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Decompositions of Regular Representations of the Canonical Commutation Relations

By

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Abstract

Every cyclic regular representation of the canonical commutation relations over any inner product space V can be decomposed into a direct integral of irreducible regular representations, where the fibers are representations over subspaces of V. An example using the so-called direct-product representations shows that generally the irreducible representations cannot be defined over the whole V. So we get a new type of decomposition having no equivalent in the decompositions of locally compact groups.

Introduction

In this paper we are concerned with the decomposition of regular representations of the canonical commutation relations over a complex inner product space into irreducible representations. This article is the abridged version of a paper ([10]) accepted as a doctoral thesis by the Technische Universität München (1988).

Let V be a complex inner product space. A mapping W of V into the group $\mathcal{U}(H)$ of the unitary operators on the complex Hilbert space H is called a representation of the canonical commutation relations (CCR) over V, iff

$$W(f)\,W(g) = \exp(i\,\mathrm{Im}\langle f, g\rangle/2)\,W(f+g) \qquad \text{for } f, g \in V.$$

W is called regular, iff

$$t \in \mathbf{R} \longmapsto W(tf) \in \mathcal{U}(H)$$

is strongly continuous for every $f \in V$. In some cases, more general spaces V are permitted (see 2.8(ii)).

If V is finite dimensional, the decomposition theory for representations of

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the CCR over V is very simple: By a famous theorem due to Stone and von Neumann, there is only one irreducible regular representation of the CCR over V, and every regular representation over V is a multiple of this representation.

In G.C. Hegerfeldt's papers [4] and [5], certain regular representations of the CCR over infinite dimensional V are decomposed into a direct integral of irreducible regular representations; for example, representations over an inner product space with countable Hamel base are allowed, but representations over the separable Hilbert space are not considered. In this paper we decompose cyclic regular representations $W: V \to \mathscr{U}(H)$ over any complex inner product space V into irreducible representations:

W can be decomposed into a direct integral $\int_X^\oplus W(\varphi)\,d\nu(\varphi)$, where for $\varphi \in X$ $W(\varphi)$ is an irreducible regular representation of the CCR over a complex subspace $V(\varphi)$ of V. (What this means precisely, see Chapter 1.) As to the proof we shall have to decompose a representation of a C^*-algebra called $\mathscr{B}(V)$. Since $\mathscr{B}(V)$ is nonseparable, we will not use the usual reduction theory. Instead, for the sake of applying a decomposition theorem due to R.W. Henrichs, we need the concept of direct integrals of Hilbert spaces introduced by W. Wils (see [12]). The main difference to the usual concept is that nonseparable Hilbert spaces are permitted.

In Chapter 1 we derive the theorem above, in Chapter 2 we discuss our result:

By the theorem the representation W can be decomposed into simpler components, namely into irreducible representations over complex subspaces of V. So all the regular representations of the CCR over V are known, if we know the irreducible representations over all the subspaces of V. In Chapter 2 we discuss the question whether it is possible to choose the representations $W(\varphi)$ of the fibers as irreducible regular representations over the whole V.

We construct some examples of irreducible regular representations of the CCR over a dense subspace of $l^2(\mathbb{N})$ which cannot be extended to regular representations of the CCR over $l^2(\mathbb{N})$. They belong to the class of direct product representations introduced by J.R. Klauder, J. McKenna, and E.J. Woods (see [7]). By forming the direct integral of suitable representations of that kind, we get a regular representation W_Y of the CCR over $l^2(\mathbb{N})$ with the following property: There is no decomposition $W_Y = \int_Z^\oplus W(\zeta)\,d\mu(\zeta)$ of W_Y into irreducible regular representations $W(\zeta)$ that are defined over $l^2(\mathbb{N})$—at least, if we assume that the diagonal algebra in the decomposition is maximal abelian in $\{W_Y(f): f \in l^2(\mathbb{N})\}'$.

So we recognize that we cannot strengthen our theorem by choosing irreducible regular representations over V in the fibers.

Throughout this paper we use the following

Notation. For a Hilbert space H and a subset M of H, let be

$H_1 := \{\psi \in H : \|\psi\| = 1\}$,
$M^{\perp} := \{\psi \in H : \langle \psi, \varrho \rangle = 0 \text{ for every } \varrho \in M\}$,
$[M]$ — the closed subspace of H, generated by M,
$\mathscr{L}(H)$ — the set of continuous linear operators in H,
$\mathscr{U}(H)$ — the set of unitary operators in H, and
$\mathbb{1}$ or $\mathbb{1}_H$ — the unit operator in H.

For a set $\mathscr{M} \subseteq \mathscr{L}(H)$, let be
$\mathscr{M}'$ — the commutant of $\mathscr{M}$.
For a C^*-algebra $\mathscr{A}$, let be
$\mathscr{A}_+$ — the set of the positive operators of $\mathscr{A}$, and
$\mathscr{S}(\mathscr{A})$ — the set of states of $\mathscr{A}$.
For $\omega \in \mathscr{S}(\mathscr{A})$, let be
$(\pi_\omega, H_\omega, \xi_\omega)$ — the corresponding GNS representation.
For a group G, a function $h: G \to \mathbb{C}$, and $s \in G$, let be
${}_sh: G \to \mathbb{C}$, — defined by $({}_sh)(t) := h(s^{-1} \circ t)$.
For a set A and a subset T of A, let χ_T be the mapping from A to $\mathbb{C}$, such that

$$\chi_T(a) = \begin{cases} 1 & \text{for } a \in T \\ 0 & \text{for } a \notin T \end{cases}.$$

For an inner product space V, let be
$\mathscr{A}(V)$ — the CCR algebra over V. (This is a C^*-algebra generated by unitary operators $w(f)$, $f \in V$, which satisfy the commutation relations.)

We note that—in an obvious manner—there is a bijective correspondence between the nondegenerate representations of $\mathscr{A}(V)$ and the representations of the CCR over V.

Chapter 1. A Decomposition Theorem for Regular Representations of the Canonical Commutation Relations

§ 1. The C^*-Algebra $\mathscr{B}(V)$ and their Representations

Let V be an inner product space. The set $\mathbb{R} \times V$ endowed with the multiplication

$$(s, f) \circ (t, g) = (s + t + \operatorname{Im} \langle f, g \rangle / 2, f + g)$$

is a group called $\mathscr{H}_V$. By the assignment

$$\pi \longmapsto W_\pi,\ W_\pi(f) = \pi(0, f), \tag{1}$$

we get a bijective correspondence between the unitary representations of $\mathscr{H}_V$ satisfying the relations $\pi(s, 0) = e^{is}\,\mathbb{1}$ for $s \in \mathbb{R}$ and the representations of the CCR over V.

If V has finite dimension m, $\mathscr{H}_V$, carrying the usual topology, is a locally compact group, namely the Heisenberg group of dimension m. A representation of the CCR over V is regular, iff the corresponding representation of $\mathscr{H}_V$ is strongly continuous.

Let $W: V \to \mathscr{U}(H)$ be a regular cyclic representation of the CCR over the inner product space V.

Let $\mathscr{N}$ be the set of the one-dimensional complex subspaces of V. For $N \in \mathscr{N}$ the restriction $W|_N$ of W to N is a regular representation of the CCR over N. By π_N we denote the corresponding representation of $\mathscr{H}_N$, and by π'_N the nondegenerate representation of $L^1(\mathscr{H}_N)$ associated with π_N (see [2], 13.3).

Definition 1.1. The C^*-algebra in $\mathscr{L}(H)$ generated by

$$\bigcup_{N \in \mathscr{N}} \pi'_N(L^1(\mathscr{H}_N)) \cup \{\mathbb{1}_H\}$$

is called $\mathscr{B}(V)$.

Remarks. (i) Let $\tilde{W}$ be another regular representation of the CCR over V. As before, we introduce the C^*-algebra $\tilde{\mathscr{B}}(V)$. Using the theorem due to Stone and von Neumann, we easily see that $\mathscr{B}(V)$ and $\tilde{\mathscr{B}}(V)$ are isomorphic in a canonical way.

(ii) For $N \in \mathscr{N}$, $L^1(\mathscr{H}_N)$ is separable. If the dimension of V is greater than 1, $\mathscr{N}$ is not countable, and it can be shown that $\mathscr{B}(V)$ is nonseparable. Thus, for decomposing the identity representation of $\mathscr{B}(V)$, we cannot use the classical reduction theory. If V has a countable Hamel base $\{b_n : n \in \mathbb{N}\}$ (but not, if V is a separable Hilbert space), we are able to apply the classical theory by considering the separable C^*-algebra, generated by

$$\bigcup_{n=1}^{\infty} \pi'_{\mathbb{C}b_n}(L^1(\mathscr{H}_{\mathbb{C}b_n})) \cup \{\mathbb{1}_H\}$$

instead of $\mathscr{B}(V)$, for example (cf. the proof of Theorem 2.1 in [4]).

(iii) We have

$$\mathscr{B}(V)' = \bigcap_{N \in \mathscr{N}} (\pi'_N(L^1(\mathscr{H}_N))' = \bigcap_{N \in \mathscr{N}} \{W(f) : f \in N\}' = \{W(f) : f \in V\}'.$$

For a two-dimensional subspace M of V we introduce the representations π_M of $\mathscr{H}_M$ and π'_M of $L^1(\mathscr{H}_M)$ as before. We note that for one- or two-dimensional subspaces L of V the product of the Lebesgue measures of $\mathbb{R}$ and L is the Haar measure of $\mathscr{H}_L$ in the normalization we will use.

Lemma 1.2. *Let $N_0, N_1 \in \mathscr{N}$ such that $N_0 \neq N_1$, and $M := N_0 + N_1$. Then the closure $\mathscr{V}(N_0, N_1)$ of the subspace in $\mathscr{L}(H)$, generated by*

$$\{\pi'_{N_0}(h)\pi'_{N_1}(k) \colon h \in L^1(\mathscr{H}_{N_0}),\ k \in L^1(\mathscr{H}_{N_1})\},$$

is equal to the closure $\overline{\pi'_M(L^1(\mathscr{H}_M))}$ of $\pi'_M(L^1(\mathscr{H}_M))$ (closures with respect to the norm topology).

Particularly, we have $\pi'_M(L^1(\mathscr{H}_M)) \subseteq \mathscr{B}(V)$.

Proof. For one- and two-dimensional subspaces L of V and $r \in L^1(L)$, we define

$$\bar{\pi}_L(r) := \int_L r(f)\, W(f)\, df,$$

By an easy consideration we get:

$$\bar{\pi}_L(L^1(L)) = \pi'_L(L^1(\mathscr{H}_L)).$$

For $r \in L^1(N_0)$ and $s \in L^1(N_1)$ we have

$$\begin{aligned} \bar{\pi}_{N_0}(r)\bar{\pi}_{N_1}(s) &= \int_{N_0}\int_{N_1} r(f)\, s(g)\, W(f)\, W(g)\, df\, dg \\ &= \int_{N_0}\int_{N_1} (\exp(i \operatorname{Im}\langle f, g\rangle/2) r(f)\, s(g))\, W(f+g)\, df dg \\ &\in \bar{\pi}_M(L^1(M)) = \pi'_M(L^1(\mathscr{H}_M)). \end{aligned} \tag{2}$$

If follows $\mathscr{V}(N_0, N_1) \subseteq \overline{\pi'_M(L^1(\mathscr{H}_M))}$. Now let us prove the other inclusion.

Using a Stone-Weierstrass argument, we easily conclude that the subspace of $L^1(N_0 \times N_1)$, generated by

$$\{(f, g) \longmapsto r(f)\, s(g) \colon r \in L^1(N_0),\ s \in L^1(N_1)\},$$

is dense in $L^1(N_0 \times N_1)$. Since the map

$$u \longmapsto \tau(u),\ \tau(u)(f, g) = \exp(i \operatorname{Im}\langle f, g\rangle/2)\, u(f, g),$$

is an isometric isomorphism in $L^1(N_0 \times N_1)$, the subspace generated by

$$\{(f, g) \longmapsto \exp(i \operatorname{Im}\langle f, g\rangle/2)\, r(f)\, s(g) \colon r \in L^1(N_0),\ s \in L^1(N_1)\}$$

is dense in $L^1(N_0 \times N_1)$, too. Now it follows from relation (2) that every element of $\pi'_M(L^1(\mathscr{H}_M))$ can be approximated by elements of the linear span of $\{\bar{\pi}_{N_0}(r)\bar{\pi}_{N_1}(s): r\in L^1(N_0),\ s\in L^1(N_1)\}$. ∎

From now on, let $\rho: \mathscr{B}(V)\to\mathscr{L}(K)$ be a nondegenerate representation of $\mathscr{B}(V)$, $\rho'_L := \rho\circ\pi'_L$ for one- or two-dimensional complex subspaces L of V,

$\mathscr{N}_\rho := \{N\in\mathscr{N}: \rho'_N$ a nondegenerate representation of $L^1(\mathscr{H}_N)\}$, and

$$V_\rho := \begin{cases} \bigcup_{N\in\mathscr{N}_\rho} N & \text{for } \mathscr{N}_\rho \neq \emptyset \\ \{0\} & \text{for } \mathscr{N}_\rho = \emptyset \end{cases} \quad (\subseteq V).$$

Proposition 1.3. *(i) For $N\in\mathscr{N}_\rho$ let ρ_N be the representation of $\mathscr{H}_N$ corresponding to ρ'_N. Then for $t\in\mathbb{R}$ $\rho_N(t,0)=e^{it}\mathbb{1}$, that means that a regular representation called $W_{\rho,N}$ of the CCR over N is associated with ρ_N.*

(ii) If $\mathscr{N}_\rho\neq\emptyset$, V_ρ is a complex subspace of V; by $W_\rho(f) := W_{\rho,N}(f)$ for $f\in N$ and $N\in\mathscr{N}_\rho$, a regular representation $W_\rho: V_\rho \longrightarrow \mathscr{U}(K)$ of the CCR over V_ρ is defined.

Additionally, if $\mathscr{N}_\rho=\emptyset$, W_ρ is defined over $V_\rho=\{0\}$ by $W_\rho(0):=\mathbb{1}_K$.

Proof. (i) Since for $h\in L^1(\mathscr{H}_N)$

$$\rho(e^{it}\pi'_N(h)) = \rho(\pi_N(t,0)\cdot\pi'_N(h)) = \rho(\pi'_N({}_{(t,0)}h)),$$

$e^{it}\mathbb{1}$ satisfies the equation $e^{it}\mathbb{1}\cdot\rho'_N(h)=\rho'_N({}_{(t,0)}h)$ for $h\in L^1(\mathscr{H}_N)$. From this we get $e^{it}\mathbb{1}=\rho_N(t,0)$.

(ii) Let $N_0, N_1\in N_\rho$ such that $N_0\neq N_1$ and put $M:=N_0+N_1$. The representations ρ'_{N_0} of $L^1(\mathscr{H}_{N_0})$ and ρ'_{N_1} of $L^1(\mathscr{H}_{N_1})$ are nondegenerate. Now, considering Lemma 1.2 we see that ρ'_M is a nondegenerate representation of $L^1(\mathscr{H}_M)$. For every one-dimensional subspace N of M, ρ'_N is nondegenerate. This follows for $N\neq N_0$ and $N\neq N_1$ by applying Lemma 1.2 once more (if we use N instead of N_0). Now it is easy to show that V_ρ is a subspace of V.

It is clear that W_ρ is well defined, and that $t\in\mathbb{R}\mapsto W_\rho(tf)$ is strongly continuous for $f\in V_\rho$. Obviously,

$$W_\rho(f_0)\,W_\rho(f_1) = \exp(i\operatorname{Im}\langle f_0, f_1\rangle/2)\,W_\rho(f_0+f_1) \tag{3}$$

is satisfied, if f_0 and f_1 belong to a common one-dimensional subspace of V_ρ. We will show (3) in case $M:=\mathbb{C}f_0+\mathbb{C}f_1$ is a two-dimensional subspace of V_ρ. Since the representations ρ'_{N_0} and ρ'_{N_1} for $N_0:=\mathbb{C}f_0$ and $N_1:=\mathbb{C}f_1$ are nondegenerate, ρ'_M is nondegenerate, too. Thus there is a representation ρ_M of $\mathscr{H}_M$ associated with ρ'_M. The same argument as in (i) implies that $f\in M\mapsto\rho_M(0,f)$ is a regular representation of the CCR over M. So we can show the relation (3) by proving

$$W_\rho(f) = \rho_M(0, f) \qquad \text{for } f \in M. \tag{4}$$

Obviously $f = 0$ satisfies (4); for $f \neq 0$ let $N := \mathbf{C}f$, and $N^\perp$ be the orthogonal complement of N in M. For $h \in L^1(\mathscr{H}_N)$ and $k \in L^1(\mathscr{H}_{N^\perp})$ we have

$$\rho'_N(h)\,\rho'_{N^\perp}(k) = \rho(\pi'_N(h)\,\pi'_{N^\perp}(k)) =$$

$$\rho\left(\int_{\mathbf{R}}\int_N\int_{\mathbf{R}}\int_{N^\perp} h(t, g)\,k(t^\perp, g^\perp)\,e^{i(t+t^\perp)}\,W(g)\,W(g^\perp)\,dg^\perp\,dt^\perp\,dg\,dt\right) =$$

$$\rho\left(\int_{\mathbf{R}}\int_{N\times N^\perp}\left(\int_{\mathbf{R}} k(t^\perp, g^\perp)e^{it^\perp}\,dt^\perp\right)h(t, g)\,e^{it}\,W(g + g^\perp)\,dg^\perp\,dg\,dt\right) = \rho'_M(l),$$

where

$$l(t, g, g^\perp) = \left(\int_{\mathbf{R}} k(t^\perp, g^\perp)e^{it^\perp}\,dt^\perp\right)h(t, g) \qquad \text{for } t \in \mathbf{R},\, g \in N, \text{ and } g^\perp \in N^\perp.$$

(Identifying $\mathscr{H}_M$ and $\mathbf{R} \times N \times N^\perp$, we get $l \in L^1(\mathscr{H}_M)$). Similarly, we conclude

$$\rho_M(0, f)\,\rho'_N(h)\,\rho'_{N^\perp}(k) = \rho_M(0, f)\,\rho'_M(l) = \rho'_M({}_{(0,f)}l) =$$

$$\rho\left(\int_{\mathbf{R}}\int_{N\times N^\perp}\left(\int_{\mathbf{R}} k(t^\perp, g^\perp)e^{it^\perp}\,dt^\perp\right)({}_{(0,f)}h)(t, g)e^{it}\,W(g + g^\perp)dg^\perp dg\,dt\right) =$$

$$\rho'_N({}_{(0,f)}h)\,\rho'_{N^\perp}(k).$$

Since $\rho_{N^\perp}$ is nondegenerate, we get $\rho_N(0, f) = \rho_M(0, f)$ and equation (4). ∎

Proposition 1.4. *Let $\rho\colon \mathscr{B}(V) \to \mathscr{L}(K)$ be a factor representation of $\mathscr{B}(V)$ in a Hilbert space $K \neq 0$.*

(*i*) *For $N \in \mathscr{N}$, ρ'_N is either the zero representation or it is a nondegenerate representation of $L^1(\mathscr{H}_N)$.*

(*ii*) *W_ρ is factorial, too. If ρ is irreducible, W_ρ is irreducible, too.*

First we show a lemma.

Lemma 1.5. *Let $\rho\colon \mathscr{B}(V) \to \mathscr{L}(K)$ be a representation of $\mathscr{B}(V)$. For $N \in \mathscr{N}$ the essential space $K_N := [\rho'_N(L^1(\mathscr{H}_N))K]$ of the representation ρ'_N is an invariant subspace of ρ.*

Proof. It suffices to show

$$\rho'_{\tilde N}(h)\,\rho'_N(k)\,\psi \in K_N \tag{5}$$

for every $\tilde N \in \mathscr{N}$, $\psi \in K$, $h \in L^1(\mathscr{H}_{\tilde N})$, and $k \in L^1(\mathscr{H}_N)$. The case $\tilde N = N$ is trivial; otherwise we set $M := N + \tilde N$; by Lemma 1.2 $\rho'_{\tilde N}(h)\rho'_N(k)$ is in the closure of $\rho'_M(L^1(\mathscr{H}_M))$. By Lemma 1.2 (using N instead of N_0 and $\tilde N$ instead of N_1) we conclude that $\rho'_{\tilde N}(h)\cdot\rho'_N(k)$ belongs to the closure of the linear span of $\rho'_N(L^1(\mathscr{H}_N))\cdot\rho'_{\tilde N}(L^1(\mathscr{H}_{\tilde N}))$. It follows (5). ∎

Proof of Proposition 1.4. (i) By Lemma 1.5 K_N and therefore $(K_N)^{\perp}$ are invariant subspaces of $\rho(\mathscr{B}(V))$. Let $\rho^{\perp}$ be the subrepresentation of ρ defined on $(K_N)^{\perp}$. Let us assume that ρ'_N is degenerate. Then $(K_N)^{\perp} \neq \{0\}$, and $\rho^{\perp}$ is quasi-equivalent to ρ, since ρ is factorial. Thus $\rho^{\perp}|_{\pi'_N(L^1(\mathscr{H}_N))} = 0$ entails $\rho|_{\pi'_N(L^1(\mathscr{H}_N))} = 0$.

(ii) The case $\mathscr{N}_\rho = \emptyset$ is trivial. For $\mathscr{N}_\rho \neq \emptyset$ from

$$\{W_\rho(f): f \in V_\rho\}' = \bigcap_{N \in \mathscr{N}_\rho} \{e^{it} W_\rho(f): f \in N,\ t \in \mathbb{R}\}' =$$

$$\bigcap_{N \in \mathscr{N}_\rho} (\rho'_N(L^1(\mathscr{H}_N)))' = (\text{using (i)})$$

$$\bigcap_{N \in \mathscr{N}} (\rho'_N(L^1(\mathscr{H}_N)))' = \rho(\mathscr{B}(V))',$$

we get the assertion. ∎

§2. The Decomposition Theorem

For the following we need the concept of direct integrals introduced by W. Wils. We recall the definition (see [12]).

Definition 2.1. Let (Z, μ) be a measure space; for $\zeta \in Z$ let $H(\zeta)$ be a complex Hilbert space. A linear subspace Γ of $\prod_{\zeta \in Z} H(\zeta)$ is called a set of μ-square integrable vector fields, iff

(i) $\zeta \in Z \to \|\eta(\zeta)\|^2$ is μ-integrable for every $\eta = (\eta(\zeta))_{\zeta \in Z} \in \Gamma$.

(ii) If for $\eta \in \prod_{\zeta \in Z} H(\zeta)$ there is an $\eta' \in \Gamma$ such that $\eta(\zeta) = \eta'(\zeta)$ for almost every ζ, then $\eta \in \Gamma$.

(iii) If $\eta \in \Gamma$ and $h \in L^\infty(Z, \mu)$, then $h \cdot \eta = (h(\zeta)\eta(\zeta))_{\zeta \in Z} \in \Gamma$.

(iv) Γ with respect to the seminorm $\|\eta\|_2 = \left(\int_Z \|\eta(\zeta)\|^2 d\mu(\zeta) \right)^{\frac{1}{2}}$ is complete.

The corresponding Hilbert space is called the direct integral of the Hilbert spaces $H(\zeta)$, denoted by

$$\int_Z^{\oplus} H(\zeta)\, d\mu(\zeta) \quad \text{or} \quad \int_Z^{\Gamma} H(\zeta)\, d\mu(\zeta).$$

In order to make a distinction we call the direct integrals defined by the usual concept, which for example is described in Dixmier's books [1] and [2], as "direct integrals in the sense of J. Dixmier". Obviously, these direct integrals also satisfy the Definition 2.1.

Definition 2.2. Let $H = \int_Z^{\Gamma} H(\zeta)\, d\mu(\zeta)$ be a direct integral of Hilbert spaces. Let $W\colon V\to\mathscr{U}(H)$ be a regular representation of the CCR over V. For $\zeta\in Z$ let $V(\zeta)$ be a complex subspace of V, and $W(\zeta)\colon V(\zeta)\to\mathscr{U}(H(\zeta))$ be a regular representation of the CCR over $V(\zeta)$. Moreover let us assume that for every $f\in V$ there is a μ-negligible set N_f such that $f\in V(\zeta)$ for every $\zeta\in Z\backslash N_f$, $\int_Z^\Gamma W(\zeta)(f)\,d\mu(\zeta)$ is well defined and equal to $W(f)$. (This precisely means that for $\eta=(\eta(\zeta))_{\zeta\in Z}\in\Gamma$ the almost everywhere unique extension $(\eta_f(\zeta))_{\zeta\in Z}$ of $(W(\zeta)(f)\eta(\zeta))_{\zeta\in Z\backslash N}$ to an element of $\prod_{\zeta\in Z}H(\zeta)$ belongs to Γ, and that the operator of $\mathscr{L}(H)$, given by

$$\int_Z^\Gamma \eta(\zeta)\,d\mu(\zeta)\longrightarrow\int_Z^\Gamma \eta_f(\zeta)\,d\mu(\zeta),$$

is equal to $W(f)$.)

Then we say:

$$W=\int_Z^\Gamma W(\zeta)\,d\mu(\zeta)$$

is a decomposition of W into regular representations $W(\zeta)$ of the CCR that are defined over subspaces $V(\zeta)$ of V.

Now we can formulate the main result of our paper.

Theorem 2.3. *Let $W\colon V\to\mathscr{U}(H)$ be a cyclic regular representation of the CCR over the complex inner product space $V\neq 0$. Then there exist a regular Borel measure ν on a compact space X and a decomposition*

$$W=\int_X^\oplus W(\varphi)\,d\nu(\varphi)$$

of W into irreducible regular representations $W(\varphi)\colon V(\varphi)\to\mathscr{U}(H(\varphi))$ of the CCR, which are defined over complex subspaces $V(\varphi)$ of V.

Supplement. (*i*) *If V is a separable inner product space, almost every $V(\varphi)$ is dense in V.*

(*ii*) *If the decomposition is constructed as in the proof, the diagonal algebra in $\int_X^\oplus H(\varphi)\,d\nu(\varphi)$ is maximal abelian in $\{W(f)\colon f\in V\}'$.*

We can formulate this result using the group $\mathscr{H}_V$: Certain representations of the group $\mathscr{H}_V$ will be decomposed into certain irreducible representations of subgroups $\mathscr{H}_{V(\varphi)}$ of $\mathscr{H}_V$. So we get a decomposition theorem for representations of a non locally compact group, having no equivalent in the theory of locally compact groups.

In the proof we shall have to decompose the identity representation $\mathbf{1}_{\mathscr{B}(V)}$ of

$\mathscr{B}(V)$. For this purpose we need a theorem due to R.W. Henrichs. First let us give a definition.

Definition 2.4. Let π be a cyclic representation of a C^*-algebra $\mathscr{A}$ and ξ a cyclic vector for π such that $\|\xi\| = 1$. We say that a decomposition

$$\pi = \int_Z^\oplus \pi(\zeta)\, d\mu(\zeta)$$

of π is normalized (with respect to ξ), iff in the corresponding decomposition $\xi = \int_Z^\oplus \xi(\zeta)\, d\mu(\zeta)$ of ξ almost every $\xi(\zeta)$ is cyclic for $\pi(\zeta)$ and $\|\xi(\zeta)\| = 1$.

Theorem 2.5 ([6]). *Let π be a cyclic representation of a unital (not necessarily separable) C^*-algebra $\mathscr{A}$ and ξ a cyclic vector for π such that $\|\xi\| = 1$. Then there is a (with respect to ξ) normalized decomposition*

$$\pi = \int_X^\oplus \pi(\varphi)\, d\mu(\varphi)$$

of π such that every $\pi(\varphi)$ is an irreducible representation of $\mathscr{A}$, and μ is a regular Borel measure on the compact space X.

Remarks. (i) Let us briefly sketch Henrichs's construction of the decomposition:
Let ω be the state of $\mathscr{A}$ belonging to π and ξ, and $\mathscr{C}$ be a maximal abelian von Neumann subalgebra of $\pi(\mathscr{A})'$. μ is chosen as the orthogonal measure associated with ω and $\mathscr{C}$. For $\varphi \in \mathscr{S}(\mathscr{A})$, $\pi(\varphi)$ is the GNS representation of a pure state $\tilde{\varphi} \in \mathscr{S}(\mathscr{A})$ such that

$$\{a \in \mathscr{A} : \varphi(a^* a) = 0\} \subseteq \{a \in \mathscr{A} : \tilde{\varphi}(a^* a) = 0\}.$$

The diagonal algebra in the direct integral can be identified with $\mathscr{C}$.

(ii) In [6], §3 R.W. Henrichs varies the decomposition of π, in which the orthogonal measure associated with ω and the center $\pi(A)' \cap \pi(\mathscr{A})''$ of $\pi(\mathscr{A})'$ is used, in a similar way in order to get a decomposition of π into factorial representations.

If we apply this decomposition for $\mathbb{1}_{\mathscr{B}(V)}$ instead of the decomposition above, we obtain a decomposition of W into factorial regular representations $W(\varphi)$, which are defined over subspaces $V(\varphi)$ of V.

Lemma 2.6. *Let ξ be a cyclic vector for W (and therefore for $\mathbb{1}_{\mathscr{B}(V)}$) such that $\|\xi\| = 1$. Let*

$$\mathbb{1}_{\mathscr{B}(V)} = \int_Z^\Gamma \rho(\zeta)\, d\mu(\zeta)$$

be a (with respect to ξ) normalized decomposition of the identical representation $\mathbb{1}_{\mathscr{B}(V)}$ of $\mathscr{B}(V)$ into factorial representations $\rho(\zeta)$. For a fixed $N \in \mathscr{N}$, $\rho(\zeta)'_N$ is nondegenerate for almost every $\zeta \in Z$.

Proof. Let $\xi = \int_Z^\Gamma \xi(\zeta)\, d\mu(\zeta)$ be the corresponding decomposition of ξ; for $\zeta \in Z$ let $\varphi(\zeta)$ be the state of $\mathscr{B}(V)$ belonging to $\rho(\zeta)$ and $\xi(\zeta)$ and ω be the state of $\mathscr{B}(V)$ belonging to $\mathbb{1}_{\mathscr{B}(V)}$ and ξ. For $N \in \mathscr{N}$ let $(u_n)_{n\in\mathbb{N}}$ be an approximate unit in $L^1(\mathscr{H}_N)$. Since $(\pi'_N(u_n))_{n\in\mathbb{N}}$ converges strongly to $\mathbb{1}_H$, we have

$$1 = \lim_{n\to\infty} \omega(\pi'_N(u_n)) = \lim_{n\to\infty} \int_Z \varphi(\zeta)(\pi'_N(u_n))\, d\mu(\zeta).$$

Since $0 \le \varphi(\zeta)(\pi'_N(u_n)) \le 1$, it follows $\lim_{n\to\infty} \varphi(\zeta)(\pi'_N(u_n)) = 1$ for almost every $\zeta \in Z$, and Proposition 1.4 (i) implies the assertion. ∎

Remark. Let us consider a decomposition $\pi = \int_Z^\oplus \pi(\zeta)\, d\mu(\zeta)$ of a nondegenerate representation π of a separable C^*-algebra. If the usual concept of direct integrals is used, then almost every $\pi(\zeta)$ is nondegenerate. For our more general concept of direct integrals such a statement is not correct.

Lemma 2.7. *Let*

$$\mathbb{1}_{\mathscr{B}(V)} = \int_Z^\Gamma \rho(\zeta)\, d\mu(\zeta)$$

be a decomposition of $\mathbb{1}_{\mathscr{B}(V)}$ such that for $\zeta \in Z$ the representation $\rho(\zeta)\colon \mathscr{B}(V) \to \mathscr{L}(H(\zeta))$ is nondegenerate and for every fixed $N \in \mathscr{N}$ $\rho(\zeta)'_N$ is nondegenerate almost everywhere. For $\zeta \in Z$, let $V(\zeta) := V_{\rho(\zeta)}$ and $W(\zeta) := W_{\rho(\zeta)}$. Then

$$W = \int_Z^\Gamma W(\zeta)\, d\mu(\zeta)$$

is a decomposition of W into the regular representations $W(\zeta)$, which are defined over the complex subspaces $V(\zeta)$ of V.

Proof. Let $f \in N$, where $N \in \mathscr{N}$. Since $\rho(\zeta)'_N$ is almost everywhere nondegenerate, f belongs to almost every $V(\zeta)$. Let $(u_n)_{n\in\mathbb{N}}$ be an approximate unit in $L^1(\mathscr{H}_N)$ and $v_n := {}_{(0,f)}u_n$ for $n \in \mathbb{N}$. Then $(\pi'_N(v_n))_{n\in\mathbb{N}}$ converges strongly to $\pi_N(0, f) = W(f)$, and $(\rho(\zeta)'_N(v_n))_{n\in\mathbb{N}}$ to $W(\zeta)(f)$ almost everywhere. Using $\pi'_N(v_n) = \int_Z^\Gamma \rho(\zeta)'_N(v_n)\, d\mu(\zeta)$ we easily see that $(W(\zeta)(f))_{\zeta\in Z}$ is a measurable, almost everywhere defined field of operators and

$$W(f) = \int_Z^\Gamma W(\zeta)(f)\, d\mu(\zeta)$$

holds. □

Proof of Theorem 2.3. The assertion follows from Proposition 1.4, Theorem 2.5 (applied to $\mathbb{1}_{\mathscr{B}(V)}$), Lemma 2.6, and Lemma 2.7. It is easy to show Supplement (i); Supplement (ii) follows from Remark (i) after Theorem 2.5. □

Remarks 2.8. (i) If V is a Hilbert space, one can get a decomposition of W with the properties of the decomposition of Theorem 2.3, for which, additionally, holds:
All the $V(\varphi)$'s are dense in V.

Let a decomposition $W = \int_X^{\oplus} W(\varphi)\, d\mu(\varphi)$ of W as in Theorem 2.3 be given; for $\varphi \in X$ we define

$$V(\varphi)^{\perp} := \{f \in V: \langle f, g \rangle = 0 \text{ for } g \in V(\varphi)\}.$$

Let $W(\varphi)^{\perp}: V(\varphi)^{\perp} \to \mathscr{U}(H(\varphi)^{\perp})$ be an irreducible regular representation of the CCR over $V(\varphi)^{\perp}$ (the Fock-representation for example); for $V(\varphi)^{\perp} = \{0\}$ let $H(\varphi)^{\perp} := \mathbb{C}$ and $W(\varphi)^{\perp}(0) := \mathbb{1}$.

Let $\tilde{V}(\varphi) := V(\varphi) + V(\varphi)^{\perp}$, $\tilde{H}(\varphi) := H(\varphi) \otimes H(\varphi)^{\perp}$, and $\tilde{W}(\varphi): \tilde{V}(\varphi) \to \mathscr{U}(\tilde{H}(\varphi))$ such that

$$\tilde{W}(\varphi)(f + f^{\perp}) = W(\varphi)(f) \otimes W(\varphi)^{\perp}(f^{\perp})$$

for $f \in V(\varphi)$ and $f^{\perp} \in V(\varphi)^{\perp}$.

$\tilde{V}(\varphi)$ is dense in V, and $\tilde{W}(\varphi)$ is an irreducible regular representation of the CCR over $\tilde{V}(\varphi)$. Considering $\tilde{W}(\varphi)$ as extension of $W(\varphi)$, we get the decomposition

$$W = \int_X^{\oplus} \tilde{W}(\varphi)\, d\mu(\varphi)$$

of W which has the desired properties.

(ii) In some cases more general spaces V are permitted in the definition of the CCR. We consider real vector spaces V, endowed with an antisymmetric bilinear form σ such that there is a countable subset $\{f_n: n \in \mathbb{N}\}$ having the following property:
For every $g \in V \setminus \{0\}$ there is an $n \in \mathbb{N}$ such that $\sigma(f_n, g) \neq 0$. (Important examples are real dense subspaces of the separable Hilbert space endowed with the imaginary part of the inner product.) In the definition of the CCR we then have to replace $\operatorname{Im}\langle f, g \rangle$ by $\sigma(f, g)$.

Using similar methods as above one can get a decomposition theorem for regular representation over such spaces, too (see [10], Theorem 2.2.6). Then the representations of the fibers are irreducible regular representations over subspaces $V(\varphi)$ of V, for which $\sigma|_{V(\varphi)\times V(\varphi)}$ is nondegenerate. (That means that for every $f\in V(\varphi)$ there is a $g\in V(\varphi)$ such that $\sigma(f, g)\neq 0$.)

Chapter 2. The Domains of the Fibers in the Decomposition of Regular Representations of the Canonical Commutation Relations

§3. Extensions of Direct-Product Representations

In this Section we construct some irreducible regular representations of the CCR over a dense subspace of the Hilbert space $l^2(\mathbb{N})$, which cannot be extended to regular representations over $l^2(\mathbb{N})$. For our purposes we need a very general concept of extension; thus the results of the literature (see [11] or [13], for example) are not sufficient.

Definition 3.1. Let V_1 and V_2 be real dense subspaces of $l^2(\mathbb{N})$ such that $V_1\subseteq V_2$. Let $W_1: V_1\to\mathscr{U}(H)$ and $W_2: V_2\to\mathscr{U}(K)$ be regular representations over V_1 and V_2, resp., such that H is a closed subspace of K (for the definition of representations of the CCR over real dense subspaces of $l^2(\mathbb{N})$ see 2.8 (ii)). Moreover, let H be invariant under W_2, and let the restriction of W_2 on H be equal to W_1. Then we say that W_2 is an extension of W_1.

Let $W_S: \mathbb{C}\to\mathscr{U}(L^2(\mathbb{R}))$ be the Schrödinger representation of the CCR over $\mathbb{C}$, defined by

$$(W_S(r+is)f)(x)=\exp(ir(x+s/2))f(x+s).$$

For $m\in\mathbb{N}$, the m-fold tensor product W_S^m of W_S is an irreducible regular representation of the CCR over $\mathbb{C}^m$, called the m-dimensional Schrödinger representation.

For $n\in\mathbb{N}$ let $\psi_n\in L^2(\mathbb{R})_1$; moreover, let $\xi:=\bigotimes_{n=1}^{\infty}\psi_n$, and let H_ξ be the incomplete infinite tensor product of countable many copies of $L^2(\mathbb{R})$, distinguished by ξ (see [8] or [3]). Furthermore, for $n\in\mathbb{N}$ let $e_n:=(\delta_{jn})_{j\in\mathbb{N}}\in l^2(\mathbb{N})$ and V_0 be the complex algebraic span of the e_n, $n\in\mathbb{N}$. By putting

$$W_\xi^0\Big(\sum_{n=1}^{m}\lambda_n e_n\Big)\bigotimes_{n=1}^{\infty}\varrho_n:=\bigotimes_{n=1}^{m}W_S(\lambda_n)\varrho_n\otimes\bigotimes_{n=m+1}^{\infty}\varrho_n,$$

we define an irreducible regular representation $W_\xi^0: V_0\to\mathscr{U}(H_\xi)$ of the CCR over V_0. It is called the direct-product representation distinguished by ξ (see [7]).

Let us recall a result of L. Streit (see [11], §4) concerning extensions of W_ξ^0. Consider the elements $\sum_{n=1}^{\infty}\lambda_n e_n$ of $l^2(\mathbb{N})$ such that for every $t\in\mathbb{R}$ $\bigotimes_{n=1}^{\infty}W_S(t\lambda_n)\psi_n$ belongs to H_ξ, that means that

$$\sum_{n=1}^{\infty}|\langle W_S(t\lambda_n)\psi_n,\psi_n\rangle-1|<\infty$$

for every $t \in \mathbb{R}$. They form a real subspace V_ξ of $l^2(\mathbb{N})$. Obviously we have $V_\xi \supseteq V_0$. By defining

$$W_\xi\Big(\sum_{n=1}^{\infty} \lambda_n e_n\Big) \bigotimes_{n=1}^{\infty} \varrho_n = \bigotimes_{n=1}^{\infty} W_S(\lambda_n)\varrho_n \qquad \text{for } \bigotimes_{n=1}^{\infty} \varrho_n \in H_\xi \text{ and } \sum_{n=1}^{\infty} \lambda_n e_n \in V_\xi,$$

we get an irreducible regular representation $W_\xi\colon V_\xi \to \mathscr{U}(H_\xi)$ of the CCR over V_ξ. W_ξ is an extension of W_ξ^0.

The following result makes it possible to find direct-product representations having no extension to regular representations of the CCR over $l^2(\mathbb{N})$.

Theorem 3.2. *Let $g = \sum_{n=1}^{\infty} \mu_n e_n \in l^2(\mathbb{N}) \setminus V_\xi$ such that for every $t \in \mathbb{R}\setminus\{0\}$ $\bigotimes_{n=1}^{\infty} W_S(t\mu_n)\psi_n$ is not weakly equivalent to $\bigotimes_{n=1}^{\infty} \psi_n$, that means that*

$$\sum_{n=1}^{\infty} \big|\,|\langle W_S(t\mu_n)\psi_n, \psi_n\rangle| - 1\big| = \infty$$

for every $t \in \mathbb{R}\setminus\{0\}$. Then W_ξ^0 cannot be extended to a regular representation of the CCR over a real subspace V of $l^2(\mathbb{N})$, containing g and V_0.

So, if such a g exists, W_ξ and W_ξ^0 have no extension to a regular representation over $l^2(\mathbb{N})$.

First let us show a lemma.

Lemma 3.3. *Let ω_ξ^0 be the pure state of the CCR algebra $\mathscr{A}(V_0)$ belonging to W_ξ^0 and the cyclic vector $\xi = \bigotimes_{n=1}^{\infty} \psi_n$. Moreover, let V be a real subspace of $l^2(\mathbb{N})$ such that $V \supseteq V_0$, and let ω be a state of $\mathscr{A}(V)$, extending ω_ξ^0 such that the corresponding representation W_ω of the CCR over V is regular. Then for $f = \sum_{n=1}^{\infty} \lambda_n e_n \in V$*

$$|\omega(w(f))| \leq \prod_{n=1}^{\infty} |\langle W_S(\lambda_n)\psi_n, \psi_n\rangle|$$

holds.

Proof. We show

$$|\omega(w(f))| \leq \prod_{n=1}^{m} |\langle W_S(\lambda_n)\psi_n, \psi_n\rangle|$$

for every $m \in \mathbb{N}$.

Let m be fixed. We identify $\mathbb{C}^m$ and the subspace $\{\sum_{n=1}^{m} \lambda_n e_n : \lambda_1, \dots, \lambda_m \in \mathbb{C}\}$ of V_0 in a canonical way.

Let $(W_\omega, H_\omega, \xi_\omega)$ be the GNS representation of the CCR over V corresponding to ω. (Instead of the representation π_ω of $\mathscr{A}(V)$ belonging to ω we apply the representation W_ω of the CCR over V that is associated with π_ω.) By the theorem of Stone and von Neumann the restriction $W_\omega|_{\mathbb{C}^m}$ of W_ω to a representation over $\mathbf{C}^m$ is a multiple of the Schrödinger representation W_S^m. Therefore, we may assume that

$$H_\omega = \bigotimes_{n=1}^{m} L^2(\mathbf{R}) \otimes l^2(I),$$

where I is a suitable index set, and

$$W_\omega(\sum_{n=1}^{m} \lambda_n e_n) = \bigotimes_{n=1}^{m} W_S(\lambda_n) \otimes \mathbf{1}$$

for $\lambda_1, \dots, \lambda_m \in \mathbf{C}$.

Next we need more information about ξ_ω. Let ω_m be the restriction of ω to $\mathscr{A}(\mathbf{C}^m)$. From

$$\omega_m(w(\sum_{n=1}^{m} \lambda_n e_n)) = \prod_{n=1}^{m} \langle W_S(\lambda_n)\psi_n, \psi_n \rangle = \langle (\bigotimes_{n=1}^{m} W_S(\lambda_n)) \bigotimes_{n=1}^{m} \psi_n, \bigotimes_{n=1}^{m} \psi_n \rangle$$

for $\lambda_1, \dots, \lambda_m \in \mathbf{C}$ we conclude that the GNS-representation for ω_m is equal to

$$(W_S^m, \bigotimes_{n=1}^{m} L^2(\mathbf{R}), \bigotimes_{n=1}^{m} \psi_n). \tag{6}$$

So ω_m is a pure state.

Let $(\eta_\iota)_{\iota \in I}$ be an orthonormal base of $l^2(I)$. We can write ξ_ω as the sum $\sum_{\iota \in I} \varrho_\iota \otimes \eta_\iota$ where $\varrho_\iota \in \bigotimes_{n=1}^{m} L^2(\mathbf{R})$ and $\sum_{\iota \in I} \|\varrho_\iota\|^2 = 1$. For $g \in \mathbf{C}^m$ we have

$$\omega_m(w(g)) = \omega(w(g)) = \langle (W_S^m(g) \otimes \mathbf{1})(\sum_{\iota \in I} \varrho_\iota \otimes \eta_\iota), (\sum_{\iota \in I} \varrho_\iota \otimes \eta_\iota) \rangle =$$

$$= \sum_{\iota \in I} \langle W_S^m(g)\varrho_\iota, \varrho_\iota \rangle = \sum_{\iota \in I} \omega^\iota(w(g)),$$

where ω^ι is the positive linear form of $\mathscr{A}(\mathbf{C}^m)$ determined by $\omega^\iota(w(g)) = \langle W_S^m(g)\varrho_\iota, \varrho_\iota \rangle$ for $g \in \mathbf{C}^m$. Since ω_m is pure, we have $\omega^\iota = t_\iota^2 \omega_m$, where t_ι is a suitable nonnegative number. From (6) we see that there is a unitary operator U in $\bigotimes_{n=1}^{m} L^2(\mathbf{R})$ such that

$$U\varrho_\iota = t_\iota \bigotimes_{n=1}^{m} \psi_n \text{ and } U W_S^m(g) U^* = W_S^m(g) \qquad \text{for } g \in \mathbf{C}^m.$$

Since W_S^m is irreducible, $U = \gamma \mathbf{1}$, where $|\gamma| = 1$, and $\varrho_\iota = \alpha_\iota \bigotimes_{n=1}^{m} \psi_n$, where $\alpha_\iota = t_\iota \bar{\gamma}$. We get

$$\xi_\omega = \bigotimes_{n=1}^{m} \psi_n \otimes \eta \qquad \text{where } \eta := \sum_{\iota \in I} \alpha_\iota \eta_\iota. \tag{7}$$

Moreover, for $f = \sum_{n=1}^{\infty} \lambda_n e_n \in V$ we get

$$W_\omega(\sum_{n=m+1}^{\infty} \lambda_n e_n) \in \{W_S^m(g) \otimes \mathbb{1} : g \in \mathbb{C}^m\}' = \mathbb{C}\mathbb{1} \otimes \mathscr{L}(l^2(I));$$

therefore we have

$$W_\omega(\sum_{n=m+1}^{\infty} \lambda_n e_n) = \mathbb{1} \otimes W_I(\sum_{n=m+1}^{\infty} \lambda_n e_n),$$

where $W_I(\sum_{n=m+1}^{\infty} \lambda_n e_n)$ is a unitary operator in $\mathscr{L}(l^2(I))$.

It follows

$$\begin{aligned} W_\omega(\sum_{n=1}^{\infty} \lambda_n e_n) &= W_\omega(\sum_{n=1}^{m} \lambda_n e_n) \cdot W_\omega(\sum_{n=m+1}^{\infty} \lambda_n e_n) \\ &= W_S^m(\sum_{n=1}^{m} \lambda_n e_n) \otimes W_I(\sum_{n=m+1}^{\infty} \lambda_n e_n). \end{aligned}$$

(7) and $\|\eta\| = 1$ finally yield

$$\begin{aligned} |\omega(w(f))| &= |\langle(\bigotimes_{n=1}^{m} W_S(\lambda_n) \otimes W_I(\sum_{n=m+1}^{\infty} \lambda_n e_n))(\bigotimes_{n=1}^{m} \psi_n \otimes \eta), (\bigotimes_{n=1}^{m} \psi_n \otimes \eta)\rangle| \\ &\leq \prod_{n=1}^{m} |\langle W_S(\lambda_n)\psi_n, \psi_n\rangle| \cdot 1. \end{aligned}$$

∎

Proof of Theorem 3.2. Let us assume that $W: V \to \mathscr{U}(K)$ is a regular extension of W_ξ^0. Let ω be the state of $\mathscr{A}(V)$ corresponding to W and ξ (considered as an element of K). By assumption,

$$\prod_{n=1}^{\infty} |\langle W_S(t\mu_n)\psi_n, \psi_n\rangle| = 0$$

holds for every $t \in \mathbb{R} \setminus \{0\}$. Lemma 3.3 implies $\omega(w(tg)) = 0$ for $t \in \mathbb{R} \setminus \{0\}$. This is a contradiction to the regularity of W, from which $\lim_{t \to 0} \omega(w(tg)) = 1$ follows. ∎

Now we discuss some examples. First let us introduce some notation, which is used in the next section, too.

$\phi_0 \in L^2(\mathbb{R})$, defined by $\phi_0(x) = \dfrac{\sqrt{2}}{\pi^{1/4}} x \exp\left(-\dfrac{1}{2}x^2\right)$ for $x \in \mathbb{R}$

(One-particle state),

$\phi_1 \in L^2(\mathbb{R})$, defined by $\phi_1(x) = \dfrac{1}{\sqrt{2}} \chi_{[-1,1]}(x)$ for $x \in \mathbb{R}$, and

$e_i(\lambda) := \langle W_S(\lambda)\phi_i, \phi_i\rangle$ for $\lambda \in \mathbb{C}$ and $i = 0, 1$.

Corollary 3.4. *Let* $\mathbb{N} = N_0 \cup N_1$ *be a decomposition of* $\mathbb{N}$, $\beta(n) = 0$ *for* $n \in N_0$, *and* $\beta(n) = 1$ *for* $n \in N_1$. *For* $\xi := \bigotimes_{n=1}^{\infty} \phi_{\beta(n)}$ *we have*

$$V_\xi = \{ \sum_{n=1}^{\infty} \lambda_n e_n \in l^2(\mathbb{N}) : \sum_{n \in N_1} |\operatorname{Im} \lambda_n| < \infty \}.$$

If $V \supseteq V_0$ is a subspace of $l^2(\mathbb{N})$ not contained in V_ξ, W_ξ^0 cannot be extended to a regular representation over V.

Proof. It is easy to calculate that

$$e_1(\lambda) = 1 - \frac{|\operatorname{Im} \lambda|}{2} - \frac{(\operatorname{Re} \lambda)^2}{6} + o(|\lambda|^2)$$

for $|\lambda| \le 2$. It follows that for $\sum_{n=1}^{\infty} \lambda_n e_n \in l^2(\mathbb{N})$

$$\sum_{n=1}^{\infty} |e_1(\lambda_n) - 1| < \infty \Longleftrightarrow \sum_{n=1}^{\infty} |\operatorname{Im} \lambda_n| < \infty$$

holds. From

$$e_0(\lambda) = \left(1 - \frac{|\lambda|^2}{2}\right) \exp\left(-\frac{1}{4}|\lambda|^2\right)$$

we see that

$$\sum_{n=1}^{\infty} |e_0(\lambda_n) - 1| < \infty$$

is satisfied for every $\sum_{n=1}^{\infty} \lambda_n e_n \in l^2(\mathbb{N})$. This shows the equation for V_ξ; applying Theorem 3.2 and observing that $e_0(\lambda) \ge 0$ and $e_1(\lambda) \ge 0$ if λ has a sufficiently small modulus, we get the assertion. ∎

§4. An Example, Part 1

In this Section we form the direct integral of irreducible regular representations of the CCR, which are defined over proper subspaces of $l^2(\mathbb{N})$ and which cannot be extended to regular representations over $l^2(\mathbb{N})$, and get a cyclic regular representation W_Y of the CCR over $l^2(\mathbb{N})$.

Notation.

$Y := \{0, 1\}^{\mathbb{N}} = \{(\alpha_n)_{n \in \mathbb{N}} : \alpha_n \in \{0, 1\} \text{ for } n \in \mathbb{N}\}$ is a metrizable compact abelian group; let μ_Y be the normalized Haar measure of Y.

In this Section we identify $\mathbb{N}$ with the disjoint union $\bigcup_{j=1}^{\infty} \{0, 1\}^j$ by using an arbitrary but fixed bijection.

Let us introduce some further notation used in the following. Let k be an element of $\mathbb{N}$.

$\alpha^k := (\alpha_1, \dots, \alpha_k) \in \{0, 1\}^k \ (\subseteq \mathbb{N})$ for $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in Y$,

$l^k \quad := (l_1, \ldots, l_k) \in \{0, 1\}^k \ (\subseteq \mathbb{N})$ for $j \geq k$ and $l = (l_1, \ldots, l_j) \in \{0, 1\}^j$,

$I_k \quad := \bigcup_{j=1}^{k} \{0, 1\}^j \ (\subseteq \mathbb{N})$,

$I_k^n \quad := I_k \backslash \{n^1, \ldots, n^k\}$ for $n \in \{0, 1\}^k$,

$\mathbb{N}_k := \mathbb{N} \backslash \{0, 1\}^k$,

$C_n^{(k)} := \{\alpha \in Y: \alpha^k = n\}$ for $n \in \{0, 1\}^k$,

$G_{n,j}^{(k)} := \{l \in \{0, 1\}^j: l^k = n\}$ for $j \geq k$ and $n \in \{0, 1\}^k$,

$\delta(m, \alpha) := \begin{cases} 1, \text{ if } m = \alpha^k \text{ for a suitable } k \in \mathbb{N} \\ 0 \text{ otherwise} \end{cases}$ for $\alpha \in Y$ and $m \in \mathbb{N}$,

$\delta(m, n) := \begin{cases} 1 \text{ for } m = n \\ 0 \text{ otherwise} \end{cases}$ for $m, n \in \{0, 1\}^k$,

$\xi_\alpha \quad := \bigotimes_{n=1}^{\infty} \phi_{\delta(n,\alpha)}$ for $\alpha \in Y$ (Definition of ϕ_0 and ϕ_1 before 3.4),

$W_\alpha \quad := W_{\xi_\alpha}, \ W_\alpha^0 := W_{\xi_\alpha}^0, \ V_\alpha := V_{\xi_\alpha}, \ H_\alpha := H_{\xi_\alpha}$.

It is $\mu_Y(C_n^{(k)}) = 1/2^k$.

§3 implies that for $\alpha \in Y$ $W_\alpha: V_\alpha \to \mathcal{U}(H_\alpha)$ is an irreducible regular representation of the CCR over

$$V_\alpha = \{ \sum_{n=1}^{\infty} \lambda_n e_n \in l^2(\mathbb{N}): \sum_{j=1}^{\infty} |\operatorname{Im} \lambda_{\alpha^j}| < \infty \},$$

and there is no regular representation of the CCR over $l^2(\mathbb{N})$ extending W_α.

Lemma 4.1. *For every $f \in l^2(\mathbb{N})$ $N_f := \{\alpha \in Y: f \notin V_\alpha\}$ is a μ_Y-negligible set.*

Proof. $N_f = \emptyset$ for $f = 0$; from now on let $f = \sum_{n=1}^{\infty} \lambda_n e_n \neq 0$. For $k \in \mathbb{N}$ we have

$$N_f = \{\alpha \in Y: \sum_{j=1}^{\infty} |\operatorname{Im} \lambda_{\alpha^j}| = \infty\} \subseteq \bigcup_{j=k}^{\infty} M_j,$$

where $M_j := \{\alpha \in Y: |\lambda_{\alpha^j}| \geq \|f\|/j^2\}$. It follows

$$N_f \subseteq \bigcap_{k=1}^{\infty} (\bigcup_{j=k}^{\infty} M_j). \tag{8}$$

It is

$$M_j = \bigcup_{n \in J_j} C_n^{(j)} \text{ where } J_j := \left\{ n \in \{0, 1\}^j: |\lambda_n| \geq \frac{\|f\|}{j^2} \right\}.$$

Therefore, $\mu_Y(M_j) = |J_j| \cdot 1/2^j$. From

$$\frac{|J_j|\cdot\|f\|^2}{j^4}\le\sum_{n\in J_j}|\lambda_n|^2\le\|f\|^2$$

we get $|J_j|\le j^4$ and $\mu_Y(M_j)\le j^4/2^j$. This implies

$$\mu_Y(\bigcap_{k=1}^{\infty}(\bigcup_{j=k}^{\infty}M_j))=\inf_{k\in\mathbf{N}}\mu_Y(\bigcup_{j=k}^{\infty}M_j)\le\inf_{k\in\mathbf{N}}\sum_{j=k}^{\infty}\frac{j^4}{2^j}=0;$$

from (8) we receive the assertion. ■

We put

$$H_Y:=\int_Y^{\oplus}H_\alpha d\alpha,$$

where a direct integral in the sense of J. Dixmier is used. For $f\in V_0$, $(W_\alpha(f))_{\alpha\in Y}$ is a measurable field of operators. For $f=\sum_{n=1}^{\infty}\lambda_n e_n\in l^2(\mathbf{N})$ and $\alpha\in Y\backslash N_f$, $(W_\alpha(\sum_{n=1}^{m}\lambda_n e_n))_{m\in\mathbf{N}}$ converges strongly to $W_\alpha(f)$ (see [11], Lemma 5); so $(W_\alpha(f))_{\alpha\in Y}$ is an almost everywhere defined measurable field of operators, and by

$$W_Y: l^2(\mathbf{N})\longrightarrow\mathscr{U}(H_Y),\ W_Y(f)=\int_Y^{\oplus}W_\alpha(f)\,d\alpha,$$

a regular representation of the CCR over $l^2(\mathbf{N})$ is given.

Let W_Y^0 be the restriction of W_Y to V_0.

Proposition 4.2. *(i)* $\xi_Y:=\int_Y^{\oplus}\xi_\alpha d\alpha$ *is a cyclic vector for* W_Y^0 *and* W_Y.

(ii) For the diagonal algebra $\mathscr{D}$ *in* $H_Y=\int_Y^{\oplus}H_\alpha d\alpha$,

$$\mathscr{D}=\{W_Y(f): f\in l^2(\mathbf{N})\}'=\{W_Y^0(f): f\in V_0\}'$$

holds.

We denote the state of $\mathscr{A}(l^2(\mathbf{N}))$ ($\mathscr{A}(V_0)$, resp.), corresponding to W_Y (W_Y^0, resp.) and ξ_Y, by ω_Y (ω_Y^0, resp.).

Proof. (i) Since $\int_Y^{\oplus}\xi_\alpha d\alpha$ is a separating vector for $\mathscr{D}$, (i) follows from (ii).

(ii) In several steps:

(a) It suffices to show

$$\int_Y^{\oplus}\chi_{C_m^{(k)}}(\alpha)\mathbf{1}\,d\alpha\in\{W_Y^0(f): f\in V_0\}'' \tag{9}$$

for $k\in\mathbf{N}$ and $m\in\{0,1\}^k$.

For: Since $\mathscr{D}$ is generated by

$$\left\{\int_Y^{\oplus} \chi_{C_m^{(k)}}(\alpha)\mathbb{1}\,d\alpha\colon k\in\mathbb{N},\ m\in\{0,1\}^k\right\},$$

$\mathscr{D}\subseteq\{W_Y^0(f)\colon f\in V_0\}''$ is satisfied. Since $\mathscr{D}\subseteq\{W_Y^0(f)\colon f\in V_0\}'$, we get

$$\mathscr{D}\subseteq\{W_Y^0(f)\colon f\in V_0\}'\cap\{W_Y^0(f)\colon f\in V_0\}''.$$

Since in the decomposition $W_Y^0=\int_Y^{\oplus} W_\alpha^0\,d\alpha$ of W_Y^0 the fibers W_α^0 are irreducible and regular, the diagonal algebra $\mathscr{D}$ is a maximal abelian von Neumann subalgebra of $\{W_Y^0(f)\colon f\in V_0\}'$ (see [9], Lemma 1.2). Now it is easy to conclude that

$$\mathscr{D}=\{W_Y^0(f)\colon f\in V_0\}'\cap\{W_Y^0(f)\colon f\in V_0\}''=\{W_Y^0(f)\colon f\in V_0\}'.$$

The considerations after the proof of Lemma 4.1 establish that for $f=\sum_{n=1}^\infty\lambda_n e_n\in l^2(\mathbb{N})$ $(W_Y(\sum_{n=1}^m\lambda_n e_n))_{m\in\mathbb{N}}$ converges strongly to $W_Y(f)$, and thus

$$\mathscr{D}=\{W_Y^0(f)\colon f\in V_0\}'=\{W_Y(f)\colon f\in l^2(\mathbb{N})\}'$$

holds.

(b) Let $j\in\mathbb{N}$ be fixed. For $\alpha\in Y$ let $H_{\alpha,j}$ be the incomplete tensor product $\bigotimes_{n\in\mathbb{N}_j}L^2(\mathbb{R})$ distinguished by $\bigotimes_{n\in\mathbb{N}_j}\phi_{\delta(n,\alpha)}$; moreover, let $H_{Y,j}:=\int_Y^{\oplus}H_{\alpha,j}\,d\alpha$ and $L_j:=\bigotimes_{n\in\{0,1\}^j}L^2(\mathbb{R})$.

Applying [1], Proposition 11, p. 175, we find that there is a unique Hilbert space isomorphism

$$U\colon L_j\otimes H_{Y,j}\longrightarrow H_Y$$

which maps

$$\psi\otimes\int_Y^{\oplus}\eta(\alpha)\,d\alpha\quad\text{into}\quad\int_Y^{\oplus}\psi\otimes\eta(\alpha)\,d\alpha$$

for $\psi\in L_j$ and $\int_Y^{\oplus}\eta(\alpha)\,d\alpha\in H_{Y,j}$. (Identifying $L_j\otimes H_{\alpha,j}$ and H_α, we consider $\psi\otimes\eta(\alpha)$ as an element of H_α.)

Let V_j be the subspace

$$\{\sum_{n\in\mathbb{N}_j}\lambda_n e_n\in l^2(\mathbb{N})\colon \lambda_n=0 \text{ for almost every } n\in\mathbb{N}_j\}$$

of V_0. For $\alpha\in Y$ we define a regular representation $W_{\alpha,j}^0\colon V_j\to\mathscr{U}(H_{\alpha,j})$ of the CCR over V_j by

$$W_{\alpha,j}^0(\sum_{n\in\mathbb{N}_j}\lambda_n e_n)=\bigotimes_{n\in\mathbb{N}_j}W_S(\lambda_n).$$

Then by putting

$$W_{Y,j}^0(\sum_{n=1}^{\infty}\lambda_n e_n)=\bigotimes_{n\in\{0,1\}^j}W_S(\lambda_n)\otimes\int_Y^{\oplus}W_{\alpha,j}^0(\sum_{n\in\mathbb{N}_j}\lambda_n e_n)\,d\alpha$$

for $\sum_{n=1}^{\infty}\lambda_n e_n\in V_0$, we obtain a regular representation

$$W_{Y,j}^0\colon V_0\longrightarrow\mathscr{U}(L_j\otimes H_{Y,j})$$

of the CCR over V_0, which is transformed by U into W_Y^0.

(c) For $j\in\mathbb{N}$ and $l\in\{0,1\}^j$ let

$$\psi_l:=\bigotimes_{n\in\{0,1\}^j}\phi_{\delta(n,l)}\in L_j.$$

From now on, let $k\in\mathbb{N}$ be fixed.

For $m\in\{0,1\}^k$ and $j\geq k$ let $P_{mj}\in\mathscr{L}(L_j)$ denote the orthogonal projection onto

$$[\psi_l\colon l\in\{0,1\}^j,\ l^k=m];$$

moreover, let $Q_{mj}:=U(P_{mj}\otimes\mathbb{1}_{H_{Y,j}})U^*$.

Since the Schrödinger representation W_S is irreducible,

$$\mathscr{L}(L_j)=\{\bigotimes_{n\in\{0,1\}^j}W_S(\lambda_n)\colon\lambda_n\in\mathbb{C}\text{ for }n\in\{0,1\}^j\}'';$$

it follows

$$P_{mj}\otimes\mathbb{1}\in\{\bigotimes_{n\in\{0,1\}^j}W_S(\lambda_n)\colon\lambda_n\in\mathbb{C}\text{ for }n\in\{0,1\}^j\}''\otimes\mathbb{C}\mathbb{1}\subseteq\{W_{Y,j}^0(f)\colon f\in V_0\}''$$

and $Q_{mj}\in\{W_Y^0(f)\colon f\in V_0\}''$ for $j\geq k$. Now we can prove (9) by showing that $(Q_{mj})_{j\geq k}$ converges strongly to $\int_Y^{\oplus}\chi_{C_m^{(k)}}(\alpha)\mathbb{1}\,d\alpha$ and by applying the density theorem due to J. von Neumann.

(d) Since $\sup_{j\geq k}\|Q_{mj}\|=1<\infty$, it suffices to show

$$\lim_{\substack{j\to\infty\\ j\geq k}}Q_{mj}\eta=\int_Y^{\oplus}\chi_{C_m^{(k)}}(\alpha)\mathbb{1}\,d\alpha\cdot\eta$$

only for those η's that belong to a suitable generating set of the Hilbert space H_Y. Therefore we only consider the case

$$\eta = \int_Y^{\oplus} h(\alpha) \bigotimes_{n=1}^{\infty} \varrho_n(\alpha)\, d\alpha,$$

where $h \in L^\infty(Y)$ and a $j_0 \geq k$ exists such that $\varrho_n(\alpha) = \phi_{\delta(n,\alpha)}$ for $j \geq j_0$, $n \in \{0, 1\}^j$, and $\alpha \in Y$ (that means that $\bigotimes_{n \in \{0,1\}^j} \varrho_n(\alpha) = \psi_l$ for $j \geq j_0$, $l \in \{0, 1\}^j$, and $\alpha \in C_l^{(j)}$). For $j \geq j_0$ we get

$$\begin{aligned}
Q_{mj}\eta &= (\text{identifying } H_\alpha \text{ and } L_j \otimes H_{\alpha,j}) \\
&= Q_{mj} \int_Y^{\oplus} \sum_{l \in \{0,1\}^j} \chi_{C_l^{(j)}}(\alpha) h(\alpha) \psi_l \otimes \bigotimes_{n \in \mathbb{N}_j} \varrho_n(\alpha)\, d\alpha = \\
&= Q_{mj} U^* \sum_{l \in \{0,1\}^j} \psi_l \otimes \int_Y^{\oplus} \chi_{C_l^{(j)}}(\alpha) h(\alpha) \bigotimes_{n \in \mathbb{N}_j} \varrho_n(\alpha)\, d\alpha = \\
&= U \sum_{l \in \{0,1\}^j} P_{mj} \psi_l \otimes \int_Y^{\oplus} \chi_{C_l^{(j)}}(\alpha) h(\alpha) \bigotimes_{n \in \mathbb{N}_j} \varrho_n(\alpha)\, d\alpha = (*) \\
&= U \sum_{l \in G_{m,j}^{(k)}} \psi_l \otimes \int_Y^{\oplus} \chi_{C_l^{(j)}}(\alpha) h(\alpha) \bigotimes_{n \in \mathbb{N}_j} \varrho_n(\alpha)\, d\alpha = \\
&= \int_Y^{\oplus} \sum_{l \in G_{m,j}^{(k)}} \chi_{C_l^{(j)}}(\alpha)\, h(\alpha) \bigotimes_{n=1}^{\infty} \varrho_n(\alpha)\, d\alpha = \int_Y^{\oplus} \chi_{C_m^{(k)}}(\alpha)\, \mathbb{1}\, d\alpha \cdot \eta\,;
\end{aligned}$$

in $(*)$ we used the fact that $\langle \phi_1, \phi_2 \rangle = 0$, and therefore $P_{mj}\psi_l = 0$ for $l^k \neq m$. ∎

Remark 4.3. It is not difficult to show that $f \in l^2(\mathbb{N}) \mapsto W_Y(f) \in \mathscr{U}(H_Y)$ is even strongly continuous (see [10] for a proof).

§5. Insertion about Decompositions with the Same Diagonal Algebra

Let $\mathscr{A}$ be a unital C^*-algebra, ω a state of $\mathscr{A}$, and $\mathscr{C}$ an abelian von Neumann subalgebra of $\pi_\omega(\mathscr{A})'$. Moreover, let ν be the orthogonal measure in $\mathscr{S}(\mathscr{A})$ associated with ω and $\mathscr{C}$.

5.1 Theorem. *Let*

$$\pi_\omega = \int_Z^{\oplus} \pi(\zeta)\, d\mu(\zeta)$$

be a (with respect to ξ_ω) normalized decomposition of π_ω into representations $\pi(\zeta)\colon \mathscr{A} \to \mathscr{L}(H(\zeta))$ of $\mathscr{A}$ such that $\mathscr{C}$ is transformed into the diagonal algebra of the direct integral $\int_Z^{\oplus} H(\zeta)\, d\mu(\zeta)$ (in the sense of Definition 2.1). Let $\xi_\omega = \int_Z^{\oplus} \xi(\zeta)\, d\mu(\zeta)$ be the corresponding decomposition of ξ_ω, for $\zeta \in Z$ let $\varphi(\zeta)$ be the state of $\mathscr{A}$ belonging to $\pi(\zeta)$ and $\xi(\zeta)$. Moreover, let Σ be the σ-field, on which *the measure* μ *is defined, let* ν_0 *be the restriction of* ν *to a Baire measure on* $\mathscr{S}(\mathscr{A})$, *and* $\mathscr{B}_0$ *the* σ*-field of Baire sets on* $\mathscr{S}(\mathscr{A})$.

Then $T\colon Z \to \mathscr{S}(\mathscr{A})$, $\zeta \mapsto \varphi(\zeta)$, *is* $\Sigma\text{–}\mathscr{B}_0$*-measurable, and* ν_0 *is the image of* μ *under* T.

Proof. In several steps:

(a) Let U be the Hilbert space isomorphism which is composed of both the canonical isomorphisms, transforming $\int_{\mathscr{S}(\mathscr{A})}^{\oplus} H_\varphi d\nu(\varphi)$ into H_ω and H_ω into $\int_Z^{\oplus} H(\zeta)\, d\mu(\zeta)$, resp.. For $a \in \mathscr{A}$,

$$U \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \varphi(a)\mathbb{1}\, d\nu(\varphi) U^* = \int_Z^{\oplus} \varphi(\zeta)(a)\mathbb{1}\, d\mu(\zeta) \tag{10}$$

holds.

For: Let P be the orthogonal projection onto $[\mathscr{C}\xi_\omega]$ in H_ω. For $C \in \mathscr{C}'$ there is a unique operator $\Phi(C) \in \mathscr{C}$ such that $PCP = \Phi(C)P$ (cf. [6], §2.). In order to get the assertion (10) we will show that for $a \in \mathscr{A}$

$$\Phi(\pi_\omega(a)) = \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \varphi(a)\mathbb{1}\, d\nu(\varphi), \tag{11}$$

if we identify H_ω and $\int_{\mathscr{S}(\mathscr{A})}^{\oplus} H_\varphi d\nu(\varphi)$, and that

$$\Phi(\pi_\omega(a)) = \int_Z^{\oplus} \varphi(\zeta)(a)\mathbb{1}\, d\nu(\zeta), \tag{12}$$

if we identify H_ω and $\int_Z^{\oplus} H(\zeta)\, d\mu(\zeta)$. It suffices to establish (12); (11) follows in the same way. So we have to prove

$$P\pi_\omega(a)\eta = \int_Z^{\oplus} \varphi(\zeta)(a)\mathbb{1}\, d\mu(\zeta)\cdot\eta \qquad \text{for every } \eta \in \mathscr{C}\xi_\omega.$$

We can easily check this relation after having shown that

$$P = \int_Z^{\oplus} P(\zeta)\, d\mu(\zeta),$$

where $P(\zeta)$ is the projection onto $\mathbf{C}\,\xi(\zeta)$ for $\zeta \in Z$.

Obviously, $\int_Z^{\oplus} P(\zeta)\, d\mu(\zeta)$ is well defined, $\left(\int_Z^{\oplus} P(\zeta)\, d\mu(\zeta)\right)(H_\omega) \subseteq [\mathscr{C}\xi_\omega]$, and

$\int_Z^{\oplus} P(\zeta)\,d\mu(\zeta)\cdot\eta = \eta$ for $\eta\in\mathscr{C}\xi_\omega$. It remains to prove that $\int_Z^{\oplus} P(\zeta)\,d\mu(\zeta)\cdot\eta\in(\mathscr{C}\xi_\omega)^\perp$ holds for $\eta = \int_Z^{\oplus}\eta(\zeta)\,d\mu(\zeta)\in(\mathscr{C}\xi_\omega)^\perp$. We get this from

$$\left\langle \int_Z^{\oplus} P(\zeta)\,d\mu(\zeta)\cdot\eta, \int_Z^{\oplus} h(\zeta)\xi(\zeta)\,d\mu(\zeta)\right\rangle =$$

$$= \int_Z \langle\langle\eta(\zeta),\, \xi(\zeta)\rangle\,\xi(\zeta),\, h(\zeta)\xi(\zeta)\rangle\,d\mu(\zeta) =$$

$$= \left\langle \int_Z^{\oplus}\eta(\zeta)\,d\mu(\zeta), \int_Z^{\oplus} h(\zeta)\xi(\zeta)\,d\mu(\zeta)\right\rangle = 0$$

for $h\in L^\infty(Z,\mu)$.

(b) For $a\in\mathscr{A}_+$ and an interval I in $\mathbb{R}_0^+$ we define $B_{a,I} := \{\varphi\in\mathscr{S}(\mathscr{A})\colon \varphi(a)\in I\}$ and $C_{a,I} := T^{-1}(B_{a,I}) = \{\zeta\in Z\colon \varphi(\zeta)(a)\in I\}$. For $I = [0, s[\,(s>0)$, $C_{a,I}$ is μ-measurable, and

$$U\int_{\mathscr{S}(\mathscr{A})}^{\oplus}\chi_{B_{a,I}}(\varphi)\,\mathbb{1}\,d\nu(\varphi)\,U^* = \int_Z^{\oplus}\chi_{C_{a,I}}(\zeta)\,\mathbb{1}\,d\mu(\zeta). \tag{13}$$

For: Since $\int_{\mathscr{S}(\mathscr{A})}^{\oplus}\chi_{B_{a,I}}(\varphi)\,\mathbb{1}\,d\nu(\varphi)$ is a projection, there is a set $D_{a,I}\in\Sigma$ satisfying (13) where $C_{a,I}$ is replaced by $D_{a,I}$. From (a) we get

$$U\int_{\mathscr{S}(\mathscr{A})}^{\oplus}\chi_{B_{a,I}}(\varphi)(s-\varphi(a))\,\mathbb{1}\,d\nu(\varphi)\,U^* = \int_Z^{\oplus}\chi_{D_{a,I}}(\zeta)(s-\varphi(\zeta)(a))\,\mathbb{1}\,d\mu(\zeta).$$

The operator on the left is positive, therefore the operator on the right is positive, too. We obtain

$$\varphi(\zeta)(a)\le s \text{ for almost every } \zeta\in D_{a,I}.$$

Since the operator

$$\int_{\mathscr{S}(\mathscr{A})}^{\oplus}(1-\chi_{B_{a,I}}(\varphi))(\varphi(a)-s)\,\mathbb{1}\,d\nu(\varphi)$$

is positive, we conclude as before that $\varphi(\zeta)(a)\ge s$ for almost every $\zeta\in Z\setminus D_{a,I}$. After a modification on a negligible set we get

$$C_{a,[0,s[}\subseteq D_{a,I}\subseteq C_{a,[0,s]}. \tag{14}$$

For $n\in\mathbb{N}$ let $I_n := [0, s(1-1/n)[$. We have

$$U\int_{\mathscr{S}(\mathscr{A})}^{\oplus}\chi_{B_{a,I}}(\varphi)\,\mathbb{1}\,d\nu(\varphi)\,U^* = U\left(\sup_{n\in\mathbb{N}}\int_{\mathscr{S}(\mathscr{A})}^{\oplus}\chi_{B_{a,I_n}}(\varphi)\,\mathbb{1}\,d\nu(\varphi)\right)U^* =$$

$$= \sup_{n\in\mathbb{N}} \int_Z^{\oplus} \chi_{D_{a,I_n}}(\zeta)\,\mathbb{1}\,d\mu(\zeta). \tag{15}$$

From (14)(applied to I_n), $C_{a,I} = \bigcup_{n=1}^{\infty} D_{a,I_n}$ follows. Thus $C_{a,I}$ is μ-measurable, and

$$\sup_{n\in\mathbb{N}} \int_Z^{\oplus} \chi_{D_{a,I_n}}(\zeta)\,\mathbb{1}\,d\mu(\zeta) = \int_Z^{\oplus} \chi_{C_{a,I}}(\zeta)\,\mathbb{1}\,d\mu(\zeta).$$

From this and (15) the assertion follows.

(c) So

$$U \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \chi_E(\varphi)\,\mathbb{1}\,d\nu(\varphi)\,U^* = \int_Z^{\oplus} \chi_{T^{-1}(E)}(\zeta)\,\mathbb{1}\,d\mu(\zeta) \tag{16}$$

holds for every $E = B_{a,[0,s[}$. Similar arguments establish (16) for $E = B_{a,[0,s]}$ $(s > 0)$. Now it is easy to see that (16) is satisfied for every

$$E \in \mathscr{E} := \{ \bigcap_{n=1}^{m} B_{a_n,I_n} : m\in\mathbb{N},\ a_n \in \mathscr{A}_+,\ I_n \text{ a relatively open interval in } \mathbb{R}_0^+,\ n = 1,\dots,m.\}.$$

Thus for $E\in\mathscr{E}$ we obtain

$$\begin{aligned}
\nu(E) &= \left\langle \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \chi_E(\varphi)\,\mathbb{1}\,d\nu(\varphi) \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \xi_\varphi d\nu(\varphi),\ \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \xi_\varphi d\nu(\varphi) \right\rangle \\
&= \left\langle \int_{\mathscr{S}(\mathscr{A})}^{\oplus} \chi_E(\varphi)\,\mathbb{1}\,d\nu(\varphi) U^* \int_Z^{\oplus} \xi(\zeta) d\mu(\zeta),\ U^* \int_Z^{\oplus} \xi(\zeta)\,d\mu(\zeta) \right\rangle \\
&= \left\langle \int_Z^{\oplus} \chi_{T^{-1}(E)}(\zeta)\,\mathbb{1}\,d\mu(\zeta) \int_Z^{\oplus} \xi(\zeta) d\mu(\zeta),\ \int_Z^{\oplus} \xi(\zeta)\,d\mu(\zeta) \right\rangle = \mu(T^{-1}(E)).
\end{aligned}$$

Since the σ-field $\mathscr{B}_0$ is generated by $\mathscr{E}$, and $\mathscr{E}$ is closed with respect to finite intersections, $\nu(B) = \mu(T^{-1}(B))$ holds for every $B\in\mathscr{B}_0$. □

We mainly need the following corollary.

Corollary 5.2. *Additionally, let us suppose that $\mathscr{A}$ is a separable C^*-algebra. Let*

$$\pi_\omega = \int_Z^{\oplus} \pi(\zeta)\,d\mu(\zeta) \quad \textit{and} \quad \pi_\omega = \int_X^{\oplus} \rho(\beta)\,d\lambda(\beta)$$

be two decompositions of π_ω with diagonal algebra $\mathscr{C}$, and $H_\omega = \int_Z^{\oplus} H(\zeta)\,d\mu(\zeta)$ and $H_\omega = \int_X^{\oplus} K(\beta)\,d\lambda(\beta)$, resp., the corresponding decompositions of H_ω. Moreover, *let the first decomposition of* π_ω *be normalized with respect to* ξ_ω; *let* $\int_X^\oplus K(\beta)\,d\lambda(\beta)$ *be a direct integral in the sense of J. Dixmier, let* X *be a standard Borel space and* λ *a* σ*-finite measure on* X.

Then for almost every $\zeta \in Z$ *there is a* $\beta \in X$ *such that* $\pi(\zeta)$ *and* $\rho(\beta)$ *are unitarily equivalent.*

Remarks. (i) From the assumptions above it follows that $\int_Z^\oplus H(\zeta)\,d\mu(\zeta)$ is a direct integral in the sense of J. Dixmier.

(ii) If Z is a standard Borel space, too, the corollary is an immediate conclusion from [2], Proposition 8.2.4, but we are interested in the general case.

Proof. Let ν be the orthogonal measure in $\mathscr{S}(\mathscr{A})$ associated with ω and $\mathscr{C}$, and let $\pi_\omega = \int_{\mathscr{S}(\mathscr{A})}^\oplus \pi_\varphi d\nu(\varphi)$ the corresponding decomposition of π_ω. Since $\mathscr{S}(\mathscr{A})$ is compact and metrizable, $\mathscr{S}(\mathscr{A})$ is a standard Borel space, and the direct integral can be interpreted in the sense of J. Dixmier. Let us compare this decomposition of π_ω with $\pi_\omega = \int_X^\oplus \rho(\beta)\,d\lambda(\beta)$. The Proposition 8.2.4 in [2] mentioned above implies that there are a ν-negligible set N, a λ-negligible set N_1, and a Borel isomorphism $S: \mathscr{S}(\mathscr{A})\backslash N \to X\backslash N_1$ such that π_φ is unitarily equivalent to $\rho(S(\varphi))$ for $\varphi \in \mathscr{S}(\mathscr{A})\backslash N$. Since the set of Baire sets in $\mathscr{S}(\mathscr{A})$ is equal to the set of Borel sets, from Theorem 5.1 we get that $T^{-1}(\mathscr{S}(\mathscr{A})\backslash N)$ is μ-measurable, $\mu(T^{-1}(\mathscr{S}(\mathscr{A})\backslash N)) = 1$, and that for $\zeta \in T^{-1}(\mathscr{S}(\mathscr{A})\backslash N)$ $\pi(\zeta)$ is unitarily equivalent to $\pi_{\varphi(\zeta)}$ and therefore to $\rho(S(\varphi(\zeta)))$. ∎

§6. The Example, Part 2

Proposition 6.1. *There is no decomposition*

$$W_Y = \int_Z^\Gamma W(\zeta)\,d\mu(\zeta)$$

of W_Y *into regular representations* $W(\zeta)$ *of the CCR which are defined over* $l^2(\mathbb{N})$ *such that the diagonal algebra of the direct integral is maximal abelian in* $\{W_Y(f): f \in l^2(\mathbb{N})\}'$.

Proof. In several steps:

(a) Let us suppose that

$$W_Y = \int_Z^\Gamma W(\zeta)\,d\mu(\zeta)$$

is a maximal abelian decomposition of W_Y into regular representations $W(\zeta): l^2(\mathbb{N}) \to \mathscr{U}(H(\zeta))$.

Let $\pi_Y(\pi(\zeta)$, resp.) be the representation of $\mathscr{A}(l^2(\mathbb{N}))$ associated with W_Y ($W(\zeta)$, resp.). Let $\mathscr{A}(V_0, \mathbf{Q})$ be the C^*-subalgebra of $\mathscr{A}(l^2(\mathbb{N}))$ generated by

$$\{w(\sum_{n=1}^{m} \lambda_n e_n): m \in \mathbb{N}, \lambda_n \in \mathbf{Q} + i\mathbf{Q}, n = 1, \ldots, m\};$$

obviously $\mathscr{A}(V_0, \mathbf{Q})$ is separable. Let $\pi_{Y,\mathbf{Q}}(\pi(\zeta)_{\mathbf{Q}}$, resp.) be the restriction of π_Y ($\pi(\zeta)$, resp.) on $\mathscr{A}(V_0, \mathbf{Q})$. We obtain the decomposition

$$\pi_{Y,\mathbf{Q}} = \int_Z^{\Gamma} \pi(\zeta)_{\mathbf{Q}} \, d\mu(\zeta) \tag{17}$$

of $\pi_{Y,\mathbf{Q}}$. Since

$$\pi_{Y,\mathbf{Q}}(\mathscr{A}(V_0, \mathbf{Q}))'' = \{W_Y^0(f): f \in V_0\}'' = \{W_Y(f): f \in l^2(\mathbb{N})\}'',$$

ξ_Y is a cyclic vector for $\pi_{Y,\mathbf{Q}}$, and the diagonal algebra in $\int_Z^{\Gamma} H(\zeta) \, d\mu(\zeta)$ is $\pi_{Y,\mathbf{Q}}(\mathscr{A}(V_0, \mathbf{Q}))'$ (see Proposition 4.2(ii)).

(b) Our next intension is to normalize the decomposition (17).

Let $\xi_Y = \int_Z^{\Gamma} \xi(\zeta) \, d\mu(\zeta)$ be the corresponding decomposition of ξ_Y. For $\zeta \in Z$ let $K(\zeta) := [\pi(\zeta)_{\mathbf{Q}}(\mathscr{A}(V_0, \mathbf{Q})) \xi(\zeta)]$. Since ξ_Y is cyclic for $\pi_{Y,\mathbf{Q}}$, almost every $\eta(\zeta)$ belongs to $K(\zeta)$ for every $\eta = (\eta(\zeta))_{\zeta \in Z} \in \Gamma$. Thus $\int_Z^{\Gamma} H(\zeta) \, d\mu(\zeta)$ is isomorphic to $\int_Z^{\tilde{\Gamma}} K(\zeta) \, d\mu(\zeta)$ in a canonical way, where $\tilde{\Gamma} = \Gamma \cap \prod_{\zeta \in Z} K(\zeta)$ is the set of square integrable vector fields in $\int_Z^{\tilde{\Gamma}} K(\zeta) \, d\mu(\zeta)$.

We can assume that all the $K(\zeta)$'s are not equal to zero. (Otherwise we remove every $\zeta \in Z$ satisfying $K(\zeta) = 0$ from Z.) Instead of μ we can use a suitable measure $\tilde{\mu}$ equivalent to μ, and transform $\int_Z^{\tilde{\Gamma}} K(\zeta) d\mu(\zeta)$ into $\int_Z^{\Delta} K(\zeta) d\tilde{\mu}(\zeta)$ such that in the corresponding decomposition

$$\xi_Y = \int_Z^{\Delta} \tilde{\xi}(\zeta) \, d\tilde{\mu}(\zeta)$$

of ξ_Y every $\tilde{\xi}(\zeta)$ is equal to $\dfrac{\xi(\zeta)}{\|\xi(\zeta)\|}$. (For details see [10].)

In this way we obtain the decomposition

$$\pi_{Y,\mathbb{Q}} = \int_Z^{\Delta} \tilde{\pi}(\zeta)_{\mathbb{Q}}\, d\mu(\zeta) \tag{18}$$

of $\pi_{Y,\mathbb{Q}}$, where $\tilde{\pi}(\zeta)_{\mathbb{Q}}$ denotes the restriction of $\pi(\zeta)_{\mathbb{Q}}$ on $K(\zeta)$. It is easy to show that the diagonal algebra is $\pi_{Y,\mathbb{Q}}(\mathscr{A}(V_0, \mathbb{Q}))'$.

(c) For $\alpha \in Y$ let π_α be the representation of $\mathscr{A}(V_\alpha)$ corresponding to W_α and $\pi_{\alpha,\mathbb{Q}}$ the restriction of π_α on $\mathscr{A}(V_0, \mathbb{Q})$. It is easy to recognize that

$$\pi_{Y,\mathbb{Q}} = \int_Y^{\oplus} \pi_{\alpha,\mathbb{Q}}\, d\alpha \tag{19}$$

is a decomposition of $\pi_{Y,\mathbb{Q}}$ such that $\pi_{Y,\mathbb{Q}}(\mathscr{A}(V_0, \mathbb{Q}))'$ is the diagonal algebra in the direct integral. Let us compare the decompositions (18) and (19): Corollary 5.2 implies that almost every $\tilde{\pi}(\zeta)_{\mathbb{Q}}$ is unitarily equivalent to a representation $\pi_{\alpha,\mathbb{Q}}$. This is a contradiction, since $\pi_{\alpha,\mathbb{Q}}$ cannot be extended to a regular representation of $\mathscr{A}(l^2(\mathbb{N}))$. □

Proposition 6.1 suggests that W_Y cannot be decomposed into irreducible regular representations of the CCR over $l^2(\mathbb{N})$. For example, our result shows that it is not possible to extend the representations of the fibers appearing in the decomposition of Theorem 2.3 to irreducible regular representations over $l^2(\mathbb{N})$, and so to get a decomposition of W_Y into irreducible regular representations of the CCR over $l^2(\mathbb{N})$.

But there remains the question whether there is a decomposition of W_Y into irreducible regular representations of the CCR over $l^2(\mathbb{N})$ such that the diagonal algebra is not maximal abelian in $W_Y(l^2(\mathbb{N}))'$.

We cannot exclude this possibility completely for the following reason: In contrast to the case of separable C^*-algebras there is a representation π of a nonseparable C^*-algebra $\mathscr{A}$ possessing a decomposition into irreducible fibers such that the diagonal algebra is not maximal abelian in $\pi(\mathscr{A})'$. A still unpublished example of such a kind was constructed by R.W. Henrichs. Is it possible that this phenomen also appears in the case of the representation π_Y of $\mathscr{A}(l^2(\mathbb{N}))$?

Remark 6.2. It can be shown (see [10], Proposition 3.4.2) that there is no decomposition fo W_Y into factorial representations $W(\zeta)$ of the CCR over $l^2(\mathbb{N})$, for which

$$f \in l^2(\mathbb{N}) \longrightarrow W(\zeta)(f)$$

is strongly continuous. So it is impossible to transfer the strong continuity of W_Y (see Remark 4.3) to the representations of the fibers.

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