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The Representations of the q-analogue of Brauer's Centralizer Algebras and the Kauffman Polynomial of Links[1]

By

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§0. Introduction

Let $\mathbf{C}(\alpha, q)$ be the two-variable rational function field over $\mathbf{C}$ with indeterminates α and q. It is shown in [3] and [8] that the Kauffman polynomial of links are associated with a sequence of $\mathbf{C}(\alpha, q)$-algebras, which are denoted by $C_n(\alpha, q)\,(n\in\mathbf{N})$. These algebras can be considered as q-analogues of Brauer's algebras $D_n(\beta)$ [4] defined over $\mathbf{C}(\beta)$, where β is an indeterminate. Let G be a group of linear transformations on a vector space V and $\pi^{\otimes n}$ the representation of G on $V^n = V\otimes\cdots\otimes V$, the n-th tensor power of V. Let $Z_n(G)$ be the centralizer algebras of $\pi^{\otimes n}$, i.e.

$$Z_n(G) = \{x\in \mathrm{End}\,(V^n) \mid x\pi^{\otimes n}(g) = \pi^{\otimes n}(g)x \qquad \text{for all } g\in G\}.$$

Let G be the symplectic group Sp_{2m} or the special orthogonal group SO_{2m+1}. Then $Z_n(G)$ is a semisimple quotient of the algebra $D_n(\beta)$. From the results of [3], [5], [8] and [10], we have an analogous result for $C_n(\alpha, q)$. Let $\mathfrak{g}$ be one of Lie algebras $\mathfrak{sp}_{2m}$ and $\mathfrak{so}_{2m+1}$ of the Lie groups Sp_{2m} and SO_{2m+1}. Let $\hat{\mathscr{U}}(\mathfrak{g})$ be the q-analogue of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ (see e.g. [5]). Then there is an integer r such that the centralizer algebras associated with the vector representation of $\hat{\mathscr{U}}(\mathfrak{g})$ are quotients of the algebras $C_n(q^r, q)$.

The aim of the present paper is to construct irreducible representations of $C_n(\alpha, q)$ explicitly. The $\mathbf{C}(\alpha, q)$-algebra $C_n(\alpha, q)\,(n\in\mathbf{N})$ with 1 is defined by the following.

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\begin{aligned}
(0.1)\qquad & C_1(\alpha, q) = \mathbb{C}(\alpha, q),\\
& C_n(\alpha, q) = \langle \tau_i, \tau_i^{-1}, \varepsilon_i (1 \le i \le n-1) | \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1},\\
& \quad \varepsilon_i\varepsilon_{i+1}\varepsilon_i = \varepsilon_i,\ \varepsilon_{i+1}\varepsilon_i\varepsilon_{i+1} = \varepsilon_{i+1},\\
& \quad \tau_i^{\pm 1}\varepsilon_{i+1}\varepsilon_i = \tau_{i+1}^{\mp 1}\varepsilon_i,\ \tau_{i+1}^{\pm 1}\varepsilon_i\varepsilon_{i+1} = \tau_i^{\mp 1}\varepsilon_{i+1},\\
& \quad \varepsilon_i\varepsilon_{i+1}\tau_i^{\pm 1} = \varepsilon_i\tau_{i+1}^{\mp 1},\ \varepsilon_{i+1}\varepsilon_i\tau_{i+1}^{\pm 1} = \varepsilon_{i+1}\tau_i^{\mp 1}\ (1 \le i \le n-2),\\
& \quad \tau_i\tau_j = \tau_j\tau_i,\ \varepsilon_i\tau_j = \tau_j\varepsilon_i,\ \varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i\ (1 \le i < j-1 \le n-2),\\
& \quad \tau_i\tau_i^{-1} = \tau_i^{-1}\tau_i = 1,\ \tau_i\varepsilon_i = \varepsilon_i\tau_i = (-\alpha^2 q)^{-1}\varepsilon_i,\\
& \quad \tau_i - \tau_i^{-1} = (q - q^{-1})(1 - \varepsilon_i)\ (1 \le i \le n-1)\rangle\ (n \ge 2).
\end{aligned}
$$

Note that (0.1) implies that $\varepsilon_i^2 = \left(1 - \dfrac{\alpha^2 q - \alpha^{-2}q^{-1}}{q - q^{-1}}\right)\varepsilon_i$. Hence the algebra $C_n(\alpha, q)$ is a one-parameter deformation of the algebra $D_n(\beta)$. More precisely, $D_n(\beta)$ is the limit $q \to 1$ of $C_n(q^{\beta/2}, q)$. The algebra $D_n(\beta)$ is semisimple and its irreducible representations are classified [3]. Hence its one-parameter deformation $C_n(\alpha, q)$ is semisimple and there is a bijection between the irreducible representations of $C_n(\alpha, q)$ and those of $D_n(\beta)$. Let ρ be an irreducible representation of $C_n(\alpha, q)$. Then the representation matrices of $\rho(\tau_i)$, $\rho(\tau_i^{-1})$ and $\rho(\varepsilon_i)(1 \le i \le n-1)$ with respect to a certain basis are given in §1. By taking the limit $q \to 1$ of the above matrices, we get the irreducible representations of the algebra $D_n(\beta)$. Our construction is based on [6].

Let B_n denote the braid group on n-strings and $\sigma_i(1 \le i \le n-1)$ its standard generators of B_n. Let p_n be the algebra homomorphism from the group ring $\mathbb{C}B_n$ to $C_n(\alpha, q)$ defined by $p_n(\sigma_i^{\pm 1}) = (-\alpha^2 q)^{\mp 1}\tau_i^{\pm 1}(1 \le i \le n-1)$. For $\rho \in C_n(\alpha, q)^\wedge$ let χ_ρ denote the character of $\rho \circ p_n$. For $b \in B_n$, let $\hat{b}$ denote the closure of b. Let $F(\hat{b})$ denote the Kauffman polynomial of the closed braid $\hat{b}$ of $b \in B_n$ with values in $C_n(\alpha, q)$. Then there are $a_\rho \in \mathbb{C}(\alpha, q)$ for $\rho \in C_n(\alpha, q)^\wedge$ such that

$$
(0.2)\qquad F(\hat{b}) = \sum_{\rho \in C_n(\alpha, q)^\wedge} a_\rho \chi_\rho(b) \qquad \text{(see [3] and [8]).}
$$

We explicitly give the coefficients a_ρ in (2.1) and Theorem 2.2. Hence (0.2) can be used to calculate the Kauffman polynomial of closed braids. The representation matrices of $\rho(\tau_i)$ and $\rho(\tau_i^{-1})(1 \le i \le n-1)$ given in §1 are all symmetric. This fact is used to show [9, Theorem 6.2.4], which claims the following. Let ν be a one-dimensional representation of $C_n(\alpha, q)$ and $F^{(r,\nu)}$ a link invariant associated with F and ν introduced in [9, Section 1.5]. Then we have $F^{(r,\nu)}(K) = F^{(r,\nu)}(K')$ for mutant knots K and K'.

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§ 1. Construction

Fix a positive integer n. Let

$$\Lambda = \{(\lambda_1, \lambda_2,\dots) | \lambda_i \geq \lambda_{i+1} \geq 0 (i \in \mathbb{N}), \lambda_j = 0 (j \gg 0)\}.$$

An element of Λ is called a partition. We also use the following notation.

$$\Lambda(n) = \left\{(\lambda_1, \lambda_2,\dots) \in \Lambda | \sum_{i \in \mathbb{N}} \lambda_i = n - 2j, 0 \leq j \leq \left[\frac{n}{2}\right]\right\}.$$

For two partitions $\lambda = (\lambda_1, \lambda_2,\dots)$ and $\lambda' = (\lambda'_1, \lambda'_2,\dots)$ in Λ, we denote $\lambda \underset{1}{\sim} \lambda'$ if there is $j \in \mathbb{N}$ such that $\lambda_i = \lambda'_i$ for $i \neq j$ and $\lambda_j = \lambda'_j \pm 1$. For a partition $\lambda \in \Lambda(n)$, let

$$\mathscr{P}(\lambda) = \{P = (\lambda^{(0)}, \lambda^{(1)},\dots, \lambda^{(n)}) | \lambda^{(0)} = (0, 0,\dots), \lambda^{(n)} = \lambda,$$
$$\lambda^{(i)} \underset{1}{\sim} \lambda^{(i+1)} \text{ for } 0 \leq i \leq n-1\}.$$

Let $V_\lambda = \bigoplus_{P \in \mathscr{P}(\lambda)} \mathbb{C}(\alpha, q) v_P$, which is a vector space over $\mathbb{C}(\alpha, q)$ with a basis $\{v_P | P \in \mathscr{P}(\lambda)\}$. In this paper, we use the following notations.

$$\{k\} = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad \{k; m\} = \frac{\alpha^m q^k - \alpha^{-m} q^{-k}}{q - q^{-1}}.$$

For $\nu = (\nu_1, \nu_2,\dots, \nu_\ell, 0, 0,\dots) \in \Lambda$, let

$$h_\nu(i, j) = \nu_i - i - j + \max\{k | \nu_k \geq j\} \ (1 \leq i \leq \ell, 1 \leq j \leq \nu_i),$$

$$g_\nu(i) = \begin{cases} \dfrac{\prod_{j=i}^{\nu_i+i-1} \{\nu_i + \nu_j + 2 - i - j; 2\}}{\prod_{j=1}^{\nu_i} \{j - i + 1; 1\}} & (\nu_i + i > \ell), \\[2ex] \dfrac{\prod_{j=i}^{\ell} \{\nu_i + \nu_j + 2 - i - j; 2\}}{\prod_{j=1}^{\nu_i} \{j - i + 1; 1\} \prod_{j=1}^{\ell-\nu_i-i+1} \{3 - 2i - j; 2\}} & (\nu_i + i \leq \ell), \end{cases}$$

and

$$G_\nu = \prod_{i=1}^{\ell} g_\nu(i) \left(\prod_{j=1}^{\nu_i} \frac{\{j - i; 1\}}{\{h_\nu(i, j) + 1\}} \right). \tag{1.1}$$

Fix i in $\{1, 2,\dots, n-1\}$. We define an element A_i in $\mathrm{End}(V_\lambda)$. We give the matrix of A_i with respect to the basis $\{v_P | P \in \mathscr{P}(\lambda)\}$. Let $P = (\lambda^{(0)}, \lambda^{(1)},\dots, \lambda^{(n)})$ and $Q = (\nu^{(0)}, \nu^{(1)},\dots, \nu^{(n)})$ be elements of $\mathscr{P}(\lambda)$. The elements of the partition $\lambda^{(r)}$ is denoted by $\lambda_1^{(r)}, \lambda_2^{(r)},\dots$. Let ℓ denote the maximal integer satisfying $\lambda_\ell^{(i-1)} \neq 0$ and $(\eta_1,\dots, \eta_\ell, 0,\dots) = \lambda^{(i-1)}$. We put

$$\eta(i) = \eta_i - i + 1.$$

Let $A_i v_P = \sum_{Q \in \mathscr{P}(\lambda)} (A_i)_{QP} v_Q$. In the following, δ_1 and δ_2 denote either 1 or -1. If there is an element $j \in \{1, 2, \dots, n-1\} \setminus \{i\}$ such that $\lambda^{(j)} \neq \nu^{(j)}$, then we put

$$(A_i)_{QP} = 0. \tag{1.2a}$$

From now on, we treat the case that $\lambda^{(j)} = \nu^{(j)}$ for $j \in \{1, 2, \dots, n-1\} \setminus \{i\}$. At first we assume that $P = Q$. If there is an r in $\mathbb{N}$ such that $\lambda_r^{(i-1)} = \lambda_r^{(i+1)} \pm 2$, then we put

$$(A_i)_{PP} = q. \tag{1.2b}$$

If $\lambda^{(i-1)} = \lambda^{(i+1)}$, then there is a unique r such that $\lambda_r^{(i-1)} = \lambda_r^{(i)} - \delta_1$. For such P, we put

$$(A_i)_{PP} = -\frac{\alpha^{-2\delta_1} q^{-2\delta_1 \eta(r) - 1}}{\{2\delta_1 \eta(r) + 1 ; 2\delta_1\}} - \frac{\alpha^{-2\delta_1} q^{-2\delta_1 \eta(r) - 1} G_{\lambda^{(i)}}}{G_{\lambda^{(i-1)}} \{2\delta_1 \eta(r) + 1 ; 2\delta_1\}}. \tag{1.2c}$$

If otherwise, there are unique r and s in $\mathbb{N}$ such that $r \neq s$, $\lambda_r^{(i-1)} = \lambda_r^{(i)} - \delta_1$ and $\lambda_s^{(i)} = \lambda_s^{(i+1)} - \delta_2$. For such P, we put

$$(A_i)_{PP} = -\frac{\alpha^{\delta_2 - \delta_1} q^{\delta_2 \eta(s) - \delta_1 \eta(r)}}{\{\delta_1 \eta(r) - \delta_2 \eta(s) ; \delta_1 - \delta_2\}}. \tag{1.2d}$$

Now, we assume that $P \neq Q$. If $\lambda^{(i-1)} \neq \lambda^{(i+1)}$, then there are unique r and s in $\mathbb{N}$ such that $r \neq s$ and $\lambda_r^{(i-1)} = \lambda_r^{(i)} - \delta_1$, $\lambda_s^{(i)} = \lambda_s^{(i+1)} - \delta_2$. For such P and Q, we put

$$(A_i)_{QP} = \frac{\sqrt{\{\delta_1 \eta(r) - \delta_2 \eta(s) + 1 ; \delta_1 - \delta_2\} \{\delta_1 \eta(r) - \delta_2 \eta(s) - 1 ; \delta_1 - \delta_2\}}}{\{\delta_1 \eta(r) - \delta_2 \eta(s) ; \delta_1 - \delta_2\}}. \tag{1.2e}$$

If $\lambda^{(i-1)} = \lambda^{(i+1)}$ then there are unique r and s in $\mathbb{N}$ such that $\lambda_r^{(i-1)} = \lambda_r^{(i)} - \delta_1$ and $\nu_s^{(i-1)} = \nu_s^{(i)} - \delta_2$. For such P and Q, we put

$$(A_i)_{QP} = -\frac{\alpha^{-\delta_1 - \delta_2} q^{-\delta_1 \eta(r) - \delta_2 \eta(s) - 1} \sqrt{G_{\lambda^{(i)}} G_{\nu^{(i)}}}}{G_{\lambda^{(i-1)}} \{\delta_1 \eta(r) + \delta_2 \eta(s) + 1 ; \delta_1 + \delta_2\}}. \tag{1.2f}$$

The above definition of A_i implies that $(A_i)_{QP} = (A_i)_{PQ}$ for $P, Q \in \mathscr{P}(\lambda)$, in other words, $A_i (1 \leq i \leq n-1)$ are symmetric matrices. Let $P = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)})$ and $Q = (\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(n)}) \in \mathscr{P}(\lambda)$ such that $\lambda^{(j)} = \nu^{(j)}$ for $j \in \{1, 2, \dots, n\} \setminus \{i\}$. By (1.2), the matrix element $(A_i)_{QP}$ does not depend on $\lambda^{(j)}$, $j \neq i, i \pm 1$. Let $A(\lambda^{(i-1)}, \lambda^{(i)}, \nu^{(i)}, \lambda^{(i+1)}) = (A_i)_{QP}$. Let $W_m^t\left(\begin{matrix} a & b \\ d & c \end{matrix} \middle| u \right)$ denote the trigonometric limit of the Boltzmann weight defined in [6] associated with the Lie algebra of Dynkin type $C_m^{(1)}$. In this case, the function $[u]$ used in the definition of the Boltzmann weight in [6] is equal to $2 \sin(\pi u / L)$, where L is an arbitrary nonzero complex parameter. Put $x = \exp(\pi u\sqrt{-1}/L)$. Recall that $W_m^t\begin{pmatrix} a & b \\ d & c \end{pmatrix}\!\!\left|\, u \right)$ is a Laurent polynomial in x and the highest degree of $W_m^t\begin{pmatrix} a & b \\ d & c \end{pmatrix}\!\!\left|\, u \right)$ with respect to x is equal to 2 ((1.5) of [6]). For dominant integral weight a of the simple Lie algebra of Dynkin type C_m given as in Table 1 of [6], let $\lambda(a) = (\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \ldots)$ be an element of Λ such that

$$\lambda_i = a_i - n - 1 + i. \tag{1.3}$$

By comparing the definition of A_i and (1.5) of [6], we have the following.

Lemma 1.4. *For dominant integral weights a, b, c and d of the simple Lie algebra of Dynkin type C_m such that $\lambda(a) \underset{1}{\sim} \lambda(b)$, $\lambda(b) \underset{1}{\sim} \lambda(c)$, $\lambda(c) \underset{1}{\sim} \lambda(d)$ and $\lambda(d) \underset{1}{\sim} \lambda(a)$, we have*

$$\lim_{u\to\infty} W_m^t\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, -\sqrt{-1}\,u\right)x^{-2} = \lim_{q\to e^{\sqrt{-1}\pi/L}} \left.\frac{q^{(m+1)}A(\lambda(a), \lambda(b), \lambda(d), \lambda(c))}{\{m+1\}\{1\}^2}\right|_{\alpha=q^m},$$

where L is the complex parameter used in [6].

Proof. For a dominant integral weight $a = (a_1, a_2, \ldots, a_m)$, let

$$G'_a = \varepsilon(a)\prod_{j=1}^{m}[2a_j]\prod_{1\le j<k\le m}[a_j - a_k][a_j + a_k] \quad (\varepsilon(a) = \pm 1).$$

Then the above lemma is proved by using the fact that the limit of $G_{\lambda(a)}|_{\alpha=q^m}$ with q to $e^{\sqrt{-1}\pi/L}$ is equal to the trigonometric limit of $\varepsilon(a)G'_{a(v)}(\prod_{1\le j<k\le m}[i - k])^{-1}(\prod_{1\le j\le k\le m}[i+k])^{-1}$. We omit the detail. ■

The following theorem is the main result of this paper.

Theorem 1.5. *Let λ be a partition in $\Lambda(n)$.*

(i) *There is a representation $(\rho_\lambda, V_\lambda)$ of $C_n(\alpha, q)$ such that the representation matrices of $\rho_\lambda(\tau_i)$ is equal to A_i defined by* (1.2).

(ii) *The representation ρ_λ is irreducible.*

(iii) *Two irreducible representations ρ_λ and ρ_ν $(\lambda, \nu \in \Lambda(n), \lambda \neq \nu)$ are not equivalent.*

(iv) *Any irreducible representation of $C_n(\alpha, q)$ is equivalent to one of $(\rho_\lambda, V_\lambda)$ $(\lambda \in \Lambda(n))$.*

A similar result holds for the Brauer's algebra $D_n(\beta)$.

Corollary 1.6. *Any irreducible representation of $D_n(\beta)$ is a limit of $\rho_\lambda|_{\alpha=q^{\beta/2}}$ with $q \to 1$.*

In the rest of this section we give proofs of the above theorem and its corollary. Some combinatorial consideration about partitions shows the following two lemmas.

Lemma 1.7. *For $\lambda \in \Lambda(n)$, let ν and ν' be two elements of $\Lambda(n-1)$ such that $\nu \underset{1}{\sim} \lambda$ and $\nu' \underset{1}{\sim} \lambda$. Then there is $\mu \in \Lambda(n-2)$ such that $\mu \underset{1}{\sim} \nu$ and $\mu \underset{1}{\sim} \nu'$.*

Lemma 1.8. *For $\nu \in \Lambda(n)$, let $L(\nu) = \{\mu \in \Lambda(n-1) \mid \mu \underset{1}{\sim} \nu\}$. For $\nu_1, \nu_2 \in \Lambda(n)$ with $n \geq 3$, assume that $L(\nu_1) = L(\nu_2)$. Then $\nu_1 = \nu_2$.*

We use the following to show the relation $E_i^2 = (1 - (\alpha^2 q - \alpha^{-2} q^{-1})(q - q^{-1})^{-1}) E_i$.

Lemma 1.9. *For $\lambda \in \Lambda$, we have*

$$\sum_{\nu \underset{1}{\sim} \lambda} \frac{G_\nu}{G_\lambda} = \frac{\alpha^2 q - \alpha^{-2} q^{-1}}{q - q^{-1}} - 1.$$

Proof. It suffices to show in the case $\alpha = q^m$ for infinitely may integers m. Fix an integer $m \geq \sum_{i \in \mathbf{N}} \lambda_i$. We use representation theory of the Lie algebra $\mathfrak{g}$ of type C_m. Let Δ be a root system corresponding to $\mathfrak{g}$ and Γ the weight lattice associated with Δ. Fix a simple root system Π in Δ and let Δ_+ be the set of positive roots associated with Π. Let $\psi = \sum_{\mu \in \Delta_+} \mu$. For a $\mathfrak{g}$-module V, we denote $m_V(\mu)$ the dimension of the weight space of V corresponding to μ. Let

$$d_q(V) = \sum_{\mu \in \Gamma} m_V(\mu) q^{(\psi, \mu)}, \tag{1.10}$$

where $(,)$ is the natural pairing of roots and weights. For $\nu \in \Lambda$, let V_ν be the simple $\mathfrak{g}$-module with highest weight corresponding to ν as in (1.3). By using the q-analogue of Weyl's character formula ([7], proof of Corollary 8.9), we have

$$G_\nu = d_q(V_\nu). \tag{1.11}$$

On the other hand, we know that $\bigoplus_{\nu \underset{1}{\sim} \lambda} V_\nu = V_\lambda \otimes V_{(1,0,0,\dots)}$, where $V_{(1,0,0,\dots)}$ is corresponding to the vector representation of $\mathfrak{g}$. Hence (1.10) and (1.11) imply $\sum_{\nu \underset{1}{\sim} \lambda} G_\nu = G_\lambda G_{(1,0,0,\dots)}$. Since $G_{(1,0,0,\dots)} = 1 - \dfrac{q^{2m+1} - q^{-2m-1}}{q - q^{-1}}$, we get the statement of the lemma in the case $\alpha = q^m$. ∎

Proof of Theorem 1.5. We first prove the statement (i). Let A_i' be the matrix obtained from A_i by substituting α^{-1} to α and q^{-1} to q. Then Lemma 1.4 and the inversion relation [6, (2.13a)] imply that $A_i A_i' = 1$. Let

$$E_i = -\frac{A_i - A_i'}{q - q^{-1}} + 1. \tag{1.12}$$

Let $P = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)})$ and $Q = (\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(n)})$ be two elements of $\mathscr{P}(\lambda)$. Then the matrix element $(E_i)_{QP}$ is given as follows. If there is an element $j \in \{1, 2, \ldots, n\} \setminus \{i\}$ such that $\lambda^{(j)} \neq \nu^{(j)}$, then we have

$$(1.13a) \qquad (E_i)_{QP} = 0.$$

From now on, we treat the case that $\lambda^{(j)} = \nu^{(j)}$ for $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. If $\lambda^{(i-1)} \neq \lambda^{(i+1)}$, then we have

$$(1.13b) \qquad (E_i)_{QP} = 0.$$

If $\lambda^{(i-1)} = \lambda^{(i+1)}$ and $P \neq Q$, then we have

$$(1.13c) \qquad (E_i)_{QP} = -\frac{\sqrt{G_{\lambda^{(i)}} G_{\nu^{(i)}}}}{G_{\lambda^{(i-1)}}}.$$

If $\lambda^{(i-1)} = \lambda^{(i+1)}$ and $P = Q$, then there is an r in $\mathbf{N}$ and $\delta_1 = \pm 1$ such that $\lambda_r^{(i-1)} = \lambda_r^{(i)} - \delta_1$ and we have

$$(1.13d) \qquad (E_i)_{PP} = -\frac{G_{\lambda^{(i)}}}{G_{\lambda^{(i-1)}}}.$$

By using Lemma 1.9, the matrix E_i satisfies

$$(1.14) \qquad E_i^2 = -\left(\frac{\alpha^2 q - \alpha^{-2} q^{-1}}{q - q^{-1}} - 1\right) E_i.$$

We have relations

$$(1.15) \qquad E_i E_{i \pm 1} E_i = E_i$$

from $(-1)^3 \frac{\sqrt{G_{\nu_1} G_{\nu_3}}}{G_{\nu_2}} \frac{\sqrt{G_{\nu_2} G_{\nu_2}}}{G_{\nu_3}} \frac{\sqrt{G_{\nu_3} G_{\nu_4}}}{G_{\nu_2}} = -\frac{\sqrt{G_{\nu_1} G_{\nu_4}}}{G_{\nu_2}}$ for $\nu_i \in \Lambda (1 \leq i \leq 4)$. We have relations

$$(1.16) \qquad A_i A_{i \pm 1} A_i = A_{i \pm 1} A_i A_{i \pm 1}$$

from Lemma 1.4 and the star-triangle relation [6, (2.2)]. Let $P = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)})$ and $Q = (\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(n)})$ be two elements of $\mathscr{P}(\lambda)$ such that $\lambda^{(j)} = \nu^{(j)}$ for $j \in \{1, 2, \ldots, n\} \setminus \{i\}$. By the definition of A_i', the matrix element $(A_i')_{QP}$ is not depend on $\lambda^{(j)}, j \neq i, \; i \pm 1$. Let $A'(\lambda^{(i-1)}, \lambda^{(i)}, \nu^{(i)}, \lambda^{(i+1)}) = (A_i')_{QP}$. Then, from [6, (2.12)] and Lemma 1.4, we have

$$(1.17) \quad A(\lambda^{(i-1)}, \lambda^{(i)}, \nu^{(i)}, \lambda^{(i+1)}) = \sqrt{\frac{G_{\lambda^{(i)}} G_{\nu^{(i)}}}{G_{\lambda^{(i-1)}} G_{\lambda^{(i+1)}}}} A'(\nu^{(i)}, \lambda^{(i-1)}, \lambda^{(i+1)}, \lambda^{(i)}).$$

This formula implies the following relations.

$$(1.18)\qquad \begin{aligned} E_iE_{i\pm1}A_i &= E_iA'_{i\pm1}, & E_iE_{i\pm1}A'_i &= E_iA_{i\pm1},\\ A_iE_{i\pm1}E_i &= A'_{i\pm1}E_i, & A'_iE_{i\pm1}E_i &= A_{i\pm1}E_i. \end{aligned}$$

From now on, we determine the eigenvalues of A_i. From (1.2b)-(1.2d), we know that A_i has eigenvalues q and $-q^{-1}$. Hence formula (1.12) implies that A_i has at most two eigenvalues except q and $-q^{-1}$. From (1.14), we know that the remaining eigenvalue is equal to $\alpha^2 q$ or $(-\alpha^2 q)^{-1}$. But we know that there is only one eigenvalue $(-\alpha^2 q)^{-1}$ of A_i except q and $-q^{-1}$, since $\lim_{\alpha\to\infty}\alpha^2 A_i$ is bounded. The above argument implies that A_i acts on $\operatorname{Im} E_i$ by the scalar $(-\alpha^2 q)^{-1}$. Since E_i is a scalar multiple of a projection, we have

$$(1.19)\qquad A_iE_i = (-\alpha^2 q)^{-1}E_i.$$

The above relations (1.14)–(1.16), (1.18) and (1.19) among A_i, A'_i and $E_i (1 \le i \le n-1)$ imply that there is a representation ρ_λ of $C_n(\alpha, q)$ with $\rho_\lambda(\tau_i) = A_i$, $\rho_\lambda(\tau_i^{-1}) = A'_i$ and $\rho_\lambda(\varepsilon_i) = E_i$. This proves the part (i) of Theorem 1.5.

From now on, we prove (ii) and (iii) inductively. We identify $C_1(\alpha, q)$ with $\mathbf{C}(\alpha, q)$. For $n = 2$, $\rho_{(2)}(\tau_1) = q$, $\rho_{(11)}(\tau_1) = q^{-1}$ and $\rho_\phi(\tau_1) = (-\alpha^2 q)^{-1}$. Hence the statements (ii) and (iii) are satisfied for the case $n = 1,2$. Now, for $n \ge 3$, assume that $C_{n-1}(\alpha, q)$ satisfies (ii) and (iii). In the following, we identify $C_{n-1}(\alpha, q)$ with the subalgebra of $C_n(\alpha, q)$ generated by $\tau_1^{\pm1}, \dots, \tau_{n-2}^{\pm1}$, $\varepsilon_1, \dots, \varepsilon_{n-2}$. Let $\lambda \in \Lambda(n)$. Let W_ν is the subspace of V_ν spanned by the vectors $\{v_{(\mu_0, \mu_1, \dots, \mu_{n-2}, \nu, \lambda)} \mid (\mu_0, \mu_1, \dots, \mu_{n-2}, \nu, \lambda) \in \mathscr{P}(n)\}$. Then W_ν is $C_{n-1}(\alpha, q)$-invariant and isomorphic to V_ν as a $C_{n-1}(\alpha, q)$-module. Hence, by the induction hypothesis, W_ν is an irreducible $C_{n-1}(\alpha, q)$-module. By the definition of V_λ, we have $V_\lambda = \bigoplus_{\substack{\nu\in\Lambda(n-1)\\ \nu \underset{1}{\sim} \lambda}} W_\nu$. Let U be a $C_n(\alpha, q)$-submodule of V_λ. Then U is also a $C_{n-1}(\alpha, q)$-submodule of V_λ. The induction hypothesis says that W_ν and $W_{\nu'}$ are not equivalent for distinct ν and ν' in $\Lambda(n-1)$ and so V_λ is multiplicity-free as an $C_{n-1}(\alpha, q)$-module. Hence, for each $\nu \in \Lambda(n-1)$ such that $\nu \underset{1}{\sim} \lambda$, $U \cap W_\lambda = W_\lambda$ or 0. Now, assume that there are ν and ν' in $\Lambda(n-1)$ such that $\nu \underset{1}{\sim} \lambda$, $\nu' \underset{1}{\sim} \lambda$, $U \cap W_\nu = W_\nu$ and $U \cap W'_\nu = 0$. Lemma 1.7 shows that there is $\mu \in \Lambda(n-2)$ such that $\mu \underset{1}{\sim} \nu$ and $\mu \underset{1}{\sim} \nu'$. Let $P = (p^{(o)}, p^{(1)}, \dots, p^{(n)})$ and $P' = (p'^{(0)}, p'^{(1)}, \dots, p'^{(n)})$ be elements of $\mathscr{P}(\lambda)$ such that $p^{(i)} = p'^{(i)}$ for $i \ne n-1$, $p^{(n-2)} = \mu$, $p^{(n-1)} = \nu$, $p'^{(n-1)} = \nu'$ and $p^{(n)} = \lambda$. The construction of the representation ρ_λ implies that

$$(1.20)\qquad \rho_\lambda(\tau_{n-1})v_P = A_{PP'}v_{P'} + \cdots$$

with respect to the basis $\{v_Q \mid Q \in \mathscr{P}(\lambda)\}$ of V_λ. Recall that the subspaces W_ν and $W_{\nu'}$ are spanned by subsets of $\{v_Q \mid Q \in \mathscr{P}(\lambda)\}$, $v_p \in W_\nu$, $v_{p'} \in W_{\nu'}$ and $A_{PP'} \ne 0$. Since U is a $C_n(\alpha, q)$-module, $A_{PP'} \ne 0$ contradicts that the assumption $U \cap W'_\nu = 0$. Hence, U must be V_λ or 0 and so V_λ is an irreducible

$C_n(\alpha, q)$-module. This proves (ii).

With the induction hypothesis, Lemma 1.8 implies that V_{λ_1} and V_{λ_2} are not isomorphic as $C_{n-1}(\alpha, q)$-modules. Hence V_{λ_1} and V_{λ_2} are not isomorphic as $C_n(\alpha, q)$-modules. This implies (iii).

We show the last statement (iv). Since the algebra $C_n(\alpha, q)$ is a one-parameter deformation of the Brauer's algebra $D_n(\beta)$, the number of irreducible representations of $C_n(\alpha, q)$ is equal to that of $D_n(\beta)$. The number of the irreducible representations of $D_n(\beta)$ is given, for example, in [12], which coincides with the number of $\mathscr{P}(n)$. Hence the representations constructed above covers the all irreducible representations of $C_n(\alpha, q)$. ■

Proof of Corollary 1.6. We noted that the limit $q \to 1$ of the algebra $C_n(q^{\beta/2}, q)$ is isomorphic to Brauer's algebra $D_n(\beta)$ in [12]. Let ρ_λ be the representation of $C_n(\alpha, q)$ constructed above and $c(\alpha, q)$ a coefficient of the representation matrix $\rho_\lambda(\tau_i)$ of a generator $\tau_i (1 \leq i \leq n-1)$. Then $c(q^{\beta/2}, q)$ has a limit with $q \to 1$ and $c(q^{\beta/2}, q) \neq 0$ if $c(\alpha, q) \neq 0$. Hence we can apply a similar argument of the proof of Theorem 1.5 to the limiting case and we get the corollary. ■

§2. The Kauffman Polynomial

In this section we give a formula (2.1) for the Kauffman polynomial in terms of the irreducible characters of the algebra $C_n(\alpha, q)$ introduced in the last section. Let α and q be indeterminates. For a positive integer n, let B_n denote the braid group on n-strings and $\mathbf{C}(\alpha, q) B_n$ the group ring of B_n over the field $\mathbf{C}(\alpha, q)$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the standard generators of B_n given as in Figure 1.

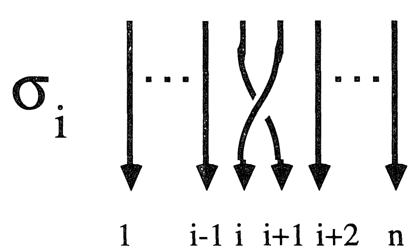

Figure 1

Let $p_n\colon \mathbf{C}(\alpha, q) B_n \to C_n(\alpha, q)$ be the algebra homomorphism defined by $p_n(\sigma_i) = (-\alpha^2 q)^{-1} \tau_i$. For $b \in B_n$, let

$$F_n(b) = \left(1 - \frac{\alpha^2 q - \alpha^{-2} q^{-1}}{q - q^{-1}}\right)^{-1} \sum_{\lambda \in \Lambda(n)} G_\lambda \chi_\lambda(p_n(b)), \tag{2.1}$$

where χ_λ be the character of the irreducible representation ρ_λ of the algebra $C_n(\alpha, q)$ introduced in §1.

Theorem 2.2. *Let b be an n-braid and $\hat{b}$ denote its closure. Let $F(\hat{b}) = F_n(b)$. Then F is the Kauffman polynomial of links (see e.g. [2] and [8]).*

Proof. First, we prove that F is an invariant of link isotopy types. To do this, we show that F is constant on a equivalence class of B with respect to the Markov relation. The Markov relation $\sim$ is the equivalence relation of $B = \{(b, n) | b \in B_n\}$ generated by the following.

(1) For $b_1, b_2 \in B_n$, $(b_1 b_2, n) \sim (b_2 b_1, n)$.

(2) For $b \in B_n$, $(b, n) \sim (b\sigma_n^{\pm 1}, n+1)$.

We have $F(\widehat{b_1 b_2}) = F(\widehat{b_2 b_1})$ for $b_1, b_2 \in B_n$ since F_n is a linear combination of characters $\chi_\lambda \circ p_n$ of B_n. We show that $F(\hat{b}) = F(\widehat{b\sigma_n^{\pm 1}})$ for $b \in B_n$. By using the construction of ρ_λ, the above equalities follow from Lemma 2.3 below. Let $w(b)$ denote the exponent sum of b. The regular isotopy invariant L defined by $L(\hat{b}) = (-\alpha^2 q)^{w(b)} F(\hat{b})$ satisfies the following relations.

$$L(K_+) - L(K_-) = (q - q^{-1})(L(K_0) - L(K_\infty)),$$

$$L(K_{\ell +}) = -\alpha^2 q L(K_\frown), \quad L(K_{\ell -}) = (-\alpha^2 q)^{-1} L(K_\frown), \quad L(\bigcirc) = 1,$$

where the K_* are identical except within a ball where they are as in Figure 2. Hence F is equal to the Kauffman polynomial by [8, Section 2]. ∎

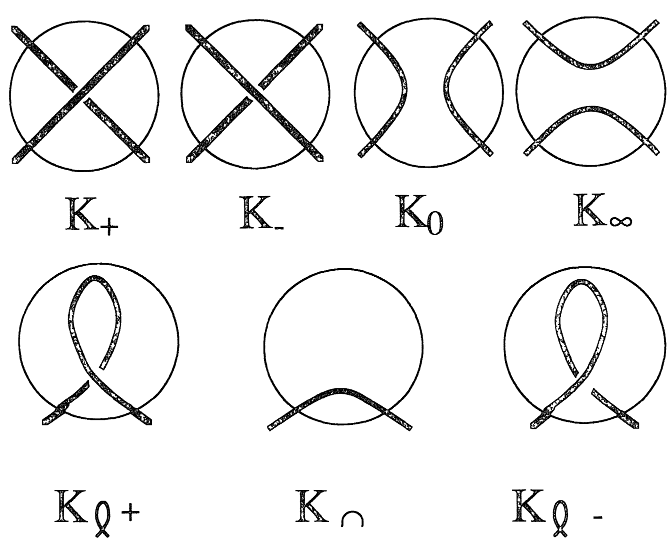

Figure 2

Lemma 2.3. *For λ_1 and λ_2 in Λ such that $\lambda_1 \underset{1}{\sim} \lambda_2$, we have*

$$\sum_{\lambda_2 \underset{1}{\sim} \lambda_3} \frac{G_{\lambda_3}}{G_{\lambda_2}} A(\lambda_1, \lambda_2, \lambda_2, \lambda_3) = -\alpha^2 q.$$

Proof. Let λ_4 be an element of Λ such that $\lambda_4 \underset{1}{\sim} \lambda_2$. From (1.19) and $A_i A_i' = 1$, we have $\sum_{\lambda_2 \underset{1}{\sim} \lambda_3} \sqrt{G_{\lambda_3} G_{\lambda_4}}\, G_{\lambda_2}^{-1} A'(\lambda_2, \lambda_3, \lambda_1, \lambda_2) = -\alpha^2 q \sqrt{G_{\lambda_1} G_{\lambda_4}}\, G_{\lambda_2}^{-1}$ and so we have $\sum_{\lambda_2 \underset{1}{\sim} \lambda_3} \sqrt{\frac{G_{\lambda_3}}{G_{\lambda_1}}} A'(\lambda_2, \lambda_3, \lambda_1, \lambda_2) = -\alpha^2 q$. Applying (1.17) to this, we get the statement of the lemma. ■

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