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Holomorphic and Singular Solutions of Nonlinear Singular First Order Partial Differential Equations

By

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Introduction

In this paper we will discuss the following types of nonlinear singular first order partial differential equations:

$$\text{(E)} \qquad t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right),$$

where $(t, x) \in \mathbb{C}_t \times \mathbb{C}_x^n$, $x = (x_1, \ldots, x_n)$, $\dfrac{\partial u}{\partial x} = \left(\dfrac{\partial u}{\partial x_1}, \ldots, \dfrac{\partial u}{\partial x_n}\right)$, $F(t, x, u, v)$ is a function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_u \times \mathbb{C}_v^n$, and $v = (v_1, \ldots, v_n)$. Our assumption is as follows:

(A_1) $\quad F(t, x, u, v)$ is holomorphic in Δ,

(A_2) $\quad F(0, x, 0, 0) = 0$ in Δ_0,

(A_3) $\quad \dfrac{\partial F}{\partial v_i}(0, x, 0, 0) = 0$ in Δ_0 for $i = 1, \ldots, n$,

where $\Delta_0 = \Delta \cap \{t = 0, u = 0 \text{ and } v = 0\}$.

The purpose of this paper is to study the following:

Problem. Investigate the structure of holomorphic and singular solutions of (E).

The most typical model of our equation (E) is the following ordinary differential equation

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$$t\frac{du}{dt}=f(t,u), \qquad f(0,0)=0 \tag{0.1}$$

which was first studied by Briot-Bouquet [2]. Nowadays it is called the Briot-Bouquet equation and the structure of solutions of (0.1) near the origin of $\mathbb{C}_t$ is well-known (see Hille [6], Hukuhara-Kimura-Matuda [8], Kimura [9], Gérard [5] etc.). In particular, when

$$\rho=\frac{\partial f}{\partial u}(0,0)$$

is in a generic position, we know the following:

Theorem. *Assume that $f(t,u)$ is a holomorphic function defined near the origin of $\mathbb{C}_t \times \mathbb{C}_u$. Then we have:*

(I) (Holomorphic solutions). *If $\rho \notin \mathbb{N}^*(=\{1,2,3,\dots\})$, the equation* (0.1) *has a unique solution $u_0(t)$ holomorphic near the origin of $\mathbb{C}_t$ satisfying $u_0(0)=0$.*

(II) (Singular solutions). *If $\rho \notin \mathbb{N}^* \cup \{a \in \mathbb{R};\, a \leqq 0\}$, the general solution $u(t)$ of* (0.1) *near the origin of $\mathbb{C}_t$ is given by*

$$u(t)=ct^{\rho}+a_{1,0}t+\sum_{i+j\geqq 2} a_{i,j}t^i(ct^{\rho})^j, \tag{0.2}$$

where $c \in \mathbb{C}$ is arbitrary, the coefficients $a_{i,j} \in \mathbb{C}$ are uniquely determined by the equation (0.1), *and the series*

$$w+a_{1,0}t+\sum_{i+j\geqq 2} a_{i,j}t^i w^j$$

is a convergent power series in $\{t,w\}$. The holomorphic solution $u_0(t)$ in (I) *is given by the case $c=0$.*

Is it possible to generalize these results (I) and (II) to our partial differential equations? What kinds of series appear in the expansion of singular solutions of (E)?

In order to answer these questions, let us make clear the meaning of our "singular solutions". Denote by:

- $\widetilde{\mathbb{C}\setminus\{0\}}$ the universal covering space of $\mathbb{C}\setminus\{0\}$;
- S_θ the sector in $\widetilde{\mathbb{C}\setminus\{0\}}$ defined by $\{t \in \widetilde{\mathbb{C}\setminus\{0\}};\, |\arg t|<\theta\}$;
- $S(\varepsilon(s))=\{t \in \widetilde{\mathbb{C}\setminus\{0\}};\, 0<|t|<\varepsilon(\arg t)\}$ for some positive-valued function $\varepsilon(s)$ defined and continuous on $\mathbb{R}_s$;
- $D(\delta) \quad =\{x \in \mathbb{C}^n;\, |x_i|<\delta,\ i=1,\dots,n\}$;
- $\mathbb{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n_x$.

Definition of $\tilde{\mathcal{O}}_+$. $\tilde{\mathcal{O}}_+$ is the set of all functions $u(t,x)$ satisfying the following conditions (i) and (ii):

(i) $u(t,x)$ is holomorphic in $S(\varepsilon(s)) \times D(\delta)$ for some $\varepsilon(s)$ and $\delta>0$;

(ii) there is an $a>0$ such that for any $\theta>0$ and any compact subset K of $D(\delta)$

$$\max_{x\in K}|u(t, x)| = O(|t|^a)$$

as t tends to zero in S_θ.

Note that $\mu(s)=s^a$, $a>0$, is a particular form of a function satisfying the following properties:

(μ_1) $\mu(s)\in C^0((0, \infty))$,

(μ_2) $\mu(s)>0$ and $\mu(s)$ is increasing in $s\in(0, \infty)$,

(μ_3) $\displaystyle\int_0^1 \frac{\mu(s)}{s}ds < +\infty$.

Definition of $\tilde{\mathcal{O}}_{\text{int}}$. $\tilde{\mathcal{O}}_{\text{int}}$ is the set of all functions $u(t, x)$ satisfying the following conditions (i) and (ii):

(i) $u(t, x)$ is holomorphic in $S(\varepsilon(s))\times D(\delta)$ for some $\varepsilon(s)$ and $\delta>0$;

(ii) there is a function $\mu(s)$ satisfying (μ_1), (μ_2), (μ_3) such that for any $\theta>0$ and any compact subset K of $D(\delta)$

$$\max_{x\in K}|u(t, x)| = O(\mu(|t|))$$

as t tends to zero in S_θ.

In this paper we will employ the space $\tilde{\mathcal{O}}_+$ or $\tilde{\mathcal{O}}_{\text{int}}$ as a framework of singular solutions. Clearly, we have $\tilde{\mathcal{O}}_+ \subset \tilde{\mathcal{O}}_{\text{int}}$.

Put

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0). \tag{0.3}$$

Then we can state our main theorem as follows:

Main Theorem. *Assume* (A_1), (A_2), (A_3) *and* $\rho(0)\notin \mathbb{N}^*$. *Then we have*:

(I) (Holomorphic solutions). *The equation* (E) *has a unique solution* $u_0(t, x)$ *holomorphic near the origin of* $\mathbb{C}_t\times\mathbb{C}_x^n$ *satisfying* $u_0(0, x)\equiv 0$.

(II) (Singular solutions). *Denote by* $\mathcal{S}_+$ [*resp.* $\mathcal{S}_{\text{int}}$] *the set of all* $\tilde{\mathcal{O}}_+$-*solutions* [*resp.* $\tilde{\mathcal{O}}_{\text{int}}$-*solutions*] *of* (E). *Then*:

$$\mathcal{S}_{\text{int}} = \mathcal{S}_+ = \begin{cases} \{u_0\}, & \textit{when } \operatorname{Re}\rho(0)\leqq 0, \\ \{u_0\}\cup\{U(\varphi);\ 0\not\equiv\varphi(x)\in\mathbb{C}\{x\}\}, & \textit{when } \operatorname{Re}\rho(0)>0, \end{cases}$$

where u_0 *is the holomorphic solution in* (I), *and* $U(\varphi)$ *is an* $\tilde{\mathcal{O}}_+$-*solution of* (E) *having the expansion of the following form*

$$\begin{cases} U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{\substack{i+2j \geq k+2 \\ j \geq 1}} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \\ \varphi_{0,1,0}(x) = \varphi(x). \end{cases}$$

Note that our result is consistent with the one for (0.1). To see this, we have only to recall that (0.1) is transformed into the equation $\left(t\frac{\partial}{\partial t} - \rho\right)w = 0$ under the relation

$$u = w + a_{1,0}t + \sum_{i+j \geq 2} a_{i,j} t^i w^j$$

and therefore in (0.2) the condition $u(t) \in \tilde{\mathcal{O}}_+$ is equivalent to the condition $ct^\rho \in \tilde{\mathcal{O}}_+$ (see Hukuhara-Kimura-Matuda [8]).

Thus, our equation (E) is quite similar to the Briot-Bouquet equation (0.1) not only in the form of the equation but also in the structure of solutions, and therefore the following definition will be reasonable:

Definition. If (E) satisfies (A_1), (A_2) and (A_3), we say that (E) is *of Briot-Bouquet type* with respect to t. Then, the holomorphic function $\rho(x)$ defined by (0.3) is called the characteristic exponent function of (E).

The paper is organized as follows. In §1 we will show the existence and uniqueness of holomorphic solutions, in §2 we will construct a family of singular solutions $U(\varphi)$ in $\tilde{\mathcal{O}}_+$, and in §3 we will discuss the uniqueness of $\tilde{\mathcal{O}}_{\mathrm{int}}$-solutions. Using the results obtained in §1 ~ §3, in §4 we will give a proof of Main Theorem. Some remarks will be stated in §5.

Throughout this paper we write $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$.

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§1. Holomorphic Solutions

In this section we will study holomorphic solutions of nonliear partial differential equations of the form:

$$(\mathrm{E}_1) \qquad \left(t\frac{\partial}{\partial t} - \rho(x)\right)u = ta(x) + G_2(x)\left(t, t\frac{\partial u}{\partial t}, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right),$$

where $\rho(x)$ and $a(x)$ are holomorphic functions defined in a polydisk D centered at the origin of $\mathbb{C}_x^n$, and

$$G_2(x)(t, z, X_0, X_1, \ldots, X_n) = \sum_{p+q+|\alpha| \geq 2} a_{p,q,\alpha}(x) t^p z^q X_0^{\alpha_0} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$$

in which the coefficients $\{a_{p,q,\alpha}(x)\}_{p+q+|\alpha|\geqq 2}$ are holomorphic in D and

$$|a_{p,q,\alpha}(x)| \leqq A_{p,q,\alpha};$$

moreover the power series

$$|G_2|(t, z, X) = \sum_{p+q+|\alpha|\geqq 2} A_{p,q,\alpha} t^p z^q X^\alpha$$

is convergent near the origin of $\mathbb{C}_t \times \mathbb{C}_z \times \mathbb{C}_X^{n+1}$.

A holomorphic solution of (E_1) means in this section a solution $u(t, x)$ holomorphic near the origin of $\mathbb{C}_t \times \mathbb{C}_x^n$ satisfying $u(0, x) \equiv 0$, and a formal solution of (E_1) means a formal power series solution of the form

$$\sum_{m\geqq 1} u_m(x) t^m$$

whose coefficients $\{u_m(x)\}_{m\geqq 1}$ are holomorphic in a same disk centered at the origin of $\mathbb{C}_x^n$.

Then we have:

Theorem 1. (1) *Each formal solution of* (E_1) *is convergent.*

(2) *If* $\rho(0) \notin \mathbb{N}^*$, (E_1) *has a unique formal solution which gives a unique holomorphic solution* $u(t, x)$ *satisfying* $u(0, x) \equiv 0$.

Note that the assertion (I) of Main Theorem easily follows from (2) of Theorem 1.

Proof. The assumption $\rho(0) \notin \mathbb{N}^*$ implies easily that (E_1) has a unique formal solution of the form

$$\sum_{m\geqq 1} u_m(x) t^m.$$

Moreover, $u_m(x)$ (for $m \geqq 1$) is determined by the following recursive formula:

$$u_1(x) = \frac{a(x)}{1-\rho(x)}$$

and for $m \geqq 2$

$$(1.1) \qquad u_m(x) = \frac{1}{m-\rho(x)} f_m\Big(u_1(x), 2u_2(x), \dots, (m-1)u_{m-1}(x), u_1(x), \dots, u_{m-1}(x),$$
$$\frac{\partial u_1}{\partial x_1}, \dots, \frac{\partial u_1}{\partial x_n}, \dots, \frac{\partial u_{m-1}}{\partial x_1}, \dots, \frac{\partial u_{m-1}}{\partial x_n}; \{a_{p,q,\alpha}(x)\}_{p+q+|\alpha|\leqq m}\Big).$$

Hence, if $\rho(0) \notin \mathbb{N}^*$, this formula shows the unicity of the formal solution and the unicity of a holomorphic solution if it does exist.

To prove the assertion (2) of Theorem 1, we have only to prove the convergence of this formal solution. From now we write

$$D_a = \{x \in \mathbb{C}^n;\ |x_i| \leqq a,\ i = 1, \dots, n\}.$$

We will make use of

Lemma 1. *If a function $u(x)$ holomorphic in D_R satisfies*

$$|u(x)| \leqq \frac{C}{(R-r)^p} \text{ on } D_r \text{ for } 0 < r < R,$$

then we have

$$\left| \frac{\partial u}{\partial x_i}(x) \right| \leqq \frac{Ce(p+1)}{(R-r)^{p+1}} \text{ on } D_r \text{ for } 0 < r < R,\ i = 1, \dots, n$$

(*where e is the real number such that* $\log e = 1$).

For the proof, see Hörmander [7, Lemma 5.1.3].

Proof of the convergence of the formal solution

$$\sum_{m \geqq 1} u_m(x) t^m. \tag{1.2}$$

First we assume $\rho(0) \notin \mathbb{N}^*$. Then by taking R sufficiently small we may assume:
1) $0 < R < 1$;
2) all the $u_m(x)$ are holomorphic in D_R;
3) we have in D_R

$$|u_1(x)| \leqq A,\ \left| \frac{\partial u_1}{\partial x_i}(x) \right| \leqq A,\ i = 1, \dots, n;$$

$$|a_{p,q,\alpha}(x)| \leqq A_{p,q,\alpha},\ p + q + |\alpha| \geqq 2;$$

$$|m - \rho(x)| \geqq \sigma m,\ m = 1, 2, 3, \dots.$$

Consider now the following analytic equation

$$\sigma Y = \sigma A t + \frac{1}{R-r} \sum_{p+q+|\alpha| \geqq 2} \frac{A_{p,q,\alpha}}{(R-r)^{p+q+|\alpha|-2}} t^p Y^q Y^{\alpha_0} (eY)^{\alpha_1} \dots (eY)^{\alpha_n}.$$

By the implicit function theorem, this equation has a unique holomorphic solution of the form

$$Y = \sum_{m \geqq 1} Y_m(r) t^m, \tag{1.3}$$

and $Y_m(r)$ (for $m \geqq 1$) is determined by the following recursive formula:

$$Y_1 = A$$

and for $m \geqq 2$

$$\sigma Y_m = \frac{1}{R-r} F_m\Big(Y_1, \dots, Y_{m-1}, eY_1, \dots, eY_{m-1};\ \Big\{ \frac{A_{p,q,\alpha}}{(R-r)^{p+q+|\alpha|-2}} \Big\}_{p+q+|\alpha| \leqq m} \Big). \tag{1.4}$$

Moreover by induction on m we can see that $Y_m(r)$ is expressed in the form

$$Y_m(r) = \frac{C_m}{(R-r)^{m-1}}, \quad m = 1, 2, \dots \tag{1.5}$$

with constants $C_1 = A$ and $C_m \geqq 0$ (for $m \geqq 2$).

Let us show that this power series (1.3) is a majorant power series for the formal solution (1.2). To do so, it is sufficient to prove the following inequalities for all m:

$$|u_m(x)| \leqq |mu_m(x)| \leqq Y_m(r) \text{ on } D_r \text{ for } 0 < r < R; \tag{1.6$_m$}$$

$$\left| \frac{\partial u_m}{\partial x_i}(x) \right| \leqq eY_m(r) \text{ on } D_r \text{ for } 0 < r < R,\ i = 1, \dots, n. \tag{1.7$_m$}$$

Since $Y_1 = A$, the case $m = 1$ is clear from the definition of A. We will prove the general case by induction on m.

Let $m \geqq 2$. Suppose that $(1.6)_p$ and $(1.7)_p$ are already known for all $p < m$. Then from (1.1) and (1.4) we have

$$\begin{aligned}
&|u_m(x)| \\
&\leqq \frac{1}{|m-\rho(x)|} f_m\Big(|u_1|, |2u_2|, \dots, |(m-1)u_{m-1}|, |u_1|, \dots, |u_{m-1}|, \\
&\qquad \left|\frac{\partial u_1}{\partial x_1}\right|, \dots, \left|\frac{\partial u_1}{\partial x_n}\right|, \dots, \left|\frac{\partial u_{m-1}}{\partial x_1}\right|, \dots, \left|\frac{\partial u_{m-1}}{\partial x_n}\right|; \{|a_{p,q,\alpha}|\}_{p+q+|\alpha| \leqq m} \Big) \\
&\leqq \frac{1}{\sigma m} f_m\Big(Y_1, Y_2, \dots, Y_{m-1}, Y_1, \dots, Y_{m-1}, \\
&\qquad eY_1, \dots, eY_1, \dots, eY_{m-1}, \dots, eY_{m-1}; \Big\{ \frac{A_{p,q,\alpha}}{(R-r)^{p+q+|\alpha|-2}} \Big\}_{p+q+|\alpha| \leqq m} \Big) \\
&= \frac{1}{\sigma m} F_m\Big(Y_1, \dots, Y_{m-1}, eY_1, \dots, eY_{m-1}; \\
&\qquad \Big\{ \frac{A_{p,q,\alpha}}{(R-r)^{p+q+|\alpha|-2}} \Big\}_{p+q+|\alpha| \leqq m} \Big) \\
&= \frac{1}{m}(R-r) Y_m(r)
\end{aligned} \tag{1.8}$$

which yields $(1.6)_m$. Since $Y_m(r)$ has the form (1.5), the above inequality (1.8) is written as

$$|u_m(x)| \leqq \frac{1}{m}\frac{C_m}{(R-r)^{m-2}} \text{ on } D_r \text{ for } 0<r<R$$

and therefore by Lemma 1 we obtain

$$\left|\frac{\partial u_m}{\partial x_i}(x)\right| \leqq \frac{m-1}{m}\frac{eC_m}{(R-r)^{m-1}} \leqq eY_m(r) \text{ on } D_r$$
$$\text{for } 0<r<R,\ i=1,\dots,n$$

which implies $(1.7)_m$.

Thus, by summing up we have obtained the convergence of the formal solution (1.2) under the assumption $\rho(0)\notin\mathbb{N}^*$.

Now assume that $\rho(0)=k\in\mathbb{N}^*$. Then the equation for $u_k(x)$ takes the form

$$(1.9)\qquad (k-\rho(x))u_k(x)=f_k\Bigg(u_1(x),\ 2u_2(x),\dots,(k-1)u_{k-1}(x),\ u_1(x),\dots,u_{k-1}(x),$$
$$\frac{\partial u_1}{\partial x_1},\dots,\frac{\partial u_1}{\partial x_n},\dots,\frac{\partial u_{k-1}}{\partial x_1},\dots,\frac{\partial u_{k-1}}{\partial x_n};\ \{a_{p,q,\alpha}(x)\}_{p+q+|\alpha|\leqq k}\Bigg).$$

When $\rho(x)\not\equiv k$ and if (E_1) has a formal solution, (1.9) implies that $u_k(x)$ is determined uniquely. When $\rho(x)\equiv k$ and if (E_1) has a formal solution, we must have from (1.9)

$$0\equiv f_k\Bigg(u_1(x),\ 2u_2(x),\dots,(k-1)u_{k-1}(x),\ u_1(x),\dots,u_{k-1}(x),$$
$$\frac{\partial u_1}{\partial x_1},\dots,\frac{\partial u_1}{\partial x_n},\dots,\frac{\partial u_{k-1}}{\partial x_1},\dots,\frac{\partial u_{k-1}}{\partial x_n};\ \{a_{p,q,\alpha}(x)\}_{p+q+|\alpha|\leqq k}\Bigg)$$
$$\text{on } D_R$$

which implies that $u_k(x)$ can be taken arbitrarily. Moreover, if $u_k(x)$ satisfies $(1.6)_k$ and $(1.7)_k$, the proof given before can be applied and the formal solution is convergent. Hence, to have the convergence of the formal solution we have only to notice the following fact: for a given $u_k(x)$ we can modify $\{A_{p,q,\alpha}\}_{p+q+|\alpha|\leqq k}$ so that $(1.6)_k$ and $(1.7)_k$ are satisfied.

Thus, the proof of Theorem 1 is completed.

Denote by $\mathbb{C}\{x\}[[t]]_0$ the ring of formal power series of the form

$$\sum_{m\geq 1} u_m(x)t^m$$

whose coefficients $u_m(x)$ are all holomorphic in a same disk centered at the origin of $\mathbf{C}_x^n$, and by $\mathbf{C}\{x\}\{t\}_0$ the subring of convergent series.

Definition 1. An operator

$$D: \mathbf{C}\{x\}[[t]]_0 \longrightarrow \mathbf{C}\{x\}[[t]]_0$$

is called singular regular, if and only if

$$u\in\mathbf{C}\{x\}[[t]]_0 \text{ and } Du\in\mathbf{C}\{x\}\{t\}_0 \text{ imply } u\in\mathbf{C}\{x\}\{t\}_0.$$

Then we have:

Corollary. *The operator*

$$D: u \longrightarrow Du = \left(t\frac{\partial}{\partial t} - \rho(x)\right)u - ta(x) - G_2(x)\left(t, t\frac{\partial u}{\partial t}, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$$

is singular regular.

Remark 1. Theorem 1 is not true for a partial differential equation containing higher derivatives with respect to x. For example, let us consider

$$\left(t\frac{\partial}{\partial t} - \rho\right)u = \frac{t}{1-x} + t\frac{\partial^2 u}{\partial x^2}, \ \rho \text{ is constant and } \rho\notin\mathbf{N}^*; \tag{1.10}$$

the equation (1.10) has the following formal solution

$$u(t, x) = \sum_{m\geq 1} \frac{(2m-2)!}{(1-\rho)\cdots(m-\rho)} \frac{t^m}{(1-x)^{2m-1}}$$

which clearly diverges.

Remark 2. (1) When $\rho(0) = k\in\mathbf{N}^*$ and $\rho(x) \not\equiv k$ hold and if (E_1) has a formal solution, then Theorem 1 gives a unique holomorphic solution.

(2) When $\rho(x) \equiv k\in\mathbf{N}^*$ holds near the origin of $\mathbf{C}_x^n$ and if (E_1) has a formal solution, then Theorem 1 gives a family of holomorphic solutions depending on the arbitrary function $u_k(x)$.

For some other results on the existence of holomorphic solutions for nonlinear partial differential equations, see Bengel-Gérard [1] and Gérard [3, 4].

§ 2. Singular Solutions

In this section we will construct a family of singular solutions in $\tilde{\mathcal{O}}_+$ of nonlinear partial differential equations of the form:

$$(\mathrm{E}_2) \qquad \left(t\frac{\partial}{\partial t} - \rho(x)\right)u = ta(x) + G_2(x)\left(t, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right),$$

where $\rho(x)$ and $a(x)$ are holomorphic functions defined in a polydisk D centered at the origin of $\mathbb{C}_x^n$, and

$$(2.1) \qquad G_2(x)(t, X_0, X_1, \dots, X_n) = \sum_{p+|\alpha|\geqq 2} a_{p,\alpha}(x) t^p X_0^{\alpha_0} X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

in which the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha|\geqq 2}$ are holomorphic in D and

$$|a_{p,\alpha}(x)| \leqq A_{p,\alpha};$$

moreover the power series

$$|G_2|(t, X) = \sum_{p+|\alpha|\geqq 2} A_{p,\alpha} t^p X^\alpha$$

is convergent near the origin of $\mathbb{C}_t \times \mathbb{C}_X^{n+1}$.

Note that (E_2) is a particular form of (E_1). Therefore, from Theorem 1 we know the following: if $\rho(0) \notin \mathbb{N}^*$, the equation (E_2) has a unique holomorphic solution $u(t, x)$ satisfying $u(0, x) \equiv 0$.

Now, let us find singular solutions of (E_2) of the form

$$(2.2) \qquad u(t, x) = w(t, t^{\rho(x)}, x, \log t)$$

with

$$w(t_1, t_2, x, y) = \sum_{m_1+m_2\geqq 1} w_{m_1,m_2}(x, y) t_1^{m_1} t_2^{m_2}.$$

To do so, we put t_1, t_2 and y as follows:

$$(2.3) \qquad t_1 = t,\ t_2 = t^{\rho(x)} \text{ and } y = \log t.$$

Then, under the relations (2.2) and (2.3) we have

$$t\frac{\partial u}{\partial t} = t_1\frac{\partial w}{\partial t_1} + \rho(x)t_2\frac{\partial w}{\partial t_2} + \frac{\partial w}{\partial y},$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial w}{\partial x_i} + \frac{\partial \rho(x)}{\partial x_i} y t_2 \frac{\partial w}{\partial t_2}, \quad i = 1, \dots, n.$$

Therefore, in order that $u(t, x)$ given by (2.2) is a solution of (E_2) the function $w(t_1, t_2, x, y)$ must be a solution of

$$(\mathrm{E}_2') \qquad \left(t_1\frac{\partial}{\partial t_1} + \rho(x)t_2\frac{\partial}{\partial t_2} - \rho(x) + \frac{\partial}{\partial y}\right)w$$
$$= t_1 a(x) + G_2(x)\left(t_1, w, \frac{\partial w}{\partial x_1} + \frac{\partial \rho(x)}{\partial x_1} y t_2 \frac{\partial w}{\partial t_2}, \dots, \frac{\partial w}{\partial x_n} + \frac{\partial \rho(x)}{\partial x_n} y t_2 \frac{\partial w}{\partial t_2}\right).$$

Proposition 1. (1) *If the condition*

$$m_1 + \rho(x)(m_2 - 1) \neq 0 \ \text{on} \ D_r \quad \text{for any} \ (m_1, m_2) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\} \tag{2.4}$$

is satisfied, (E_2') *has a formal solution of the form*

$$\sum_{m \geqq 1} u_m(x) t_1^m + \sum_{\substack{m_1 + 2m_2 \geqq k+2 \\ m_2 \geqq 1}} \varphi_{m_1, m_2, k}(x) t_1^{m_1} t_2^{m_2} y^k,$$

where $\varphi_{0,1,0}(x)$ *is arbitrary, and all the other coefficients* $u_m(x)$, $\varphi_{m_1,m_2,k}(x)$ *are uniquely determined by* $\varphi_{0,1,0}(x)$. *If* $\varphi_{0,1,0}(x)$ *is holomorphic in* D_r, *then all the coefficients* $u_m(x)$, $\varphi_{m_1,m_2,k}(x)$ *are also holomorphic in* D_r.

(2) *If* $\rho(0) \notin \mathbb{N}^*$ *and* $\operatorname{Re} \rho(0) > 0$, *then the formal solution given in* (1) *is convergent on a region of the form*

$$\{(t_1, t_2, x, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C};\ |t_1| < \varepsilon,\ |t_2| < \varepsilon,$$
$$|yt_1| < \varepsilon,\ |y^2 t_2| < \varepsilon \ \text{and} \ |x_i| < r \ (i = 1, \dots, n)\}$$

for some $\varepsilon > 0$ *and* $r > 0$ (*depending on* $\varphi_{0,1,0}$).

Before the proof, let us present some preparatory discussions.

First we note that under the relation

$$w(t_1, t_2, x, y) = \sum_{m_1 + m_2 \geqq 1} w_{m_1, m_2}(x, y) t_1^{m_1} t_2^{m_2}$$

the equation (E_2') is equivalent to a recursive family of equations as follows:

$$\left(1 - \rho(x) + \frac{\partial}{\partial y}\right) w_{1,0} = a(x), \tag{2.5}$$

$$\left(\frac{\partial}{\partial y}\right) w_{0,1} = 0, \tag{2.6}$$

and for $m_1 + m_2 \geqq 2$

$$\begin{aligned} &\left(m_1 + \rho(x)(m_2 - 1) + \frac{\partial}{\partial y}\right) w_{m_1, m_2} \\ &= f_{m_1, m_2}\Bigg(\{w_{i,j}\}_{\substack{i \leqq m_1, j \leqq m_2 \\ i+j \leqq m_1 + m_2 - 1}}, \left\{ \frac{\partial w_{i,j}}{\partial x_1} + \frac{\partial \rho}{\partial x_1} yj w_{i,j} \right\}_{\substack{i \leqq m_1, j \leqq m_2 \\ i+j \leqq m_1 + m_2 - 1}}, \dots, \\ &\qquad \left\{ \frac{\partial w_{i,j}}{\partial x_n} + \frac{\partial \rho}{\partial x_n} yj w_{i,j} \right\}_{\substack{i \leqq m_1, j \leqq m_2 \\ i+j \leqq m_1 + m_2 - 1}} ; \{a_{p,\alpha}\}_{p + |\alpha| \leqq m_1 + m_2} \Bigg). \end{aligned} \tag{2.7}$$

Next let us consider the operator

$$L_{m_1,m_2} = m_1 + \rho(x)(m_2 - 1) + \frac{\partial}{\partial y}.$$

Denote by $\mathbb{C}_r\{x\}$ the ring of all holomorphic functions in D_r, and by $P_{m,r}\{x\}[y]$ the set of all polynomials $\sum_{i=0}^{m} \varphi_i(x)y^i$ of degree m in y with coefficients $\varphi_i(x)$ belonging to $\mathbb{C}_r\{x\}$. Then, if the condition (2.4) is satisfied, we can easily see the following properties 1) and 2):

1) $L_{0,1}(P_{0,r}\{x\}[y]) = 0$;
2) for any $(m_1, m_2) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}$ and $m \in \mathbb{N}$ the operator

$$L_{m_1,m_2}: P_{m,r}\{x\}[y] \longrightarrow P_{m,r}\{x\}[y]$$

is invertible.

In order to estimate the operator norm of the inverse of $L_{m_1,m_2}: P_{m,r}\{x\}[y] \longrightarrow P_{m,r}\{x\}[y]$, for $\varphi(x) \in \mathbb{C}_r\{x\}$ we define the norm $\|\varphi\|_r$ by

$$\|\varphi\|_r = \max_{x \in D_r} |\varphi(x)|, \tag{2.8}$$

and for $w(x, y) = \sum_{i=0}^{m} \varphi_i(x)y^i \in P_{m,r}\{x\}[y]$ we define the norm $\|w\|_{r,\lambda}$ by

$$\|w\|_{r,\lambda} = \sum_{i=0}^{m} \|\varphi_i\|_r \lambda^i.$$

Then we have:

Lemma 2. *If the condition*

$$|m_1 + \rho(x)(m_2 - 1)| \geqq \sigma(m_1 + m_2) \text{ on } D_r$$
$$\text{for any } (m_1, m_2) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}$$

is satisfied for some $\sigma > 0$, *we have the following properties* 1) *and* 2) *for any* $(m_1, m_2) \in \mathbb{N}^2 \setminus \{(0, 0), (0, 1)\}$ *and* $m \in \mathbb{N}$:

1) $L_{m_1,m_2}: P_{m,r}\{x\}[y] \longrightarrow P_{m,r}\{x\}[y]$ *is an invertible operator*;
2) *for any* $w(x, y) \in P_{m,r}\{x\}[y]$ *we have*

$$\frac{\sigma}{2}(m_1 + m_2)\|w\|_{r,\lambda} \leqq \|L_{m_1,m_2}(w)\|_{r,\lambda}, \text{ if } \lambda \geqq \frac{2m}{\sigma(m_1 + m_2)}.$$

Proof. 1) is clear. Put $w(x, y) = \sum_{i=0}^{m} \varphi_i(x)y^i \in P_{m,r}\{x\}[y]$. Then, 2) is verified by the following:

$$\begin{aligned} &\|L_{m_1,m_2}(w)\|_{r,\lambda} \\ &\geqq \sum_{i=0}^{m} \|(m_1 + \rho(m_2 - 1))\varphi_i\|_r \lambda^i - \sum_{i=0}^{m} i\|\varphi_i\|_r \lambda^{i-1} \\ &\geqq \sigma(m_1 + m_2)\sum_{i=0}^{m} \|\varphi_i\|_r \lambda^i - \frac{m}{\lambda}\sum_{i=0}^{m} \|\varphi_i\|_r \lambda^i \end{aligned}$$

$$= \left(\sigma(m_1 + m_2) - \frac{m}{\lambda} \right) \| w \|_{r,\lambda}.$$

This lemma guarantees that the same argument as in § 1 can be also applied to (E_2').

Proof of Proposition 1. The existence of a formal solution of the form

$$\sum_{m_1+m_2 \geqq 1} w_{m_1,m_2}(x, y) t_1^{m_1} t_2^{m_2} \tag{2.9}$$

with

$$\begin{cases} w_{m,0} \in P_{0,r}\{x\}[y] = \mathbf{C}_r\{x\}, & m \geqq 1, \\ w_{m_1,m_2} \in P_{m_1+2m_2-2,r}\{x\}[y], & m_1 + m_2 \geqq 1 \text{ and } m_2 \geqq 1 \end{cases} \tag{2.10}$$

is a consequence of (2.5), (2.6), (2.7) and 1) of Lemma 2.

Let us show the convergence of the formal solution (2.9) with (2.10). The assumptions $\rho(0) \notin \mathbf{N}^*$ and $\operatorname{Re} \rho(0) > 0$ imply that

$$|m_1 + \rho(x)(m_2 - 1)| \geqq \sigma(m_1 + m_2) \text{ on } D_R$$
$$\text{for any } (m_1, m_2) \in \mathbf{N}^2 \setminus \{(0, 0), (0, 1)\}$$

is satisfied for some $\sigma > 0$ and $R > 0$. Choosing $R > 0$ sufficiently small, we may assume:

1) $0 < R < 1$;
2) $w_{m,0} \in \mathbf{C}_R\{x\}$, $m \geqq 1$;
3) $w_{m_1,m_2}(x, y) \in P_{m_1+2m_2-2,R}\{x\}[y]$, $m_1 + m_2 \geqq 1$ and $m_2 \geqq 1$;
4) we have the following estimates:

$$\| w_{1,0} \|_R \leqq A_1, \quad \left\| \frac{\partial w_{1,0}}{\partial x_i} \right\|_R \leqq A_1, \qquad i = 1, \dots, n;$$

$$\| w_{0,1} \|_R \leqq A_2, \quad \left\| \frac{\partial w_{0,1}}{\partial x_i} \right\|_R \leqq A_2, \qquad i = 1, \dots, n;$$

$$\left\| \frac{\partial \rho}{\partial x_i} \right\|_R \leqq c, \qquad i = 1, \dots, n;$$

$$\| a_{p,\alpha} \|_R \leqq A_{p,\alpha}, \qquad p + |\alpha| \geqq 2.$$

Consider now the following analytic equation

$$\frac{\sigma}{2} Y = \frac{\sigma}{2}(A_1 t_1 + A_2 t_2)$$
$$+ \frac{1}{R - r} \sum_{p+|\alpha| \geqq 2} \frac{A_{p,\alpha}}{(R - r)^{p+|\alpha|-2}} t_1^p Y^{\alpha_0} (eY + c\lambda Y)^{\alpha_1} \cdots (eY + c\lambda Y)^{\alpha_n}.$$

The implicit function theorem tells us that this equation has a unique holomorphic solution of the form

$$Y = \sum_{m_1+m_2 \geqq 1} Y_{m_1,m_2}(r, \lambda) t_1^{m_1} t_2^{m_2}. \tag{2.11}$$

Moreover it is easy to see that the coefficients $Y_{m_1,m_2}(r, \lambda)$ has the form

$$Y_{m_1,m_2}(r, \lambda) = \frac{C_{m_1,m_2}(\lambda)}{(R-r)^{m_1+m_2-1}}$$

with $C_{1,0}(\lambda) = A_1$, $C_{0,1}(\lambda) = A_2$ and $C_{m_1,m_2}(\lambda) \geqq 0$ (for $m_1 + m_2 \geqq 2$). Hence, by induction on (m_1, m_2) we can see in the same way as in § 1 that the following inequalities hold for any $(m_1, m_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, $0 < r < R$ and $\lambda > (4/\sigma)$:

$$\| w_{m_1,m_2} \|_{r,\lambda} \leqq Y_{m_1,m_2}(r, \lambda),$$

$$\left\| \frac{\partial w_{m_1,m_2}}{\partial x_i} \right\|_{r,\lambda} \leqq e Y_{m_1,m_2}(r, \lambda), \qquad i = 1, \dots, n,$$

$$\left\| \frac{\partial \rho}{\partial x_i} y m_2 w_{m_1,m_2} \right\|_{r,\lambda} \leqq c\lambda Y_{m_1,m_2}(r, \lambda), \qquad i = 1, \dots, n.$$

This implies that the power series Y in (2.11) is a majorant series for the formal solution (2.9).

If we fix r and λ as above, by the convergence of Y we have

$$\| w_{m_1,m_2} \|_{r,\lambda} \leqq \frac{M}{\varepsilon^{m_1+m_2}}, \qquad (m_1, m_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}$$

for some $M > 0$ and $\varepsilon > 0$. Therefore, if $|x| \leqq r$ and $\lambda \geqq 1$, by denoting $\eta(y) = \max\{1, |y|\}$ we have

$$\begin{aligned} &\sum_{m_1+m_2 \geqq 1} |w_{m_1,m_2}(x, y)| \, |t_1|^{m_1} |t_2|^{m_2} \\ &\leqq \sum_{m_1+m_2 \geqq 1} \| w_{m_1,m_2} \|_{r,\lambda} (\eta(y))^{m_1+2m_2} |t_1|^{m_1} |t_2|^{m_2} \\ &\leqq \sum_{m_1+m_2 \geqq 1} \frac{M}{\varepsilon^{m_1+m_2}} (\eta(y)|t_1|)^{m_1} (\eta(y)^2 |t_2|)^{m_2} \end{aligned}$$

which converges for any (t_1, t_2, y) satisfying $\eta(y)|t_1| < \varepsilon$ and $\eta(y)^2 |t_2| < \varepsilon$. This proves (2) of Proposition 1.

Summing up, we have

Theorem 2. *If* $\rho(0) \notin \mathbb{N}^*$ *and* $\operatorname{Re} \rho(0) > 0$, *the equation* (E_2) *has a family of* $\tilde{\mathcal{O}}_+$*-solutions of the form*

(2.12) $$\sum_{i\geqq 1} u_i(x)t^i + \sum_{\substack{i+2j\geqq k+2\\ j\geqq 1}} \varphi_{i,j,k}(x)t^{i+j\rho(x)}(\log t)^k,$$

where $\varphi_{0,1,0}(x)\in\mathbf{C}\{x\}$ *can be taken arbitrarily, and all the coefficients* $u_i(x)$, $\varphi_{i,j,k}(x)$ *are holomorphic in a same disk centered at the origin of* $\mathbf{C}_x^n$; *moreover the solution* (2.12) *is uniquely determined by* $\varphi_{0,1,0}\in\mathbf{C}\{x\}$. *If we take* $\varphi_{0,1,0}=0\in\mathbf{C}\{x\}$, *the above* $\tilde{\mathcal{O}}_+$*-solution is reduced to the unique holomorphic solution of* (E_2).

Proof. Let

$$\sum_{i\geqq 1} u_i(x)t_1^i + \sum_{\substack{i+2j\geqq k+2\\ j\geqq 1}} \varphi_{i,j,k}(x)t_1^i t_2^j y^k$$

be a holomorphic solution of (E_2') obtained in Proposition 1. Then, by the assumption $\operatorname{Re}\rho(0)>0$ and (2) of Proposition 1 we can easily see that the series

$$\sum_{i\geqq 1} u_i(x)t^i + \sum_{\substack{i+2j\geqq k+2\\ j\geqq 1}} \varphi_{i,j,k}(x)t^{i+j\rho(x)}(\log t)^k$$

is convergent in $\tilde{\mathcal{O}}_+$ and therefore it gives an $\tilde{\mathcal{O}}_+$-solution of (E_2).

Since the existence of the unique holomorphic solution of (E_2) is already known, to complete the proof of Theorem 2 it is sufficient to prove that the formal solution of (E_2) of the form (2.12) is uniquely determined by $\varphi_{0,1,0}\in\mathbf{C}\{x\}$.

Let $\hat{u}(t, x)$ be a formal solution of (E_2) of the form

(2.13) $$\hat{u}(t, x) = \sum_{i\geqq 1} u_i(x)t^i + \sum_{\substack{i+2j\geqq k+2\\ j\geqq 1}} \varphi_{i,j,k}(x)t^{i+j\rho(x)}(\log t)^k,$$

where $u_i(x)$, $\varphi_{i,j,k}(x)\in\mathbf{C}_r\{x\}$. When $\rho(x)$ is not a constant function in $\mathbf{C}\{x\}$ or when $\rho(x)\equiv$ constant $\notin\mathbf{Q}$, then any finite subset of

$$\{t^{i+j\rho(x)}(\log t)^k;\ i+j+k\geqq 1\}$$

is linearly independent over the ring $\mathbf{C}_r\{x\}$ as a family of functions and therefore all the coefficients $u_i(x)$, $\varphi_{i,j,k}(x)\in\mathbf{C}_r\{x\}$ of the formal solution $\hat{u}(t, x)$ are uniquely determined by $\varphi_{0,1,0}(x)$.

When $\rho(x)\equiv$ constant $\in\mathbf{Q}$, $\rho(x)$ is expressed as $\rho(x)\equiv p/q$ for some $p, q\in\mathbf{N}^*$ satisfying $(p, q)=1$. Since $\rho(0)\notin\mathbf{N}^*$ is assumed, we have $q\geqq 2$. Then, $\hat{u}(t, x)$ in (2.13) can be rewritten in the form

(2.14) $$\hat{u}(t, x) = \sum_{\substack{l\geqq 1\\ 0\leqq k\leqq 2(l-1)}} \psi_{l,k}(x)t^{l/q}(\log t)^k,$$

where $\psi_{p,0}(x)=\varphi_{0,1,0}(x)$. Since $\hat{u}(t, x)$ is a formal solution of (E_2) with $\rho(x)\equiv p/q$ and since any finite subset of $\{t^{l/q}(\log t)^k;\ l+k\geqq 1\}$ is linearly independent over the ring $\mathbb{C}_r\{x\}$, an easy calculation shows that logarithmic terms do not appear in the formal solution of the form (2.14) and therefore $\hat{u}(t, x)$ is expressed as

$$\hat{u}(t, x) = \sum_{l \geqq 1} \psi_{l,0}(x) t^{l/q}. \tag{2.15}$$

Thus, if we rewrite $\hat{u}(t, x)$ into the form (2.15) as above, we can conclude that all the coefficients $\psi_{l,0}(x)$ are uniquely determined by $\psi_{p,0}(x) = \varphi_{0,1,0}(x)$.

Thus, the proof of Theorem 2 is completed.

§ 3. Uniqueness of the Solution

In this section we will discuss the uniqueness of the $\tilde{\mathcal{O}}_{\text{int}}$-solution of the equation (E_2).

As is proved in Theorem 2, we already know the following uniqueness result: if $\rho(0) \notin \mathbb{N}^*$ and $\operatorname{Re} \rho(0) > 0$, if $u_1(t, x) \in \tilde{\mathcal{O}}_+$ and $u_2(t, x) \in \tilde{\mathcal{O}}_+$ are solutions of (E_2) having the expansion of the form

$$u_p(t, x) = \sum_{i \geqq 1} u_i^{(p)}(x) t^i + \sum_{\substack{i+2j \geqq k+2 \\ j \geqq 1}} \varphi_{i,j,k}^{(p)}(x) t^{i+j\rho(x)} (\log t)^k, \tag{3.1}$$

$$p = 1, 2$$

with $u_i^{(p)}(x)$, $\varphi_{i,j,k}^{(p)}(x) \in \mathbb{C}_r\{x\}$ for some $r > 0$, and if $\varphi_{0,1,0}^{(1)}(x) = \varphi_{0,1,0}^{(2)}(x)$ holds in $\mathbb{C}_r\{x\}$, then we have $u_1(t, x) = u_2(t, x)$ in $\tilde{\mathcal{O}}_+$.

The purpose of this section is to remove the assumption that $u_p(t, x)$ has an expansion of the form (3.1).

Theorem 3. *Let $u_1(t, x) \in \tilde{\mathcal{O}}_{\text{int}}$ and $u_2(t, x) \in \tilde{\mathcal{O}}_{\text{int}}$ be solutions of* (E_2). *Then*:

(1) *For any $a < \operatorname{Re} \rho(0)$ we have $t^{-a}(u_1 - u_2) \in \tilde{\mathcal{O}}_+$.*

(2) *If $t^{-b}(u_1 - u_2) \in \tilde{\mathcal{O}}_{\text{int}}$ holds for some $b \geqq \operatorname{Re} \rho(0)$, we have $u_1(t, x) = u_2(t, x)$ in $\tilde{\mathcal{O}}_{\text{int}}$.*

Proof. Let $u_1 \in \tilde{\mathcal{O}}_{\text{int}}$ and $u_2 \in \tilde{\mathcal{O}}_{\text{int}}$ be two solutions of (E_2). Set $w = (u_1 - u_2) \in \tilde{\mathcal{O}}_{\text{int}}$. Then $w(t, x)$ is a solution of

$$\left(t\frac{\partial}{\partial t} - \rho(x) \right) w = A_0(t, x) w + \sum_{i=1}^{n} A_i(t, x) \frac{\partial w}{\partial x_i} \tag{3.2}$$

with

$$A_i(t, x) = \int_0^1 \frac{\partial G_2}{\partial X_i}(x) \left(t, sw + u_2, s\frac{\partial w}{\partial x_1} + \frac{\partial u_2}{\partial x_1}, \dots, s\frac{\partial w}{\partial x_n} + \frac{\partial u_2}{\partial x_n} \right) ds,$$

$$i = 0, 1, \dots, n,$$

where $G_2(x)(t, X_0, X_1,\dots, X_n)$ is as in (2.1). Since w and u_2 belong to $\tilde{\mathcal{O}}_{\rm int}$, from (2.1) we have $A_i(t, x)\in\tilde{\mathcal{O}}_{\rm int}$, $i = 0, 1,\dots, n$. Therefore, by the definition of $\tilde{\mathcal{O}}_{\rm int}$ there are $\varepsilon(s)$, $R > 0$ and $\mu(s)$ satisfying (μ_1), (μ_2), (μ_3) such that for $i = 0, 1,\dots, n$ we have:

1) $A_i(t, x)$ is holomorphic in $S(\varepsilon(s)) \times D_R$;
2) for any $\theta > 0$ there are $\delta > 0$ and $M > 0$ such that

$$\max_{x\in D_R} |A_i(t, x)| \leqq M\mu(|t|) \tag{3.3}$$

holds for any $t\in S_\theta(\delta) = \{t\in S_\theta;\ 0 < |t| \leqq \delta\}$; in addition, by (μ_3) we may assume

$$\int_0^\delta \frac{\mu(s)}{s}ds < \frac{R}{3M}.$$

Take any $\theta > 0$ and fix it. Let $R > 0$, $\mu(s)$, $M > 0$ and $\delta > 0$ be as above. For $t_0\in S_\theta(\delta)$ we write

$$L(t_0, \delta) = \{t;\ t = st_0,\ 0 < s \leqq \delta/|t_0|\}.$$

Lemma 3. *Let $0 < R_2 < R_1 < R$ be such that $R_2 + (R/3) < R_1$ and $R_1 + (R/3) < R$. Then:*

(1) *For any $(t_0, x^0)\in S_\theta(\delta) \times D_{R_1}$ the equation*

$$\begin{cases} \dfrac{dx(t)}{dt} = -\dfrac{1}{t}(A_1(t, x(t)),\dots, A_n(t, x(t))), \\ x(t_0) = x^0 \end{cases} \tag{3.4}$$

has a unique holomorphic solution $x(t)$ defined near $L(t_0, \delta)$ such that $x(t)\in D_R$ holds for any $t\in L(t_0, \delta)$ and that $x(t)$ converges to some point in D_R as t tends to zero in $L(t_0, \delta)$.

(2) *Denote by $x(t; t_0, x^0)$ the unique solution of* (3.4) *and put*

$$\Gamma(t_0, x^0) = \{(t, x(t; t_0, x^0));\ t\in L(t_0, \delta)\}.$$

Then we have for any $t_0\in S_\theta(\delta)$

$$\bigcup_{x^0\in D_{R_1}} \Gamma(t_0, x^0) \supset L(t_0, \delta) \times D_{R_2}.$$

To see this, we have only to notice the conditions $R_1 + (R/3) < R$, $R_2 + (R/3) < R_1$ and the following fact: for any $t\in L(t_0, \delta)$ we have

$$\begin{aligned} |x_i(t) - x_i^0| &\leqq \left| \int_t^{t_0} \frac{A_i(\tau, x(\tau))}{\tau} d\tau \right| \\ &\leqq M\left| \int_{|t|}^{|t_0|} \frac{\mu(s)}{s}ds \right| \leqq M\int_0^\delta \frac{\mu(s)}{s}ds < \frac{R}{3}. \end{aligned}$$

By using this lemma we can investigate the behavior of $w(t, x(t))$ as t tends to zero in $L(t_0, \delta)$. In fact, if we put

$$W(t) = w(t, x(t; t_0, x^0)), \quad t \in L(t_0, \delta)$$

for $(t_0, x^0) \in S_\theta(\delta) \times D_{R_1}$, by (3.2) and (3.4) we have

$$t\frac{dW}{dt} - (\rho(x(t)) + A_0(t, x(t)))W = 0 \tag{3.5}$$

with $x(t) = x(t; t_0, x^0)$ and therefore by integrating (3.5) we obtain

$$W(t) = W(t_0)\exp\left[-\int_t^{t_0} \frac{\rho(x(\tau)) + A_0(\tau, x(\tau))}{\tau} d\tau \right]. \tag{3.6}$$

Now let us prove (1) of Theorem 3. Take any $a < \operatorname{Re}\rho(0)$. We may assume that

$$a + \varepsilon < \operatorname{Re}\rho(x) \text{ on } D_R$$

holds for some $\varepsilon > 0$. Then, by (3.3) and (3.6) we have for any $t \in L(t_0, \delta)$ satisfying $|t| \leqq |t_0|$

$$\begin{aligned} &|t^{-a-\varepsilon}W(t)| \\ &\leqq |t_0|^{-a-\varepsilon}|W(t_0)|\exp\left[\left|\int_t^{t_0} \frac{A_0(\tau, x(\tau))}{\tau} d\tau\right|\right] \times \\ &\quad \times \exp\left[-\int_{|t|/|t_0|}^1 \frac{\operatorname{Re}\rho(x(st_0)) - (a+\varepsilon)}{s} ds \right] \\ &\leqq |t_0|^{-a-\varepsilon}|W(t_0)|\exp\left[M\int_0^\delta \frac{\mu(s)}{s} ds \right]. \end{aligned}$$

Hence, by putting $t_0 = \delta e^{i\eta}$, $|\eta| \leqq \theta/2$ and by using (2) of Lemma 3 we obtain for any $(t, x) \in S_{\theta/2}(\delta) \times D_{R_2}$

$$\begin{aligned} &|t^{-a-\varepsilon}w(t, x)| \\ &\leqq \delta^{-a-\varepsilon}\exp\left[M\int_0^\delta \frac{\mu(s)}{s} ds \right] \times \max_{\substack{|\eta| \leqq \theta/2 \\ x \in D_{R_1}}} |w(\delta e^{i\eta}, x)|. \end{aligned}$$

This implies that $t^{-a}w \in \tilde{\mathcal{O}}_+$ holds, since $\theta > 0$ is taken arbitrarily. Thus, (1) of Theorem 3 is proved.

Next let us prove (2) of Theorem 3. Assume that $t^{-b}w = t^{-b}(u_1 - u_2) \in \tilde{\mathcal{O}}_{\text{int}}$ holds for some $b \geqq \operatorname{Re}\rho(0)$. Then, by the definition of $\tilde{\mathcal{O}}_{\text{int}}$ we see the following: if $R > 0$ is sufficiently small, we have

$$\max_{x \in D_R} |t^{-b}w(t, x)| = o(1) \text{ as } t \text{ tends to zero in } S_\theta \tag{3.7}$$

for any $\theta > 0$.

When $b > \operatorname{Re}\rho(0)$ or when $b \equiv \operatorname{Re}\rho(x)$, we may assume that

$$b \geqq \operatorname{Re}\rho(x) \text{ on } D_R.$$

Then by (3.6) and (3.7) we have

$$\begin{aligned}
&|w(t_0, x^0)| \\
&\leqq |t_0|^b |t^{-b} w(t, x(t; t_0, x^0))| \exp\left[M \int_0^\delta \frac{\mu(s)}{s} ds \right] \times \\
&\quad \times \exp\left[\int_{|t|/|t_0|}^1 \frac{\operatorname{Re}\rho(x(st_0)) - b}{s} ds \right] \\
&= o(1), \text{ as } t \text{ tends to zero in } L(t_0, \delta),
\end{aligned}$$

and therefore we obtain $w(t_0, x^0) = 0$. This implies that $w \equiv 0$ in $\tilde{\mathcal{O}}_{\text{int}}$, since $(t_0, x^0) \in S_\theta(\delta) \times D_{R_1}$ is taken arbitrarily.

When $b = \operatorname{Re}\rho(0)$ and $b \not\equiv \operatorname{Re}\rho(x)$ hold, we can find an $\mathring{x} \in \mathbb{C}^n$ sufficiently close to the origin such that $b > \operatorname{Re}\rho(\mathring{x})$ holds. Then by the above discussion we have $w(t, x) = 0$ on $S_\theta(\delta) \times \{x; |x - \mathring{x}| \leqq R\}$ for sufficiently small $R > 0$. Hence, we can obtain $w \equiv 0$ in $\tilde{\mathcal{O}}_{\text{int}}$, since w is a holomorphic function.

Thus, (2) of Theorem 3 is also proved.

§4. Proof of Main Theorem

Assume that (E) satisfies the conditions (A_1), (A_2) and (A_3) in the introduction. Then, (E) can be rewritten into an equation of the form (E_2). Therefore, we already know the following results.

(C_1)(by Theorem 1). If $\rho(0) \notin \mathbb{N}^*$, the equation (E) has a unique holomorphic solution $u_0(t, x)$ satisfying $u_0(0, x) \equiv 0$.

(C_2)(by Theorem 2). If $\rho(0) \notin \mathbb{N}^*$ and $\operatorname{Re}\rho(0) > 0$, for any $\varphi(x) \in \mathbb{C}\{x\}$ the equation (E) has a unique $\tilde{\mathcal{O}}_+$-solution $U(\varphi)(t, x)$ having the expansion of the following form:

$$\text{(4.1)} \qquad \begin{cases} U(\varphi) = \sum_{i \geqq 1} u_i(x) t^i + \sum_{\substack{i + 2j \geqq k + 2 \\ j \geqq 1}} \varphi_{i,j,k}(x) t^{i + j\rho(x)} (\log t)^k, \\ \varphi_{0,1,0}(x) = \varphi(x), \end{cases}$$

where all the coefficients $u_i(x)$, $\varphi_{i,j,k}(x)$ are holomorphic in a same disk centered at the origin of $\mathbb{C}_x^n$. If we take $\varphi = 0 \in \mathbb{C}\{x\}$, $U(\varphi)(t, x)$ is reduced to the unique holomorphic solution u_0 in (C_1).

(C_3)(by a calculation). If $\rho(0) \notin \mathbb{N}^*$ and $\operatorname{Re}\rho(0) > 0$, if $b > 0$ satisfies

$$\operatorname{Re}\rho(0) < b < \min\{2\operatorname{Re}\rho(0), \operatorname{Re}\rho(0) + 1\},$$

and if a solution $u(t, x) \in \tilde{\mathcal{O}}_{\mathrm{int}}$ of (E) is expressed in the form

$$t^{-b}\Big(u(t, x) - \sum_{1 \leqq i \leqq b} u_i(x) t^i - \varphi(x) t^{\rho(x)}\Big) \in \tilde{\mathcal{O}}_+,$$

then the coefficients $\{u_i(x);\ 1 \leqq i \leqq b\}$ are uniquely determined by the equation (E) and they are independent of $\varphi(x)$.

Now, denote by $\mathscr{S}_+$ [resp. $\mathscr{S}_{\mathrm{int}}$] the set of all $\tilde{\mathcal{O}}_+$-solutions [resp. $\tilde{\mathcal{O}}_{\mathrm{int}}$-solutions] of (E). If $\rho(0) \notin \mathbb{N}^*$, by ($C_1$) and ($C_2$) we have

$$\mathscr{S}_{\mathrm{int}} \supset \mathscr{S}_+ \supset \begin{cases} \{u_0\}, & \text{when } \operatorname{Re}\rho(0) \leqq 0, \\ \{u_0\} \cup \{U(\varphi);\ 0 \neq \varphi(x) \in \mathbb{C}\{x\}\}, & \text{when } \operatorname{Re}\rho(0) > 0. \end{cases} \tag{4.2}$$

Hence, to complete the proof of Main Theorem it is sufficient to prove the following proposition.

Proposition 2. *Assume* (A_1), (A_2) *and* (A_3). *Let* u_0, $U(\varphi)$ *and* $\mathscr{S}_{\mathrm{int}}$ *be as above. Then*:

(1) *If* $\operatorname{Re}\rho(0) \leqq 0$ *and* $u \in \mathscr{S}_{\mathrm{int}}$, *we have* $u \equiv u_0$ *in* $\tilde{\mathcal{O}}_{\mathrm{int}}$.

(2) *If* $\rho(0) \notin \mathbb{N}^*$, $\operatorname{Re}\rho(0) > 0$ *and* $u \in \mathscr{S}_{\mathrm{int}}$, *we can find a* $\varphi(x) \in \mathbb{C}\{x\}$ *such that* $u \equiv U(\varphi)$ *holds in* $\tilde{\mathcal{O}}_{\mathrm{int}}$.

Proof. First let us show (1). Assume $\operatorname{Re}\rho(0) \leqq 0$ and $u \in \mathscr{S}_{\mathrm{int}}$. Then, by putting $b = 0$ we have $t^{-b}(u - u_0) = (u - u_0) \in \tilde{\mathcal{O}}_{\mathrm{int}}$ and $\operatorname{Re}\rho(0) \leqq b$. Hence, by (2) of Theorem 3 we obtain $u \equiv u_0$ in $\tilde{\mathcal{O}}_{\mathrm{int}}$.

Next let us show (2). Assume $\rho(0) \notin \mathbb{N}^*$, $\operatorname{Re}\rho(0) > 0$ and $u \in \mathscr{S}_{\mathrm{int}}$. Set $w = (u - u_0) \in \tilde{\mathcal{O}}_{\mathrm{int}}$. Since u and u_0 are solutions of (E), w satisfies the following equation

$$\left(t \frac{\partial}{\partial t} - \rho(x)\right) w = f(t, x)$$

with

$$f(t, x) = F\left(t, x, w + u_0, \frac{\partial w}{\partial x} + \frac{\partial u_0}{\partial x}\right) - F\left(t, x, u_0, \frac{\partial u_0}{\partial x}\right). \tag{4.3}$$

Take $0 < a < b$ such that

$$a < \operatorname{Re}\rho(0) < b < \min\{2a, a + 1\}. \tag{4.4}$$

Then, by (1) of Theorem 3 we have $t^{-a} w = t^{-a}(u - u_0) \in \tilde{\mathcal{O}}_+$ and therefore by (4.3) we obtain $t^{-b} f \in \tilde{\mathcal{O}}_+$. It is easy to see that $W(t, x)$ defined by

$$W(t, x) = \int_0^1 s^{-\rho(x) - 1} f(st, x)\, ds$$

satisfies $t^{-b}W\in\tilde{\mathcal{O}}_+$ and $\left(t\frac{\partial}{\partial t}-\rho(x)\right)W=f$. Hence, by solving the equation $\left(t\frac{\partial}{\partial t}-\rho(x)\right)(w-W)=0$ we can see that $w-W$ is expressed in the form

$$(w-W)(t,x)=\varphi(x)t^{\rho(x)}$$

for some $\varphi(x)\in\mathbb{C}\{x\}$. Thus, by summing up we obtain

$$t^{-b}(u(t,x)-u_0(t,x)-\varphi(x)t^{\rho(x)})=t^{-b}W\in\tilde{\mathcal{O}}_+.$$

On the other hand, if we use the same $\varphi(x)$ as above, by (4.1) and (4.4) we have

$$t^{-b}(U(\varphi)(t,x)-\sum_{1\leqq i\leqq b}u_i(x)t^i-\varphi(x)t^{\rho(x)})\in\tilde{\mathcal{O}}_+.$$

Hence, by (C_3) we have $t^{-b}(u-U(\varphi))\in\tilde{\mathcal{O}}_+$ and therefore by (2) of Theorem 3 we obtain $u\equiv U(\varphi)$ in $\tilde{\mathcal{O}}_+$.

Thus, by (4.2) and Proposition 2 we can easily obtain Main Theorem. See also Tahara [10, 11].

§5. Remarks

In this paper, we restricted ourselves to the study of singular solutions in $\tilde{\mathcal{O}}_+$ or $\tilde{\mathcal{O}}_{\mathrm{int}}$. But, there seems to be a possibility that (E) has singular solutions which do not belong to the class $\tilde{\mathcal{O}}_{\mathrm{int}}$, as is seen in the following example.

Let us consider

$$t\frac{\partial u}{\partial t}=\rho u+u\frac{\partial u}{\partial x}, \tag{5.1}$$

where $(t,x)\in\mathbb{C}^2$ and $\rho\in\mathbb{C}$ is constant. By the separation of variables we can easily see that (5.1) has a family of singular solutions of "meromorphic type" as follows:

$$u_{a,b,c}(t,x)=\begin{cases}\dfrac{\rho t^{\rho}(ax+b)}{\rho c-at^{\rho}}, & \text{when } \rho\neq 0,\\[2ex] \dfrac{ax+b}{c-a\log t}, & \text{when } \rho=0,\end{cases}$$

where $a,b,c\in\mathbb{C}$ are arbitrary. Note that in the case $\mathrm{Re}\,\rho\leqq 0$ and $a\neq 0$ the solution $u_{a,b,c}$ does not belong to the class $\tilde{\mathcal{O}}_{\mathrm{int}}$.

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