

The Moyal Product and Spectral Theory for a Class of Infinite Dimensional Matrices

By

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Abstract

We study tempered distributions that are multipliers of the Schwartz space relative to the Moyal product. They form an algebra N under the Moyal product containing the polynomials. The elements of N are represented as infinite dimensional matrices with certain growth properties of the entries. The representation transforms the Moyal product into matrix multiplication. Each real element of N allows a resolvent map with values in tempered distributions and an associated spectral resolution. This gives a tool to study distributions associated with symmetric, but not necessarily self-adjoint operators.

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§1. Introduction and Main Results

Moyal considered in 1949 the problem of describing quantum mechanics in a semi-classical setting by making use of such classical notions as phase space and Hamiltonian function. The purpose was to gain a better understanding of

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the relationship between classical and quantized systems. In particular, Moyal aimed at understanding inherent quantum mechanical behaviour in terms of classical concepts. He showed that the transition probabilities associated with a quantum system can be calculated from the classical Hamiltonian by integrating with respect to a certain measure on phase space. The measure is not positive, but only real. Nevertheless, it assigns the correct positive probabilities to the quantum system.

Moyal's discovery has inspired many authors. The real measure on phase space induces a non-commutative product of functions on phase space, now known as the Moyal product. The associated left regular representation reduced by a certain ideal is equivalent to the Weyl mapping.

The Moyal product is a well-defined composition in both the Schwartz space and the space of square-integrable functions. These algebras, however, are far too small to contain observables of physical interest and the associated spectral theory just becomes a reformulation of the Weyl formalism.

In order to better exploit Moyal's basic idea, we must identify algebras under the Moyal product which are large enough to contain all conceivable Hamiltonians. The Moyal product of tempered distributions on phase space may fail to exist in any reasonable sense of the word and even when it does exist, the product may not be associative.

However, the Moyal product of a tempered distribution with a function in the Schwartz space does always exist as a tempered distribution. We can therefore consider the set N of left-multipliers of the Schwartz space. It is an (associative) algebra under the Moyal product and contains the polynomials together with many other distributions of physical interest.

We show that N can be realized as an algebra of infinite-dimensional matrices satisfying certain growth conditions on the entries. This representation induces a natural topology on N making it a reflexive, nuclear space. It is the dual space of a certain subalgebra N_* that also can be characterized by growth conditions on the matrix entries. We show that the algebra obtained by adjoining the unit element to N_* is invariant under (Moyal) exponentiation of even non-real elements.

A real, tempered distribution T is said to be positive, if $T(\xi \circ \bar{\xi}) \geq 0$ for each element ξ of the Schwartz space (the Moyal product is denoted by \circ). The associated order structure is an important tool in the construction of the spectral theory.

Each real element of N allows a resolvent mapping defined on complex numbers with non-vanishing imaginary part and with values in bounded tempered distributions. This gives a tool to study distributions associated with symmetric, but not necessarily self-adjoint operators.

The resolvent distribution allows a spectral decomposition with respect to a positive measure on the real line with values in positive, idempotent, tempered

distributions. We construct finally for each real element of N a functional calculus for bounded and measurable functions.

§2. The Moyal Product and Other Preliminaries

Symplectic forms and complex structures.

We shall consider the even-dimensional real vector space $\mathbf{E} = \mathbf{R}^{2d}$, $d \in \mathbf{N}$. A symplectic form on \mathbf{E} is a bilinear mapping $\sigma: \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{R}$ which is non-degenerate and antisymmetric. The pair (\mathbf{E}, σ) is an example of a symplectic space. A σ -allowable complex structure on (\mathbf{E}, σ) is a linear operator J on \mathbf{E} such that

$$\begin{aligned}
 J^2 &= -1, \\
 \sigma(Ja, Jb) &= \sigma(a, b) \quad \forall a, b \in \mathbf{E}, \\
 \sigma(a, Ja) &\geq 0.
 \end{aligned}
 \tag{2.1}$$

Relative to this complex structure, one can introduce a real, positive definite and symmetric bilinear form s on \mathbf{E} by setting

$$s(a, b) = \sigma(a, Jb). \tag{2.2}$$

Finally, a positive definite inner product $h: \mathbf{E} \times \mathbf{E} \rightarrow \mathbf{C}$ is defined by setting

$$h(a, b) = s(a, b) + i\sigma(a, b). \tag{2.3}$$

A linearly independent set $\{e_i, f_i\}_{i=1}^d$ of vectors in \mathbf{E} satisfying

$$\begin{aligned}
 \sigma(e_i, e_j) &= \sigma(f_i, f_j) = 0 \\
 \sigma(e_i, f_j) &= \delta_{i,j},
 \end{aligned}
 \tag{2.4}$$

$i, j = 1, \dots, d$

is called a symplectic basis for (\mathbf{E}, σ) . If we to a given symplectic basis $\{e_i, f_i\}_{i=1}^d$ define a linear operator J on \mathbf{E} such that

$$\begin{aligned}
 J e_i &= f_i, \\
 J f_i &= -e_i,
 \end{aligned}
 \tag{2.5}$$

$i, j = 1, \dots, d$

then it follows that J is a σ -allowable complex structure on (\mathbf{E}, σ) . Conversely, suppose that such a structure J is given. If we choose a set of vectors $\{e_i\}_{i=1}^d$ orthonormal with respect to the inner product h defined in (2.3) and put $f_i = J e_i$ for $i = 1, \dots, d$, then $\{e_i, f_i\}_{i=1}^d$ becomes a symplectic basis for (\mathbf{E}, σ) satisfying (2.5).

We will subsequently assume that a σ -allowed complex structure J has been

fixed and consider the triplet (\mathbf{E}, σ, J) . Explicit coordinates relative to a symplectic basis satisfying (2.5) will be referred to as canonical variables. We obtain

$$v = (p, q), \quad \text{where } p = \sum_{i=1}^d p_i e_i \quad \text{and } q = \sum_{i=1}^d q_i f_i,$$

$$(2.6) \quad \sigma(v, v') = pq' - p'q = \sum_{i=1}^d p_i q'_i - p'_i q_i,$$

$$J(p, q) = (-q, p),$$

$$s(v, v') = \sigma(v, Jv') = pp' + qq'.$$

The expressions in (2.6) furthermore constitute an explicit realization of the various notions introduced in this subsection.

The symplectic Fourier transformation.

We let dv denote the Haar-measure on $\mathbf{E} = \mathbf{R}^{2d}$ and normalize the Haar-measure such that

$$(2.7) \quad \int_{\mathbf{E}} e^{-1/2s(v,v)} dv = 1.$$

If we express the measure dv in terms of the explicit coordinates given in (2.6) above, then we obtain $dv = dp_1 \cdots dp_d dq_1 \cdots dq_d$, where the Lebesgue measure dx on \mathbf{R} is normalized such that an interval of unit length has the measure $(2\pi)^{-1/2}$.

The symplectic Fourier transformation F is defined by

$$(2.8) \quad (Ff)(v) = \tilde{f}(v) = \int_{\mathbf{E}} e^{is(v,v')} f(v') dv',$$

and extends with the normalization of Haar-measure given in (2.7) to a self-adjoint unitary operator on $L^2(\mathbf{E}, dv)$. In particular, $F^2 = 1$.

The Moyal product.

Let f, g be elements in $\mathcal{S}(\mathbf{E})$, the space of Schwartz functions on \mathbf{E} . We define the Moyal product (or twisted product) $f \circ g$ of f and g by setting

$$(2.9) \quad (f \circ g)(v) = \int_{\mathbf{E}} \int_{\mathbf{E}} f(v') g(v'') e^{i\varphi(v, v', v'')} dv' dv'' \quad v \in \mathbf{E},$$

where φ is the antisymmetric affine function defined by

$$(2.10) \quad \varphi(a, b, c) = \sigma(a, c) + \sigma(c, b) + \sigma(b, a) \quad a, b, c \in \mathbf{E}.$$

The twisted convolution of Schwartz functions on \mathbf{E} is defined by

$$(2.11) \quad f \times g = F(Ff \circ Fg),$$

and has the explicit form

$$(2.12) \quad (f \times g)(v) = \int_{\mathbf{E}} e^{-i\sigma(v,v')} f(v') g(v - v') dv' \quad v \in \mathbf{E}.$$

Twisted multiplication and twisted convolution are associative compositions in $S(\mathbf{E})$. The Moyal product satisfies

$$(2.13) \quad \overline{f \circ g} = \bar{g} \circ \bar{f},$$

$$\int_{\mathbf{E}} (f \circ g)(v) dv = \int_{\mathbf{E}} f(v)g(v) dv.$$

Even if only one of the functions in (2.9) belongs to the Schwartz space $S(\mathbf{E})$, the twisted product can make sense as a function or distribution on \mathbf{E} . This is a major topic in this branch of mathematics and will be addressed later on. At this point we shall only notice that

$$(2.14) \quad 1 \circ f = f.$$

We state without proof that

$$(2.15) \quad \|f \circ g\|_2 \leq 2^{-d/2} \|f\|_2 \|g\|_2,$$

and refer to [20, Theorem 3.2 (iii)]. Twisted multiplication and twisted convolution can consequently be extended to continuous compositions in $L^2(\mathbf{E}, dv)$.

The two properties expressed in (2.13) imply that the adjoint operator on $L^2(\mathbf{E}, dv)$ to twisted multiplication from the left (right) with a function $f \in L^2(\mathbf{E})$ is twisted multiplication from the left (right) with the complex conjugate \bar{f} .

The Gaussian function.

Of special interest in physics is the Gaussian function Ω defined by

$$(2.16) \quad \Omega(v) = 2^d e^{-1/2s(v,v)} \quad v \in \mathbf{E}.$$

The normalization of Haar-measure made in (2.7) implies that

$$(2.17) \quad \|\Omega\|_2 = 2^d \left(\int_{\mathbf{E}} e^{-s(v,v)} dv \right)^{1/2} = 2^{d/2}.$$

The Gaussian function Ω receives its prominence by satisfying the following formula

$$(2.18) \quad \Omega \circ f \circ \Omega = 2^{-d} (\Omega | f)_2 \Omega \quad \forall f \in L^2(\mathbf{E}, dv).$$

The proof is somewhat tricky and makes use of a transformation in the complex

extension C^{2d} of \mathbf{E} . We leave the proof to the reader and conclude from (2.14) and (2.18) that $\Omega \circ \Omega = \Omega$. The twisted multiplications from left and right with Ω are hence projections on $L^2(\mathbf{E}, dv)$.

The left regular representation.

We define a representation of $L^2(\mathbf{E}, dv)$ on the closed left ideal

$$(2.19) \quad I_\Omega = \{f \circ \Omega \mid f \in L^2(\mathbf{E}, dv)\},$$

by setting

$$(2.20) \quad \pi(f)g = f \circ g \quad \forall g \in I_\Omega,$$

for every $f \in L^2(\mathbf{E}, dv)$. We notice from (2.13), that $\pi(\bar{f}) = \pi(f)^*$ and that π is faithful. Furthermore, the Hilbert-Schmidt norm of $\pi(f)$ equals $2^{-d/2} \|f\|_2$ for every $f \in L^2(\mathbf{E}, dv)$, cf. [20, Theorem 3.2 (iii)].

Proposition 2.1. *The representation π is irreducible, and the finite rank operators in $B(I_\Omega)$ are exactly the operators of the form $\pi(a)$, where*

$$(2.21) \quad a = \sum_{i=1}^n \xi_i \circ \bar{\eta}_i \quad \xi_i, \eta_i \in I_\Omega, n \in \mathbb{N}.$$

Proof. Taking $f \in I_\Omega$ and applying (2.13) and (2.18), we obtain

$$(2.22) \quad \begin{aligned} \pi(a)f &= \sum_{i=1}^n \xi_i \circ \bar{\eta}_i \circ f \\ &= \sum_{i=1}^n \xi_i \circ \Omega \circ \bar{\eta}_i \circ f \circ \Omega \\ &= 2^{-d} \sum_{i=1}^n (\Omega \mid \bar{\eta}_i \circ f)_2 \xi_i \circ \Omega \\ &= 2^{-d} \sum_{i=1}^n (\eta_i \mid f)_2 \xi_i, \end{aligned}$$

and the proof is complete. ■

It follows, that the minimal projections in $B(I_\Omega)$ are exactly the operators of the form $\pi(\xi \circ \bar{\xi})$ with $\xi \in I_\Omega$ and $\|\xi\| = 2^{d/2}$. We collect a number of useful identities in the following

Proposition 2.2. *Let $\xi, \eta \in L^2(\mathbf{E}, dv)$, then*

$$F(\xi \times \eta) = (F\xi) \times \eta,$$

$$\xi \circ \eta = \xi \times (F\eta),$$

$$(\overline{F\xi}) = (F\bar{\xi})^\vee,$$

$$\begin{aligned}
 (2.23) \quad & F\Omega = \Omega, \\
 & F\xi = \check{\xi} \quad \forall \xi \in I_\Omega, \\
 & F\bar{\xi} = \bar{\xi} \quad \forall \xi \in I_\Omega,
 \end{aligned}$$

where $\check{\xi}(v) = \xi(-v)$, $v \in \mathbf{E}$.

The proofs are straightforward and are left to the reader.

§3. The Resolution of the Identity

A Hilbert space of holomorphic functions.

The additive group \mathbf{C}^d is identified with the additive group \mathbf{R}^{2d} by setting

$$(3.1) \quad z = p + iq \in \mathbf{C}^d \quad \text{for} \quad v = (p, q) \in \mathbf{R}^{2d}.$$

We can thus consider σ , s and h to be bilinear forms on \mathbf{C}^d and obtain the expressions

$$\begin{aligned}
 (3.2) \quad & \sigma(z, z') = \text{Im } h(z, z'), \\
 & s(z, z') = \text{Re } h(z, z'), \\
 & h(z, z') = \bar{z}z'.
 \end{aligned}$$

In particular, the Gaussian function Ω takes the form

$$(3.3) \quad \Omega(z) = 2^d e^{-1/2|z|^2} \quad z \in \mathbf{C}^d.$$

The Haar-measure $dz = dpdq$ on \mathbf{C}^d corresponds to the abovementioned identification. Let $A(\mathbf{C}^d)$ denote the space of holomorphic functions on \mathbf{C}^d and set

$$(3.4) \quad H^2(\mathbf{C}^d) = \left\{ f \in A(\mathbf{C}^d) \mid \int_{\mathbf{C}^d} |f(z)|^2 \Omega(z)^2 dz < \infty \right\}.$$

The vector space $H^2(\mathbf{C}^d)$ is a Hilbert space with inner product given by

$$(3.5) \quad (f \mid g) = \int_{\mathbf{C}^d} \overline{f(x)} g(z) \Omega(z)^2 dz.$$

Proposition 3.1. *The set of vectors*

$$(3.6) \quad u_n(z) = 2^{-d/2} (n!)^{-1/2} z^n, \quad n = (n_1, \dots, n_d) \in \mathbf{N}_0^d, \quad \text{where}$$

$$z^n = \prod_{i=1}^d z_i^{n_i} \quad \text{and} \quad n! = \prod_{i=1}^d n_i!,$$

is an orthonormal basis for $H^2(\mathbb{C}^d)$.

Proof. The proof is basically found in [1], and we just have to keep track of the normalization. ■

The transformation A.

We define a linear mapping $A: L^2(\mathbb{E}, dv) \rightarrow A(\mathbb{C}^d)$ by setting

$$(3.7) \quad (A\xi)(z) = 2^{-d} \int_{\mathbb{C}^d} e^{z\bar{z}'} \Omega(z') \xi(z') dz',$$

where ξ is considered as a function on \mathbb{C}^d through the identification made in (3.1).

Lemma 3.2.

$$(3.8) \quad \xi \circ \Omega = \Omega(A\xi) \quad \forall \xi \in L^2(\mathbb{E}, dv),$$

where $A\xi$ is considered as a function on \mathbb{E} , cf. (3.1).

Proof. By making use of the identity

$$(3.9) \quad \Omega(v - v') = 2^{-d} \Omega(v) \Omega(v') e^{s(v,v')},$$

and (2.23), we obtain

$$\begin{aligned} (\xi \circ \Omega)(v) &= \int_{\mathbb{E}} e^{-i\sigma(v,v')} \xi(v') \Omega(v - v') dv' \\ &= 2^{-d} \Omega(v) \int_{\mathbb{E}} e^{-i\sigma(v,v')} \xi(v') \Omega(v') e^{s(v,v')} dv' \\ &= \Omega(v)(A\xi)(v), \end{aligned}$$

for each $\xi \in L^2(\mathbb{E}, dv)$ and $v \in \mathbb{E}$. ■

It follows from (3.8) that A is a contraction from $L^2(\mathbb{E}, dv)$ on $H^2(\mathbb{C}^d)$.

Theorem 3.3. *The transformation A is a partial isometry which maps $L^2(\mathbb{E}, dv)$ onto $H^2(\mathbb{C}^d)$. The projection on I_Ω is the initial projection and the identity operator on $H^2(\mathbb{C}^d)$ is the final projection. The adjoint operator is given by*

$$(3.10) \quad A^*u = u\Omega \quad \forall u \in H^2(\mathbb{C}^d).$$

Proof. Take $\xi \in L^2(\mathbb{E}, dv)$, $u \in H^2(\mathbb{C}^d)$ and make use of the identification in (3.1) when appropriate. We obtain

$$\begin{aligned} (A\xi|u) &= \int_{\mathbf{C}^d} \overline{(A\xi)(z)} u(z) \Omega(z)^2 dz \\ &= \int_{\mathbf{C}^d} 2^{-d} \int_{\mathbf{C}^d} e^{z\bar{z}'} \Omega(z') \overline{\xi(z')} dz' u(z) \Omega(z)^2 dz \\ &= (\xi|A^*u)_2, \end{aligned}$$

where

$$\begin{aligned} (A^*u)(v) &= 2^{-d} \Omega(v) \int_{\mathbf{E}} e^{h(v',v)} u(v') \Omega(v')^2 dv' \\ &= \Omega(v) \int_{\mathbf{E}} e^{h(v',v) - 1/2s(v',v')} \Omega(v') u(v') dv'. \end{aligned}$$

The identity

$$\begin{aligned} h(v', v) - 1/2s(v', v) &= s(v', v) + i\sigma(v', v) - 1/2s(v', v) \\ &= -1/2s(v' - v, v' - v) + 1/2s(v, v) + i\sigma(v', v), \end{aligned}$$

entails that

$$\begin{aligned} (A^*u)(v) &= 2^d \int_{\mathbf{E}} e^{-1/2s(v' - v, v' - v) + i\sigma(v', v)} \Omega(v') u(v') dv' \\ &= \int_{\mathbf{E}} e^{-i\sigma(v, v')} \Omega(v') u(v') \Omega(v' - v) dv' \\ &= (u\Omega \times \Omega)(v), \end{aligned}$$

hence

$$(3.11) \quad A^*u = (u\Omega) \times \Omega = (u\Omega) \circ \Omega.$$

It thus follows that A^* maps $H^2(\mathbf{C}^d)$ into I_Ω . Furthermore

$$A^*A(\xi \circ \Omega) = A^*\left(\frac{\xi \circ \Omega}{\Omega}\right) = \xi \circ \Omega.$$

We have established, that A is a partial isometry with I_Ω as initial projection, and that A^* acts as multiplication with Ω when restricted to functions in the final projection of A . The assertion follows by showing that each of the vectors in (3.6) belongs to the range of A . We first notice that

$$2 \int_{\mathbf{C}} (z')^n e^{-|z'|^2} e^{z\bar{z}'} dz'$$

$$\begin{aligned}
 &= 2 \int_{\mathbf{C}} (z')^n e^{-|z'|^2} \sum_{k=0}^{\infty} \frac{(\bar{z}')^k}{k!} z^k dz' \\
 &= 2 \sum_{k=0}^{\infty} \frac{z^k}{k!} \int_{\mathbf{C}} (z')^n (\bar{z}')^k e^{-|z'|^2} dz' \\
 &= \frac{2z^n}{n!} \int_{\mathbf{C}} |z'|^{2n} e^{-|z'|^2} dz' \\
 &= z^n.
 \end{aligned}$$

For $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$, we thus obtain

$$\begin{aligned}
 (A(u_n \Omega))(z) &= 2^{-d} \int_{\mathbf{E}} e^{z\bar{z}'} \Omega(z') 2^{-d/2} (n!)^{-1/2} \Omega(z') \prod_{i=1}^d (z_i)^{n_i} dz' \\
 &= 2^{-d/2} \prod_{i=1}^d 2 \int_{\mathbf{C}} e^{z_i \bar{z}'_i} e^{-|z'_i|^2} (n_i!)^{-1/2} (z'_i)^{n_i} dz'_i \\
 &= 2^{-d/2} \prod_{i=1}^d (n_i!)^{-1/2} z_i^{n_i} \\
 &= u_n(z),
 \end{aligned}$$

which proves the assertion. ■

Matrix units for $B(I_\Omega)$.

Let $\{e_i, f_i\}_{i=1}^d$ be a symplectic basis for \mathbf{E} satisfying (2.5). We assume explicite coordinates as in (2.6) and the identification (3.1). It follows that

$$(3.12) \quad z_i = p_i + iq_i = h(e_i, v) \quad \text{for } i = 1, \dots, d.$$

We introduce the functions

$$(3.13) \quad a_n(v) = (n!)^{-1/2} \Omega(v) \prod_{i=1}^d h(e_i, v)^{n_i}, \quad v \in \mathbf{E},$$

for $n = (n_1, \dots, n_d) \in \mathbb{N}_0^d$. We conclude from Theorem 3.3 that $\{2^{-d/2} a_n\}_{n \in \mathbb{N}_0^d}$ is an orthonormal basis for I_Ω and put

$$(3.14) \quad a_{n,m} = a_n \circ \bar{a}_m \quad \text{for } n, m \in \mathbb{N}_0^d.$$

Theorem 3.4. *The vectors $\{a_{n,m}\}_{n,m \in \mathbb{N}_0^d}$ have each length $2^{d/2}$ and constitute an orthogonal basis for $L^2(\mathbf{E}, dv)$. Furthermore, they satisfy*

$$(3.15) \quad \begin{aligned} a_{n,m} \circ a_{k,l} &= \delta_{m,k} a_{n,l} \\ \overline{a_{n,m}} &= a_{m,n}, \end{aligned}$$

for $n, m, k, l \in \mathbb{N}_0^d$ and are mapped by the left regular representation π onto a system of matrix units for $B(I_\Omega)$.

Proof. Proposition 2.1 and equation (2.22) entails that $\pi(a_{n,m})$ is the matrix unit mapping $2^{-d/2} a_m$ onto $2^{-d/2} a_n$. Since the vectors $\{a_n\}_{n \in \mathbb{N}_0^d}$ are mutually orthogonal, we have

$$\begin{aligned} \pi(a_{n,m} \circ a_{k,l}) &= \pi(a_{n,m})\pi(a_{k,l}) \\ &= \delta_{m,k} \pi(a_{n,l}). \end{aligned}$$

Since π is faithful, the first part of (3.15) follows. The second is a consequence of (2.13). Next

$$\begin{aligned} (a_{n,m} | a_{k,l})_2 &= \int_{\mathbf{E}} \overline{a_{n,m}(v)} a_{k,l}(v) dv \\ &= \int_{\mathbf{E}} (\overline{a_{n,m}} \circ a_{k,l})(v) dv \\ &= \delta_{n,k} \int_{\mathbf{E}} a_{m,l}(v) dv \\ &= \delta_{n,k} (a_m | a_l)_2 \\ &= \delta_{n,k} \delta_{m,l} 2^d, \end{aligned}$$

which shows that the vectors $\{a_{n,m}\}_{n,m \in \mathbb{N}_0^d}$ are mutually orthogonal and have length $2^{d/2}$. They generate $L^2(\mathbf{E}, dv)$ because $\pi(a_{n,m})$, for $n, m \in \mathbb{N}_0^d$, constitute a full set of matrix units for $B(I_\Omega)$ according to Proposition 2.1. ■

We shall furthermore notice that

$$\begin{aligned} (3.16) \quad F a_{n,m} &= (-1)^{|n|} a_{n,m}, \\ a_{n,m} &= a_n \times \overline{a_m}, \end{aligned}$$

for $n, m \in \mathbb{N}_0^d$, where $|n| = n_1 + \dots + n_d$. This follows readily from Proposition (2.2).

§ 4. Distributions as Matrices

Generating the Schwartz space.

We shall in a number of steps relate the vectors $a_{n,m}$ to the Schwartz space $S(\mathbf{E})$. Let us to every $n \in \mathbb{N}_0^d$ and $k = 1, \dots, d$ introduce the tuples $\alpha_k(n)$, $\beta_k(n) \in \mathbb{N}_0^d$ defined by

$$(4.1) \quad \begin{aligned} \alpha_k(n)_j &= \begin{cases} n_j, & \text{for } j \neq k \\ n_j + 1, & \text{for } j = k \end{cases} \\ \beta_k(n)_j &= \begin{cases} n_j, & \text{for } j \neq k \\ 0, & \text{for } j = k \text{ and } n_k = 0 \\ n_j - 1, & \text{for } j = k \text{ and } n_k > 0 \end{cases} \end{aligned}$$

for $j = 1, \dots, d$.

Lemma 4.1.

$$(4.2) \quad \begin{aligned} \frac{\partial a_n}{\partial p_k} &= \sqrt{n_k} a_{\beta_k(n)} - p_k a_n, \\ \frac{\partial a_n}{\partial q_k} &= i\sqrt{n_k} a_{\beta_k(n)} - q_k a_n, \end{aligned}$$

for $n \in \mathbb{N}_0^d$ and $k = 1, \dots, d$.

Proof. We obtain by direct calculation that

$$\begin{aligned} &\frac{\partial a_n(v)}{\partial p_k} \\ &= (n!)^{-1/2} \left(\prod_{j \neq k} h(e_j, v)^{n_j} \right) \frac{\partial}{\partial p_k} (\Omega(v) h(e_k, v)^{n_k}) \\ &= (n!)^{-1/2} \left(\prod_{j \neq k} h(e_j, v)^{n_j} \right) (\Omega(v) n_k h(e_k, v)^{\beta_k(n)_k} - p_k \Omega(v) h(e_k, v)^{n_k}) \\ &= \sqrt{n_k} a_{\beta_k(n)}(v) - p_k a_n(v). \end{aligned}$$

The other part of the assertion is similarly proven. ■

We shall use the symbols p_i, q_i to denote not only the coordinates of a vector $v \in \mathbb{E}$, cf. (2.6), but also the operator acting as multiplication with the corresponding coordinate function. This is a common and useful, but also slightly confusing notation. We obtain the following lemma by using (3.16) and a straightforward calculation.

Lemma 4.2. For $n, m \in \mathbb{N}_0^d$ and $k = 1, \dots, d$ we have the following relations:

$$(4.3) \quad \begin{aligned} \frac{\partial a_{n,m}}{\partial p_k} &= a_n \times \frac{\partial \overline{a_m}}{\partial p_k} - i(q_k a_n) \times \overline{a_m}, \\ \frac{\partial a_{n,m}}{\partial q_k} &= a_n \times \frac{\partial \overline{a_m}}{\partial q_k} + i(p_k a_n) \times \overline{a_m}, \end{aligned}$$

$$p_k a_{n,m} = p_k a_n \times \overline{a_m} + a_n \times p_k \overline{a_m},$$

$$q_k a_{n,m} = q_k a_n \times \overline{a_m} + a_n \times q_k \overline{a_m}.$$

Let V_0 denote the complex vector space $\text{span}\{a_{n,m} | n, m \in \mathbb{N}_0^d\}$ and let $R_k: V_0 \rightarrow V_0$ for $k = 1, \dots, d$ be the linear operator defined by

$$(4.4) \quad R_k a_{n,m} = \sqrt{m_k + 1} a_{n, \alpha_k(m)} \quad \forall n, m \in \mathbb{N}_0^d.$$

If we let S denote the antiunitary operator on $L^2(\mathbb{E}, dv)$ for which $S\xi = \bar{\xi}$ and put $L_k = SR_k S$, then we have

$$(4.5) \quad L_k a_{n,m} = \sqrt{n_k + 1} a_{\alpha_k(n), m} \quad \forall n, m \in \mathbb{N}_0^d.$$

Theorem 4.3. *For $k = 1, \dots, d$, we have*

$$R_k = \frac{1}{2} \left(p_k - i q_k - \left(\frac{\partial}{\partial p_k} - i \frac{\partial}{\partial q_k} \right) \right),$$

$$L_k = \frac{1}{2} \left(p_k + i q_k - \left(\frac{\partial}{\partial p_k} + i \frac{\partial}{\partial q_k} \right) \right).$$

Proof. Taking $n, m \in \mathbb{N}_0^d$ and applying the lemmas above, we obtain

$$\begin{aligned} & \left(\frac{\partial}{\partial p_k} - i \frac{\partial}{\partial q_k} \right) a_{n,m} \\ &= a_n \times \frac{\partial \overline{a_m}}{\partial p_k} - i(q_k a_n) \times \overline{a_m} - i a_n \times \frac{\partial \overline{a_m}}{\partial q_k} + (p_k a_n) \times \overline{a_m} \\ &= a_n \times (\sqrt{m_k} \overline{a_{\beta_k(m)}} - p_k \overline{a_m}) - i(q_k a_n) \times \overline{a_m} \\ &\quad - i a_n \times (-i \sqrt{m_k} \overline{a_{\beta_k(m)}} - q_k \overline{a_m}) + (p_k a_n) \times \overline{a_m} \\ &= -(a_n \times \overline{(p_k + i q_k) a_m}) + (p_k - i q_k) a_n \times \overline{a_m}, \end{aligned}$$

and consequently

$$\begin{aligned} & \left(p_k - i q_k - \left(\frac{\partial}{\partial p_k} - i \frac{\partial}{\partial q_k} \right) \right) a_{n,m} \\ &= p_k a_n \times \overline{a_m} + a_n \times p_k \overline{a_m} - i q_k a_n \times \overline{a_m} - i a_n \times q_k \overline{a_m} \\ &\quad + (a_n \times \overline{(p_k + i q_k) a_m}) - (p_k - i q_k) a_n \times \overline{a_m} \\ &= 2(a_n \times \overline{(p_k + i q_k) a_m}). \end{aligned}$$

Since

$$\begin{aligned}
 (p_k + iq_k)a_m(v) &= h(e_k, v)a_m(v) \\
 &= h(e_k, v)(m!)^{-1/2}\Omega(v)\prod_{j=1}^d h(e_j, v)^{m_j} \\
 &= (m!)^{-1/2}\Omega(v)\prod_{j=1}^d h(e_j, v)^{\alpha_k(m)_j} \\
 &= \sqrt{m_k + 1} a_{\alpha_k(m)}(v),
 \end{aligned}$$

the first statement of the theorem follows. The second is similarly proven. ■

Corollary 4.4. For $k = 1, \dots, d$, we have

$$\begin{aligned}
 p_k &= \frac{1}{2}(L_k + R_k^* + R_k + L_k^*), \\
 q_k &= \frac{1}{2i}(L_k + R_k^* - R_k - L_k^*), \\
 \frac{\partial}{\partial p_k} &= \frac{1}{2}(R_k^* - L_k + L_k^* - R_k), \\
 \frac{\partial}{\partial q_k} &= \frac{1}{2i}(R_k^* - L_k - L_k^* + R_k).
 \end{aligned}$$

Proof. Make use of Theorem 4.3 and reduce the right hand sides. ■

For each $p \in \mathbf{Z}^d$ and $\xi \in L^2(\mathbf{E}, dv)$, we define

$$(4.6) \quad r_p(\xi) = 2^{-d} \left(\sum_{n, m \in \mathbf{N}_0^d} |a_{n, m}(\xi)|^2 (1+n)^p (1+m)^p \right)^{1/2},$$

where 1 is the d -tuple with 1 as entry on every site. The set

$$(4.7) \quad \mathbf{H}_p = \{ \xi \in L^2(\mathbf{E}, dv) \mid r_p(\xi) < \infty \}$$

is for $p \in \mathbf{N}_0^d$ a Hilbert space which is dense in $L^2(\mathbf{E}, dv)$.

Theorem 4.5. A vector $\xi \in L^2(\mathbf{E}, dv)$ belongs to the Schwartz space $S(\mathbf{E})$, if and only if $r_p(\xi) < \infty$ for every $p \in \mathbf{N}_0^d$. That is

$$(4.8) \quad S(\mathbf{E}) = \bigcap_{p \in \mathbf{N}_0^d} \mathbf{H}_p.$$

The topology on $S(\mathbf{E})$ is the Fréchet topology given by the norms r_p , $p \in \mathbf{N}_0^d$.

Proof. The operators R^p , L^p defined for every $p \in \mathbf{N}_0^d$ by

$$R^p = \prod_{k=1}^d R_k^{p_k}, \quad L^p = \prod_{k=1}^d L_k^{p_k},$$

and their adjoints leave $S(\mathbf{E})$ invariant according to Theorem 4.3. If we assume that $\xi \in S(\mathbf{E})$, we thus have $R^p L^p \xi \in L^2(\mathbf{E}, dv)$ for every $p \in \mathbf{N}_0^d$. By making use of (4.4) and (4.5) to express $R^p L^p \xi$ in terms of the vectors in the basis, and by applying Parseval's formula, we conclude that $r_p(\xi) < \infty$. If on the other hand this condition is assumed for every $p \in \mathbf{N}_0^d$, then we apply Corollary 4.4 and express any operator written as a product of powers of the operators $p_k, q_k, \partial/\partial p_k, \partial/\partial q_k$ in terms of the operators R_k, R_k^*, L_k, L_k^* . Any such operator will therefore map ξ into $L^2(\mathbf{E}, dv)$. It follows that $\xi \in S(\mathbf{E})$. ■

The Moyal product as matrix multiplication.

Take $\xi \in S(\mathbf{E})$ with the expansion

$$(4.9) \quad \xi = \sum_{n, m \in \mathbf{N}_0^d} \alpha_{n, m} a_{n, m}.$$

The coefficients are given by

$$(4.10) \quad \alpha_{n, m} = 2^{-d} (a_{n, m} | \xi)_2 \quad \forall n, m \in \mathbf{N}_0^d.$$

If the operator $\pi(\xi)$ acts on a vector $\eta \in I_\Omega$ with expansion

$$(4.11) \quad \eta = \sum_{k \in \mathbf{N}_0^d} \alpha_k a_k, \quad \alpha_k = 2^{-d} (a_k | \eta)_2 \quad \forall k \in \mathbf{N}_0^d,$$

then we obtain

$$(4.12) \quad \begin{aligned} \pi(\xi)\eta &= \sum_{n, m \in \mathbf{N}_0^d} \alpha_{n, m} a_{n, m} \circ \left(\sum_{k \in \mathbf{N}_0^d} \alpha_k a_k \right) \\ &= \sum_{n \in \mathbf{N}_0^d} \left(\sum_{m \in \mathbf{N}_0^d} \alpha_{n, m} \alpha_m \right) a_n. \end{aligned}$$

The operator $\pi(\xi)$ thus acts as the matrix $\{\alpha_{n, m} | n, m \in \mathbf{N}_0^d\}$ on the Hilbert space I_Ω . It is a Hilbert-Schmidt operator with Hilbert-Schmidt norm $\text{Tr}(\pi(\xi)^* \pi(\xi))^{1/2} = 2^{-d/2} \|\xi\|_2$. Furthermore, the twisted product of two functions $\xi, \eta \in S(\mathbf{E})$ with coefficient sets $(\alpha_{n, m})$ and $(\beta_{n, m})$, cf. (4.10), is given by

$$(4.13) \quad \xi \circ \eta = \sum_{n, m \in \mathbf{N}_0^d} \left(\sum_{k \in \mathbf{N}_0^d} \alpha_{n, k} \beta_{k, m} \right) a_{n, m}.$$

The twisted product of functions in $S(\mathbf{E})$ is thus given by matrix multiplication of the corresponding coefficient sets. An application of Cauchy-Schwarz inequality yields

$$\begin{aligned}
 & r_p(\xi \circ \eta) \\
 &= \left(\sum_{n,m \in \mathbb{N}_0^d} \left| \sum_{k \in \mathbb{N}_0^d} \alpha_{n,k} \beta_{k,m} \right|^2 (1+n)^p (1+m)^p \right)^{1/2} \\
 (4.14) \quad & \leq \left(\sum_{n,k \in \mathbb{N}_0^d} |\alpha_{n,k}|^2 (1+n)^p (1+k)^p \right)^{1/2} \left(\sum_{k,m \in \mathbb{N}_0^d} |\beta_{k,m}|^2 (1+k)^{-p} (1+m)^p \right)^{1/2} \\
 & \leq r_p(\xi) r_p(\eta),
 \end{aligned}$$

for every $p \in \mathbb{N}_0^d$. We conclude that the Schwartz space $S(\mathbb{E})$ is a topological algebra under the Moyal product.

Tempered distributions.

Let for every $n, m \in \mathbb{N}_0^d$ the tempered distribution $t_{n,m}$ be defined by

$$(4.15) \quad t_{n,m}(\xi) = 2^{-d} \int_{\mathbb{E}} a_{n,m}(v) \xi(v) dv \quad \forall \xi \in S(\mathbb{E}).$$

We notice that $t_{n,m}(\xi) = \text{Tr}(\pi(a_{n,m})\pi(\xi))$, cf. [20, Theorem 3.2].

Theorem 4.6. *Let T be a tempered distribution on \mathbb{E} and set*

$$(4.16) \quad \alpha_{n,m} = T(a_{n,m}) \quad \forall n, m \in \mathbb{N}_0^d.$$

Then there exists a $p \in \mathbb{N}_0^d$ such that

$$(4.17) \quad \sum_{n,m \in \mathbb{N}_0^d} |\alpha_{n,m}|^2 n^{-p} m^{-p} < \infty.$$

Furthermore, we have that

$$(4.18) \quad T = \sum_{n,m \in \mathbb{N}_0^d} \alpha_{n,m} t_{n,m},$$

where the sum converges in the sense of distribution.

Proof. Let η be an arbitrary vector in $S(\mathbb{E})$ with expansion

$$\eta = \sum_{n,m \in \mathbb{N}_0^d} \beta_{n,m} a_{n,m}.$$

Since the sum converges in $S(\mathbb{E})$ and $T \in S'(\mathbb{E})$, we obtain

$$(4.19) \quad T(\eta) = \sum_{n,m \in \mathbb{N}_0^d} \beta_{n,m} T(a_{n,m}) = \sum_{n,m \in \mathbb{N}_0^d} \alpha_{m,n} \beta_{n,m}.$$

The family of norms $\{r_p | p \in \mathbb{N}_0^d\}$ defining the topology on $S(\mathbb{E})$ is upward filtering. There exists consequently a single $p \in \mathbb{N}_0^d$ and a positive constant C such that

$$|T(\eta)| \leq C r_p(\eta) \quad \forall \eta \in \mathbf{H}_p \cap S(\mathbb{E}).$$

It follows, that T can be extended by continuity to the Hilbert space \mathbf{H}_p and therefore it is represented by a vector in \mathbf{H}_p . We derive, that the representing

vector has coefficient set $\{n^{-p}m^{-p}\alpha_{m,n}\}$. This implies the condition in (4.17). Finally, we notice that $\beta_{n,m} = t_{m,n}(\eta)$, and the rest of the theorem follows. \square

We have thus transformed the study of tempered distributions on \mathbb{E} to the study of matrices with a growth property for the entries as given by (4.17).

The Schwartz space $S(\mathbb{E})$ can be regarded as a subspace of $S'(\mathbb{E})$, and one may naturally ask, how this embedding is reflected with regard to the expansion (4.18). A tempered distribution T is of the form

$$(4.20) \quad T(\eta) = 2^{-d} \int_{\mathbb{E}} \xi(v)\eta(v) dv \quad \forall \eta \in S(\mathbb{E}),$$

for a $\xi \in S(\mathbb{E})$, if and only if the matrix $\{\alpha_{n,m} | n, m \in \mathbb{N}_0^d\}$, cf. (4.16), defines an element in \mathbb{H}_p for every $p \in \mathbb{N}_0^d$. The function ξ is then given by

$$\xi = \sum_{n,m} \alpha_{n,m} a_{n,m}.$$

We shall freely use this identification of $S(\mathbb{E})$ with a subspace of $S'(\mathbb{E})$. We notice that if $\pi(T)$ is a bounded operator on I_Ω , then

$$(4.21) \quad |T(\xi)| \leq \|\pi(T)\| \|\pi(\xi)\|_{\text{Tr}},$$

for every $\xi \in S(\mathbb{E})$. The trace norm $\|\pi(\xi)\|_{\text{Tr}} = \text{Tr}((\pi(\xi)^* \pi(\xi))^{1/2})$ is finite for every $\xi \in S(\mathbb{E})$, cf. [27].

§5. The Multiplier Algebra

We have so far considered twisted multiplication of functions belonging to the Schwartz space $S(\mathbb{E})$. We also noticed that the definition of the twisted product can be extended to functions belonging to $L^2(\mathbb{E}, dv)$. This is done by continuity, cf. equation (2.15). It is furthermore possible to define the twisted product of a tempered distribution with a function in $S(\mathbb{E})$ in a very natural way. Let T be a tempered distribution on \mathbb{E} and let $\xi \in S(\mathbb{E})$. We define $T \circ \xi$ and $\xi \circ T$ as the tempered distributions given by

$$(5.1) \quad \begin{aligned} (T \circ \xi)(\eta) &= T(\xi \circ \eta), \\ (\xi \circ T)(\eta) &= T(\eta \circ \xi). \end{aligned} \quad \forall \eta \in S(\mathbb{E})$$

Suppose that T is given by an integral kernel $f \in S(\mathbb{E})$, then we apply equation (2.13) and obtain

$$(5.2) \quad (T \circ \xi)(\eta) = \int_{\mathbb{E}} f(v)(\xi \circ \eta)(v) dv = \int_{\mathbb{E}} (f \circ \xi)(v)\eta(v) dv,$$

$$(\xi \circ T)(\eta) = \int_{\mathbf{E}} f(v)(\eta \circ \xi)(v) dv = \int_{\mathbf{E}} (\xi \circ f)(v)\eta(v) dv.$$

Thus the tempered distributions $T \circ \xi$ and $\xi \circ T$ are given by the integral kernels $f \circ \xi$ and $\xi \circ f$ respectively.

Proposition 5.1. *Let $T \in S'(\mathbf{E})$ and $\xi \in S(\mathbf{E})$ have expansions*

$$T = \sum_{n,m \in \mathbb{N}_0^d} \alpha_{n,m} t_{n,m}, \quad \xi = \sum_{n,m \in \mathbb{N}_0^d} \beta_{n,m} a_{n,m},$$

cf. (4.18) and (4.9). *The tempered distributions $T \circ \xi$ and $\xi \circ T$ are then given by the expansions*

$$T \circ \xi = \sum_{n,m \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{N}_0^d} \alpha_{n,k} \beta_{k,m} \right) t_{n,m}, \quad \xi \circ T = \sum_{n,m \in \mathbb{N}_0^d} \left(\sum_{k \in \mathbb{N}_0^d} \beta_{n,k} \alpha_{k,m} \right) t_{n,m}.$$

Proof. The coefficients for the tempered distribution $T \circ \xi$ with respect to $t_{n,m}$ is according to Theorem 4.6 given by

$$\begin{aligned} (T \circ \xi)(a_{m,n}) &= T(\xi \circ a_{m,n}) \\ &= T\left(\sum_{k,l \in \mathbb{N}_0^d} \beta_{k,l} a_{k,l} \circ a_{m,n}\right) \\ &= T\left(\sum_{k \in \mathbb{N}_0^d} \beta_{k,m} a_{k,n}\right) \\ &= \sum_{k \in \mathbb{N}_0^d} \alpha_{n,k} \beta_{k,m}. \end{aligned}$$

The other assertion is similarly proven. ■

It follows from Proposition 5.1 that

$$(5.3) \quad \begin{aligned} t_{n,m} \circ a_{k,l} &= \delta_{m,k} t_{n,l}, \\ a_{k,l} \circ t_{n,m} &= \delta_{l,n} t_{k,m}. \end{aligned}$$

Várilly and Gracia-Bondia considered in [33] the set

$$(5.4) \quad N = \{T \in S'(\mathbf{E}) \mid T \circ \xi \in S(\mathbf{E}) \quad \forall \xi \in S(\mathbf{E})\}$$

of left multipliers. Since $S(\mathbf{E})$ is invariant under complex conjugation it follows, that \bar{N} is the set of right multipliers, cf. (2.13). The intersection $M = N \cap \bar{N}$ is the set of twosided multipliers of $S(\mathbf{E})$. We define the twisted product $T \circ S$ of elements in $T, S \in N$ by setting

$$(5.5) \quad (T \circ S)(\xi) = T(S \circ \xi) \quad \forall \xi \in S(\mathbf{E}).$$

The embedding of $S(\mathbf{E})$ in $S'(\mathbf{E})$ is implicitly used, cf. (4.20). The product $T \circ S$ is

a tempered distribution, cf. the inequality (5.8). If $T \in N$ and $\xi, \eta, \zeta \in S(\mathbf{E})$, then we obtain

$$\begin{aligned} ((T \circ \xi) \circ \eta)(\zeta) &= (T \circ \xi)(\eta \circ \zeta) \\ &= T(\xi \circ (\eta \circ \zeta)) \\ &= T((\xi \circ \eta) \circ \zeta) \\ &= (T \circ (\xi \circ \eta))(\zeta), \end{aligned}$$

which shows that $(T \circ \xi) \circ \eta = T \circ (\xi \circ \eta)$. If furthermore $T, S \in N$ and $\xi, \eta \in S(\mathbf{E})$, then we make use of this equality and obtain

$$\begin{aligned} ((T \circ S) \circ \xi)(\eta) &= (T \circ S)(\xi \circ \eta) \\ &= T(S \circ (\xi \circ \eta)) \\ &= T((S \circ \xi) \circ \eta) \\ &= (T \circ (S \circ \xi))(\eta), \end{aligned}$$

which implies that

$$(5.6) \quad (T \circ S) \circ \xi = T \circ (S \circ \xi) \quad \forall T, S \in N \forall \xi \in S(\mathbf{E}).$$

We conclude that the tempered distribution $T \circ S$ is an element of N . Furthermore, the property (5.6) implies associativity of the twisted product of elements in N thus making N an algebra. Finally, we notice that M is an involutive algebra with twisted product as multiplication and complex conjugation as involution.

Let T be a tempered distribution and set $\alpha_{n,m} = T(a_{m,n})$ for $n, m \in \mathbb{N}_0^d$. We define

$$(5.7) \quad L_{p,q}(T) = \left(\sum_{n,m \in \mathbb{N}_0^d} |\alpha_{n,m}|^2 (1+n)^q (1+m)^{-p} \right)^{1/2},$$

for $p, q \in \mathbb{N}_0^d$ and notice that the value plus infinity may be attained. $L_{p,q}$ is for each $p, q \in \mathbb{N}_0^d$ a norm on a subspace of $S'(\mathbf{E})$ which contains $S(\mathbf{E})$.

Theorem 5.2. *A tempered distribution T belongs to N , if and only if to each $q \in \mathbb{N}_0^d$ there exists a $p \in \mathbb{N}_0^d$ such that $L_{p,q}(T) < \infty$.*

Proof. Assume that $T \in S'(\mathbf{E})$ belongs to N , that is $T \circ \xi \in S(\mathbf{E})$ for each $\xi \in S(\mathbf{E})$. We first show that the map

$$S(\mathbf{E}) \ni \xi \mapsto T \circ \xi \in S(\mathbf{E})$$

is closeable. Let $\xi_i \rightarrow 0$ and suppose that $T \circ \xi_i \rightarrow \zeta$, both in the Fréchet topology on $S(\mathbf{E})$. Since the twisted product is continuous, we conclude that

$(T \circ \xi_i)(\eta) = T(\xi_i \circ \eta) \rightarrow 0$ for each $\eta \in S(\mathbb{E})$, so $\zeta = 0$. Since $S(\mathbb{E})$ is complete and metrizable, Banach's closed graph theorem can be applied. We obtain that the above map is actually continuous.

Let now $q \in \mathbb{N}_0^d$ be given. We may well assume that $q_i > 0$ for $i = 1, \dots, d$. The family of norms $\{r_q | q \in \mathbb{N}_0^d\}$ defining the topology on $S(\mathbb{E})$ is upward filtering. There exists therefore a single $s \in \mathbb{N}_0^d$ and a positive constant C such that

$$r_q(T \circ \xi) \leq Cr_s(\xi) \quad \forall \xi \in S(\mathbb{E}).$$

Let $j \in \mathbb{N}$ and set

$$\xi_j = \sum_{\substack{n \in \mathbb{N}_0^d \\ |n| \leq j}} (1+n)^{-q-s} a_{n,n}.$$

We notice that $\xi_j \in S(\mathbb{E})$ and calculate that

$$r_s(\xi_j) \leq \left(\sum_{n \in \mathbb{N}_0^d} (1+n)^{-2q} \right)^{1/2} = K < \infty \quad \forall j \in \mathbb{N}.$$

Consequently, we obtain that $r_q(T \circ \xi_j) \leq CK$ for every $j \in \mathbb{N}$. Finally, we observe that $r_q(T \circ \xi_j) \rightarrow L_{p,q}(T)$ for $j \rightarrow \infty$, where $p = 2s + q$. This proves the necessity of the condition.

We apply Cauchy-Schwarz inequality to prove the converse. Let $\alpha_{n,m} = T(a_{m,n})$ and $\beta_{n,m} = 2^{-d}(a_{n,m}|\xi)_2$ for $n, m \in \mathbb{N}_0^d$. We obtain for $\xi \in S(\mathbb{E})$ that

$$\begin{aligned} r_q(T \circ \xi)^2 &= \sum_{n,m \in \mathbb{N}_0^d} \left| \sum_{k \in \mathbb{N}_0^d} \alpha_{n,k} \beta_{k,m} \right|^2 (1+n)^q (1+m)^{-p} \\ &= \sum_{n,m \in \mathbb{N}_0^d} \left| \sum_{k \in \mathbb{N}_0^d} \alpha_{n,k} (1+k)^{-p/2} \beta_{k,m} (1+k)^{p/2} \right|^2 (1+n)^q (1+m)^{-p} \\ &\leq L_{p,q}(T)^2 r_p(\xi)^2. \end{aligned}$$

Consequently,

$$(5.8) \quad r_q(T \circ \xi) \leq L_{p,q}(T) r_p(\xi) \quad \forall \xi \in S(\mathbb{E}).$$

The condition is thus sufficient. ■

Corollary 5.3. *A tempered distribution T belongs to M , if and only if there to each $q \in \mathbb{N}_0^d$ exists a $p \in \mathbb{N}_0^d$ such that $L_{p,q}(T) < \infty$ and $L_{p,q}(\bar{T}) < \infty$.*

Proof. Let the tempered distribution T be in M and let $q \in \mathbb{N}_0^d$ be given. Then there exist $p', p'' \in \mathbb{N}_0^d$ such that $L_{p',q}(T) < \infty$ and $L_{p'',q}(\bar{T}) < \infty$ respectively. Choosing $p \in \mathbb{N}_0^d$ with $p_i = \max\{p'_i, p''_i\}$ for $i = 1, \dots, d$, we observe that $L_{p,q}(T) \leq L_{p',q}(T)$ and $L_{p,q}(\bar{T}) \leq L_{p'',q}(\bar{T})$. ■

We can give N a natural topology. Define for each pair $p, q \in \mathbb{N}_0^d$ the

Hilbert space

$$(5.9) \quad \mathbf{H}(p, q) = \{T \in S'(\mathbf{E}) \mid L_{p,q}(T) < \infty\}.$$

It follows from Theorem 5.2 that

$$(5.10) \quad N = \bigcap_{q \in \mathbf{N}_0^d} \bigcup_{p \in \mathbf{N}_0^d} \mathbf{H}(p, q).$$

This identity induces a locally convex topology upon N making it a complete, nuclear and reflexive space. A net $(T_i)_{i \in I}$ of elements in N is convergent to zero in this topology, if and only if there to each $q \in \mathbf{N}_0^d$ is a $p \in \mathbf{N}_0^d$ such that $\lim_{i \in I} L_{p,q}(T_i) = 0$. The inequality (5.8) hence entails that the mapping

$$N \times S(\mathbf{E}) \ni (T, \xi) \longmapsto T \circ \xi \in S(\mathbf{E})$$

is jointly continuous.

Lemma 5.4. *Let T, S be tempered distributions and take $p, q, s \in \mathbf{N}_0^d$. Then we have*

$$(5.11) \quad L_{p,q}(T \circ S) \leq L_{s,q}(T) L_{p,s}(S).$$

Proof. Put $\alpha_{n,m} = T(a_{m,n})$ and $\beta_{n,m} = S(a_{m,n})$ for every pair $n, m \in \mathbf{N}_0^d$. We obtain by Cauchy-Schwarz inequality that

$$\begin{aligned} L_{p,q}(T \circ S)^2 &= \sum_{n,m \in \mathbf{N}_0^d} \left| \sum_{k \in \mathbf{N}_0^d} \alpha_{n,k} \beta_{k,m} \right|^2 (1+n)^q (1+m)^{-p} \\ &= \sum_{n,m \in \mathbf{N}_0^d} \left| \sum_{k \in \mathbf{N}_0^d} \alpha_{n,k} (1+k)^{-s/2} \beta_{k,m} (1+k)^{s/2} \right|^2 (1+n)^q (1+m)^{-p} \\ &\leq \sum_{n,m \in \mathbf{N}_0^d} \left(\sum_{k \in \mathbf{N}_0^d} |\alpha_{n,k}|^2 (1+k)^{-s} \right) \left(\sum_{k \in \mathbf{N}_0^d} |\beta_{k,m}|^2 (1+k)^s \right) (1+n)^q (1+m)^{-p} \\ &= L_{s,q}(T)^2 L_{p,s}(S)^2, \end{aligned}$$

and the assertion is proved. ■

The above lemma shows that N is a topological algebra. Suppose that $(T_i)_{i \in I}$ and $(S_i)_{i \in I}$ are two nets in N converging to zero. We can to a given $q \in \mathbf{N}_0^d$ choose $s \in \mathbf{N}_0^d$ such that $\lim_{i \in I} L_{s,q}(T_i) = 0$, and again choose $p \in \mathbf{N}_0^d$ such that $\lim_{i \in I} L_{p,s}(S_i) = 0$. It then follows from the lemma that $\lim_{i \in I} L_{p,q}(T_i \circ S_i) = 0$.

Lemma 5.5. *Let \mathbf{K} be a subspace of finite codimension of a Hilbert space \mathbf{H} . A dense subspace $\mathbf{D} \subseteq \mathbf{H}$ intersects \mathbf{K} in a subspace which is dense in \mathbf{K} .*

Proof. We first prove the assertion when \mathbf{K} is of codimension 1. The general case then follows by repeated application of this result. We therefore

assume that $\mathbb{K} = \{\xi | (a|\xi) = 0\}$ for some vector $a \in \mathbb{H}$. We furthermore choose a vector $d \in \mathbb{D}$ such that $(a|d) = 1$. The linear and continuous mapping given by

$$\Phi(\xi) = \xi - (a|\xi)d$$

maps \mathbb{H} onto \mathbb{K} and hence \mathbb{D} into a dense subset of \mathbb{K} . The assertion follows by noticing that Φ maps \mathbb{D} into itself. ■

Proposition 5.6. *The elements of M mapped by π into finite rank operators in $B(I_\Omega)$ are exactly the elements of the form*

$$(5.12) \quad a = \sum_{i=1}^n \xi_i \circ \bar{\eta}_i \quad \xi_i, \eta_i \in I_\Omega \cap S(\mathbb{E}), \quad n \in \mathbb{N},$$

where the sets of vectors (ξ_1, \dots, ξ_n) and (η_1, \dots, η_n) are linearly independent. The representation π is defined in equation (2.20).

Proof. The finite rank operators on I_Ω are according to Proposition 2.1 exactly of the form (5.12) with the vectors $\xi_i, \eta_i \in I_\Omega, i = 1, \dots, d$. We can in finite many steps rewrite (5.12) such that both sets of vectors are linearly independent. If each of the vectors belong to $S(\mathbb{E})$, then so does a . If on the other hand it is known that $a \in M$, then we consider for each $i = 1, \dots, n$ the orthogonal complement to $\text{span}\{\xi_j | j \neq i\}$ which is of finite codimension. We can apply Lemma 5.5 to find a $f \in S(\mathbb{E})$ orthogonal to η_j for $j \neq i$, but not orthogonal to η_i . We have that $\pi(a)f = 2^{-d}(\eta_i|f)_2 \xi_i$ according to (2.22) and derive that $\xi_i \in S(\mathbb{E})$ for $i = 1, \dots, d$. Similarly, by considering \bar{a} , we derive that $\eta_i \in S(\mathbb{E})$ for $i = 1, \dots, d$. ■

We have in particular proved that every element $a \in M$ for which $\pi(a)$ is a finite rank operator on I_Ω belongs to $S(\mathbb{E})$.

We defined the twisted product of a tempered distribution with a Schwartz function in equation (5.1). If S is a tempered distribution and $T \in N$, then we can similarly define the twisted product $S \circ T$ by setting

$$(5.13) \quad (S \circ T)(\xi) = S(T \circ \xi) \quad \forall \xi \in S(\mathbb{E}).$$

It follows from (5.8) that the linear functional $S \circ T$ defined on $S(\mathbb{E})$ in this way is continuous and thus a tempered distribution. Furthermore, the mapping

$$S'(\mathbb{E}) \times N \ni (S, T) \longmapsto S \circ T \in S'(\mathbb{E})$$

is jointly continuous.

Likewise, if $T \in \bar{N}$, then the product $T \circ S$ can be defined by setting

$$(5.14) \quad (T \circ S)(\xi) = S(\xi \circ T) \quad \forall \xi \in S(\mathbb{E}).$$

We obtain that $T \circ S$ is a tempered distribution, and that the product is jointly continuous in $\bar{N} \times S'(\mathbb{E})$.

The question of associativity of the Moyal product is not obvious outside the algebras N and \bar{N} . In fact, the property may fail to be true. However, we have the following results.

Proposition 5.7. *Let S be a tempered distribution, then*

- (1) $(S \circ T_1) \circ T_2 = S \circ (T_1 \circ T_2) \quad \forall T_1, T_2 \in N,$
- (2) $T_1 \circ (T_2 \circ S) = (T_1 \circ T_2) \circ S \quad \forall T_1, T_2 \in \bar{N},$
- (3) $T_1 \circ (\xi \circ T_2) = T_1 \circ (\xi \circ T_2) \quad \forall T_1 \in N \forall T_2 \in \bar{N} \forall \xi \in S(\mathbf{E}),$
- (4) $(T_2 \circ S) \circ T_1 = T_2 \circ (S \circ T_1) \quad \forall T_1 \in N \forall T_2 \in \bar{N}.$

Proof. We apply definition (5.13) and make use of (5.6) to obtain

$$\begin{aligned} ((S \circ T_1) \circ T_2)(\eta) &= (S \circ T_1)(T_2 \circ \eta) \\ &= S(T_1 \circ (T_2 \circ \eta)) \\ &= S((T_1 \circ T_2) \circ \eta) \\ &= (S \circ (T_1 \circ T_2))(\eta) \quad \forall \eta \in S(\mathbf{E}). \end{aligned}$$

The second part follows by taking the complex conjugates. Since $\xi \circ T_2 \in S(\mathbf{E})$ and \bar{N} is an associative algebra, we obtain

$$\begin{aligned} (T_1 \circ (\xi \circ T_2))(\eta) &= T_1((\xi \circ T_2) \circ \eta) \\ &= T_1(\xi \circ (T_2 \circ \eta)) \\ &= ((T_2 \circ \eta) \circ T_1)(\xi) \\ &= (T_2 \circ \eta)(T_1 \circ \xi) \\ &= T_2(\eta \circ (T_1 \circ \xi)) \\ &= ((T_1 \circ \xi) \circ T_2)(\eta) \quad \forall \eta \in S(\mathbf{E}), \end{aligned}$$

which gives the third assertion. Finally,

$$\begin{aligned} ((T_2 \circ S) \circ T_1)(\eta) &= (T_2 \circ S)(T_1 \circ \eta) \\ &= S((T_1 \circ \eta) \circ T_2) \\ &= S(T_1 \circ (\eta \circ T_2)) \\ &= (S \circ T_1)(\eta \circ T_2) \\ &= (T_2 \circ (S \circ T_1))(\eta) \quad \forall \eta \in S(\mathbf{E}), \end{aligned}$$

where we have used (3) in the calculation. ■

A tempered distribution $S \in S'(\mathbf{E})$ is said to be bounded, if the set

$$\{S(\xi \circ \bar{\eta}) \mid \xi, \eta \in S(\mathbf{E}), \|\xi\|_2 \leq 1, \|\eta\|_2 \leq 1\}$$

is a bounded subset of the complex plane. The norm $\|S\|$ of a bounded distribution S is defined as the radius in the smallest closed disk with center in 0 that contains the above set. Since, according to (5.1),

$$(5.15) \quad \begin{aligned} S(\xi \circ \bar{\eta}) &= (S \circ \xi)(\bar{\eta}) \\ &= 2^{-d}(\eta | S \circ \xi)_2 \quad \forall \xi, \eta \in S(\mathbb{E}), \end{aligned}$$

it follows that a tempered distribution S is bounded, if and only if $\pi(S)$ is a well-defined bounded operator on I_Ω . The set of bounded distributions is thus an algebra under the Moyal product.

Take $T \in \bar{N}$, $\xi \in L^2(\mathbb{E}, dv)$ and let S be a bounded distribution. Notice that the product $T \circ S$ is well-defined according to (5.14). We define the Moyal product of $T \circ S$ and ξ by setting

$$\begin{aligned} ((T \circ S) \circ \xi)(\eta) &= (T \circ S)(\xi \circ \eta) \\ &= S((\xi \circ \eta) \circ T) \\ &= S(\xi \circ (\eta \circ T)) \quad \forall \eta \in S(\mathbb{E}). \end{aligned}$$

Since the map $\eta \rightarrow \eta \circ T$ is continuous from $S(\mathbb{E})$ into itself, and hence in particular continuous from $S(\mathbb{E})$ into $L^2(\mathbb{E}, dv)$, we deduce from (5.15) that $(T \circ S) \circ \xi$ is a well-defined tempered distribution. We furthermore obtain that

$$\begin{aligned} ((T \circ S) \circ \xi)(\eta) &= S(\xi \circ (\eta \circ T)) \\ &= (S \circ \xi)(\eta \circ T) \\ &= (T \circ (S \circ \xi))(\eta) \quad \forall \eta \in S(\mathbb{E}). \end{aligned}$$

We have thus proved the identity

$$(5.16) \quad (T \circ S) \circ \xi = T \circ (S \circ \xi),$$

for $T \in \bar{N}$, bounded distributions S , and $\xi \in L^2(\mathbb{E}, dv)$.

§6. The Pre-Dual Algebra

Let $T \in S'(\mathbb{E})$ be a tempered distribution and suppose that there exists a $p \in \mathbb{N}_0^d$ such that

$$(6.1) \quad L_{p,q}(T) < \infty \quad \forall q \in \mathbb{N}_0^d.$$

We denote by N_* the set of all such tempered distributions. It is clear that N_* is a proper subset of N .

Proposition 6.1. *N_* is an algebra with twisted product as multiplication. It is left invariant under symplectic Fourier transformation.*

Proof. If a tempered distribution satisfies (6.1) with respect to some $p \in \mathbb{N}_0^d$, then it will satisfy the same condition with respect to any $p' \in \mathbb{N}_0^d$ with $p'_i \geq p_i$. It therefore follows that N_* is a subspace of N . Lemma 5.4 states that

$$L_{p,q}(T \circ S) \leq L_{s,q}(T) L_{p,s}(S),$$

for arbitrary $T, S \in S'(\mathbb{E})$. If T, S are in N_* , then there exists a $p \in \mathbb{N}_0^d$ such that $L_{p,s}(S) < \infty$ for every $s \in \mathbb{N}_0^d$. Choosing $s \in \mathbb{N}_0^d$ such that $L_{s,q}(T) < \infty$ for every $q \in \mathbb{N}_0^d$, we obtain that $L_{p,q}(T \circ S) < \infty$ for every $q \in \mathbb{N}_0^d$. The statements of the proposition now follow readily. ■

It follows from the definition in (6.1) and (5.9) that

$$(6.2) \quad N_* = \bigcup_{p \in \mathbb{N}_0^d} \bigcap_{q \in \mathbb{N}_0^d} \mathbf{H}(p, q).$$

Consequently,

$$(6.3) \quad N_* = \bigcup_{p \in \mathbb{N}_0^d} N_*(p), \quad \text{where } N_*(p) = \bigcap_{q \in \mathbb{N}_0^d} \mathbf{H}(p, q).$$

Each $N_*(p)$ is naturally given the Fréchet topology induced by the upward filtering set of norms $\{L_{p,q} | q \in \mathbb{N}_0^d\}$. Since $N_*(p) \subseteq N_*(p')$ whenever $p_i \leq p'_i$ for $i = 1, \dots, d$, the union N_* can be given the limes topology. A functional ϕ on N_* is thus continuous, if and only if there to each $p \in \mathbb{N}_0^d$ exists a $q \in \mathbb{N}_0^d$ and a constant C such that

$$(6.4) \quad |\phi(\xi)| \leq C L_{p,q}(\xi) \quad \forall \xi \in N_*(p).$$

We notice that $N_*(p)' \subset S'(\mathbb{E})$ for every $p \in \mathbb{N}_0^d$.

Theorem 6.2.

- (1) Let $T \in S'(\mathbb{E})$ and take $p \in \mathbb{N}_0^d$. Then $T \in N_*(p)'$ if and only if $L_{q,p}(T) < \infty$ for some $q \in \mathbb{N}_0^d$.
- (2) N is the topological dual of N_* .
- (3) $T \circ N_*(p) \subseteq N_*(p) \quad \forall T \in N \forall p \in \mathbb{N}_0^d$.

Proof. Suppose $T \in N_*(p)'$ and put $\alpha_{n,m} = T(a_{n,m})$ for $n, m \in \mathbb{N}_0^d$, cf. (4.16) and (4.19). We choose $q \in \mathbb{N}_0^d$ according to equation (6.4) and obtain that

$$|T(\xi)| \leq C L_{p,q}(\xi) \quad \forall \xi \in N_*(p).$$

Hence T is continuous on $\mathbf{H}(p, q)$ and thus given by

$$T(\xi) = (\eta | \xi)_{\mathbf{H}(p,q)} \quad \forall \xi \in N_*(p),$$

for an $\eta \in \mathbf{H}(p, q)$. We set $\beta_{n,m} = 2^{-d}(a_{n,m} | \eta)_2$ for $n, m \in \mathbb{N}_0^d$ and conclude that $\alpha_{m,n} = \overline{\beta_{m,n}}(1+n)^{-p}(1+m)^q$ for each $n, m \in \mathbb{N}_0^d$. Consequently, we have

$$\begin{aligned} L_{q,p}(T)^2 &= \sum_{n,m \in \mathbb{N}_0^d} |\alpha_{n,m}|^2 (1+n)^p (1+m)^{-q} \\ &= \sum_{n,m \in \mathbb{N}_0^d} |\beta_{m,n}|^2 (1+m)^q (1+n)^{-p} \\ &= \|\eta\|_{\dot{H}(p,q)}^2 < \infty. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |T(\xi)| &= \left| \sum_{n,m \in \mathbb{N}_0^d} \alpha_{m,n} \beta_{n,m} \right| \\ &= \left| \sum_{n,m \in \mathbb{N}_0^d} \alpha_{m,n} (1+m)^{p/2} (1+n)^{-q/2} \beta_{n,m} (1+n)^{q/2} (1+m)^{-p/2} \right| \\ &\leq L_{q,p}(T) L_{p,q}(\xi) \quad \forall \xi \in N_*(p), \end{aligned}$$

for every $q \in \mathbb{N}_0^d$. We thus conclude that T is continuous on $N_*(p)$ provided that $L_{q,p}(T)$ is finite for some $q \in \mathbb{N}_0^d$. This proves part (1) of the theorem. Part (2) is an immediate consequence of part (1).

Take finally $T \in N$, $p \in \mathbb{N}_0^d$ and $\xi \in N_*(p)$. According to Lemma 5.4 we have that

$$L_{p,q}(T \circ \xi) \leq L_{s,q}(T) L_{p,s}(\xi) \quad \forall q, s \in \mathbb{N}_0^d.$$

If we to a given $q \in \mathbb{N}_0^d$ choose $s \in \mathbb{N}_0^d$ such that $L_{s,q}(T) < \infty$, then we can conclude that $L_{p,q}(T \circ \xi) < \infty$. ■

Let $T \in N_*$ satisfy condition (6.1) with respect to $p \in \mathbb{N}_0^d$. Again applying Lemma 5.4 we have

$$(6.5) \quad L_{p,q}(T \circ T) \leq L_{p,p}(T) L_{p,q}(T) < \infty \quad \forall q \in \mathbb{N}_0^d.$$

That is, the same $p \in \mathbb{N}_0^d$ can be used for $T \circ T$ as for T in condition (6.1). Repetition of this argument yields that

$$(6.6) \quad L_{p,q}((T \circ)^n) \leq L_{p,p}(T)^{n-1} L_{p,q}(T) < \infty \quad \forall q \in \mathbb{N}_0^d \forall n \in \mathbb{N}_0^d.$$

Theorem 6.3. *Let $T \in N_*$. Then*

$$(6.7) \quad \sum_{n=1}^{\infty} \frac{1}{n!} (T \circ)^n \in N_*.$$

Proof. Let $T \neq 0$ satisfy (6.1) with respect to $p \in \mathbb{N}_0^d$. It follows from equation (6.6) that

$$L_{p,q} \left(\sum_{n=1}^{\infty} \frac{1}{n!} (T \circ)^n \right) \leq \sum_{n=1}^{\infty} \frac{1}{n!} L_{p,q}((T \circ)^n)$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n!} L_{p,p}(T)^{n-1} L_{p,q}(T) = \frac{L_{p,q}(T)}{L_{p,p}(T)} (\exp(L_{p,p}(T)) - 1) < \infty,$$

for each $q \in \mathbb{N}_0^d$. ■

We naturally denote and define the twisted exponential of T by

$$(6.8) \quad \exp^\circ(T) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (T^\circ)^n.$$

Theorem 6.3 therefore states that the twisted exponential $\exp^\circ(T)$ of an element $T \in N_*$ exists as an element of N and that $\exp^\circ(T) - 1 \in N_*$.

Theorem 6.4. *Let $T \in N_*$ and choose a $p \in \mathbb{N}_0^d$ such that equation (6.1) is satisfied for every $q \in \mathbb{N}_0^d$. If $L_{p,p}(T) < 1$, then $1 - T$ has an inverse in the algebra with unit obtained from N_* by adjoining the constant function 1.*

Proof. Suppose $T \neq 0$ and p are chosen as in the assumptions of the theorem. We define the element

$$(6.9) \quad (1 - T)^\circ^{-1} = 1 + \sum_{n=1}^{\infty} (T^\circ)^n,$$

and applying equation (6.9) we obtain

$$\begin{aligned} L_{p,q}(\sum_{n=1}^{\infty} (T^\circ)^n) &\leq \sum_{n=1}^{\infty} L_{p,q}((T^\circ)^n) \leq \sum_{n=1}^{\infty} L_{p,p}(T)^{n-1} L_{p,q}(T) \\ &= \frac{L_{p,q}(T)}{L_{p,p}(T)(1 - L_{p,p}(T))} < \infty, \end{aligned}$$

for each $q \in \mathbb{N}_0^d$. It follows that $(1 - T)^\circ^{-1} - 1 \in N_*$. Finally, we observe that

$$(1 - T)^\circ (1 - T)^\circ^{-1} = (1 - T)^\circ (1 + \sum_{n=1}^{\infty} (T^\circ)^n) = 1,$$

and similarly that $(1 - T)^\circ^{-1} \circ (1 - T) = 1$. ■

§7. The Order Structure

For arbitrary tempered distributions $T \in S'(\mathbb{E})$, we introduce the real and imaginary part of T by setting

$$(7.1) \quad \operatorname{Re} T = \frac{T + \bar{T}}{2}, \quad \operatorname{Im} T = \frac{T - \bar{T}}{2i}.$$

The real and imaginary parts of T are real, tempered distributions, and $T = \operatorname{Re} T + i \operatorname{Im} T$. A tempered distribution $S \in S'(\mathbb{E})$ is said to be positive, if

$$(7.2) \quad S(\xi \circ \bar{\xi}) \geq 0 \quad \forall \xi \in S(\mathbf{E}).$$

If S is real, and $S(\xi \circ \bar{\xi}) = 0$ for every $\xi \in S(\mathbf{E})$, then we conclude by polarisation that $S \circ \eta = 0$ for every $\xi \in S(\mathbf{E})$, and hence $S = 0$. We can therefore define a partial order relation \leq on the real vector space of real, tempered distributions on \mathbf{E} by setting $T \leq S$, if $S - T$ is positive.

Lemma 7.1. *Let ξ and η be positive, tempered distributions on \mathbf{E} and suppose that $\xi \leq \eta$.*

- (1) *If η is a square-integrable function, then so is ξ , and $\|\xi\|_2 \leq \|\eta\|_2$.*
- (2) *$\eta \in S(\mathbf{E})$ implies that $\xi \in S(\mathbf{E})$.*

Proof. The operator $\pi(\eta)$ is of the Hilbert-Schmidt class with $2^{-d/2}\|\eta\|_2$ as Hilbert-Schmidt norm, and $0 \leq \pi(\xi) \leq \pi(\eta)$. The operator $\pi(\xi)$ is hence of the form $a\pi(\eta)a^*$ with $\|a\| \leq 1$, and

$$\begin{aligned} \text{Tr}(\pi(\xi)^2)^{1/2} &= \text{Tr}(a\pi(\eta)a^*a\pi(\eta)a^*)^{1/2} \\ &\leq \text{Tr}(a\pi(\eta)^2a^*)^{1/2} \\ &= \text{Tr}(\pi(\eta)a^*a\pi(\eta))^{1/2} \\ &\leq \text{Tr}(\pi(\eta)^2)^{1/2}. \end{aligned}$$

The first part of the Lemma thus follows from operator theory. If $\eta \in S(\mathbf{E})$, then it follows that

$$\|S^{p/2} \circ \eta \circ S^{p/2}\|_2 = r_p(\eta) < \infty \quad \forall p \in \mathbb{N}_0^d,$$

cf. (4.6), where S is defined by

$$(7.3) \quad S = \sum_{n \in \mathbb{N}_0^d} (1+n)a_{n,n}.$$

It then follows from (1) that $S^{p/2} \circ \xi \circ S^{p/2} \in L^2(\mathbf{E}, dv)$ for each $p \in \mathbb{N}_0^d$ which implies (2), cf. (4.8). ■

Lemma 7.2.

- (1) *Let ξ be a positive, tempered distribution belonging to the Schwartz space $S(\mathbf{E})$. There exists a uniquely defined, positive, tempered distribution η in $S(\mathbf{E})$ such that*

$$\eta \circ \eta = \xi.$$

- (2) *Let ξ be a real, tempered distribution belonging to the Schwartz space $S(\mathbf{E})$. There exists positive, tempered distributions ξ_+ and ξ_- in $S(\mathbf{E})$ such that $\xi = \xi_+ - \xi_-$.*
- (3) *$\text{span}\{S(\mathbf{E}) \circ S(\mathbf{E})\} = S(\mathbf{E})$.*

Proof. Let $p \in \mathbb{N}_0^d$ be arbitrary. We observe that

$$\begin{aligned} \|\pi(S^{p/2} \circ \xi \circ S^{p/2})\| &\leq 2^{-d/2} \|S^{p/2} \circ \xi \circ S^{p/2}\|_2 \\ &= 2^{-d/2} r_p(\xi) < \infty, \end{aligned}$$

where S is the tempered distribution defined in (7.3). Since the distribution $S^{p/2} \circ \xi \circ S^{p/2}$ is positive, we have

$$S^{p/2} \circ \xi \circ S^{p/2} \leq 2^{-d/2} r_p(\xi),$$

and consequently

$$\xi \leq 2^{-d/2} r_p(\xi) S^{-p}.$$

Let η be the uniquely defined, positive tempered distribution for which $\pi(\eta)^2 = \pi(\xi)$. Then $\eta \leq 2^{-d/4} r_p(\xi)^{1/2} S^{-p/2}$, and

$$S^{(p-2)/4} \circ \eta \circ S^{(p-2)/4} \leq 2^{-d/4} r_p(\xi)^{1/2} S^{-1}.$$

It follows, that $r_{(p-2)/2}(\eta) \leq 2^{-d/4} r_p(\xi)^{1/2} \|S^{-1}\|_2$. Since p is arbitrary, we conclude that $\eta \in S(\mathbf{E})$ which proves (1). We take, in order to prove (2), a real element ξ of $S(\mathbf{E})$ and consider $|\xi| = (\xi^2)^{1/2}$ which according to (1) belongs to $S(\mathbf{E})$. Possibly by considering the representation π , we conclude that $-|\xi| \leq \xi \leq |\xi|$. The tempered distributions ξ_+ and ξ_- defined by

$$\xi_+ = \frac{1}{2}(|\xi| + \xi), \quad \xi_- = \frac{1}{2}(|\xi| - \xi),$$

are thus positive elements of $S(\mathbf{E})$, and $\xi = \xi_+ - \xi_-$. We conclude from (2) that each Schwartz function on \mathbf{E} is a linear combination of at most four elements from $S(\mathbf{E}) \circ S(\mathbf{E})$. ■

The proof of Lemma 7.2 (1) actually shows that the square root is a continuous mapping of the positive part of the Schwartz space into itself. We also conclude that the representation π is a continuous mapping of $S(\mathbf{E})$ into the space of trace-class operators on I_Ω , cf. also [27].

We notice from Lemma 7.2 (2) that a positive, tempered distribution on \mathbf{E} is real. However, even when given by a kernel, cf. equation (5.2), the kernel may not be pointwise positive. We say that a real, tempered distribution T is strictly positive, if there exists a positive real number ε such that $\varepsilon \leq T$.

Proposition 7.3. *Let $(T_j)_{j \in J}$ be a downward filtering net of positive, tempered distributions on \mathbf{E} . There exists a positive, tempered distribution T on \mathbf{E} such that*

$$T_j \succcurlyeq T.$$

The net $(T_j)_{j \in J}$ is converging to T in the topology of $S'(\mathbf{E})$.

Proof. Take $\xi \in S(\mathbb{E})$. The net of positive numbers $(T_j(\xi \circ \bar{\xi}))_{j \in J}$ is downward filtering and hence convergent. By polarisation we conclude that the net of numbers $(T_j(\xi \circ \bar{\eta}))_{j \in J}$ is convergent for each $\xi, \eta \in S(\mathbb{E})$. We can consequently define a linear functional T on $S(\mathbb{E})$, cf. Lemma 7.2 (3), by setting

$$T(\xi) = \lim_{j \in J} T_j(\xi) \quad \forall \xi \in S(\mathbb{E}).$$

The so-defined linear functional T on $S(\mathbb{E})$ is the pointwise limit of a net of continuous linear functionals. We put

$$\alpha_{n,m} = T(a_{m,n}), \quad \text{and} \quad \alpha_{n,m}^j = T_j(a_{m,n}) \quad \forall j \in J \forall n, m \in \mathbb{N}_0^d.$$

Define for $k \in \mathbb{N}$ the projection P_k by setting

$$(7.4) \quad P_k = \sum_{|n| \leq k} t_{n,n}.$$

We consider the linear functionals ϕ and ϕ_j defined by setting $\phi(\xi) = T(P_k \circ \xi \circ P_k)$, and $\phi_j(\xi) = T_j(P_k \circ \xi \circ P_k)$ for every $\xi \in S(\mathbb{E})$ and obtain

$$\sum_{|n| \leq k, |m| \leq k} \alpha_{n,m} t_{n,m} = \phi \leq \phi_j = \sum_{|n| \leq k, |m| \leq k} \alpha_{n,m}^j t_{n,m}.$$

Take $j \in J$ and choose $p \in \mathbb{N}_0^d$ such that $r_{-p}(T_j) < \infty$. By applying $S^{p/2}$ from the left and the right hand side in the inequality above and by making use of Lemma 7.1 (1), we conclude that

$$\sum_{|n| \leq k, |m| \leq k} |\alpha_{n,m}|^2 (1+n)^{-p} (1+m)^{-p} \leq r_{-p}(T_j)^2,$$

for every $k \in \mathbb{N}$. This implies the continuity of T . We furthermore notice that the net eventually converges pointwise inside a bounded subset of $S'(\mathbb{E})$. Since $S'(\mathbb{E})$ is the dual space of a Montel space, its topology coincides with the weak topology on bounded subsets. We hence conclude that the net converges to T in the topology of $S'(\mathbb{E})$. We finally notice that the net also filters downward to T . ■

Lemma 7.4. *Let T be a positive, bounded tempered distribution. Then $1 + T$ is invertible, and the inverse*

$$B(T) = \frac{1}{1 + T}$$

is a tempered distribution for which $0 \leq B(T) \leq 1$. Furthermore,

$$T \circ (B(\bar{T} \circ T) - 1) = (B(T \circ \bar{T}) - 1) \circ \bar{T},$$

for each bounded, tempered distribution T .

Proof. Let K be a positive constant such that $0 \leq T \leq K - 1$ and put $S = K^{-1}(K - 1 - T)$. Then S satisfies $0 \leq S \leq K^{-1}(K - 1) < 1$, and the sequence

$$A_n = \sum_{k=0}^n S^k, \quad n \in \mathbb{N}$$

of positive, tempered distributions is increasing and bounded by $\sum_{k=0}^{\infty} K^{-n} (K - 1)^n = K$. The sequence therefore increases to a positive element $A \in S'(\mathbb{E})$ bounded by the constant K , cf. Proposition 7.3. It follows, that A is the inverse of $1 - S$, and that $B(T) = K^{-1}A$ is the inverse of $K - KS = 1 + T$. To prove the latter part of the Theorem, we notice that $B(\bar{T} \circ T) - 1$ is the limit of an increasing, but bounded sequence of tempered distributions written on the form $p_n(\bar{T} \circ T)$, where p_n for each $n \in \mathbb{N}$ is a polynomium. The constant terms $p_n(0)$ are converging to zero as n tends to infinity. The assertion now follows by observing that $T \circ p(\bar{T} \circ T) = p(T \circ \bar{T}) \circ \bar{T}$ for any polynomium p with vanishing constant term. ■

Theorem 7.5. *Let T be a real element of N . The element $1 + T^2$ is invertible, and the inverse*

$$B(T^2) = \frac{1}{1 + T^2}$$

is a tempered distribution for which $0 \leq B(T^2) \leq 1$. Furthermore, $T \circ B(T^2) = B(T^2) \circ T$.

Proof. We consider the projection P_k defined in (7.4) and notice that it is a Schwartz function. Since $T \in N$, it follows that $T \circ P_k \circ T$ is a Schwartz function, in particular it is a positive, bounded distribution. We can thus apply Lemma 7.4 and conclude that $1 + T \circ P_k \circ T$ is invertible with a positive inverse $B(T \circ P_k \circ T) \in S'(\mathbb{E})$ bounded by 1. Since $(T \circ P_k \circ T)_k$ is increasing (towards T^2), we conclude that the sequence $(B(T \circ P_k \circ T))_k$ is decreasing. By applying Proposition 7.3, we obtain that the sequence decreases to a positive, tempered distribution B . Observe that

$$\lim_{k \rightarrow \infty} B(T \circ P_k \circ T)(\xi \circ (1 + T^2)) = B(\xi \circ (1 + T^2)) \quad \forall \xi \in S(\mathbb{E}),$$

cf. Proposition 7.3. We split $1 + T^2$ into $1 + T \circ P_k \circ T + T \circ (1 - P_k) \circ T$ and write

$$B(T \circ P_k \circ T)(\xi \circ (1 + T^2)) = 2^{-d} \int_{\mathbb{E}} \xi(v) dv + (T \circ (1 - P_k) \circ T)(B(T \circ P_k \circ T) \circ \xi).$$

The last term is evaluated by

$$\begin{aligned} |(T \circ (1 - P_k) \circ T)(B(T \circ P_k \circ T) \circ \xi)| &= |B(T \circ P_k \circ T)(\xi \circ T \circ (1 - P_k) \circ T)| \\ &\leq \|\pi(\xi \circ T \circ (1 - P_k) \circ T)\|_{\text{Tr}} \end{aligned}$$

where we have used (4.21). Since $T \circ (1 - P_k) \circ T$ converges to zero in N as k tends to infinity, we obtain that $\xi \circ T \circ (1 - P_k) \circ T$ converges to zero in $S(\mathbb{E})$. The embedding of $S(\mathbb{E})$ into trace-class operators on L_Ω is continuous so that the last term converges to 0 as k tends to infinity. We conclude that

$$B(\xi \circ (1 + T^2)) = 2^{-d} \int_{\mathbb{E}} \xi(v) dv \quad \forall \xi \in S(\mathbb{E}),$$

hence $(1 + T^2) \circ B = 1$. By taking the complex conjugate, we obtain that B is the inverse of $1 + T^2$. The last part of the statement follows by applying the equality in Theorem 7.4 to the element $P_k \circ T, k \in \mathbb{N}$. We obtain for each $k \in \mathbb{N}$ that

$$(7.5) \quad P_k \circ T \circ (B(T \circ P_k \circ T) - 1) = (B(P_k \circ T^2 \circ P_k) - 1) \circ T \circ P_k.$$

The factor $P_k \circ T$ converges to T in the topology of \bar{N} and $B(T \circ P_k \circ T)$ converges to $B(T^2)$ in $S'(\mathbb{E})$. The joint continuity of the twisted product, cf. (5.14), thus ensures that the left hand side of equation (7.5) converges to $T \circ (B(T^2) - 1)$ in $S'(\mathbb{E})$ for k going to infinity. To examine the right hand side, we first evaluate $B(P_k \circ T^2 \circ P_k)$ taken in an element of the form $\xi \circ (1 + T^2)$, where $\xi \in S(\mathbb{E})$. We split $1 + T^2$ into $1 + P_k \circ T^2 \circ P_k + (T^2 - P_k \circ T^2 \circ P_k)$ and obtain

$$\begin{aligned} &B(P_k \circ T^2 \circ P_k)(\xi \circ (1 + T^2)) \\ &= B(P_k \circ T^2 \circ P_k)(\xi \circ (1 + P_k \circ T^2 \circ P_k) + \xi \circ (T^2 - P_k \circ T^2 \circ P_k)) \\ &= 2^{-d} \int_{\mathbb{E}} \xi(v) dv + B(P_k \circ T^2 \circ P_k)(\xi \circ (T^2 - P_k \circ T^2 \circ P_k)). \end{aligned}$$

The last term is evaluated by

$$|B(P_k \circ T^2 \circ P_k)(\xi \circ (T^2 - P_k \circ T^2 \circ P_k))| \leq \|\pi(\xi \circ (T^2 - P_k \circ T^2 \circ P_k))\|_{\text{Tr}}.$$

Since $P_k \circ T^2 \circ P_k$ converges to T^2 in N , it follows, that the last term converges to zero as k tends to infinity, cf. the argument above. This shows that

$$(7.6) \quad \lim_{k \rightarrow \infty} B(P_k \circ T^2 \circ P_k)(\xi \circ (1 + T^2)) = 2^{-d} \int_{\mathbb{E}} \xi(v) dv \quad \forall \xi \in S(\mathbb{E}).$$

Since $P_k \circ T^2 \circ P_k$ is not increasing towards T^2 , but merely converging in N , we cannot argue that the sequence of positive, tempered distributions $B(P_k \circ T^2 \circ P_k)$ is at all convergent. However, the sequence is bounded by 1 and thus contained in a weakly compact subset of $S'(\mathbb{E})$. A subnet will hence converge weakly (and strongly because $S'(\mathbb{E})$ is a Montel space) towards a positive and bounded,

tempered distribution B' . It follows from (7.6) that $(1 + T^2) \circ B' = 1$. By taking the complex conjugate, we conclude that $B' = B(T^2)$. Finally, we make use of the joint continuity of the twisted product to conclude that a subnet of the right hand side of (7.5) tends to $(B(T^2) - 1) \circ T$ and the proof is complete. \square

Suppose that $T \in \bar{N}$. Since the twisted product of T with an arbitrary L^2 -function is well-defined, cf. (5.14), it makes sense to define the domain $\mathcal{D}_2(T)$ by setting

$$\mathcal{D}_2(T) = \{ \xi \in L^2(\mathbb{E}, dv) \mid T \circ \xi \in L^2(\mathbb{E}, dv) \}.$$

Choose arbitrary $\xi \in \mathcal{D}_2(T)$ and $\eta \in L^2(\mathbb{E}, dv)$. We can apply (5.16) and obtain

$$T \circ (\xi \circ \eta) = (T \circ \xi) \circ \eta \in L^2(\mathbb{E}, dv),$$

hence $\mathcal{D}_2(T)$ is a right ideal of $L^2(\mathbb{E}, dv)$ containing $S(\mathbb{E})$. We have thus constructed an extension of the operator $\pi(T)$ a priori defined only on the intersection of $S(\mathbb{E})$ with I_Ω . If T is real, then $\pi(T)$ is a symmetric operator. The extension, however, may not be, cf. example (8.5).

Proposition 7.6. *If $T \in \mathbb{N}$ is real, then*

$$B(T^2) \circ \mathcal{D}_2(T) \subseteq \mathcal{D}_2(T).$$

Proof. We first notice that

$$\begin{aligned} (\eta \circ B(T^2)) \circ T &= \eta \circ (B(T^2) \circ T) \\ &= \eta \circ (T \circ B(T^2)) \\ &= (\eta \circ T) \circ B(T^2) \in L^2(\mathbb{E}, dv) \quad \forall \eta \in S(\mathbb{E}), \end{aligned}$$

cf. Proposition 5.7 (4) and (2). For every $\xi \in \mathcal{D}_2(T)$ we thus obtain

$$\begin{aligned} (T \circ \xi)(\eta \circ B(T^2)) &= \xi((\eta \circ B(T^2)) \circ T) \\ &= \xi((\eta \circ T) \circ B(T^2)) \\ &= (B(T^2) \circ \xi)(\eta \circ T) \\ &= (T \circ (B(T^2) \circ \xi))(\eta) \quad \forall \eta \in S(\mathbb{E}). \end{aligned}$$

That is,

$$(7.7) \quad B(T^2) \circ (T \circ \xi) = T \circ (B(T^2) \circ \xi) \quad \forall \xi \in \mathcal{D}_2(T),$$

from which the assertion follows. \square

Suppose that $T \in N$ is real and take arbitrary $\eta \in S(\mathbb{E})$. Then $\xi = B(T^2) \circ \eta \in \mathcal{D}_2(T)$, according to Proposition (7.6), and

$$\begin{aligned}
(1 + T^2) \circ \xi &= (1 + T^2) \circ (B(T^2) \circ \eta) \\
&= ((1 + T^2) \circ B(T^2)) \circ \eta \\
&= \eta,
\end{aligned}$$

so $S(\mathbf{E}) \subset (1 + T^2) \circ \mathcal{D}_2(T)$. The range of the action by $1 + T^2$ on $\mathcal{D}_2(T)$ is thus dense in $L^2(\mathbf{E}, dv)$.

§8. The Resolvent Distribution

Lemma 8.1. *Let $T \in N$ be a real distribution and take $\lambda \in \mathbf{C}$ with $\text{Im } \lambda \neq 0$. The linear space $(T - \lambda) \circ N_*$ is dense in N_* .*

Proof. We may without loss of generality assume that $\lambda = i$. We notice that $(T - i) \circ N_*$ is a subspace of N_* , cf. Theorem 6.2. Suppose that it is not dense. Then there exists, according to Hahn-Banach's Theorem, a non-zero element $S \in N'_* = N$ such that

$$(8.1) \quad S((T - i) \circ \xi) = 0 \quad \forall \xi \in N_*.$$

The twisted product $S \circ (T - i)$ is an element of N , and

$$(S \circ (T - i))(\xi) = S((T - i) \circ \xi) = 0 \quad \forall \xi \in N_*.$$

Consequently, $S \circ (T - i) = 0$. We obtain from Theorem 7.5 and Proposition 5.7 (2) that

$$\begin{aligned}
S &= S \circ 1 = S \circ ((1 + T^2) \circ B(T^2)) \\
&= (S \circ (1 + T^2)) \circ B(T^2) \\
&= (S \circ (T - i) \circ (T + i)) \circ B(T^2) \\
&= 0.
\end{aligned}$$

This is a contradiction. ■

Theorem 8.2. *Let $T \in N$ be a real distribution. There is for each $\lambda \in \mathbf{C}$ with $\text{Im } \lambda \neq 0$ a bounded distribution $R(T, \lambda) \in S'(\mathbf{E})$ for which*

$$R(T, \lambda) \circ (T - \lambda) = (T - \lambda) \circ R(T, \lambda) = 1.$$

We denote it as the resolvent distribution for T at the point λ . It satisfies

$$\overline{R(T, \lambda)} = R(T, \bar{\lambda}),$$

$$\|R(T, \lambda) \circ \xi\|_2 \leq |\text{Im } \lambda|^{-1} \|\xi\|_2 \quad \forall \xi \in S(\mathbf{E}),$$

for each $\lambda \in \mathbf{C}$ with $\text{Im } \lambda \neq 0$.

Proof. We first define the resolvent distribution for T at the point i by setting

$$R(T, i) = B(T^2) \circ (T + i),$$

cf. Theorem 7.5. The product is a well-defined tempered distribution, cf. (5.13), and

$$\begin{aligned} R(T, i) \circ (T - i) &= [B(T^2) \circ (T + i)] \circ (T - i) \\ &= B(T^2) \circ [(T + i) \circ (T - i)] \\ &= B(T^2) \circ (1 + T^2) \\ &= 1, \end{aligned}$$

cf. Proposition 5.7 (i). In the general case we define

$$R(T, \lambda) = \frac{1}{\text{Im } \lambda} R\left(\frac{T - \text{Re } \lambda}{\text{Im } \lambda}, i\right)$$

and obtain that $R(T, \lambda) \circ (T - \lambda) = 1$. Since T commutes with $B(T^2)$, it follows that $\overline{R(T, \lambda)} = R(T, \bar{\lambda})$ and consequently that $(T - \lambda) \circ R(T, \lambda) = 1$.

To prove that last part of the statement we first notice that

$$(B(T^2) \circ T) \circ \xi = B(T^2) \circ (T \circ \xi) \in \mathcal{D}_2(T)$$

for every $\xi \in S(\mathbb{E})$, cf. Propositions 5.7 and 7.6. The square of the distribution $B(T^2) \circ T$ can hence be calculated, and we obtain

$$\begin{aligned} ((B(T^2) \circ T) \circ (B(T^2) \circ T))(\xi) &= B(T^2)(T \circ (B(T^2) \circ (T \circ \xi))) \\ &= (B(T^2) \circ (1 - B(T^2)))(\xi) \quad \forall \xi \in S(\mathbb{E}). \end{aligned}$$

It follows that $R(T, i)$ is bounded with square $(B(T^2) \circ T)^2 + B(T^2)^2 = B(T^2) \leq 1$. ■

Proposition 8.3. *Let $T \in N$ be a real distribution. The resolvent equation*

$$(8.3) \quad R(T, \lambda) - R(T, \mu) = (\lambda - \mu)R(T, \mu) \circ R(T, \lambda)$$

is valid for any $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$.

Proof. We obtain by calculation that

$$\begin{aligned} (R(T, \lambda) - R(T, \mu)) \circ (T - \lambda) &= 1 - R(T, \mu) \circ (T - \lambda) \\ &= 1 - R(T, \mu) \circ (T - \mu - (\lambda - \mu)) \\ &= (\lambda - \mu)R(T, \mu). \end{aligned}$$

We can multiply with $R(T, \lambda)$ from the right, cf. Proposition 7.6, and obtain the desired result. ■

Notice that the resolvent equation (8.3) entails that the resolvent

In particular the case where

$$\lambda_n = n(n + 1) \sum_{k=1}^n \frac{1}{k^2} \quad \forall n \in \mathbb{N}.$$

Since $0 < \lambda_n < n(n + 1)\pi^2/6$ for every $n \in \mathbb{N}$, it follows that T is a real element of N . The range $R(T - i)$ is not dense in $L^2(\mathbb{E}, dv)$. The condition for a vector $x \in I_\Omega$ written on the form

$$x = \sum_{n=0}^{\infty} (-i)^{n+1} r_n a_n$$

to be orthogonal to $R(T - i)$ is that

$$\begin{cases} \lambda_1 r_1 &= r_0, \\ \lambda_{n+1} r_{n+1} &= \lambda_n r_{n-1} + r_n \quad \forall n > 0. \end{cases}$$

The solution is uniquely determined by the value of r_0 and is given by the sequence $r_n = r_0/(n + 1)$, $n \in \mathbb{N}_0$. The analogous condition for the vector $x \in I_\Omega$ to be orthogonal to the range $R(T + i)$ is that

$$\begin{cases} \lambda_1 r_1 &= -r_0, \\ \lambda_{n+1} r_{n+1} &= \lambda_n r_{n-1} - r_n \quad \forall n > 0. \end{cases}$$

The solution is uniquely determined by the value of r_0 and is given by the sequence $r_n = (-1)^n r_0/(n + 1)$, $n \in \mathbb{N}_0$.

The resolvent distribution $R(T, i) = B(T^2) \circ (T + i)$ is only unique because we have defined it explicitly in terms of a certain invers, $B(T^2)$, of $1 + T^2$. If we try to solve the equation

$$(1 + T^2) \circ \xi = a_0,$$

we derive that

$$\begin{aligned} (1 + \lambda_1^2)\xi_0 + \lambda_1 \lambda_2 \xi_2 &= 1 \\ \lambda_1 \lambda_2 \xi_0 + (1 + \lambda_2^2 + \lambda_3^2)\xi_2 + \lambda_3 \lambda_4 \xi_4 &= 0 \\ &\vdots \\ \lambda_{2n-1} \lambda_{2n} \xi_{2n-2} + (1 + \lambda_{2n}^2 + \lambda_{2n+1}^2)\xi_{2n} + \lambda_{2n+1} \lambda_{2n+2} \xi_{2n+2} &= 0 \\ &\vdots \end{aligned} \tag{8.4}$$

There is a similar, but homogeneous system of equations for the odd variables ξ_{2n+1} . It is obvious that the above system of equations have infinite many solutions each characterised by the value of ξ_0 . To actually find the solutions, we first notice that

$$\frac{1 + \lambda_{2n}^2 + \lambda_{2n+1}^2}{2n + 1} = \frac{\lambda_{2n-1} \lambda_{2n}}{2n - 1} + \frac{\lambda_{2n+1} \lambda_{2n+2}}{2n + 3}$$

for every $n \in \mathbb{N}$. By setting

$$\xi_{2n} = (-1)^n \frac{p_{2n}}{2n + 1},$$

$$v_n = \frac{\lambda_{2n-1} \lambda_{2n} (2n + 1)}{(2n - 1)(1 + \lambda_{2n}^2 + \lambda_{2n+1}^2)},$$

we obtain that $0 \leq v_n \leq 1$ for every $n \in \mathbb{N}$, and furthermore

$$p_0 = (1 - v_0)p_2 + \frac{1}{1 + \lambda_1^2}$$

$$p_2 = v_1 p_0 + (1 - v_1)p_4$$

$$\vdots$$

$$p_{2n} = v_n p_{2n-2} + (1 - v_n)p_{2n+2}$$

$$\vdots$$

For $n \geq 1$, we derive that

$$p_{2n-2} - p_{2n} = \frac{v_n}{1 - v_n} (p_{2n} - p_{2n-2})$$

$$= \frac{v_n}{1 - v_n} \frac{v_{n-1}}{1 - v_{n-1}} \cdots \frac{v_1}{1 - v_1} (p_2 - p_0)$$

$$= \frac{5}{4(n + 1)^2} \frac{p_2 - p_0}{\theta(2n + 1)\theta(2n + 2)},$$

where

$$\theta(n) = \sum_{i=1}^n \frac{1}{i^2}.$$

It follows that

$$p_{2n} = p_0 - \frac{1}{4} \sum_{i=0}^{n-1} \frac{1}{(i + 1)^2} \frac{1}{\theta(2i + 1)\theta(2i + 2)} \quad \forall n \in \mathbb{N}.$$

The solutions to (8.4) are thus parametrised by the value ξ_0 and given by

$$\xi_{2n} = \frac{(-1)^n}{2n + 1} \xi_0 - \frac{1}{4} \frac{(-1)^n}{2n + 1} \sum_{i=0}^{n-1} \frac{1}{(i + 1)^2} \frac{1}{\theta(2i + 1)\theta(2i + 2)} \quad \forall n \in \mathbb{N}.$$

The first term corresponds to the part of the solution that belongs to the null space of $1 + T^2$. We want to calculate ξ_0 of the particular solution $\xi = B(T^2) \circ a_0$ and therefore examine the equation

$$(1 + T \circ P_k \circ T) \circ \xi = a_0$$

for arbitrary $k \in \mathbb{N}$. This truncates the system of equations (8.4) in the following way

$$\begin{aligned}
 & (1 + \lambda_1^2)\xi_0 + \lambda_1\lambda_2\xi_2 = 1 \\
 & \lambda_1\lambda_2\xi_0 + (1 + \lambda_2^2 + \lambda_3^2)\xi_2 + \lambda_3\lambda_4\xi_4 = 0 \\
 & \qquad \qquad \qquad \vdots \\
 (8.5) \quad & \lambda_{2k-1}\lambda_{2k}\xi_{2k-2} + (1 + \lambda_{2k}^2 + \lambda_{2k+1}^2)\xi_{2k} + \lambda_{2k+1}\lambda_{2k+2}\xi_{2k+2} = 0 \\
 & \lambda_{2k+1}\lambda_{2k+2}\xi_{2k} + (1 + \lambda_{2k+2}^2)\xi_{2k+2} = 0 \\
 & \qquad \qquad \qquad \xi_{2k+4} = 0 \\
 & \qquad \qquad \qquad \vdots
 \end{aligned}$$

There is now a unique solution of ξ_0 for each value of $k \in \mathbb{N}$ given by

$$\begin{aligned}
 \xi_0 = & \frac{1 + (2k + 2)^2(2k + 3)^2\theta(2k + 2)^2}{4(k + 1)^2\theta(2k + 1)\theta(2k + 2)(1 + (2k + 3)^2\theta(2k + 2))} \\
 & + \frac{1}{4} \sum_{i=0}^{k-1} \frac{1}{(i + 1)^2} \frac{1}{\theta(2i + 1)\theta(2i + 2)}.
 \end{aligned}$$

We obtain ξ_0 of the particular solution $\xi = B(T^2) \circ a_0$ by letting k tend to infinity. It is given by

$$\xi_0 = \frac{6}{\pi^2} + \frac{1}{4} \sum_{i=1}^{\infty} \frac{1}{(i + 1)^2} \frac{1}{\theta(2i + 1)\theta(2i + 2)}.$$

Finally we realize that the all the odd variables ξ_{2n+1} vanish for this particular solution. We can proceed in this way and construct the particular inverse $B(T^2)$ and then the resolvent $R(T, i)$

The above example shows that the resolvent distribution cannot simply be constructed from the operator $\pi(T) - \lambda$, because the inverse may not be densely defined in I_Ω .

§9. Measure and Integration Theory

Positive measures with values in $S'(\mathbf{E})$.

Let (X, \mathcal{S}) be a measure space X with a σ -algebra \mathcal{S} . A positive measure μ with values in $S'(\mathbf{E})$ is a countable additive set map

$$(9.1) \quad \mu: \mathcal{S} \longrightarrow S'(\mathbf{E})_+,$$

mapping the σ -algebra \mathcal{S} into positive, tempered distributions. The measure is

said to be a probability measure, if $\mu(X) = 1$. The range $\mu(\mathcal{S})$ of a probability measure μ is mapped by the representation π into bounded, positive operators on I_{Ω} . We shall only be concerned with two measure spaces, the closed interval $[0, 2\pi]$ and the real line \mathbb{R} , both equipped with the system of Borel sets as σ -algebra. We need the following generalization of Helly's theorem:

Lemma 9.1. *The set of probability measures on the interval $[0, 2\pi]$ with values in $S'(\mathbb{E})$ is compact in the topology of weak convergence.*

Proof. Let $(\mu_j)_{j \in J}$ be a universal net of probability measures on $[0, 2\pi]$ with values in $S'(\mathbb{E})$. We consider for each $\xi \in S(\mathbb{E})$ and $j \in J$ the ordinary measure μ_j^ξ with mass $\|\xi\|_2^2$ given by

$$(9.2) \quad \mu_j^\xi(B) = \mu_j(B)(\xi \circ \bar{\xi}) \quad \forall B \in \mathcal{S}.$$

According to Helly's theorem this net of measures is weakly convergent towards a positive measure μ^ξ on $[0, 2\pi]$ with total mass $\|\xi\|_2^2$. For each positive continuous function f on $[0, 2\pi]$, we define a quadratic form on $S(\mathbb{E})$ given by

$$(9.3) \quad \begin{aligned} \langle \xi | \xi \rangle_f &= \mu^\xi(f) = \int_0^{2\pi} f(\theta) d\mu^\xi(\theta) \\ &= \lim_{j \in J} \mu_j^\xi(f) \quad \forall \xi \in S(\mathbb{E}). \end{aligned}$$

The quadratic form is the limit of quadratic forms satisfying the parallelogram identity and is therefore the diagonal of a positive definite, sesqui-linear form. Furthermore,

$$|\langle \xi | \xi \rangle_f| \leq \sup \{f(t) | t \in [0, 2\pi]\} \|\xi\|_2^2 \quad \forall \xi \in S(\mathbb{E}).$$

There exists thus a bounded, tempered distribution $\mu(f)$ such that

$$(9.4) \quad \begin{aligned} \mu(f)(\xi \circ \bar{\xi}) &= \langle \xi | \xi \rangle_f \\ &= \lim_{j \in J} \mu_j(f)(\xi \circ \bar{\xi}) \quad \forall \xi \in S(\mathbb{E}). \end{aligned}$$

We have defined a Radon measure μ on $[0, 2\pi]$ with values in $S'(\mathbb{E})$. It is the weak limit of the net $(\mu_j)_{j \in J}$ and maps positive, continuous and bounded functions into positive, tempered distributions. The Radon measure μ satisfies $\mu(1) = 1$. We have to show that it is induced by a probability measure on $[0, 2\pi]$ with values in $S'(\mathbb{E})$. For each compact $K \subseteq [0, 2\pi]$, we denote by $\mathcal{C}(K)$ the set of continuous functions $f: [0, 2\pi] \mapsto [0, 1]$ for which $f(t) = 1$ for every t in K . We define

$$(9.5) \quad \mu(K) = \inf \{ \mu(f) | f \in \mathcal{C}(K) \}.$$

Since the net of functions in $\mathcal{C}(K)$ is downward filtering and the corresponding operators $\pi(\mu(f))$ are positive and bounded, it follows that $\mu(K)$ is a well-defined, positive definite tempered distribution. Let K_1 and K_2 be disjoint compact subsets of $[0, 2\pi]$. Since every open set G with $K_1 \cup K_2 \subset G$ contains disjoint open subsets G_1 and G_2 with $K_1 \subset G_1$ and $K_2 \subset G_2$, it follows that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. We have thus defined a finitely additive set function μ on the system of compact subsets of $[0, 2\pi]$ for which $\mu([0, 2\pi]) = 1$. For each open set $G \subseteq [0, 2\pi]$, we define

$$(9.6) \quad \mu_*(G) = \sup\{\mu(K) \mid K \subset G, K \text{ compact}\},$$

and notice that $\mu_*(G)$ is a positive, tempered distribution bounded by 1. The essential property for μ to satisfy in order to be extendable to a probability measure on $[0, 2\pi]$ with values in $S'(\mathbb{E})$ is that

$$(9.7) \quad \mu(K) = \inf\{\mu_*(G) \mid K \subset G, G \text{ open}\},$$

for every compact $K \subset [0, 2\pi]$. Suppose that (9.7) is not satisfied for some compact $K \subseteq [0, 2\pi]$. Then there is a $\xi \in S(\mathbb{E})$ and an $\varepsilon > 0$ such that

$$(9.8) \quad \mu^\xi(K) < \varepsilon + \sup\{\mu^\xi(F) \mid F \subset G, F \text{ compact}\},$$

for every open set $G \subseteq [0, 2\pi]$ with $K \subset G$. But this contradicts the regularity of the measure μ^ξ . We have thus established (9.7) and can proceed as is usually done in measure theory to define

$$(9.9) \quad \mu(B) = \inf\{\mu_*(G) \mid B \subseteq G \text{ open}\},$$

for each Borel set $B \subseteq [0, 2\pi]$. We have that $\mu(B) \in S'(\mathbb{E})$ is positive and bounded. The property (9.7) readily implies that μ is a probability measure on $[0, 2\pi]$ with values in $S'(\mathbb{E})$. ■

We are mostly interested in measures on $X = [0, 2\pi]$ or $X = \mathbb{R}$ with values in $S'(\mathbb{E})$. Such a measure can be constructed from a map $f: X \mapsto S'(\mathbb{E})_+$ satisfying

- (1) $x \rightarrow f(x)(\xi)$ is continuous for every $\xi \in S(\mathbb{E})$,
- (2) $\int_X f(x)(\bar{\xi} \circ \xi) dx \leq K \|\xi\|_2^2 \quad \forall \xi \in S(\mathbb{E})$ for some constant K .

We set $\mu(B)(\xi) = \int_B f(x)(\xi) dx$ for every $B \in \mathcal{S}$, and observe that μ is a bounded measure on X . Since $\mu(B)$ can be attained as the supremum of an upward filtering but bounded net of finite sums of elements in $S'(\mathbb{E})$, we can appeal to Proposition 7.3 and conclude that the measure μ takes values in $S'(\mathbb{E})$.

§10. The Spectral Theorem

Proposition 10.1. *Let T be a real element of N . The imaginary part of the resolvent distribution maps the upper half plane $\{\lambda \in \mathbb{C} \mid \text{Im } \lambda > 0\}$ into positive,*

tempered distributions. Furthermore

$$0 \leq \text{Im } R(T, \lambda) \leq (\text{Im } \lambda)^{-1} \quad \forall \lambda \in \mathbb{C}, \text{Im } \lambda > 0.$$

Proof. Let $\text{Im } \lambda > 0$. Theorem 8.2 and the resolvent equation yield that

$$\begin{aligned} \text{Im } R(T, \lambda) &= \frac{1}{2i} (R(T, \lambda) - \overline{R(T, \lambda)}) \\ &= \frac{1}{2i} (R(T, \lambda) - R(T, \bar{\lambda})) \\ &= (\text{Im } \lambda) R(T, \bar{\lambda}) \circ R(T, \lambda). \end{aligned}$$

Consequently, we have

$$(10.1) \quad \begin{aligned} (\text{Im } R(T, \lambda))(\xi \circ \bar{\xi}) &= (\text{Im } \lambda) \|R(T, \lambda) \circ \xi\|^2 \\ &\leq (\text{Im } \lambda)^{-1} \|\xi\|^2 \end{aligned}$$

for every element $\xi \in S(\mathbb{E})$. ■

Notice the difference between the distribution $\text{Im } R(T, \lambda)$ taken in some vector ξ and the imaginary part of $R(T, \lambda)(\xi)$. A map $z \rightarrow f(z)$ defined on a complex domain and with values in $S'(\mathbb{E})$ is said to be analytic or harmonic, if the complex function $z \rightarrow f(z)(\xi)$ is analytic or harmonic for each $\xi \in S(\mathbb{E})$.

Theorem 10.2. *Let $u: \{z \in \mathbb{C} \mid |z| < 1\} \mapsto S'(\mathbb{E})$ be a harmonic map with values in the positive part of $S'(\mathbb{E})$. There exists a positive measure μ on $[0, 2\pi]$ with values in $S'(\mathbb{E})$ such that*

$$u(re^{i\theta}) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)} d\mu(\phi),$$

for $0 \leq r < 1$ and $\theta \in [0, 2\pi]$.

Proof. We first suppose that $u(z) = u(re^{i\theta})$ is harmonic in a disk of radius greater than 1. We may determine a harmonic conjugate $v(z)$ in such a way that $v(0)(\xi) = 0 \quad \forall \xi \in S(\mathbb{E})$. The map $f(z) = u(z) + iv(z)$ is analytic in a disk of radius greater than 1 and hence represented by a power series

$$(10.2) \quad f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

which converges absolutely and uniformly on the circle $|z| \leq 1$ for every $\xi \in S(\mathbb{E})$. The real part is thus given by

$$(10.3) \quad u(z) = u(re^{i\theta}) = c_0 + \frac{1}{2} \sum_{n=1}^{\infty} (c_n e^{in\theta} + \bar{c}_n e^{-in\theta}) r^n,$$

and this function also converges absolutely and uniformly on the circle $|z| \leq 1$ for every $\xi \in S(\mathbb{E})$. Integrating the function $z = e^{in\phi} \mapsto e^{-in\phi} u(e^{i\phi})(\xi)$ around the circle $|z| = 1$ for each $n \in \mathbb{Z}$ and $\xi \in S(\mathbb{E})$ gives

$$\begin{aligned}
 \int_0^{2\pi} e^{-in\phi} u(e^{i\phi})(\xi) d\phi &= \int_0^{2\pi} (c_0(\xi) + \frac{1}{2} \sum_{k=1}^{\infty} (c_k(\xi)e^{ik\phi} + \bar{c}_k(\xi)e^{-ik\phi}))e^{-in\phi} d\phi \\
 (10.4) \qquad &= \pi \begin{cases} 2c_0(\xi) & \text{for } n = 0 \\ c_n(\xi) & \text{for } n > 0, \\ \bar{c}_{-n}(\xi) & \text{for } n < 0. \end{cases}
 \end{aligned}$$

We insert the values (10.4) in equation (10.3) and notice that the order of integration and summation can be interchanged for $r < 1$. This gives the representation

$$\begin{aligned}
 (10.5) \qquad u(re^{i\theta})(\xi) &= \frac{1}{2\pi} \int_0^{2\pi} (1 + \sum_{n=1}^{\infty} (e^{in(\theta-\phi)} + e^{-in(\theta-\phi)})r^n)u(e^{i\phi})(\xi) d\phi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r \cos(\theta-\phi)} u(e^{i\phi})(\xi) d\phi,
 \end{aligned}$$

which is valid for $r < 1$. The measure $d\mu(\phi)$ is given by the density $(1/2\pi)u(e^{i\phi})$ with respect to Lebesgue measure on the interval $[0, 2\pi]$. The density is bounded and maps the interval $[0, 2\pi]$ into positive, tempered distributions. The function $\phi \rightarrow (1/2)u(e^{i\phi})(\xi)$ is continuous for each $\xi \in S(\mathbb{E})$. The total mass of the measure is given by

$$(10.6) \qquad u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\phi}) d\phi.$$

We return now to the general case and assume only that $u(z) = u(re^{i\theta})$ is defined in the open disk $|z| < 1$. We consider, for each $\varepsilon > 0$, the map $u_\varepsilon(z) = u((1 + \varepsilon)^{-1}z)$ which is positive and harmonic in a disk of radius greater than 1. Each of these maps thus allow the representation (10.5) with respect to the measure $(1/2\pi)u_\varepsilon(e^{i\phi})$. The total mass of this measure is $u_\varepsilon(0) = u(0)$, cf. (10.6), and is independent of $\varepsilon > 0$. We can apply Lemma 9.1 to obtain a weak contact point μ for the net of measures $((1/2\pi)u_\varepsilon(e^{i\phi}))_\varepsilon$ for $\varepsilon \rightarrow 0$, and then pass to a subnet that is weakly converging to μ . Since the kernel in (10.5) is continuous, the integrals with respect to the measures in the subnet converge for $r < 1$ towards the integral of the kernel with respect to μ . Likewise, $u_\varepsilon(re^{i\theta})$ converges to $u(re^{i\theta})$ for any subnet of $\varepsilon \rightarrow 0$, and it follows that

$$u(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r \cos(\theta-\phi)} d\mu(\phi),$$

for $0 \leq r < 1$ and $\theta \in [0, 2\pi]$. ■

Corollary 10.3. *Let $F: \{\lambda \in \mathbb{C} | \operatorname{Im} z > 0\} \mapsto S'(\mathbb{E})$ be an analytic map. If the imaginary part of F is positive, then there exists a positive measure ν on the real line with values in $S'(\mathbb{E})$ and finite total mass such that*

$$F(\lambda) = \alpha\lambda + \beta + \int_{-\infty}^{\infty} \frac{\lambda t + 1}{t - \lambda} d\nu(t),$$

where $\alpha \geq 0$ and β is real.

Proof. We make use of the conformal mappings

$$(10.7) \quad \lambda(z) = \frac{1}{i} \frac{z+1}{z-1} \quad \text{and} \quad z(\lambda) = \frac{\lambda-i}{\lambda+i}.$$

The first maps $\{z \in \mathbb{C} | |z| < 1\}$ onto $\{\lambda \in \mathbb{C} | \operatorname{Im} \lambda > 0\}$ and the other is the inverse transformation. We set

$$(10.8) \quad f(z) = -iF(\lambda(z)) \quad \forall z \in \mathbb{C}, |z| < 1,$$

and notice that f is an analytic map with values in $S'(\mathbb{E})$. The real part

$$(10.9) \quad u(z) = \operatorname{Re} f(z) \quad \forall z \in \mathbb{C}, |z| < 1,$$

is hence harmonic with values in positive, tempered distributions. We can thus apply Theorem 10.2 to obtain a positive measure ω on $[0, 2\pi]$ with values in $S'(\mathbb{E})$ such that

$$(10.10) \quad u(re^{i\theta}) = \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} d\omega(\phi),$$

for $0 \leq r < 1$ and $\theta \in [0, 2\pi]$. The harmonic conjugate of the integrand is determined up to an imaginary constant. Since

$$(10.11) \quad \frac{e^{i\phi} + z}{e^{i\phi} - z} = \frac{1-r^2-2ir\sin(\phi-\theta)}{1+r^2-2r\cos(\phi-\theta)}$$

is analytic, it follows that

$$(10.12) \quad f(z) = -i\beta + \int_0^{2\pi} \frac{e^{i\phi} + z}{e^{i\phi} - z} d\omega(\phi),$$

where β is a real constant. Let α be the sum of the masses of the measure ω in the points 0 and 2π and let ω' be the measure on $]0, 2\pi[$ obtained from ω by leaving out these two points. We obtain that

$$(10.13) \quad f(z) = \alpha \frac{1+z}{1-z} - i\beta + \int_{]0, 2\pi[} \frac{e^{i\phi} + z}{e^{i\phi} - z} d\omega'(\phi),$$

for $|z| < 1$. By inserting $z = z(\lambda)$ and multiplying by i , we derive that

$$(10.14) \quad F(\lambda) = \alpha\lambda + \beta + \int_{]0, 2\pi[} \frac{\lambda \cos \phi/2 - \sin \phi/2}{\lambda \sin \phi/2 + \cos \phi/2} d\omega'(\phi).$$

We introduce the change of variable $t = -\cot(\phi/2)$ which carries the circle onto the real axis, mapping the deleted points $\phi = 0$ and $\phi = 2\pi$ into infinity and the measure ω' onto a measure ν of finite mass on the real line. This transforms (10.14) into the expression in the corollary. We furthermore notice that $\beta = \operatorname{Re} F(i)$. ■

We shall here notice that the mass which the measure μ in Theorem 10.2 puts in the points $\phi = 0$ and $\phi = 2\pi$ obviously can be freely distributed between the two points. But this is also the only ambiguity in the definition of μ up to equivalence of measure. That this is so follows by considering the complex function $u(z)(\xi \circ \bar{\xi})$ for each $\xi \in S(\mathbf{E})$ and making use of the integral expression in the theorem. The unicity up to equivalence of the corresponding ordinary measure μ^ξ except for a possible redistribution of mass between the points $\phi = 0$ and $\phi = 2\pi$ is well known. The assertion thus follows by polarisation. It also readily follows that a positive measure on $[0, 2\pi]$ with values in $S'(\mathbf{E})$ through the formula in Theorem 10.2 gives rise to a harmonic map on the open unit disk in the complex plane with values in positive, tempered distributions. The same remarks apply to the measure ν in Corollary 10.3 except that it is uniquely defined up to equivalence of measure.

Theorem 10.4. *Let T be a real element of N . There exists a probability measure μ on the real line with values in $S'(\mathbf{E})$, unique up to equivalence of measure, such that the resolvent distribution is given by*

$$R(T, \lambda) = \int_{-\infty}^{\infty} \frac{1}{t - \lambda} d\mu(t) \quad \forall \lambda \in \mathbf{C}, \operatorname{Im} \lambda > 0.$$

Proof. The resolvent distribution maps the complex upper half plane into $S'(\mathbf{E})$, cf. Theorem 8.2. It is analytic according to Corollary 8.5 and positive, cf. Proposition 10.1. Hence we can apply Corollary 10.3 to obtain a positive measure ν on the real line with values in $S'(\mathbf{E})$ of finite total mass, a positive constant α , and a real constant β such that

$$(10.15) \quad R(T, \lambda) = \alpha\lambda + \beta + \int_{-\infty}^{\infty} \frac{\lambda t + 1}{t - \lambda} d\nu(t) \quad \forall \lambda \in \mathbf{C}, \operatorname{Im} \lambda > 0.$$

The measure ν is unique up to equivalence of measure. Setting $\lambda = is, s > 0$ and taking the imaginary part, we derive that

$$(10.16) \quad \operatorname{Im} R(T, is) = \alpha s + \int_{-\infty}^{\infty} \frac{s(t^2 + 1)}{t^2 + s^2} d\nu(t) \quad \forall s \in \mathbf{R}, s > 0.$$

The imaginary part of $R(T, is)$ is positive and bounded by s^{-1} according to Proposition 10.1, so we must conclude that $\alpha = 0$. We define a positive measure μ on the real line with values in $S'(\mathbb{E})$ by setting $d\mu(t) = (t^2 + 1)dv(t)$ and obtain

$$(10.17) \quad s \operatorname{Im} R(T, is) = \int_{-\infty}^{\infty} \frac{s^2}{t^2 + s^2} d\mu(t) \leq 1 \quad \forall s > 0.$$

It follows, that the measure μ has finite total mass. Furthermore,

$$(10.18) \quad R(T, \lambda) = \beta + \int_{-\infty}^{\infty} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\mu(t) \quad \forall \lambda \in \mathbb{C}, \operatorname{Im} \lambda \neq 0.$$

Again setting $\lambda = is, s > 0$, but this time calculating the real part, we obtain

$$(10.19) \quad s \operatorname{Re} R(T, is) = s\beta + \int_{-\infty}^{\infty} \left(\frac{ts}{t^2 + s^2} - \frac{ts}{t^2 + 1} \right) d\mu(t) \quad \forall s > 0.$$

Since $-1 \leq s \operatorname{Re} R(T, is) \leq 1$ for every $s > 0$ and the first half of the integral is bounded uniformly in s , we conclude that

$$(10.20) \quad \beta = \int_{-\infty}^{\infty} \frac{t}{t^2 + 1} d\mu(t).$$

Inserting this in (10.18), we obtain

$$(10.21) \quad R(T, \lambda) = \int_{-\infty}^{\infty} \frac{1}{t - \lambda} d\mu(t) \quad \forall \lambda \in \mathbb{C}, \operatorname{Im} \lambda > 0.$$

To prove that μ is a probability measure, we first notice that

$$(10.22) \quad -isR(T, is) = 1 - T \circ R(T, is) \quad \forall s > 0.$$

It then follows from (4.21) that

$$(10.23) \quad |(T \circ R(T, is))(\xi)| = |R(T, is)(\xi \circ T)| \leq s^{-1} \|\xi \circ T\|_{\operatorname{Tr}}$$

for every $s > 0$ and $\xi \in S(\mathbb{E})$. Consequently

$$(10.24) \quad \lim_{s \rightarrow \infty} sR(T, is)(\xi \circ \bar{\xi}) = i \|\xi\|_2^2 \quad \forall \xi \in S(\mathbb{E}).$$

Since

$$(10.25) \quad s \operatorname{Im} R(T, is) = \int_{-\infty}^{\infty} \frac{s^2}{t^2 + s^2} d\mu(t) \quad \forall s > 0,$$

we conclude that the measure μ^ξ has total mass $\|\xi\|_2^2$ for each $\xi \in S(\mathbb{E})$. Hence μ is a probability measure. ■

The support $\sigma(T)$ of the measure μ is a closed subset of the real line. We denote it as the spectrum of T . Let $M^\infty(\sigma(T))$ denote the set of bounded, measurable complex functions on $\sigma(T)$.

Theorem 10.5. *Let T be a real element of N , and let μ be the measure associated with T in Theorem 10.4. The map*

$$(10.26) \quad \Phi(f) = \int_{\sigma(T)} f(t) d\mu(t)$$

is an algebra homomorphism of $M^\infty(\sigma(T))$ into $S'(\mathbf{E})$.

Proof. The map Φ is linear and maps $M^\infty(\sigma(T))$ into a self-adjoint, weakly closed subspace of $S'(\mathbf{E})$. Let $f \in M^\infty(\sigma(T))$ be of the form

$$(10.27) \quad f(t) = \sum_{i \in \Lambda} \alpha_i \frac{1}{t - \lambda_i},$$

where Λ is a finite set and $\lambda_i \in \mathbf{C} \setminus \mathbf{R}$ for every $i \in \Lambda$. It follows from Theorem 10.4 that

$$(10.28) \quad \Phi(f) = \sum_{i \in \Lambda} \alpha_i R(T, \lambda_i).$$

We denote by $A_0(\sigma(T))$ the linear span of functions on the form (10.27) and notice that it is weakly dense in $M^\infty(\sigma(T))$. An application of the resolvent equation shows that $\Phi(fg) = \Phi(f)\Phi(g)$ for all functions f, g in $A_0(\sigma(T))$ from which the statement follows. ■

Let T be a real element of N , and let μ be the measure in Theorem 10.4. For each $t \in \mathbf{R}$, we set

$$(10.29) \quad E(t) = \int_{-\infty}^t d\mu(s).$$

We conclude from Theorem 10.5 that $t \mapsto E(t)$ is a spectral function with values in real, idempotent, tempered distributions (corresponding to projections on I_Ω). The evolution group associated with T is defined by setting

$$(10.30) \quad \begin{aligned} U(t) &= \exp^\circ(itT) \\ &= \int_{\sigma(T)} e^{its} d\mu(s). \end{aligned}$$

It follows, that $t \mapsto U(t)$ is a group representation of $(\mathbf{R}, +)$ into $S'(\mathbf{E})$.

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