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G-Central States of Almost Periodic Type on C^*-Algebras

By

Masaharu KUSUDA*

§ 1. Introduction

Let (A, G, α) be a C^*-dynamical system, namely a triple consisting of a C^*-algebra A, a locally compact group G and a homomorphism α from G into the automorphism group of A such that $G \ni t \to \alpha_t(x) \in A$ is continuous for each x in A. Now assume that the C^*-algebra A is unital. Then the set of α-invariant states of A, denoted by S^G, forms a weak* compact convex subset in the state space of A, and each extremal point in S^G is called an ergodic state (occasionally, G-ergodic state or α-ergodic state). When we attempt to decompose an α-invariant state into ergodic states, it is most important to investigate the existence and uniqueness of the decomposition, and in order to carry out such investigation we need to require some "abelianness" of C^*-dynamical systems, in particular, of invariant states. Now denote by $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ the GNS covariant representation associated with an α-invariant state φ, that is, π_φ is a representation of A on the Hilbert space H_φ with the canonical cyclic vector ξ_φ and u^φ is a strongly continuous unitary representation of G on H_φ defined by

$$u_t^\varphi(\pi_\varphi(x))\,\xi_\varphi = \pi_\varphi(\alpha_t(x))\,\xi_\varphi$$

for $x \in A$ and $t \in G$. Note that

$$\pi_\varphi(\alpha_t(x)) = u_t^\varphi(\pi_\varphi(x))\,u_t^{\varphi *}$$

for all $x \in A$ and all $t \in G$. Given an α-invariant state φ, it is well-known that there is a bijective correspondence between the orthogonal measures, over the state space of A, with barycenter φ which satisfy the invariance condition with respect to G and the abelian von Neumann subalgebras of $\{\pi_\varphi(A) \cup u_G^\varphi\}'$ (see [1] or [9] for the details). This fact plays a fundamental role in decomposition theory of invariant states. In fact, the above correspondence allows us to

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* Department of Applied Mathematics, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka 560, Japan.

connect a few kinds of "abelianness" conditions of invariant states with decompositions, such as the ergodic decomposition or the subcentral decomposition of invariant states. Now recall that an α-invariant state φ is said to be *G-abelian* if, for each x, y in A and u^{φ}-invariant vector ξ in H_{φ},

$$\inf|(\pi_{\varphi}([x', y])\xi|\xi)| = 0$$

where the infimum is taken over all x' in the convex hull of $\{\alpha_t(x)|t\in G\}$. Then G-abelianness of φ yields abelianness of $\{\pi_{\varphi}(A)\cup u_G^{\varphi}\}'$ and the corresponding maximal measure with barycenter φ is uniquely determined. This implies that the notion of G-abelianness is most suitable for carrying out the ergodic decomposition (see [1] or [9] for the details). Recall also that an α-invariant state φ is said to be *G-central* if, for each x, y in A, z in $\pi_{\varphi}(A)'$ and u^{φ}-invariant vector ξ in H_{φ},

$$\inf|(\pi_{\varphi}([x', y])z\xi|\xi)| = 0$$

where the infimum is taken over all x' in the convex hull of $\{\alpha_t(x)|t\in G\}$. Then G-centrality of φ implies G-abelianness of φ and yields that

$$\pi_{\varphi}(A)'\cap u_G^{\varphi'} = \pi_{\varphi}(A)''\cap\pi_{\varphi}(A)'\cap u_G^{\varphi'}.$$

Thus the corresponding maximal measure is subcentral. This implies that the notion of G-centrality is most suitable for carrying out the subcentral decomposition of invariant states (see [1] for the details).

Though we have so far concentrated on the problem of decomposing an invariant state into ergodic states, we now turn our attention to decomposing an α-invariant state, in particular an ergodic state, into states which are not necessarily α-invariant. More precisely, we restrict G to locally compact abelian groups and consider the decomposition into almost periodic states. Here note that a state ψ of A is said to be *almost periodic* if, for each x in A, the function $G\ni t\to\psi(\alpha_t(x))$ is the uniform limit of a family of finite linear combinations of characters of G (cf. [1] or [2]). In the almost periodic decomposition, the notion corresponding to G-abelianness in the ergodic decomposition was originated by [6], explicitly introduced by [1] and is called G_{Γ}-abelianness, which is defined as follows. For given x in A and γ in the dual group $\hat{G}$ of G, let $\mathrm{Con}(\gamma\alpha(x))$ denote the convex hull of $\{\langle t, \gamma\rangle\alpha_t(x)|t\in G\}$. Then an α-invariant state φ is said to be *G_{Γ}-abelian* if, for all $x, y\in A$, $\gamma\in\hat{G}$, and u_G^{φ}-almost periodic vectors $\xi, \eta\in H_{\varphi}$,

$$\inf|(\pi_{\varphi}([x', y])\xi|\eta)| = 0$$

where the infimum is taken over all x' in $\mathrm{Con}(\gamma\alpha(x))$. It is well-known that G_{Γ}-abelianness characterizes uniqueness of maximal measures, over the state space of A, which are supported by an appropriate class of almost periodic states (see

[1, 4.3.41]).

In this paper, we introduce the notion corresponding to G-centrality in order to consider the subcentral decomposition into almost periodic states, and we shall refer to such notion not as G_Γ-centrality but as G-centrality of almost periodic type (occasionally, almost periodic G-centrality) in order to emphasize "almost periodicity".

Section 2 is devoted to the preliminaries to establish our main results given in Section 3. More precisely, for a G-central state of almost periodic type, we describe some results corresponding to the spectral results in G_Γ-abelian case [1, 4.3.30–31].

In Section 3, we shall give some necessary and sufficient conditions for an invariant state to be a G-central state of almost periodic type. In particular, we shall show that, an invariant state φ is a G-central state of almost periodic type if

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'$$

and $\{p_\varphi \pi_\varphi(A) p_\varphi\}''$ is abelian, where p_φ is the projection from H_φ onto the subspace of u_G^φ-almost periodic vectors. Conversely, we shall also prove that if the canonical cyclic vector ξ_φ is separating for $\pi_\varphi(A)''$, almost periodic G-centrality of φ then yields the above equality and abelianness of $\{p_\varphi \pi_\varphi(A) p_\varphi\}''$.

§2. Almost Periodic G-Centrality

Let (A, G, α) be a C^*-dynamical system. Throughout this paper, we assume that a locally compact group G is always abelian. We denote by $\hat{G}$ the dual group of G. We generally use additive notation for group operations of $\hat{G}$. But for brevity we shall occasionally use multiplicative notation, i.e., $\gamma_1\gamma_2$ in place of $\gamma_1 + \gamma_2$ and γ^{-1} in place of $-\gamma$. Let φ be an α-invariant state of A and $(\pi_\varphi, u^\varphi, H_\varphi, \xi_\varphi)$ be the GNS covariant representation associated with φ. Then the spectral decomposition of u^φ is given by

$$u_t^\varphi = \int_{\hat{G}} \overline{\langle t, \gamma \rangle}\, dP_\varphi(\gamma),$$

where dP_φ denotes the projection-valued measure on $\hat{G}$. For simplicity, we use the notation

$$p_\varphi(\gamma) = P_\varphi(\{\gamma\}).$$

Then the point spectrum $\sigma(u^\varphi)$ of u^φ is defined by

$$\sigma(u^\varphi) = \{\gamma \in \hat{G} \,|\, p_\varphi(\gamma) \neq 0\},$$

and this definition implies that $\gamma \in \sigma(u^\varphi)$ if and only if there exists a non-zero eigenvector η_γ in H_φ such that

$$u_t^{\varphi}\eta_\gamma = \overline{\langle t, \gamma \rangle}\, \eta_\gamma$$

for all t in G. Define the projection p_φ on H_φ by

$$p_\varphi = \sum_{\gamma \in \hat{G}} p_\varphi(\gamma),$$

and we refer to $p_\varphi H_\varphi$ as the subspace of *u^φ-almost periodic vectors.*

We obtain the group of automorphisms $\bar{\alpha}$ of $\pi_\varphi(A)''$ by the canonical extension

$$\bar{\alpha}_t(x) = u_t^\varphi x u_t^{\varphi *}$$

for all x in $\pi_\varphi(A)''$ and t in G.

For each γ in $\hat{G}$, we define a unitary representation γu^φ of G on H_φ by

$$(\gamma u^\varphi)_t = \langle t, \gamma \rangle u_t^\varphi$$

for all t in G. Similarly, we define a family of bijective linear maps $\gamma\alpha$ of A by

$$(\gamma\alpha)_t(x) = \langle t, \gamma \rangle \alpha_t(x)$$

for all x in A and t in G.

We are now in a position to introduce the notion of a G-central state of almost periodic type.

Definition 2.1. An α-invariant state φ is said to be a *G-central state of almost periodic type* if for each $x, y \in A$, $z \in \pi_\varphi(A)'$, $\gamma \in \hat{G}$ and $\xi, \eta \in p_\varphi H_\varphi$, the following is satisfied:

$$\inf |(\pi_\varphi([x', y]) z\xi | \eta)| = 0,$$

where x' runs over $\mathrm{Con}(\gamma\alpha(x))$.

We remark that a G-central state of almost periodic type is always a G-central state. But the converse is false in general (see Example 3.15).

Since G is amenable, there exists an invariant mean m on G. We can rephrase the definition of almost periodic G-centrality by using an invariant mean m as follows.

Proposition 2.2. *Let (A, G, α) be a C^*-dynamical system, where G is a locally compact abelian group. Let φ be an α-invariant state of A. Then the following conditions* (1)–(3) *are equivalent*:

(1) *φ is a G-central state of almost periodic type.*

(2) $m((\pi_\varphi([\gamma\alpha(x), y]) z\xi | \eta)) = 0$ *for all $x, y \in A$, $z \in \pi_\varphi(A)'$, $\gamma \in \hat{G}$ and all $\xi, \eta \in p_\varphi H_\varphi$ and for some invariant mean m on G.*

(3) $p_\varphi(\gamma_1)\pi_\varphi(x)p_\varphi(\gamma_1 - \gamma)\pi_\varphi(y) z p_\varphi(\gamma_2) = p_\varphi(\gamma_1) z \pi_\varphi(y) p_\varphi(\gamma + \gamma_2)\pi_\varphi(x) p_\varphi(\gamma_2)$

for all $x, y \in A$, $z \in \pi_\varphi(A)'$ *and* $\gamma, \gamma_1, \gamma_2 \in \hat{G}$.

This proposition corresponds to [1, Proposition 4.3.30] in G_Γ-abelian case and the proofs are almost similar. Hence it is left to the reader to check the details.

In statement (3) of Proposition 2.2, put $\gamma_1 = \gamma$ and $\gamma_2 = 0$. Then we have the following lemma from the modification of the proof of [1, Theorem 4.3.31 (2)–(3)].

Lemma 2.3. *Let (A, G, α) be a C^*-dynamical system, where G is a locally compact abelian group. Assume that a G-central state of almost periodic type φ is α-ergodic. Let γ be an element in $\sigma(u^\varphi)$ and let x be an element in $\pi_\varphi(A)''$ with $p_\varphi(\gamma)x\xi_\varphi \neq 0$. Then, for finite subsets $\{y_i\}$ in $\pi_\varphi(A)''$ and $\{z_i\}$ in $\pi_\varphi(A)'$, we have*

$$\| \sum_i y_i z_i p_\varphi(\gamma) x \xi_\varphi \| = \| p_\varphi(\gamma) x \xi_\varphi \| \, \| \sum_i y_i z_i \xi_\varphi \| .$$

Theorem 2.4. *Let (A, G, α) be a C^*-dynamical system, where G is a locally compact abelian group. Assume that a G-central state of almost periodic type φ on A is α-ergodic.*

(1) For $\gamma \in \sigma(u^\varphi)$, *there exist a unit vector η_γ in H_φ and a unitary element v_γ in $\pi_\varphi(A)'' \cap \pi_\varphi(A)'$ such that*

$$u_t^\varphi \eta_\gamma = \overline{\langle t, \gamma \rangle}\, \eta_\gamma, \qquad \bar{\alpha}_t(v_\gamma) = \overline{\langle t, \gamma \rangle}\, v_\gamma$$

for all t in G. Moreover, η_γ and v_γ are uniquely determined up to scalar multiples of modulas one.

(2) *The point spectrum $\sigma(u^\varphi)$ of u^φ is a subgroup of $\hat{G}$. Let N_φ denote the annihilator of $\sigma(u^\varphi)$.* Then we have

$$\begin{aligned}\{v_\gamma \mid \gamma \in \sigma(u^\varphi)\}'' &= \pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}' \\ &\subset \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{e_{N_\varphi}\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap u_{N_\varphi}^{\varphi}{}',\end{aligned}$$

where p_φ is the projection on the subspace spanned by u_G^φ-almost periodic vectors and e_{N_φ} is the projection onto the subspace of $u_{N_\varphi}^\varphi$-invariant vectors.

This is shown in the same way as [1, Theorem 4.3.31] by using Lemma 2.3. The details are left to the reader.

We have just obtained that if an α-ergodic state φ of A is a G-central state of almost periodic type, then we have

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'.$$

Conversely, it would be very interesting to consider the question whether every α-ergodic state φ satisfying the above formula is always a G-central state of almost periodic type. But we can not expect the affirmative answer to this question in general. In fact, we will give a counterexample in the next section (see Example 3.15). In the case where φ is not necessarily α-ergodic, we will describe some conditions equivalent to the condition that φ be a G-central state of almost periodic type or to the above formula under some assumption, in Theorem 3.9, Corollaries 3.10–3.11 and Propositions 3.12–3.13.

Note here that if a G-central state φ is factorial, then it is ergodic. In fact, an α-invariant factorial state is always centrally ergodic. It therefore follows from [1, Theorem 4.3.14] that φ is ergodic.

Corollary 2.5. *Let (A, G, α) be as in Theorem 2.4 and let φ be a G-central state of almost periodic type on A. Assume that φ is factorial. Then $\sigma(u^{\varphi})$ consists only of the identity of $\hat{G}$.*

Proof. Since φ is factorial, the center of $\pi_{\varphi}(A)''$ is trivial. Hence, the above theorem shows that $\{v_{\gamma} | \gamma \in \sigma(u^{\varphi})\}''$ consists only of scalars, which implies that $\alpha_t(v_{\gamma}) = v_{\gamma}$ for all t in G. We thus obtain that $\langle t, \gamma \rangle = 1$ for all t in G, and therefore $\gamma = 0$. Q.E.D.

Remark 2.6. Assume that a G-central state of almost periodic type φ is N_{φ}-ergodic. Then φ is also G-ergodic. In this case, it follows from Theorem 2.4 and N_{φ}-ergodicity that $\{v_{\gamma} | \gamma \in \sigma(u^{\varphi})\}''$ consists only of scalars. Thus, we obtain the same conclusion as in the above corollary.

§3. Some Conditions Equivalent to Almost Periodic G-Centrality

Let (A, G, α) be a C^*-dynamical system with a locally compact abelian group G and let φ be an α-invariant state of A. Using the invariant mean m on G in Proposition 2.2, we define a linear map Q_{γ} from $\pi_{\varphi}(A)''$ onto the closed subspace

$$\{x \in \pi_{\varphi}(A)'' | \bar{\alpha}_t(x) = \overline{\langle t, \gamma \rangle}\, x \text{ for all } t \in G\}$$

by

$$\langle Q_{\gamma}(x), \psi \rangle = m(\langle (\gamma\bar{\alpha})(x), \psi \rangle)$$

for $x \in \pi_{\varphi}(A)''$ and $\psi \in \pi_{\varphi}(A)''_{*}$, where $\gamma\bar{\alpha}$ is defined by

$$(\gamma\bar{\alpha})_t(x) = \langle t, \gamma \rangle \bar{\alpha}_t(x)$$

for all $x \in \pi_{\varphi}(A)''$. It is easy to verify that

$$Q_{\gamma}(x)^* = Q_{-\gamma}(x^*)$$

for all x in $\pi_\varphi(A)''$ and γ in $\hat{G}$. This formula will be used to prove the next lemma. Moreover, we will see from the proof of the next lemma that if ξ_φ is separating for $\pi_\varphi(A)''$, then $\sum_{\gamma\in\hat{G}} Q_\gamma$ is the identity map of $\pi_\varphi(A)''\cap\{p_\varphi\}'$. Lemmas 3.1, 3.2 and 3.4 are keys to establish our main results in this paper.

Lemma 3.1. *Let (A, G, α) be a C^*-dynamical system and let φ be an α-invariant state of A. Then we have:*

(1) $Q_\gamma(\pi_\varphi(A)'')\subset\pi_\varphi(A)''\cap\{p_\varphi\}'$ *for all* γ *in* $\hat{G}$.

(2) *If ξ_φ is separating for $\pi_\varphi(A)''$, then*

$$\{\bigcup_\gamma Q_\gamma(\pi_\varphi(A)'')\}''=\pi_\varphi(A)''\cap\{p_\varphi\}'.$$

Proof. (1) Let $\xi_{\gamma'}$ be an eigenvector for γ' in $\sigma(u^\varphi)$. Take any element x from $\pi_\varphi(A)''$. We then have

$$u_t^\varphi Q_\gamma(x)\,\xi_{\gamma'}=(u_t^\varphi Q_\gamma(x)\,u_t^{\varphi*})\,u_t^\varphi\,\xi_{\gamma'}$$
$$=\overline{\langle t,\gamma+\gamma'\rangle}\,Q_\gamma(x)\,\xi_{\gamma'}.$$

Hence we see that

$$Q_\gamma(x)\,\xi_{\gamma'}=(\gamma\gamma' u^\varphi)_t\,Q_\gamma(x)\,\xi_{\gamma'},$$

and thus the Alaoglu-Birkhoff mean ergodic theorem yields that

$$Q_\gamma(x)\,\xi_{\gamma'}=p_\varphi(\gamma+\gamma')\,Q_\gamma(x)\,\xi_{\gamma'}.$$

This means that

$$Q_\gamma(x)\,p_\varphi(\gamma')=p_\varphi(\gamma+\gamma')\,Q_\gamma(x)p_\varphi(\gamma').$$

Passing to the *-operation and using the formula established before this lemma, we obtain that

$$p_\varphi(\gamma')\,Q_\gamma(x^*)=p_\varphi(\gamma')\,Q_\gamma(x^*)\,p_\varphi(\gamma'-\gamma).$$

Since this formula holds for all x in $\pi_\varphi(A)''$, we have

$$p_\varphi(\gamma')\,Q_\gamma(x)=p_\varphi(\gamma')\,Q_\gamma(x)\,p_\varphi(\gamma'-\gamma).$$

We therefore have

$$p_\varphi Q_\gamma(x)=\sum_{\gamma'}p_\varphi(\gamma')\,Q_\gamma(x)=\sum_{\gamma'}p_\varphi(\gamma')\,Q_\gamma(x)p_\varphi(\gamma'-\gamma)$$
$$=\sum_{\gamma'}p_\varphi(\gamma+\gamma')\,Q_\gamma(x)\,p_\varphi(\gamma')=\sum_{\gamma'}Q_\gamma(x)p_\varphi(\gamma')$$
$$=Q_\gamma(x)\,p_\varphi.$$

This means that $Q_\gamma(x)\in\pi_\varphi(A)''\cap\{p_\varphi\}'$.

(2) Take any element x from $\pi_\varphi(A)''$. For any vector ξ and η in H_φ, we have

$$\begin{aligned}(p_\varphi(\gamma+\gamma')xp_\varphi(\gamma')\xi|\eta) &= m(((\gamma\gamma' u^\varphi)xp_\varphi(\gamma')\xi|\eta)) = m((\gamma u^\varphi x u^{\varphi *} p_\varphi(\gamma')\xi|\eta))\\ &= m((\gamma\bar{\alpha}(x)p_\varphi(\gamma')\xi|\eta)) = (Q_\gamma(x)p_\varphi(\gamma')\xi|\eta).\end{aligned}$$

We thus obtain that

$$p_\varphi(\gamma+\gamma')xp_\varphi(\gamma') = Q_\gamma(x)p_\varphi(\gamma')$$

and in particular

$$p_\varphi(\gamma)x\xi_\varphi = Q_\gamma(x)\xi_\varphi.$$

If $xp_\varphi = p_\varphi x$, for any element y in $\pi_\varphi(A)'$ we then have

$$\begin{aligned}xy\xi_\varphi &= yxp_\varphi\xi_\varphi = yp_\varphi x\xi_\varphi = \sum_\gamma yp_\varphi(\gamma)x\xi_\varphi\\ &= \sum_\gamma yQ_\gamma(x)\xi_\varphi = \sum_\gamma Q_\gamma(x)y\xi_\varphi.\end{aligned}$$

Take any element z from $\{\bigcup_\gamma Q_\gamma(\pi_\varphi(A)'')\}'$. Then, for y_1 and y_2 in $\pi_\varphi(A)'$, we have

$$\begin{aligned}([x,z]y_1\xi_\varphi|y_2\xi_\varphi) &= (xzy_1\xi_\varphi|y_2\xi_\varphi) - (zxy_1\xi_\varphi|y_2\xi_\varphi)\\ &= (zy_1\xi_\varphi|x^*y_2\xi_\varphi) - (zxy_1\xi_\varphi|y_2\xi_\varphi)\\ &= \sum_\gamma(zy_1\xi_\varphi|Q_\gamma(x^*)y_2\xi_\varphi) - \sum_\gamma(zQ_\gamma(x)y_1\xi_\varphi|y_2\xi_\varphi)\\ &= \sum_\gamma(Q_\gamma(x)zy_1\xi_\varphi|y_2\xi_\varphi) - \sum_\gamma(zQ_\gamma(x)y_1\xi_\varphi|y_2\xi_\varphi)\\ &= \sum_\gamma([Q_\gamma(x),z]y_1\xi_\varphi|y_2\xi_\varphi) = 0.\end{aligned}$$

This means that $x\in\{\cup Q_\gamma(\pi_\varphi(A)'')\}''$. Q.E.D.

Lemma 3.2. *Let φ be an α-invariant state on A. Then we obtain*

$$p_\varphi\pi_\varphi(A)''p_\varphi = \{\cup Q_\gamma(\pi_\varphi(A)'')\}''p_\varphi.$$

Proof. Note that $p_\varphi(\gamma+\gamma')xp_\varphi(\gamma') = Q_\gamma(x)p_\varphi(\gamma')$ for all $x\in\pi_\varphi(A)''$ and $\gamma,\gamma'\in\hat{G}$ (see the proof of Lemma 3.1 (2)). By summation over γ, we see that

$$p_\varphi xp_\varphi(\gamma') = (\sum_\gamma Q_\gamma(x))p_\varphi(\gamma').$$

Moreover by summation over γ', we obtain that

$$p_\varphi x p_\varphi = (\sum_\gamma Q_\gamma(x)) p_\varphi.$$

We therefore conclude that

$$p_\varphi \pi_\varphi(A)'' p_\varphi \subset \{\bigcup_\gamma Q_\gamma(\pi_\varphi(A)'')\}'' p_\varphi.$$

Since the reverse inclusion is clear by Lemma 3.1, we complete the proof.
Q.E.D.

Lemma 3.3. *Let (A, G, α) be a C^*-dynamical system and let φ be an α-invariant state of A. If ξ_φ is separating for $\pi_\varphi(A)''$, then $Q_\gamma(\pi_\varphi(A))$ is strongly dense in $Q_\gamma(\pi_\varphi(A)'')$ for all γ in $\hat{G}$.*

Proof. Take any element x from $\pi_\varphi(A)''$. It then follows from Kaplansky's density theorem [8, Theorem 2.3.3] that there exists a net $\{x_i\}$, with $\|x_i\| \leqq \|x\|$, in $\pi_\varphi(A)$ which strongly converges to x. Take any element η from H_φ. Given $\varepsilon > 0$, we choose an element z in $\pi_\varphi(A)'$ such that $\|\eta - z\xi_\varphi\| < \varepsilon$. Since $Q_\gamma(y)\xi_\varphi = p_\varphi(\gamma) y \xi_\varphi$ for all $y \in \pi_\varphi(A)''$, we have

$$\begin{aligned}\|Q_\gamma(x)z\xi_\varphi - Q_\gamma(x_i)z\xi_\varphi\| &= \|zQ_\gamma(x)\xi_\varphi - zQ_\gamma(x_i)\xi_\varphi\| \\ &= \|zp_\varphi(\gamma)x\xi_\varphi - zp_\varphi(\gamma)x_i\xi_\varphi\| \\ &\leqq \|zp_\varphi(\gamma)\|\,\|x\xi_\varphi - x_i\xi_\varphi\| \longrightarrow 0.\end{aligned}$$

If we choose an index i such that $\|Q_\gamma(x)z\xi_\varphi - Q_\gamma(x_i)z\xi_\varphi\| < \varepsilon$, we then have

$$\begin{aligned}\|Q_\gamma(x)\eta - Q_\gamma(x_i)\eta\| &\leqq \|Q_\gamma(x)\eta - Q_\gamma(x)z\xi_\varphi\| + \|Q_\gamma(x)z\xi_\varphi - Q_\gamma(x_i)z\xi_\varphi\| \\ &\quad + \|Q_\gamma(x_i)z\xi_\varphi - Q_\gamma(x_i)\eta\| < \|Q_\gamma(x)\|\varepsilon + \varepsilon + \|Q_\gamma\|\,\|x\|\varepsilon.\end{aligned}$$

This establishes the desired result. Q.E.D.

Let φ be an α-invariant state of A. From now on, we denote by q_φ the support projection of p_φ, in $\pi_\varphi(A)''$, which is directly defined as the projection from H_φ onto the closed subspace generated by $\pi_\varphi(A)' p_\varphi H_\varphi$. In the case when the canonical cyclic vector ξ_φ is separating for $\pi_\varphi(A)''$, i.e., cyclic for $\pi_\varphi(A)'$, we remark that $q_\varphi = 1$.

Lemma 3.4. *Let φ be an α-invariant state of A. Assume that*

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'.$$

If $\{p_\varphi \pi_\varphi(A) p_\varphi\}''$ is abelian, or if ξ_φ is separating for $\pi_\varphi(A)''$, then we have

$$\{\pi_\varphi(A)'' \cap \{p_\varphi\}'\} q_\varphi = \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} q_\varphi.$$

Proof. Assume that $\{p_\varphi \pi_\varphi(A) p_\varphi\}''$ is abelian. Let μ be an orthogonal measure corresponding to $\pi_\varphi(A)' \cap \{p_\varphi\}'$. It then follows from [1, Theorem 4.1.25] that

$$p_\varphi \pi_\varphi(a) p_\varphi = \kappa_\mu(\hat{a}) p_\varphi,$$

for all a in A, where $\kappa_\mu(\hat{a})$ is defined as

$$\kappa_\mu(\hat{a}) \pi_\varphi(b) \xi_\varphi = \pi_\varphi(b) p_\varphi \pi_\varphi(a) \xi_\varphi$$

for all b in A. On the other hand, for any x in $\pi_\varphi(A)'' \cap \{p_\varphi\}'$, there exists a net $\{a_i\}$ in A such that $\pi_\varphi(a_i)$ weakly converges to x. We then have

$$x p_\varphi = p_\varphi x p_\varphi = \operatorname{Lim} p_\varphi \pi_\varphi(a_i) p_\varphi = \operatorname{Lim} \kappa_\mu(\hat{a}_i) p_\varphi.$$

Since $\kappa_\mu(\hat{a}_i)$ lies in $\{\pi_\varphi(A) \cup \{p_\varphi\}\}'$ (cf. [1, Theorem 4.1.25 and Proposition 4.3.40]), we conclude that $x p_\varphi$ belongs to $\{\pi_\varphi(A) \cup \{p_\varphi\}\}' p_\varphi$. It therefore follows, from the assumption, that

$$\begin{aligned}\{\pi_\varphi(A)'' \cap \{p_\varphi\}'\} p_\varphi &\subset \{\pi_\varphi(A) \cup \{p_\varphi\}\}' p_\varphi \\ &= \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} p_\varphi.\end{aligned}$$

Multiplying the above inclusion by $\pi_\varphi(A)'$ from the left-hand side, we obtain that

$$\{\pi_\varphi(A)'' \cap \{p_\varphi\}'\} q_\varphi \subset \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} q_\varphi.$$

Since the reverse inclusion is clear, we obtain the desired result.

Next we consider the case when ξ_φ is separating for $\pi_\varphi(A)''$. Let S be the antilinear operator on H_φ defined by

$$S x \xi_\varphi = x^* \xi_\varphi$$

for $x \in \pi_\varphi(A)''$. We then have

$$S u_t^\varphi x \xi_\varphi = S u_t^\varphi x u_t^{\varphi *} \xi_\varphi = u_t^\varphi x^* u_t^{\varphi *} \xi_\varphi = u_t^\varphi S x \xi_\varphi.$$

Since $\pi_\varphi(A)'' \xi_\varphi$ is a core for S, we obtain that $S u_t^\varphi = u_t^\varphi S$. Hence the uniqueness of the polar decomposition of S shows that $J u_t^\varphi = u_t^\varphi J$, which means that $J p_\varphi(\gamma) = p_\varphi(\gamma) J$ for all $\gamma \in \hat{G}$, *i.e.*, $J p_\varphi = p_\varphi J$, where J denotes the modular conjugation associated with ξ_φ (cf. [1, 2.5.2, or 8, 8.13]). Since $J \pi_\varphi(A)'' J = \pi_\varphi(A)'$, we have

$$\begin{aligned}\pi_\varphi(A)'' \cap \{p_\varphi\}' &= J\{\pi_\varphi(A)' \cap \{p_\varphi\}'\} J \\ &= J\{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} J \\ &= \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'.\end{aligned}$$

We thus obtain the desired result. Q.E.D.

Let φ be an α-invariant state of A. Then we put

$$\mathfrak{M} = \pi_\varphi(A)'' \cap \{p_\varphi\}'.$$

This notation will be used in the next lemma and the proof of Lemma 3.7.

Lemma 3.5. *Let φ be an α-invariant state of A. If we have*

$$([x, q_\varphi y q_\varphi] z\xi | \xi) = 0$$

for all $x \in \mathfrak{M}$, $y \in \pi_\varphi(A)''$, $z \in \pi_\varphi(A)'$, $\xi \in p_\varphi H_\varphi$, then we have

$$\{\pi_\varphi(A)'' \cap \{p_\varphi\}'\} q_\varphi = \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} q_\varphi.$$

Proof. Since $q_\varphi H_\varphi$ is invariant under $\pi_\varphi(A)'$ and $q_\varphi p_\varphi = p_\varphi q_\varphi = p_\varphi$, q_φ belongs to $\mathfrak{M}$. On the other hand, for all $x \in \mathfrak{M}$ and all $z \in \pi_\varphi(A)'$, we have

$$xzp_\varphi = zxp_\varphi = zp_\varphi x.$$

Hence, we see that $q_\varphi \in \mathfrak{M} \cap \mathfrak{M}'$. By assumption, we have

$$\mathfrak{M} q_\varphi = q_\varphi \mathfrak{M} q_\varphi \subset \{q_\varphi \pi_\varphi(A)'' q_\varphi\}' = \pi_\varphi(A)' q_\varphi.$$

Therefore we obtain that

$$\mathfrak{M} q_\varphi \subset \{\pi_\varphi(A)' \cup \{p_\varphi\} \cup \pi_\varphi(A)''\}' q_\varphi = \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} q_\varphi.$$

Since the reverse inclusion is clear, we complete the proof. Q.E.D.

Lemma 3.6. *Let φ be an α-invariant state of A. The following conditions are equivalent:*

(1) *φ is a G-central state of almost periodic type.*

(2) *$[Q_\gamma(\pi_\varphi(x)), q_\varphi \pi_\varphi(y) q_\varphi] = 0$ for all x, y in A and γ in $\hat{G}$.*

Proof. Let m be an invariant mean on G. For all $x, y \in A$, $z \in \pi_\varphi(A)'$, $\gamma \in \hat{G}$ and $\xi, \eta \in p_\varphi H_\varphi$, we have

$$\begin{aligned} & m((\pi_\varphi([\gamma\alpha(x), y]) z\xi | \eta)) \\ &= m(([\gamma\bar{\alpha}(\pi_\varphi(x)), \pi_\varphi(y)] z\xi | \eta)) \\ &= ([Q_\gamma(\pi_\varphi(x)), \pi_\varphi(y)] z\xi | \eta). \end{aligned}$$

From Lemma 3.1 and the proof of Lemma 3.5, we easily see that q_φ lies in the commutant of $Q_\gamma(\pi_\varphi(A)'')$. Then Proposition 2.2 (1)–(2) easily yields the equivalence of conditions (1) and (2). Q.E.D.

Lemma 3.7. *Let φ be an α-invariant state of A. Consider the following conditions.*

(1) $\{\pi_\varphi(A)'' \cap \{p_\varphi\}'\} q_\varphi = \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\} q_\varphi$.

(2) *φ is a G-central state of almost periodic type.*

Then it follows that (1) $\Rightarrow$ (2) *and if ξ_φ is separating for $\pi_\varphi(A)''$, the conditions* (1) *and* (2) *are equivalent.*

Proof. (1)$\Rightarrow$(2). Using Lemma 3.1, we have $Q_\gamma(\pi_\varphi(A)'') \subset \pi_\varphi(A)'' \cap \{p_\varphi\}'$ for γ in $\hat{G}$. By condition (1), we have

$$Q_\gamma(\pi_\varphi(A)'')q_\varphi \subset \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\}q_\varphi$$

for all γ in $\hat{G}$. As seen in the proof of Lemma 3.5, q_φ is an element in $\mathfrak{M} \cap \mathfrak{M}'$. We therefore obtain that

$$[Q_\gamma(\pi_\varphi(x)),\ q_\varphi\pi_\varphi(y)q_\varphi] = 0$$

for all x, y in A and γ in $\hat{G}$. Thus we see from Lemma 3.6 that φ is a G-central state of almost periodic type.

(2)$\Rightarrow$(1). Assume that ξ_φ is separating for $\pi_\varphi(A)''$. Combining Lemma 3.6 with Lemmas 3.1 (2) and 3.3, we obtain that

$$[x,\ q_\varphi y q_\varphi] = 0$$

for all $x \in \mathfrak{M}$ and $y \in \pi_\varphi(A)$. Thus we see the desired result from Lemma 3.5.

Q.E.D.

Lemma 3.8. *Let φ be a G-central state of almost periodic type. Assume that the canonical cyclic vector ξ_φ is separating for $\pi_\varphi(A)''$. Then we have*

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'.$$

Proof. Put

$$\mathfrak{N} = \{\pi_\varphi(A) \cup \{p_\varphi\}\}''.$$

We now assert that

$$p_\varphi \mathfrak{N} p_\varphi = p_\varphi \pi_\varphi(A)'' p_\varphi.$$

In fact, it easily follows from Lemma 3.2 that

$$p_\varphi \mathfrak{N} p_\varphi \subset p_\varphi \pi_\varphi(A)'' p_\varphi$$

and the reverse inclusion is clear.

Since $p_\varphi \pi_\varphi(A)'' p_\varphi$ is abelian by [1, Proposition 4.3.30] and since ξ_φ is also cyclic for $p_\varphi \pi_\varphi(A)'' p_\varphi$ in $p_\varphi H_\varphi$, $p_\varphi \mathfrak{N} p_\varphi$ is maximal abelian in $p_\varphi H_\varphi$. Hence we have

$$p_\varphi \pi_\varphi(A)'' p_\varphi = (p_\varphi \mathfrak{N} p_\varphi)' p_\varphi = p_\varphi \mathfrak{N}' p_\varphi,$$

from which it follows that

$$\begin{aligned}\{\pi_\varphi(A)' \cap \{p_\varphi\}'\}p_\varphi &= p_\varphi \mathfrak{N}' p_\varphi = p_\varphi \pi_\varphi(A)'' p_\varphi = \{\bigcup_\gamma Q_\gamma(\pi_\varphi(A)'')\}'' p_\varphi \\ &= \{\pi_\varphi(A)'' \cap \{p_\varphi\}'\}p_\varphi = \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\}p_\varphi.\end{aligned}$$

Here the third equality follows from Lemma 3.2 and the fifth equality from Lemma 3.7. Since ξ_φ is also separating for $\pi_\varphi(A)'$, we conclude that

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'.$$

This completes the proof. Q.E.D.

Now we may summarize what we have obtained in the following theorem.

Theorem 3.9. *Let (A, G, α) be a C*-dynamical system, where G is a locally compact abelian group. Let φ be an α-invariant state of A. Consider the following conditions:*

(1) *φ is a G-central state of almost periodic type.*

(2) $[Q_\gamma(\pi_\varphi(x)), q_\varphi\pi_\varphi(y)q_\varphi] = 0$ *for all x, y in A and γ in $\hat{G}$, where q_φ is the support projection of p_φ in $\pi_\varphi(A)''$.*

(3) $\{\pi_\varphi(A)'' \cap \{p_\varphi\}'\}q_\varphi = \{\pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'\}q_\varphi$.

(4) $\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'$ *and* $\{p_\varphi\pi_\varphi(A)p_\varphi\}''$ *is abelian.*

(5) $\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)''\pi_\varphi(A)' \cap \{p_\varphi\}'$.

It follows that $(1)\Leftrightarrow(2)\Leftarrow(3)\Leftarrow(4)\Rightarrow(5)$ *and if ξ_φ is separating for $\pi_\varphi(A)''$, then the five conditions* (1)–(5) *are equivalent.*

Proof. $(1)\Leftrightarrow(2)$. This follows from Lemma 3.6.

$(3)\Rightarrow(1)$. This follows from Lemma 3.7.

$(4)\Rightarrow(3)$. This follows from Lemma 3.4.

Now we assume that ξ_φ is separating for $\pi_\varphi(A)''$. Then the implication $(1)\Rightarrow(5)$ follows from Lemma 3.8. Since condition (1) implies that $\{p_\varphi\pi_\varphi(A)p_\varphi\}''$ is abelian (see [1, Proposition 4.3.30]), the four conditions (1)–(4) are equivalent. Moreover the implication $(5)\Rightarrow(3)$ follows from Lemma 3.4. Thus the five conditions (1)–(5) are equivalent. Q.E.D.

In the above theorem, we remark that condition (5) does not necessarily implies condition (1), hence condition (4), in general (see Example 3.15). Thus the condition that ξ_φ be separating for $\pi_\varphi(A)''$ is necessary for the implications $(5)\Rightarrow(4)$ and $(1)\Rightarrow(5)$.

Corollary 3.10. *Let (A, G, α) be as in Theorem* 3.9 *and let φ be an α-ergodic state of A. Then the four conditions* (1)–(4) *in Theorem* 3.9 *are equivalent.*

Proof. By the above theorem, we have only to prove the implication $(1)\Rightarrow(4)$. But this immediately follows from statement (2) of Theorem 2.4. Q.E.D.

Corollary 3.11. *Let (A, G, α) be as in Theorem* 3.9. *Assume that an α-invariant state φ of A is factorial. Then the four conditions* (1)–(4) *in Theorem* 3.9 *are equivalent.*

Proof. If φ is a G-central state of almost periodic type, then φ is G-central. Hence, φ is α-ergodic from the note preceding Corollary 2.5. By Corollary 3.10, we complete the proof. Q.E.D.

Consider the case when A is the C^*-algebra of all compact operators on a Hilbert space. Since every state on A is factorial, Corollary 3.11 will be useful for such a C^*-dynamical system (A, G, α).

As mentioned before, almost periodic G-centrality implies G-centrality and the converse is not true in general (see Example 3.15). On the other hand, if we consider N-centrality for some closed subgroup N of G, that condition is usually stronger than G-centrality. Hence it is very natural to consider whether or not we can find a closed subgroup N of G such that N-centrality is equivalent to almost periodic G-centrality. We now give an approach to this "duality" problem.

Proposition 3.12. *Let (A, G, α) be as in Theorem 3.9. Let φ be an α-invariant state of A such that the canonical cyclic vector ξ_φ is separating for $\pi_\varphi(A)''$. Assume that the point spectrum $\sigma(u^\varphi)$ of u^φ is a countable closed subgroup in $\hat{G}$. We denote by N_φ the annihilator of $\sigma(u^\varphi)$. Then the following conditions are equivalent.*

(1) *φ is a G-central state of almost periodic type.*

(2) *φ is N_φ-central.*

Proof. Note that under the above assumptions, p_φ is equal to the projection e onto the subspace of $u^\varphi_{N_\varphi}$-invariant vectors in H_φ (see [1, Theorem 4.3.27]).

(1)$\Rightarrow$(2). By Theorem 3.9, $\{e\pi_\varphi(A)e\}''(=\{p_\varphi\pi_\varphi(A)p_\varphi\}'')$ is abelian and we have

$$\pi_\varphi(A)'\cap u^{\varphi\,\prime}_{N_\varphi}=\pi_\varphi(A)'\cap\{e\}'=\pi_\varphi(A)'\cap\{p_\varphi\}'\subset\pi_\varphi(A)''.$$

Using [4, Corollary 2], then we easily see that

$$e\{\pi_\varphi(A)''\cup\pi_\varphi(A)'\cup u^\varphi_{N_\varphi}\}''e=e\pi_\varphi(A)''e.$$

Let x and y be elements in A, z be an element in $\pi_\varphi(A)'$, and ξ be a vector in eH_φ. We then see that

$$(\pi_\varphi(y)z\xi\,|\,e\pi_\varphi(x)^*\xi)=(e\pi_\varphi(x)\xi\,|\,\pi_\varphi(y)^*z^*\xi).$$

For any $\varepsilon>0$, using the Alaoglu-Birkhoff mean ergodic theorem, we choose a convex combination $C_\lambda(u^\varphi)$ of u^φ,

$$C_\lambda(u^\varphi)=\sum_{i=1}^n\lambda_i u^\varphi_{t_i}\qquad(t_i\in N_\varphi),$$

such that

$$\|(C_\lambda(u^\varphi)-e)\pi_\varphi(x)\xi\|,\ \|(C_\lambda(u^\varphi)-e)\pi_\varphi(x^*)\xi\| < \varepsilon/2\|y\|\|z\|\|\xi\|.$$

Now put

$$C_\lambda(\alpha(x)) = \sum_{i=1}^{n} \lambda_i \alpha_{t_i}(x).$$

We then have

$$\begin{aligned}
&|(\pi_\varphi([C_\lambda(\alpha(x)),\ y])z\xi|\xi)| \\
&\quad \leqq |(\pi_\varphi(C_\lambda(\alpha(x))y)z\xi|\xi) - (\pi_\varphi(y)z\xi|e\pi_\varphi(x^*)\xi)| \\
&\qquad + |(\pi_\varphi(y)z\xi|e\pi_\varphi(x^*)\xi) - (\pi_\varphi(yC_\lambda(\alpha(x)))z\xi|\xi)| \\
&\quad \leqq \|\pi_\varphi(y)z\xi\|\,\|(C_\lambda(u^\varphi)-e)\pi_\varphi(x^*)\xi\| \\
&\qquad + \|(e-C_\lambda(u^\varphi))\pi_\varphi(x)\xi\|\,\|\pi_\varphi(y^*)z^*\xi\| < \varepsilon.
\end{aligned}$$

(2)$\Rightarrow$(1). N_φ-centrality of φ shows that

$$\pi_\varphi(A)' \cap u^{\varphi\,\prime}_{N_\varphi} = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap u^{\varphi\,\prime}_{N_\varphi}$$

(cf. [1, Theorem 4.3.14]). Since $p_\varphi = e$, it then follows that

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}'.$$

On the other hand, since $\{p_\varphi \pi_\varphi(A) p_\varphi\}'' = \{e\pi_\varphi(A)e\}''$, N_φ-centrality of φ implies that $\{p_\varphi \pi_\varphi(A) p_\varphi\}''$ is abelian (cf. [1. Theorem 4.3.14]). By Theorem 3.9, we complete the proof. Q.E.D.

We remark that in the implication (2)$\Rightarrow$(1) of the above proposition, the condition that ξ_φ be separating for $\pi_\varphi(A)''$ is not necessary. Note also that the proof of Proposition 3.12 did not use this condition except to show that $\pi_\varphi(A)' \cap \{p_\varphi\}' \subset \pi_\varphi(A)''$. And this inclusion is valid for every α-ergodic state (see Theorem 2.4 (2), or Corollary 3.10). Thus we have the following.

Proposition 3.13. *Let (A, G, α) be as in Theorem 3.9. Let φ be an α-ergodic state of A. Assume that the point spectrum $\sigma(u^\varphi)$ of u^φ is a countable closed subgroup of $\hat{G}$. We denote by N_φ the annihilator of $\sigma(u^\varphi)$. Then the following conditions are equivalent.*

(1) *φ is a G-central state of almost periodic type.*

(2) *φ is N_φ-central.*

Note that an α-invariant factorial state φ is α-ergodic if it satisfies either of conditions (1) and (2) in Proposition 3.13. Hence the result of Proposition 3.13 is valid under the assumption that φ is factorial.

This paper ends by stating simple examples.

Example 3.14. Let u be any unitary operator, on an infinite-dimensional Hilbert space H, with a unique unit eigenvector ξ corresponding to the eigenvalue zero. Let A be a C^*-algebra on H such that $uAu^* = A$ and $A \neq \{0\}$. Then we consider a C^*-dynamical system $(A, \mathbb{Z}, \alpha)$ where $\mathbb{Z}$ denotes the set of integers and the action α is defined by $\alpha_n(x) = u^n x u^{-n}$. Furthermore, define an α-invariant state φ on A by

$$\varphi(x) = (x\xi | \xi).$$

Then $\{p_\varphi \pi_\varphi(A) p_\varphi\}''$ is abelian and $\pi_\varphi(A)' \cap \{p_\varphi\}' = \mathbb{C} \cdot 1$, from which it follows that φ is a G-central state of almost periodic type (cf. Theorem 3.9). Note also that φ is an extremal point in the weak* closure of the convex set of almost periodic states of A. Some results related to extremal almost periodic states will be discussed in [7].

Next we give an example of a G-central state which is not a G-central state of almost periodic type.

Example 3.15. Let A be the C^*-algebra of all 2×2 complex matrices and let a group G be the one-dimensional torus group. We define an action of G by

$$\alpha_t = \mathrm{Ad}\begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$$

for all $t \in G$. Then the α-invariant state φ of A defined by $\varphi(x) = x_{11}$ for $x = (x_{ij}) \in A$ is G-central. Note also that

$$\pi_\varphi(A)' \cap \{p_\varphi\}' = \pi_\varphi(A)'' \cap \pi_\varphi(A)' \cap \{p_\varphi\}' \qquad (= \mathbb{C} \cdot 1).$$

But φ is *not* a G-central state of almost periodic type.

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