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The Structure and Representations of a C*-Algebra Associated to the Super-Poincaré Group.

By

Keith C. HANNABUSS,* and Winfried R. E. WEISS,**

Abstract

The representation theory of a class of algebras associated to certain graded Lie groups is investigated. To a group whose even part is central is associated a natural involutive algebra all of whose *-representations factor through a quotient algebra of continuous Clifford algebra-valued fields. The irreducible representations of crossed products of the algebra by a Lie algebra, such as the super-Poincaré group, are then constructed by Takesaki's method. It is then shown that they may also be constructed by Rieffel's C*-algebraic induction. Tensor product decompositions are briefly discussed.

§1. Introduction

In this paper we shall show how the theory of induced representations of C*-algebras, [1], [2], [3], can be used to construct the irreducible representations of the super-Poincaré group. This is directly comparable to the Wigner-Mackey theory for the ordinary Poincaré group, [4], [5].

We shall work with the semi-direct product of a group L and a graded Lie algebra, $\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1$, whose even subalgebra, $\mathfrak{n}_0$ is central in $\mathfrak{n}$ and is the Lie algebra of a vector group N_0. In the case of the super-Poincaré group, L is the Lorentz covering group $SL(2, \mathbb{C})$ and $\mathfrak{n}$ is the supertranslations, which we shall describe in more detail in the next section.

Although the structure of the supertranslations is most transparent in the Lie algebras, the global group-theoretic structure is needed for the inducing construction. This suggests that the representations should be considered as Harish-Chandra modules on which both N_0 and $\mathfrak{n}$ act in a compatible way. (So that the action of $\mathfrak{n}_0$ is the derivative of the action of N_0.) In fact, for the algebraic inducing we shall not directly consider the action of N_0, but rather that

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* Balliol College, Oxford OX1 3BJ, England.
** Wolfson College, Oxford OX2 6UD, England.

of its convolution algebra of Schwartz functions, $\mathscr{S}(N_0)$. The Schwartz functions are an obvious choice since they allow us both to perform harmonic analysis using the Fourier transform and to obtain an action of $\mathfrak{n}_0$ by differentiation. To be precise, we define for $X \in \mathfrak{n}_0$, $a \in \mathscr{S}(N_0)$, $v \in N_0$,

$$(X.a)(v) = \frac{d}{dt} a(\exp(-tX)v)|_{t=0}.$$

The supertranslations will be incorporated through the (complex) universal enveloping algebra $\mathscr{U}(\mathfrak{n})$.

Both $\mathscr{U}(\mathfrak{n})$ and $\mathscr{S}(N_0)$ can be equipped with involutions. On $\mathscr{S}(N_0)$ we have the natural involution

$$a^*(v) = \overline{a(-v)}.$$

For $\mathscr{U}(\mathfrak{n})$ matters are a little more delicate and will be examined in more detail in the next section. The involution will play a crucial role both in determining the structure of the algebra and in picking out the physically important representations.

In order that a representation M of the *-algebra $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ describe compatible representations of $\mathfrak{n}$ and N_0 we shall require that

$$M(AX \otimes a) = M(A \otimes X.a), \quad A \in \mathscr{U}(\mathfrak{n}),\ X \in \mathfrak{n}_0,\ a \in \mathscr{S}(N_0).$$

Any representation satisfying this requirement clearly vanishes on the subspace

$$I = \mathrm{span}\{AX \otimes a - A \otimes X.a : A \in \mathscr{U}(\mathfrak{n}),\ X \in \mathfrak{n}_0,\ a \in \mathscr{S}(N_0)\}.$$

We shall show in Section 3 that I is actually a *-ideal, so that compatible representations are lifted from the quotient *-algebra $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)/I$. We shall then show that the involution is only becomes compatible with a C* structure when we pass to another quotient which is isomorphic to a pre-C*-algebra of continuous fields.

Once these structural results have been established the representation theory is described in Section 4. Then in Section 5 we construct the *-representations of the crossed product of the *-algebra by L using the algebraic generalisation of Mackey's method due to Takesaki, [2]. In Section 6 a more fully algebraic method is used to induce irreducible representations from representations of $U(2).N_0$, thus tightening the analogy with the Poincaré group. Finally in Section 7 the reduction of tensor products of irreducible representations is sketched.

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§2. Space-time Supertranslations

Although the general theory does not depend on the detailed structure of $(N_0, \mathfrak{n})$ it is useful to bear the physical supertranslations in mind. There one takes N_0 to be the additive group $\mathbf{R}^4$. Its Lie algebra $\mathfrak{n}_0$ is also isomorphic to $\mathbf{R}^4$, and it is taken to have vanishing Lie bracket with the whole of $\mathfrak{n}$. The dual group $\hat{N}_0$ and dual space $\mathfrak{n}_0^*$ (with which it can be identified using the dual of the exponential map) are also copies of $\mathbf{R}^4$, on which we have the Minkowski inner product

$$g(\xi, \eta) = -\xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3,$$

$\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$, $\eta = (\eta_0, \eta_1, \eta_2, \eta_3) \in \mathfrak{n}_0^*$. We have chosen the sign for this product so that the Clifford algebra $\mathrm{Cliff}(\mathfrak{n}_0^*, g)$, generated by elements $\gamma(\xi)$ such that

$$\gamma(\xi)^2 = g(\xi, \xi),$$

has a real irreducible representation on the four-dimensional space of Majorana spinors. We denote this representation space by $\mathfrak{n}_1$.

It is well-known that $\mathfrak{n}_1$ can be equipped with a symplectic form c such that

$$c(A, \gamma(\xi)B) = -c(\gamma(\xi)A, B) = c(B, \gamma(\xi)A),$$

$A, B \in \mathfrak{n}_1$, $\xi \in \mathfrak{n}_0^*$. (In matrix form one takes $c(A, B) = -A^t\gamma^0 B$ where $\gamma^0 = \gamma(1, 0, 0, 0)$. Since the representation is irreducible c is unique up to scalar multiples.)

The symmetry of the above expression in A and B allows us to define the Lie bracket in $\mathfrak{n}$ such that if $A, B \in \mathfrak{n}_1$, then $[A, B]$ is the unique element of $\mathfrak{n}_0$ with

$$\xi([A, B]) = 2c(A, \gamma(\xi)B),$$

for all ξ in $\mathfrak{n}_0^*$. (In matrix notation the j-th component of $[A, B]$ is $-2(A^t\gamma^0\gamma^j B)$, which is the form used by physicists.) Since $\mathfrak{n}_0$ is, by definition, central the Jacobi identity is automatic.

For future reference we note the following consequence of our definitions.

Proposition 2.1. *If c is normalised so that $c(A, \gamma^0 A) > 0$, for all A (as happens with the physicists' convention) then $c(A, \gamma(\xi)A)$ is non-negative for all A in $\mathfrak{n}_1$ if and only if ξ is timelike or lightlike and $\xi_0 > 0$. When ξ is forward-pointing and timelike the quadratic form is positive definite; when ξ is forward-pointing and lightlike the quadratic form is positive semi-definite of rank two.*

Proof. Clearly the quadratic form $c(A, \gamma(\xi)A) = -c(A, \gamma^0\gamma^0\gamma(\xi)A)$ is non-negative if and only if the operator $-\gamma^0\gamma(\xi)$ is non-negative with respect to the inner product $A\ B \to c(A, \gamma^0 B)$ on $\mathfrak{n}_1$. Now

$$(\xi_0 + \gamma^0\gamma(\xi))^2 = \xi_0^2 + g(\xi, \xi) = \xi_1^2 + \xi_2^2 + \xi_3^2,$$

so the eigenvalues of $-\gamma^0\gamma(\xi)$ are $\xi_0 \pm \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$, and the positivity condition follows immediately. It also follows that when ξ is forward-pointing timelike all four eigenvalues are positive and so the form is positive definite.

In fact, since $\gamma^0\gamma(\xi) + \xi_0 = 2^{-1}[\gamma^0, \gamma(\xi)]$, its trace must vanish and each possible eigenvalue must appear twice. This immediately tells us that when ξ is lightlike two eigenvalues vanish and the rank is two.

Finally we consider the involution. For unitary representations we want the elements of the even subalgebra $\mathfrak{n}_0$ to be represented by i times self-adjoint operators. Since $[\mathfrak{n}_1, \mathfrak{n}_1] \subset \mathfrak{n}_0$, this suggests that $\mathfrak{n}_1$ should be represented by $\exp(i\pi/4)$ times self-adjoint operators so that the appropriate involution on its complexification is

$$(e^{-i\pi/4}A)^* = e^{-i\pi/4}\bar{A}$$

where $\bar{A}$ denotes the natural conjugation in the complexification. Rearranging this we obtain

$$A^* = -i\bar{A}.$$

This is then extended to an anti-automorphism on $\mathscr{U}(\mathfrak{n})$. We note that, up to sign this is the only involution consistent with the other data.

§3. The C*-Algebra of Continuous Clifford Fields

We start by proving the result promised in the introduction:

Lemma 3.1. *The linear span I of $\{AX \otimes a - A \otimes (X.a)\colon A \in \mathscr{U}(\mathfrak{n}), X \in \mathfrak{n}_0, a \in \mathscr{S}(N_0)\}$ is a *-ideal in $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$.*

Proof. For $B \otimes b \in \mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ we have

$$(AX \otimes a - A \otimes (X.a))(B \otimes b) = (AXB \otimes ab - AB \otimes (X.a)b).$$

Now $\mathfrak{n}_0$ is central in $\mathfrak{n}$ so $XB = BX$, and also by definition $X.(ab) = (X.a)b$, so

$$(AX \otimes a - A \otimes (X.a))(B \otimes b) = (ABX \otimes ab - AB \otimes X.(ab)) \in I.$$

Similarly $(B \otimes b)(AX \otimes a - A \otimes (X.a)) \in \mathrm{I}$. Finally,

$$\begin{aligned}(AX \otimes a - A \otimes (X.a))^* &= (X^*A^* \otimes a^* - A^* \otimes (X.a)^*) \\ &= (-XA^* \otimes a^* + A^* \otimes (X.a^*)) \\ &= -(A^*X \otimes a^* - A^* \otimes (X.a^*)) \in I,\end{aligned}$$

using the centrality of $\mathfrak{n}_0$ once more.

Remark. By induction we may replace the coset $(AX_1 \dots X_k \otimes a) + I$ by $A \otimes (X_1 \dots X_k).a + I$ for any $X_1, \dots, X_k \in \mathfrak{n}_0$. Since $\mathscr{U}(\mathfrak{n}) \cong \Lambda(\mathfrak{n}_1) \otimes \mathscr{U}(\mathfrak{n}_0)$ as a vector space (by the Poincaré-Birkhoff-Witt Theorem), we see that as vector spaces

$$\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)/I \cong \Lambda(\mathfrak{n}_1) \otimes \mathscr{S}(N_0),$$

where $\Lambda(\mathfrak{n}_1)$ denotes the exterior algebra of $\mathfrak{n}_1$.

We introduced the convolution algebra $\mathscr{S}(N_0)$ to facilitate comparison with standard group representation theory, but for further analysis it is useful to Fourier transform this. We shall identify N_0 with $\mathfrak{n}_0$ and $\hat{N}_0$ with $\mathfrak{n}_0^*$ using the exponential map, and write

$$\hat{a}(\xi) = \int_{N_0} e^{i\xi(x)} a(x)dx$$

where $\xi(x)$ denotes the natural pairing between duals. It is well-known that the Fourier transform provides an isomorphism between the convolution algebra $\mathscr{S}(N_0)$ and the algebra $\mathscr{S}(\hat{N}_0) = \mathscr{S}(\mathfrak{n}_0^*)$ of Schwartz functions under pointwise multiplication. Using the relation

$$(X.a)\hat{}\,(\xi) = i\xi(X)\hat{a}(\xi), \quad X \in \mathfrak{n}_0,$$

we may identify $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ with $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(\mathfrak{n}_n^*)$ which is factored out by the ideal $\hat{I}$ generated by the relation

$$(AX \otimes \hat{a})(\xi) = i\xi(X)(A \otimes \hat{a})(\xi).$$

As a vector space this is clearly isomorphic to $\Lambda(\mathfrak{n}_1) \otimes \mathscr{S}(\mathfrak{n}_0^*)$. We shall abbreviate the notation by writing $\hat{a}$ instead of $(1 \otimes \hat{a})$.

Lemma 3.2. *The element $A \otimes \hat{a}$ in $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(\mathfrak{n}_0^*)$ with the above relation is of the form*

$$(A \otimes \hat{a})(\xi) = \hat{a}(\xi)A_\xi$$

for a suitable $A_\xi \in \Lambda(\mathfrak{n}_1)$. For $A, B \in \mathfrak{n}_1$, we have

$$[A_\xi, B_\xi] = i\xi([A, B]).$$

Proof. Let $\hat{a}$ and $\hat{b}$ be nowhere vanishing Schwartz functions. Then the identities

$$(A \otimes \hat{a})(\xi)\hat{b}(\xi) = (A \otimes \hat{a}\hat{b})(\xi) = (A \otimes \hat{b})(\xi)\hat{a}(\xi)$$

show that $A_\xi = \hat{a}(\xi)^{-1}(A \otimes \hat{a})(\xi)$ is independent of the particular choice of $\hat{a}$. We therefore have

$$(A \otimes \hat{a})(\xi) = \hat{a}(\xi) A_\xi.$$

Multiplication by a general Schwartz function $\hat{f}$ shows that this identity is valid even when $\hat{a}$ is allowed to have zeroes.

Finally we note that when A and B are in $\mathfrak{n}_1$ so are A_ξ and B_ξ, and

$$\begin{aligned}[A_\xi, B_\xi] &= \hat{a}(\xi)^{-1}([A, B] \otimes \hat{a})(\xi) \\ &= \hat{a}(\xi)^{-1}(1 \otimes ([A, B].a)\hat{})(\xi) \\ &= i\xi([A, B]).\end{aligned}$$

This result tells us that the quotient algebra $\mathscr{U}(\mathfrak{n}_1) \otimes \mathscr{S}(N_0)/\hat{I}$ looks like an algebra of fields over $\mathfrak{n}_0^*$ in which the fibre at ξ is a Clifford algebra. This can be made more precise when we consider the involution as well. Before we can state the results we need a little more notation. Let us introduce

$$\mathfrak{n}_0^+ = \{\xi \in \mathfrak{n}_0^* : \xi([A, A]) \geq 0,\ A \in \mathfrak{n}_1\},$$

$$\mathscr{S}(\mathfrak{n}_0^+) = \mathscr{S}(\mathfrak{n}_0^*)/\{\hat{a} \in \mathscr{S}(\mathfrak{n}_0^*) : \hat{a}|\pi_0^+ = 0\}.$$

For $\xi \in \mathfrak{n}_0^+$ the quadratic form $\xi([A, A])$ is positive and the Clifford algebra, C_ξ of $\mathfrak{n}_1$ factored out by the radical has a natural C*-norm which for $A \in \mathfrak{n}_1$ is given by

$$\| A_\xi \| = 2^{-1/2} \xi([A, A])^{1/2}.$$

Theorem 3.3. *Any *-homomorphism from $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ to a C^*-algebra must factor through the pre-C*-algebra of continuous Clifford algebra-valued fields over $\mathfrak{n}_0^+$.*

Remark. By a pre-C*-algebra of fields we mean one satisfying the first three conditions in [6] Ch. 10, but which requires completion to obtain the fourth.

Proof. For $A \in \mathfrak{n}_1$ we have, by definition,

$$[A_\xi^*, A_\xi] = -i[A_\xi, A_\xi] = \xi([A, A]).$$

For any homomorphism to a C*-algebra this must map to a non-negative element. This means that only when ξ in $\mathfrak{n}_0^+$ can we obtain non-zero elements. The homomorphism therefore factors through $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(\mathfrak{n}_0^+)/\hat{I}$. The same identity shows that elements of $\mathfrak{n}_1$ which lie in the radical of the quadratic form also map to 0 and the result now follows.

We shall denote the resulting quotient algebra of fields by $\mathscr{S}(\mathfrak{n}, N_0)$. It may be given the usual C* norm

$$\| \Sigma(A_j \otimes a_j) \| = \sup\{(2^{-1/2} \Sigma |\hat{a}_j(\xi)|^2 \xi([A_j, A_j]))^{1/2} : \xi \in \mathfrak{n}_0^+\}.$$

Since, apart from an occasional choice of sign (as for the involution) all the constructions have been canonical it seems that this is the most natural C*-algebra to associate to the super-translation subgroup.

There are other ways of arriving at the same structure. One of the most elegant is to exploit the natural conditional expectation $P: \mathscr{U}(\mathfrak{n}) \to \mathscr{U}(\mathfrak{n}_0)$. This is uniquely characterised by the following conditions:

(i) P vanishes on the odd part of $\mathscr{U}(\mathfrak{n})$;
(ii) $P(AX) = P(A)X$, $A \in \mathscr{U}(\mathfrak{n})$, $X \in \mathfrak{n}_0$;
(iii) $P(A^*) = P(A)^*$, $A \in \mathscr{U}(\mathfrak{n})$.

We may then provide $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ with the inner product:

$$\langle (A \otimes a), (B \otimes b) \rangle = \int_{\mathfrak{n}_0^+} \overline{\hat{a}(\xi)}\, (P(A^*B)b)\hat{}\,(\xi)\, d\xi.$$

The radical of this inner product clearly contains all functions which vanish on $\mathfrak{n}_0^+$ as well as I, since

$$\langle (A \otimes a), (BX \otimes b - B \otimes Xb) \rangle = \int \bar{\hat{a}}\, ((P(A^*BX) - P(A^*B)X)b)\hat{} = 0.$$

The radical also contains the other elements by which it is necessary to factor out to obtain a C*-algebra. One can actually uncover further structure by this method, but since it will play no further role we leave the matter with that observation.

§4. Representation Theory of the Supertranslation Algebra

It is now a straightforward matter to find the irreducible *-representations of the algebra $\mathscr{S}(\mathfrak{n}, N_0)$.

Theorem 4.1. *For $\xi \in \mathfrak{n}_0^+$ let T_ξ be an irreducible representation of the Clifford algebra C_ξ. Then*

$$M_\xi(A \otimes a) = \hat{a}(\xi)\, T_\xi(A_\xi)$$

*defines an irreducible *-representation of $\mathscr{S}(\mathfrak{n}, N_0)$. Conversely every non-degenerate irreducible *-representation is of this form.*

Proof. We shall start with the converse. The subalgebra $1 \otimes \mathscr{S}(N_0) \subset \mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ projects onto a central subalgebra of $\mathscr{S}(\mathfrak{n}, N_0)$. By Schur's Lemma this subalgebra must be represented by scalars in any irreducible representation, and it is known that the only non-zero *-homomorphisms from $\mathscr{S}(N_0)$ to $\mathbb{C}$ take the form $a \to \hat{a}(\xi)$. For a map to be induced on the quotient subalgebra of $\mathscr{S}(\mathfrak{n}, N_0)$ we require $\xi \in \mathfrak{n}_0^+$. Now if M_ξ is any irreducible *-representation of $\mathscr{S}(\mathfrak{n}, N_0)$ whose restriction to this central subalgebra is given by ξ then one shows as in the proof of Lemma 3.2 that

$$T_\xi(A_\xi) = \hat{a}(\xi)^{-1} M_\xi(A \otimes a)$$

is a representation of C_ξ, whose irreducibility is forced by that of M_ξ. It is, on the other hand, easy to see that $M_\xi(A \otimes a) = T_\xi(A_\xi)\hat{a}(\xi)$ does always define an irreducible *-representation so that the proof is complete.

Remark. If C_ξ is the Clifford algebra of an even dimensional space then there is a unique irreducible representation T_ξ. It follows from Proposition 2.1 that this is the case for the supertranslations. For odd-dimensional Clifford algebras there are two inequivalent irreducibles, determined by the eigenvalue of the central element. This can be made to depend continuously of ξ.

Corollary 4.2. *Every non-degenerate irreducible *-representation of* $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ *takes the form*

$$M_\xi(A \otimes a) = \hat{a}(\xi) T_\xi(A_\xi)$$

for some $\xi \in n_0^+$.

Proof. Since a *-representation is a homomorphism into a C*-algebra it must factorise through $\mathscr{S}(\mathfrak{n}, N_0)$.

Corollary 4.3. *The C*-algebra obtained by completing* $\mathscr{S}(\mathfrak{n}, N_0)$ *with respect to the norm defined after Theorem* 3.3 *is of type I.*

Proof. According to the theorem the dual of $\mathscr{S}(\mathfrak{n}, N_0)$ is either $\mathfrak{n}_0^+$ or a double cover of $\mathfrak{n}_0^+$, according to whether the Clifford algebra is even or odd dimensional. In either case this is smooth. The result therefore follows by Glimm's Theorem, [7].

Remark. The above results refer to ungraded representations. When considering graded representations one notes that whenever a quotient has been taken the ideal by which we factored out was invariant under the grading operator. (The grading operator takes the value -1 on odd terms and $+1$ on even terms.) This means that the gradings on $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$, on $\mathscr{S}(\mathfrak{n}, N_0)$ and on the Clifford algebra C_ξ are all consistent. Thus we obtain the graded irreducibles simply by taking T_ξ to be a graded irreducible representation of C_ξ. The graded irreducibles of an even-dimensional C_ξ can be represented on the ungraded space but there are two inequivalent ways of introducing a grading. (The easiest way to see this is to note that the grading operator anticommutes with the generators of C_ξ and has square 1. Together the grading operator and C_ξ generate the Clifford algebra of a space of one higher dimension, which, being odd-dimensional, has two inequivalent irreducibles.) In the case of odd-dimensional Clifford algebras the two inequivalent ungraded irreducibles combine into a single graded irreducible.

§5. The Super-Poincaré Group and Its C*-Algebra

By assumption the group L acts as automorphisms of $\mathfrak{n}$ and of N_0. It therefore acts on the algebra $\mathscr{U}(\mathfrak{n}) \otimes \mathscr{S}(N_0)$ in a natural way. (The action on N_0 being equivalent under the exponential map to that on $\mathfrak{n}_0$ is linear so that we can define an action on $\mathscr{S}(N_0)$ by

$$(\lambda. a)(v) = a(\lambda^{-1} v),$$

or equivalently on $\mathscr{S}(\mathfrak{n}_0^*)$ by

$$(\lambda. a)\hat{}\,(\xi) = \hat{a}(\lambda^{-1}\xi),$$

for $\lambda \in L$, where $(\lambda^{-1}\xi)(v) = \xi(\lambda v)$.) The action clearly leaves the ideal I invariant and also respects the * operation so that it will pass to the quotients and provide automorphisms of $\mathscr{S}(\mathfrak{n}, N_0)$. We may therefore form the C*-crossed product $\bar{\mathscr{S}}(\mathfrak{n}, N_0) \ltimes L$ where $\bar{\mathscr{S}}(\mathfrak{n}, N_0)$ is the C*-algebra obtained by completing $\mathscr{S}(\mathfrak{n}, N_0)$. (When L is the Lorentz covering group and $\mathfrak{n}$ is the supertranslation algebra the C*-crossed product is the C*-algebra which we associate with the super-Poincaré group.) We are now in a position to apply Takesaki's theory of induced *-representations of C*-crossed products, [2].

First we need some notation. For $\xi \in \mathfrak{n}_0^+$ we let L_ξ be the subgroup of L which stabilises the equivalence class of the representation M_ξ of $\mathscr{S}(\mathfrak{n}, N_0)$. That is, L_ξ consists of those $\lambda \in L$ for which there exists a unitary operator $D(\lambda)$ such that

$$M_\xi(\lambda, \alpha) = D(\lambda) M_\xi(\alpha) D(\lambda)^{-1}, \quad \alpha \in \mathscr{S}(\mathfrak{n}, N_0).$$

(As a matter of fact L_ξ injects into the spin group which is a subgroup of the invertible elements in C_ξ and D is just the composition of this injection with M_ξ.) By a standard argument the irreducibility of M_ξ means that $\lambda \to D(\lambda)$ defines a projective representation of L_ξ. If we assume that $H^2(L, T) = 0$, as happens for when $L = SL(2, \mathbb{C})$ then the multiplier is trivial and we may as well assume that D is an ordinary representation.

Theorem 5.1. *Suppose that the action of L on $\hat{N}_0$ is smooth and that $H^2(L, T) = 0$. Let E be an irreducible unitary representation of L_ξ, so that $(D \otimes E)(\lambda)(M_\xi(\alpha) \otimes 1)(D \otimes E)(\lambda)^{-1} = (M_\xi(\lambda. \alpha) \otimes 1)$ for $\alpha \in \mathscr{S}(\mathfrak{n}, N_0)$. Then the covariant *-representation of $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L$ induced from the covariant *-representation $(M_\xi \otimes 1,\ D \otimes E)$ of the C*-crossed product $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L_\xi$ is irreducible. Conversely, every irreducible *-representation of $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L$ is of this form, and two such representations are equivalent if and only if the ξ's lie on the same L-orbit and the E's are conjugate.*

Proof. Theorem 4.1 and Corollary 4.3 have already told us that $\bar{\mathscr{S}}(\mathfrak{n}, N_0)$

is of type I and that its dual is essentially $\hat{N}_0$. The assumption that L acts smoothly on $\hat{N}_0$ is therefore all we need to be able to apply Takesaki's Theorem 6.1, [2]. Clearly $(M_\xi \otimes 1, D \otimes E)$ is an irreducible covariant representation of $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L_\xi$ and so induces an irreducible covariant representation of $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L$. On the other hand any primary representation of $\mathscr{S}(\mathfrak{n}, N_0)$ which is quasi-equivalent to M_ξ has the form $M_\xi \otimes 1$. If $(M_\xi \otimes 1, F)$ forms a covariant representation of $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L_\xi$ then $F(\lambda)(D(\lambda)^{-1} \otimes 1)$ $\lambda \in L_\xi$ commutes with $M_\xi(\alpha) \otimes 1$, $\alpha \in \mathscr{S}(\mathfrak{n}, N_0)$ and so takes the form $1 \otimes E(\lambda)$. In other words $F = (D \otimes E)$, and if $(M_\xi \otimes 1, D \otimes E)$ is irreducible then so is E. Takesaki's Theorem tells us that every irreducible representation of $\mathscr{S}(\mathfrak{n}, N_0) \ltimes L$ is induced from such a covariant representation.

Remarks. 1. It is easy to see that a grading may also be induced if one wishes to work with graded representations. (The trick of regarding the grading operator as enlarging the Clifford algebra is again useful here.)
2. One readily checks from the covariance relation for the crossed product that the derivative of the induced representation furnishes a representation of the super-Poincaré Lie algebra.

In the case of the super-Poincaré group we know that any timelike ξ can be Lorentz transformed to the form $(m, 0, 0, 0)$ on the same orbit. Since the Clifford algebra has only one irreducible representation up to equivalence the isotropy subgroup of $(m, 0, 0, 0)$ must be $SU(2)$. In fact if we give $\mathfrak{n}_1$ the complex structure defined by γ^0 then $\mathfrak{n}_1 \cong \mathbb{C}^2$ and $SU(2)$ acts naturally. The representation which implements the equivalence on the Clifford algebra representation is therefore the graded exterior algebra $\Lambda(D^{1/2}) \cong D^0 \oplus D^{1/2} \oplus D^0$. If we take $E = D^j$ for the other representation then $D \otimes E \cong (D^0 \oplus D^{1/2} \oplus D^0) \otimes D^j \cong D^j \oplus D^{j+1/2} \oplus D^{j-1/2} \oplus D^j$. We must therefore induce $(M_{(m,0,0,0)} \otimes 1, D^j \oplus D^{j+1/2} \oplus D^{j-1/2} \oplus D^j)$ to obtain an irreducible. (These are called the massive irreducibles.)

Similarly when ξ is lightlike it is on the same orbit as $(1, 0, 0, 1)$, whose isotropy subgroup is the solvable group of upper triangular matrices.

§6. Algebraically Induced Representations

We have now shown that the irreducible covariant representations of $\bar{\mathscr{S}}(\mathfrak{n}, N_0) \ltimes L$ are induced from covariant representations of $\bar{\mathscr{S}}(\mathfrak{n}, N_0) \ltimes L_\xi$. In this section we shall show how these can in turn be induced by Rieffel's algebraic method, [1], [3]. We shall do this just for the super-Poincaré group itself, where N_0 is the Minkowski translation group and $\mathfrak{n}_1$ the Majorana spinors.

The first step is to notice that for timelike ξ the representation M_ξ of C_ξ can be algebraically induced. We first consider the group $U(1)$ acting on $\mathfrak{n}_1$ by

$$e^{it} \longrightarrow \exp(t\gamma(\xi)(-g(\xi, \xi))^{-1/2}).$$

Since these are orthogonal transformations each induces an automorphism of C_ξ. Since even dimensional C_ξ is simple and each of these automorphisms is inner, implemented by say, $R_\xi(t) \in C_\xi$. Thus C_ξ can be regarded as a left C_ξ-right $U(1)$-bimodule. There is, moreover, a conditional expectation from C_ξ to $C(U(1))$

$$P(\mathrm{A})(t) = \mathrm{Tr}_s(AR_\xi(t)),$$

where Tr_s denotes the canonical supertrace, so any representation μ of $U(1)$ can be induced to give a representation of C_ξ by Rieffel's procedure. (Equivalently one may use the procedure of [3] using the operator-valued inner product

$$\langle A, B\rangle = \int \mu(t)P(A^*B)(t)dt.)$$

If $\mu(t)$ is chosen to be the lowest possible weight (e^{-it} with the usual conventions), then the induced representation is the irreducible M_ξ. (This is because $\int\mu(t)R_\xi(t)dt$ is then a rank one projection.)

Before moving on to the general case we note that the action of $\kappa \in L_\xi$ commutes with $\gamma(\xi)$ and gives rise to an inner automorphism of C_ξ implemented by $\delta_\xi(\kappa)$ which commutes with $R_\xi(t)$. (In the notation of Section 5 the representation $D(\kappa) = M_\xi(\delta_\xi(\kappa))$.)

We now extend this idea to the C*-crossed product $\bar{\mathscr{S}}(\mathrm{n}, N_0) \ltimes L$. We recall that this may be considered as a completion of the algebra of continuous functions of compact support from L to $\mathscr{S}(\mathrm{n}. N_0)$. These may also be regarded as function on $L \times N_0$, and we shall write them as $\alpha(\lambda, \xi)$.

Theorem 6.1. *For* $\xi = (m, 0, 0, 0)$ *and a representation* E *of* $SU(2)$ *the *-representation induced algebraically using the operator-valued inner product*

$$\langle\alpha, \beta\rangle = \int \mu(\mathrm{t})\mathrm{Tr}(\alpha^*\beta(\kappa, \xi)\delta_\xi(\kappa)R_\xi(t))E(\kappa)dtd\kappa,$$

$\alpha, \beta \in \bar{\mathscr{S}}(\mathrm{n}, N_0) \ltimes L$ *is irreducible, and every massive irreducible *-representation is equivalent to one of this form.*

Proof. The integration over t gives M_ξ, whilst the integration over κ gives the inner product needed for inducing from $\mathscr{S}(\mathrm{n}, N_0) \ltimes L_\xi$, viewed algebraically ([1], Theorem 5.12), so that overall we recover the same irreducible as before. In some ways this makes the analogy with the Wigner-Mackey construction of the Poincaré irreducibles even stronger.

§7. Tensor Products of Irreducible Representations

In this section we shall just outline how one can decompose the tensor product of two massive irreducible graded representations of the super-Poincaré group. We denoe by $\hat{\otimes}$ the graded tensor product of two graded algebras, [8], [9].

Let E_1 and E_2 be two representations of L_ξ and M_ξ^1, M_ξ^2 graded representations of C_ξ. Using Takesaki's Theorem 7.1, [2], we may restrict the induced covariant representation $\mathrm{ind}(\xi M_\xi^1,\ D\otimes E_1)\otimes\mathrm{ind}(\xi M_\xi^2,\ D\otimes E_2)$ of $(\mathscr{S}(\mathfrak{n},\ N_0)\hat{\otimes}\mathscr{S}(\mathfrak{n},\ N_0),\ L\otimes L)$ to $(\mathscr{S}(\mathfrak{n},\ N_0)\hat{\otimes}\mathscr{S}(\mathfrak{n},\ N_0),\ L)$ where L is regarded as the diagonal subgroup of $L\times L$. Just as for the Poincaré group the double cosets of $L\backslash(L\times L)/(L_\xi\times L_\xi)$ are in one-one correspondence with those of $L_\xi\backslash L/L_\xi$ and the restricted representation decomposes as

$$\int_{L_\xi\backslash L/L_\xi}\mathrm{ind}\{\xi M_\xi^1\otimes(\lambda\xi)M_{\lambda\xi}^2,\ (D\otimes E_1)\otimes(D\otimes E_2)^\lambda\}\,d\lambda.$$

To achieve further reduction we must note that $\mathscr{S}(\mathfrak{n}, N_0)$ has a natural Hopf algebra structure with comultiplication $\delta(A\otimes\hat{a})(\xi,\ \eta)=\hat{a}(\xi+\eta)(A_\xi\hat{\otimes}1+1\hat{\otimes}A_\eta)$. (Compare [8] Theorem 1.6.) Writing $\eta=\lambda\xi$ we may restrict the representation $\xi M_\xi^1\otimes\eta M_\eta^2$ of $\mathscr{S}(\mathfrak{n},\ N_0)\hat{\otimes}\mathscr{S}(\mathfrak{n},\ N_0)$ to the diagonal subalgebra $\delta\mathscr{S}(\mathfrak{n},\ N_0)$. This gives

$$(A\otimes\hat{a})\longrightarrow\hat{a}(\xi+\eta)(M_\xi^1(A_\xi)\hat{\otimes}1+1\hat{\otimes}M_\eta^2(A_\eta)).$$

This cannot be irreducible for dimensional reasons, but is a sum of four irreducibles. Combining this with Takesaki's direct integral one obtains an explicit expression for the tensor product of graded irreducible representations of the super-Poincaré group.

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