=1}^{p-1}\Big(\sum_{s=0}^{l_i}K^h_{-\pi,i,s}(z, t'', \lambda)\Big).$$

It holds for $K^h_{-\pi,i,s}(z, t'', \lambda)$ that

Proposition 8.8. *The following estimates hold*:

$$(8.9)\qquad |K^h_{-\pi,i,0}(z, t'', \lambda)|\leqq A\exp(B|\lambda|^{\alpha_{i-1}})\qquad \textit{for}\quad |\arg\lambda-\pi|<\pi/2,$$

and

$$(8.10)\qquad |K^h_{-\pi,i,0}(z, t'', \lambda)|\leqq A\exp(-c|\lambda|^{\alpha_i})\ (c>0)\quad \textit{for } \lambda \textit{ with } \arg\lambda=\pi,$$

for $1\leqq s\leqq l_i$

$$(8.11)\qquad |K^h_{-\pi,i,s}(z, t'', \lambda)|\leqq A\exp(B|\lambda|^{\alpha_i}),$$

$$(8.12)\qquad |K^h_{-\pi,i,s}(z, t'', \lambda)|\leqq A\exp(-c|\lambda|^{\alpha_i})\ (c>0)\ \textit{for } \lambda \textit{ with } \arg\lambda=\psi_{i,s}.$$

Proof. (8.9) and (8.11) are obvious. We have (8.10) by Proposition 8.4 and (8.12) by Proposition 8.7.

We remark that (8.11) is valid without the condition of the argument of λ. As we said, $(z, t'', \lambda)\in\Xi'$ are assumed. But the estimates (8.9)–(8.11) are also valid for $(z, t'', \lambda)\in\Xi$. The condition $(z, t'', \lambda)\in\Xi'$ is required to show (8.12).

(III) Decomposition of $u^h_{-\pi}(z)$. By using $K^h_{-\pi,i,s}(z, t'', \lambda)$, we divide $u^h_{-\pi}(z)$ into the sum of $u^h_{-\pi,i,s}(z)$. $K^h_{-\pi}(z, t'', \lambda)$ is single valued by Proposition 2.4, and $\hat{u}^h_{-\pi}(\lambda, t'')$ is an entire function of λ by Lemma 2.1. Hence we have from (5.10)

$$(8.13)\quad u^h_{-\pi}(z)=\int_{\Lambda_0 e^{i\pi}}^{\infty e^{i\psi}}\exp(\lambda z_0)d\lambda\int_{T''}K^h_{-\pi}(z,t'',\lambda)\hat{u}^h_{-\pi}(\lambda,t'')dt''+v^h(z)\,,$$

where $v^h(z)=\dfrac{1}{2\pi i}\displaystyle\int_{|\lambda|=\Lambda_0}\exp(\lambda z_0)\log\lambda\,d\lambda\int_{T''}K^h_{-\pi}(z,t'',\lambda)\hat{u}^h_{-\pi}(\lambda,t'')dt''+v^h_{-\pi}(z)$, $|\psi-\pi|<\pi/2$, and $|\lambda|=\Lambda_0$ is a path on the circle starting at $-\Lambda_0=\Lambda_0e^{i\pi}$ and going around once. Set

$$(8.14)\quad u^h_{-\pi,i,s}(z)=\int_{\Lambda_0 e^{i\pi}}^{\infty e^{i\psi}}\exp(\lambda z_0)d\lambda\int_{T''}K^h_{-\pi,i,s}(z,t'',\lambda)\hat{u}^h_{-\pi}(\lambda,t'')dt''\,.$$

Hence we have in $\tilde{U}(\pi)=\{z\in C^{n+1};\ 0<|z_0|\leqq r,\ |z'|\leqq r,\ |\arg z_0|<\pi\}$

$$(8.15)\qquad u(z)=\sum_{(h,i,s)}u^h_{-\pi,i,s}(z)+v(z)\,,\quad v(z)=\sum_h v^h(z)\in\mathcal{O}(U)\,,$$

and from Proposition 8.8

Proposition 8.9. *$u^h_{-\pi,i,s}(z)$ $(1\leqq s\leqq l_i)$ are holomorphically extensible to $\tilde{U}(\theta_0)$ and for any θ_1 with $0<\theta_1<\theta_0$*

$$(8.16)\qquad |u^h_{-\pi,i,s}(z)|\leqq A_{\theta_1}\exp(c_{\theta_1}|z_0|^{-\gamma_i})\quad \textit{for}\ \ z\in\tilde{U}(\theta_1)\,.$$

Proof. In the representation of $u^h_{-\pi,i,s}(z)$ ((8.14)) we can deform the path of integration in λ by (8.11). Namely we change ψ in (8.14). In doing so we have to replace $\hat{u}^h_{-\pi}(\lambda,t'')$ by $\hat{u}^h_{\theta}(\lambda,t'')$ and take another holomorphic function $w^h_\theta(z)$ (see Lemmas 2.2 and 6.12). Consequently if $\theta<\arg z_0<\theta+2\pi$ $(-\theta_0<\theta<\theta_0-2\pi)$, we have

$$(8.17)\quad u^h_{-\pi,i,s}(z)=\int_{\Lambda_0 e^{i\pi}}^{\infty e^{i\psi}}\exp(\lambda z_0)d\lambda\int_{T''}K^h_{-\pi,i,s}(z,t'',\lambda)\hat{u}^h_{\theta}(\lambda,t'')dt''+w^h_\theta(z)\,,$$

ψ being $|\theta-\psi|<\pi/2$.
Thus we get holomorphic extension of $u^h_{-\pi,i,s}(z)$ and (8.16) from (8.11).

Now we use the decay estimate (8.10) and (8.12) and obtain the asymptotic expansion of $u^h_{-\pi,i,s}(z)$. In the representation of $u^h_{-\pi,i,s}(z)$ ((8.17)) we choose $\psi=\psi_{i,s}$ in Proposition 8.8 $(\psi_{i,0}=\pi)$. Then we can show that it has the asymptotic expansion with respect to z_0. Namely,

Proposition 8.10. *$u^h_{-\pi,i,s}(z)$ $(0\leqq s\leqq l_i)$ have the asymptotic expansion with bounds with respect to z_0 in $\{z_0;\ |\arg z_0+\psi_{i,s}-\pi|<\pi/2\}$, that is, there are holomorphic functions $u^h_{-\pi,i,s,n}(z')$ $(n=0,1,\cdots)$ in $\{|z'|\leqq r,\ |z_1-\hat{z}_1|<\varepsilon_1\}$ and constants A_8 and B_8 such that*

$$(8.18)\qquad \Big|u^h_{-\pi,i,s}(z)-\sum_{n=0}^{N-1}u^h_{-\pi,i,s,n}(z')(z_0)^n\Big|\leqq A_8B_8^N|z_0|^N\Gamma(N/\gamma_i+1)$$

in $\{z;\ 0<|z_0|\leqq r,\ |\arg z_0+\psi_{i,s}-\pi|<\pi/2-\delta,\ |z'|\leqq r,\ |z_1-\hat{z}_1|<\varepsilon_1\}$.

Proof. We apply Proposition 10.10 in § 10 to $u^h_{-\pi,i,s}(z)$. The inequalities (8.10) and (8.12) imply the condition of Proposition 10.10. Hence we have (8.18).

Since $\gamma=\gamma_{p-1}\leqq\gamma_i$, we can say $u^h_{-\pi,i,s}(t^{-1}, z')$ has the γ-asymptotic expansion with respect to t in $\pi/2<-\arg t+\psi_{i,s}<3\pi/2$.

(IV) Laplace transform of $u(z)$. Now we proceed to complete Theorem 1.3. Here we use the assumption concerning the growth property of $u(z)$. Let us write it again:

For any $\varepsilon>0$ there is a $C_\varepsilon>0$ such that

$$|u(z)|\leqq C_\varepsilon \exp(\varepsilon|z_0|^{-\gamma_{p-1}}) \text{ in } \tilde{\Omega}(\theta_0), \qquad \theta_0>\pi(1/2\gamma_{p-1}+1),\ \gamma_{p-1}=\sigma_{p-1}-1\,. \tag{8.19}$$

For simplicity we denote $u(z)=u(z_0, z_1, \cdots, z_n)$ by $u(z_0)$ and γ_{p-1} by γ because other variables are not important. By setting $z_0=t^{-1/\gamma}$, we have from (8.19)

$$|u(t^{-1/\gamma})|\leqq C_\varepsilon \exp(\varepsilon|t|) \text{ in } \{t;\ |\arg t|<\gamma\theta_0,\ |t|>C^\gamma\}\,. \tag{8.20}$$

Define the γ-Laplace transform of $u(z_0)$, $v(z_0)$ and $u^h_{-\pi,i,s}(z_0)$ by

$$\begin{cases} \hat{u}(\xi)=\displaystyle\int_a^{+\infty}\exp(\xi t)u(t^{-1/\gamma})t^{-1}dt \qquad (a>C^\gamma)\,. \\ \hat{v}(\xi)=\displaystyle\int_a^{+\infty}\exp(\xi t)v(t^{-1/\gamma})t^{-1}dt \qquad (a>C^\gamma)\,. \\ \hat{u}^h_{-\pi,i,s}(\xi)=\displaystyle\int_a^{\infty e^{i\varphi}}\exp(\xi t)u^h_{-\pi,i,s}(t^{-1/\gamma})t^{-1}dt \qquad (a>C^\gamma)\,, \end{cases} \tag{8.21}$$

where $\pi/2<-\dfrac{\varphi}{\gamma}+\psi_{i,s}<3\pi/2$ (see Proposition 8.10). We have from (8.21) and Proposition 8.10.

Proposition 8.11. (a) $\hat{u}(\xi)\in\mathcal{O}\left(\widetilde{\left\{\xi;\ |\arg\xi-\pi|<\frac{\pi}{2}+\gamma\theta_0\right\}}\right)$,

(b) $\hat{u}^h_{-\pi,i,s}(\xi)\in\mathcal{O}\left(\widetilde{\left\{\xi;\ -\frac{\pi}{2}+\frac{\pi}{2}\gamma<\arg\xi-\pi+\gamma\psi_{i,s}<\frac{3}{2}\pi\gamma+\frac{\pi}{2}\right\}}\right)$,

(c) $\hat{v}(\xi)\in\mathcal{O}(\widetilde{\{0<|\xi|<\infty\}})$.

By Proposition 8.10 each $u^h_{-\pi,i,s}(t^{-1})$ has the γ-asymptotic expansion with respecte to t in $\{t\in\boldsymbol{C}^1;\ |t|>C,\ \pi/2<-\arg t+\psi_{i,s}<3\pi/2\}$ and $v(t^{-1})$ is holomorphic at $t=\infty$, $v(t^{-1})=\sum_{n=0}^{\infty}v_n(z')t^{-n}$. This gives informations of the behaviours of $\hat{u}^h_{-\pi,i,s}(\xi)$ and $\hat{v}(\xi)$ near $\xi=0$, that is, by Theorem 1.7 we have

Proposition 8.12. (a) *$\hat{u}^h_{-\pi,i,s}(\xi)$ has the holomorphic prolongation onto $\{0<|\xi|<c\}$ for some $c>0$ such that $\hat{u}^h_{-\pi,i,s}(\xi)\in\mathcal{O}(\widetilde{\{0<|\xi|<c\}})$, $|\hat{u}^h_{-\pi,i,s}(\xi)|\leqq A_\Theta|\log\xi|$ in $\{\xi;\ 0<|\xi|<c,\ |\arg\xi|<\Theta\}$ and*

$$\hat{u}^h_{-\pi,i,s}(\xi)-\hat{u}^h_{-\pi,i,s}(\xi e^{2\pi i})=\textstyle\sum_{n=0}^{\infty} u^h_{-\pi,i,s,n}(z')\xi^{n/\gamma}/\Gamma(n/\gamma+1)\,, \tag{8.22}$$

(b) *For $\hat{v}(\xi)$, $|v(\xi)|\leqq A_{\Theta,0}|\log\xi|$ in $\{\xi;\ 0<|\xi|<c,\ |\arg\xi|<\theta\}$*

$$\hat{v}(\xi)-\hat{v}(\xi e^{2\pi i}) = \textstyle\sum_{n=0}^{\infty} v_n(z')\xi^{n/\gamma}/\Gamma(n/\gamma+1)\,. \tag{8.23}$$

Now let us study the relation between $\hat{u}(\xi)$, $\hat{u}^h_{-\pi,i,s}(\xi)$ and $\hat{v}(\xi)$. For this purpose we employ a limitting method (see (8.25)). Firstly we give

Lemma 8.13. *There are $\varphi_{i,s}$ $(1\leqq i\leqq p-1,\ 0\leqq s\leqq l_i)$ such that*
(1) *$\varphi_{i,0}=0$ and $|\varphi_{i,s}|<\pi\gamma/2\gamma_i$ for $1\leqq s\leqq l_i$,*
(2) *$\pi/2<(\varphi_{i,s}/\gamma)+\psi_{i,s}<3\pi/2$,*
where $\psi_{i,0}=\pi$ and $\psi_{i,s}$ $(1\leqq s\leqq l_i)$ are those in Proposition 8.7.

Proof. Let us note $|\psi_{i,s}-\pi|<\pi/2\alpha_i$. In order that there exist $\varphi_{i,s}$ satisfying (1) and (2), it is necessary and sufficient that $3\pi/2-\psi_{i,s}>-\pi/2\gamma_i$ and $\pi/2\gamma_i>\pi/2-\psi_{i,s}$. This conditions are satisfied by $\psi_{i,s}$.

Choose $\nu_i>0$ $(i=1,2,\cdots,p-1)$ such that $|\varphi_{i,s}((\gamma_i/\gamma)+\nu_i)|<\pi/2$ for all $1\leqq s\leqq l_i$. Set

$$\delta_0 = \min\{\pi/2+\varphi_{i,s},\ \pi/2-\varphi_{i,s};\ 1\leqq i\leqq p-1,\ 1\leqq s\leqq l_i\}>0\,. \tag{8.24}$$

Let $\kappa_i>0$ $(i=1,2,\cdots,p-1)$. Then we have

$$\hat{u}(\xi) = \lim_{\kappa_{p-1}\to 0}(\lim_{\kappa_{p-2}\to 0}\cdots(\lim_{\kappa_1\to 0}\int_a^{+\infty}\exp(\xi t-\textstyle\sum_{j=1}^{p-1}\kappa_j t^{(\gamma_j/\gamma)+\nu_j})u(t^{-1/\gamma})t^{-1}dt)\cdots). \tag{8.25}$$

We have from (8.25)

Proposition 8.14. *Let $|\arg\xi-\pi|<\delta_0$. Then*

$$\hat{u}(\xi) = \textstyle\sum_{h=0}^{s_m-1}\sum_{i=1}^{p-1}\sum_{s=0}^{l_i}\hat{u}^h_{-\pi,i,s}(\xi)+\hat{v}(\xi)\,. \tag{8.26}$$

Proof. Let $s\geqq 1$. Since $|u^h_{-\pi,i,s}(t^{-1/\gamma})|\leqq A_{\theta_1}\exp(c_{\theta_1}|t|^{\gamma_i/\gamma})$ in $\{|\arg t|<\gamma\theta_1\}$ $(0<\theta_1<\theta_0)$ by Proposition 8.9, we have

$$\begin{aligned}&\bar{u}^h_{-\pi,i,s}(\xi)\\&=\lim_{\kappa_{p-1}\to 0}(\lim_{\kappa_{p-2}\to 0}(\cdots(\lim_{\kappa_i\to 0}\int_a^{\infty}\exp(\xi t-\textstyle\sum_{j=1}^{p-1}\kappa_j t^{\gamma_j/\gamma+\nu_j})u^h_{-\pi,i,s}(t^{-1/\gamma})dt)\cdots)\end{aligned}$$

$$=\lim_{\kappa_{p-1}\to 0}(\lim_{\kappa_{p-2}\to 0}(\cdots(\lim_{\kappa_i\to 0}\int_a^{\infty e^{i\varphi_{i,s}}}\exp(\xi t-\sum_{j=i}^{p-1}\kappa_j t^{\gamma_j/\gamma+\nu_j})u^h_{-\pi,i,s}(t^{-1/\gamma})t^{-1}dt\,.$$

Here we use $|\varphi_{i,s}(r_{i,s}/\gamma+\nu_i)|<\pi/2$ and $\pi/2\leqq-\delta+\varphi_{i,s}<\arg\xi-\pi+\varphi_{i,s}<\delta+\varphi_{i,s}\leqq\pi/2$. Since $u^h_{-\pi,i,s}(t^{-1/\gamma})$ is bounded on $\arg t=\varphi_{i,s}$, we have

$$\bar{u}^h_{-\pi,i,s}(\xi)=\int_a^{\infty e^{i\varphi_{i,s}}}\exp(\xi t)u^h_{-\pi,i,s}(t^{-1/\gamma})t^{-1}dt=\hat{u}^h_{\pi,i,s}(\xi)\,. \tag{8.27}$$

This means (8.26).

Combining Proposition 8.14 with Propositions 8.11 and 8.12, we have

Proposition 8.15. $\hat{u}(\xi)\in\mathcal{O}\left(\widetilde{\left\{\xi;\,|\arg\xi-\pi|<\frac{\pi}{2}+\gamma\theta_0\right\}}\right)$ *and it has the holomorphic prolongation around* $\xi=0$, *that is,* $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{0<|\xi|<c\}})$ *with* $|\hat{u}(\xi)|\leqq A_\Theta|\log\xi|$ $(|\arg\xi|<\Theta)$ *and it holds that*

$$\hat{u}(\xi)-\hat{u}(\xi e^{2\pi i})=\sum_{n=0}^{\infty}u_n(z')\xi^{n/\gamma}/\Gamma(n/\gamma+1)\,, \tag{8.28}$$

where $u_n(z')=\sum_{(h,i,s)}u^h_{-\pi,i,s,n}(z')+v_n(z')$.

It follows from Proposition 8.15, (8.20) and $\theta_0>\pi(1/2\gamma+1)$ that the conditions in Theorem 1.9 are satisfied. So $u(t^{-1})=u(t^{-1},z')$ is holomorphic at $t=\infty$. This means that $u(z)$ is holomorphic on $\{z_0=0\}$ in $\{|z|\leqq r,\ |z_1-\hat{z}_1|<\varepsilon_1\}$. Hence $\{z_0=0\}$ is removable singularity. Thus we complete the proof of Theorem 1.3.

§ 9. Miscellaneous Results-(II)

In § 9 we show lemmas and propositions used in the previous sections, but their proofs are not yet given. We give estimates of $v_p(z,t'',\lambda,\tau)$ and $h_p(z,t'',\lambda,\tau)$, and existence of $h_p(z,t'',\lambda,\tau)$ in (I)–(II). For these purposes lemmas about holomorphic functions are given in (II). By proving that $K^h_\theta(z,\lambda,\tau)$ is single valued, we complete the proof of Proposition 2.4 in (III).

(I) Estimates. We obtain estimates of $v_p(z,t'',\lambda,\tau)$ and $h_p(z,t'',\lambda,\tau)$ by the method of majorant power series. Let $A(z)=\sum A_\alpha z^\alpha$ and $B(z)=\sum B_\alpha z^\alpha$ be formal power series. Then $A(z)\ll B(z)$ means $|A_\alpha|\leqq B_\alpha$ for all multi-indices α. We state elementary properties of majorant power series without the proof, which will be often used. For the proof we refer to [3], [5] and [16].

Lemma 9.1. (*Wagschal*). *Let* $\Theta(s)$ *be a formal power series of one variable* s *such that* $\Theta(s)\gg 0$ *and*

(9.1) $$(R'-s)\Theta(s)\gg 0\,.$$

Then for derivatives, $\Theta^{(j)}(s)=(d/ds)^j\Theta(s)$ $(j=0, 1, \cdots)$ *we have*

(9.2) $$(R'-s)\Theta^{(j)}(s)\gg 0\,,\quad \Theta^{(j)}(s)\ll R'\Theta^{(j+1)}(s)$$

and

(9.3) $$(R_0-s)^{-1}\Theta^{(j)}(s)\ll(R_0-R')^{-1}\Theta^{(j)}{}_0(s)(R_0>R')\,.$$

In the following we assume $r<R'<R_0<R_1<R$, $R_1\leqq|t_i|\leqq R$ $(i\geqq 2)$, and $|\lambda|\geqq A_0$ and try to obtain estimates of holomorphic functions of z, considering λ, τ, t'' to be parameters. We set $s=z_0+z_1+\cdots+z_n$ and

(9.4) $$\theta(s) = (R'-s)^{-1}\,.$$

$\theta(s)$ satisfies the conditions in Lemma 9.1. From Proposition 6.6 we have

Lemma 9.2. *Let* $\tau\in\tau(i)$. *Then*

(9.5) $$G_0(z, \lambda, \tau)^{-1}\ll A|\lambda|^{\beta_{k_i-1}}|\tau|^{-s_{k_i-1}}(R_0-s)^{-1}\,.$$

Let us note $v_p(z, t'', \lambda, \tau)=0$ for $p\leqq h-2$ and proceed to obtain estimates of $v_p(z, t'', \lambda, \tau)$.

Lemma 9.3. *Assume for* $\tau\in\tau(1)$ *and* $j\geqq 1$

(9.6) $$v_{p-j}(z, t'', \lambda, \tau)\ll AB^{p-j}|\tau|^{-p+j-2}\theta^{(p-j+1)}(s)\,.$$

Then

(9.7) $$G_0(z, \lambda, \tau)^{-1}g_p(z, t'', \lambda, \tau)\ll ACB^{p-1}|\tau|^{-p-2}\theta^{(p+1)}(s)\,.$$

Proof. In view of proposition 6.7 and the definition of $G_j(z, \lambda, \tau, \partial_z)$ ((3.9)), we have

$$\begin{aligned}&G_0(z, \lambda, \tau)^{-1}G_j(z, \lambda, \tau, \partial_z)v_{p-j}(z, t'', \lambda, \tau)\\&\quad\ll A_1(R_0-s)^{-1}AB^{p-j}|\tau|^{-p-2}\theta^{(p+1)}(s)\ll ACB^{p-1}|\tau|^{-p-2}\theta^{(p+1)}(s)\,.\end{aligned}$$

From the definition of $g_p(z, t'', \lambda, \tau)$ (see (3.10)), we have (9.7).

Lemma 9.4. *Under the same assumptions as in Lemma* 9.3,

(9.8) $$\int_\gamma \frac{g_p(z, t'', \lambda, \tau)}{G_0(z, \lambda, \tau)}\tau^l\,d\tau\ll ACB^{p-1}\,b_0^{-p}|\lambda|^{(\alpha-1)(p+1-l)}\,\theta^{(p+1)}(s)\,.$$

Proof. We can choose the circle $|\tau|=b_0|\lambda|^{1-\alpha}$ as the integration path γ. So (9.8) follows from Lemma 9.3.

For $C_{p,l}(z_0, z'', t'', \lambda)$ defined by (4.3),

Lemma 9.5. *Under the same assumption as in Lemma* 9.3,

$$C_{p,l}(z_0, z'', t'', \lambda) \ll ACB^{p-1} b_0^{-p} |\lambda|^{(\alpha-1)(p+1-l)} \theta^{(p+1)}(s). \tag{9.9}$$

Proof. We have

$$v_{p-i}^{(i)}(z_0, 0, z'', t'', \lambda, \tau)|\tau|^{l-i} \ll ACB^{p-1}|\tau|^{-p-2+l} \theta^{(p+1)}(s).$$

Hence, we get in the same way as in Lemma 9.4

$$\int_\gamma v_{p-i}^{(i)}(z_0, 0, z'', t'', \lambda, \tau)|\tau|^{l-i} d\tau \ll ACB^{p-1}|\lambda|^{(\alpha-1)(p+1-l)} b_0^{-p} \theta^{(p+1)}(s).$$

Therefore (9.9) follows from the above estimate and Lemma 9.4.

For $h_p(z, t'', \lambda, \tau)$ determined by $v_{p-i}(z, t'', \lambda, \tau)$ $(i \geqq 1)$ we have

Lemma 9.6. *Let* $\tau \in \tau(1)$. *Then*

$$h_p(z, t'', \lambda, \tau) \ll ACB^{p-1}|\tau|^{-p-2}|\lambda|^{-\kappa(1)} \theta^{(p+1)}(s), \tag{9.10}$$

where $\kappa(1) = (\alpha-1)s_m$.

The existence of $h_p(z, t'', \lambda, \tau)$ and the proof of Lemma 9.6 will be given in (II) by using Lemma 9.5.

We can show Proposition 7.1.

Proof of Proposition 7.1. It follows from Lemma 9.3 and Lemma 9.6 that

$$\begin{aligned} v_p(z, t'', \lambda, \tau) &= \{g_p(z, t'', \lambda, \tau) + h_p(z, t'', \lambda, \tau)\}/G_0(z, \lambda, \tau) \\ &\ll ACB^{p-1}|\tau|^{-p-2} \theta^{(p+1)}(s) \ll AB^p|\tau|^{-p-2} \theta^{(p+1)}(s). \end{aligned}$$

Thus there is an r such that for $|z| \leqq r$

$$|v_p(z, t'', \lambda, \tau)| \leqq AB^p|\tau|^{-p-2}(p+1)!. \tag{9.11}$$

(II) Lemmas on holomorphic functions. We give some lemmas concerning holomorphic functions and show Lemma 9.6. In (II) we always assume that $f(\tau)$ is a holomorphic functions of one variable τ in $\{\tau \in C^1; |\tau| \leqq R\}$, $f(\tau) \neq 0$ on $|\tau| = R$ and the number of zeros of $f(\tau)$ in $\{|\tau| < R\}$ is s, the multiplicity being counted.

Lemma 9.7. *Let* $g(\tau)$ *be holomorphic on* $\{|\tau| \leqq R\}$ *such that*

$$\int_{|\tau|=R} \frac{g(\tau)}{f(\tau)} \tau^l d\tau = 0 \qquad \text{for} \quad 0 \leqq l \leqq s-1. \tag{9.12}$$

Then $g(\tau)/f(\tau)$ is holomorphic on $\{|\tau|\leqq R\}$. In particular if $g(\tau)$ is a polynomial with degree $\leqq s-1$, then $g(\tau)\equiv 0$.

Proof. Let $\tau_i (1\leqq i\leqq q)$ be the distinct zeros of $f(\tau)$ with the multiplicity δ_i, $\sum_{i=1}^q \delta_i = s$. Hence we have

$$\frac{g(\tau)}{f(\tau)} = g_1(\tau)+\sum_{i=1}^q \{\sum_{p=1}^{\delta_i} \frac{A_{i,p}}{(\tau-\tau_i)^p}\}, \tag{9.13}$$

where $g_1(\tau)$ is holomorphic on $\{|\tau|\leqq R\}$. Put

$$u(t) = \int_{|\tau|=R} \exp(t\tau)\frac{g(\tau)}{f(\tau)}\, d\tau = \sum_{i=1}^q \sum_{p=1}^{\delta_i} \frac{A_{i,p}}{(p-1)!}\, t^{p-1}\exp(t\tau_i), \tag{9.14}$$

and $f_1(\tau)=\prod_{i=1}^p(\tau-\tau_i)^{\delta_i}$. Then we have $f_1(d/dt)\, u(t)=0$ and from the assumption $(d/dt)^l u(0)=0$ for $0\leqq l\leqq s-1$. Thus it follows from the uniqueness of the Cauchy problem of ordinary differential equations that $u(t)=0$, that is, $A_{i,p}=0$ for all i and p. So $g(\tau)/f(\tau)=g_1(\tau)$ is holomorphic on $\{|\tau|\leqq R\}$. Now assume $g(\tau)=g_1(\tau)f(\tau)$ is a polynomial with degree $\leqq s-1$. $g(\tau)$ has s zeros. Hence $g(\tau)\equiv 0$.

Lemma 9.8. *For any complex numbers c_l $(0\leqq l\leqq s-1)$, there exists uniquely a polynomial $h(\tau)$ with degree $\leqq s-1$ such that*

$$\frac{1}{2\pi i}\int_{|\tau|=R}\frac{h(\tau)}{f(\tau)}\tau^l\, d\tau = c_l \qquad \text{for} \quad 0\leqq l\leqq s-1\,. \tag{9.15}$$

Proof. Put $c=(c_0, c_1, \cdots, c_{s-1})\in \boldsymbol{C}^s$. Then the linear mapping defined by (9.15), that is, from the space of all polynomials with degree $\leqq s-1$ to $\boldsymbol{C}^s$, is injective by Lemma 9.7. Since the dimensions of these linear spaces are equal, this linear mapping is surjective.

Lemma 9.9. *Let $g(\tau)$ be a holomorphic function on $\{|\tau|\leqq R\}$. Then there exists uniquely a polynomial $h(\tau)$ with degree $\leqq s-1$ such that $(g(\tau)-h(\tau))/f(\tau)$ is holomorphic on $\{|\tau|\leqq R\}$.*

Proof. Let $h(\tau)$ be a polynomial with degree $\leqq s-1$ such that

$$\int_{|\tau|=R}\frac{h(\tau)}{f(\tau)}\tau^l\, d\tau = \int_{|\tau|=R}\frac{g(\tau)}{f(\tau)}\tau^l\, d\tau \qquad \text{for} \quad 0\leqq l\leqq s-1\,,$$

whose existence and uniqueness follow from Lemma 9.8. By Lemma 9.7 $(h(\tau)-g(\tau))/f(\tau)$ is holomorphic on $\{|\tau|\leqq R\}$.

We apply Lemmas 9.7–9.9 to holomorphic functions of τ with holomorphic parameters (z, t'', λ) and we can easily show the existence of $h_p(z, t'', \lambda, \tau)$ in §4 and $h_p^i(z, t'', \lambda, \tau)$ in Proposition 7.4.

Now let us proceed to obtain the estimate of $h_p(z, t'', \lambda, \tau)$, namely, the proof of Lemma 9.6. Let $G(z, \lambda, \tau)$ be a holomorphic function of (z, λ, τ) in $X=\{(z, \lambda, \tau)\in \boldsymbol{C}^{n+1}\times\boldsymbol{C}^1\times\boldsymbol{C}^1;\ |z|\leqq R_0,\ |\tau|\leqq b|\lambda|^\delta,\ |\lambda|\geqq A_0\}$. We assume that there is a $\kappa\neq 0$ such that

$$G(z, \lambda, b\lambda^\delta \mu) = \lambda^{-\kappa}\,\hat{G}(z, \lambda, \mu)\,, \tag{9.16}$$

and $\lim_{\lambda\to\infty} \hat{G}(z, \lambda, \mu)=\hat{G}(z, \infty, \mu)$ exists uniformly in $\{(z, \mu);\ |z|\leq R_0,\ |\mu|\leq 1\}$. We also denote by $\hat{G}(z, \lambda, \mu)$ this extension to $\lambda=\infty$. We add assumptions on $\hat{G}(z, \lambda, \mu)$

$$|\hat{G}(z, \lambda, \mu)|\geqq c>0 \quad \text{on} \quad |\mu|=1\,, \tag{9.17}$$

and $\hat{G}(z, \lambda, \mu)=0$ has exactly s zeros in $\{|\mu|<1\}$ for any (z, λ). Consider the equation

$$\int_\gamma \frac{h(z, \lambda, \tau)}{G(z, \lambda, \tau)}\,\tau^l\,d\tau = C_l(z, \lambda) \qquad \text{for} \quad 0\leqq l\leqq s-1\,, \tag{9.18}$$

where γ is a circle starting at $b\lambda^\delta$ and ending at $b\lambda^\delta e^{2\pi i}$. Assume that $C_l(z, \lambda)$ is holomorphic in $\{(z, \lambda);\ |z|\leqq R_0,\ |\lambda|\geqq A_0\}$ and satisfies

$$C_l(z, \lambda)\ll AB(\lambda)\,|\lambda|^{\delta l}\,\Theta(s)\,, \tag{9.19}$$

where $\Theta(s)$ satisfying the conditions in Lemma 9.1.

Lemma 9.10. *There is a unique polynomial $h(z, \lambda, \tau)$ of τ with degree$\leqq s-1$ satisfying* (9.18) *such that for $|\tau|\leqq b|\lambda|^\delta$*

$$h(z, \lambda, \tau)\ll A_1 B(\lambda)\,|\lambda|^{-\kappa-\delta}\,\Theta(s)\,, \tag{9.20}$$

where A_1 is independent of λ and $B(\lambda)$ is that in (9.19).

Proof. Set $h(z, \lambda, \tau)=\sum_{k=0}^{s-1} A_k(z, \lambda)\,\tau^k$. Then (9.18) is equivalent to the algebraic equation

$$\sum_{k=0}^{s-1} A_k(z, \lambda)\int_\gamma \frac{\tau^{k+l}}{G(z, \lambda, \tau)}\,d\tau = C_l(z, \lambda) \qquad \text{for} \quad 0\leqq l\leqq s-1\,.$$

By putting $\tau=b\lambda^\delta\mu$, we have

$$\sum_{k=0}^{s-1} A_k(z, \lambda)\,(b\lambda^\delta)^{k+1}\int_{|\mu|=1} \frac{\mu^{k+l}}{\hat{G}(z, \lambda, \mu)}\,d\mu = C_l(z, \lambda)\,(b\lambda^\delta)^{-l}\,\lambda^{-\kappa}\,.$$

It follows from Lemma 9.8 that

$$D(z, \lambda) = \det. \left(\int_{|\mu|=1} \frac{\mu^{k+l}}{\hat{G}(z, \lambda, \mu)} d\mu, 0 \leqq k, l \leqq s-1\right) \neq 0 .$$

From (9.17) there is a $c_0 > 0$ such that $|D(z, \lambda)| \geqq c_0$ in $\{(z, \lambda); |z| \leqq R_0, |\lambda| \geqq A_0\}$. Thus $D(z, \lambda)^{-1} \ll C(R_0 - s)^{-1}$. By the Cramer's formula we have $A_k(z, \lambda)(b\lambda^\delta)^{k+1} \ll CB(\lambda)|\lambda|^{-\kappa} \Theta(s)$. Hence, if $|\tau| \leqq b|\lambda|^\delta$, we have $h(z, \lambda, \tau) = \sum_{k=0}^{s-1} A_k(z, \lambda)\tau^k \ll A_1 B(\lambda)|\lambda|^{-\kappa-\delta} \Theta(s)$.

Thus we can show Lemma 9.6.

Proof of Lemma 9.6. From Lemma 9.5, by putting $\delta = 1 - \alpha$ and $B(\lambda) = B^{p-1} b_0^{-p} |\lambda|^{-\delta(p+1)}$, we have $C_{p,l}(z_0, z'', \lambda, \tau) \ll AB(\lambda)|\lambda|^{\delta l} \theta^{(p+1)}(s)$. Hence by Lemma 9.10, by putting $G(z, \lambda, \tau) = G_0(z, \lambda, \tau)$, we have $h(z, \lambda, \tau) \ll A_1 B(\lambda) |\lambda|^{-\kappa(1)-\delta} \theta^{(p+1)}(s)$ for $|\tau| \leqq b_0 |\lambda|^\delta$. Therefore, if $|\tau| \leqq b_0|\lambda|^\delta$, we have $\tau^{p+2} h(z, \lambda, \tau) \ll AB^p |\lambda|^{-\kappa(1)} \theta^{(p+1)}(s)$. This implies (9.10).

Thus we complete the proof of Proposition 7.1. We can also show Proposition 5.1 in the same way. Because the estimates in Propositions 6.6 and 6.7 and Lemma 9.6 are valid for $\tau \in \{a_1 \leqq |\tau| \leqq b_0 |\lambda|^{1-\alpha}\}$ under the condition $A_{m,s_m}(0, \hat{\xi}') \neq 0$.

By the same method we can obtain Lemma 7.5, the estimate of $v_p^i(z, t'', \lambda, \tau)$ in $X(i)$. In that case, instead of Lemma 9.6, we adopt:

Lemma 9.11. *Let* $\tau \in \tau(i)$. *Then*

$$h_p^i(z, t'', \lambda, \tau) \ll ACB^{p-1} |\tau|^{-p-2} \lambda^{-\kappa(i)} \theta^{(p+1)}(s) , \tag{9.21}$$

where $\kappa(i) = \beta_{k_{i-1}} + (\alpha - \alpha_{i-1}) s_{k_{i-1}}$.

The proof of Lemma 9.11 is similar to that of Lemma 9.6.

(III) Single valued function $K_\theta^h(z, t'', \lambda)$. In (III) we complete the proof of Proposition 2.4, by showing that $K_\theta^h(z, t'', \lambda)$ is single valued, which is not yet proved. In the following the upper suffix * of a function of λ, for example $f^*(\lambda, \cdot)$, means it is a single valued holomorphic function of λ on $\{\lambda; |\lambda| \geqq A_0\}$. Set $v_p(\lambda, \rho) = v_p(z, t'', \lambda, \zeta, \rho\lambda^{1-\alpha})$. Then we have

Proposition 9.12. $v_p(\lambda, \rho)$ *has the form*

$$v_p(\lambda, \rho) = \lambda^{\alpha(p+2)} v_p^*(\lambda, \rho) . \tag{9.22}$$

Let us show (9.22) by induction on p. $v_k(\lambda, \rho) \equiv 0$ for $k \leqq h-2$. We need lemmas.

Lemma 9.13. *Assume $v_k(\lambda, \rho)$ has the form* (9.22) *for $k\leqq p-1$. Then*

$$G_j(z, \lambda, \tau; \partial_z)|_{\tau=\rho\lambda^{1-\alpha}} = \lambda^{\alpha(j-s_m)} G_j^*(z, \lambda, \rho; \partial_z), \tag{9.23}$$

$$g_p(\lambda, \rho) \equiv \textstyle\sum_{j=1}^m G_j(z, \lambda, \tau; \partial_z)|_{\tau=\rho\lambda^{1-\alpha}} v_{p-j}(\lambda, \rho) = \lambda^{\alpha(p+2-s_m)} g_p^*(\lambda, \rho), \tag{9.24}$$

where the coefficients of $G_j^*(z, \lambda, \rho; \partial_z)$ are single valued with respect to λ.

Proof. (9.23) and (9.24) follow from (3.9) and (3.10)

For $C_{p,l}(z_0, z'', t'', \lambda)$ defined by (4.3) we have

Lemma 9.14. *Under the same assumption as in Lemma* 9.13,

$$C_{p,l}(z_0, z'', t'', \lambda) = \lambda^{\alpha(p-l+1)} C_{p,l}^*(z_0, z'', t'', \lambda). \tag{9.25}$$

Proof. We have $v_{p-i}^{(i)}(\lambda, \rho) \lambda^{(1-\alpha)(l-i)} = \lambda^{\alpha(p+2-l)} v_{p,i}^*(\lambda, \rho)$. Hence

$$C_{p,l}(z_0, z'', t'', \lambda) = \lambda^{\alpha(p+1-l)} c_{p,l}^*(\lambda) + \delta_{p,l-1} \delta_{l,n}(2\pi i)^{-n+1} \textstyle\prod_{i=2}^n (t_i - z_i)^{-1}.$$

If $p \neq l-1$, we have $C_{p,l}(z_0, z'', t'', \lambda) = \lambda^{\alpha(p+1-l)} c_{p,l}^*(\lambda)$. If $p=l-1$, then the assertion is also valid.

Set $h_p(\tau) = h_p(z, t'', \lambda, \tau) = \sum_{k=0}^{s-1} A_{p,k}(\lambda) \tau^k$, where $s=s_m$ and $A_{p,k}(\lambda) = A_{p,k}(z, t'', \lambda)$. We have

Lemma 9.15. *Under the same assumption as in Lemma* 9.13,

$$h_p(\tau)|_{\tau=\rho\lambda^{1-\alpha}} = \lambda^{\alpha(p-s+2)} H_p^*(\lambda). \tag{9.26}$$

Proof. $h_p(\tau)$ is determined by the equation

$$\int_\gamma \frac{h_p(\tau)}{G_0(\tau)} \tau^l \, d\tau = C_{p,l}(\lambda).$$

Since

$$\int_\gamma \frac{\tau^{k+l}}{G_0(\tau)} d\tau = \lambda^{(1-\alpha)(k+l-s+1)} \int_{\gamma(b)} \frac{\rho^{k+l}}{G_0^*(\lambda, \rho)} d\rho = \lambda^{-\alpha(k+l-s+1)} H_{k,l}^*(\lambda),$$

the equation becomes $\sum_{k=0}^{s-1} A_{p,k}(\lambda) \lambda^{-\alpha(k+l-s+1)} H_{k,l}^*(\lambda) = \lambda^{\alpha(p-l+1)} C_{p,l}^*(\lambda)$. Put $A_{p,k}(\lambda) = \lambda^{\alpha(p+k-s+2)} A'_{p,k}(\lambda)$. Then $\sum_{k=0}^{s-1} A'_{p,k}(\lambda) H_{k,l}^*(\lambda) = C_{p,l}^*(\lambda)$. Hence $A'_{p,k}(\lambda)$ is a single valued holomorphic function of λ. Thus we have (9.26).

Proof of Proposition 9.12. We have (9.22) from Lemmas 9.13 and 9.15 and $v_p(z, t'', \lambda, \tau) = \{g_p(z, t'', \lambda, \tau) + h_p(z, t'', \lambda, \tau)\}/G_0(z, \lambda, \tau)$.

Set

$$I_p(\lambda) = \int_{C(\theta)} \exp(-\lambda^\alpha \zeta) \, d\zeta \int_{\gamma(b\lambda^{1-\alpha})} v_p(\lambda, \tau) f_p(\zeta + \tau z_1) \, d\tau. \tag{9.27}$$

For $C(\theta)=C(de^{i\theta}\lambda^{1-\alpha})$ see §2–III. Then

Lemma 9.16. *$I_p(\lambda)$ is a single valued holomorphic function on $\{\lambda;\ |\lambda|\geqq\Lambda_0\}$.*

Proof. Put $\tau=\rho\lambda^{1-\alpha}$. Then

$$(9.28)\quad I_p(\lambda)=\int_{C(\theta)}\exp(-\lambda^\alpha\zeta)\,d\zeta\int_{\gamma(b)}v_p(\lambda,\rho\lambda^{1-\alpha})f_p(\zeta+\rho\lambda^{1-\alpha}z_1)\,\lambda^{1-\alpha}\,d\rho\,.$$
$$=\int_{\gamma(b)}v_p(\lambda,\rho\lambda^{1-\alpha})\,\lambda^{1-\alpha}\,d\rho\int_{C(\theta)}\exp(-\lambda^\alpha\zeta)f_p(\zeta+\rho\lambda^{1-\alpha}z_1)\,d\zeta$$
$$=\int_{\gamma(b)}v_p(\lambda,\rho\lambda^{1-\alpha})\,\lambda^{1-\alpha}\,d\rho\int_{-\rho\lambda^{1-\alpha}z_1}^{de^{i\theta}\lambda^{1-\alpha}}\exp(-\lambda^\alpha\zeta)\{(\zeta+\rho\lambda^{1-\alpha}z_1)^p/p!\}\,d\zeta\,.$$

Since

$$\int_{-\rho\lambda^{1-\alpha}z_1}^{de^{i\theta}\lambda^{1-\alpha}}\exp(-\lambda^\alpha\zeta)\{(\zeta+\rho\lambda^{1-\alpha}z_1)^p/p!\}\,d\zeta$$
$$=\lambda^{(1-\alpha)(p+1)}\exp(\lambda\rho z_1)\int_0^{de^{i\theta}+\rho z_1}\exp(-\lambda\eta)\,\eta^p/p!\,d\eta\,,$$

we have

$$I_p(\lambda)=\int_{\gamma(b)}v_p(\lambda,\rho\lambda^{1-\alpha})\,\lambda^{(1-\alpha)(p+2)}\exp(\lambda\rho z_1)\,d\rho\int_0^{de^{i\theta}+\rho z_1}\exp(-\lambda\eta)\,\eta^p/p!\,d\eta\,.$$

This means that $I_p(\lambda)$ is single valued.

From Proposition 9.12 $K_\theta^h(z,t'',\lambda)=\sum_{p=h-1}^{+\infty}I_p(\lambda)$ is also single valued with respect to λ.

§ 10. Function with Asymptotic Expansion

In §10 we consider functions with asymptotic expansions and give the proofs of Theorems 1.7 and 1.9. Let $u(t)$ be a continuous function on $[A,+\infty)$ $(A>0)$ such that $|u(t)|\leqq C\exp(B|t|^\gamma)$ $(\gamma>0)$. We have defined the γ-Laplace transform $\hat{u}(\xi)$ of $u(t)$ by (1.12) and its inversion formula by (1.13) in §1. Let us recall the notation $\tilde{S}(a,b)=\{t\in C^1;\ |t|\geqq A,\ a<\arg t<b\}$ $(A>0)$ and $\tilde{S}(a)=\tilde{S}(-a,a)$ $(a>0)$.

Now suppose that $u(t)$ has the γ-asymptotic expansion (1.14),

$$(10.1)\qquad u(t)\sim\sum_{k=0}^{+\infty}c_k\,t^{-k}\quad\text{on}\quad[A,+\infty)\,.$$

We have $|c_N|\leqq A_1R^{-N}\,\Gamma(N/\gamma+1)$ (see Definition 1.6). By using the sequence $\{c_k\}$ $(k=0,1,\cdots)$, define

$$(10.2)\qquad g(z)=\sum_{k=0}^{+\infty}\frac{c_k\,z^k}{\Gamma(k/\gamma+1)}$$

and

(10.3) $$g_n(z) = \sum_{k=0}^{n} \frac{c_k z^k}{\Gamma(k/\gamma+1)}.$$

We have

Lemma 10.1. (i) *$g(z)$ is holomorphic in $\{|z|<R\}$ and $|g(z)|\leq A_1 (1-|z/R|)^{-1}$.*
(ii) *Let $0<r<R$. Then $|g_n(z)|\leqq A_r|z/r|^{n+1}$ for $|z|\geqq r$ and $|g(z)-g_n(z)|\leqq A_r |z/R|^{n+1}$ for $|z|\leqq r$.*

Proof. It follows from the estimates of c_k that $g(z)$ is holomorphic in $\{|z|<R\}$ and $|g(z)|\leq A_1(1-|z/R|)^{-1}$. Since $|g_n(z)|\leqq A_1 \sum_{k=0}^{n}|z/R|^k$, if $|z|\geqq r$, $|g_n(z)|\leqq A_r|z/r|^{n+1}$. For $|z|\leq r$, $|g(z)-g_n(z)|\leqq A_1 \sum_{k=n+1}^{\infty}|z|^k R^{-k} \leqq A_r|z/R|^{n+1}$.

Set

(10.4) $$v(t) = t^{\gamma}\int_0^c \exp(-t^{\gamma} z)\, g(z^{1/\gamma})\, dz, \quad 0<c^{1/\gamma}<R.$$

Proposition 10.2. (i) $v(t)\in\mathcal{O}(\widetilde{C^1-\{0\}})$.
(ii) *$v(t)$ has the γ-asymptotic expansion as $t\to\infty$ in $\tilde{S}(\pi/2\gamma)$, that is, there is an $A(c)$ such that for $t\in\tilde{S}(\pi/2\gamma)$, $t=|t|\, e^{i\varphi}$,*

(10.5) $$|v(t)-\sum_{k=0}^{N-1} c_k t^{-k}|\leqq A(c)\, c^{-N/\gamma}(\cos(\gamma\varphi))^{-N/\gamma-1}\,\Gamma(N/\gamma+1)|t|^{-N}$$

holds for each N.

Proof. Obviously $v(t)\in\mathcal{O}(\widetilde{C^1-\{0\}})$. Let us show $v(t)$ has the asymptotic expansion (10.5). We have

(10.6) $$\begin{aligned} v(t) &= t^{\gamma}\int_0^{+\infty} \exp(-t^{\gamma} z)\, g_{N-1}(z^{1/\gamma})\, dz \\ &\quad + t^{\gamma}\int_0^c \exp(-t^{\gamma} z)\, \{g(z^{1/\gamma})-g_{N-1}(z^{1/\gamma})\}\, dz \\ &\quad - t^{\gamma}\int_c^{+\infty} \exp(-t^{\gamma} z)\, g_{N-1}(z^{1/\gamma})\, dz. \end{aligned}$$

By a simple calculation we have $t^{\gamma}\int_0^{+\infty} \exp(-t^{\gamma} z)\, g_{N-1}(z^{1/\gamma})\, dz=\sum_{k=0}^{N-1} c_k t^{-k}$. From (ii) in Lemma 10.1, we have

$$\begin{aligned} &|t^{\gamma}\int_0^c \exp(-t^{\gamma} z)\, \{g(z^{1/\gamma})-g_{N-1}(z^{1/\gamma})\}\, dz| \\ &\leqq A_c R^{-N}(\cos(\gamma\varphi))^{-N/\gamma-1}\,\Gamma(N/\gamma+1)|t|^{-N}. \end{aligned}$$

For the third term in the right hand side in (10.6),

$$|t^\gamma \int_c^{+\infty} \exp(-t^\gamma z)\, g_{N-1}(z^{1/\gamma})\, dz|$$
$$\leqq A|t^\gamma| \int_c^{+\infty} |\exp(-t^\gamma z)|\, |\frac{z}{c}|^{N/\gamma} |dz| \leqq \frac{A}{\cos(\gamma\varphi)} \frac{\Gamma(N/\gamma+1)}{(c^{1/\gamma}|t|\cos(\gamma\varphi)^{1/\gamma})^N}.$$

Thus we have

$$(10.7)\quad |v(t)-\textstyle\sum_{k=0}^{N-1} c_k t^{-k}| \leqq A(c)\, c^{-N/\gamma} (\cos(\gamma\varphi))^{-N/\gamma-1}\, \Gamma(N/\gamma+1)\, |t|^{-N}.$$

Set $w(t)=u(t)-v(t)$. From the assumption on $u(t)$ and Proposition 10.2 $w(t)\sim 0$ as $t\to+\infty$ on the positive real axis. More precisely $|w(t)|\leqq A(c)$ $c^{-n/\gamma}\,\Gamma(n/\gamma+1)|t|^{-n}$ for each n. This implies

Lemma 10.3. *The estimate* $|w(t)|\leqq C(ct^\gamma)^{1/2}\exp(-ct^\gamma)$ *holds for* $A\leqq t<+\infty$, *where* C *depends only on* c.

Proof. By Stirling's formula, we have

$$|w(t)|\leqq B(c)|c^{1/\gamma} t|^{-n} \exp(-n/\gamma)\,(n/\gamma)^{(n/\gamma)+1/2}.$$

So, if $n/\gamma\leqq ct^\gamma\leqq(n+1)/\gamma$, $|w(t)|\leqq C(ct^\gamma)^{1/2}\exp(-ct^\gamma)$.

Let us investigate the γ-Laplace transform $\hat{u}(\xi)$ of $u(t)$ with the asymptotic expansion (10.1). Since $\hat{u}(\xi)=\hat{v}(\xi)+\hat{w}(\xi)$, we study $\hat{v}(\xi)$ and $\hat{w}(\xi)$. By Lemma 10.3 we have

Lemma 10.4. $\hat{w}(\xi)\in\mathcal{O}(\{\xi;\ \mathrm{Re}\,\xi<c\})$.

On the other hand $\hat{v}(\xi)$ is represented as follows:

$$(10.8)\quad \hat{v}(\xi)=\int_a^{+\infty}\exp(\xi t)\,dt\int_0^c \exp(-tz)\,g(z^{1/\gamma})\,dz=\int_0^c \frac{e^{-a(z-\xi)}\,g(z^{1/\gamma})}{z-\xi}\,dz.$$

Hence we have

Proposition 10.5. (i) $\hat{v}(\xi)\in\mathcal{O}(\{\xi;\ \xi\in C^1-[0,c]\})$.

(ii) $\hat{v}(\xi)\in\widetilde{\mathcal{O}}(\{\xi\in C^1;\ 0<|\xi|<c\})$ and

$$(10.9)\quad \{\hat{v}(\xi)-\hat{v}(\xi e^{2\pi i})\}/2\pi i=g(\xi^{1/\gamma}),\ \xi^{1/\gamma}=|\xi|^{1/\gamma}e^{i(\arg\xi)/\gamma}.$$

Proof. From (10.8) we have (i). Since $g(z)$ is holomorphic in $\{|z|<R\}$, by deforming the integration path, we have the first assertion in (ii). Let $0<\xi<c$. Then, considering the holomorphic extension of $\hat{v}(\xi)$, we have

$$\{\hat{v}(\xi)-\hat{v}(\xi e^{2\pi i})\}/2\pi i=\frac{1}{2\pi i}\int_Z \frac{e^{-a(z-\xi)}\,g(z^{1/\gamma})}{z-\xi}\,dz=g(\xi^{1/\gamma}),$$

where $Z=\{z(s);\ 0\leqq s\leqq 1\}$ is a piecewise smooth contour such that $z(0)=z(1)=0$, $z(1/2)=c$, $\mathrm{Im}\, z(s)<0$ for $0<s<1/2$ and $\mathrm{Im}\, z(s)>0$ for $1/2<s<1$ (see Fig. 10.1).

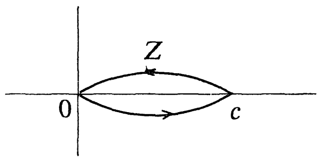

Fig. 10.1.

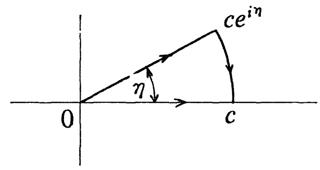

Fig. 10.2.

Remark 10.6. $\hat{v}(\xi)$ is represented for $\eta<\arg\xi<\eta+2\pi$ in the form,

$$(10.10)\qquad \begin{cases} \hat{v}(\xi)=\hat{v}_\eta(\xi)+\text{a holomorphic function at } \xi=0\,, \\ \hat{v}_\eta(\xi)=\displaystyle\int_0^{ce^{i\eta}} \frac{e^{-a(z-\xi)}\, g(z^{1/\gamma})}{z-\xi}\, dz\,. \end{cases}$$

This follows from the deformation of the integration path. The holomorphic part of $\hat{v}(\xi)$ in (10.10) corresponds to the integration from $ce^{i\eta}$ to c in Fig. 10.2. Set

$$(10.11)\qquad v_\eta(t)=t^\gamma\int_0^{ce^{i\eta}} \exp(-t^\gamma z)\, g(z^{1/\gamma})\, dz, \quad 0<c^{1/\gamma}<R\,.$$

Then $v_\eta(t)$ has the γ-asymptotic expansion

$$(10.12)\qquad v_\eta(t)\sim \textstyle\sum_{k=0}^{+\infty} c_k\, t^{-k} \quad \text{in} \quad \tilde{S}(-(\pi/2+\eta)/\gamma,\ (\pi/2-\eta)/\gamma)$$

and its γ-Laplace transform is $\hat{v}_\eta(\xi)$,

$$(10.13)\qquad \hat{v}_\eta(\xi)=\int_a^{\infty e^{-i\eta}} \exp(\xi\, t)\, v_\eta(t^{1/\gamma})\, dt\,.$$

Thus we obtain for $\hat{u}(\xi)=\hat{v}(\xi)+\hat{w}(\xi)$, the γ-Laplace transform of $u(t)$ with the asymptotic expansion (10.1),

Proposition 10.7. (i) *$\hat{u}(\xi)$ is holomorphic in $\{\xi;\ \mathrm{Re}\,\xi<c,\ \xi\notin[0, c)\}$.*
(ii) *$\hat{u}(\xi)$ can be holomorphically extended into $\{\xi;\ 0<|\xi|<c\}$ such that $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ 0<|\xi|<c\}})$ and $\{\hat{u}(\xi)-\hat{u}(\xi\, e^{2\pi i})\}/2\pi i=g(\xi^{1/\gamma})$.*

Next consider holomorphic functions on a sector. We have

Proposition 10.8. *Let $u(t)\in\mathcal{O}(\tilde{S}(\theta_0))$ and suppose that for any $\varepsilon>0$ there is a $C_\varepsilon>0$ such that*

$$(10.14)\qquad |u(t)|\leqq C_\varepsilon \exp(\varepsilon|t|^\gamma) \quad (\gamma>0)\,.$$

Then the followings hold:

(i) $\hat{u}(\xi)\in\mathcal{O}(\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2\})$.

(ii) *For any* δ *and* $\varepsilon>0$ $|\hat{u}(\xi)|\leqq M_{\varepsilon,\delta}\exp(a|\xi|)$ *in* $\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2-\delta,\ |\xi|>\varepsilon\}$.

(iii) *If* $u(t)$ *is bounded on any closed subsector in* $\tilde{S}(\theta_0)$, *then* $|\hat{u}(\xi)|\leqq C_{c,\delta}|\log\xi|$ *in* $\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2-\delta,\ 0<|\xi|<c\}$.

Proof. By deforming the integration path in (1.12) in §1 we have (i) and the estimate in (ii). Let us show (iii). For $\xi\in\{\xi;\ |\arg\xi-\pi|<\gamma\theta_0+\pi/2-\delta\}$ we can choose ω such that $|\arg\xi+\omega-\pi|<\pi/2-\delta/2$ and $|\omega|<\gamma\theta_0-\delta/2$. Putting $\arg t=\omega$, we have

$$|\hat{u}(\xi)|\leqq M_\delta\left(\int_a^{ae^{i\omega}}+\int_{ae^{i\omega}}^{\infty e^{i\omega}}\right)|\exp(\xi\, t)\,t^{-1}|\,|dt|\leqq$$

$$M_\delta\{(\exp(a|\xi|)+\int_{ae^{i\omega}}^{\infty e^{i\omega}}\exp(-c_\delta|\xi\, t|)|t|^{-1}|dt|)\leqq C_{c,\delta}|\log\xi|\ .$$

For $\hat{v}(\xi)$ defined by (10.8), the γ-Laplace transform of $v(t)$, we have

Corollary 10.9. *For any* $\Theta>0$ *there is a* M_Θ *such that*

$$|\hat{v}(\xi)|\leqq M_\Theta|\log\xi|\quad in\quad \{\xi;\ |\arg\xi|<\Theta,\ 0<|\xi|<c/2\}\ . \tag{10.15}$$

Proof. $v_\eta(t)$ in Remark 10.6 has the γ-asymptotic expansion in $\tilde{S}(-(\pi/2+\eta)/\gamma,(\pi/2-\eta)/\gamma)$. The difference $\hat{v}(\xi)-\hat{v}_\eta(\xi)$ is holomorphic at $\xi=0$. So the assertion follows from Proposition 10.8–(iii).

Proof of Theorem 1.7. We can choose c $(0<c<R^\gamma)$ in Proposition 10.7 as close to R^γ as possible (see (10.4)). So $\hat{u}(\xi)$ is holomorphic in $\{\xi;\ \mathrm{Re}\,\xi<R^\gamma,\ \xi\notin[0,R^\gamma)\}$, $\hat{u}(\xi)\in\mathcal{O}(\widetilde{\{\xi;\ 0<|\xi|<R^\gamma\}})$ and $\{\hat{u}(\xi)-\hat{u}(\xi\, e^{2\pi i})\}/2\pi i=g(\xi^{1/\gamma})$. From Corollary 10.9 $\hat{u}(\xi)$ has the logarithmic growth at $\xi=0$.

Proof of Theorem 1.9. The only if part follows from Theorem 1.7. We show the conditions in Theorem 1.9 are sufficient. From the assumption, $\hat{u}(\xi)$ has at most the logarithmic growth at $\xi=0$. So by the deformation of the integration path (see Fig. 10.3) the inverse transform is given by

$$\begin{aligned} u(t) &= \frac{t^\gamma}{2\pi i}\left(\int_{\infty e^{i(\pi+\varphi)}}^0+\int_0^{\infty e^{i(\pi-\varphi)}}\right)\exp(-\xi\, t^\gamma)\,\hat{u}(\xi)\,d\xi \\ &= \frac{t^\gamma}{2\pi i}\left(\int_{ce^{2\pi i}}^0+\int_0^c\right)\exp(-\xi\, t^\gamma)\,\hat{u}(\xi)\,d\xi+s_\varphi(t) \\ &= t^\gamma\int_0^c\exp(-\xi\, t^\gamma)\,F(\xi)\,d\xi+s_\varphi(t)\ , \end{aligned} \tag{10.16}$$

where

$$(10.17)\qquad \begin{cases} F(\xi) = \hat{u}(\xi) - \hat{u}(\xi\, e^{2\pi i}), \\ s_\varphi(t) = \dfrac{t^\gamma}{2\pi i}\left(\displaystyle\int_{\infty e^{i(\pi+\varphi)}}^{ce^{2\pi i}} + \int_c^{\infty e^{i(\pi-\varphi)}}\right) \exp(-\xi\, t^\gamma)\, \hat{u}(\xi)\, d\xi \end{cases}$$

and φ is a constant with $\pi/2 < \varphi < \gamma\theta_0 + \pi/2$ and $\varphi \leqq \pi$

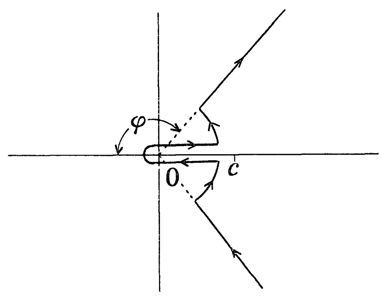

Fig. 10.3.

For $t \in \tilde{S}(\pi/2\gamma)$ we have the γ-asymptotic expansion

$$t^\gamma \int_0^c \exp(-\xi\, t^\gamma)\, F(\xi)\, d\xi \sim \textstyle\sum_{k=0}^{+\infty} c_k\, t^{-k}$$

and for any $\delta > 0$ in $\{t;\ |t| \geqq A',\ \pi/2 - \varphi + \delta < \gamma \arg t < -\pi/2 + \varphi - \delta\}$

$$|s_\varphi(t)| \leqq A_{\varphi,\delta} \exp(-c_\delta |t|^\gamma) \quad (c_\delta > 0)\,.$$

Hence if $\theta_0 \leqq \pi/2\gamma$, since we can choose φ arbitrarily in $(\pi/2,\ \gamma\theta_0 + \pi/2)$, $u(t)$ has the expansion (1.20). If $\theta_0 > \pi/2\gamma$, we can put $\varphi = \pi$. We also have, choosing ω with $|\omega + \gamma \arg t| < \pi/2$ and $|\omega| < \gamma\theta_0 - \pi/2$,

$$(10.18)\qquad u(t) = t^\gamma \int_0^{\infty e^{i\omega}} \exp(-\xi\, t^\gamma)\, F(\xi)\, d\xi\,.$$

Hence we get also the γ-asymptotic expansion (1.20). Further assume $\theta_0 > \pi/2\gamma + \pi$. Then this means $u(te^{\pi i}) - u(te^{-\pi i})$ ($|\arg t| < \theta_0 - \pi$) has the zero γ-asymptotic expansion at $t = \infty$ in $\tilde{S}(\theta_0 - \pi)$, that is, for $z \in S(\theta_1)$ $(0 < \theta_1 < \theta_0 - \pi)$

$$|u(te^{\pi i}) - u(te^{-\pi i})| \leqq AB^{-N}\, \Gamma(N/\gamma + 1)\, |t|^{-N} \qquad \text{for any} \quad N\,.$$

This implies that there is a $C = C(\theta_1) > 0$ such that for $t \in \tilde{S}(\theta_1)$

$$(10.19)\qquad |u(te^{\pi i}) - u(te^{-\pi i})| \leqq A \exp(-C|t|^\gamma)\,.$$

Since $\theta_0 - \pi > \pi/2\gamma$, $u(te^{\pi i}) - u(te^{-\pi i}) \equiv 0$. Thus $u(t)$ is single valued in $\{t;\ |t| >$

$A\}$ and bounded at $t=\infty$. Hence $u(t)$ is holomorphic at $t=\infty$. This completes the proof of Theorem 1.9.

Finally let us give

Proposition 10.10. *Let $K(z, \lambda)$ be a continuous function defined on $\{(z, \lambda)\in \boldsymbol{C}^{n+1}\times\boldsymbol{C};\ |z|\leqq r, \lambda=\lambda e^{i\psi}, |\lambda|\geqq A\}$, which is holomorphic in z and fulfills $|K(z, \lambda)|\leqq A \exp(-c|\lambda|^{\alpha})$ $(c>0, 0<\alpha<1)$. Then*

$$k(z)=\int_{Ae^{i\psi}}^{\infty e^{i\psi}} \exp(\lambda z_0)\, K(z, \lambda)\, d\lambda \tag{10.20}$$

has the asymptotic expansion with respect to z_0 in $U=\{|z|\leqq r; \pi/2<\arg z_0+\psi<3\pi/2\}$, that is,

$$|k(z)-\textstyle\sum_{n=0}^{N-1} k_n(z')\,(z_0)^n/n!|\leqq AB^N\,\Gamma(N/\gamma+1)\,|z_0|^N, \tag{10.21}$$

where $k_n(z')=\lim_{z_0\to 0\ z\in U}(\partial/\partial z_0)^n k(z)$ and $\gamma=\alpha/(1-\alpha)$

Proof. We have

$$(\partial/\partial z_0)^n k(z)=\int_{Ae^{i\psi}}^{\infty e^{i\psi}} \textstyle\sum_{k=0}^{n}\binom{n}{k}\lambda^k \exp(\lambda z_0)\,(\partial/\partial z_0)^{n-k} K(z, \lambda)\, d\lambda\,.$$

Hence if $\pi/2<\arg z_0+\psi<3\pi/2$, we have

$$|(\partial/\partial z_0)^n k(z)|\leqq A\textstyle\sum_{k=0}^{n}\int_{Ae^{i\psi}}^{\infty e^{i\psi}} (n!/k!)\, r^{k-n}|\lambda^k|\exp(-c|\lambda|^{\alpha})\,|d\lambda|$$
$$\leqq AB^n\,\Gamma\left(\frac{n+1}{\alpha}\right).$$

By the Taylor's formula, we have

$$|k(z)-\textstyle\sum_{n=0}^{N-1} k_n(z')\,(z_0)^n/n!|\leqq AB^N\dfrac{\Gamma(N+1/\alpha)}{N!}|z_0|^N\leqq AB^N\,\Gamma(N/\gamma+1)\,|z_0|^N,$$

where $k_n(z')=\lim_{z_0\to 0\ z\in U}(\partial/\partial z_0)^n k(z)$ and $\gamma=\alpha/(1-\alpha)$.

We can say for $k(z)$ defined by (10.20) that $k(t^{-1}, z')$ has the γ-asymptotic expansion with respect to t in $\pi/2<-\arg t+\psi<3\pi/2$.

References

[1] De Paris, J.C. Problème de Cauchy analytique à données singulières pour un opérateur différentiel bien decomposable, *J. Math. Pures Appl.*, **51** (1972), 465–488.

[2] Hamada, Y., The singularities of solutions of Cauchy problem, *Publ. RIMS, Kyoto Univ.*, **5** (1969), 21–40.

[3] Hamada, Y., Leray, J. et C. Wagschal, Systèmes d'équations aux derivées partielles à caractéristiques multiples; problème de Cauchy ramifié; hyperbolicité partielle, *J. Math. Pures Appl.*, **55** (1976), 297–352.

[4] Kashiwara, M. and Schapira, P., Problème de Cauchy pour les systèmes microdifférentiels dans le domain complexe, *Inv. Math.*, **46** (1978), 17–38.

[5] Komatsu, H., Irregularity of characteristic elements and construction of null solutions, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.*, **23** (1976), 297–342.

[6] Ōuchi, S., Asymptotic behavior of singular solutions of linear partial differential equations in the complex domain, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.*, **27** (1980), 1–36.

[7] ———, An integral representation of singular solutions of linear partial differential equations in the complex domain, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.*, **27** (1980), 37–85.

[8] ———, Characteristic Cauchy problems and solution of formal power series, *Ann. Inst. Fourier*, **33** (1983), 131–176.

[9] ———, Characteristic indices and subcharacteristic indices of surfaces for linear partial differential operators, *Proc. Jap. Acad. Ser.*, **57A** (1981), 481–484.

[10] ———, Index, localization and classification of characteristic surfaces for linear partial differential operators, *Proc. Jap. Acad.*, **60A** (1984), 189–192.

[11] ———, Existence of singular solutions and null solutions for linear partial differential operators, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.*, **32** (1985), 457–498.

[12] ———, Solutions with singularities on a surface of linear partial differential equations, *Proc. Hyperbolic equations and related topics, Taniguchi Symp.* 1984, Kinokuniya Tokyo, 307–316.

[13] Persson, J., On the Cauchy problem in $\boldsymbol{C}^n$ with singular data, *Mathematiche*, **30** (1975), 339–362.

[14] ———, Singular holomorphic solutions of linear partial differential equations with holomorphic coefficients and nonanalytic solutions of equations with analytic coefficients, *Astérisque* **89–90**, Soc. Math. France, (1981) 223–247.

[15] Ramis, J.P., Dévissage Gevrey, *Astérisque* **59–60**, Soc. Math. France, (1978) 173–204.

[16] Wagschal, C., Problème de Cauchy analytique à données méromorphes, *J. Math. Pures Appl.*, **51** (1972), 375–397.