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On Multi-valued Analytic Solutions of First Order Non-linear Cauchy Problems

Dedicated to Professor Shigetake Matsuura on the sixtieth anniversary of his birthday

By

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Introduction

In this article we study the first order non-linear Cauchy problems of the form

$$(1)\qquad \begin{cases} F(x\,;\,du(x),\,u(x))=0 \\ u|_S=\phi \end{cases}$$

where (1) are defined in a complex domain M in C^n, $n\geqq 2$. Our aims are to find analytic solutions of (1) multi-valued in general, which ramify around a *fixed point* x^0 in M, and to calculate their ramification degrees there.

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We always consider (1) in the following situation (2):

(2)
a) $F(x;\xi,z)$ is holomorphic in an open neighborhood of a *fixed base point*

$$e^0=(x^0;\xi^0,z^0)\in J^1M\cap F^{-1}(0)$$

where we denote by $J^1M=\bigcup_{x\in M}J^1_xM$ the first order complex jet bundle over M, which can be identified to the product space $T^*M\times\boldsymbol{C}$.

b) S is a non-singular complex hypersurface of M passing through the point $x^0\in M$. S is defined by a holomorphic germ $s\in\mathcal{O}_{M,x^0}$ locally at x^0, that is, $S=\{x\in M;\ s(x)=0\}$ near x^0.

c) $\phi\in\mathcal{O}_{S,x^0}$ is a holomorphic Cauchy data on S at x^0 satisfying

$$(d\phi(x^0),\ \phi(x^0))=(\iota^*_{x^0}\xi^0,\ z^0).$$

In c) of (2), $\iota^*: T^*M|_S\to T^*S$ denotes the dual bundle map of the injective tangent map $\iota_*: TS\to TM|_S$ induced by the inclusion map $\iota: S\hookrightarrow M$.

We assume the following three conditions [A.1], [A.2] and [A.3]:

The *first condition* is

$$\text{[A.1]}\qquad \sum_{j=1}^n|\partial_{\xi_j}F(e^0)|\neq 0$$

where $(\xi_1,\cdots,\xi_n)$ is the dual coordinate system of a local coordinate system $(x_1,\cdots,x_n)$ of M around the point $x^0\in M$. We note that $(x_1,\cdots,x_n;\xi_1,\cdots\xi_n,z)$ forms a local coordinate system of J^1M around $e^0=(x^0;\xi^0,z^0)$. We remark that the condition [A.1] is independent of a choice of local coordinate systems.

The *second condition* is

[A.2] The function $\tau\mapsto F(x^0;\tau ds(x^0)+\xi^0,z^0)$ of the one variable τ vanishes with a finite vanishing order $p\geqq 1$ at $\tau=0$.

We note that the special case $p=1$ is nothing but the case the following condition holds:

$$0\neq\partial_\tau\{F(x^0;\tau ds(x^0)+\xi^0,z^0)\}|_{\tau=0}=\sum_{j=1}^n\partial_{\xi_j}F(e^0)\partial_{x_j}s(x^0).\tag{3}$$

We call S is *non-characteristic for F micro-locally at e^0*, if the condition (3) holds. Thus our condition [A.2] involves the non-characteristic case.

The *third condition* is, roughly speaking, stated as the following form:

[A.3] There exists a holomorphic approximate solution $\Phi\in\mathcal{O}_{M,x^0}$ of the Cauchy problem (1) such that Φ has several "good" properties.

These "good" properties of Φ in [A.3] are stated by means of the Newton polygon of the function

$$f^\Phi(y,\tau):=F(y;\tau ds(y)+d\Phi(y),\ \Phi(y))\in\mathcal{O}_{S\times C,(x^0,0)}.$$

In this article such an approximate solution Φ with "good" properties is called by the name of a *good extension* of the Cauchy data ϕ.

This naming comes from the following definition: We call a germ $\Phi \in \mathcal{O}_{M,x^0}$ an *approximate solution of* (1) *at* e^0 *of the approximation order* $k \in \boldsymbol{N} \cup \{\infty\}$ if

$$
\text{(4)} \qquad \begin{cases} \operatorname{ord}_{x^0}[F(x\,;\, d\Phi(x),\, \Phi(x))] = k \\ \Phi|_S = \phi \quad (\Phi \text{ is an extension of } \phi) \\ (x^0\,;\, d\Phi(x^0),\, \Phi(x^0)) = e^0 \end{cases}
$$

where the notation $\operatorname{ord}_{x^0}[f(x)]$ denotes the vanishing order of f at $x = x^0$.

Note that the condition [A.3] can be said for short the following form:

[A.3]′ There exists a good extension Φ of the Cauchy data ϕ.

For a precise definition of the good extensions, see § 2 (Definition 2.16).

Now we assume the conditions [A.1]-[A.3]. Let Φ be a good extension of the Cauchy data ϕ of (1). We consider a map germ

$$
\text{(5)} \qquad \begin{cases} \gamma_\Phi : (S \times \boldsymbol{C},\, (x^0,\, 0)) \longrightarrow (J^1M,\, e^0) \\ \gamma_\Phi(y,\, \tau) := (y\,;\, \tau ds(y) + d\Phi(y),\, \Phi(y)) \end{cases}
$$

and the pull-back f^Φ of F by γ_Φ:

$$
\text{(6)} \qquad f^\Phi(y,\, \tau) := (\gamma_\Phi^* F)(y,\, \tau) = F(y\,;\, \tau ds(y) + d\Phi(y),\, \Phi(y)).
$$

Taking the Taylor expansion

$$
f^\Phi(y,\, \tau) = \sum_{\nu=0}^{\infty} c_\nu(y)\tau^\nu \qquad (c_\nu \in \mathcal{O}_{S,x^0} \text{ for } \nu = 0, 1, 2, \cdots)
$$

of f^Φ along $\{\tau = 0\}$, we define the *Newton polygon* $\mathrm{N}(f^\Phi)$ *of* f^Φ *at* $(x^0, 0)$ by

$$
\text{(7)} \qquad \mathrm{N}(f^\Phi) := \operatorname{ch}[\bigcup_{c_\nu \not\equiv 0} \{(\operatorname{ord}_{x^0}[c_\nu],\, \nu) + \bar{\boldsymbol{R}}_+^2\}]
$$

where the notation $\operatorname{ch}[A]$ for a subset A of $\boldsymbol{R}^2$ denotes the convex hull of A, and where $\bar{\boldsymbol{R}}_+$ denotes the set of non-negative real numbers.

In order to construct solutions of the Cauchy problem (1), we utilize the classical theory of characteristic curves. Let

$$
\text{(8)} \qquad f^\Phi(y,\, \tau) = \prod_{j=1}^{r} f_j^\Phi(y,\, \tau)^{\nu(j)}
$$

be the irreducible decomposition of f^Φ in the local ring $\mathcal{O}_{S\times C(x^0,0)}$. We set germs V_j $(1 \leqq j \leqq r)$ of analytic sets of $(\boldsymbol{C}, 0)_t \times (S \times \boldsymbol{C}, (x^0, 0))_{(y,\tau)}$ by

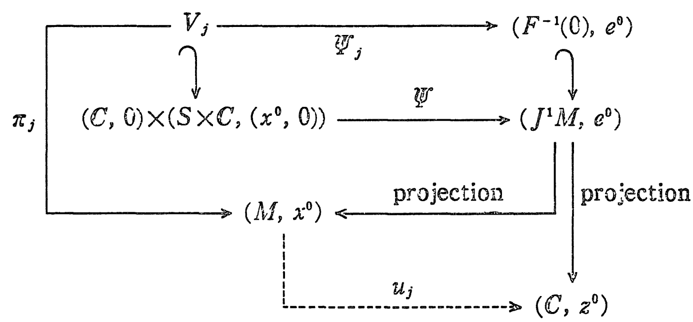
$$
\text{(9)} \qquad V_j := \{(t,\, y,\, \tau)\,;\, f_j^\Phi(y,\, \tau) = 0\}.
$$

Let $t \mapsto \Psi(t, y, \tau)$ be the *characteristic curve of* F (the integral curve of the characteristic vector field Y_F associated to F) satisfying the initial condition $\Psi(0, y, \tau)=\gamma_\Phi(y, \tau)\in(J^1M, e^0)$, where Y_F is given by

(10) $$Y_F=\sum_{j=1}^{n}(\partial_{\xi_j}F)\partial_{x_j}-\sum_{j=1}^{n}\{\xi_j(\partial_z F)+\partial_{x_j}F\}\partial_{\xi_j}+\Big(\sum_{j=1}^{n}\xi_j\partial_{\xi_j}F\Big)\partial_z\,.$$

Then we have the following induced map germs Ψ_j and π_j for $1\leqq j\leqq r$:

(11)



Indeed, the following property of the characteristic curves

(12) $$\partial_t\{F(\Psi(t, y, \tau)\}\equiv 0$$

yields $\Psi(V_j)\subset F^{-1}(0)$. Thus we have the induced map germs Ψ_j $(1\leqq j\leqq r)$.

Our main result is the following

Theorem 4.2. *Assume the conditions* [A.1], [A.2] *and* [A.3]. *Then, for any* $1\leqq j\leqq r$, *the following statements* 1) *and* 2) *hold:*

1) *The map germ* π_j *is a germ of an analytic covering of* (M, x^0) *such that its ramification degree at* x^0 *is the positive integer* ν_j *which can be obtained from the Newton polygon* $\mathrm{N}(f^\Phi)$ *by means of the formula* (4.7) *stated in* § 4.

2) *Let* Σ_j *be the critical locus of the germ* π_j *of an analytic covering of* (M, x^0) (*see* § 3). *Then there exists a multi-valued germ* u_j *on* $(M-\Sigma_j, x^0)$ *which makes the diagram* (11) *commute, such that*

(a) $$F(x\,;du_j(x), u_j(x))=0 \quad \textit{and}$$

(b) u_j *is exactly* ν_j*-valued on* $(M-\Sigma_j, x^0)$
(*the ramification degree of* u_j *at the point* x^0 *is equal to* ν_j).

We remark that the assertion of the main theorem involves the classical result in the case $p=1$ (Theorem 1.6), which says that if $p=1$ then the ramification degree is equal to one (unramified), see Remark 4.4.

Our program proceeds as follows:

In § 0, we give an example in C^2, which is a prototype of our theory.

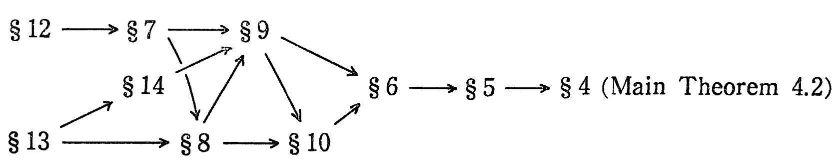
In Chapter I, we give preliminaries to state the main result. In § 1, we summarize the classical theory of characteristic curves from our view point. In § 2, we give a precise definition of the good extensions. In § 3, we prepare several geometric notions such as finite holomorphic maps and germs of analytic (ramified) coverings.

In chapter II, we state the main theorem and its direct corollaries. The first corollary is related to the analytic continuation of holomorphic local solutions of the Cauchy problem (1) at generic points y in $S-\{x^0\}$. The second corollary asserts the neccessity of the non-charactericity (3) for the existence of holomorphic local solution of (1) at x^0, under [A. 1], [A. 2] and [A. 3].

In chapter III, we give a proof of the main theorem. In § 5, we carry out a reduction of the main theorem to a simpler Theorem 5.1. In § 6, we introduce map germs $\pi_j^{\sim}$ $(1\leqq j\leqq r)$ and their decompositions. By virtue of these decompositions, our proof of Theorem 5.1 can be reduced to those of Theorems 6.10 and 6.11. In §§ 7-10, we prove these theorems.

In chapter IV, we give proofs of several basic facts which are assumed in chapter III.

The logical relations among the sections in Chapters III and IV except for § 11 are as follows (the content of § 11 is used almost everywhere):



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§ 0. A Typical Example

In this section we give a simple example which is a prototype of our general theory.

Example 0.1. In C^2, we consider a Cauchy problem

$$(0.1)\qquad \begin{cases} \prod_{\mu=1}^{m}((\partial_{x_1}u)^{p(\mu)}-x_2^{q(\mu)})-\partial_{x_2}u=0 \\ u(0,\ x_2)=\phi(x_2)\equiv 0 \end{cases}$$

under the following assumptions (a), (b) and (c) for the positive integers $p(\mu)$ and $q(\mu)$:

(a) $$p(1)/q(1)>\cdots>p(m)/q(m)\,.$$

(b) $p(\mu)$ and $q(\mu)$ are coprime for $1\leqq\mu\leqq m$.

(c) $\begin{cases} \text{If we put } a(0):=0,\ a(\mu):=q(1)+\cdots+q(\mu) & \text{for } 1\leqq\mu\leqq m\,, \\ \text{then } q(\mu) \text{ and } a(\mu-1)+1 \text{ are coprime for } 1\leqq\mu\leqq m\,. \end{cases}$

We fix a base point $e^0:=(0, 0; 0, 0, 0)\in J^1\boldsymbol{C}^2\cap F^{-1}(0)$, where F is given by

$$F(x;\xi, z)=\prod_{\mu=1}^{m}(\xi_1^{p(\mu)}-x_2^{q(\mu)})-\xi_2\,. \tag{0.2}$$

We note that the assumptions [A.1] and [A.2] of the main theorem are satisfied in this example, since we have

$$\begin{cases} \partial_{\xi_2}F\equiv-1 \quad \text{and} \\ \mathrm{ord}_0[F(0, 0; \tau dx_1, 0)]=p(1)+\cdots+p(m)<\infty\,. \end{cases}$$

We take an extension $\Phi(x)\equiv 0$ of the Cauchy data $\phi(x_2)\equiv 0$, and consider the function $f(y, \tau)$ defined by

$$\begin{aligned} f(y, \tau):=f^{\Phi}(y, \tau)&=F(0, y; \tau dx_1+d\Phi(0, y), \Phi(0, y)) \\ &=F(0, y; \tau, 0, 0)=\prod_{\mu=1}^{m}(\tau^{p(\mu)}-y^{q(\mu)})\,. \end{aligned} \tag{0.3}$$

Since Newton polygons have the additivity property

$$\mathrm{N}(gh)=\mathrm{N}(g)+\mathrm{N}(h) \qquad (\text{see §11, Proposition 11.3})$$

we have

$$\mathrm{N}(f)=\sum_{\mu=1}^{m}\mathrm{N}(\tau^{p(\mu)}-y^{q(\mu)})\,. \tag{0.4}$$

Lemma 0.2. *For positive integers $p(\mu)$, $q(\mu)$, we put*

$$\mathrm{N}_{q(\mu), p(\mu)}:=\{(c, d); (c/q(\mu))+(d/p(\mu))\geqq 1\}.$$

Let $\mathrm{N}:=\sum_{\mu=1}^{m}\mathrm{N}_{q(\mu), p(\mu)}$ be the vector sum of these $\{\mathrm{N}_{q(\mu), p(\mu)}\}$ in $\boldsymbol{R}^2$. If the finite sequences $\{p(\mu)\}_{\mu=1,2,\dots,m}$ and $\{q(\mu)\}_{\mu=1,2,\dots,m}$ satisfy the condition (a), *then the vertices of* N *are given by*

$$\{(a(\mu), p-b(\mu));\ 0\leqq\mu\leqq m\}\subset\overline{\boldsymbol{R}}_+^2 \tag{0.5}$$

where we define $a(\mu)$ as in the assumption (c), *and we put*

$$\begin{cases} b(0):=0, \quad \textit{and} \quad b(\mu):=p(1)+\cdots+p(\mu) \qquad \text{for } \mu\geqq 1\,. \\ p:=b(m)\,. \end{cases}$$

Proof. Let $(c, d)=\sum_{\mu=1}^{m}(c_\mu, d_\mu)\in\mathrm{N}$ with $(c_\mu, d_\mu)\in\mathrm{N}_{q(\mu), p(\mu)}$ for $1\leqq\mu\leqq m$. We can write (c_μ, d_μ) as follows:

$$(0.6)\qquad \begin{cases} (c_\mu, d_\mu) \in (c_\mu^{\sim}, d_\mu^{\sim}) + \bar{\boldsymbol{R}}_+^2 \\ (c_\mu^{\sim}, d_\mu^{\sim}) \in [\text{the segment joining } (0, p(\mu)) \text{ and } (q(\mu), 0)]\,. \end{cases}$$

For our aim, it suffices to show the following facts:

$$(0.7)\qquad p(\mu)c + q(\mu)d \geqq p(\mu)a(\mu) + q(\mu)\{p - b(\mu)\} \qquad \text{for } 1 \leqq \mu \leqq m$$

and the equality holds in (0.7) if and only if

$$(0.8)\qquad (c_\lambda, d_\lambda) \begin{cases} = (q(\lambda), 0) & \text{if } \lambda < \mu \\ = (0, p(\lambda)) & \text{if } \lambda > \mu \\ \in [\text{the segment joining } (0, p(\mu)) \text{ and } (q(\mu), 0)] & \text{if } \lambda = \mu. \end{cases}$$

By the expression (0.6), we have

$$\begin{aligned} (0.9)\qquad p(\mu)c + q(\mu)d &= \sum_{\lambda=1}^{m} \{p(\mu)c_\lambda + q(\mu)d_\lambda\} \geqq \sum_{\lambda=1}^{m} \{p(\mu)c_\lambda^{\sim} + q(\mu)d_\lambda^{\sim}\} \\ &= \sum_{\lambda=1}^{m} [p(\mu)c_\lambda^{\sim} + q(\mu)\{-(p(\lambda)/q(\lambda))c_\lambda^{\sim} + p(\lambda)\}] \\ &= q(\mu) \sum_{\lambda=1}^{m} [\{(p(\mu)/q(\mu)) - (p(\lambda)/q(\lambda))\} c_\lambda^{\sim} + p(\lambda)]\,. \end{aligned}$$

Since the assumption (a) yields

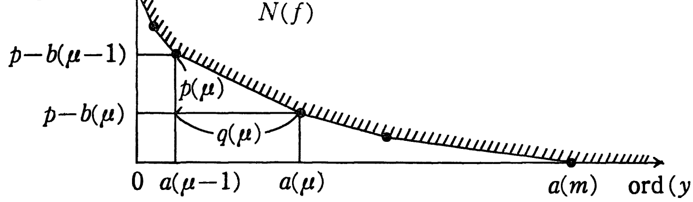
$$(p(\mu)/q(\mu)) - (p(\lambda)/q(\lambda)) \begin{cases} < 0 & \text{if } \lambda < \mu \\ = 0 & \text{if } \lambda = \mu \\ > 0 & \text{if } \lambda > \mu \end{cases}$$

the rightest hand of (0.9) (hence also $p(\mu)c + q(\mu)d$) is minimized only if the condition (0.8) holds. Conversely, if (0.8) holds then we have

$$\begin{aligned} p(\mu)c + q(\mu)d &= p(\mu)\{a(\mu-1) + c_\mu^{\sim}\} + q(\mu)\{p - b(\mu) + d_\mu^{\sim}\} \\ &= p(\mu)a(\mu) + q(\mu)(p - b(\mu))\,. \end{aligned}$$

Hence we get Lemma 0.2. Q.E.D.

Since it is clear that $\mathrm{N}(\tau^{p(\mu)} - y^{q(\mu)}) = \mathrm{N}_{q(\mu), p(\mu)}$ for $1 \leqq \mu \leqq m$, the equality (0.4) and Lemma 0.2 yield the following figure of $\mathrm{N}(f)$:



The aim of this section is to show the following

Proposition 0.3. *For* $1 \leqq \mu \leqq m$, *we define a positive integer* $v(\mu)$ *by*

$$v(\mu):=p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}\,. \tag{0.10}$$

Then the Cauchy problem (0.1) *has a* $v(\mu)$*-valued analytic solution* $u_\mu(x)$ *around the origin of* $\boldsymbol{C}^2$ *for* $1 \leqq \mu \leqq m$.

To show Proposition 0.3, we utilize the classical theory of characteristic curves. Let $t \mapsto (X\,;\Xi,Z)(t,y,\tau)$ be the characteristic curve of $F(x\,;\xi,z)$ given by (0.2), passing through a point $(0,y\,;\tau,0,0) \in F^{-1}(0)$ at the initial time $t=0$. Note that the definition (0.3) of $f(y,\tau)$ yields

$$(0,y\,;\tau,0,0) \in F^{-1}(0) \Longleftrightarrow (y,\tau) \in f^{-1}(0)\,.$$

We set complex curves $D(\mu)$ $(1 \leqq \mu \leqq m)$ by

$$D(\mu):=\{(y,\tau)\,;\ \tau^{p(\mu)}-y^{q(\mu)}=0\}. \tag{0.11}$$

Then we have the following irreducible decomposition:

$$f^{-1}(0)=\bigcup_{\mu=1}^{m} D(\mu)\,.$$

We construct solution $u_\mu(x)$ of (0.1) by the following diagram:

$$\begin{array}{ccc}
(\boldsymbol{C},0)_t \times (D(\mu),(0,0))_{y,\tau} & \xrightarrow{(X;\,\Xi,\,Z)} & (F^{-1}(0),e^0)_{x:\xi,z} \\
X \downarrow & & \downarrow \\
(\boldsymbol{C}^2,(0,0))_x & \xleftarrow{\text{projection}} & (J^1\boldsymbol{C}^2,e^0)_{x:\xi,z} \\
\vdots\, u_\mu & & \text{projection} \downarrow \\
\dashrightarrow & \dashrightarrow & (\boldsymbol{C},0)_z
\end{array} \tag{0.12}$$

We must show that the diagram (0.12) determines a multi-valued germ $u_\mu(x)$ around the origin. It suffices to show the map

$$X\colon (\boldsymbol{C},0)_t \times (D(\mu),(0,0))_{y,\tau} \longrightarrow (\boldsymbol{C}^2,(0,0))_x \tag{0.13}$$

is a *germ of an analytic covering* of $(\boldsymbol{C}^2,(0,0))_x$ (for the terminology, see § 3).

To verify this fact, we observe the components (X_1, X_2, Ξ_1) which satisfy

$$\begin{cases} \partial_t X_1=\partial_{\xi_1}F=(\partial_\tau f)(X_2,\Xi_1)=\partial_{\Xi_1}\Big[\prod_{\lambda=1}^{m}(\Xi_1^{p(\lambda)}-X_2^{q(\lambda)})\Big] \\ \partial_t X_2=\partial_{\xi_2}F=-1 \quad \text{and} \quad \partial_t \Xi_1=-\partial_{x_1}F-\Xi_1\partial_z F=0 \end{cases} \tag{0.14}$$

$$(X_1,X_2,\Xi_1)(0,y,\tau)=(0,y,\tau) \qquad (y,\tau)\in D(\mu)\,. \tag{0.15}$$

We solve (0.14)-(0.15) explicitly as follows. First we obviously have

$$(0.16)\qquad \begin{cases} X_2(t, y, \tau)=y-t & \text{and} \\ \Xi_1(t, y, \tau)=\tau . \end{cases}$$

Then the first equation of (0.14) can be written as the form

$$(0.17)\qquad \begin{cases} \partial_t X_1=\partial_\tau\left[\prod_{\lambda=1}^{m}\{\tau^{p(\lambda)}-(y-t)^{q(\lambda)}\}\right], \\ X_1(0, y, \tau)=0 . \end{cases}$$

We put

$$(0.18)\qquad g(y, \tau ; t):=\prod_{\lambda=1}^{m}\{\tau^{p(\lambda)}-(y-t)^{q(\lambda)}\}$$

$$=f(y-t, \tau)=\sum_{i,j=0}^{\infty} c_{i,j}(y-t)^i\tau^j$$

where the coefficients $c_{i,j}\in \boldsymbol{C}$ satisfy

$$(0.19)\qquad c_{i,j}\neq 0 \quad \text{only if } (i, j)\in \mathrm{N}(f) .$$

Using the function g and its expansion, we solve (0.17) as

$$(0.20)\qquad X_1(t, y, \tau)=\sum_{i,j=0}^{\infty}\{jc_{i,j}/(i+1)\}\{y^{i+1}-(y-t)^{i+1}\}\tau^{j-1} .$$

Note that, by virtue of the assumption b), the curve $D(\mu)$ defined by (0.11) has a *resolution of singularity* of the form

$$(0.21)\qquad \rho : (\boldsymbol{C}, 0)\ni\theta \longmapsto (y, \tau)=(\theta^{p(\mu)}, \theta^{q(\mu)})\in D(\mu) .$$

We define $(X^\sim ; \Xi^\sim, Z^\sim)(t, \theta)$ as the pull-back of $(X; \Xi, Z)$ by the mapping

$$(\boldsymbol{C}, 0)_t\times(\boldsymbol{C}, 0)_\theta \xrightarrow{1\times\rho} (\boldsymbol{C}, 0)_t\times(D(\mu), (0, 0))_{y,\tau} .$$

Then we have the following expressions $(0.16)^\sim$ and $(0.20)^\sim$:

$$(0.16)^\sim\qquad \begin{cases} X_2^\sim(t, \theta)=\theta^{p(\mu)}-t \\ \Xi_1^\sim(t, \theta)=\theta^{q(\mu)} \end{cases}$$

$$(0.20)^\sim\qquad X_1^\sim(t, \theta)=\sum_{i,j=0}^{\infty}\{jc_{i,j}/(i+1)\}\{\theta^{p(\mu)(i+1)}-X_2^\sim(t, \theta)^{i+1}\}\theta^{q(\mu)(j-1)} .$$

In order to count the ramification degrees of (t, θ) as a multi-valued function of (x_1, x_2), we consider the following equation in (t, θ):

$$(0.22)\qquad \begin{cases} X_1^\sim(t, \theta)=x_1 \\ X_2^\sim(t, \theta)=x_2 . \end{cases}$$

Note that $X_1^\sim(t, \theta)$ involves the variable t only of the form $X_2^\sim(t, \theta)$. Thus we can write $X_1^\sim(t, \theta)$ as the form

$$X_1^{\sim}(t,\theta)=H_\mu(\theta, X_2^{\sim}(t,\theta)) \tag{0.23}$$

where $H_\mu(\theta, x_2)$ is given by

$$H_\mu(\theta, x_2):=\sum_{i,j=0}^{\infty}\{jc_{i,j}/(i+1)\}\{\theta^{p(\mu)(i+1)}-x_2^{i+1}\}\theta^{q(\mu)(j-1)}.$$

In this situation we have the following

Lemma 0.4. *Assume* (a), (b) *and* (c). *We set*

$$h_\mu(x_1, x_2, \theta):=H_\mu(\theta, x_2)-x_1 \tag{0.24}$$

Then it follows that

$$\operatorname{ord}[h_\mu(0,0,\theta)]=v(\mu)\underset{\text{def.}}{=}p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}.$$

Proof. Note that

$$h_\mu(0,0,\theta)=H_\mu(\theta,0)=\sum_{i,j=0}^{\infty}\{jc_{i,j}/(i+1)\}\theta^{p(\mu)(i+1)+q(\mu)(j-1)} \tag{0.25}$$

which yields the inequality

$$\operatorname{ord}[h_\mu(0,0,\theta)]\geqq p(\mu)-q(\mu)+\min\{p(\mu)i+q(\mu)j\,;\,c_{i,j}\neq 0,\ j\geqq 1\}. \tag{0.26}$$

For Lemma 0.4, it suffices to show the following (0.27) and (0.28):

$$\min\{p(\mu)i+q(\mu)j\,;\,c_{i,j}\neq 0,\ j\geqq 1\}=p(\mu)a(\mu-1)+q(\mu)\{p-b(\mu-1)\}. \tag{0.27}$$

$$\sum_{(i,j)\in I} jc_{i,j}/(i+1)\neq 0 \tag{0.28}$$

where $I\subset \boldsymbol{Z}^2$ denotes the set of (i, j) attaining the minimum value (0.27).

By virtue of (0.19), we consider the linear functional

$$k_\mu: \boldsymbol{R}^2\ni(i,j)\longmapsto p(\mu)i+q(\mu)j\in\boldsymbol{R} \tag{0.29}$$

and observe that the minimum value of k_μ on $\mathrm{N}(f)\cap\{(i,j);\ j\geqq 1\}$ is given by the right hand side of (0.27). Note that, for any $c\in\boldsymbol{R}$, the level set $k_\mu^{-1}(c)$ forms a line with the slope $-p(\mu)/q(\mu)$. Thus the minimum value is attained if and only if the level set $k_\mu^{-1}(c)$ coincides with the line joining $(a(\mu-1), p-b(\mu-1))$ and $(a(\mu), p-b(\mu))$. Hence the assumption (b) and the condition $j\geqq 1$ yield that the minium value can be attained by (i, j) if and only if the following (0.30) holds:

$$(i,j)=\begin{cases}(a(\mu-1),\ p-b(\mu-1)) \quad\text{or}\quad (a(\mu),\ p-b(\mu)) & \text{if } \mu<m.\\ (a(m-1),\ p-b(m-1)) & \text{if } \mu=m.\end{cases} \tag{0.30}$$

Thus we have

$$\min k_\mu=k_\mu(a(\mu-1),\ p-b(\mu-1))=p(\mu)a(\mu-1)+q(\mu)\{p-b(\mu-1)\}$$

which shows (0.27).

Now we prove (0.28). First we claim

$$(0.31)\quad \sum_{(i,j)\in I} jc_{i,j}/(i+1)$$
$$=\begin{cases} (-1)^{\mu-1}\{(p-b(\mu-1))/(a(\mu-1)+1)-(p-b(\mu))/(a(\mu)+1)\} & \text{if } \mu<m. \\ (-1)^{m-1}(p-b(m-1))/(a(m-1)+1) & \text{if } \mu=m. \end{cases}$$

Proof of (0.31). Since $c_{i,j}$ is the Taylor coefficient of $y^i\tau^j$ in

$$f(y, \tau)=\prod_{\lambda=1}^{m}\{\tau^{p(\lambda)}-y^{q(\lambda)}\}$$

we especialy have

$$(0.32)\qquad c_{a(\mu),\, p-b(\mu)}=(-1)^{\mu} \qquad \text{for } 0\leqq\mu\leqq m\,.$$

Indeed, Lemma 0.2 yields that $(a(\mu), p-b(\mu))$ is a vertex of $\mathrm{N}(f)$. Thus the condition (0.8) in the proof of Lemma 0.2 holds, which derives the following implications:

$$\begin{cases} \exists(i_\lambda, j_\lambda)\in\mathrm{N}(\tau^{p(\lambda)}-y^{q(\lambda)}), \qquad 1\leqq\lambda\leqq m, \text{ such that} \\ (a(\mu), p-b(\mu))=\sum_{\lambda=1}^{m}(i_\lambda, j_\lambda) \end{cases}$$

$$\Longrightarrow (i_\lambda, j_\lambda)=\begin{cases} (q(\lambda), 0) & \text{if } \lambda<\mu \\ (0, p(\lambda)) & \text{if } \lambda>\mu \end{cases} \Longrightarrow (i_\mu, j_\mu)=(q(\mu), 0).$$

Hence we get

$$c_{a(\mu),\, p-b(\mu)}=1^{\#\{\lambda;\lambda>\mu\}}(-1)^{\#\{\lambda;\lambda\leqq\mu\}}=(-1)^{\mu}\,.$$

Thus we have (0.32). Then it is obvious that (0.30) and (0.32) yield the desired (0.31). Q. E. D.

We continue the proof of (0.28). But this is easily verified from (0.31) since $a(\mu)$ and $b(\mu)$ satisfy the inequalities

$$a(\mu-1)<a(\mu), \qquad \text{and} \qquad p-b(\mu-1)>p-b(\mu)\,.$$

The proof of (0.28), hence of Lemma 0.4, is complete. Q. E. D.

Since $h_\mu(x_1, x_2, \theta)=H_\mu(\theta, x_2)-x_1$ vanishes at $(0, 0, 0)$ with order one, $h_\mu(x_1, x_2, \theta)$ is irreducible at the origin. Hence Lemma 0.4 yields that the function $\theta(x_1, x_2)$ determined by the equation

$$(0.33)\qquad h_\mu(x_1, x_2, \theta)=0$$

is exactly $v(\mu)$-valued. We therefore have the following at most $v(\mu)$-valued inverse $(t(x), \theta(x))$ of the mapping $X^\sim: (\boldsymbol{C}, 0)_t\times(\boldsymbol{C}, 0)_\theta\to(\boldsymbol{C}^2, 0)_x$, which gives the solutions of the equation (0.22): Indeed, if we put

$$t(x):=\theta(x)^{p(\mu)}-x_2$$

then we have

$$\begin{cases} X_1^{\sim}(t(x),\ \theta(x))=H_\mu(\theta(x),\ X_2^{\sim}(t(x),\ \theta(x)))=h_\mu(x_1,\ x_2,\ \theta(x))+x_1=x_1 \\ X_2^{\sim}(t(x),\ \theta(x))=\theta^{p(\mu)}-t(x)=x_2\,. \end{cases}$$

Thus we have the following diagram:

$$(0.34)\qquad \begin{array}{ccc} (\boldsymbol{C},\ 0)_t\times(D(\mu),\ (0,\ 0))_{y,\tau} & \longrightarrow\!\!\!\!\longrightarrow & \\ \uparrow 1\times\rho & & \downarrow (X;\ \Xi,\ Z) \\ (\boldsymbol{C},\ 0)_t\times(\boldsymbol{C},\ 0)_\theta & \xrightarrow{(X^{\sim};\ \Xi^{\sim},\ Z^{\sim})} & (F^{-1}(0),\ e^0) \\ X^{\sim}\downarrow\ \ \vdots\!\uparrow (t(x),\ \theta(x)) & & \downarrow \\ (\boldsymbol{C}^2,\ (0,\ 0))_x & \xleftarrow{\text{projection}} & (J^1\boldsymbol{C}^2,\ e^0)_{x:\xi,z} \\ \vdots\ u_\mu(x) & & \downarrow \text{projection} \\ \cdots\cdots\cdots & \dashrightarrow & (\boldsymbol{C},\ 0)_z \end{array}$$

Hence the multi-valued germ $u_\mu(x)$ can be defined by the diagram (0.34), or by the diagram (0.12), such that $u_\mu(x)$ is at most $v(\mu)$-valued and satisfies the equation

$$F(x\,;\ du_\mu(x),\ u_\mu(x))=\prod_{\lambda=1}^{m}((\partial_{x_1}u_\mu)^{p(\lambda)}-x_2^{q(\mu)})-\partial_{x_2}u_\mu=0\,.$$

It remains to show that the germ $u_\mu(x)$ is exactly $v(\mu)$-valued.

To verify this we use the well-known relation

$$(0.35)\qquad \partial_{x_1}u_\mu(x)=\xi_1(x)$$

where the function $\xi_1(x)$ is given by the following diagram (0.36):

$$(0.36)\qquad \begin{array}{ccccc} (\boldsymbol{C},\ 0)_t\times(\boldsymbol{C},\ 0)_\theta & \xrightarrow{(X_1^{\sim},\ X_2^{\sim},\ \Xi_1^{\sim})} & (\boldsymbol{C}^2,\ (0,\ 0))_x\times(\boldsymbol{C},\ 0)_{\xi_1} & \longrightarrow & (\boldsymbol{C},\ 0)_{\xi_1} \\ \vdots\!\uparrow & & \downarrow & & \uparrow\!\vdots\ \xi_1(x) \\ \cdots & \xleftarrow{(t(x),\ \theta(x))} & (\boldsymbol{C}^2,\ (0,\ 0))_x & \cdots & \cdots \end{array}$$

Lemma 0.5. *The function $\xi_1(x)$ defined by* (0.36) *is also $v(\mu)$-valued.*

Proof. By the definition of $\Xi_1^{\sim}(t,\ \theta)$, we have

$$\Xi_1^{\sim}(t,\ \theta):=\Xi_1(t,\ y,\ \tau)|_{(y,\tau)=(\theta^{p(\mu)},\ \theta^{q(\mu)})}=\theta^{q(\mu)}$$

which implies

$$(0.37)\qquad \xi_1(x)=\theta(x)^{q(\mu)}\,.$$

Thus when $\theta(x)$ rounds its singular points one-time, $\xi_1(x)$ rounds its singular points $q(\mu)$-times.

Note that, for $1\leqq\mu\leqq m$, the assumptions (b), (c) yield that

$$q(\mu) \text{ and } v(\mu) \text{ are coprime.} \tag{0.38}$$

Indeed, if we denote the greatest common divisor of $a, b\in \mathbf{Z}$ by (a, b), then we have

$$\begin{aligned}(q(\mu), v(\mu)) &= (q(\mu), p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}) \\ &= (q(\mu), a(\mu-1)+1) \qquad [\because (q(\mu), p(\mu))=1] \\ &= 1. \qquad [\because \text{the condition (c)}]\end{aligned}$$

Hence we have the following implication:

$$q(\mu)k\in v(\mu)\mathbf{Z} \Longrightarrow k\in v(\mu)\mathbf{Z}. \tag{0.39}$$

Hence we conclude that the function $\xi_1(x)$ is exactly $v(\mu)$-valued. Q.E.D.

As a consequence of Lemma 0.5 and the relation (0.35) we get that the multi-valued solution u_μ is exactly $v(\mu)$-valued as desired.

The proof of Proposition 0.3 is complete. Q.E.D.

We conclude this section to give the simple

Corollary 0.6. *We consider a very special case that*

$$m=1, \quad p=p(1)=2 \quad \textit{and} \quad q(1)=1 \tag{0.40}$$

hold in Example 0.1:

$$\begin{cases} (\partial_{x_1}u)^2-x_2-\partial_{x_2}u=0. \\ u(0, x_2)\equiv 0. \end{cases} \tag{0.41}$$

Then the Cauchy problem (0.41) *has a* 3*-valued analytic solution.*

Proof. We only have to verify $v(1)=3$. The assumption (0.40) yields

$$\begin{aligned}v(1) &= p(1)\{a(0)+1\}+q(1)\{p-b(0)-1\} \\ &= p(1)+q(1) \\ &= 3. \end{aligned}$$

Q.E.D.

Remark 0.7. By a direct computation, we have the following explicit expressions of the functions $X^\sim(t, \theta)$ and $\theta(x)$ of the Cauchy problem (0.41):

$$X_1^\sim(t, \theta)=2\theta t, \qquad X_2^\sim(t, \theta)=\theta^2-t$$

$$h_1(x_1, x_2, \theta)=X_1^{\sim}(t, \theta)-x_1|_{t=\theta^2-x_2}=2\theta(\theta^2-x_2)-x_1=0\,.$$

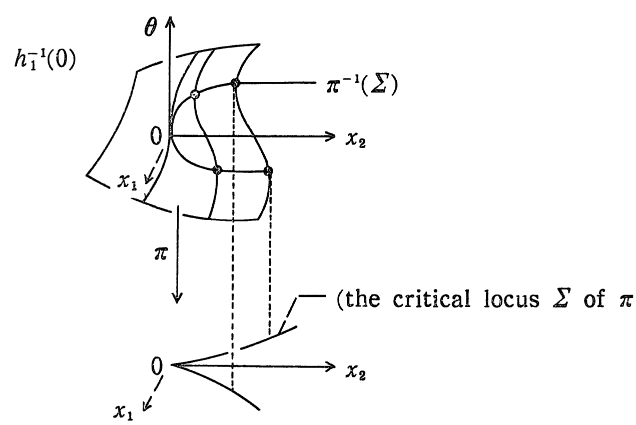
Thus the 3-valuedness of the function

$$\partial_{x_1}u(x)=\xi_1(x)=\theta(x)^{q(1)}=\theta(x)$$

is a consequence of the 3-sheetedness of the following mapping π:

$$\pi: h_1^{-1}(0)\hookrightarrow(C^3, 0)_{x,\theta}\xrightarrow[\text{projection}]{}(C^2, 0)_x\,. \tag{0.42}$$

We give an illustration of the surface $h_1^{-1}(0)\cap R^3$ as follows:



Note that this kind of singularity of the map germ π is called by the name of "*Whitney's tuck*" (see e.g. [Ar: Appendix 12, Lagrangian singularities]).

Chapter I. Preliminaries

§1. Classical Theory of Chracteristic Curves

In this section we give a summary of the classical theory of characteristic curves from our view point, by introducing an *affine bundle* $E=E(\phi)$ over the initial hypersurface S. This bundle E is, roughly speaking, a space of jets

$$\bigcup_{\Phi}\{(y\,; d\Phi|_S(y), \Phi(y))\in J^1M\,;\, y\in S\}$$

where $\Phi\in\mathcal{O}_{M,x^0}$ runs through all holomorphic extensions of the data $\phi\in\mathcal{O}_{S,x^0}$ of the Cauchy problem (1).

Let us recall the Cauchy problem (1) with the condition c) in (2):

$$(d\phi(x^0), \phi(x^0))=(\iota_{x^0}^*\xi^0, z^0)\in J_{x^0}^1S$$

where $e^0=(x^0\,; \xi^0, z^0)$ is the base point lying in a neighborhood the equation $F(x, \xi, z)$ is defined, and where $\iota^*: T^*M|_S\to T^*S$ denotes the dual bundle map of the tangent bundle map $\iota_*: TS \to TM|_S$ induced by $\iota: S \hookrightarrow M$.

Definition 1.1. We define a *subset* $E=E(\phi)$ by setting

$$E=E(\phi):=\{(y\,;\xi,\phi(y))\in J^1M;\ y\in S,\ \iota_y^*\xi=d\phi(y)\} \tag{1.1}$$

and we also set the *fiber* of E by

$$E_y:=E\cap J_y^1M \quad \text{for } y\in S. \tag{1.2}$$

The meaning of E is clarified by the

Lemma 1.2. 1) *For any local extension $\Phi\in\mathcal{O}_{M,y^0}$ of the data ϕ at $y^0\in S$ we have*

$$(d\Phi(y),\phi(y))\in E_y \quad \textit{for } \forall y\in(S,y^0) \tag{1.3}$$

where (S,y^0) denotes the germ of S at y^0, that is, the set which consists of all $y\in S$ sufficiently near y^0.

2) *For $y\in S$, the set E_y forms an one-dimensional affine subspace of the $(n+1)$-dimensional complex vector space $J_y^1M=T_y^*M\times\boldsymbol{C}$. More precisely, for any local extension $\Phi\in\mathcal{O}_{M,y}$ of ϕ at y, the following equality holds:*

$$E_y=\{(\tau ds(y)+d\Phi(y),\phi(y))\in J_y^1M;\tau\in\boldsymbol{C}\}. \tag{1.4}$$

Proof. The first assertion (1.3) is a direct consequence of the commutativity of the pull-back ι^* and the exterior derivativation d:

$$\iota_y^*d\Phi(y)=d(\iota^*\Phi)(y)=d\phi(y) \quad \text{if } \Phi|_S=\iota^*\Phi=\phi.$$

Note that (1.3) and $s=0$ on S imply the inclusion

$$E_y\supset\{(\tau ds(y)+d\Phi(y),\phi(y));\tau\in\boldsymbol{C}\} \tag{1.4$^\sim$}$$

for any holomorphic extension Φ of ϕ. Hence it suffices for (1.4) to show the converse inclusion of (1.4)$^\sim$. Let $(\xi,\phi(y))\in E_y$. Then (1.3) yields

$$\iota_y^*(\xi-d\Phi(y))=d\phi(y)-d\phi(y)=0, \quad \text{that is, } \langle\xi-d\Phi(y),\iota_{*y}(T_yS)\rangle=0.$$

Hence we get $\xi-d\Phi(y)\in\boldsymbol{C}ds(y)$ which shows the equality (1.4). Q.E.D.

Corollary 1.3. *Let $u\in\mathcal{O}_{M,y^0}$ be a holomorphic local solution of the Cauchy problem* (1). *Then it follows*

$$(y\,;du(y),u(y))\in E\cap F^{-1}(0) \quad \textit{for } \forall y\in(S,y^0). \tag{1.5}$$

We shall give a summary of the theory of characteristic curves, by concerning geometric nature of $E\cap F^{-1}(0)$ as a hypersurface of E.

Let us recall the *characteristic vector field* Y_F on the germ (J^1M,e^0) associate with F, which can be written as the form

$$Y_F = \sum_{j=1}^{n} (\partial_{\xi_j} F)\partial_{x_j} - \sum_{j=1}^{n} (\xi_j \partial_z F + \partial_{x_j} F)\partial_{\xi_j} + \Big(\sum_{j=1}^{n} \xi_j \partial_{\xi_j} F\Big)\partial_z \tag{1.6}$$

by means of any local coordinate system of the form $(x_1, \cdots, x_n; \xi_1, \cdots, \xi_n, z)$.

Notation 1.4. 1) We denote by

$$(\boldsymbol{C}, 0) \ni t \longmapsto \Psi^{\sim}(t, e) = (X; \Xi, Z)(t, e) \in (J^1M, e^0)$$

the *characteristic curve* of F, passing through a point $e \in (E, e^0)$ at the initial time $t=0$, that is, a uniquely determined integral curve of Y_F passing through e. This family of characteristic curves determines a holomorphic map germ

$$\Psi^{\sim} : (\boldsymbol{C}, 0) \times (E, e^0) \longrightarrow (J^1M, e^0). \tag{1.7}$$

2) We define an analytic set V by

$$V := (\boldsymbol{C}, 0) \times (E \cap F^{-1}(0), e^0). \tag{1.8}$$

Restricting the map germ $\Psi^{\sim}$ on V, we have the induced map germ

$$\Psi : (V, (0, e^0)) \longrightarrow (F^{-1}(0), e^0) \tag{1.9}$$

since the characteristic curve $\Psi^{\sim}$ satisfies

$$\partial_t \{F(\Psi^{\sim}(t, e))\} \equiv 0. \tag{1.10}$$

Note that the induced map Ψ is holomorphic as a map between analytic sets (see Definition 3.1).

3) We define a tangent vector $L_F(e^0) \in T_{x^0}M$ by

$$L_F(e^0) := \sum_{j=1}^{n} \partial_{\xi_j} F(e^0) \partial_{x_j}. \tag{1.11}$$

Note that this vector is nothing but the image vector of the characteritic vector $Y_F(e^0) \in T_{x^0}(J^1M)$ at e^0 under the tangent bundle map $\pi_{*, x^0} : \boldsymbol{T}_{e^0}(J^1M) \to T_{x^0}M$ of the natural projection $\pi : J^1M \to M$.

Using these notations, our conditions [A.1] [A.2] can be written as

[A.1]′ $$L_F(e^0) \neq 0.$$

[A.2]′ $$\mathrm{ord}_{e^0}[F|_{E_{x^0}}] =: p \in [1, \infty).$$

Note that [A.2]′ implies $F|_E \not\equiv 0$, hence the intersection $E \cap F^{-1}(0)$ is a complex hypersurface of E. Thus the germ $(V, (0, e^0))$ defined by (1.8) is a n-dimensional hypersurface of $(\boldsymbol{C}, 0) \times (E, e^0)$, which has singular points V_{sing} containing x^0 if

$$1 < \mathrm{ord}_{e^0}[F|_E] \leqq p \ (:= \mathrm{ord}_{e^0}[F|_{E_{x^0}}]).$$

The classical theory of characteristic curves is based on the fact (1.10) and the following well-known

Lemma 1.5. *We denote by* $V_{\mathrm{reg}}:=V-V_{\mathrm{sing}}$ *the regular part of* V *which forms a n-dimensional complex manifold. Then it follows that the pull-back of the fundamental* 1*-form* $dz-\sum_{j=1}^{n}\xi_j dx_j$ *on* J^1M *by* Ψ *vanishes on* V_{reg}:

$$\Psi^*\Big(dz-\sum_{j=1}^{n}\xi_j dx_j\Big)\equiv 0 \qquad \text{on } V_{\mathrm{reg}}. \tag{1.12}$$

Let $\pi_V:(V,(0,e^0))\to(M,x^0)$ be a holomorphic map germ determined by the following diagram:

$$\begin{array}{ccc} (V,(0,e^0)) & \xrightarrow{\quad\Psi\quad} & (F^{-1}(0),e^0) \\ {\scriptstyle\pi_V}\big\downarrow & & \big\downarrow \\ (M,x^0) & \xleftarrow{\text{projection}} & (J^1M,e^0) \end{array}$$

The above properties (1.10) and (1.12) of characteristic curves derive the following classical existence theorem:

Theorem 1.6 (see, for example, [Ar: Appendix 4 M]). *There are the implications* 1) ⇒ 2) ⇒ 3) *for the following conditions*:

1) *The induced map germ*

$$\pi_V|_{t=0}:(E\cap F^{-1}(0),e^0)\hookrightarrow(J^1M|_S,e^0)\longrightarrow(S,x^0)$$

is locally biholomorphic at e^0, *that is, the vanishing order* p *in* [A.2] *is equal to one* (S *is non-characteristic for* F *micro-locally at* e^0).

2) *The map germ* π_V *is locally biholomorphic at* $(0,e^0)$.

3) *There exists a unique holomorphic local solution* $u\in\mathcal{O}_{M,x^0}$ *of the Cauchy problem* (1) *at* x^0 *satisfying*

$$(x^0\,;\,du(x^0),\,u(x^0))=e^0. \tag{1.13}$$

This unique holomorphic solution u *is determined by the following diagram*:

$$\begin{array}{ccc} (V,(0,e^0)) & \xrightarrow{\quad\Psi\quad} & (F^{-1}(0),e^0) \\ {\scriptstyle\pi_V^{-1}}\big\uparrow & & \big\downarrow \\ (M,x^0) & \xleftarrow{\text{projection}} & (J^1M,e^0) \\ & {\scriptstyle u}\searrow & \big\downarrow{\scriptstyle\text{projection}} \\ & & (\mathbf{C}_z,z^0) \end{array} \tag{1.14}$$

It is our starting point of this article to consider the

Problem 1.7. *If we weaken the condition* 1) *in Theorem* 1.6 *to our condition* [A.2] (*or* [A.2]′) *then what kind of solutions of the Cauchy problem* (1) *do appear around the point* $x^0 \in M$?

§2. Definition of Good Extensions

In this section we give a precise definition of "good extension" in our condition [A.3]. First we recall the

Definition 2.1. A holomorphic germ $\Phi \in \mathcal{O}_{M,x^0}$ is called a *holomorphic approximate solution of* the Cauchy problem (1) *of the approximation order k at* $e^0=(x^0, \xi^0, z^0)$ if Φ satisfies the condition (4) in the introduction.

From now on we call such a Φ an *approximate solution* for short.

Notation 2.2. Let Φ be an approximate solution of (1). 1) We set a map germ $\gamma_\Phi : (S \times \boldsymbol{C}, (x^0, 0)) \to (E, e^0)$ by

$$\gamma_\Phi(y, \tau) := (y ; \tau ds(y) + d\Phi(y), \phi(y)) . \tag{2.1}$$

By virtue of Lemma 1.2, the map germ γ_Φ is locally biholomorphic.

2) We define a germ f^Φ of a function as the pull-back of the restriction $F|_E$ by the biholomorphic map germ γ_Φ:

$$f^\Phi := \gamma_\Phi^*(F|_E) \in \mathcal{O}_{S\times \boldsymbol{C}, (x^0, 0)} \qquad (\gamma_\Phi^* : \mathcal{O}_{E, e^0} \xrightarrow{\sim} \mathcal{O}_{S\times \boldsymbol{C}, (x^0, 0)}) \tag{2.2}$$

Definition 2.3. Let $f^\Phi(y, \tau) = \sum_{\nu=0}^{\infty} c_\nu(y)\tau^\nu$ be the Taylor expansion along $\tau=0$.

1) We define a *Newton polygon* $\mathrm{N}(f^\Phi)$ of f^Φ at $(x^0, 0)$ by

$$\mathrm{N}(f^\Phi) := \text{convex hull} \bigcup_{c_\nu \not\equiv 0} [(\mathrm{ord}(c_\nu), \nu) + \bar{\boldsymbol{R}}_+^2] \tag{2.3}$$

where $\mathrm{ord}(c_\nu)$ denotes the vanishing order of $c_\nu(y)$ at $y = x^0$ and we put

$$\bar{\boldsymbol{R}}_+ := \{t \in \boldsymbol{R} ; t \geqq 0\}.$$

2) For a Newton polygon N we define its *strict boundary* $\partial^\circ \mathrm{N}$ by

$$\partial^\circ \mathrm{N} := \{A \in \mathrm{N} ; [A + (-\bar{\boldsymbol{R}}_+)^2] \cap \mathrm{N} = \{A\}\}. \tag{2.4}$$

Note that $\partial^\circ \mathrm{N}$ consists of either only one point or a union of finitely many segments, where we call a subset $\sigma \subset \partial^\circ \mathrm{N}$ by the name of a *segment* of N if there exists a line $\sigma^\sim$ in $\boldsymbol{R}^2$ such that $\sigma = \sigma^\sim \cap \partial^\circ \mathrm{N}$ with $\sharp \sigma \geqq 3$.

3) A point $A \in \partial^\circ \mathrm{N}$ is called a *vertex* of N if the following implication (2.5) holds for any $B, C \in \mathrm{N}$ with $B \neq C$ and for $t \in [0, 1]$:

$$A = tB + (1-t)C \Longrightarrow t = 0 \quad \text{or} \quad t = 1 . \tag{2.5}$$

Note that the assumption [A.2] yields the point $(0, p) \in \boldsymbol{R}^2$ is always a vertex of $\mathrm{N}(f^\Phi)$, since we have $\mathrm{ord}_{\tau=0}[f^\Phi(x^0, \tau)] = \mathrm{ord}_{e^0}[F|_{E_{x^0}}] = p$.

Notation 2.4. 1) We denote by Seg N [or Ver N respectively] the set of all segments [vertices] of a Newton polygon N.

2) Let $\Phi\in\mathcal{O}_{M,x^0}$ be an apploximate solution of (1). We set

$$(2.6)\qquad \begin{cases} m:=\#\,\mathrm{Seg}\,\mathrm{N}(f^{\Phi}) \\ \{A(\mu)=(a(\mu),\ p-b(\mu))\,;\ 0\leqq\mu\leqq m\}:=\mathrm{Ver}\,\mathrm{N}(f^{\Phi}) \end{cases}$$

where sequences $\{a(\mu)\}$ $\{b(\mu)\}$ are arranged as strictly monotone increasing:

$$(2.7)\qquad \begin{aligned} &0=a(0)<a(1)<\cdots<a(m)\\ &0=b(0)<b(1)<\cdots<b(m)\leqq p\,. \end{aligned}$$

3) For $1\leqq\mu\leqq m$, we define positive integers $q(\mu)$, $p(\mu)$ and a positive rational number $\kappa(\mu)$ as follows:

$$(2.8)\qquad \begin{cases} q(\mu):=a(\mu)-a(\mu-1),\ p(\mu):=b(\mu)-b(\mu-1) \quad \text{and} \\ \kappa(\mu):=p(\mu)/q(\mu)\,. \end{cases}$$

Note that $-\kappa(\mu)$ represents the slope of the μ-th segment of $\mathrm{N}(f^{\Phi})$, thus we get

$$\kappa(1)>\kappa(2)>\cdots>\kappa(m)>0\,.$$

Definition 2.5. We say a Newton polygon $\mathrm{N}(f^{\Phi})$ satisfies the *coprimeness condition* if

1) $f^{\Phi}(y,0)\neq0$, that is, the Newton polygon $\mathrm{N}(f^{\Phi})$ intersects the horizontal axis $\boldsymbol{R}\times0$.

2) For $1\leqq\mu\leqq m$, the integers $p(\mu)$ and $q(\mu)$ are coprime.

Definition 2.6. Let $c\in\mathcal{O}_{S,x^0}$ be a germ. For a local coordinate system $(y_1,\cdots,y_{n-1})$ of S at x^0, let $c(y)=\sum a_\alpha y^\alpha$ be the Taylor expansion of $c(y)$ with respect to the coordinate system. We define the *localization* $\mathrm{Loc}[c]:T_{x^0}S\to\boldsymbol{C}$ of the germ c at $x^0\in S$ by

$$(2.9)\qquad \mathrm{Loc}[c]\Big(\sum_{j=1}^{n-1}Y_j\partial_{y_j}\Big):=\sum_{|\alpha|=\mathrm{ord}(c)}a_\alpha Y^\alpha\,.$$

Note that the localization $\mathrm{Loc}[c]$ is determined, as a homogeneous polynomial function on $T_{x^0}S$, independently of a choice of local coordinate systems.

Remark 2.7. If $p:=\mathrm{ord}_{e^0}[F|_{E_{x^0}}]\geqq2$, then the tangent vector $L_F(e^0)$ defined by (1.11) can be regarded as a non-zero vector in $T_{x^0}S$.

Indeed, we have

$$\langle ds(x^0),\ L_F(e^0)\rangle=\partial_\tau\{F(x^0\,;\tau ds(x^0)+\xi^0,\ z^0)\}|_{\tau=0}=0$$

if $p\geqq2$. Hence it follows $L_F(e^0)\in\iota_*(T_{x^0}S)\underset{\iota_*}{\xleftarrow{\ \sim\ }}T_{x^0}S$.

Definition 2.8. Let L be a non-zero vector in $T_{x^0}S$. We say a Newton polygon $\mathrm{N}(f^{\Phi})$ is *stable in a direction of* L, if

$$\mathrm{Loc}[c_\nu](L)\neq 0 \qquad \text{for all } \nu \text{ satisfying } (\mathrm{ord}(c_\nu),\ \nu)\in \mathrm{Ver}\,\mathrm{N}(f^{\Phi}) \tag{2.10}$$

where $c_\nu(y)$ is the ν-th Taylor coefficient of $f^{\Phi}(y, \tau)$.

Remark 2.9. Let $(\boldsymbol{C}, 0)\ni\theta\to y(\theta)\in(S, x^0)$ be a complex curve satisfying

$$y(0)=x^0, \qquad \text{and} \qquad y'(0)\in(\boldsymbol{C}-\{0\})L$$

where L is a non-zero vector in Definition 2.8. For such a curve $y(\theta)$ we set

$$g(\theta, \tau):=f^{\Phi}(y(\theta), \tau).$$

Then the condition (2.10) is equivalent to

$$\mathrm{N}(f^{\Phi})=\mathrm{N}(g). \tag{2.10$'$}$$

Proof. Since $y(\theta)$ can be expanded as

$$y(\theta)=y'(0)\theta+O(\theta^2)=kL\theta+O(\theta^2) \qquad \exists k\in\boldsymbol{C}-\{0\}$$

we have

$$c_\nu(y(\theta))=(k\theta)^{\mathrm{ord}(c_\nu)}\mathrm{Loc}[c_\nu](L)+O(\theta^{\mathrm{ord}(c_\nu)+1}),$$

which yields the equivalence between (2.10) and (2.10)′. Q.E.D.

Notation 2.10. 1) Now we denote an *irreducible decomposition* of $F|_E$ in the local ring $\mathcal{O}_{E,e^0}$ by

$$F|_E=\prod_{j=1}^{r}F_j^{\nu(j)} \tag{2.11}$$

where r, $\nu(j)$ are positive integers and where $F_j\in\mathcal{O}_{E,e^0}$ are irreducible such that $F_j\neq gF_k$ for any germ $g\in\mathcal{O}_{E,e^0}$ if $j\neq k$.

2) For an approximate solution Φ of the Cauchy problem (1), we set

$$f_j^{\Phi}:=\gamma_{\Phi}^{*}F_j\in\mathcal{O}_{S\times C,(x^0,0)} \qquad \text{for } 1\leqq j\leqq r. \tag{2.12}$$

3) For positive integers $p(\mu)$, $q(\mu)$ in Notation 2.4, we put

$$\mathrm{N}_{q(\mu),p(\mu)}:=\{(s, t)\in\boldsymbol{R}^2:\ s, t\geqq 0,\ (s/q(\mu))+(t/p(\mu))\geqq 1\}.$$

Proposition 2.11. *Under Notations* 2.4 *and* 2.10, *it follows that*

$$\mathrm{N}(f^{\Phi})=\sum_{j=1}^{r}\nu(j)\mathrm{N}(f_j^{\Phi})=\sum_{\mu=1}^{m}\mathrm{N}_{q(\mu),p(\mu)}. \tag{2.13}$$

Proposition 2.12. *Assume that* $\mathrm{N}(f^{\Phi})$ *satisfies the coprimeness condition. Then we have*

1) $\nu(j)=1$ *for all* j.

2) $N(f_j^{\Phi})$ *satisfies the coprimeness condition for all* j.

3) *There exist subsets* M_j *of* $\{1, 2, \cdots, m\}$, $1 \leqq j \leqq r$, *suth that the following* (2.14)-(2.16) *hold*:

$$M_j \cap M_k = \emptyset \qquad \text{if } j \neq k\,. \tag{2.14}$$

$$\{1, 2, \cdots, m\} = \bigcup_{j=1}^{r} M_j \qquad (\text{disjoint union}). \tag{2.15}$$

$$N(f_j^{\Phi}) = \sum_{\mu \in M_j} N_{q(\mu), p(\mu)} \qquad \text{for } 1 \leqq j \leqq r. \tag{2.16}$$

The proofs of Propositions 2.11 and 2.12 are given in § 11.

Remark 2.13. In Proposition 2.12, the assertion 2) follows from the assertion 3). Indeed, the equality (2.16) and the coprimeness of $N(f^{\Phi})$ imply the coprimeness of $N(f_j^{\Phi})$ for $1 \leqq j \leqq r$.

Definition 2.14. We say a subset M_j defined by 3) in Proposition 2.12 is a *nice* subset if the following condition (2.17) holds:

$$\mathrm{GCD}\Big[\bigcup_{\mu \in M_j} \{a(\mu-1)+1,\ q(\mu)\}\Big] = 1 \tag{2.17}$$

where $\mathrm{GCD}[B]$ denotes the greatest common divisor of a finite subset $B \subset \boldsymbol{Z}$.

Remark 2.15. Let $\hat{j}$ be the integer satisfying $1 \in M_{\hat{j}}$. Then $M_{\hat{j}}$ is a nice subset, since

$$\bigcup_{\mu \in M_{\hat{j}}} \{a(\mu-1)+1,\ q(\mu)\} \ni a(0)+1 = 1\,.$$

In particular, if the germ $F|_E$ is irreducible, we have that $M_1 = \{1, \cdots, m\}$ is a nice subset.

Now we can give a precise definition of "good extension" as follows:

Definition 2.16. Let $\Phi \in \mathcal{O}_{M, x^0}$ be an approximate solution of the Cauchy problem (1) at e^0 of a finite approximation order $k \in \boldsymbol{N}$.

1° In *the case* $p=1$, we say Φ is a *good extention* of the Cauchy data ϕ if the Newton polygon $N(f^{\Phi})$ satisfies the coprimeness condition (Definition 2.5).

2° In *the case* $p \geqq 2$, we say Φ is a *good extention* of the Cauchy data ϕ if the following conditions 1)∼4) holds:

1) *The Newton polygon* $N(f^{\Phi})$ *satisfies the coprimeness condition.*
2) $N(f^{\Phi})$ *is stable in the direction of the tangent vector* $L_F(e^0) \in T_{x^0}S$ (*Definition* 2.8 *and Remark* 2.7).
3) *The subsets* M_j *are all nice for* $1 \leqq j \leqq r$ (*Definition* 2.14).
4) *The approximation order* $k \in \boldsymbol{N}$ *of* Φ *is greater than* $\kappa(m)^{-1}$:

$$k := \mathrm{ord}_{x^0}[F(x\,;\, d\Phi(x),\ \Phi(x))] > \kappa(m)^{-1} \tag{2.18}$$

where $-\kappa(m)$ *is the slope of the rightest segment of* $\mathrm{N}(f^{\Phi})$.

Remark 2.17. In the case that $p=1$, there exists an good extention of the Cauchy data ϕ. Thus our assumption [A. 3] is trivial if $p=1$ in [A. 2].

Proof. Since $p=1$ it suffices to find Φ satisfying the first condition of Definition 2.5, that is, $\mathrm{ord}_{x^0}[f^{\Phi}(y, 0)]<\infty$. Recall that, by Theorem 1.6, we can find a unique holomorphic solution $u\in\mathcal{O}_{M,x^0}$ of (1) such that

$$(x^0; du(x^0), u(x^0))=e^0. \tag{1.13}$$

We construct a desired approximate solution Φ of (1) of the following form:

$$\Phi(x)=u(x)+s(x)w(x) \qquad (w\in\mathcal{O}_{M,x^0}).$$

Since $d\Phi(y)=du(y)+w(y)ds(y)$ on S, if we choose w as $w(x^0)=0$ then Φ is an approximate solution of (1) at e^0. By the Taylor expansion and the equation $F(x; du(x), u(x))=0$, we have

$$\begin{aligned}F(y; d\Phi(y), \phi(y))&=\sum_{|\alpha|\geqq 1}(\partial_{\xi}^{\alpha}F)(y, du(y), \phi(y))w(y)^{|\alpha|}\{ds(y)\}^{\alpha}\\ &=\langle L_F(y; du(y), \phi(y)), ds(y)\rangle w(y)+O(w(y)^2) \qquad (y\in S).\end{aligned}$$

Recall that $\langle L_F(e^0), ds(x^0)\rangle\neq 0$ if $p=1$, which yields that

$$\mathrm{ord}_{x^0}[f^{\Phi}(y, 0)]=\mathrm{ord}_{x^0}[F(y: d\Phi(y), \phi(y)]=\mathrm{ord}_{x^0}[w(y)].$$

Hence we get a desired Φ if we choose w as $\mathrm{ord}[w|_S]<\infty$.

The proof of Remark 2.17 is complete. Q. E. D.

Example 2.18. *Recall the typical Example* 0.1 *under the assumptions* (a), (b) *and* (c). *Then* $\Phi(x_1, x_2):=0$ *is a good extension of the data* $\phi(x_2):=0$.

Proof. By Remark 2.17 we may assume $p\geqq 2$. Since

$$\begin{aligned}F(x_1, x_2; \partial_{x_1}\Phi, \partial_{x_2}\Phi)&=\prod_{\mu=1}^{m}(\xi_1^{p(\mu)}-x_2^{q(\mu)})-\xi_2|_{(\xi_1,\xi_2)=(0,0)}\\ &=\prod_{\mu=1}^{m}(-x_2)^{q(\mu)},\end{aligned}$$

the approximation order of $\Phi\equiv 0$ is $q:=q(1)+\cdots+q(m)$. Note that the inequality $q\leqq\kappa(m)^{-1}$ implies $p:=p(1)+\cdots+p(m)=1$. Hence we have the inequality (2.18) in the case $p\geqq 2$. By the definition, we also have

$$f^{\Phi}(y, \tau)=F(0, y; \tau, 0)=\prod_{\mu=1}^{m}(\tau^{p(\mu)}-y^{q(\mu)}).$$

Then the assumption (b) [or (c) resp.] means that $\mathrm{N}(f^{\Phi})$ satisfies the coprimeness condition [or, that $\mathrm{M}_{\mu}=\{\mu\}$ is all nice for $1\leqq\mu\leqq m$]. On the other hand, the stability of $\mathrm{N}(f^{\Phi})$ in the direction of $L_F(e^0)=-\partial_{x_2}$ is trivial because $S=\{x_1=0\}$

is one-dimensional. Q. E. D.

Example 2.19. *In $\boldsymbol{C}^3$, we consider the following Cauchy problem*

$$(2.19)\qquad \begin{cases} F(y, z; u_x, u_y) := u_x^5 - y(1+y^2)u_x^3 + (y^4+z^4)u_x^2 + y^4 u_x - u_y = 0 \\ u(0, y, z) = \phi(y, z) := (1/8)y^8 + (1/4)y^4 z^4 \end{cases}$$

with a base point $e^0 := (0; 0, 0) \in J^1\boldsymbol{C}^3 \cap F^{-1}(0)$ *Then, a local extension*

$$\Phi(x, y, z) = xA(x, y, z) + \phi(y, z) \qquad (A \in \mathcal{O}_{C^3, 0})$$

is a good extension of the data ϕ if and only if the germ

$$(2.20)\qquad A(x, y, z) = x^3 \Gamma(x, y, z) + \sum_{i=0}^{2} x^i \{\alpha_i(z) + y\beta_i(y, z)\}$$

satisfies

$$(2.21)\qquad \begin{cases} \beta_i \in (y, z)^{3-i} \quad \text{for } i = 0, 1, 2 \text{ and} \\ \alpha_0 \in (z)^3. \end{cases}$$

Proof. We put $a(y, z) := A(0, y, z)$. Then we have

$$f^{\Phi}(y, z, \xi) = \{(\xi + a)^2 - y^3\}\{(\xi + a)^3 - y(\xi + a) + y^4 + z^4\}$$

which gives an irreducible decomposition of f^{Φ}, since

$$\begin{cases} f_1^{\Phi} := (\xi + a)^2 - y^3 \\ f_2^{\Phi} := (\xi + a)^3 - y(\xi + a) + y^4 + z^4 \end{cases}$$

are both irreducible in $\mathcal{O}_{S \times C, (0, 0)}$.

Claim (1). $\mathrm{N}(f^{\Phi})$ *satisfies the coprimeness condition if and only if*

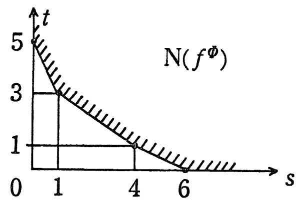
$$(2.22)\qquad a \in (y, z)^2.$$

Indeed, the necessity of (2.22) is obtained since, if we assume that (2.22) is not true, then it follows

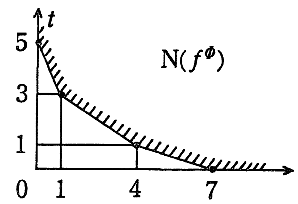
$$\mathrm{N}(f_1^{\Phi}) = \{(s, t); s + t \geqq 2, s \geqq 0, t \geqq 0\}$$

which does not satisfy the coprimeness condition. This contradicts the assertion 2) in Proposition 2.12. Conversely if we assume (2.22), then the Newton polygon $\mathrm{N}(f^{\Phi}) = \mathrm{N}(f_1^{\Phi}) + \mathrm{N}(f_2^{\Phi})$ is given by one of the following (2.23). Hence we have the coprimeness condition of $\mathrm{N}(f^{\Phi})$:

(2.23)



The case $a\in(y, z)^2-(y, z)^3$



The case $a\in(y, z)^3$

Note that, under the condition (2.22), the subsets $M_1=\{2\}$, $M_2=\{1, 3\}$ of $\{1, 2, 3\}$ are both nice subsets.

Claim (2). *Under the condition* (2.22), $N(f^{\Phi})$ *is stable in the direction of* $L_F(e^0)=-\partial_y$ *if and only if one of the following* (2.24)-(2.26) *holds*:

$$a\in(y, z)^2-(y, z)^3 \quad \textit{and} \ \mathrm{Loc}[a](\partial_y)\neq 0. \tag{2.24}$$

$$a\in(y, z)^3-(y, z)^4 \quad \textit{and} \ \mathrm{Loc}[a](\partial_y)\neq 1. \tag{2.25}$$

$$a\in(y, z)^4. \tag{2.26}$$

Indeed, since we easily observe

$$\mathrm{Loc}[\partial_\xi^i f^{\Phi}(y, z, 0)](\partial_y)\neq 0 \qquad \text{for } i=1, 3, 5 \tag{2.27}$$

we only have to consider (2.27) for $i=0$. Note that

$$\begin{aligned}
&\mathrm{Loc}[f^{\Phi}(y, z, 0)](Y\partial_y+Z\partial_z)\\
&\quad=\{\mathrm{Loc}[a^2-y^3]\,\mathrm{Loc}[a^3-ya+y^4+z^4]\}(Y\partial_y+Z\partial_z)\\
&\quad=\begin{cases} Y^4\,\mathrm{Loc}[a](Y\partial_y+Z\partial_z) & \text{if } a\in(y, z)^2-(y, z)^3.\\ Y^3\{Y\,\mathrm{Loc}[a](Y\partial_y+Z\partial_z)-Y^4-Z^4\} & \text{if } a\in(y, z)^3-(y, z)^4.\\ -Y^3(Y^4+Z^4) & \text{if } a\in(y, z)^4.\end{cases}
\end{aligned}$$

Thus we get Claim (2) as desired.

Claim (3). *An extension* $\Phi=xA+\phi$ *has an approximation order greater than* $\kappa(3)^{-1}$ *if and only if either the following* (2.28) *or* (2.29) *holds*:

$$a\in(y, z)^2-(y, z)^3 \quad \textit{and} \quad \partial_y A\in(x, y, z)^2. \tag{2.28}$$

$$a\in(y, z)^3 \qquad\qquad \textit{and} \quad \partial_y A\in(x, y, z)^3. \tag{2.29}$$

Indeed, we easily observe

$$F(y, z\,; \partial_x\Phi, \partial_y\Phi)\equiv -x\partial_y A \qquad \mathrm{mod}(x, y, z)^4.$$

On the other hand, by the figure (2.23), we have

$$1/\kappa(3)=\begin{cases} 2 & \text{if } a\in(y,z)^2-(y,z)^3. \\ 3 & \text{if } a\in(y,z)^3. \end{cases}$$

Hence we get Claim (3).

Note that (2.24) and (2.28) are not compatible since, under (2.24), we have $\mathrm{ord}[\partial_y A]\leqq\mathrm{ord}[\partial_y a]=1$. Thus it suffices for the conclusion (2.21) to consider the compatibility conditions of (2.25) and (2.29), or of (2.26) and (2.29). Note that, for a germ $A(x, y, z)$ of the form (2.20), we have

$$(2.30)\qquad \begin{cases} (2.29) \Leftrightarrow a\in(y,z)^3 \ \text{ and } \ y\partial_y\beta_i+\beta_i\in(y,z)^{3-i} \quad (i=0, 1, 2) \\ \qquad\quad \Leftrightarrow a\in(y,z)^3 \ \text{ and } \ \beta_i\in(y,z)^{3-i} \quad (i=0, 1, 2) \end{cases}$$

since the operator $y\partial_y+1$ preserves the vanishing order of β_i. In particular we have $\beta_0\in(y, z)^3$ which yields the equivalence

$$(2.31)\qquad a=\alpha_0(z)+y\beta_0(y, z)\in(y, z)^3-(y, z)^4 \Leftrightarrow \alpha_0\in(z)^3-(z)^4.$$

Hence we get

$$(2.32)\qquad \mathrm{Loc}[a](\partial_y)=0\ (\neq 1) \qquad \text{if } a\in(y, z)^3-(y, z)^4.$$

From these (2.30)-(2.32) we conclude that Φ is a good extension if and only if the condition (2.21) holds, as desired.

Thus the assertion of Example 2.19 is proved. Q. E. D.

§ 3. Germs of Analytic Coverings

In this section we prepare several geometric notions such as finite holomorphic maps, germs of analytic coverings, which are needed to state our main result in § 4. We refer [Gr-Re] for this section.

Definition 3.1. Let X [or, Y resp.] be an analytic set of a domain D [D'] in $\boldsymbol{C}^N$ [$\boldsymbol{C}^{N'}$], that is, locally at any $x\in X$ [$y\in Y$], X [Y] is defined as a common zero set of finitely many holomorphic germs

$$g_1, \cdots, g_i\in\mathcal{O}_{D,x}\ [h_1, \cdots, h_j\in\mathcal{O}_{D',y}].$$

A continuous map $f: X\to Y$ is called a *holomorphic map* if there exists a holomorphic map $g: D\to D'$ in the sense of theory of complex manifolds such that the map f is induced by g, that is, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ \cap\downarrow & & \cap\downarrow \\ D & \xrightarrow{\ g\ } & D' \end{array}$$

Example 3.2. The map germ $\Psi:(V,(0,e^0))\to(F^{-1}(0),e^0)$ defined by (1.9) is a holomorphic map since it is induced by the map germ $\Psi^{\sim}$ defined by (1.7).

Definition 3.3. Let X, Y be analytic sets, and let $f:X\to Y$ be a holomorphic map. We call f is a *finite map* if f is a closed map and each fiber $f^{-1}(y)$ of $y\in Y$ is a finite subset in X.

Lemma 3.4 [Gr-Re; Proposition 3.1.2, p. 63]. *Let $f:X\to Y$ be a holomorphic map. Suppose $x\in X$ is an isolated point of the fiber $f^{-1}(f(x))$. Then there exist open neighborhoods U of x in X and V of y in Y with $f(U)\subset V$ such that the induced map*

$$f_{U,V}:U\longrightarrow V$$

is a finite map.

This Lemma 3.4 asserts that the notion of finite maps is localizable, that is, a notion of finite holomorphic map germs makes sense:

Definition 3.5. A holomorphic map germ $f:(X,x)\to(Y,y)$ is called a *finite holomorphic map germ* if there exist open neighborhoods U of x in X and V of y in Y with $f(U)\subset V$ such that the germ f has a finite holomorphic representative $f_{U,V}:U\to V$.

Criterion 3.6. *A holomorphic map germ $f:(X,x)\to(Y,y)$ is a finite map germ if and only if x is isolateted in the fiber $f^{-1}(y)$.*

Proof. The "only if" part is trivial since any finite holomorphic representative $f_{U,V}$ of f has a finite fiber $(f_{U,V})^{-1}(y)$. The "if" part is easily obtained by applying Lemma 3.4 to a holomorphic representative of f. Q. E. D.

Definition 3.7. A holomorphic map germ $f:(X,x)\to(Y,y)$ is called an *open holomorphic map germ* if there exist open neighborhoods U of x in X and V of y in Y with $f(U)\subset V$ such that the germ f has a holomorphic representative $f_{U,V}:U\to V$ which is open at x, that is, for any open neighborhood $U^{\sim}$ of x in X with $U^{\sim}\subset U$, the image $f_{U,V}(U^{\sim})$ is an open neighborhood of y in Y.

Example 3.8. Let

$$w(x,z):=z^k+\sum_{j=1}^{k}w_j(x)z^{k-j}\in\mathcal{O}_{C^n,0}[z]$$

be a Weierstrass polynomial in z of degree k, that is, $w_j(0)=0$ $(1\leqq j\leqq k)$. We set $X:=w^{-1}(0)$ and $Y:=\boldsymbol{C}^n$. Then the holomorphic map germ f defined by

$$\begin{array}{ccc} (X,(0,0)) & \hookrightarrow & (Y\times \boldsymbol{C},(0,0)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \text{projection}} \\ & \longrightarrow & (Y,0) \end{array}$$

is a finite open holomorphic map germ.

Definition 3.9. Let $f:(X, x)\to(Y, y)$ be a finite open holomorphic map germ. The germ f is called a *germ of an analytic covering* of (Y, y), if there exists a germ (Σ, y) of a nowhere dense analytic subset of Y at y such that

1) $(f^{-1}(\Sigma), x)$ is a germ of a nowhere dense analytic subset of X at x.
2) The induced map germ

$$f^{\sim}:(X-f^{-1}(\Sigma), x)\longrightarrow(Y-\Sigma, y)$$

is a locally biholomorphic map germ.

Remark 3.10. Let $f:(X, x)\to(Y, y)$ be a germ of an analytic covering of (Y, y), and let (Σ_i, y) be germs of nowhere dense analytic subsets of Y at y satisfying the conditions 1) and 2) in Definition 3.9 for $i=1, 2$. Then the intersection germ $(\Sigma_1\cap\Sigma_2, y)$ also satisfies the conditions 1) and 2).

Definition 3.11. By virtue of Remark 3.10 and of the Noether property of the ring $\mathcal{O}_{Y,y}$ [Gr-Re; Corollary 2.2.1, p. 44], there exists a unique germ (Σ_0, y) of a nowhere dense analytic subset of Y at y such that (Σ_0, y) is minimal in such germs (Σ, y) satisfying the conditions 1) and 2) in Definition 3.9. This germ (Σ_0, y) is called the *critical locus* of the germ f of an analytic covering.

Definition 3.12. Let $f:(X, x)\to(Y, y)$ be a germ of an analytic covering of (Y, y) with a critical locus (Σ, y). Then the following germ $\nu(z)$ of a function determined by

$$\nu(z):=\sharp f^{-1}(z) \qquad \text{for } z\in(Y-\Sigma, y)$$

is locally constant. In particular, if $(Y-\Sigma, y)$ is connected, then $\nu(z)$ is constant there:

$$\nu(z)\equiv\exists\nu\in\boldsymbol{N} \qquad \text{on } (Y-\Sigma, y). \tag{3.1}$$

When this (3.1) occurs, we call that f is a *ν-sheeted* germ of an analytic covering of (Y, y).

Remark 3.13. It is known that if (Y, y) is a germ of a complex *manifold* then, for any germ (Σ, y) of a nowhere dense analytic subset of Y at y, it follows that $(Y-\Sigma, y)$ is connected. This fact is a direct consequence of the following

Riemann's Extension Theorem [Gr-Re; Theorem 7.1.3, p. 132]. *Let X be a complex manifold, let A be a nowhere dence analytic subset of X and let $f\in\mathcal{O}(X-A)$ be a holomorphic function on $X-A$. Assume that f is bounded near A. Then f has a unique holomorphic extension $f^\wedge$ to X.*

Example 3.14. Let $X:=\boldsymbol{C}$, and let Y be a complex curve in $\boldsymbol{C}^2$ defined by

$$Y:=\{(y, z);\ y^p-z^q=0\}.$$

If the integers p and q are coprime, then the following map germ

$$f: X\longrightarrow Y \quad f(x):=(x^q, x^p)$$

is a one-sheeted germ of an analytic covering of $(Y, (0, 0))$ with a critical locus $\Sigma=\{(0, 0)\}$.

Proof. It suffices to show the existence of an inverse map germ

$$g: Y-\{(0, 0)\}\longrightarrow X-\{0\}$$

of $f|_{X-\{0\}}$. We first remark that

$$Y-\{(0, 0)\}\subset Y\cap\{(y, z);\ yz\neq 0\}. \tag{3.2}$$

Since p and q are coprime, we can find integers a, b such that

$$ap+bq=1. \tag{3.3}$$

We define $g(y, z):=y^b z^a$. Then (3.2) yields $g\in\mathcal{O}(Y-\{(0, 0)\})$. Moreover (3.3) implies that

$$(g\circ f)(x)=(x^q)^b(x^p)^a=x^{ap+bq}=x \qquad \text{for any } x\in X-\{0\}.$$

On the other hand, since $y^p=z^q$ on Y, we also have

$$\begin{aligned}(f\circ g)(y, z)&=((y^b z^a)^q, (y^b z^a)^p))=(y^{bq}(z^q)^a, (y^p)^b z^{ap})\\&=(y^{bq}(y^p)^a, (z^q)^b z^{ap})=(y^{ap+bq}, z^{ap+bq})\\&=(y, z) \qquad \text{for any } (y, z)\in Y-\{(0, 0)\}.\end{aligned}$$

Thus the assertion is proved as desired. Q. E. D.

Chapter II. Results

§ 4. Statement of the Main Result

In this section we state our main result (Main Theorem 4.2) and show its corollaries.

We return to the situation at where the classical Theorem 1.6 is stated. We assume the conditions [A. 1, 2, 3] and recall the diagram in Theorem 1.6:

$$(4.1)\qquad \begin{array}{ccc} (V,(0,e^0)):=(\boldsymbol{C},0)\times(E\cap F^{-1}(0),e^0) & \xrightarrow{\Psi} & (F^{-1}(0),e^0) \\ \downarrow \pi_V & & \downarrow \\ (M,x^0) & \xleftarrow{\text{projection}} & (J^1M,e^0) \end{array}$$

We also recall the irreducible decomposition

$$(4.2)\qquad F|_E=\prod_{j=1}^{r} F_j^{\nu(j)}=\prod_{j=1}^{r} F_j \qquad (\nu(j)=1, \text{ see Proposition 2.12})$$

locally at $e^0\in E$.

We define germs $(V_j, (0, e^0))$ for $1\leqq j\leqq r$, of analytic hypersurfaces of $\boldsymbol{C}\times E$ at $(0, e^0)$ by setting

$$(4.3)\qquad (V_j,(0,e^0)):=(\boldsymbol{C},0)\times(F_j^{-1}(0),e^0).$$

Then the germ $(V, (0, e^0))$ can be decomposed into the following union of irreducible components at $(0, e^0)$:

$$(4.4)\qquad (V,(0,e^0))=\bigcup_{j=1}^{r}(V_j,(0,e^0)).$$

Now we consider the following r-diagrams instead of (4.1):

$$(4.5)_j\qquad \begin{array}{ccc} (V_j,(0,e^0)) & \xrightarrow{\Psi_j:=\Psi|_{V_j}} & (F^{-1}(0),e^0) \\ \downarrow \pi_j:=\pi_V|_{V_j} & & \downarrow \\ (M,x^0) & \longleftarrow & (J^1M,e^0) \end{array} \qquad \text{for } 1\leqq j\leqq r.$$

Our result asserts that the map germs π_j: $(V_j, (0, e^0))\to(M, x^0)$ are germs of analytic coverings of (M, x^0) and that their numbers of sheets are calculable by means of the Newton polygon $\mathrm{N}(f^{\Phi})$, where Φ is a good extension at e^0 of the Cauchy data ϕ.

Definition 4.1. 1) For $1\leqq\mu\leqq m:=\#\,\mathrm{Seg}\,\mathrm{N}(f^{\Phi})$ we set

$$(4.6)\qquad v(\mu):=p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}$$

where $a(\mu)$, $b(\mu)$, $p(\mu)$ and $q(\mu)$ are the integers determined by Notation 2.4.

2) We define integers v_j for $1\leqq j\leqq r$ by

$$(4.7)\qquad v_j:=\sum_{\mu\in\mathrm{M}_j} v(\mu)$$

where M_j $(\subset\{1, 2, \cdots, m\})$ is the subset which is defined by the assertion 3) in Proposition 2.12.

Now the time has come to state our main result:

Main Theorem 4.2. *Assume the conditions* [A.1], [A.2] *and* [A.3]. *Then, for* $1 \leqq j \leqq r$, *the following statements hold:*

1) *The holomorphic map germ* $\pi_j : (V_j, (0, e^0)) \to (M, x^0)$ *determined by the diagram* $(4.5)_j$ *is a* v_j*-sheeted germ of an analytic covering of* (M, x^0).

2) *Let* (Σ_j, x^0) *be the critical locus of* π_j. *We define a germ* $u_j(x)$ *on* $(M-\Sigma_j, x^0)$ *of a multi-valued analytic function by the following diagram* $(4.8)_j$:

$$(4.8)_j \qquad \begin{array}{ccccc} (V_j-\pi_j^{-1}(\Sigma_j), (0, e^0)) & \hookrightarrow & (V_j, (0, e^0)) & \xrightarrow{\Psi_j} & (F^{-1}(0), e^0) \\ \downarrow \pi_j^{\sim} & & \downarrow \pi_j & & \cap\downarrow \\ (M-\Sigma_j, x^0) & \hookrightarrow & (M, x^0) & \xleftarrow{\text{projection}} & (J^1M, e^0) \\ \vdots\, u_j & & & & \mid \\ \dashrightarrow & \dashrightarrow & (\boldsymbol{C}_z, z^0) & \xleftarrow{\text{projection}} & \lrcorner \end{array}$$

Then the germ u_j *is exactly* v_j*-valued, that is, for* $\forall x \in (M-\Sigma_j, x^0)$ *the multi-valued germ* u_j *has* v_j*-branches* $u_{j,x}^{(i)} \in \mathcal{O}_{M,x}$ $(1 \leqq i \leqq v_j)$ *such that any two branches of* u_j *can be continued each other along a path in* $(M-\Sigma_j, x^0)$.

Remark 4.3. If the assertion 1) of Main Theorem 4.2 is established then the multi-valued germ u_j is well-defined by the diagram $(4.8)_j$, since the induced map germ $\pi_j^{\sim} : (V_j-\pi_j^{-1}(\Sigma_j), (0, e^0)) \to (M-\Sigma_j, x^0)$ is a locally biholomorphic map germ.

Remark 4.4. Main Theorem 4.2 includes Theorem 1.6 as a special case. Indeed, if $p=1$ then there exists an approximate solution Φ of (1) such that $q := \mathrm{ord}_{x^0}[f^{\Phi}] < \infty$ (Remark 2.17). Thus we have $\mathrm{N}(f^{\Phi}) = \mathrm{N}_{q,1}$ (Notation 2.10). Hence Theorem 4.2 yields that the ramification degree is given by

$$\begin{aligned} v_1 = v(1) &= p(1)\{a(0)+1\} + q(1)\{p-b(0)-1\} \\ &= p \cdot 1 + q(p-1) \\ &= 1\,. \end{aligned}$$

In the remaining part of this section, we state and show the following Corollaries 4.6 and 4.7.

The first one is related to the analytic continuations of holomorphic local solutions of the Cauchy problem (1). To state this, we prepare the

Lemma 4.5. *Assume* [A.1, 2, 3]. *We define a germ* Ω *by*

$$(4.9) \qquad \Omega := \{e = (y\,;\, \xi, \phi(y)) \in (E \cap F^{-1}(0), e^0)\,;\, \mathrm{ord}_e[F|_{E_y}] = 1\}.$$

Then the following 1) *and* 2) *hold:*

1) Ω *is a non-empty germ at* e^0.

2) *For any* $e\in\Omega$ *there exists a unique* j $(1\leqq j\leqq r)$ *such that*

$$e\in F_j^{-1}(0)-\bigcup_{i\neq j}F_i^{-1}(0)\,. \tag{4.10}$$

Lemma 4.5, Theorem 1.6 and Main Theorem 4.2 immediately yield the

Corollary 4.6. *Let* $e=(y\,;\,\xi,\,\phi(y))\in\Omega$ *and let* j *be the unique number satisfying* (4.10). *Then the following* 1) *and* 2) *hold:*

1) *There exists a unique holomorphic local solution* $u_y^{\sim}\in\mathcal{O}_{M,y}$ *of the Cauchy problem* (1) *satisfying* $(y\,;\,du_y^{\sim}(y),\,u_y^{\sim}(y))=e$.

2) *The holomorphic local solution* $u_y^{\sim}\in\mathcal{O}_{M,y}$ *mentioned in* 1) *can be continued analitically to the multi-valued germ* u_j *determined by* $(4.8)_j$ *on* $(M-\Sigma_j,\,x^0)$. *Hence the analytic continuation of* $u_y^{\sim}$ *around the point* $x^0\in M$ *is exactly* v_j*-valued.*

Proof of Lemma 4.5. We only have to verify the assertion 1), since the assertion 2) is a consequence of the fact

$$\Omega\cap\bigcup_{i\neq j}(F_i^{-1}(0)\cap F_j^{-1}(0))=\emptyset\,.$$

It suffices for the assertion 1) to show that, for any open neighborhood U of $(x^0,\,0)$ in $S\times\boldsymbol{C}$, and for $1\leqq j\leqq r$, it follows that

$$f_j^{-1}(0)\cap U-[(\partial_\tau f_j)^{-1}(0)\cup\bigcup_{i\neq j}f_i^{-1}(0)]\neq\emptyset \tag{4.11}$$

where we set

$$f_j(y,\,\tau):=(\gamma_\Phi^*F_j)(y,\,\tau)=F_j(y\,;\,\tau ds(y)+d\Phi(y),\,\phi(y))\,.$$

Recall that the coprimeness condition yields that the germ $F|_E$ has no multiple factor F_j (see Proposition 2.12). Therefore the factorization

$$f^{\Phi}(y,\,\tau)=\prod_{j=1}^{r}f_j(y,\,\tau) \tag{4.12}$$

is an irreducible decomposition of the germ $f^{\Phi}:=\gamma_\Phi^*(F|_E)$.

We show (4.11) by contradiction. If we assume that (4.11) is not true then we can find an open neighborhood U and a number j such that

$$f_j^{-1}(0)\cap U\subset(\partial_\tau f_j)^{-1}(0)\cup\bigcup_{i\neq j}f_i^{-1}(0)\,. \tag{4.13}$$

We set $h(y,\,\tau):=\partial_\tau f_j(t,\,\tau)\times\prod_{i\neq j}f_i(y,\,\tau)$. Then (4.13) yields

$$h\Big|_{f_j^{-1}(0)}\equiv 0\,. \tag{4.13$'$}$$

Hence, by virtue of the *Rückert's Nullstellensatz* (see §13), we have

$$h\in\text{Rad}\,[(f_j)]:=\{g\in\mathcal{O}_{S\times C,(x^0,0)}\,;\,\exists k\in\boldsymbol{N},\,g^k\in(f_j)\}.$$

Since the ideal (f_j) is a prime ideal, it follows that $h\in(f_j)$ thus we have

$$\partial_\tau f_j \in (f_j) \qquad \text{or} \qquad \exists i(\neq j)\ f_i \in (f_j).$$

This contradicts the facts that f_j is irreducible and that (4.12) is an irreducible decomposition. Hence (4.11) follows. Q.E.D.

Our second corollary (Corollary 4.7 below) asserts the converse of the classical Theorem 1.6 under the assumptions [A.1, 2, 3].

Corollary 4.7. *Assume the conditions* [A.1, 2, 3]. *If there exists a holomorphic local solution* $u \in \mathcal{O}_{M,x^0}$ *of the Cauchy problem* (1) *satisfying* $(x^0; du(x^0), u(x^0)) = e^0$, *then it follows that*

$$\mathrm{ord}_{e^0}[F|_{E_{x^0}}]=1.$$

Proof. By virtue of Main Theorem 4.2, the assumption of Corollary 4.7 yields that there exists a number j $(1 \leqq j \leqq r)$ such that

$$v_j = \sum_{\mu \in \mathrm{M}_j} v(\mu) = 1. \tag{4.14}$$

By the definition (4.6) of $v(\mu)$ we have

$$\begin{aligned} v(\mu) &= p(\mu)\{a(\mu-1)+1\} + q(\mu)\{p-b(\mu-1)-1\} \\ &\geqq p(\mu)\{a(\mu-1)+1\} \\ &\geqq p(\mu) \\ &\geqq 1 \end{aligned}$$

because $p(\mu), q(\mu) > 0$ and $a(\mu-1),\ p-b(\mu-1)-1 \geqq 0$. Hence (4.14) implies

$$\begin{cases} \sharp \mathrm{M}_j = 1, \text{ that is, } \mathrm{M}_j = \{\exists \mu^0\} \text{ and} \\ p-b(\mu^0-1)-1 = a(\mu^0-1) = 0 \text{ and } p(\mu^0)=1 \end{cases}$$

which yield

$$\mu^0 = m = 1 \quad \text{and} \quad p = p(1) = 1.$$

Hence we get Corollary 4.7 as desired. Q.E.D.

Example 4.8. Let us recall Example 2.19. We calculate the ramification degrees v_j $(j=1, 2)$ of multi-valued analytic solutions of the Cauchy problem (2.19) as follows: By the condition (2.21), any good extension $\Phi = xA(x, y, z) + \phi(y, z)$ of the data ϕ satisfies

$$a(y, z) := A(0, y, z) \in (y, z)^3.$$

Hence, by the right figure of (2.23), it follows that

$$v(1)=6, \quad v(2)=10 \quad \text{and} \quad v(3)=5.$$

Since we have $M_1=\{2\}$, $M_2=\{1, 3\}$, we get

$$v_1=v(2)=10, \qquad v_2=v(1)+v(3)=11$$

which shows that

(4.15) $\left\{\begin{array}{l}\textit{the Cauchy problem (2.19) has two sorts of multi-valued analytic} \\ \textit{solutions } u_1 \textit{ and } u_2 \textit{ of 10-valued and 11-valued respectively.}\end{array}\right.$

Chapter III. Proof

§ 5. Reduction of the Main Theorem to Theorem 5.1

In this section we reduce the proof of Main Theorem 4.2 to that of the following Theorem 5.1.

We consider the following Cauchy problem (5.1) which is defined in an *open neighborhood* M *of the origin* of $\boldsymbol{C}^n$ with the *zero Cauchy data*:

$$\text{(5.1)} \qquad \left\{\begin{array}{l} F(x\,;du(x),\,u(x)):=G(x\,;\partial_{x_1}u,\,\partial_{x''},\,u,\,u)-\partial_{x_n}u=0 \\ u\,|_{x_1=0}\equiv 0 \end{array}\right.$$

where $(x_1, x'', x_n)\in\boldsymbol{C}\times\boldsymbol{C}^{n-2}\times\boldsymbol{C}$. We treat (5.1) with a *base point*

$$e^0=(0\,;0,0)\in J^1M\cap F^{-1}(0)$$

and with the following assumptions [B.1]-[B.4]:

[B.1] $\qquad G(0\,;\xi_1,\xi'',0)\in(\xi_1,\xi'')^2\,.$

[B.2] $\qquad \mathrm{ord}_0[G(0\,;\xi_1,0,0)]=:p\in[2,\infty)$

[B.3] $\qquad \Phi(x):=0$ *is a good extension of the data* $\phi(x'):=0$.

[B.4] $\left\{\begin{array}{l}\textit{The approximation order } \mathrm{ord}_0[F(x\,;0,0)] \textit{ of } \Phi\equiv 0 \textit{ is equal} \\ \textit{to the order } q:=\mathrm{ord}_0[f^{\Phi}(x',0)]=\mathrm{ord}_0[F(0,x'\,;0,0)].\end{array}\right.$

Note that the condition [B.4] is stronger than the inequality

$$\mathrm{ord}_0[F(x\,;0,0)]>\kappa(m)^{-1} \qquad \text{(the fourth condition of [B.3])}$$

since the condition $p\geqq 2$ implies $q>\kappa(m)^{-1}$.

Theorem 5.1. *Under the assumptions* [B.1]-[B.4], *the conclusions* 1) *and* 2) *of Main Theorem* 4.2 *hold for the reduced Cauchy problem* (5.1).

Let us return the situation of Main Theorem 4.2.

We must show that Theorem 5.1 implies Main Theorem 4.2. Note that, since in the case $p=1$ the assertion of Main Theorem 4.2 is contained in that of Theorem 1.6, we may assume $p\geqq 2$, where p is the vanishing order in the condition [A.2], that is, $p=\mathrm{ord}_{\tau=0}[F(x^0\,;\tau ds(x^0)+\xi^0,z^0)]$.

Our reduction starts from a simple

Lemma 5.2. *There exists a local coordinate system* $(x_1, \cdots, x_n)$ *of* M *around* x^0 *such that*

(5.2) $$S=\{x_1=0\} \quad \text{and}$$

(5.3) $$L_F(e^0)=-\partial_{x_n}.$$

Proof. Taking a coordinate system as $x_1=s(x)$, we may assume that a system satisfying (5.2) is already chosen. The assumption $p\geqq 2$ yields $\partial_{\xi_1}F(e^0)=0$, hence, by the condition [A.1], we may assume $\partial_{\xi_n}F(e^0)\neq 0$. We take a linear coordinate transformation of the form

$$x \longmapsto x^{\sim}:=\left[\begin{array}{c|c} I_{n-1} & \begin{matrix} 0 \\ \lambda'' \end{matrix} \\ \hline 0 & \lambda_n \end{array}\right] x$$

and choose (λ'', λ_n) as

$$\lambda_n:=-[\partial_{\xi_n}F(e^0)]^{-1}, \quad \lambda_j:=\lambda_n\partial_{\xi_j}F(e^0) \qquad \text{for } 2\leqq j\leqq n-1.$$

Then it is easily verified that the coordinate system $(x_1^{\sim}, \cdots, x_n^{\sim})$ satisfies the conditions (5.2), (5.3) as desired. Q.E.D.

The next step of our reduction is to show the existence of another good extension $\Phi^{\sim}$ with a "better" approximation order than that of the original Φ:

Proposition 5.3. *Let* Φ *be a good extension of* ϕ. *For the irreducible decomposition* (4.2) *of* $F|_E$ *locally at* e^0, *we set*

(5.4) $$\begin{cases} f^{\Phi}(x', \xi_1):=F(0, x'; \xi_1 dx_1+d\Phi(0, x'), \phi(x'))=\sum_{\nu=0}^{\infty} c_\nu(x')\xi_1^\nu. \\ f_j^{\Phi}(x', \xi_1):=F_j(0, x'; \xi_1 dx_1+d\Phi(0, x'), \phi(x'))=\sum_{\nu=0}^{\infty} c_{\nu,j}(x')\xi_1^\nu. \end{cases}$$

Then there exists a good extension $\Phi^{\sim}$ *of* ϕ *such that*

1) $\mathrm{N}(f_j^{\Phi})=\mathrm{N}(f_j^{\Phi^{\sim}})$ *for all* $j=1, 2, \cdots, r$.

2) $\mathrm{Loc}[c_\nu]=\mathrm{Loc}[c_\nu^{\sim}]$

for $\forall\nu$ *satisfying* $(\mathrm{ord}[c_\nu], \nu)\in \mathrm{Ver}\,\mathrm{N}(f^{\Phi})$ $(=\mathrm{Ver}\,\mathrm{N}(f^{\Phi^{\sim}}))$.

3) $\mathrm{ord}_{x^0}[F(x; d\Phi^{\sim}(x), \Phi^{\sim}(x))]=q$ $(:=\mathrm{ord}_{x^0}[f^{\Phi}(x', 0)])$

where we use the analogous expression of $f^{\Phi^{\sim}}$ [*or* $f_j^{\Phi^{\sim}}$ *resp.*] *which is gained by replacement of* (Φ, c_ν) $[(\Phi, c_{\nu,j})]$ *in* (5.4) *with* $(\Phi^{\sim}, c_\nu^{\sim})$ $[(\Phi^{\sim}, c_{\nu,j}^{\sim})]$.

Proof. We seek the desired "better" extension $\Phi^{\sim}$ as the form

$$\Phi^{\sim}(x)=x_1 w(x)+\Phi(x) \quad (w\in\mathcal{O}_{M,x^0})$$

where $(x_1, \cdots, x_n)$ is a local coordinate system of M at x^0, which is obtained by Lemma 5.2. Note that the assumption $p\geqq 2$ implies

$$q>\kappa(m)^{-1}. \tag{5.5}$$

We denote the approximation order of Φ by k:

$$k:=\mathrm{ord}_{x^0}[F(x; d\Phi(x), \Phi(x))]>\kappa(m)^{-1}. \tag{5.6}$$

Since Proposition 5.3 clearly holds for $\Phi^{\sim}:=\Phi$ if $q\leqq k$, we may assume

$$q>k. \tag{5.7}$$

We first give a sufficient condition of w for the assertions of Proposition 5.3 except for the assertion 3):

Lemma 5.4. *If $w\in\mathcal{O}_{M,x^0}$ satisfies the condition*

$$\mathrm{ord}_{x^0}[w]\geqq k \tag{5.8}$$

then the assertions of Proposition 5.3 *except for the assertion* 3) *follow.*

Proof. Note that $d\Phi^{\sim}(0, x')=w(0, x')dx_1+d\Phi(0, x')$ yields that the germ $\Phi^{\sim}$ is a holomorphic approximate solution of the Cauchy problem (1), since the inequalities (5.6) and (5.8) imply $w(0, 0)=0$. Moreover we have

$$\begin{aligned} f^{\Phi^{\sim}}(x', \xi_1)&=f^{\Phi}(x', \xi_1+w(0, x'))=\sum_{\nu=0}^{\infty} c_\nu(x')\{\xi_1+w(0, x')\}^\nu \\ &=\sum_{\nu=0}^{\infty} c_\nu(x')\sum_{\lambda=0}^{\nu} C(\nu, \lambda)w(0, x')^{\nu-\lambda}\xi_1^\lambda \\ &=\sum_{\lambda=0}^{\infty}\left[\sum_{\nu=\lambda}^{\infty} C(\nu, \lambda)c_\nu(x')w(0, x')^{\nu-\lambda}\right]\xi_1^\lambda \end{aligned} \tag{5.9}$$

where $C(\nu, \lambda)$ denotes $C(\nu, \lambda):=\nu!/(\lambda!(\nu-\lambda)!)$. We similarly have

$$f_j^{\Phi^{\sim}}(x', \xi_1)=\sum_{\lambda=0}^{\infty}\left[\sum_{\nu=\lambda}^{\infty} C(\nu, \lambda)c_{\nu,j}(x')w(0, x')^{\nu-\lambda}\right]\xi_1^\lambda. \tag{5.9$'$}$$

We fix j and first show the following inclusion:

$$\mathrm{N}(f_j^{\Phi^{\sim}})\subset\mathrm{N}(f_j^{\Phi}). \tag{5.10}$$

We define positive integers p_j, $a_j(\mu)$ and $b_j(\mu)$ for $\mu\in\mathrm{M}_j$ by

$$\begin{cases} p_j:=\mathrm{ord}[f_j^{\Phi}(0, \xi_1)] \quad \text{and} \\ a_j(\mu):=\sum_{\lambda\in\mathrm{M}_j,\, \lambda\leqq\mu} q(\lambda), \quad b_j(\mu):=\sum_{\lambda\in\mathrm{M}_j,\, \lambda\leqq\mu} p(\lambda). \end{cases}$$

Then Lemma 0.2 and the assertion 3) of Proposition 2.12 yield that

$$\text{Ver } N(f_j^{\Phi})=\{(a_j(\mu),\ p_j-b_j(\mu))\,;\ \mu\in M_j\}\cup\{(0,\ p_j)\}.$$

Hence $(\operatorname{ord}[c_{\nu,j}],\ \nu)\in N(f_j^{\Phi})$ implies the following inequality:

$$\nu\geqq-\kappa(\mu)\{\operatorname{ord}[c_{\nu,j}]-a_j(\mu)\}+p_j-b_j(\mu) \qquad \text{for } \mu\in M_j\,;\ \nu=0,\ 1,\ \cdots \tag{5.11}$$

where $\kappa(\mu)=p(\mu)/q(\mu)$. Thus (5.9)′ yields the following inequality:

$$\begin{aligned}
&-\kappa(\mu)\{\operatorname{ord}[c_{\lambda,j}^{\sim}]-a_j(\mu)\}+p_j-b_j(\mu)\\
&\quad\leqq\max_{\nu\geqq\lambda}\,[-\kappa(\mu)\{\operatorname{ord}[c_{\nu,j}]+(\nu-\lambda)\operatorname{ord}[w(0,\ x')]-a_j(\mu)\}+p_j-b_j(\mu)]\\
&\quad\leqq\max_{\nu\geqq\lambda}\,[\nu-\kappa(\mu)(\nu-\lambda)\operatorname{ord}[w(0,\ x')]]\\
&\quad=\{1-\kappa(\mu)\operatorname{ord}[w(0,\ x')]\}\ \min_{\nu\geqq\lambda}\{\nu\}+\kappa(\mu)\operatorname{ord}[w(0,\ x')]\lambda\\
&\quad\leqq\lambda
\end{aligned}$$

since (5.8), (5.6) imply $\kappa(\mu)\operatorname{ord}[w(0,\ x')]\geqq\kappa(m)k>1$. Hence (5.10) follows.

Interchanging the roles of Φ and $\Phi^{\sim}$, we also have the converse inclusion of (5.10). Hence we conclude the assertion 1) of Proposition 5.3.

Note that this assertion 1) implies that

$$N(f^{\Phi})=\sum_{j=1}^{r}N(f_j^{\Phi})=\sum_{j=1}^{r}N(f_j^{\Phi^{\sim}})=N(f^{\Phi^{\sim}}) \quad\text{and}\quad M_j=M_j^{\sim}\quad(1\leqq j\leqq r)$$

which show that the Newton polygon $N(f^{\Phi^{\sim}})$ satisfies the coprimeness condition and that all the subsets $M_j^{\sim}$ $(1\leqq j\leqq r)$ are nice subsets of $\{1,\ 2,\ \cdots,\ m\}$.

For the proof of the assertion 2), it suffices to show

$$\operatorname{ord}[c_{p-b(\mu)}]=a(\mu)<\min_{\nu>p-q(\mu)}\{\operatorname{ord}[c_\nu]+(\nu-p+b(\mu))\operatorname{ord}[w(0,\ x')]\} \qquad \text{for } 1\leqq\mu\leqq m \tag{5.12}$$

since we have

$$c_{p-b(\mu)}^{\sim}(x')=c_{p-b(\mu)}(x')+\sum_{\nu>p-b(\mu)}C(\nu,\ p-b(\mu))c_\nu(x')w(0,\ x')^{\nu-p+b(\mu)}$$

by (5.9). By virtue of $\operatorname{ord}[w(0,\ x')]\geqq k$, it suffices for (5.12) to derive

$$a(\mu)<\min_{\nu>p-b(\mu)}\{\operatorname{ord}[c_\nu]+(\nu-p+b(\mu))k\} \qquad \text{for } 1\leqq\mu\leqq m\,. \tag{5.13}$$

From $(\operatorname{ord}[c_\nu],\ \nu)\in N(f^{\Phi})$, we have the inequality

$$\nu\geqq-\kappa(\mu)\{\operatorname{ord}[c_\nu]-a(\mu)\}+p-b(\mu) \qquad \text{for } 1\leqq\mu\leqq m\,;\ \nu=0,\ 1,\ \cdots$$

which is equivalent to $\operatorname{ord}[c_\nu]\geqq a(\mu)-\kappa(\mu)^{-1}(\nu-p+b(\mu))$. Hence it follows

the right hand side of (5.13) $\geqq a(\mu)+\min\limits_{\nu>p-b(\mu)}\{\nu-p+b(\mu)\}\{k-\kappa(\mu)^{-1}\}.$

Thus, by virtue of $k>\kappa(m)^{-1}\geqq\kappa(\mu)^{-1}$, we get (5.13).

Note that the assertion 2) of Proposition 5.3 implies the stability of the

Newton polygon $\mathrm{N}(f^{\Phi^{\sim}})$ in the direction of the tangent vector $L_F(e^0)$. Thus if we establish the assertion 3) of Proposition 5.3, then the holomorphic approximate solution $\Phi^{\sim}$ is a good extension of the Cauchy data $\phi(x')$.

The proof of Lemma 5.4 is complete. Q.E.D.

We continue the proof of Proposition 5.3. If suffices to find a germ $w(x)$ satisfying (5.8) and the assertion 3) in Proposition 5.3 when we set $\Phi^{\sim}$ as

$$\Phi^{\sim}:=x_1 w(x)+\Phi .$$

Taking the Taylor expansion of F at $(x;\xi, z)=(x; d\Phi(x), \Phi(x))$, we have

$$\begin{aligned}(5.14)\quad F(x; d\Phi^{\sim}(x), \Phi^{\sim}(x))=&\sum_{\alpha_1=0}^{\infty}(\alpha_1!)^{-1}\partial_{\xi_1}^{\alpha_1}F(x; d\Phi(x), \Phi(x))(x_1\partial_{x_1}w+w)^{\alpha_1}\\ &+\sum_{|\alpha'|+i\geqq 1,\ \alpha_1\geqq 0}(\alpha!i!)^{-1}\partial_\xi^\alpha\partial_z^i F(x; d\Phi(x), \Phi(x))\\ &\times(x_1\partial_{x_1}w+w)^{\alpha_1}(x_1\partial_{x'}w)^{\alpha'}(x_1 w)^i .\end{aligned}$$

Note that the coefficients of the first term in (5.14) have the following expressions (5.15): Since there exists a holomorphic germ $g(x, \xi_1)$ such that

$$\begin{aligned}F(x;\xi_1 dx_1+d\Phi(x), \Phi(x))&=F(0, x';\xi_1 dx_1+d\Phi(0, x'), \phi(x'))+x_1 g(x, \xi_1)\\ &=f^{\Phi}(x', \xi_1)+x_1 g(x, \xi_1),\end{aligned}$$

we have

$$\begin{aligned}(5.15)\qquad \partial_{\xi_1}^{\alpha_1}F(x; d\Phi(x), \Phi(x))&=\partial_{\xi_1}^{\alpha_1}f^{\Phi}(x', 0)+x_1\partial_{\xi_1}^{\alpha_1}g(x, 0)\\ &=(\alpha_1!)\{c_{\alpha_1}(x')+x_1\exists g_{\alpha_1}(x)\}\qquad \text{for } \alpha_1\geqq 0 .\end{aligned}$$

In particular, taking $\alpha_1=0$, we note

$$(5.16)\qquad \mathrm{ord}[g_0]=\mathrm{ord}[F(x; d\Phi(x), \Phi(x))-c_0(x')]-1\geqq k-1 .$$

On the other hand, the second term in the right hand side of (5.14) can be divided by x_1, thus it can be written as the form

$$\begin{aligned}(5.17)\qquad x_1\Big[\sum_{j=2}^{n}\partial_{\xi_j}F(x; d\Phi(x), \Phi(x))\partial_{x_j}w+\partial_z F(x; d\Phi(x), \Phi(x))w\\ +\exists K(x; x_1\partial_{x_1}w+w, \partial_{x'}w, w)\Big]\end{aligned}$$

where K satisfies

$$(5.18)\qquad K(x; x_1\xi_1+z, \xi', z)\in(x_1\xi_1+z)+(x_1)(\xi', z)^2 .$$

Hence, from (5.15) and (5.17), we can write (5.14) as the following form:

$$(5.19)\quad F(x\,;d\Phi^{\sim}(x),\,\Phi^{\sim}(x))=\sum_{\alpha_1=0}^{\infty}\{c_{\alpha_1}(x')+x_1g_{\alpha_1}(x)\}(x_1\partial_{x_1}w+w)^{\alpha_1}$$
$$+x_1\Big[\sum_{j=2}^{n}\partial_{\xi_j}F(x\,;d\Phi(x),\,\Phi(x))\partial_{x_j}w+\partial_zF(x\,;d\Phi(x),\,\Phi(x))w$$
$$+K(x\,;x_1\partial_{x_1}w+w,\,\partial_{x'}w,\,w)\Big].$$

We claim that the following inequality holds under (5.8):

$$(5.20)\qquad \operatorname{ord}[c_{\alpha_1}(x')(x_1\partial_{x_1}w+w)^{\alpha_1}]\geqq q \qquad \text{for all } \alpha_1\geqq 0\,.$$

Indeed, from the inclusion

$$(\operatorname{ord}[c_{\alpha_1}],\,\alpha_1)\in \mathrm{N}(f^{\Phi})\subset\{(s,\,t)\,;\,t\geqq-\kappa(m)(s-q)\}$$

and (5.8), we easily have

$$\textit{the left hand side of } (5.20)\geqq q-\alpha_1\kappa(m)^{-1}+\alpha_1\operatorname{ord}[w]$$
$$\geqq q+\alpha_1\{k-\kappa(m)^{-1}\}$$
$$\geqq q\,.$$

By virtue of (5.19) and (5.20), it suffices for Proposition 5.3 to show the existence of a holomorphic germ $w(x)$ satisfying (5.8) and the following equation (5.21) for some $h_1(x)\in(x)^{q-1}$:

$$(5.21)\qquad \sum_{\alpha_1=0}^{\infty}g_{\alpha_1}(x)(x_1\partial_{x_1}w+w)^{\alpha_1}+\sum_{j=2}^{n}\partial_{\xi_j}F(x\,;d\Phi(x),\,\Phi(x))\partial_{x_j}w$$
$$+\partial_zF(x\,;d\Phi(x),\,\Phi(x))w+K(x\,;x_1\partial_{x_1}w+w,\,\partial_{x'}w,\,w)=\exists h_1(x)\,.$$

We set

$$(5.22)\qquad \begin{cases} h(x):=g_0(x)-h_1(x) \qquad \text{and} \\ H(x\,;\xi,\,z):=\displaystyle\sum_{\alpha_1=1}^{\infty}g_{\alpha_1}(x)(x_1\xi_1+z)^{\alpha_1}+\sum_{j=2}^{n}\partial_{\xi_j}F(x\,;d\Phi(x),\,\Phi(x))\xi_j \\ \qquad\qquad +\partial_zF(x\,;d\Phi(x),\,\Phi(x))z+K(x\,;x_1\xi_1+z,\,\xi',z)+h(x). \end{cases}$$

Note that (5.16) yields

$$(5.23)\qquad \operatorname{ord}[h]\geqq \min\{q,\,k\}-1=k-1\,.$$

We seek a solution w of the non-linear equation

$$(5.21)'\qquad H(x\,;\partial_xw,\,w)=0$$

with the condition (5.8), that is, $\operatorname{ord}[w]\geqq k$. Since the condition (5.18) yields $K\in(x_1,\,z)$, we note

$$(5.24)\qquad H(0\,;\xi,\,0)=\sum_{j=2}^{n}\partial_{\xi_j}F(e^0)\xi_j+h(0)=-\xi_n+h(0)\,.$$

Then the Weierstrass's preparation theorem leads us to

(5.25) $$H(x;\xi,z)=\{H^{\sim}(x;\xi_1,\xi'',z)-\xi_n\}e(x;\xi,z)$$

locally at $(x;\xi,z)=(0;0,\cdots,0,h(0),0)$ for some holomorphic germs

$$H^{\sim}\in\mathcal{O}_{C^n\times C^{n-1}\times C,(0;0,0)},\ e\in\mathcal{O}_{C^n\times C^n\times C,(0;0,\cdots,0,h(0),0)}$$

with $H^{\sim}(0;0,0)=h(0)$, $e(0;0,\cdots,0,h(0),0)\neq 0$. From (5.25) we have the

Claim. *For any germ* $w(x)$ *satisfying* $(dw(0),w(0))=(0,\cdots 0,h(0),0)$, *the equation* (5.21)′ *is equivalent to the following normal formed one:*

(5.26) $$\partial_{x_n}w=H^{\sim}(x;\partial_{x_1}w,\partial_{x''}w,w)$$

where $H^{\sim}(x;\xi_1,\xi'',z)$ *satisfies*

(5.27) $$\mathrm{ord}[H^{\sim}(x;0,0,0)-h(0)]\geqq\max\{k-1,1\}.$$

Proof. It only remains to verify (5.27). Setting $(\xi,z)=(0,\cdots,0,h(0),0)$ in (5.25), we have

(5.28) $$H(x;0,\cdots,0,h(0),0)=\{H^{\sim}(x;0,0,0)-h(0)\}e(x;0,\cdots,0,h(0),0).$$

On the other hand, the definition (5.22) of H yields that

(5.29) $$H(x;0,\cdots,0,h(0),0)=\partial_{\xi_n}F(x;d\Phi(x),\Phi(x))h(0)+h(x)$$
$$+K(x;x_1 0+0,0,\cdots,0,h(0),0).$$

Since $H(0;0,\cdots,0,h(0),0)=0$ by (5.24), it follows that

$$\mathrm{ord}[H(x;0,\cdots,0,h(0),0)]\geqq 1=\max\{k-1,1\}\quad\text{if}\quad k=1.$$

If $k\geqq 2$ then $h(0)=0$ since $h\in(x)^{k-1}$. Therefore, (5.18) and (5.29) yield

$$H(x;0,\cdots,0,h(0),0)=h(x)+K|_{(\xi,z)=(0,0)}=h(x).$$

Thus we have

$$\mathrm{ord}[H(x;0,\cdots,0,h(0),0)]=\mathrm{ord}[h]\geqq k-1=\max\{k-1,1\}.$$

Hence, from (5.28), we get (5.27). The proof of Claim is complete. Q.E.D.

Since the equation (5.26) can be solved with any holomorphic data on $\{x_n=0\}$, we can find a unique holomorphic solution $w(x;w_0)$ of (5.26) satisfying the following data:

(5.30) $$w(x;w_0)|_{x_n=0}=w_0(x_1,x'')\in(x_1,x'')^k.$$

It suffices for the proof of Proposition 5.3 to show

(5.31) $$\mathrm{ord}[w(x;w_0)]\geqq k.$$

We write $H^{\sim}(x; \xi_1, \xi'', z)$ as the form

$$H^{\sim}(x; \xi_1, \xi'', z)=\sum_{j=1}^{n-1} a_j(x; \xi_1, \xi'', z)\xi_j+b(x; \xi_1, \xi'', z)z$$
$$+H^{\sim}(x; 0, 0, 0)-h(0)+h(0).$$

Then the solution $w(x; w_0)$ satisfies the following linear equation

$$\partial_{x_n}w=\sum_{j=1}^{n-1} A_j(x)\partial_{x_j}w+B(x)w+C(x)+h(0) \tag{5.32}$$

where we put

$$\begin{cases} A_j(x):=a_j(x; \partial_{x_1}w(x; w_0), \partial_{x''}w(x; w_0), w(x; w_0)) \quad 1\leqq j\leqq n-1 \\ B(x):=b(x; \partial_{x_1}w(x; w_0), \partial_{x''}w(x; w_0), w(x; w_0)) \\ C(x):=H^{\sim}(x; 0, 0, 0)-h(0). \end{cases}$$

By virtue of (5.27), it suffices for (5.31) to verify the

Lemma 5.5. *Let $w(x)$ be the unique holomorphic solution of the linear Cauchy problem* (5.32) *with the data* (5.30). *Let $k\geqq 1$ and assume*

$$\mathrm{ord}_{x^0}[C]\geqq \max\{k-1, 1\}. \tag{5.33}$$

Then it follows that

$$\mathrm{ord}_{x^0}[w]\geqq k. \tag{5.8}$$

Proof. We expand $w(x)$ in x_n as $w(x)=\sum_{\mu=0}^{\infty} w_\mu(x_1, x'')x_n^\mu$. Then the equation (5.32) yields the following equalities:

$$\begin{cases} w_1(x_1, x'')\equiv C(x_1, x'', 0)+h(0) \quad \mathrm{mod} \sum_{j=1}^{n-1}(\partial_{x_j}w_0)+(w_0). \\ \mu w_\mu(x_1, x'')\equiv(\mu-1)!^{-1}\partial_{x_n}^{\mu-1}C(x_1, x'', 0) \quad \mathrm{mod} \sum_{\lambda=0}^{\mu-1}\left\{\sum_{j=1}^{n-1}(\partial_{x_j}w_\lambda)+(w_\lambda)\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } \mu\geqq 2. \end{cases} \tag{5.34}$$

We prove

$$\mathrm{ord}[w_\mu(x_1, x'')]\geqq k-\mu \qquad \text{for } 0\leqq\mu\leqq k \tag{5.35}$$

by induction on μ. Note that if $k=1$ then there is nothing to prove. Thus we may assume that $k\geqq 2$, hence $h(0)=0$. In the case $\mu=1$, the equality (5.34) with the assumptions (5.33) and (5.30) yields that $\mathrm{ord}[w_1]\geqq k-1$.

Now let $k\geqq\mu\geqq 2$, and assume

$$\mathrm{ord}[w_\lambda]\geqq k-\lambda \qquad \text{for } 0\leqq\lambda\leqq\mu-1. \tag{5.36}$$

Then (5.34) with the assumptions (5.33) and (5.36) yields that

$$\begin{aligned}\operatorname{ord}[w_\mu] &\geqq \min\{\operatorname{ord}[\partial_{x_n}^{\mu-1}C(x_1, x'', 0)], \min_{\substack{0\leqq\lambda\leqq\mu-1\\1\leqq j\leqq n-1}}\{\operatorname{ord}[\partial_{x_j}w_\lambda], \operatorname{ord}[w_\lambda]\}\}\\ &\geqq \min\{k-1-(\mu-1), \min_{0\leqq\lambda\leqq\mu-1}\{k-1-\lambda, k-\lambda\}\}\\ &\geqq k-\mu.\end{aligned}$$

Hence the proof of Lemma 5.5 is complete. Q. E. D.

By virtue of Lemmata 5.4 and 5.5, we get Proposition 5.3. Thus the proof of Proposition 5.3 is complete. Q. E. D.

The third step of our reduction is the changing of unknown function from $u(x)$ to the following $\hat{u}(x)$. We denote by Φ the good extension $\Phi^{\sim}$ which is gained by Proposition 5.3. We set $\hat{u}(x)$ and $F^{\wedge}(x; \xi^{\wedge}, z^{\wedge})$ by

$$(5.37)\qquad \begin{cases}\hat{u} := u-\Phi.\\ F^{\wedge}(x; \xi^{\wedge}, z^{\wedge}) := F(x; \xi^{\wedge}+d\Phi(x), z^{\wedge}+\Phi(x)).\end{cases}$$

Note that if $\hat{u}$ is a solution of the Cauchy problem

$$(5.38)\qquad \begin{cases}F^{\wedge}(x; d\hat{u}(x), \hat{u}(x))=0\\ \hat{u}(0, x')\equiv 0\end{cases}$$

then $u=\hat{u}+\Phi$ is a solution of the original Cauchy problem (1).

Lemma 5.6. *Let* $t\mapsto\Psi^{\wedge}(t, \hat{e})=(X^{\wedge}(t, \hat{e}); \Xi^{\wedge}(t, \hat{e}), Z^{\wedge}(t, \hat{e}))$ *be the characteristic curve of* $F^{\wedge}$ *which passes through*

$$\hat{e} := (0, y'; \eta_1^{\wedge}dx_1, 0)\in\hat{E} := T_S^*M\times\{0\}$$

at the initial time $t=0$. *We set a biholomorphic map* λ: $\hat{E}\to E$ *by*

$$\lambda(\hat{e}) := (0, y'; \eta_1^{\wedge}dx_1+d\Phi(0, y'), \Phi(0, y'))$$

and we put

$$\Psi_{\#}(t, \lambda(\hat{e})) := (X^{\wedge}; \Xi^{\wedge}+d\Phi(X^{\wedge}), Z^{\wedge}+\Phi(X^{\wedge}))(t, \hat{e}).$$

Then $t\mapsto\Psi_{\#}(t, \lambda(\hat{e}))$ *is a characteristic curve of* F *passing through* $\lambda(\hat{e})$ *at the initial time* $t=0$.

Proof. Since we have

$$\begin{cases}\partial_{x_j}F^{\wedge}=\partial_{x_j}F+\sum_{i=1}^{n}(\partial_{\xi_i}F)\partial_{x_i}\partial_{x_j}\Phi+(\partial_z F)\partial_{x_j}\Phi\\ \partial_{\xi_j^{\wedge}}F^{\wedge}=\partial_{\xi_j}F, \quad\text{and}\quad \partial_{z^{\wedge}}F^{\wedge}=\partial_z F\end{cases}$$

it follows that

$$\begin{cases} \partial_t \hat{X}_j = \partial_{\xi_j} F(\Psi_{\#}(t, \lambda(\hat{e}))) \\ \partial_t \{\hat{\Xi}_j + \partial_{x_j} \Phi(\hat{X})\} = -\hat{\Xi}_j \partial_{\hat{z}} \hat{F} - \partial_{x_j} \hat{F} + \sum_{i=1}^{n} (\partial_{x_i} \partial_{x_j} \Phi) \partial_t \hat{X}_i \\ \qquad = \{-(\hat{\Xi}_j + \partial_{x_j} \Phi) \partial_z F - \partial_{x_j} F\}(\Psi_{\#}(t, \lambda(\hat{e}))) \\ \partial_t \{\hat{Z} + \Phi(\hat{X})\} = \sum_{j=1}^{n} \hat{\Xi}_j \partial_{\hat{\xi}_j} \hat{F} + \sum_{j=1}^{n} (\partial_{x_j} \Phi) \partial_t \hat{X}_j \\ \qquad = \sum_{j=1}^{n} \{\hat{\Xi}_j + \partial_{x_j} \Phi\} \partial_{\xi_j} F(\Psi_{\#}(t, \lambda(\hat{e}))). \end{cases}$$

Hence we get Lemma 5.6. Q.E.D.

Let us recall the irreducible decomposition

(2.11) $$F|_E = \prod_{j=1}^{r} F_j^{\nu(j)} = \prod_{j=1}^{r} F_j$$

and we set

$$\hat{F}_j := \lambda^* F_j \qquad (1 \leqq j \leqq r)$$

where λ: $\hat{E} \ni (0, y'; \eta_1 dx_1, 0) \mapsto (0, y'; \eta_1 dx_1 + d\Phi(0, y'), \Phi(0, y')) \in E$ is the map germ which is introduced in Lemma 5.6. Since the map germ λ: $\hat{E} \to E$ is a biholomorphic map germ, we have the following irreducible decompositions:

(5.39) $$\hat{F}|_{\hat{E}} = \prod_{j=1}^{r} \hat{F}_j$$

(5.40) $$\hat{V} := (\boldsymbol{C}, 0) \times (\hat{E} \cap \hat{F}^{-1}(0), \hat{e}^0) = \bigcup_{j=1}^{r} \hat{V}_j$$

where we set $\hat{V}_j := (\boldsymbol{C}, 0) \times (\hat{F}_j^{-1}(0), \hat{e}^0)$ and $\hat{e}^0 := (x^0; 0, 0)$.

Definition 5.7. Let f_i: $(X_i, x_i) \to (Y, y)$ be germs of analytic coverings of (Y, y) for $i=1, 2$. We call f_1 and f_2 are *equivalent* if there exists a biholomorphic map germ g: $(X_1, x_1) \rightleftarrows (X_2, x_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} (X_1, x_1) & \xrightarrow[\sim]{g} & (X_2, x_2) \\ \downarrow f_1 & & \downarrow f_2 \\ (Y, y) & = & (Y, y) \end{array}$$

Note that if f_1 and f_2 are equivalent then they have the same critical locus $\Sigma \subset Y$. On each connected component of $(Y - \Sigma, y)$, we have

$$\# f_1^{-1}(z) = \# f_2^{-1}(z).$$

Using this terminology, we have:

Corollary 5.8. *We define the map germs* $\hat{\pi}_j$: $(\hat{V}_j, (0, \hat{e}^0)) \to (M, x^0)$ *from* $\hat{\Psi}_j := \hat{\Psi}|_{\hat{V}_j}$ *like as the constraction of* π_j *from* Ψ_j. *Then the following diagrams*

(5.41)$_j$ *commute for* $1 \leqq j \leqq r$:

$$(5.41)_j \qquad \begin{array}{ccccc} (\boldsymbol{C}, 0)\times(\hat{E}, \hat{e}^0) & \hookleftarrow & (V_j^{\wedge}, (0, \hat{e}^0)) & \longrightarrow & \downarrow \pi_j^{\wedge} \\ 1\times\lambda \downarrow \wr & & \downarrow & & (M, x^0) \\ (\boldsymbol{C}, 0)\times(E, e^0) & \hookleftarrow & (V_j, (0; e^0)) & \longrightarrow & \uparrow \pi_j \end{array}$$

Hence the germs π_j *and* $\pi_j^{\wedge}$ *of analytic coverings of* (M, x^0) *are equivalent. In particular, if the conclusion of Main theorem* 4.2 *holds for* $\pi_j^{\wedge}$, *then it also does for* π_j.

Remark 5.9. The function $F^{\wedge}(x; \xi^{\wedge}, z^{\wedge}) = F(x; \xi^{\wedge} + d\Phi(x), z^{\wedge} + \Phi(x))$ satisfies the following conditions [A.1]$^{\wedge}$-[A.3]$^{\wedge}$ and [A.4]:

[A.1]$^{\wedge}$ $\qquad L_{F^{\wedge}}(\hat{e}^0) = -\partial_{x_n}$

[A.2]$^{\wedge}$ $\qquad \mathrm{ord}_{\hat{e}^0}[F^{\wedge}|_{\hat{E}_{x^0}}] = p := \mathrm{ord}_{e^0}[F|_{E_{x^0}}]$

[A.3]$^{\wedge}$ $\qquad \Phi^{\wedge}(x) \equiv 0$ is a good extension of $\phi^{\wedge}(x') \equiv 0$.

[A.4] $\qquad \mathrm{ord}[F^{\wedge}(x; 0, 0)] = q := \mathrm{ord}[F^{\wedge}(0, x'; 0, 0)]$

where $\hat{e}^0 = \lambda^{-1}(e^0) = (x^0; 0, 0) \in \hat{E} \cap F^{\wedge -1}(0)$.

By virtue of Corollary 5.8 and Remark 5.9, our main theorem for the Cauchy problem (1) is reduced to that for the Cauchy problem (5.38) under the assumptions [A.1]$^{\wedge}$–[A.3]$^{\wedge}$ and [A.4].

Now we proceed *the fourth step* of our reduction, that is, we reduce $F^{\wedge}$ to the form of (5.1). We denote by F the function $F^{\wedge}$ satisfying [A.1]$^{\wedge}$–[A.3]$^{\wedge}$ and [A.4]. By virtue of [A.1]$^{\wedge}$ and of the Weierstrass's preparation theorem, we can find a holomorphic germ $G(x; \xi_1, \xi'', z)$ and a unit $\varepsilon(x; \xi, z)$ such that

$$F(x; \xi, z) = [G(x; \xi_1, \xi'', z) - \xi_n]\varepsilon(x; \xi, z) \tag{5.42}$$

holds locally at $e^0 := (x^0; 0, 0)$. We set

$$F^{\sim}(x; \xi, z) := G(x; \xi_1, \xi'', z) - \xi_n.$$

Note that $-F^{\sim}$ is a Weierstrass polynomial in ξ_n of degree one.

Notation 5.10. We denote by $t \mapsto \Psi^{\sim}(t, e^{\sim}) = (X^{\sim}; \Xi^{\sim}, Z^{\sim})(t, e^{\sim})$ the characteristic curve of $F^{\sim}$ passing through a point $e^{\sim} = (0, y'; \xi_1 dx_1, 0)$ in the analytic set $E \cap F^{-1}(0)$, where $E := T_S^*M \times \{0\}$.

We consider the relation between the characteristic curves $\Psi^{\sim}$ of $F^{\sim}$ and Ψ of F, and have the

Lemma 5.11. *Let* $g(t, e) \in \mathcal{O}_{C\times E, (0, e^0)}$ *be the unique solution of*

$$
(5.43) \qquad \begin{cases} \partial_t g = \varepsilon(\Psi^{\sim}(g, e)) \\ g(0, e) = 0. \end{cases}
$$

Then the following 1), 2) *hold:*

1) *The map germ* $g \times id_E : (\boldsymbol{C}, 0) \times (E, e^0) \to (\boldsymbol{C}, 0) \times (E, e^0)$ *is a biholomorphic map germ.*

2) *We set*

$$
(5.44) \qquad \Psi_{\#}(t, e) := [\Psi^{\sim} \circ (g \times id_E)](t, e) = \Psi^{\sim}(g(t, e), e).
$$

If $e \in E \cap F^{-1}(0)$, *then* $t \mapsto \Psi_{\#}(t, e)$ *is the characteristic curve of* F *passing through* e *at* $t=0$, *that is, we have* $\Psi_{\#} = \Psi$ *on* $(\boldsymbol{C}, 0) \times (E \cap F^{-1}(0), e^0)$.

Proof. Since g is a solution of (5.43) it follows that

$$
\frac{\partial(g(t, e), e)}{\partial(t, e)} = \det \begin{bmatrix} \varepsilon(\Psi^{\sim}(g, e)) & \partial_e g \\ 0 & I_n \end{bmatrix} = \varepsilon(\Psi^{\sim}(g, e)) \neq 0
$$

because $\varepsilon(x ; \xi, z)$ is a unit. Hence the assertion 1) follows.

Since $t \mapsto \Psi^{\sim}(t, e)$ is a characteristic curve of $F^{\sim}$, we have

$$
\begin{aligned}
\partial_t \{\Psi^{\sim}(g(t, e), e)\} &= (\partial_t \Psi^{\sim})(g(t, e), e) \partial_t g \\
&= [(\partial_\xi F^{\sim} ; -\Xi^{\sim} \partial_z F^{\sim} - \partial_x F^{\sim}, \Xi^{\sim} \partial_\xi F^{\sim}) \varepsilon](\Psi^{\sim}(g(t, e), e)).
\end{aligned}
$$

The assumption $e \in F^{-1}(0) = F^{\sim -1}(0)$ yields $\Psi^{\sim}(s, e) \in F^{\sim -1}(0)$ for $\forall s$, hence we have

$$
d_{x, \xi, z} F(\Psi^{\sim}(g(t, e), e)) = [(d_{x, \xi, z} F^{\sim}) \varepsilon](\Psi^{\sim}(g(t, e), e))
$$

which implies

$$
\partial_t \{\Psi^{\sim}(g(t, e), e)\} = (\partial_\xi F ; -\Xi^{\sim} \partial_z F - \partial_x F, \Xi^{\sim} \partial_\xi F)(\Psi^{\sim}(g(t, e), e)).
$$

Since $g(0, e) = 0$, we also have $\Psi_{\#}(0, e) = \Psi^{\sim}(g(0, e), e) = \Psi^{\sim}(0, e) = e$. Thus the proof of Lemma 5.11 is complete. Q.E.D.

Corollary 5.12. *We define the map germs* $\pi_j^{\sim} : (V_j, (0, e^0)) \to (M, x^0)$ *from* $\Psi_j^{\sim} := \Psi^{\sim}|_{V_j}$ *like as the constraction of* π_j *from* Ψ_j. *Then the following diagrams* $(5.45)_j$ *commute for* $1 \leq j \leq r$:

$$
(5.45)_j \qquad \begin{array}{ccccc}
(\boldsymbol{C}, 0) \times (E, e^0) & \hookleftarrow & (V_j, (0, e^0)) & \xrightarrow{\pi_j^{\sim}} & \\
{\scriptstyle g \times \mathrm{id}_E} \downarrow \wr & & \downarrow & & (M, x^0) \\
(\boldsymbol{C}, 0) \times (E, e^0) & \hookleftarrow & (V_j, (0, e^0)) & \xrightarrow{\pi_j} &
\end{array}
$$

Hence the map germs π_j *and* $\pi_j^{\sim}$ *are equivalent as germs of analytic coverings of* (M, x^0). *Thus if the conclusion of Main Theorem* 4.2 *holds for* $\pi_j^{\sim}$ *then it also does for* π_j.

To complete the reduction of Main Theorem 4.2 to Theorem 5.1, it remains to verify the

Lemma 5.13. *Let* $F=F^{\sim}\varepsilon$ *be a germ satisfies the conditions* $[A.1]^{\wedge}-[A.3]^{\wedge}$ *and* [A.4] *such that* $-F^{\sim}$ *is the Weierstrass polynomial of* F *in* ξ_n *of degree one:*

$$F^{\sim}(x\,;\,\xi,\,z)=G(x\,;\,\xi_1,\,\xi'',\,z)-\xi_n\,.$$

Then $F^{\sim}$ *satisfies the assumptions* [B.1]−[B.4] *of Theorem* 5.1.

Proof. Since [B.2] and [B.4] are trivial by $[A.2]^{\wedge}$ and [A.4], we only have to show [B.1] and [B.3]. By virtue of $[A.1]^{\wedge}$, we have

$$-d\xi_n=d_\xi F(e^0)=\varepsilon(e^0)d_\xi F^{\sim}(e^0)=\varepsilon(e^0)\left[\sum_{j=1}^{n-1}\partial_{\xi_j}G(x^0\,;\,0,\,0,\,0)d\xi_j-d\xi_n\right],$$

which shows

$$\partial_{\xi_j}G(x^0\,;\,0,\,0,\,0)=0 \qquad \text{for} \quad 1\leqq j\leqq n-1\,.$$

Thus we have the condition [B.1].

Now we check the condition [B.3]. We set

$$(5.46)\qquad \begin{cases} f(x',\,\xi_1):=f^{\Phi}(x',\,\xi_1)=F(0,\,x'\,;\,\xi_1dx_1,\,0) \\ f^{\sim}(x',\,\xi_1):=f^{\sim\Phi}(x',\,\xi_1)=F^{\sim}(0,\,x'\,;\,\xi_1dx_1,\,0)=G(0,\,x'\,;\,\xi_1dx_1,\,0) \\ \varepsilon^0(x',\,\xi_1):=\varepsilon(0,\,x'\,;\,\xi_1dx_1,\,0)\,. \end{cases}$$

Then we have $f=f^{\sim}\varepsilon^0$ (ε^0 is a unit) which yields

$$(5.47)\qquad \mathrm{N}(f)=\mathrm{N}(f^{\sim})+\mathrm{N}(\varepsilon^0)=\mathrm{N}(f^{\sim})$$

since Newton polygons have the additivity $\mathrm{N}(gh)=\mathrm{N}(g)+\mathrm{N}(h)$ (see § 11).

By (5.47), we know that $\mathrm{N}(f^{\sim})$ satisfies the coprimeness condition, and that each subset $\mathrm{M}_j^{\sim}=\mathrm{M}_j$ of $\{1,\,2,\,\cdots m\}$ is a nice subset for $1\leqq j\leqq r$.

Note that the condition [B.4] implies the inequality

$$\mathrm{ord}[F^{\sim}(x\,;\,0,\,0)]=\mathrm{ord}[f^{\sim}(x',\,0)]=q>\kappa(m)^{-1}$$

since we assume $p\geqq 2$.

It remains to show

$$(5.48)\qquad \mathrm{N}(f^{\sim}) \text{ is stable in the direction of } L_{F^{\sim}}(e^0)=-\partial_{x_n}.$$

To show (5.48) we utilize the following

Claim 5.14. *Let*

$$f(x',\,\xi_1)=\sum_{\nu=0}^{\infty}c_\nu(x')\xi_1^\nu, \qquad f^{\sim}(x',\,\xi_1)=\sum_{\nu=0}^{\infty}c_\nu^{\sim}(x')\xi_1^\nu$$

be the Taylor expansions of the germs f *and* $f^{\sim}$ *given by* (5.46). *Then we have*

$$\text{(5.49)} \qquad \mathrm{Loc}[c_\nu]=\varepsilon(e^0)\,\mathrm{Loc}[c_\nu^\sim]$$
$$\textit{for } \forall\nu \textit{ satisfying } (\mathrm{ord}[c_\nu],\ \nu)\in \mathrm{Ver}\,\mathrm{N}(f).$$

Proof. Let $\varepsilon^0(x', \xi_1)=\sum_{\nu=0}^{\infty}\varepsilon_\nu(x')\xi_1^\nu$ be the Taylor expansion. Then we have $c_\nu=\sum_{\lambda=0}^{\nu}c_\lambda^\sim\varepsilon_{\nu-\lambda}$. Assume $(\mathrm{ord}[c_\nu^\sim],\ \nu)\in \mathrm{Ver}\,\mathrm{N}(f)=\mathrm{Ver}\,\mathrm{N}(f^\sim)$ then it follows that

$$\mathrm{ord}[c_\lambda^\sim]>\mathrm{ord}[c_\nu^\sim] \qquad \text{for} \quad \forall\lambda<\nu.$$

Hence we have

$$\mathrm{ord}[c_\nu^\sim\varepsilon_0]=\mathrm{ord}[c_\nu^\sim]<\mathrm{ord}\left[\sum_{\lambda=0}^{\nu-1}c_\lambda^\sim\varepsilon_{\nu-\lambda}\right]$$

which shows (5.49) as desired. Q.E.D.

The assertion (5.48) immediately follows from Claim 5.14. Thus we complete the proof of Lemma 5.13.

Our reduction of the proof of Main Theorem 4.2 to that of Theorem 5.1 is also complete.

§6. Decomposition of Map Germs $\pi_j^\sim$

We begin to prove Theorem 5.1. In this section we consider map germs $\pi_j^\sim$: $\mathrm{graph}(\Psi_j)\to(M, 0)$ which are equivalent to π_j: $(V_j, e^0)\to(M, 0)$ for $1\leqq j\leqq r$. Recall that we have reduced x^0 [or e^0 resp.] to the origin of $\boldsymbol{C}^n[(x\,;\ \xi, z)=(0\,;\ 0, 0)]$ in §5.

An advantage of this consideration comes from the fact that $\pi_j^\sim$ is decomposable to a composition of three map germs as the form $\pi_{j3}\circ\pi_{j2}\circ\pi_{j1}$. The aim of this section is to show that the first map germ π_{j1} is a biholomorphic map germ for $1\leqq j\leqq r$. This fact yields that the analysis of π_j can be reduced to those of π_{j2} and π_{j3} (see Theorems 6.10 and 6.11 at the end of this section).

Let us recall the diagrams

$$\text{(6.1)}_j \qquad \begin{array}{ccc} (V_j,\ (0,\ e^0)) & \xrightarrow{\ \Psi_j\ } & (F^{-1}(0),\ e^0) \\ {\scriptstyle \pi_j}\downarrow & & \downarrow \\ (M,\ 0) & \longleftarrow & (J^1M,\ e^0) \end{array}$$

where $(V_j,\ (0,\ e^0))=(\boldsymbol{C},\ 0)\times(F_j^{-1}(0),\ e^0)$ is an irreducible component of

$$(V,\ (0,\ e^0))=(\boldsymbol{C},\ 0)\times(E\cap F^{-1}(0),\ e^0), \qquad \text{for} \quad 1\leqq j\leqq r.$$

Definition 6.1. We define map germs $\pi_j^\sim$ for $1\leqq j\leqq r$ by the following diagram:

$$\text{(6.2)} \qquad \begin{array}{ccccc} (V_j,\ (0,\ e^0)) & \xrightarrow[\sim]{} & \mathrm{graph}(\Psi_j) & \overset{\iota}{\hookrightarrow} & (\boldsymbol{C},\ 0)\times(E,\ e^0)\times(J^1M,\ e^0) \\ & {\scriptstyle \pi_j}\searrow & {\scriptstyle \pi_j^\sim}\downarrow & & \downarrow{\scriptstyle \text{projection}} \\ & & (M,\ 0) & \xleftarrow{\ \text{projection}\ } & (J^1M,\ e^0) \end{array}$$

Since π_j and $\pi_j^{\sim}$ are equivalent as germs of analytic coverings of $(M, 0)$ in the sense of Definition 5.7, we may consider $\pi_j^{\sim}$ instead of π_j.

Definition 6.2. 1) We decompose the composite projection

$$P\colon (\boldsymbol{C}, 0)\times(E, e^0)\times(J^1M, e^0) \longrightarrow (J^1M, e^0) \longrightarrow (M, 0)$$

as follows:

$$\begin{array}{ccc} (\boldsymbol{C}, 0)_t\times(E, e^0)_{y', \eta_1}\times(J^1M, e^0)_{x, \xi, z} & \xrightarrow{P_1} & (\boldsymbol{C}, 0)_t\times(M, 0)_x\times(\boldsymbol{C}, 0)_{\xi_1} \\ P\big\downarrow & & P_2\big\downarrow \\ (M, 0)_x & \xleftarrow{\quad P_3 \quad} & (M, 0)_x\times(\boldsymbol{C}, 0)_{\xi_1} \end{array} \tag{6.3}$$

2) We define subsets V_{j1} and V_{j2} respectively by

$$\begin{cases} V_{j1} := (P_1\circ\iota)(\operatorname{graph}(\Psi_j)) \subset (\boldsymbol{C}, 0)_t\times(M, 0)_x\times(\boldsymbol{C}, 0)_{\xi_1} \\ V_{j2} := (P_2\circ\iota^{\sim})(V_{j1}) \subset (M, 0)_x\times(\boldsymbol{C}, 0)_{\xi_1} \end{cases}$$

where ι, $\iota^{\sim}$ denote the following inclusion maps:

$$\begin{cases} \iota\colon \operatorname{graph}(\Psi_j) \hookrightarrow (\boldsymbol{C}, 0)_t\times(E, e^0)_{y', \eta_1}\times(J^1M, e^0)_{x, \xi, z} \\ \iota^{\sim}\colon V_{j1} \hookrightarrow (\boldsymbol{C}, 0)_t\times(M, 0)_x\times(\boldsymbol{C}, 0)_{\xi_1}. \end{cases}$$

3) According to the decomposition (6.3) of P, we have the following decomposition (6.4) of the map germ $\pi_j^{\sim}$:

$$\begin{array}{ccccc} (\boldsymbol{C}, 0)\times(E, e^0)\times(J^1M, e^0) & & \xrightarrow{\qquad P_1 \qquad} & & (\boldsymbol{C}, 0)\times(M, 0)\times(\boldsymbol{C}, 0) \\ & \nwarrow^{\iota} & & {}^{\iota^{\sim}}\nearrow & \\ & \operatorname{graph}(\Psi_j) & \xrightarrow{\ \pi_{j1}\ } & V_{j1} & \\ P\Big\downarrow & \big\downarrow \pi_j^{\sim} & & \big\downarrow \pi_{j2} & \Big\downarrow P_2 \\ & (M, 0) & \xleftarrow{\ \pi_{j3}\ } & V_{j2} & \\ & \| & & \searrow & \\ (M, 0) & & \xleftarrow{\qquad P_3 \qquad} & & (M, 0)\times(\boldsymbol{C}, 0) \end{array} \tag{6.4}$$

Notation 6.3. We denote a characteristic curve of F, which passes through a point $e=(0, y'; \eta_1 dx_1, 0)\in E=T^*_{\xi}M\times\{0\}$, by

$$\Psi^{\sim}(t, y', \eta_1)=(X(t, y', \eta_1); \Xi(t, y', \eta_1), Z(t, y', \eta_1)).$$

Of course, by the definition $\Psi_j := \Psi^{\sim}|_{Vj}$, we use the same expression $(X; \Xi, Z)$ to denote the value $\Psi_j(t, y', \eta_1)$ for $1\leqq j\leqq r$.

From now on, we show that the map germ $\pi_{j1}\colon \operatorname{graph}(\Psi_j)\to V_{j1}$ is a biholomorphic map germ for $1\leqq j\leqq r$. First we prove

Claim 6.4. *There exists a holomorphic map germ*

$$(Y', H_1): (\boldsymbol{C}, 0)_t \times (\boldsymbol{C}^{n-1}, 0)_{x'} \times (\boldsymbol{C}, 0)_{\xi_1} \longrightarrow (T_S^* M, (0, 0))_{y', \eta_1}$$

such that

$$\begin{bmatrix} X' \\ \Xi_1 \end{bmatrix}(t, Y'(t, x', \xi_1), H_1(t, x', \xi_1)) \equiv \begin{bmatrix} x' \\ \xi_1 \end{bmatrix} \quad \textit{and} \tag{6.5}$$

$$\begin{bmatrix} Y' \\ H_1 \end{bmatrix}(t, X'(t, y', \eta_1), \Xi_1(t, y', \eta_1)) \equiv \begin{bmatrix} y' \\ \eta_1 \end{bmatrix}. \tag{6.6}$$

Proof. We consider the following map germ

$$\begin{cases} \chi: (\boldsymbol{C}, 0)_t \times (T_S^* M, (0, 0))_{y', \eta_1} \longrightarrow (\boldsymbol{C}, 0) \times (\boldsymbol{C}^{n-1}, 0)_{x'} \times (\boldsymbol{C}, 0)_{\xi_1} \\ \chi(t, y', \eta_1) := (t, X'(t, y', \eta_1), \Xi_1(t, y', \eta_1)). \end{cases} \tag{6.7}$$

Since $(X', \Xi_1)|_{t=0} = (y', \eta_1)$ holds, we have

$$\frac{\partial \chi}{\partial(t, y', \eta_1)}(0, y', \eta_1) = \det \begin{bmatrix} 1 & 0 \\ * & I_n \end{bmatrix} \neq 0.$$

Hence, by the inverse mapping theorem, there exists the inverse map germ

$$\chi^{-1}(t, x', \xi_1) = (t, Y'(t, x', \xi_1), H_1(t, x', \xi_1))$$

which has the desired properties (6.5), (6.6). Q.E.D.

Notation 6.5. 1) Since $X_1|_{t=0} = 0$, we can write X_1 as the form

$$X_1(t, y', \eta_1) = t X_1^{\sim}(t, y', \eta_1).$$

2) We set $f_j(y', \eta_1) := (\gamma_\phi^* F_j)(y', \eta_1)$ where $\gamma_\phi(y', \eta_1) = (0, y'; \eta_1 dx_1, 0)$.

3) Using these $X_1^{\sim}$, f_j and the germ (Y', H_1) mentioned in Claim 6.4, we define germ A and B_j as follows:

$$\begin{cases} A(t, x', \xi_1) := X_1^{\sim}(t, Y'(t, x', \xi_1), H_1(t, x', \xi_1)) \\ B_j(t, x', \xi_1) := f_j(Y'(t, x', \xi_1), H_1(t, x', \xi_1)) \quad \text{for } 1 \leqq j \leqq r. \end{cases} \tag{6.8}$$

Lemma 6.6. *Set a germ H of a hypersurface of $\boldsymbol{C}_t \times M_x \times \boldsymbol{C}_{\xi_1}$ by*

$$H := \{(t, x, \xi_1); x_1 = tA(t, x', \xi_1)\}.$$

Then the projection P_1 induces a biholomorphic map germ $h^{\sim}$: $\mathrm{graph}(\Psi^{\sim}) \to H$ *as follows:*

$$\begin{array}{ccc} \mathrm{graph}(\Psi^{\sim}) & \hookrightarrow & (\boldsymbol{C}, 0)_t \times (E, e^0)_{y', \eta_1} \times (J^1 M, e^0)_{x, \xi, z} \\ h^{\sim} \downarrow & & \downarrow P_1 \\ H & \hookrightarrow & (\boldsymbol{C}, 0)_t \times (M, 0)_x \times (\boldsymbol{C}, 0)_{\xi_1} \end{array} \tag{6.9}$$

Proof. We first observe that P_1 in fact induces the map $h^{\sim}$. Assume that $(t\,;\,y',\eta_1\,;\,x,\xi,z)=(t\,;\,y',\eta_1\,;\,(X,\Xi,Z)(t,y',\eta_1))\in \mathrm{graph}(\Psi^{\sim})$. It suffices to verify

$$X_1(t,y',\eta_1)=tA(t,X'(t,y',\eta_1),\Xi_1(t,y',\eta_1)). \tag{6.10}$$

By the definition (6.8) of A, the right hand side of (6.10) is equal to

$$X_1(t,Y'(t,X'(t,y',\eta),\Xi_1(t,y',\eta)),\ H_1(t,X'(t,y',\eta_1),\Xi_1(t,y',\eta_1)))\,.$$

Thus the identity (6.6) yields (6.10).

Note that (6.10) shows that the following diagram commutes:

$$\begin{array}{ccc}
(\boldsymbol{C},0)\times(T^*_{\xi}M,(0,0))\cong(\boldsymbol{C},0)\times(E,e^0) & \xrightarrow{\ \sim\ } & \mathrm{graph}(\Psi^{\sim}) \\
\chi \big\downarrow & & \big\downarrow h^{\sim} \\
(\boldsymbol{C},0)\times(\boldsymbol{C}^{n-1},0)\times(\boldsymbol{C},0) & \xrightarrow{\ \sim\ } & H
\end{array}$$

where χ is the map germ defined by (6.7). Since we know χ is a biholomorphic germ, $h^{\sim}$ also is. The proof of Lemma 6.6 is complete. Q.E.D.

Proposition 6.7. 1) *The map germ* $\pi_{j1}:\mathrm{graph}(\Psi_j)\to V_{j1}$ *is a biholomorphic map germ for* $1\leqq j\leqq r$.

2) *The germ* $(V_{j1},(0,0,))$ *is defined by the following two equations as a germ of analytic subset in* $(\boldsymbol{C},0)_t\times(M,0)_x\times(\boldsymbol{C},0)_{\xi_1}$:

$$\begin{cases} tA(t,x',\xi_1)-x_1=0 \\ B_j(t,x',\xi_1)=0\,. \end{cases} \tag{6.11$_j$}$$

Proof. Since $\Psi_j=\Psi^{\sim}|_{V_j}$ implies that the map germ π_{j1} is the restriction of the biholomorphic germ $h^{\sim}:\mathrm{graph}(\Psi^{\sim})\rightleftarrows H$ on $\mathrm{graph}(\Psi_j)$, the assertion 1) follows.

By virtue of the identity (6.6), we have

$$B_j(\chi(t,y',\eta_1))=f_j(y',\eta_1)\,. \tag{6.12}$$

Indeed, by the definition of B_j, the left hand side of (6.12) is equal to

$$f_j(Y'(\chi(t,y',\eta_1)),H_1(\gamma(t,y',\eta_1)))\,.$$

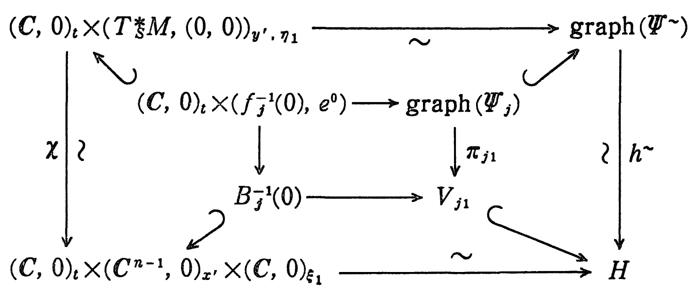
Then the identity (6.6) which is equivalent to

$$(Y'(\chi(t,y',\eta_1)),H_1(\chi(t,y',\eta_1)))=(y',\eta_1)$$

yields (6.12) as desired.

From (6.12), we have the following commutative diagram:

(6.13)



The desired equations $(6.11)_j$ immediately follows from this diagram. The proof of Proposition 6.7 is complete. Q. E. D.

Remark 6.8. Let us define ideals $\mathcal{I}_j$ of the ring $\mathcal{O}_{C\times M\times C,(0,0,0)}$ by

$$\mathcal{I}_j:=(tA(t, x', \xi_1)-x_1,\ B_j(t, x', \xi_1)).$$

Then these ideals are prime for $1\leqq j\leqq r$.

Proof. Assume $g_1g_2\in\mathcal{I}_j$, $g_1\notin\mathcal{I}_j$. We set $g_i^0(t, x', \xi_1):=g_i|_H$ $(i=1, 2)$. Then we have $g_1^0g_2^0\in(B_j)$, $g_1^0\notin(B_j)$. We claim

(6.14) $\quad (B_j)$ is a prime ideal of the ring $\mathcal{O}_{C\times C^{n-1}\times C,(0,(0,0))}$.

If (6.14) is established then we have $g_2^0\in(B_j)$ which yields the desired fact $g_2\in\mathcal{I}_j$. Note that, by virtue of (6.12), it suffices for (6.14) to show that

(6.15) $\quad (f_j(y', \eta_1))$ is a prime ideal of the ring $\mathcal{O}_{C_t\times T^*_S M,(0,0,0))}$.

Recall that $f_j(y', \eta_1)$ is irreducible in $\mathcal{O}_{T^*_S M,(0,0)}$. Hence it suffices for (6.15) to show the following

Claim 6.9. *Let* $f\in\mathcal{O}_{T^*_S M,(0,0)}$ *be an irreducible germ, and let* $\rho: C\times T^*_S M\to T^*_S M$ *be the projection. Then the pull-back germ* ρ^*f *is irreducible in the ring* $\mathcal{O}_{C\times T^*_S M,(0,(0,0))}$.

To show Claim 6.9, we note the following simple fact for $f\in\mathcal{O}_{T^*_S M,(0,0)}$:

(6.16) $$a(t, e)=\sum_{\nu=0}^{\infty}a_\nu(e)t^\nu\in(\rho^*f)\Longleftrightarrow a_\nu\in(f) \quad \text{for all } \nu.$$

Indeed, by the definition, $a\in(\rho^*f)$ means that there exists a germ $b(t, e)=\sum_{\nu=0}^{\infty}b_\nu(e)t^\nu$ such that $a(t, e)=[\rho^*f](t, e)b(t, e)=f(e)b(t, e)$. This is clearly equivalent to $a_\nu(e)=f(e)b_\nu(e)$ for all ν. Hence we get (6.16).

We continue the proof of Claim 6.9. For $a_i(t, e)=\sum_{\nu=0}^{\infty}a_{i,\nu}(e)t^\nu$ $(i=1, 2)$ we assume $a_1a_2\in(\rho^*f)$, $a_1\notin(\rho^*f)$. By (6.16), it follows that

$$\sum_{\lambda=0}^{\gamma} a_{1,\gamma-\lambda}a_{2,\lambda}\in(f) \qquad \text{for all } \gamma=0, 1, 2, \cdots \tag{6.17}$$

and that there exists μ such that

$$a_{1,\mu}\notin(f). \tag{6.18}$$

We choose μ as the minimum value of such μ. We want to show

$$a_{2,\nu}\in(f) \qquad \text{for all } \nu. \tag{6.19}$$

We prove (6.19) by induction on ν as follows:

1) The case $\nu=0$. Taking γ in (6.17) as μ, we have

$$a_{1,\mu}a_{2,0}+\sum_{\lambda=1}^{\mu} a_{1,\mu-\lambda}a_{2,\lambda}\in(f).$$

Since $a_{1,\mu-\lambda}\in(f)$ for $\lambda\geqq1$, we have $a_{1,\mu}a_{2,0}\in(f)$. Then, by (6.18), it follows that $a_{2,0}\in(f)$ since the ideal (f) is prime.

2) The case $\nu\geqq1$. Taking γ in (6.17) as $\nu+\mu$, we have

$$\sum_{\lambda=0}^{\nu-1} a_{1,\nu+\mu-\lambda}a_{2,\lambda}+a_{1,\mu}a_{2,\nu}+\sum_{\lambda=\nu+1}^{\nu+\mu} a_{1,\nu+\mu-\lambda}a_{2,\lambda}\in(f).$$

Since $a_{2,\lambda}\in(f)$ for $\lambda\leqq\nu-1$ by the inductive assumption, and since $a_{1,\nu+\mu-\lambda}\in(f)$ for $\lambda\geqq\nu+1$, we have $a_{1,\mu}a_{2,\nu}\in(f)$. Thus (6.18) implies $a_{2,\nu}\in(f)$ as desired. Hence we get (6.19).

Since (6.16) and (6.19) yield $a_2\in(\rho^*f)$, we get Claim 6.9. Hence the proof of Remark 6.8 is complete. Q.E.D.

We conclude this section to show that, if we establish the following Theorems 6.10 and 6.11, then Theorem 5.1 follows:

We consider the following diagram under the assumptions [B.1]-[B.4] of Theorem 5.1:

$$\begin{array}{ccccc}
 & & V_{j1}=\{tA-x_1=B_j=0\} & \hookrightarrow & (\boldsymbol{C},0)_t\times(M,0)_x\times(\boldsymbol{C},0)_{\xi_1} \\
 & & \downarrow \pi_{j2} & & \downarrow P_2 \\
(M,0) & \xleftarrow{\pi_{j3}} & V_{j2} & \hookrightarrow & \\
\| & & & \searrow & \\
(M,0) & & \xleftarrow{\quad P_3 \quad} & & (M,0)_x\times(\boldsymbol{C},0)_{\xi_1}
\end{array} \tag{6.20}$$

For the diagram (6.20), we state the following two theorems which imply Theorem 5.1:

Theorem 6.10. *Under the assumptions* [B.1]-[B.4], *there exists an irreducible*

Weierstrass polynomial $w_j(x, \xi_1) \in \mathcal{O}_{M,0}[\xi_1]$ *of the degree*

$$v_j = \sum_{\mu \in \mathrm{M}_j} v(\mu)$$

such that the following equality holds locally at $(x, \xi_1) = (0, 0)$:

$$(V_{j2}, (0, 0)) = (w_j^{-1}(0), (0, 0)).$$

Theorem 6.11. *The map germ* $\pi_{j2}: V_{j1} \to V_{j2}$ *is a germ of one-sheeted analytic covering of* $V_{j2} = (V_{j2}, (0, 0))$.

We must show that Theorems 6.10 and 6.11 imply Theorem 5.1. Note that Theorem 6.10 yields that the map germ $\pi_{j3}: V_{j2} \to (M, 0)$ is a germ of a v_j-sheeted analytic covering of $(M, 0)$. Hence, by Theorem 6.11 and the first assertion of Proposition 6.7, we conclude that

(6.21) $\quad \pi_j^{\sim} = \pi_{j3} \circ \pi_{j2} \circ \pi_{j1}$ is a germ of a v_j-sheeted analytic covering of $(M, 0)$

which is the assertion 1) of Theorem 5.1.

To show the assertion 2) of Theorem 5.1, let u_j be the multi-valued germ defined by the diagram $(4.8)_j$. Then, by our constraction of the maps π_j it follows

$$\text{(6.22)} \qquad \begin{cases} \partial_{x_1} u_j(X(t, y', \eta_1)) = \Xi_1(t, y', \eta_1) \\ \qquad\qquad \text{for } \forall(t; 0, y'; \eta_1 dx_1, 0) \in (V_j, (0, e^0)). \end{cases}$$

Then, the irreducibility of the defining germ $w_j(x, \xi_1)$ of V_{j2} yields that the germ $\xi_1 = \Xi_1(t, y', \eta_1)$ is exactly v_j-valued as a germ of a function in x. On the other hand, (6.21) shows that the germ u_j is at most v_j-valued around $x=0$. Hence the relation (6.22) shows that the germ u_j itself is exactly v_j-valued.

Thus Theorem 5.1 follows if we establish Theorems 6.10 and 6.11.

We shall prove Theorem 6.10 in §9, and Theorem 6.11 in §10. Before to prove these theorems we need some preparation which is done in §§7 and 8.

§7. Newton Polygons of $A(t, 0, \xi_1)$ and $B_j(t, 0, \xi_1)$

In this section we decide the "principal part" of the Newton polygons of the restrictions $A|_{x'=0}$ and $B_j|_{x'=0}$ of $A(t, x', \xi_1)$ and $B_j(t, x', \xi_1)$ for $1 \leqq j \leqq r$, where A and B_j are defined by (6.8) in Notation 6.5.

Definition 7.1. 1) Let N_i $(i=1, 2)$ be Newton polygons. We say N_1 is *properly contained in* N_2 (we denote this by $\mathrm{N}_1 \Subset \mathrm{N}_2$) if

$$\text{(7.1)} \qquad \mathrm{N}_1 \subset \mathrm{N}_2 \quad \text{and} \quad \mathrm{N}_1 \cap \partial^0 \mathrm{N}_2 = \emptyset$$

where $\partial^0 \mathrm{N}$ denotes the strict boundary of a Newton polygon N, which is defined in 2) in Definition 2.3.

2) Let (Σ, σ) be a germ of a complex manifold, and let $f, g \in \mathcal{O}_{\Sigma\times C,(\sigma,0)}$ be germs of functions. We say $\mathrm{N}(f)$ and $\mathrm{N}(g)$ *have a same principal part* if

$$\mathrm{N}(f-g) \Subset \mathrm{N}(f)=\mathrm{N}(g)\,. \tag{7.2}$$

3) Let $f \in \mathcal{O}_{\Sigma\times C,(\sigma,0)}$ be a germ and let $f(x, y)=\sum_{\nu=0}^{\infty} c_\nu(x) y^\nu$ be the Taylor expansion of f along $y=0$ ($c_\nu \in \mathcal{O}_{\Sigma,\sigma}$ for $\nu=0, 1, \cdots$). We define the *characteristic polynomial function* $\mathrm{ch}(f)$ by

$$\mathrm{ch}(f)(X, y):=\sum_{(\mathrm{ord}[c_\nu],\nu)\in\partial^0\mathrm{N}(f)} \mathrm{Loc}[c_\nu](X) y^\nu \qquad ((X, y)\in T_\sigma\Sigma\times \boldsymbol{C}) \tag{7.3}$$

where $\mathrm{Loc}[c_\nu]: T_\sigma\Sigma\to\boldsymbol{C}$ is the localization of c_ν at σ (see Definition 2.6).

Note that it clearly follows that $\mathrm{N}(f)$ and $\mathrm{N}(\mathrm{ch}(f))$ have a same principal part. Moreover $\mathrm{N}(f)$ and $\mathrm{N}(g)$ have a same principal part if and only if

$$\mathrm{ch}(f)=\mathrm{ch}(g)\,.$$

For this reason, we call the characteristic polynomial function $\mathrm{ch}(f)$ by the name of the *principal part* of $\mathrm{N}(f)$.

Let us recall the irreducible decomposition of f locally at $(0, 0)\in T_S^*M$:

$$f(y', \eta_1):=F(0, y'; \eta_1 dx_1, 0)=\prod_{j=1}^{r} f_j(y', \eta_1) \quad (f_j\in\mathcal{O}_{T_S^*M,(0,0)}).$$

The aim of this section is to show the following

Proposition 7.2. *The principal parts of the Newton polygons* $\mathrm{N}(A(t, 0, \xi_1))$ *and* $\mathrm{N}(B_j(t, 0, \xi_1))$, $1\leqq j\leqq r$, *satisfy the following* (7.4) *and* (7.5):

$$\mathrm{ch}(B_j(t, 0, \xi_1))=\mathrm{ch}(f_j)(-tL_F(e^0), \xi_1) \qquad \text{for } 1\leqq j\leqq r\,. \tag{7.4}$$

$$\mathrm{N}\Big[\mathrm{ch}(A(t, 0, \xi_1))-t^{-1}\int_0^t \partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0), \xi_1)d\theta\Big]+\mathrm{N}(\xi_1)\Subset\mathrm{N}(f)\,. \tag{7.5}$$

Remark 7.3. By the assumptions [B.1] and [B.3] of Theorem 5.1, we have

$$\mathrm{N}(f)=\mathrm{N}(f(0, \cdots, 0, y_n, \eta_1))\,. \tag{7.6}$$

$$\mathrm{N}(f_j)=\mathrm{N}(f_j(0, \cdots, 0, y_n, \eta_1))\,. \tag{7.7}$$

Proof. The first equality (7.6) is nothing but the stability of $\mathrm{N}(f)$ in the direction of $L_F(e^0)=-\partial_{x_n}$. Since $\Phi(x)\equiv 0$ is a good extension of the Cauchy data $\phi(x')\equiv 0$, (7.6) follows.

Note that there is a trivial inclusion

$$\mathrm{N}(f_j(y', \eta_1))\supset\mathrm{N}(f_j(0, \cdots, 0, y_n, \eta_1))\,. \tag{7.8}$$

On the other hand, the additivity property of Newton polygons yields

$$\mathrm{N}(f)=\sum_{j=1}^{r}\mathrm{N}(f_j). \tag{7.9}$$

Hence we have, by (7.6) and (7.9), the following equality:

$$\sum_{j=1}^{r}\mathrm{N}(f_j)=\mathrm{N}(f)=\mathrm{N}(f(0, \cdots, 0, y_n, \eta_1))=\sum_{j=1}^{r}\mathrm{N}(f_j(0, \cdots, 0, y_n, \eta_1)). \tag{7.10}$$

By (7.8) and (7.10), we conclude (7.7) as desired. Q.E.D.

Let us recall the map germ

$$\Psi^{\sim}=(X; \Xi, Z): (\boldsymbol{C}, 0)\times(E, e^0)\longrightarrow(J^1M, e^0)$$

which is induced by the following family of characteristic curves of F:

$$\{t\mapsto\Psi^{\sim}(t, e); \Psi^{\sim}(0, e)=e=(0, y'; \eta_1 dx_1, 0)\in(E, e^0)\}.$$

We expand $\Psi^{\sim}$ with respect to t as the following form:

$$\begin{cases} X(t, y', \eta_1)=(0, y')+\sum_{i=1}^{\infty}(i!)^{-1}t^i\partial_t^i X(0, y', \eta_1). \\ \Xi(t, y', \eta_1)=(\eta_1, 0)+\sum_{i=1}^{\infty}(i!)^{-1}t^i\partial_t^i \Xi(0, y', \eta_1). \\ Z(t, y', \eta_1)=\sum_{i=1}^{\infty}(i!)^{-1}t^i\partial_t^i Z(0, y', \eta_1). \end{cases} \tag{7.11}$$

Let $q:=\mathrm{ord}[F(0, x'; 0, 0)]$, which is equal to the approximation order $\mathrm{ord}[F(x, 0, 0)]$ of the good extension $\Phi\equiv 0$, by virtue of the assumption [B.4].

Notation 7.4. For germs $g, h\in\mathcal{O}_{S\times\boldsymbol{C},(0,0)}$ and for an ideal $\mathcal{J}$ of the ring $\mathcal{O}_{S\times\boldsymbol{C},(0,0)}$, we denote by $\mathrm{N}(\mathcal{J}g)\subset\mathrm{N}(h)$ [or $\mathrm{N}(\mathcal{J}g)\Subset\mathrm{N}(h)$ resp.] if

$$\mathrm{N}(ag)\subset\mathrm{N}(h)\quad[\mathrm{N}(ag)\Subset\mathrm{N}(h)]\qquad\text{for any } a\in\mathcal{J}.$$

Lemma 7.5. *Let* $f(y', \eta_1)=F(0, y'; \eta_1 dx_1, 0)$, *and let* $(y')\subset\mathcal{O}_{S\times\boldsymbol{C},(0,0)}$ *be the defining ideal of* $\{0\}\times\boldsymbol{C}\subset S\times\boldsymbol{C}$. *Assume the conditions* [B.1]-[B.4] *of Theorem* 5.1. *Then the following* 1)-6) *hold for all* $i\geqq 1$:

1) $\mathrm{N}[(y')^{i-1}(\eta_1)\{\partial_t^i X_1(0, y', \eta_1)-(-1)^{i-1}\partial_{y_n}^{i-1}\partial_{\eta_1}f(y', \eta_1)\}]\Subset\mathrm{N}(f).$

2) $\mathrm{N}[(y')^i\partial_t^i\Xi_j(0, y', \eta_1)]\subset\mathrm{N}(f)$ *for* $2\leqq j\leqq n$.

3) $\mathrm{N}[(y')^{i-1}\partial_t^i Z(0, y', \eta_1)]\subset\mathrm{N}(f).$

4) $(y')^{i-1}(\partial_t^i X_j(0, y', \eta_1))\subset(y', \eta_1)$ *for* $2\leqq j\leqq n-1$.

5) $(y')^{i-1}(\partial_t^i\Xi_1(0, y', \eta_1))\subset(y')^{q-1}+(\eta_1).$

6) $X_n(t, y', \eta_1)=y_n-t.$

The proof of Lemma 7.5 will be given in § 12.

In this section we shall prove Proposition 7.2, under the assumption that Lemma 7.5 is true.

Lemma 7.6. *Let* $(Y', H_1): (\boldsymbol{C}, 0)_t \times (\boldsymbol{C}^{n-1}, 0)_{x'} \times (\boldsymbol{C}, 0)_{\xi_1} \to (T^*_S M, (0, 0))_{y', \eta_1}$ *be the map germ determined by Claim* 6.4. *Then* (Y', H_1) *has the following expression*:

$$\left\{\begin{array}{l} Y'(t, x', \xi_1) = x' + tY'^{\sim}(t, x', \xi_1) = \begin{bmatrix} x'' \\ x_n \end{bmatrix} + t\begin{bmatrix} Y''^{\sim} \\ 1 \end{bmatrix}(t, x', \xi_1) \\ H_1(t, x', \xi_1) = \xi_1 + tH_1^{\sim}(t, x', \xi_1) \end{array}\right. \tag{7.12}$$

such that

$$\left\{\begin{array}{l} Y''^{\sim}(t, 0, \xi_1) \in (t, \xi_1) \\ H_1^{\sim}(t, 0, 0) \in (t)^{q-1}. \end{array}\right. \tag{7.13}$$

Proof. By the identity (6.5) and the expansion (7.11), it follows that

$$\begin{aligned} \begin{bmatrix} x' \\ \xi_1 \end{bmatrix} &= \begin{bmatrix} X' \\ \Xi_1 \end{bmatrix}(t, Y'(t, x', \xi_1), H_1(t, x', \xi_1)) \\ &= \begin{bmatrix} Y' \\ H^1 \end{bmatrix}(t, x', \xi_1) + \sum_{i=1}^{\infty} (i!)^{-1} t^i \partial_t^i \begin{bmatrix} X' \\ \Xi_1 \end{bmatrix}(0, Y'(t, x', \xi_1), H_1(t, x', \xi_1)). \end{aligned} \tag{7.14}$$

Setting $t=0$ in (7.14), we have $(Y', H_1)|_{t=0} = (x', \xi_1)$. Hence we get

$$\begin{bmatrix} Y' \\ H_1 \end{bmatrix}(t, x', \xi_1) = \begin{bmatrix} x' \\ \xi_1 \end{bmatrix} + t^3 \begin{bmatrix} Y'^{\sim} \\ H_1^{\sim} \end{bmatrix}(t, x', \xi_1). \tag{7.15}$$

Note that the assertion 6) in Lemma 7.5 implies

$$x_n \equiv Y_n(t, x', \xi_1) - t = x_n + tY_n^{\sim}(t, x', \xi_1) - t$$

which shows

$$Y_n^{\sim}(t, x', \xi_1) \equiv 1. \tag{7.16}$$

Substituting (7.15) into (7.14), with using (7.16) and

$$\partial_t(X', \Xi_1)(0, y', \eta_1) = (\partial_{\xi'} F, -\eta_1 \partial_z F - \partial_{x_1} F)(0, y'; \eta_1 dx_1, 0)$$

we have the following identities:

$$\left\{\begin{array}{l} \begin{bmatrix} x'' \\ x_n \end{bmatrix} \equiv \begin{bmatrix} x'' \\ x_n \end{bmatrix} + t\begin{bmatrix} Y''^{\sim} \\ 1 \end{bmatrix} + t\begin{bmatrix} \partial_{\xi''} F \\ -1 \end{bmatrix}(0, Y'; H_1 dx_1, 0) \\ \qquad\qquad + \sum_{i=2}^{\infty} (i!)^{-1} t^i \begin{bmatrix} \partial_t^i X'' \\ 0 \end{bmatrix}(0, Y', H_1). \end{array}\right. \tag{7.17}$$

$$\left\{\begin{array}{l} \xi_1 \equiv \xi_1 + tH_1^{\sim} + t\{-\eta_1 \partial_z F - \partial_{x_1} F\}(0, Y'; H_1 dx_1, 0) \\ \qquad\qquad + \sum_{i=2}^{\infty} (i!)^{-1} t^i \partial_t^i \Xi_1(0, Y', H_1). \end{array}\right. \tag{7.18}$$

Setting $x'=0$ in (7.17) and dividing it by t, we get

$$-Y''^{\sim}(t, 0, \xi_1)=\partial_{\xi''}F(0, tY''^{\sim}(t, 0, \xi_1), t; \{\xi_1+tH_1^{\sim}(t, 0, \xi_1)\}dx_1, 0)$$
$$+\sum_{i=2}^{\infty}(i!)^{-1}t^{i-1}\partial_t^i X''(0, tY''^{\sim}(t, 0, \xi_1), t, \xi_1+tH_1^{\sim}(t, 0, \xi_1)).$$

Thus, by $\partial_{\xi''}F\in(x, \xi, z)$ which is a consequence of [B.1], the first assertion of (7.13) follows.

Setting $(x', \xi_1)=(0, 0)$ in (7.18) and dividing it by t, it follows

$$(7.19)\qquad -H_1^{\sim}(t, 0, 0)=-tH_1^{\sim}(t, 0, 0)\partial_z F(0, tY'^{\sim}(t, 0, 0); tH_1^{\sim}(t, 0, 0)dx_1, 0)$$
$$-\partial_{x_1}F(0, tY'^{\sim}(t, 0, 0); tH_1^{\sim}(t, 0, 0)dx_1, 0)$$
$$+\sum_{i=2}^{\infty}(i!)^{-1}t^{i-1}\partial_t^i \Xi_1(0, tY'^{\sim}(t, 0, 0), tH_1^{\sim}(t, 0, 0)).$$

Then the assumption [B.4], that is, $F(x; 0, 0)\in(x)^q$ yields

$$(7.20)\qquad \partial_{x_1}F(0, tY'^{\sim}(t, 0, 0); 0, 0)\in(t)^{q-1}.$$

By the assertion 5) in Lemma 7.5, we also have

$$(7.21)\qquad t^{i-1}\partial_t^i \Xi_1(0, tY'^{\sim}(t, 0, 0), 0)\in(t)^{q-1}.$$

From (7.19)-(7.21), we get

$$-H_1^{\sim}(t, 0, 0)\in(t)^{q-1}+(tH_1^{\sim}(t, 0, 0))$$

that is, there exist germs $a(t)$, $b(t)$ such that

$$-H_1^{\sim}(t, 0, 0)=t^{q-1}a(t)+tH_1^{\sim}(t, 0, 0)b(t).$$

Since $1+tb(t)$ is an invertible germ, we get the second assertion of (7.13). The proof of Lemma 7.6 is complete. Q.E.D.

Corollary 7.7. *There exist map germs* $y''^{\sim}(t, \xi_1)$, $\eta_1^{\sim}(t)$ *and* $\sigma(t, \xi_1)$ *such that*

$$\begin{cases} Y'(t, 0, \xi_1)=t(y''^{\sim}(t, \xi_1), 1) \\ H_1(t, 0, \xi_1)=\xi_1\sigma(t, \xi_1)+\eta_1^{\sim}(t) \end{cases}$$

with the following properties:

$$(7.22)\qquad \begin{cases} y''^{\sim}(t, \xi_1)\in(t, \xi_1). \\ \eta_1^{\sim}(t)\in(t)^q. \\ \sigma(0, \xi_1)\equiv 1. \end{cases}$$

Proof. We set

$$y''^{\sim}(t, \xi_1):=Y''^{\sim}(t, 0, \xi_1),\ \eta_1^{\sim}(t):=tH_1^{\sim}(t, 0, 0).$$

Then, Lemma 7.6 yields that $y''^{\sim}\in(t, \xi_1)$, $\mathrm{ord}[\eta_1^{\sim}]\geq q$ and $Y'|_{x'=0}=t(y''^{\sim}, 1)$. On the other hand, it follows

$$\begin{aligned} H_1(t, 0, \xi_1) &= \xi_1 + tH_1^{\sim}(t, 0, \xi_1) \\ &= \xi_1 + t\{H_1^{\sim}(t, 0, 0) + \xi_1{}^3\sigma^{\sim}(t, \xi_1)\} \\ &= \xi_1(1 + t\sigma^{\sim}(t, \xi_1)) + \eta_1^{\sim}(t). \end{aligned}$$

Thus it suffices to set $\sigma(t, \xi_1) := 1 + t\sigma^{\sim}(t, \xi_1)$. Q.E.D.

Now we prove *the assertion* (7.4) of Proposition 7.2.

For $1 \leqq j \leqq r$, Corollary 7.7 yields

$$\begin{aligned} B_j(t, 0, \xi_1) &= f_j(Y'(t, 0, \xi_1), H_1(t, 0, \xi_1)) \\ &= f_j(t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1) + \eta_1^{\sim}(t)). \end{aligned}$$

Let $f_j(y', \eta_1) = \sum_{\nu=0}^{\infty} c_{\nu j}(y')\eta_1^{\nu}$ be the Taylor expansion of f_j. Then we have

$$B_j(t, 0, \xi_1) = \sum_{\nu=0}^{\infty} c_{\nu j}(t(y''^{\sim}(t, \xi_1), 1))\{\xi_1\sigma(t, \xi_1) + \eta_1^{\sim}(t)\}^{\nu}. \tag{7.23}$$

Claim 7.8. *Let us put* $q_j := \mathrm{ord}[f_j(y', 0)] = \mathrm{ord}[c_{0j}]$. *Then we have*

$$B_j(t, 0, 0) = \mathrm{Loc}[c_{0j}](\partial_{y_n})t^{q_j}(1 + O(t)).$$

Proof. Set $\xi_1 = 0$ in (7.23), then we have

$$\begin{aligned} B_j(t, 0, 0) &= \sum_{\nu=0}^{\infty} c_{\nu j}(t(y''^{\sim}(t, 0), 1))\eta_1^{\sim}(t)^{\nu} \\ &\underset{(\mathrm{mod}\,(t)^{q+1})}{\equiv} \begin{cases} c_{0j}(0, \cdots, 0, t) \\ + \sum_{|\alpha| \geqq 1} (\alpha!)^{-1}\partial_{y''}^{\alpha} c_{0j}(0, \cdots, 0, t)\{ty''^{\sim}(t, 0)\}^{\alpha} \\ + c_{1j}(t(y''^{\sim}(t, 0), 1))\eta_1^{\sim}(t) \end{cases} \end{aligned} \tag{7.24}$$

since $\mathrm{ord}[\eta_1^{\sim}] \geqq q$ by (7.22). Since $\mathrm{ord}[c_{0j}] = q_j$ and $\mathrm{ord}[y''^{\sim}(t, 0)] \geqq 1$, it follows that the second term in the right hand side of (7.24) has a vanishing order at least

$$\min\{q_j - |\alpha| + 2|\alpha|\,;\ |\alpha| \geqq 1\} > q_j.$$

On the third term in the right hand side of (7.24), we note:

$$q_j < q \text{ if } c_{1j}(y') \text{ is a unit in } \mathcal{O}_{S,0}. \tag{7.25}$$

Indeed, if c_{1j} is a unit then $p_j := \mathrm{ord}[f_j(0, \eta_1)] = 1$. Since we assume $p = \sum_{j=1}^{r} p_j \geqq 2$, we get $r \geqq 2$. Hence (7.25) follows. By (7.25) we have

$$\mathrm{ord}[c_{1j}(t(y''^{\sim}(t, 0), 1))\eta_1^{\sim}(t)] \geqq \mathrm{ord}[c_{1j}] + q > q_j.$$

Hence we conclude

$$B_j(t, 0, 0)-c_{0j}(0, \cdots, 0, t)\in(t)^{q_j+1}$$

which shows Claim 7.8. Q.E.D.

From (7.23), we can find a germ $g_j(t, \xi_1)$ such that

$$B_j(t, 0, \xi_1)=\sum_{\nu=0}^{\infty} c_{\nu j}(t(y''^{\sim}(t, \xi_1), 1))\{\xi_1\sigma(t, \xi_1)\}^{\nu}+\eta_1^{\sim}(t)g_j(t, \xi_1). \tag{7.26}$$

Claim 7.9. *Let us put*

$$B_j^{\sim}(t, \xi_1):=\sum_{\nu=0}^{\infty} c_{\nu j}(t(y''^{\sim}(t, \xi_1), 1))\{\xi_1\sigma(t, \xi_1)\}^{\nu}. \tag{7.27}$$

Then it follows that

$$\mathrm{N}(B_j^{\sim})\subset\mathrm{N}(f_j) \quad \textit{for} \quad 1\leqq j\leqq r.$$

Proof. We write $B_j^{\sim}(t, \xi_1)$ as $B_j^{\sim}(t, \xi_1)=\sum_{\mu=0}^{\infty} c_{\mu j}^{\sim}(t)\xi_1^{\mu}$. Then it suffices for Claim 7.9 to derive

$$(\mathrm{ord}[c_{\mu j}^{\sim}], \mu)\in\mathrm{N}(f_j) \quad \text{for all} \quad \mu. \tag{7.28}$$

From the definition (7.27) of $B_j^{\sim}$, the coefficients $c_{\mu j}^{\sim}(t)$ are given by

$$c_{\mu j}^{\sim}(t)=\sum_{\nu=0}^{\mu}((\mu-\nu)!)^{-1}\partial_{\xi_1}^{\mu-\nu}\{c_{\nu j}(t(y''^{\sim}(t, \xi_1), 1))\sigma(t, \xi_1)^{\nu}\}|_{\xi_1=0} \tag{7.29}$$

which yields

$$\mathrm{ord}[c_{\mu j}^{\sim}]\geqq \min_{0\leqq\nu\leqq\mu} \mathrm{ord}[c_{\nu j}].$$

Hence the fact

$$(\mathrm{ord}[c_{\nu j}], \mu)=(\mathrm{ord}[c_{\nu j}], \nu)+(0, \mu-\nu)\in\mathrm{N}(f_j) \quad \text{for} \quad 0\leqq\nu\leqq\mu$$

implies the desired (7.28). Thus Claim 7.9 follows. Q.E.D.

Claim 7.10. *For* $1\leqq j\leqq r$, *it follows that*

$$\mathrm{ch}(B_j^{\sim}(t, \xi_1))=\mathrm{ch}(f_j)(-tL_F(e^0), \xi_1).$$

Proof. Let us denote the set $\mathrm{Ver}\,\mathrm{N}(f_j)$ of vertices of $\mathrm{N}(f_j)$ as

$$\mathrm{Ver}\,\mathrm{N}(f_j)=\{(a_j(\lambda), p_j-b_j(\lambda)); 0\leqq\lambda\leqq m_j\} \tag{7.30}$$

where the sequences $\{a_j(\lambda)\}$, $\{b_j(\lambda)\}$ are arranged as monotonely increasing in λ.

Since the Newton polygon $\mathrm{N}(f_j)$ satisfies the coprimeness condition, it follows that

$$\begin{aligned}\mathrm{ch}(f_j)(-tL_F(e^0), \xi_1)&=\mathrm{ch}(f_j)(0, \cdots, 0, t, \xi_1)\\ &=\sum_{\lambda=0}^{m_j}\mathrm{Loc}[c_{p_j-b_j(\lambda), j}](\partial_{y_n})t^{a_j(\lambda)}\xi_1^{p_j-b_j(\lambda)}.\end{aligned} \tag{7.31}$$

Let $e_{j\lambda}\in\boldsymbol{C}$ be the coefficient of $t^{a_j(\lambda)}\xi_1^{p_j-b_j(\lambda)}$ in $B_j^{\sim}(t, \xi_1)$. Since

$$\mathrm{N}(B_j^{\sim})\subset \mathrm{N}(f_j)=\mathrm{N}[\mathrm{ch}(f_j)(-tL_F(e^0), \xi_1)]$$

by Claim 7.9 and Remark 7.3, the expression (7.31) yields that it suffices for Claim 7.10 to verify

$$e_{j\lambda}=\mathrm{Loc}[c_{p_j-b_j(\lambda),j}](\partial_{y_n}) \qquad \text{for} \quad 0\leqq\lambda\leqq m_j. \tag{7.32}$$

We fix λ. It is clear from (7.27) that $e_{j\lambda}$ depends only on the terms

$$c_{\nu j}(t(y''^{\sim}(t, \xi_1), 1))\{\xi_1\sigma(t, \xi_1)\}^{\nu} \qquad \text{for} \quad 0\leqq\nu\leqq p_j-b_j(\lambda).$$

Since $(a_j(\lambda), p_j-b_j(\lambda))\in \mathrm{Ver}\,\mathrm{N}(f_j)$ it follows that

$$\mathrm{ord}[c_{\nu j}]>\mathrm{ord}[c_{p_j-b_j(\lambda),j}]=a_j(\lambda) \qquad \text{if} \quad \nu<p_j-b_j(\lambda).$$

Thus $e_{j\lambda}$ depends only on the following term:

$$\begin{aligned} &c_{p_j-b_j(\lambda),j}(t(y''^{\sim}(t, \xi_1), 1))\{\xi_1\sigma(t, \xi_1)\}^{p_j-b_j(\lambda)} \\ &=\xi_1^{p_j-b_j(\lambda)}\{\sigma(0, \xi_1)^{p_j-b_j(\lambda)}+O(t)\}[c_{p_j-b_j(\lambda),j}(0, \cdots, 0, t) \\ &\quad +\sum_{|\alpha|\geqq 1}(\alpha !)^{-1}\partial_{y''}^{\alpha}c_{p_j-b_j(\lambda),j}(0, \cdots, 0, t)\{ty''^{\sim}(t, \xi_1)\}^{\alpha}]. \end{aligned}$$

Then, using the inequality

$$\mathrm{ord}[\partial_{y''}^{\alpha}c_{p_j-b_j(\lambda),j}(0, \cdots, 0, t)\{ty''^{\sim}(t, 0)\}^{\alpha}]\geqq a_j(\lambda)-|\alpha|+2|\alpha|>a_j(\lambda)$$

for $|\alpha|\geqq 1$, we get

$$e_{j\lambda}t^{a_j(\lambda)}\xi_1^{p_j-b_j(\lambda)}=\mathrm{Loc}[c_{p_j-b_j(\lambda),j}](\partial_{y_n})t^{a_j(\lambda)}\xi_1^{p_j-b_j(\lambda)}\sigma(0, \xi_1)^{p_j-b_j(\lambda)}.$$

Thus, by $\sigma(0, \xi_1)\equiv 1$ (Corollary 7.7), we conclude (7.32).

The proof of Claim 7.10 is complete. Q.E.D.

Proof of (7.4). By virtue of Claim 7.10, it suffices to show

$$\mathrm{ch}[B_j(t, 0, \xi_1)]=\mathrm{ch}(B_j^{\sim}(t, \xi_1)). \tag{7.33}$$

By the definition (7.27), the equality (7.26) can be written as

$$B_j(t, 0, \xi_1)=B_j^{\sim}(t, \xi_1)+\eta_1^{\sim}(t)g_j(t, \xi_1). \tag{7.34}$$

Since $\mathrm{N}(B_j^{\sim})=\mathrm{N}(f_j)$ which is a consequence of Claims 7.9 and 7.10, it suffices for (7.33) to verify

$$\mathrm{N}(\eta_1^{\sim}(t)g_j(t, \xi_1))\Subset\mathrm{N}(f_j). \tag{7.35}$$

Putting $\xi_1=0$ in (7.34), we have $B_j(t, 0, 0)=B_j^{\sim}(t, 0)+\eta_1^{\sim}(t)g_j(t, 0)$. Since Claims 7.8 and 7.10 lead us to

$$\mathrm{Loc}[B_j(t, 0, 0)]=\mathrm{Loc}[B_j^{\sim}(t, 0)]=\mathrm{Loc}[c_{0j}](\partial_{y_n})t^{q_j},$$

we have $\eta_1^{\sim}(t)g_j(t, 0)\in(t)^{q_j+1}$. Then, by $q\geqq q_j$, we have

$$\eta_1^{\sim}(t)g(t, \xi_1)\in(t)^{qj+1}+(t)^q(\xi_1)\subset(t)^{qj}(t, \xi_1),$$

which shows the desired (7.35).

The proof of (7.4) in Proposition 7.2 is complete. Q.E.D.

Now we prove *the second assertion* (7.5) in Proposition 7.2.

It suffices for (7.5) to show

$$\mathrm{N}\left[\xi_1\left\{A(t, 0, \xi_1)-t^{-1}\int_0^t \partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0), \xi_1)d\theta\right\}\right]\in\mathrm{N}(f). \tag{7.36}$$

We recall the definition

$$A(t, x', \xi_1):=X_1^{\sim}(t, Y'(t, x', \xi_1), H_1(t, x', \xi_1))$$

where $X_1(t, y', \eta_1)=tX_1^{\sim}(t, y', \xi_1)$ is the first component of the chracteristic curve $t\mapsto\Psi^{\sim}(t, y', \xi_1)$ of F. Since $X_1^{\sim}$ can be expanded as the form

$$X_1^{\sim}(t, y', \eta_1)=\partial_{\eta_1}F(0, y'; \eta_1 dx_1, 0)+\sum_{i=2}^{\infty}(i!)^{-1}t^{i-1}\partial_t^i X_1(0, y', \eta_1)$$

$$=\partial_{\eta_1}f(y', \eta_1)+\sum_{i=2}^{\infty}(i!)^{-1}t^{i-1}\partial_t^i X_1(0, y', \eta_1),$$

Corollary 7.7 yields that $A(t, 0, \xi_1)$ can be written as the form:

$$\begin{aligned}A(t, 0, \xi_1)&=X_1^{\sim}(t, t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1)+\eta_1^{\sim}(t))\\&=\partial_{\eta_1}f(t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1)+\eta_1^{\sim}(t))\\&\quad+\sum_{i=2}^{\infty}(i!)^{-1}t^{i-1}\partial_t^i X_1(0, t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1)+\eta_1^{\sim}(t)).\end{aligned} \tag{7.37}$$

Note that the expression (7.37) and the inclusion

$$\mathrm{N}(\eta_1^{\sim}(t)\xi_1)\subset\mathrm{N}(t^q\xi_1) \tag{7.38}$$

imply the following equality:

(7.39) $\xi_1 A(t, 0, \xi_1)$

$$\underset{(\mathrm{mod}(t)^q(\xi_1))}{\equiv}\left\{\begin{aligned}&\xi_1\sum_{i=1}^{q}(i!)^{-1}t^{i-1}(-1)^{i-1}\partial_{y_n}^{i-1}\partial_{\eta_1}f(t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1))\\&+\xi_1\sum_{i=2}^{q}(i!)^{-1}t^{i-1}[\partial_t^i X_1(0, t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1))\\&\qquad-(-1)^{i-1}\partial_{y_n}^{i-1}\partial_{\eta_1}f(t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1))].\end{aligned}\right.$$

By virtue of the assertion 1) in Lemma 7.5, we have

(7.40) N[*the second term in the right hand side of* (7.39)]$\in\mathrm{N}(f)$.

Hence, if we define $\Gamma(t, \xi_1)$ by

$$\Gamma(t, \xi_1):=\xi_1\sum_{i=1}^{q}(i!)^{-1}t^{i-1}(-1)^{i-1}\partial_{y_n}^{i-1}\partial_{\eta_1}f(t(y''^{\sim}(t, \xi_1), 1), \xi_1\sigma(t, \xi_1)) \tag{7.41}$$

then (7.39), (7.40) and the inclusion $\mathrm{N}(t^q\xi_1)\Subset\mathrm{N}(f(y', \eta_1))$ imply

$$\mathrm{N}(\xi_1 A(t, 0, \xi_1)-\Gamma(t, \xi_1))\Subset\mathrm{N}(f). \tag{7.42}$$

Thus it suffices for (7.36) to derive

$$\mathrm{N}[\Gamma(t, \xi_1)-\xi_1 t^{-1}\int_0^t \partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(t^0), \xi_1)d\theta]\Subset\mathrm{N}(f). \tag{7.43}$$

Let $f(y', \eta_1)=\sum_{\nu=0}^{\infty} c_\nu(y')\eta_1^\nu$ be the Taylor expansion of f. Then (7.41) can be written as

$$\Gamma(t, \xi_1)=\sum_{i=1}^{q}(i\,!)^{-1}t^{i-1}(-1)^{i-1}\sum_{\nu=1}^{\infty}\{\partial_{y_n}^{i-1}c_\nu(t(y''^{\sim}(t, \xi_1), 1))\}\nu\xi_1\{\xi_1\sigma(t, \xi_1)\}^{\nu-1}.$$

Put $\Gamma(t, \xi_1)=:\sum_{\lambda=1}^{\infty}k_\lambda(t)\xi_1^\lambda$. Then $k_\lambda(t)$ can be written as the form

$$\begin{cases} k_\lambda(t)=\sum_{\nu=1}^{\lambda}k\tilde{_{\lambda\nu}}(t) \qquad \text{where} \\ k\tilde{_{\lambda\nu}}(t):=\sum_{i=1}^{q}(i\,!)^{-1}t^{i-1}(-1)^{i-1}((\lambda-\nu)\,!)^{-1}\nu \\ \qquad \times\partial_{\xi_1}^{\lambda-\nu}[\{\partial_{y_n}^{i-1}c_\nu(t(y''^{\sim}(t, \xi_1), 1))\}\sigma(t, \xi_1)^{\nu-1}]|_{\xi_1=0}. \end{cases} \tag{7.44}$$

Since $\mathrm{ord}[k\tilde{_{\lambda\nu}}]\geqq i-1+\mathrm{ord}[c_\nu]-(i-1)=\mathrm{ord}[c_\nu]$, if follows

$$(\mathrm{ord}[k\tilde{_{\lambda\nu}}], \lambda)\in(\mathrm{ord}[c_\nu], \nu)+\bar{\boldsymbol{R}}_+^2\subset\mathrm{N}(f) \qquad \text{for} \quad 1\leqq\nu\leqq\lambda. \tag{7.45}$$

$$\begin{cases} (\mathrm{ord}[k\tilde{_{\lambda\nu}}], \lambda)\in\mathrm{N}(f)-\partial^0\mathrm{N}(f) \\ \qquad\qquad \text{if either } 1\leqq\nu<\lambda \text{ or } (\mathrm{ord}[c_\nu], \nu)\notin\mathrm{Ver}\,\mathrm{N}(f). \end{cases} \tag{7.46}$$

Note that (7.42) and (7.45) yield

$$\begin{aligned}\mathrm{N}(\xi_1 A(t, 0, \xi_1))&\subset\text{convex hull }\{\mathrm{N}(\xi_1 A(t, 0, \xi_1)-\Gamma(t, \xi_1))\cup\mathrm{N}(\Gamma(t, \xi_1))\} \\ &\subset\mathrm{N}(f).\end{aligned} \tag{7.47}$$

Moreover (7.46) yields that, if we put

$$\Gamma_1(t, \xi_1):=\sum_{\lambda\geqq1,\,(\mathrm{ord}[c_\lambda],\,\lambda)\in\mathrm{Ver}\,\mathrm{N}(f)}k\tilde{_{\lambda\lambda}}(t)\xi_1^\lambda \tag{7.48}$$

then we have

$$\mathrm{N}(\Gamma(t, \xi_1)-\Gamma_1(t, \xi_1))\Subset\mathrm{N}(f). \tag{7.49}$$

For λ satisfying $(\mathrm{ord}[c_\lambda], \lambda)\in\mathrm{Ver}\,\mathrm{N}(f)$, we consider $k\tilde{_{\lambda\lambda}}(t)$: Setting $\nu=\lambda$ in (7.44), we have

$$k\tilde{_{\lambda\lambda}}(t)=\sum_{i=1}^{q}(i\,!)^{-1}t^{i-1}(-1)^{i-1}\lambda\{\partial_{y_n}^{i-1}c_\lambda(t(y''^{\sim}(t, 0), 1))\}\sigma(t, 0)^{\lambda-1}. \tag{7.50}$$

We take the expansion

$$\partial_{y_n}^{i-1}c_\lambda(t(y''^{\sim}(t, 0), 1))$$
$$=\partial_{y_n}^{i-1}c_\lambda(0, \cdots, 0, t)+\sum_{|\alpha|\geqq 1}(\alpha!)^{-1}\partial_{y_n}^{i-1}\partial_{y''}^{\alpha}c_\lambda(0, \cdots, 0, t)\{ty''^{\sim}(t, 0)\}^{\alpha},$$

and we put

$$k_{\lambda\lambda}^{0}(t):=\sum_{i=1}^{q}(i!)^{-1}t^{i-1}(-1)^{i-1}\lambda\{\partial_{y_n}^{i-1}c_\lambda(0, \cdots, 0, t)\}\sigma(t, 0)^{\lambda-1}. \tag{7.51}$$

Then the inequality $\mathrm{ord}[ty''^{\sim}(t, 0)]\geqq 2$ implies

$$\mathrm{ord}[k_{\lambda\lambda}^{\sim}(t)-k_{\lambda\lambda}^{0}(t)]\geqq i-1+\mathrm{ord}[c_\lambda]-(i-1)+\min_{|\alpha|\geqq 1}\{-|\alpha|+2|\alpha|\}$$
$$>\mathrm{ord}[c_\lambda].$$

Thus, if we put

$$\Gamma_2(t, \xi_1):=\sum_{\lambda\geqq 1,\ (\mathrm{ord}[c_\lambda],\ \lambda)\in \mathrm{Ver}\,\mathrm{N}(f)}k_{\lambda\lambda}^{0}(t)\xi_1^{\lambda} \tag{7.52}$$

then we have

$$\mathrm{N}(\Gamma_1(t, \xi_1)-\Gamma_2(t, \xi_1))\subseteq \mathrm{N}(f). \tag{7.53}$$

By virtue of (7.49) and (7.53), it suffices for (7.43) to show

$$\mathrm{N}(\Gamma_2(t, \xi_1)-t^{-1}\xi_1\int_0^t\partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0), \xi_1)d\theta)\subseteq \mathrm{N}(f). \tag{7.54}$$

We denote the set $\mathrm{Ver}\,\mathrm{N}(f)$ by $\{(a(\mu), p-b(\mu)\,;\ 0\leqq\mu\leqq m\}$. Then, substituting $\lambda=p-b(\mu)$ in (7.51), (7.52) can be written as

$$\Gamma_2(t, \xi_1)=\sum_{\mu=0}^{m-1}k_{p-b(\mu),\,p-b(\mu)}^{0}(t)\xi_1^{p-b(\mu)}$$
$$=\sum_{\mu=0}^{m-1}\sum_{i=1}^{q}(i!)^{-1}t^{i-1}(-1)^{i-1}(p-b(\mu))$$
$$\times\{\partial_{y_n}^{i-1}c_{p-b(\mu)}(0, \cdots, 0, t)\}\sigma(t, 0)^{p-b(\mu)-1}\xi_1^{p-b(\mu)}.$$

Note that

$$c_{p-b(\mu)}(y')\equiv y_n^{a(\mu)}\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})\{1+O(y_n)\}\qquad \mathrm{mod}(y'')$$

which implies

$$\partial_{y_n}^{i-1}c_{p-b(\mu)}(0, \cdots, 0, t)=\partial_t^{i-1}[e_\mu t^{a(\mu)}\{1+O(t)\}]$$

where $e_\mu\in \boldsymbol{C}$ denotes the non-zero constant $\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})$.

Since $\mathrm{N}(t^{a(\mu)+1}\xi_1^{p-b(\mu)})\subseteq \mathrm{N}(f)$, if we put

$$\Gamma_3(t, \xi_1):=\sum_{\mu=0}^{m-1}\sum_{i=1}^{q}(i!)^{-1}t^{i-1}(-1)^{i-1}(p-b(\mu))\partial_t^{i-1}e_\mu t^{a(\mu)}\xi_1^{p-b(\mu)}$$

then we have

$$\mathrm{N}(\Gamma_2(t, \xi_1)-\Gamma_3(t, \xi_1))\subseteq \mathrm{N}(f). \tag{7.55}$$

Hence it suffices for (7.54) to show

$$\Gamma_3(t, \xi_1)=t^{-1}\xi_1\int_0^t \partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0), \xi_1)d\theta\,. \tag{7.56}$$

We verify (7.56): Since $\partial_t^{i-1}t^{a(\mu)}\equiv 0$ if $i>a(\mu)+1$, we have the following expression of $\Gamma_3(t, \xi_1)$:

$$\begin{aligned}\Gamma_3(t, \xi_1)&=\sum_{\mu=0}^{m-1}\sum_{i=1}^{a(\mu)+1}(i\,!)^{-1}t^{i-1}(-1)^{i-1}(p-b(\mu))\\ &\quad\times e_\mu\frac{a(\mu)\,!}{(a(\mu)-i+1)\,!}t^{a(\mu)-i+1}\xi_1^{p-b(\mu)}\\ &=t^{-1}\xi_1\sum_{\mu=0}^{m-1}e_\mu(p-b(\mu))\xi_1^{p-b(\mu)-1}\frac{1}{a(\mu)+1}t^{a(\mu)+1}\\ &\quad\times\sum_{i=1}^{a(\mu)+1}C(a(\mu)+1, i)(-1)^{i-1}\end{aligned}$$

where we set $C(N, i):=\dfrac{N!}{i\,!(N-i)\,!}$ for $0\leqq i\leqq N$. Note that

$$\sum_{i=1}^{N}C(N, i)(-1)^{i-1}=-(1-1)^N+C(N, 0)=1$$

which implies

$$\begin{aligned}\Gamma_3(t, \xi_1)&=t^{-1}\xi_1\sum_{\mu=0}^{m-1}e_\mu(p-b(\mu))\xi_1^{p-b(\mu)-1}(a(\mu)+1)^{-1}t^{a(\mu)+1}\\ &=t^{-1}\xi_1\int_0^t\partial_{\xi_1}\Big[\sum_{\mu=0}^{m}e_\mu\theta^{a(\mu)}\xi_1^{p-b(\mu)}\Big]d\theta\\ &=t^{-1}\xi_1\int_0^t\partial_{\xi_1}\mathrm{ch}(f)(0, \cdots, 0, \theta, \xi_1)d\theta\\ &=t^{-1}\xi_1\int_0^t\partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0), \xi_1)d\theta\,.\end{aligned}$$

Hence (7.56) follows.

The proof of (7.5) in Proposition 7.2 is complete. Q.E.D.

§8. Proof of $V_{j2}=R_j^{-1}(0)$ as Germs of Hypersurfaces

In this section we prove that the image set $V_{j2}=\pi_{j2}(V_{j1})$ has only one irreducible component locally at $(0, 0)\in(M, 0)_x\times(\boldsymbol{C}, 0)_{\xi_1}$, by means of the theory of resultant. We also show that this irreducible component is given by a zero set of a resultant $R_j(x, \xi_1)$ (see Proposition 8.9 and Theorem 8.10).

We first replace the defining germs $tA-x_1$ and B_j of V_{j1}, which are obtained by Proposition 6.7, with suitable polynomials $tP-x_1$ and Q_j with respect to the variable t:

Lemma 8.1. 1) *For $1\leqq j\leqq r$, there exist a Weierstrass polynomial Q_j with* *respect to* t *of degree* $q_j:=\mathrm{ord}[f_j(y', 0)]$, *and a unit* ε_j *such that*

$$B_j(t, x', \xi_1)=Q_j(t, x', \xi_1)\varepsilon_j(t, x', \xi_1) \quad in \quad \mathcal{O}_{C\times S\times C,(0,0,0)} \tag{8.1}$$

where Q_j *and* ε_j *are uniquely determined by* B_j.

2) *There exist a germ* $\tilde{A}$ *and a polynomial* P *with respect to* t *of degree at most* $q-1=\sum_{j=1}^{r} q_j-1$ *such that*

$$A(t, x', \xi_1)=\tilde{A}(t, x', \xi_1)\sum_{j=1}^{r} Q_j(t, x', \xi_1)+P(t, x', \xi_1) \tag{8.2}$$

where $\tilde{A}$ *and* P *are uniquely determined by* A.

Proof. By Proposition 7.2, we have $\mathrm{N}(B_j(t, 0, \xi))=\mathrm{N}(f_j)$ which yields

$$\mathrm{ord}[B_j(t, 0, 0)]=\mathrm{ord}[f_j(y', 0)]=q_j<\infty .$$

Then the assertion 1) follows from the Weierstrass's preparation theorem. The assertion 2) is a consequence of the Weierstrass's division theorem. Q. E. D.

Corollary 8.2. *It follows that*

$$(tA-x_1, B_j)=(tP-x_1, Q_j) \tag{8.3}$$

as ideals in the ring $\mathcal{O}_{C\times M\times C,(0,0,0)}$ *for* $1\leqq j\leqq r$.

Proof. Let f, g be germs in $\mathcal{O}_{C\times M\times C,(0,0,0)}$. Then we have

$$\begin{aligned} f(tA-x_1)+gB_j&=f\left\{t\left(\tilde{A}\prod_{i=1}^{r} Q_i+P\right)-x_1\right\}+gQ_j\varepsilon_j \\ &=f(tP-x_1)+(g\varepsilon_j+tf\tilde{A}\prod_{i\neq j} Q_i)Q_j . \end{aligned}$$

Note that a transformation

$$(f, g)\longmapsto(f, g\varepsilon_j+tf\tilde{A}\prod_{i\neq j} Q_i)$$

is invertible, since ε_j is a unit. Hence Corollary 8.2 follows. Q. E. D.

By this corollary we can take $tP-x_1$ and Q_j as defining functions of V_{j1} locally at $(0, 0, 0)\in \boldsymbol{C}\times M\times \boldsymbol{C}$:

$$(V_{j1}, (0, 0, 0))=\{(t, x, \xi_1);\ tP(t, x', \xi_1)-x_1=Q_j(t, x', \xi_1)=0\}. \tag{8.4}$$

Lemma 8.3. *The principal part of* $\mathrm{N}(P(t, 0, \xi_1))$ *has the same property* (7.5) *of the principal part of* $\mathrm{N}(A(t, 0, \xi_1))$, *that is, it follows that*

$$\mathrm{N}[\mathrm{ch}(P(t, 0, \xi_1))-t^{-1}\int_0^t \partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0), \xi_1)d\theta]+\mathrm{N}(\xi_1)\in\mathrm{N}(f). \tag{8.5}$$

Proof. By virtue of (7.5) in Proposition 7.2, it suffices to derive

$$\mathrm{N}(\xi_1\{A(t, 0, \xi_1)-P(t, 0, \xi_1)\})\in\mathrm{N}(f). \tag{8.6}$$

By the definition of P we have

$$A(t,0,\xi_1)-P(t,0,\xi_1)=\tilde{A}(t,0,\xi_1)\prod_{j=1}^{r}Q_j(t,0,\xi_1). \tag{8.2)'$$

Note that the additivity of Newton polygons (Proposition 11.3) yields

$$\mathrm{N}(B_j)=\mathrm{N}(Q_j\varepsilon_j)=\mathrm{N}(Q_j)+\mathrm{N}(\varepsilon_j)=\mathrm{N}(Q_j)$$

since ε_j is a unit. Hence it follows that

$$\mathrm{N}\Big(\prod_{j=1}^{r}Q_j(t,0,\xi_1)\Big)=\sum_{j=1}^{r}\mathrm{N}(Q_j(t,0,\xi_1))=\sum_{j=1}^{r}\mathrm{N}(B_j(t,0,\xi_1)).$$

Then, by (7.4) in Proposition 7.2, we have

$$\mathrm{N}\Big(\prod_{j=1}^{r}Q_j(t,0,\xi_1)\Big)=\sum_{j=1}^{r}\mathrm{N}(f_j)=\mathrm{N}\Big(\prod_{j=1}^{r}f_j\Big)=\mathrm{N}(f).$$

Thus the equality (8.2)′ yields

$$\mathrm{N}(\xi_1\{A(t,0,\xi_1)-P(t,0,\xi_1)\})\subset\mathrm{N}(\xi_1)+\mathrm{N}(f)\Subset\mathrm{N}(f).$$

The proof of Lemma 8.3 is complete. Q. E. D.

Remark 8.4. It follows that

$$q-1\geqq\deg_t(P)\geqq a(m-1)=q(1)+\cdots+q(m-1)$$

where $(a(m-1),\ p-b(m-1))$ is the rightest vertex of $\mathrm{N}(f)$ except for $(a(m),0)$.

Proof. Note that

$$(a(m-1),\ p-b(m-1)-1)\in\mathrm{Ver}\,\mathrm{N}\Big(t^{-1}\int_0^t\partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0),\xi_1)d\theta\Big). \tag{8.7}$$

Indeed, for a germ $g\in\mathcal{O}_{\mathbf{C}^2,(0,0)}$ and for a vertex $(a,b)\in\mathrm{Ver}\,\mathrm{N}(g)$ with $b\geqq1$, it follows that $(a,b-1)\in\mathrm{Ver}\,\mathrm{N}(\partial_{\xi_1}g)$, and that the operator

$$g\longmapsto t^{-1}\int_0^t g(\theta,\xi_1)d\theta$$

preserves the Newton polygon $\mathrm{N}(g)$. Hence (8.7) follows. Then, by Lemma 8.3, we conclude $\deg_t(P)\geqq a(m-1)$. Since the other inequality $q-1\geqq\deg_t(P)$ is trivial by the definition of P, the proof of Remark 8.4 is complete. Q. E. D.

Now we recall the

Definition 8.5. Let $\mathcal{O}$ be an integral domain, and let $f(t),g(t)\in\mathcal{O}[t]$ be polynomials with $\mathcal{O}$-coefficients of degree m, n respectively as follows:

$$f(t)=\sum_{i=0}^{m}a_it^i,\qquad g(t)=\sum_{j=0}^{n}b_jt^j.$$

We define a $(n+m)$-*square matrix* $D(f,g)$ by

$$(8.8)\qquad D(f,g):=\left[\begin{array}{cccc|cccc} a_m & \cdots & a_0 & & & & 0 & \\ & \ddots & & \ddots & & & & \\ & & \ddots & & a_0 & & & \\ 0 & & & a_m & a_{m-1} & \cdots & a_0 & \\ \hline b_n & \cdots & \cdots & b_1 & b_0 & & 0 & \\ 0 & \ddots & & & & \ddots & & \\ & & b_n & \cdots & \cdots & \cdots & b_0 & \end{array}\right]\begin{array}{l} \}\, n \\ \\ \\ \\ \}\, m \end{array}$$

and we also define a *resultant* $r(f, g)\in\mathcal{O}$ *of* $f(t)$ and $g(t)$ by

$$(8.9)\qquad r(f, g):=\det D(f, g).$$

The following Proposition 8.6 is well-known (see, for example: [Na: Theorem 4.11.1, p. 164]):

Proposition 8.6. *Let* $\mathcal{O}, f, g$ *be as same as in Definition* 8.5, *and* δ_j *be the* $(j, n+m)$*-cofactor of the matrix* $D(f, g)$. *Then*:

1) *If either* f *or* g *is a monic polynomial then the following* (a) *and* (b) *are equivalent as statements for an element* $c\in\mathcal{O}$:

(a) $c\in(f, g)\mathcal{O}[t]$.

(b) *There exist* $e_j\in\mathcal{O}$ $(1\leqq j\leqq n+m)$, $e\in\mathcal{O}$, *and* $d\in\mathcal{O}-\{0\}$ *such that*

$$(8.10)\qquad c=a_0e_n+b_0e_{n+m}$$

$$(8.11)\qquad e_j=(e/d)\delta_j \quad \text{for} \quad 1\leqq j\leqq n+m.$$

2) *Let* $\mathcal{K}$ *be an algebraic closed field containing the ring* $\mathcal{O}$. *Let*

$$f(t)=a_m\prod_{i=1}^{m}(t-\alpha_i), \qquad g(t)=b_n\prod_{j=1}^{n}(t-\beta_j)$$

be the factorizations of f *and* g *in the ring* $\mathcal{K}[t]$. *Then it follows that*

$$(8.12)\qquad r(f, g)=a_m^n b_n^m\prod_{i,j}(\beta_j-\alpha_i).$$

Corollary 8.7. *It follows that*

$$(8.13)\qquad r(f, g)\in(f, g)\mathcal{O}[t].$$

Proof. Let $D:=D(f, g)$ be the $(n+m)$-square matrix defined by (8.8), and let D^c be the cofactor matrix of D. Since $D^cD=(\det D)I_{n+m}$, we have

$$\delta_n a_0+\delta_{n+m}b_0=\det D=r(f, g).$$

Hence the condition (b) of the assertion 1) in Proposition 8.6 holds if we take

$$e_j:=\delta_j\ (1\leqq j\leqq n+m) \quad \text{and} \quad d=e:=1.$$

Thus Corollary 8.7 follows. Q.E.D.

Now we return to our problem. The first aim in this section is to show Proposition 8.9 stated below. Before stating this, we introduce the

Notation 8.8. We set D_j, R_j respectively as

(8.14) $$D_j := D(Q_j, tP - x_1).$$

(8.15) $$R_j := r(Q_j, tP - x_1) = \det D_j \qquad \text{for} \quad 1 \leqq j \leqq r.$$

By Corollary 8.7, it follows that

$$R_j \in \mathcal{O} \cap (Q_j, tP - x_1)\mathcal{O}[t]$$

where we put $\mathcal{O} := \mathcal{O}_{M\times C,(0,0)}$. Hence we conclude

(8.16) $$(V_{j2}, (0, 0)) \subset (R_j^{-1}(0), (0, 0)).$$

Proposition 8.9. *The germ* $(R_j^{-1}(0), (0, 0))$ *of a hypersurface of* $M\times \boldsymbol{C}$ *at* $(0, 0)$ *has only one irreducible component, that is, if*

(8.17) $$R_j(x, \xi_1) = \prod_{\lambda=1}^{k(j)} S_{\lambda j}(x, \xi_1)^{\nu(\lambda, j)}$$

is an irreducible decomposition of R_j *at* $(0, 0)$*, then* $k(j) = 1$ *follows.*

Proof. Since (8.17) is an irreducible decomposition, we note

(8.18) $$\begin{cases} \text{a)} & \nu(\lambda, j) \geqq 1, \\ \text{b)} & S_{\lambda j}\ (1 \leqq \lambda \leqq k(j)) \text{ are irreducible in } \mathcal{O}_{M\times C,(0,0)}, \text{ and} \\ \text{c)} & \text{there is no germ } g(x, \xi_1) \text{ satisfying } S_{\lambda j} = g S_{\lambda' j}\ (\lambda \neq \lambda'). \end{cases}$$

Claim 1). *Set* $X_\lambda := (S_{\lambda j}^{-1}(0), (0, 0))$. *Then, for any* $1 \leqq \lambda \leqq k(j)$ *and for any sufficiently small open neighborhood* U *of* $(0, 0)$ *in* $M\times \boldsymbol{C}$*, it follows that*

$$[X_\lambda - \bigcup_{\lambda' \neq \lambda} X_{\lambda'}] \cap U \neq \emptyset.$$

Proof. If we assume that Claim 1) is not true then there exist λ and an open neighborhood U such that

(8.19) $$X_\lambda \cap U \subset \bigcup_{\lambda' \neq \lambda} X_{\lambda'}.$$

We set $T := \prod_{\lambda' \neq \lambda} S_{\lambda' j}$. Then (8.19) yields $T|_{X_\lambda} \equiv 0$. Hence the Rückert's Nullstellensatz (§ 13) implies that

$$T \in \mathrm{Rad}[(S_{\lambda j})] := \{g \in \mathcal{O}_{M\times C,(0,0)}\,;\ \exists i,\ g^i \in (s_{\lambda j})\}.$$

Since $(S_{\lambda j})$ is a prime ideal, we have $T \in (S_{\lambda j})$ which implies that there exists λ' $(\neq \lambda)$ such that $S_{\lambda'} \in (S_\lambda)$. This contradicts the condition c) in (8.18). Hence Claim 1) follows.

Claim 2). *Let* $d:=\deg_t(P)$, *and let us write*

$$P(t,\ x',\ \xi_1)=\sum_{i=0}^{d}P_i(x',\ \xi_1)t^i\,.$$

Then $P_d|_{X_\lambda}\not\equiv 0$ *for any* $1\leqq\lambda\leqq k(j)$ *where we set* $X_\lambda:=(S_{\lambda j}^{-1}(0),\ (0,\ 0))$.

Proof. If Claim 2) is not true then the Rückert's Nullstellensatz yields $P_d\in\mathrm{Rad}[(S_{\lambda j})]=(S_{\lambda j})$ for some λ. We note that

$$\mathrm{ord}[S_{\lambda j}(x_1,\ 0,\ \cdots,\ 0)]=:s(\lambda,\ j)<\infty\,. \tag{8.20}$$

Indeed, since Q_j is a Weierstrass polynomial, we have

$$R_j(x_1,\ 0,\ \cdots,\ 0)=\det\left[\begin{array}{c|cccc} I_{d+1} & & & 0 & \\ \hline & -x_1 & & & 0 \\ * & & \ddots & & \\ & * & & \ddots & \\ & & & & -x_1 \end{array}\right]=(-x_1)^{q_j}$$

$$\overset{\longleftrightarrow}{d+1}\qquad\overset{\longleftrightarrow}{q_j}$$

which yields

$$\sum_{\lambda=1}^{k(j)}s(\lambda,\ j)\nu(\lambda,\ j)=q_j<\infty\,.$$

Hence (8.20) follows.

Since $P_d(x',\ \xi_1)$ is independent of the variable x_1, the following division

$$P_d(x',\ \xi_1)=0\times S_{\lambda j}+P_d$$

of P_d by $S_{\lambda j}(x,\ \xi_1)$ with respect to the variable x_1 is a Weierstrass division. On the other hand, the condition $P_d\in(S_{\lambda j})$ yields

$$P_d(x',\ \xi_1)=\exists a(x,\ \xi_1)S_{\lambda j}+0$$

which is also a Weierstrass division of P_d by $S_{\lambda j}$ with respect to x_1. Then the uniqueness of Weierstrass divisions implies $a\equiv 0$ thus we have $P_d\equiv 0$. This contradicts the definition of P_d. Hence Claim 2) follows.

According to Claims 1) and 2), we have:

$$[X_\lambda-(\bigcup_{\lambda'\neq\lambda}X_{\lambda'}\cup P_d^{-1}(0))]\cap U\neq\emptyset \qquad \text{for any } \lambda,\ U\,. \tag{8.21}$$

On the other hand, the assertion 2) in Proposition 8.6 yields

$$[X_\lambda-P_d^{-1}(0)]\cap U\subset\pi_{j2}(V_{j1})=V_{j2} \qquad \text{for any } \lambda,\ U\,. \tag{8.22}$$

Indeed, for any fixed $(x^0,\ \xi_1^0)\in[X_\lambda-P_d^{-1}(0)]\cap U$, it follows that

$$0=R_j(x^0,\ \xi_1^0)=P_d(x'^0,\ \xi_1^0)^{q_j}\prod_{i,k}(\beta_k-\alpha_i) \tag{8.23}$$

where $\{\alpha_\iota\}$ [or $\{\beta_k\}$ resp.] denotes the roots of $Q_j(t, x'^0, \xi_1^0)$ $[tP(t, x'^0, \xi_1^0)-x_1^0]$ in the algebraic closed field $\boldsymbol{C}$. Since $P_d(x'^0, \xi_1^0)\neq 0$, (8.23) yields that there exists a common root $t^0:=\alpha_\iota=\beta_k$. Hence we have $(x^0, \xi_1^0)\in V_{j2}$.

Note that (8.21) and (8.22) yield

(8.24) $$\pi_{j2}^{-1}(X_\lambda)\neq\emptyset \quad\text{and}\quad \pi_{j2}^{-1}(X_\lambda)\not\subset\pi_{j2}^{-1}(X_{\lambda'}) \qquad \text{for any } \lambda,\ \lambda'\ (\lambda\neq\lambda').$$

On the other hand the inclusion (8.16) implies

(8.25) $$V_{j1}=\bigcup_{\lambda=1}^{k(j)}\pi_{j2}^{-1}(X_\lambda).$$

Thus we conclude that if $k(j)\geqq 2$ then the analytic set V_{j1} is reducible at $(0, 0, 0)\in\boldsymbol{C}\times M\times\boldsymbol{C}$. But we have already shown that V_{j1} is irreducible by Corollary 8.2 and Remark 6.8. Hence we get $k(j)=1$ as desired.

The proof of Proposition 8.9 is complete. Q.E.D.

By virtue of Proposition 8.9 and of (8.16), for any j $(1\leqq j\leqq r)$ and for any open sufficiently small neighborhood U of $(0, 0)$ in $M\times\boldsymbol{C}$, we have:

1) There exist an irreducible germ $S_j(x, \xi_1)$ and an integer $\nu(j)\geqq 1$ such that

(8.26) $$R_j(x, \xi_1)=S_j(x, \xi_1)^{\nu(j)} \qquad \text{on } U.$$

2) Let $P_d(x', \xi_1)$ be the leading coefficient in the polynomial $P(t, x', \xi_1)$. Then the following inclusions hold:

(8.27) $$\emptyset\neq[R_j^{-1}(0)-P_d^{-1}(0)]\cap U\subset V_{j2}\cap U\subset R_j^{-1}(0)\cap U.$$

The second aim of this section is to show

Theorem 8.10. *It follows that*

(8.28) $$(V_{j2}, (0, 0))=(R_j^{-1}(0), (0, 0))$$

as germs of analytic subsets of $M\times\boldsymbol{C}$ *at* $(0, 0)$.

Our proof of Theorem 8.10 is based on the local dimension theory of analytic sets which is summarized in § 13. We first remark a simple

Claim 8.11. *The map germ* $\pi_{j2}: (V_{j1}, (0, 0, 0))\to(V_{j2}, (0, 0))$ *is a finite holomorphic map germ.*

Proof. Since $(V_{j1}, (0, 0, 0))=\{(t, x, \xi_1);\ tP-x_1=Q_j=0\}$, the map germ π_{j2} is a restriction of a map germ $\hat{\pi_{j2}}$ defined by the diagram:

$$\begin{array}{ccc} (Q_j^{-1}(0), (0, 0, 0)) & \hookrightarrow & (\boldsymbol{C}\times M\times\boldsymbol{C}, (0, 0, 0)) \\ {\scriptstyle\hat{\pi_{j2}}}\big\downarrow & & \big\downarrow{\scriptstyle\text{projection}} \\ & \searrow & (M\times\boldsymbol{C}, (0, 0)) \end{array}$$

Since $Q_j(t, x', \xi_1)$ is a Weierstrass polynomial in t, the map germ $\hat{\pi_{j2}}$ is finite. Hence Criterion 3.6 implies Claim 8.11. Q.E.D.

Now we recall a way of regarding an analytic subset X of a domain D in $\boldsymbol{C}^N$ as a reduced complex space $(X, \mathcal{O}_X)$ in the sense of [Gr-Re]. According to the summary of this way in § 13, we set

$$\mathcal{O}_X := (\mathcal{O}_D / i(X))|_X \tag{8.29}$$

where $i(X)$ is the ideal sheaf of the analytic set X. Note that the Rückert's Nullstellensatz asserts that, if X is defined as a common zero set of $f_i \in \mathcal{O}_D$ $(1 \leqq i \leqq m)$ (we denote this by $X = \mathrm{Null}(f_1, \cdots, f_m)$) then

$$i(X) = i(\mathrm{Null}(f_1, \cdots, f_m)) = \mathrm{Rad}[(f_1 \cdots, f_m)] \quad (\text{see § 13}).$$

Hence any stalk $\mathcal{O}_{X,x}$ is a reduced ring, that is, $\mathcal{O}_{X,x}$ has no nilpotent element.

Lemma 8.12. *Let* $(V_{j1}, \mathcal{O}_{V_{j1}})$ *[or* $(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)})$ *resp.] be the reduced complex space which is obtained from* $V_{j1}[\mathrm{Null}(R_j)]$ *by the above way. Then these complex spaces are irreducible locally at* $(0, 0, 0)$ *or at* $(0, 0)$ *respectively, that is, the following* (8.30) *holds:*

$$\mathcal{O}_{V_{j1},(0,0,0)} \text{ and } \mathcal{O}_{\mathrm{Null}(R_j),(0,0)} \text{ are integral domains.} \tag{8.30}$$

Proof. By virtue of $V_{j1} = \mathrm{Null}(tP - x_1, Q_j)$ and of the irreducibility of the ideal $(tP - x_1, Q_j)$ at $(0, 0, 0)$, we have

$$\begin{aligned} i(V_{j1})_{(0,0,0)} &= i(\mathrm{Null}(tP - x_1, Q_j))_{(0,0,0)} \\ &= \mathrm{Rad}[(tP - x_1, Q_j)]_{(0,0,0)} \\ &= (tP - x_1, Q_j)_{(0,0,0)}. \end{aligned}$$

Hence we have

$$\mathcal{O}_{V_{j1},(0,0,0)} = \mathcal{O}_{\boldsymbol{C} \times M \times \boldsymbol{C},(0,0,0)} / (tP - x_1, Q_j)_{(0,0,0)}$$

which shows the first assertion of (8.30).

By virtue of Proposition 8.9, we similarly have

$$i(\mathrm{Null}(R_j))_{(0,0)} = \mathrm{Rad}[(R_j)]_{(0,0)} = (S_j)_{(0,0)}$$

where R_j and S_j are related as (8.26). Since $S_j(x, \xi_1)$ is irreducible at $(0, 0)$ we conclude that

$$\mathcal{O}_{\mathrm{Null}(R_j),(0,0)} = \mathcal{O}_{M \times \boldsymbol{C},(0,0)} / (S_j)_{(0,0)}$$

is also an integral domain. The proof of Lemma 8.12 is complete. Q.E.D.

Let X, Y be analytic sets and $f: X \to Y$ be a holomorphic map in the sense of Definition 3.1. If we regard X [or Y resp.] as a reduced complex space $(X, \mathcal{O}_X)$ $[(Y, \mathcal{O}_Y)]$ then the map f can be regarded as a morphism of complex spaces $(f, f^{\sim}):(X, \mathcal{O}_X)\to(Y, \mathcal{O}_Y)$ where $f^{\sim}:\mathcal{O}_Y\to f_*\mathcal{O}_X$ is a sheaf map on Y, which is defined in the canonical way mentioned in § 13 (Lemma 13.5).

Now we recall the definition of analytic subset Z of a complex space $(X, \mathcal{O}_X)$ and its local dimension (Definitions 13.9 and 13.10). We regard the map germ $\pi_{j2}: V_{j1}\to V_{j2}\subset \mathrm{Null}(R_j)=R_j^{-1}(0)$ as a germ of a finite morphism

$$\pi_{j2}=(\pi_{j2}, \pi_{j2}^{\sim}):(V_{j1}, \mathcal{O}_{V_{j1}})\longrightarrow(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)})$$

and apply the local dimension theory to the finite morphism π_{j2}:

We first note that the finite mapping theorem (Theorem 13.14) yields that the image $V_{j2}=\pi_{j2}(V_{j1})$ is an analytic set of the complex space $(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)})$. Then we have

$$\dim_{(0,0,0)}(V_{j1}, \mathcal{O}_{V_{j1}})\leqq\dim_{(0,0)}V_{j2}\leqq\dim_{(0,0)}(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)}) \tag{8.31}$$

by virtue of Proposition 13.11. On the other hand, Proposition 13.12 yields

$$\dim_{(0,0)}(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)})=\dim_{(0,0)}(M\times \boldsymbol{C})-1=n. \tag{8.32}$$

By the isomorphic map germs

$$(V_{j1}, (0, 0, 0))\xleftarrow[\pi_{j1}]{\sim}(\mathrm{graph}(\Psi_j), (0, e^0, e^0))\xleftarrow{\sim}(\boldsymbol{C}, 0)\times(\mathrm{Null}(F_j), e^0)$$

we also have

$$\dim_{(0,0,0)}(V_{j1}, \mathcal{O}_{V_{j1}})=\dim \boldsymbol{C}+\dim E-1=n. \tag{8.33}$$

Combining these (8.31)-(8.33), we get

$$\dim_{(0,0)}V_{j2}=\dim_{(0,0)}(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)}). \tag{8.34}$$

Since we have observed that the complex space $(\mathrm{Null}(R_j), \mathcal{O}_{\mathrm{Null}(R_j)})$ is irreducible at (0, 0) in Lemma 8.12, we conclude that there exists an open neighborhood U of (0, 0) in $M\times\boldsymbol{C}$ such that

$$V_{j2}\cap U=\mathrm{Null}(R_j)\cap U$$

as a consequence of Proposition 13.13.

The proof of Theorem 8.10 is complete. Q.E.D.

§ 9. Irreducibility of $R_j(x, \xi_1)$

In this section we complete the proof of Theorem 6.10. By virtue of Theorem 8.10, it suffices to show the following

Theorem 9.1. *Under the assumptions* [B.1]-[B.4], *for the resultants* $R_j(x, \xi_1)=r(Q_j(t, x', \xi_1), tP(t, x', \xi_1)-x_1)$, $1\leqq j\leqq r$, *it follows*:

1) $R_j(x, \xi_1)$ *is locally irreducible at* (0, 0).

2) $R_j(x, \xi_1)$ *has the finite order* $\nu_j:=\sum_{\mu\in M_j}\nu(\mu)$ *with respect to* ξ_1:

$$\text{ord}_0[R_j(0, \xi_1)]=v_j. \tag{9.1}$$

We derive Theorem 9.1 from the following

Proposition 9.2. *The Newton polygon* $\mathrm{N}(R_j(x_1, 0, \cdots, 0, \xi_1))$ *of the restriction* $R_j|_{x'=0}$ *is given by*

$$\mathrm{N}(R_j(x_1, 0, \cdots, 0, \xi_1))=\sum_{\mu\in M_j} \mathrm{N}_{q(\mu), v(\mu)} \tag{9.2}$$

where we denote by $\mathrm{N}_{\alpha,\beta}$ *the following Newton polygon (see Notation* 2.10)*:*

$$\mathrm{N}_{\alpha,\beta}:=\{(s, t);\ s\geqq 0,\ t\geqq 0,\ (s/\alpha)+(t/\beta)\geqq 1\}. \tag{9.3}$$

Lemma 9.3. *Proposition* 9.2 *implies Theorem* 9.1.

Proof. Since the assertion 2) in Theorem 9.1 is a direct consequence of (9.2), we only have to show the assertion 1) in Theorem 9.1.

By virtue of Proposition 8.9, we have known that $R_j(x, \xi_1)$ has only one irreducible component at (0, 0), that is, there exists an irreducible germ S_j at (0, 0) such that the following (8.26) holds:

$$R_j(x, \xi_1)=S_j(x, \xi_1)^{\nu(j)}. \tag{8.26}$$

Thus it suffices for Lemma 9.3 to show

$$\nu(j)=1 \qquad \text{for } 1\leqq j\leqq r. \tag{9.4}$$

We first observe the

Claim 9.4. *The finite sequence* $\{v(\mu)/q(\mu);\ 1\leqq\mu\leqq m:=\#\,\mathrm{Seg}\,\mathrm{N}(f)\}$ *is monotonely decreasing in* μ:

$$v(1)/q(1)>v(2)/q(2)>\cdots>v(m)/q(m). \tag{9.5}$$

Proof (of Claim 9.4). Recall the definition

$$v(\mu)=p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\} \tag{4.6}$$

in Definition 4.1, which yields

$$v(\mu)/q(\mu)=\kappa(\mu)\{a(\mu-1)+1\}+p-b(\mu-1)-1$$

where $\kappa(\mu):=p(\mu)/q(\mu)$. Then we have

$$\begin{aligned}
&v(\mu)/q(\mu)-v(\mu+1)/q(\mu+1)\\
&\quad=\kappa(\mu)\{a(\mu-1)+1\}-\kappa(\mu+1)\{a(\mu)+1\}+b(\mu)-b(\mu-1)\\
&\quad=\kappa(\mu)\{a(\mu)+1-q(\mu)\}-\kappa(\mu+1)\{a(\mu)+1\}+p(\mu)\\
&\quad=\{\kappa(\mu)-\kappa(\mu+1)\}\{a(\mu)+1\}-\kappa(\mu)q(\mu)+p(\mu)
\end{aligned}$$

Since $p(\mu)=\kappa(\mu)q(\mu)$ and $\kappa(\mu)-\kappa(\mu+1)>0$, we get

$$v(\mu)/q(\mu)-v(\mu+1)/q(\mu+1)>0$$

which shows Claim 9.4. Q.E.D.

For $1\leqq j\leqq r$, we denote the subset $M_j\subset\{1, 2, \cdots, m\}$ by

$$M_j=:\{\mu_j(k);\ 1\leqq k\leqq m_j:=\sharp M_j\}\quad\text{with}\quad 1\leqq\mu_j(1)<\cdots<\mu_j(m_j)\leqq m.$$

Then it follows that

$$\text{(9.6)}\qquad \begin{aligned}&\mathrm{Ver}\,N(R_j(x_1, 0, \cdots, 0, \xi_1))\\ &=\{(s, t);\ s=q(\mu_j(1))+\cdots+q(\mu_j(k)),\\ &\qquad t=v(\mu_j(k+1))+\cdots+v(\mu_j(m_j)),\ 0\leqq k\leqq m_j\}\end{aligned}$$

Indeed, (9.6) easily follows from (9.5) and Lemma 0.2, under the assumption that Proposition 9.2 is true.

Proof of Lemma 9.3 (continued). Setting $x'=0$ in (8.26) we have

$$R_j(x_1, 0, \cdots, 0, \xi_1)=S_j(x_1, 0, \cdots, 0, \xi_1)^{\nu(j)}.$$

Thus, by the additivity of Newton polygons, it follows

$$\text{(9.8)}\qquad N(R_j(x_1, 0, \cdots, 0, \xi_1))=\nu(j)N(S_j(x_1, 0, \cdots, 0, \xi_1)).$$

Note that the Newton polygon $N(S_j(x_1, 0, \cdots, 0, \xi_1))$ can be written as the form

$$N(S_j(x_1, 0, \cdots, 0, \xi_1))=\sum_{i=1}^{i(j)} N_{\alpha(i),\beta(i)}\quad (N_{\alpha,\beta}\text{ is defined by (9.3))}$$

for some positive integers $\alpha(i)$, $\beta(i)$ satisfying

$$\beta(1)/\alpha(1)>\cdots>\beta(i(j))/\alpha(i(j))>0.$$

Thus, from (9.8), we have

$$\text{(9.9)}\qquad N(R_j(x_1, 0, \cdots, 0, \xi_1))=\sum_{i=1}^{i(j)} N_{\nu(j)\alpha(i),\nu(j)\beta(i)}.$$

Comparing (9.9) with (9.2) we get

$$\sum_{\mu\in M_j} N_{q(\mu),v(\mu)}=\sum_{i=1}^{i(j)} N_{\nu(j)\alpha(i),\nu(j)\beta(i)}$$

which implies

$$\begin{cases} i(j)=m_j\ (:=\sharp M_j)\quad\text{and}\\ \nu(j)\alpha(i)=q(\mu_j(i)),\ \nu(j)\beta(i)=v(\mu_j(i))\qquad\text{for } 1\leqq i\leqq m_j\end{cases}$$

since $\{v(\mu)/q(\mu)\}$ and $\{\beta(i)/\alpha(i)\}$ are monotonely decreasing. Hence we have

(9.10) $\nu(j)$ is a common divisor of $\bigcup_{\mu\in M_j}\{q(\mu),\ v(\mu)\}$.

Note that the greatest common divisor $(q(\mu),\ v(\mu))$ is given by

$$(q(\mu),\ v(\mu))=(q(\mu),\ p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\})$$
$$=(q(\mu),\ p(\mu)\{a(\mu-1)+1\}).$$

Hence the coprimeness condition $(q(\mu),\ p(\mu))=1$ yields

$$(q(\mu),\ v(\mu))=(q(\mu),\ a(\mu-1)+1).$$

Thus the niceness of the subset M_j implies

$$\mathrm{GCD}\bigcup_{\mu\in M_j}\{q(\mu),\ v(\mu)\}=\mathrm{GCD}\bigcup_{\mu\in M_j}\{q(\mu),\ a(\mu-1)+1\}=1.$$

Then (9.10) yields (9.4) as desired. The proof of Lemma 9.3 is complete.

Q. E. D.

It remains to show Proposition 9.2. We note that it follows

$$N(Q_j(t,\ 0,\ \xi_1))=N(f_j(-tL_F(e^0),\ \xi_1))=N(f_j(y',\ \eta_1))=\sum_{\mu\in M_j}N_{q(\mu),\,p(\mu)} \tag{9.11}$$

by virtue of Proposition 7.2, Remark 7.3 and Proposition 2.12.

Lemma 9.5. *Under the following condition*

$$N(Q_j(t,\ 0,\ \xi_1))=\sum_{\mu\in M_j}N_{q(\mu),\,p(\mu)} \qquad \textit{for } 1\leqq j\leqq r \tag{9.11'}$$

there exist irreducible Weierstrass polynomials $Q_\mu^\sim(t,\ \xi_1)\in\mathcal{O}_{C,0}[t]$ $(1\leqq\mu\leqq m=\#\mathrm{Seg}\,N(f))$ *such that*

$$Q_j(t,\ 0,\ \xi_1)=\prod_{\mu\in M_j}Q_\mu^\sim(t,\ \xi_1) \qquad \textit{for } 1\leqq j\leqq r, \qquad \textit{and} \tag{9.12}$$

$$N(Q_\mu^\sim)=N_{q(\mu),\,p(\mu)}. \tag{9.13}$$

The proof of Lemma 9.5 will be given in § 14.

Let $g_1(t)$, $g_2(t)$ and $h(t)$ be polynomials with coefficients in an integral domain $\mathcal{O}$. Let $r(g,\ h)$ be the resultant of g and h defined by (8.9). Then the assertion 2) in Proposition 8.6 yields $r(g_1g_2,\ h)=r(g_1,\ h)r(g_2,\ h)$. Thus the assertion (9.12) in Lemma 9.5 implies

$$\begin{aligned} R_j(x_1,\ 0,\ \xi_1)&=r(Q_j(t,\ 0,\ \xi_1),\ tP(t,\ 0,\ \xi_1)-x_1)\\ &=r(\prod_{\mu\in M_j}Q_\mu^\sim(t,\ \xi_1),\ tP(t,\ 0,\ \xi_1)-x_1)\\ &=\prod_{\mu\in M_j}r(Q_\mu^\sim(t,\ \xi_1),\ tP(t,\ 0,\ \xi_1)-x_1). \end{aligned} \tag{9.14}$$

By virtue of (9.14) and of the additivity of Newton polygons, it suffices for Proposition 9.2 to show the following

Proposition 9.6. *For* $1 \leqq \mu \leqq m$, *we put*

$$r_\mu(x_1, \xi_1) := r(Q_\mu^\sim(t, \xi_1), tP(t, 0, \xi_1) - x_1).$$

Then it follows that

$$\mathrm{N}(r_\mu) = \mathrm{N}_{q(\mu), v(\mu)}. \tag{9.15}$$

The first step of the proof of Proposition 9.6 is to show the

Lemma 9.7. *There exists the following inclusion:*

$$\mathrm{N}(r_\mu) \subset \mathrm{N}_{q(\mu), v(\mu)} \qquad \textit{for } 1 \leqq \mu \leqq m.$$

Proof. We write the Weierstrass polynomial $Q_\mu^\sim$ as the form

$$Q_\mu^\sim(t, \xi_1) =: \sum_{\nu=0}^{q(\mu)} w_\nu(\xi_1) t^\nu \qquad (w_{q(\mu)}(\xi_1) \equiv 1 \text{ and } w_\nu(0) = 0 \text{ for } 0 \leqq \nu < q(\mu)).$$

We also write

$$\begin{cases} P(t, x', \xi_1) =: \sum_{\nu=0}^{q-1} P_\nu(x', \xi_1) t^\nu & \text{and} \\ tP(t, 0, \xi_1) - x_1 =: \sum_{\nu=0}^{q-1} s_\nu(\xi_1) t^{\nu+1} - x_1 \end{cases}$$

where we set

$$s_\nu(\xi_1) := P_\nu(0, \xi_1) \qquad \text{for } 0 \leqq \nu \leqq q-1.$$

Remark that it is not necessarily that $s_{q-1} \not\equiv 0$. But if we define a $(q+q(\mu))$-square matrix $D_\mu(x_1, \xi_1)$ by

$$D_\mu(x_1, \xi_1) := \left[\begin{array}{cccc|ccc} 1 & w_{q(\mu)-1} & \cdots & w_0 & & 0 & \\ & 1 & \ddots & & w_0 & & \\ & 0 & & \ddots & & \ddots & \\ & & & 1 & \cdots & & w_0 \\ \hline s_{q-1} & \cdots & \cdots & s_0 & -x_1 & & 0 \\ & \ddots & & & & \ddots & \\ & 0 & s_{q-1} & \cdots & \cdots & & -x_1 \end{array}\right] \begin{array}{l} \updownarrow q \\ \\ \updownarrow q(\mu) \end{array} \tag{9.16}$$

then we always have

$$r_\mu(x_1, \xi_1) = \det[D_\mu(x_1, \xi_1)] \tag{9.17}$$

since $Q_\mu^\sim(t, \xi_1)$ is a Weierstrass, hence a monic, polynomial.

We fix μ and denote the (i, k)-component of $D_\mu(x_1, \xi_1)$ by $d_{ik}(x_1, \xi_1)$. Then $d_{i,k}$ is given by the following (9.18) [or (9.19) resp.] for $1 \leqq i \leqq q$ [for $q+1 \leqq i \leqq$

$q+q(\mu)]$:

$$d_{i,k}=\begin{cases} w_{q(\mu)+i-k}(\xi_1) & \text{for } i\leqq k\leqq i+q(\mu) \\ 0 & \text{for } k<i \text{ or } k>i+q(\mu) \end{cases} \tag{9.18}$$

$$d_{i,k}=\begin{cases} -x_1 & \text{for } i=k \\ s_{i-k-1}(\xi_1) & \text{for } i-q\leqq k\leqq i-1 \\ 0 & \text{for } k<i-q \text{ or } k>i \end{cases} \tag{9.19}$$

By the expressions $Q_{\mu}^{\sim}(t, \xi_1)=\sum_{\nu=0}^{q(\mu)} w_{\nu}(\xi_1)t^{\nu}$ and $P(t, 0, \xi_1)=\sum_{\nu=0}^{q-1} s_{\nu}(\xi_1)t^{\nu}$, we have

$$(\nu, \operatorname{ord}[w_{\nu}])\in \mathrm{N}(Q_{\mu}^{\sim}) \quad \text{and} \quad (\nu, \operatorname{ord}[s_{\nu}])\in \mathrm{N}(P(t, 0, \xi_1)).$$

Thus (9.13) in Lemma 9.5 and Lemma 8.3 yield the following inequalities:

$$\begin{cases} \operatorname{ord}[w_{\nu}]\geqq -\kappa(\mu)\nu+p(\mu). \\ \operatorname{ord}[s_{\nu}]\geqq -\kappa(\mu)\{\nu-a(\mu)\}+p-b(\mu)-1. \end{cases} \tag{9.20}$$

Now we estimate the Newton polygon $\mathrm{N}(r_{\mu})$. Since $r_{\mu}(x_1, \xi_1)$ can be written as

$$r_{\mu}(x_1, \xi_1)=\sum_{\pi\in\mathfrak{S}[q+q(\mu)]} \operatorname{sgn}(\pi)\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}(x_1, \xi_1)$$

where $\mathfrak{S}[n]$ denotes the symmetric group of order n, it suffices to estimate

$$\mathrm{N}\Big(\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}(x_1, \xi_1)\Big) \qquad \text{for } \pi\in\mathfrak{S}[q+q(\mu)].$$

Note that we may assume that $\pi\in\mathfrak{S}[q+q(\mu)]$ satisfies

$$\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}\not\equiv 0. \tag{9.21}$$

Under the conditions (9.18)-(9.21), we have

$$\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}=\Big(\prod_{i=1}^{q} w_{q(\mu)+i-\pi(i)}\Big)(-x_1)^{\alpha(\pi)}\prod_{\substack{i\geqq q+1\\ \pi(i)\neq i}} s_{i-\pi(i)-1}\in(x_1)^{\alpha(\pi)}(\xi_1)^{\beta(\pi)}$$

such that

$$\alpha(\pi)=\#\{i\,;\, i\geqq q+1,\ \pi(i)=i\} \tag{9.22}$$

and that $\beta(\pi)$ is estimated from below as follows:

$$\begin{aligned}\beta(\pi)\geqq &\sum_{i=1}^{q}[-\kappa(\mu)\{q(\mu)+i-\pi(i)\}+p(\mu)] \\ &+\sum_{\substack{i\geqq q+1\\ \pi(i)\neq i}}[-\kappa(\mu)\{i-\pi(i)-1-a(\mu)\}+p-b(\mu)-1].\end{aligned} \tag{9.23}$$

Note that (9.23) is equivalent to

$$\beta(\pi)\geqq\{v(\mu)/q(\mu)\}\{q(\mu)-\alpha(\pi)\}. \tag{9.24}$$

Indeed, the right hand side of (9.23) is equal to

$$-\kappa(\mu)\sum_{i=1}^{q+q(\mu)}\{i-\pi(i)\}+\sum_{i=1}^{q}\{-\kappa(\mu)q(\mu)+p(\mu)\}$$
$$+\sum_{\substack{i\geqq q+1\\ \pi(i)\neq i}}[\kappa(\mu)\{a(\mu)+1\}+p-b(\mu)-1].$$

Since the first and second terms vanish, (9.23) is equivalent to

$$(9.23)' \qquad \beta(\pi)\geqq[\kappa(\mu)\{a(\mu)+1\}+p-b(\mu)-1]\times\#\{i\,;\, i\geqq q+1,\ \pi(i)\neq i\}$$
$$=[\kappa(\mu)\{a(\mu)+1\}+p-b(\mu)-1]\{q(\mu)-\alpha(\pi)\}.$$

Note that the identity $\kappa(\mu)q(\mu)=p(\mu)$ implies

$$\kappa(\mu)a(\mu)-b(\mu)=\kappa(\mu)a(\mu-1)-b(\mu-1).$$

Hence (9.23)$'$ is equivalent to

$$\beta(\kappa)\geqq[\kappa(\mu)\{a(\mu-1)+1\}+p-b(\mu-1)-1]\{q(\mu)-\alpha(\pi)\}$$
$$=q(\mu)^{-1}[p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}]\{q(\mu)-\alpha(\pi)\}$$
$$=\{v(\mu)/q(\mu)\}\{q(\mu)-\alpha(\pi)\}$$

which shows that (9.23) is equivalent to (9.24).

Note that (9.24) is also equivalent to

$$(9.25) \qquad \{\alpha(\pi)/q(\mu)\}+\{\beta(\pi)/v(\mu)\}\geqq 1$$

which shows

$$\mathrm{N}\Big(\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}\Big)\subset\mathrm{N}(x_1^{\alpha(\pi)}\xi_1^{\beta(\pi)})\subset\mathrm{N}_{q(\mu),v(\mu)} \qquad \text{for } \pi\in\mathfrak{S}[q+q(\mu)].$$

Hence the proof of Lemma 9.7 is complete. Q. E. D.

In order to complete the proof of Proposition 9.6, we must show the converse inclusion of Lemma 9.7. For this purpose it suffices to show

$$(9.26) \qquad (q(\mu),\ 0)\in\mathrm{N}(r_\mu) \qquad \text{and}$$

$$(9.27) \qquad (0,\ v(\mu))\in\mathrm{N}(r_\mu).$$

Since $Q_\mu^\sim(t,\ \xi_1)$ is a Weierstrass polynomial in t, it follows

$$r_\mu(x_1,\ 0)=\det\left[\begin{array}{c|ccc} I_q & & 0 & \\ \hline & -x_1 & & 0 \\ * & & \ddots & \\ & * & & -x_1 \end{array}\right]=(-x_1)^{q(\mu)}$$
$$\underbrace{\qquad}_{q}\quad\underbrace{\qquad\qquad}_{q(\mu)}$$

which shows (9.26).

To show (9.27) we must consider the case that the equality holds in the inequality (9.23) with $\alpha(\pi)=\#\{i\,;\,i\geqq q+1,\ \pi(i)=i\}=0$. Thus we consider

$$\operatorname{ord}\Big[\sum_{\pi}{}' \operatorname{sgn}(\pi)\prod_{i=1}^{q+q(\mu)} d_{i,\pi(i)}(x_1,\xi_1)\Big] \tag{9.28}$$

where $\sum'$ denotes the sum of $\pi\in\mathfrak{S}[q+q(\mu)]$ satisfying

$$\begin{cases}\operatorname{ord}[w_{q(\mu)+i-\pi(i)}]=-\kappa(\mu)\{q(\mu)+i-\pi(i)\}+p(\mu)\\ i\leqq\pi(i)\leqq i+q(\mu) \qquad \text{for all } 1\leqq i\leqq q\,.\end{cases} \tag{9.29}$$

$$\begin{cases}\operatorname{ord}[s_{i-\pi(i)-1}]=-\kappa(\mu)\{i-\pi(i)-1-a(\mu)\}+p-b(\mu)-1\\ i-q\leqq\pi(i)\leqq i \qquad \text{for all } q+1\leqq i\leqq q+q(\mu)\,.\end{cases} \tag{9.30}$$

Note that the coprimeness condition of $\mathrm{N}(f)$ yields that (9.29), (9.30) are equivalent respectively to

$$q(\mu)+i-\pi(i)=0 \quad\text{or}\quad =q(\mu) \qquad (1\leqq i\leqq q)\,. \tag{9.29$'$}$$

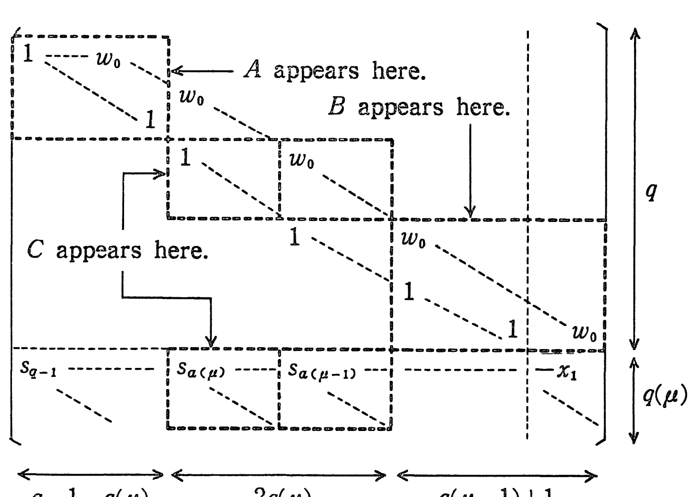
$$i-\pi(i)-1=\begin{cases} a(\mu-1) \text{ or } a(\mu) & \text{if } 1\leqq\mu<m\\ a(m-1) & \text{if } \mu=m\end{cases} \qquad (q+1\leqq i\leqq q+q(\mu)). \tag{9.30$'$}$$

Case 1. When $\mu<m$. In this case the sum (9.28) contributes to $r_\mu=\det D_\mu$ as the following form:

$$(9.28)=\operatorname{ord}[\varepsilon(\det A)(\det B)(\det C)] \qquad \text{where } \varepsilon=\pm1 \tag{9.31}$$

such that A, B and C are determined by the following (9.32)-(9.35):

(9.32) $D_\mu=$



(9.33) $$A=I_{q-1-a(\mu)}+(\text{nilpotent})\in \mathrm{Mat}(q-1-a(\mu)\,;\,\mathcal{O}_{C,0})$$

(9.34) $$B=w_0 I_{a(\mu-1)+1}+(\text{nilpotent})\in \mathrm{Mat}(a(\mu-1)+1\,;\,\mathcal{O}_{C,0})$$

(9.35) $$C=\begin{bmatrix} I_{q(\mu)} & w_0 I_{q(\mu)} \\ s_{a(\mu)} I_{q(\mu)} & s_{a(\mu-1)} I_{q(\mu)} \end{bmatrix}\in \mathrm{Mat}(2q(\mu)\,;\,\mathcal{O}_{C,0}).$$

From (9.31)-(9.35) we have

$$\mathrm{ord}[(\det A)(\det B)(\det C)]$$
$$=\{a(\mu-1)+1\}\mathrm{ord}[w_0]+\mathrm{ord}[\det\{(s_{a(\mu-1)}-s_{a(\mu)}w_0)I_{q(\mu)}\}]\,.$$

Since $\mathrm{ord}[w_0]=p(\mu)$, the right hand side can be written as

$$\{a(\mu-1)+1\}p(\mu)+q(\mu)\mathrm{ord}[s_{a(\mu-1)}-s_{a(\mu)}w_0]\,.$$

Hence, by $v(\mu):=p(\mu)\{a(\mu-1)+1\}+q(\mu)\{p-b(\mu-1)-1\}$, it suffices for (9,27) to show

(9.36) $$\mathrm{ord}[s_{a(\mu-1)}-s_{a(\mu)}w_0]=p-b(\mu-1)-1\,.$$

Recall that Lemma 8.3 asserts

(8.5) $$\mathrm{N}\Big[P(t,\,0,\,\xi_1)-t^{-1}\int_0^t \partial_{\xi_1}\mathrm{ch}(f)(-\theta L_F(e^0),\,\xi_1)d\theta\Big]+\mathrm{N}(\xi_1)\in \mathrm{N}(f)$$

which implies

(9.37) $$s_{a(\mu)}(\xi_1)=\frac{p-b(\mu)}{a(\mu)+1}\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})\xi_1^{p-b(\mu)-1}\{1+O(\xi_1)\}\qquad \text{for } \mu<m$$

where $c_\nu(y')$ denotes the ν-*th* Taylor coefficient of $f(y',\,\eta_1)$.

To derive (9.36) we need to write the value $A_\mu\in \boldsymbol{C}-\{0\}$ defined by

(9.38) $$w_0(\xi_1)=Q_{\mu}^{\sim}(0,\,\xi_1)=A_\mu \xi_1^{p(\mu)}(1+O(\xi_1))\,,$$

by means of the terms $\{\mathrm{Loc}[c_{p-b(\lambda)}](\partial_{y_n})\,;\;0\leqq\lambda\leqq m\}$.

We shall prove the following

Lemma 9.8. *It follows that*

(9.39) $$A_\mu=\mathrm{Loc}[c_{p-b(\mu-1)}](\partial_{y_n})/\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})\qquad \textit{for } \mu\geqq 1\,.$$

In order to show Lemma 9.8, we use the following properties of characteristic polynomial functions:

Lemma 9.9. *Let* $(\Sigma,\,\sigma)$ *be a germ of* $(n-1)$*-dimensional complex manifold at a point* σ *with* $n\geqq 2$. *Let* $f,\,g$ *be holomorphic germs on* $\Sigma\times\boldsymbol{C}$ *at* $(\sigma,\,0)$, *and let* $\mathrm{ch}(f)$, $\mathrm{ch}(g)$ *be the characteristic polynomial functions of* $f,\,g$ *which are defined by Definition* 7.1. *Then we have*

1) $\mathrm{ch}(\mathrm{ch}(f))=\mathrm{ch}(f)$.

2) $\mathrm{ch}(fg)=\mathrm{ch}(f\,\mathrm{ch}(g))=\mathrm{ch}(g\,\mathrm{ch}(f))$.

Proof. Recall the following equivalence

$$\mathrm{ch}(g)=\mathrm{ch}(h) \Longleftrightarrow \mathrm{N}(g-h)\Subset\mathrm{N}(g)=\mathrm{N}(h).$$

Then the assertion 1) immediately follows from

$$\mathrm{N}(f-\mathrm{ch}(f))\Subset\mathrm{N}(f)=\mathrm{N}(\mathrm{ch}(f)). \tag{9.40}$$

Note that (9.40) and the additivity property of Newton polygons yield

$$\begin{aligned}\mathrm{N}(fg-f\,\mathrm{ch}(g))&=\mathrm{N}(f)+\mathrm{N}(g-\mathrm{ch}(g))\\ &\Subset\mathrm{N}(f)+\mathrm{N}(g)=\mathrm{N}(f)+\mathrm{N}(\mathrm{ch}(g)).\end{aligned}$$

Hence we have

$$\mathrm{N}(fg-f\,\mathrm{ch}(g))\Subset\mathrm{N}(fg)=\mathrm{N}(f\,\mathrm{ch}(g)).$$

Thus the assertion 2) also holds. Q.E.D.

Corollary 9.10. *For holomorphic germs* $g_1, \cdots, g_k$ *on* $\Sigma\times C$ *at* $(\sigma, 0)$, *it follows that*

$$\mathrm{ch}\Big(\prod_{j=1}^{k} g_j\Big)=\mathrm{ch}\Big(\prod_{j=1}^{k}\mathrm{ch}(g_j)\Big). \tag{9.41}$$

Proof. When $k=1$, (9.41) is nothing but the assertion 1) in Lemma 9.9. Thus we may assume $k\geqq 2$. Then the assertion 2) in Lemma 9.9 implies

$$\mathrm{ch}\Big(\prod_{j=1}^{k} g_j\Big)=\mathrm{ch}\Big[\Big(\prod_{j=1}^{k-1} g_j\Big)\mathrm{ch}(g_k)\Big].$$

Regarding $\Big(\prod_{j=1}^{k-1} g_j\Big)\mathrm{ch}(g_k)=g_{k-1}\times\Big(\prod_{j=1}^{k-2} g_j\Big)\mathrm{ch}(g_k)$, and applying 2) in Lemma 9.9, we have

$$\mathrm{ch}\Big(\prod_{j=1}^{k} g_j\Big)=\mathrm{ch}\Big[\Big(\prod_{j=1}^{k-2} g_j\Big)\mathrm{ch}(g_{k-1})\mathrm{ch}(g_k)\Big].$$

Repeating such processes, we get (9.41) as desired. Q.E.D.

Proof of Lemma 9.8. Applying Corollary 9.10 to

$$f(0, \cdots, 0, t, \xi_1)=\prod_{j=1}^{r} f_j(0, \cdots, 0, t, \xi_1),$$

we have

$$\mathrm{ch}[f(0, \cdots, 0, t, \xi_1)]=\mathrm{ch}\Big[\prod_{j=1}^{r}\mathrm{ch}\{f_j(0, \cdots, 0, t, \xi_1)\}\Big]. \tag{9.42}$$

Note that Remark 7.3 implies

$$(9.43)\qquad \begin{cases} \mathrm{ch}[f(0, \cdots, 0, t, \xi_1)]=\mathrm{ch}(f)(0, \cdots, 0, t, \xi_1) \\ \mathrm{ch}[f_j(0, \cdots, 0, t, \xi_1)]=\mathrm{ch}(f_j)(0, \cdots, 0, t, \xi_1) \qquad \text{for } 1\leqq j\leqq r. \end{cases}$$

Recall that Proposition 7.2 asserts

$$(7.4)\qquad \mathrm{ch}(f_j)(0, \cdots, 0, t, \xi_1)=\mathrm{ch}(B_j(t, 0, \xi_1)).$$

Since $B_j=Q_j\varepsilon_j$ with $\varepsilon_j(0)\neq 0$, Corollary 9.10 yields

$$(9.44)\qquad \begin{aligned} \mathrm{ch}(B_j(t, 0, \xi_1))&=\mathrm{ch}[\mathrm{ch}(\varepsilon_j(t, 0, \xi_1))\mathrm{ch}(Q_j(t, 0, \xi_1))] \\ &=\mathrm{ch}[\varepsilon_j(0)\mathrm{ch}(Q_j(t, 0, \xi_1))]. \end{aligned}$$

Since, for a non-zero constant $a\in \boldsymbol{C}$, it is easily verified that

$$\mathrm{ch}(ag)=a\ \mathrm{ch}(g)$$

by the definition (7.3) of characteristic polynomial functions, (9.44) derives

$$(9.45)\qquad \mathrm{ch}(B_j(t, 0, \xi_1))=\varepsilon_j(0)\,\mathrm{ch}(Q_j(t, 0, \xi_1)).$$

By virtue of (9.43), (9.45) and (7.4), we can write (9.42) as

$$(9.46)\qquad \begin{aligned} \mathrm{ch}(f)(0, \cdots, 0, t, \xi_1)&=\mathrm{ch}\Big[\prod_{j=1}^{r}\mathrm{ch}(B_j(t, 0, \xi_1))\Big] \\ &=\mathrm{ch}\Big[\prod_{j=1}^{r}\{\varepsilon_j(0)\,\mathrm{ch}(Q_j(t, 0, \xi_1))\}\Big] \\ &=\varepsilon\ \mathrm{ch}\Big[\prod_{j=1}^{r}\mathrm{ch}(Q_j(t, 0, \xi_1))\Big] \end{aligned}$$

where we put $\varepsilon:=\prod_{j=1}^{r}\varepsilon_j(0)\in \boldsymbol{C}-\{0\}$.

Now, using the irreducible decomposition

$$(9.12)\qquad Q_j(t, 0, \xi_1)=\prod_{\mu\in \mathrm{M}_j}Q_\mu^{\sim}(t, \xi_1)$$

in Lemma 9.5, we claim

$$(9.47)\qquad \mathrm{ch}(f)(0, \cdots, 0, t, \xi_1)=\varepsilon\ \mathrm{ch}\Big[\prod_{\mu=1}^{m}\mathrm{ch}(Q_\mu^{\sim}(t, \xi_1))\Big].$$

Indeed, by (9.46), (9.12) and Corollary 9.10, we have

$$\begin{aligned} \mathrm{ch}(f)(0, \cdots, 0, t, \xi_1)&=\varepsilon\ \mathrm{ch}\Big[\prod_{j=1}^{r}Q_j(t, 0, \xi_1)\Big]=\varepsilon\ \mathrm{ch}\Big[\prod_{j=1}^{r}\prod_{\mu\in \mathrm{M}_j}Q_\mu^{\sim}(t, \xi_1)\Big] \\ &=\varepsilon\ \mathrm{ch}\Big[\prod_{\mu=1}^{m}Q_\mu^{\sim}(t, \xi_1)\Big]=\varepsilon\ \mathrm{ch}\Big[\prod_{\mu=1}^{m}\mathrm{ch}(Q_\mu^{\sim}(t, \xi_1))\Big]. \end{aligned}$$

Hence (9.47) follows.

Since $\mathrm{N}(Q_\mu^\sim)=\mathrm{N}_{q(\mu),p(\mu)}$, the coprimeness condition yields

$$\mathrm{ch}(Q_\mu^\sim)=t^{q(\mu)}+A_\mu\xi_1^{p(\mu)} \tag{9.48}$$

where A_μ is the non-zero constant defined by (9.38).

We note that Lemma 0.2 implies

$$\begin{aligned}\mathrm{ch}\left[\prod_{\mu=1}^{m}\mathrm{ch}(Q_\mu^\sim(t,\xi_1)\right]&=\mathrm{ch}\left[\prod_{\mu=1}^{r}(t^{q(\mu)}+A_\mu\xi_1^{p(\mu)})\right]\\&=\sum_{\mu=0}^{m}\left(\prod_{\lambda=1}^{\mu}t^{q(\lambda)}\right)\left(\prod_{\lambda=\mu+1}^{m}A_\lambda\xi_1^{p(\lambda)}\right).\end{aligned} \tag{9.49}$$

On the other hand, we have the following expression:

$$\mathrm{ch}(f)(0,\cdots,0,t,\xi_1)=\sum_{\mu=0}^{m}\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})t^{a(\mu)}\xi_1^{p-b(\mu)} \tag{9.50}$$

where $c_\nu(y')$ is the ν-*th* Taylor coefficient of $f(y',\eta_1)$.

By virtue of (9.49) and (9.50), we conclude that (9.47) is equivalent to

$$\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})t^{a(\mu)}\xi_1^{p-b(\mu)}=\varepsilon\left(\prod_{\lambda=1}^{\mu}t^{q(\lambda)}\right)\left(\prod_{\lambda=\mu+1}^{m}A_\lambda\xi_1^{p(\lambda)}\right),$$

that is,

$$\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})=\varepsilon A_{\mu+1}A_{\mu+2}\cdots A_m \qquad \text{for } 0\leqq\mu\leqq m\,.$$

Hence we get the formula (9.39) as desired.

The proof of Lemma 9.8 is complete. Q.E.D.

Proof of Proposition 9.6 (continued). By virtue of the expression (9.37) of $s_{a(\mu)}(\xi_1)$ and of Lemma 9.8, we calculate the left hand side of (9.36) as follows:

$$\begin{aligned}&s_{a(\mu-1)}-s_{a(\mu)}w_0\\&=\frac{p-b(\mu-1)}{a(\mu-1)+1}\mathrm{Loc}[c_{p-b(\mu-1)}](\partial_{y_n})\xi_1^{p-b(\mu-1)-1}\{1+O(\xi_1)\}\\&\quad-\frac{p-b(\mu)}{a(\mu)+1}\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})\xi_1^{p-b(\mu)-1}A_\mu\xi_1^{p(\mu)}\{1+O(\xi_1)\}\\&=\xi_1^{p-b(\mu-1)-1}\{1+O(\xi_1)\}\left\{\frac{p-b(\mu-1)}{a(\mu-1)+1}\mathrm{Loc}[c_{p-b(\mu-1)}](\partial_{y_n})\right.\\&\quad\left.-\frac{p-b(\mu)}{a(\mu)+1}\mathrm{Loc}[c_{p-b(\mu)}](\partial_{y_n})(\mathrm{Loc}[c_{p-b(\mu-1)}]/\mathrm{Loc}[c_{p-b(\mu)}])(\partial_{y_n})\right\}\\&=\xi_1^{p-b(\mu-1)-1}\{1+O(\xi_1)\}\left\{\frac{p-b(\mu-1)}{a(\mu-1)+1}-\frac{p-b(\mu)}{a(\mu)+1}\right\}\mathrm{Loc}[c_{p-b(\mu-1)}](\partial_{y_n}).\end{aligned}$$

Thus it suffices for (9.36) to verify

$$(p-b(\mu-1))(a(\mu)+1)-(p-b(\mu))(a(\mu-1)+1)\neq0\,.$$

But this is trivial since

$$p-b(\mu-1)>p-b(\mu) \quad \text{and} \quad a(\mu)+1>a(\mu-1)+1\,.$$

Hence we get (9.36) which implies (9.27) as desired in the case 1.

Case 2. When $\mu=m$. In this case the sum (9.28) contributes to r_μ as the form

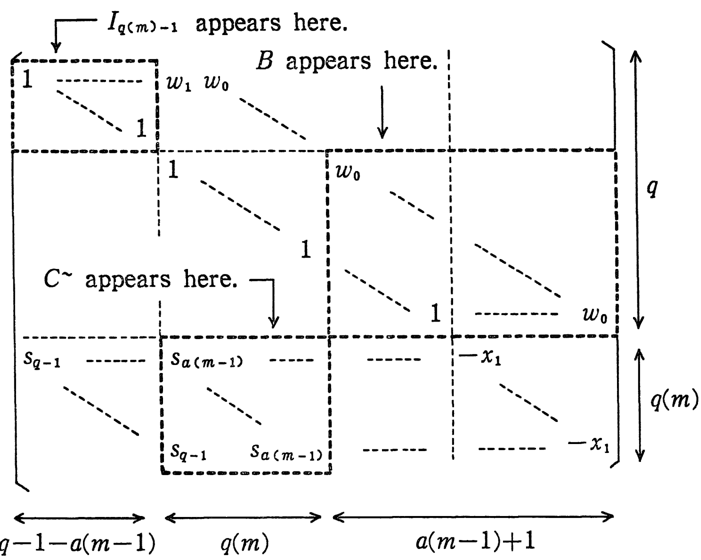
$$(9.51)\qquad (9.28)=\operatorname{ord}[\varepsilon(\det I_{q(m)-1})(\det B)(\det C^{\sim})] \qquad (\varepsilon=\pm 1)$$

such that B and $C^{\sim}$ are determined by the following (9.52)-(9.54):

$$(9.52)\qquad B=w_0 I_{a(m-1)+1}+(\text{nilpotent})$$

$$(9.53)\qquad C^{\sim}=s_{a(m-1)}I_{q(m)}$$

(9.54) $D_m=$



From (9.52) and (9.53), we have

$$\begin{aligned}
&\operatorname{ord}[(\det I_{q(m)-1})(\det B)(\det C^{\sim})]\\
&\quad=\{a(m-1)+1\}\operatorname{ord}[w_0]+q(m)\operatorname{ord}[s_{a(m-1)}]\\
&\quad=\{a(m-1)+1\}p(m)+q(m)\{p-b(m-1)-1\}\\
&\quad=v(m)\,.
\end{aligned}$$

Thus (9.51) implies (9.27) as desired also in the case 2.

The proof of Proposition 9.6, hence that of Proposition 9.2, is complete.

Q. E. D.

§ 10. One-sheetedness of Map Germs π_{j2}

In this section we prove Theorem 6.11 which asserts that the finite map germs $\pi_{j2}: V_{j1} \to V_{j2}$ are germs of one-sheeted analytic coverings of V_{j2} for $1 \leqq j \leqq r$. Our proof starts from the following Euclidean algorithm:

Definition 10.1. Put $\mathcal{O} := \mathcal{O}_{M \times C, (0,0)}$ and let $\mathcal{K}$ be the quotient field of $\mathcal{O}$, that is, the field which consists of germs of meromorphic functions at $(x, \xi_1) = (0, 0)$. We fix j and we define a finite sequence $\{s_\nu^\sim(t); 1 \leqq \nu \leqq k\}$ of polynomials in $\mathcal{K}[t]$ as follows: For $\nu = 1, 2$, we put

$$(10.1) \quad \begin{cases} s_1^\sim(t) := tP(t, x', \xi_1) - x_1, \quad s_2^\sim(t) := Q_j(t, x', \xi_1), & \text{if } 1 + \deg_t P \geqq q_j = \deg_t Q_j. \\ s_1^\sim(t) := Q_j(t, x', \xi_1), \quad s_2^\sim(t) := tP(t, x', \xi_1) - x_1, & \text{if } 1 + \deg_t P < q_j. \end{cases}$$

For $\nu \geqq 3$, we inductively define $s_\nu^\sim(t), \sigma_\nu(t) \in \mathcal{K}[t]$ by the following division in $\mathcal{K}[t]$ (the Euclidean algorithm):

$$(10.2) \quad s_{\nu-2}^\sim = \sigma_\nu s_{\nu-1}^\sim + s_\nu^\sim \quad \text{such that } \deg(s_\nu^\sim) < \deg(s_{\nu-1}^\sim).$$

Since $\mathcal{K}[t]$ is a Euclidean ring, the division (10.2) determines $\{(s_\nu^\sim, \sigma_\nu)\}$ for $3 \leqq \nu \leqq k$ where k is the integer satisfying

$$(10.3) \quad \deg(s_k^\sim) \leqq 0 < \deg(s_{k-1}^\sim) < \cdots < \deg(s_2^\sim) \leqq \deg(s_1^\sim).$$

Since $\deg(s_2^\sim) = \min\{q_j, 1 + \deg_t P\} \geqq 1$, it follows that $k \geqq 3$.

Lemma 10.2. *For $3 \leqq \nu \leqq k$, there exist $f_\nu^\sim, g_\nu^\sim \in \mathcal{K}[t]$ such that*

$$(10.4) \quad s_\nu^\sim = f_\nu^\sim s_2^\sim + g_\nu^\sim s_1^\sim \quad \textit{and}$$

$$(10.5) \quad \begin{cases} \deg(f_\nu^\sim) = \deg(s_1^\sim) - \deg(s_{\nu-1}^\sim) \\ \deg(g_\nu^\sim) = \deg(s_2^\sim) - \deg(s_{\nu-1}^\sim). \end{cases}$$

Proof. By induction on ν: For $\nu = 3$, since

$$s_3^\sim = -\sigma_3 s_2^\sim + s_1^\sim$$

we can take $f_3^\sim = -\sigma_3$, $g_3^\sim = 1$. Indeed, $\deg(g_3^\sim) = \deg(1) = 0 = \deg(s_2^\sim) - \deg(s_2^\sim)$ is trivial, and the inequality

$$\deg(s_3^\sim) < \deg(s_2^\sim) \leqq \deg(s_1^\sim)$$

implies

$$\deg(f_3^\sim) = \deg(\sigma_3) = \deg(s_1^\sim) - \deg(s_2^\sim).$$

Next, for $\nu = 4$, since

$$s_4^\sim = s_2^\sim - \sigma_4 s_3^\sim = s_2^\sim - \sigma_4(-\sigma_3 s_2^\sim + s_1^\sim) = (\sigma_3\sigma_4 + 1)s_2^\sim - \sigma_4 s_1^\sim$$

We can take $f_4^{\sim}=\sigma_3\sigma_4+1$, $g_4^{\sim}=-\sigma_4$. Indeed, we have

$$\deg(f_4^{\sim})=\deg(\sigma_3\sigma_4+1)=\deg(s_1^{\sim})-\deg(s_3^{\sim}) \quad \text{and}$$
$$\deg(g_4^{\sim})=\deg(-\sigma_4)=\deg(s_2^{\sim})-\deg(s_3^{\sim}).$$

Now let $\nu\geqq 5$. By the inductive assumption it follows that

$$\begin{aligned} s_\nu^{\sim}&=s_{\nu-2}^{\sim}-\sigma_\nu s_{\nu-1}^{\sim}\\ &=f_{\nu-2}^{\sim}s_2^{\sim}+g_{\nu-2}^{\sim}s_1^{\sim}-\sigma_\nu(f_{\nu-1}^{\sim}s_2^{\sim}+g_{\nu-1}^{\sim}s_1^{\sim})\\ &=(f_{\nu-2}^{\sim}-\sigma_\nu f_{\nu-1}^{\sim})s_2^{\sim}+(g_{\nu-2}^{\sim}-\sigma_\nu g_{\nu-1}^{\sim})s_1^{\sim}. \end{aligned}$$

Thus it suffices for (10.4) to put

$$f_\nu^{\sim}:=f_{\nu-2}^{\sim}-\sigma_\nu f_{\nu-1}^{\sim}, \quad g_\nu^{\sim}:=g_{\nu-2}^{\sim}-\sigma_\nu g_{\nu-1}^{\sim}.$$

Then the inductive assumption yields

$$\deg(f_{\nu-2}^{\sim})=\deg(s_1^{\sim})-\deg(s_{\nu-3}^{\sim}) \quad \text{and}$$
$$\begin{aligned} \deg(\sigma_\nu f_{\nu-1}^{\sim})&=\deg(\sigma_\nu)+\deg(f_{\nu-1}^{\sim})\\ &=\deg(s_{\nu-2}^{\sim})-\deg(s_{\nu-1}^{\sim})+\deg(s_1^{\sim})-\deg(s_{\nu-2}^{\sim})\\ &=\deg(s_1^{\sim})-\deg(s_{\nu-1}^{\sim}). \end{aligned}$$

Since the inequalities (10.3) and $\nu-3\geqq 2$ imply

$$\deg(s_{\nu-1}^{\sim})<\deg(s_{\nu-2}^{\sim})<\deg(s_{\nu-3}^{\sim}),$$

we get

$$\deg(f_\nu^{\sim})=\deg(\sigma_\nu f_{\nu-1}^{\sim})=\deg(s_1^{\sim})-\deg(s_{\nu-1}^{\sim}).$$

The similar argument also yields

$$\deg(g_\nu^{\sim})=\deg(\sigma_\nu g_{\nu-1}^{\sim})=\deg(s_2^{\sim})-\deg(s_{\nu-1}^{\sim}).$$

Hence we get Lemma 10.2. Q.E.D.

Lemma 10.3. *For* $3\leqq\nu\leqq k$, *there exist* $f_\nu, g_\nu\in\mathcal{O}[t]$ *and* $c_\nu, d_\nu\in\mathcal{O}-\{0\}$ *such that if we set* $s_\nu:=(d_\nu/c_\nu)s_\nu^{\sim}$ *then the following* (10.6)-(10.8) *hold*:

(10.6) $\quad s_\nu=f_\nu s_2^{\sim}+g_\nu s_1^{\sim}.$

(10.7) $\quad \deg(f_\nu)=\deg(s_1^{\sim})-\deg(s_{\nu-1}^{\sim}),\quad \deg(g_\nu)=\deg(s_2^{\sim})-\deg(s_{\nu-1}^{\sim}).$

(10.8) *The polynomial* $h_\nu:=t^{q_j^{\sim}}f_\nu+g_\nu$ *is a* *primitive* *polynomial, that is, there exists no non-unit common divisor of all coefficients of* $h_\nu\in\mathcal{O}[t]$, *where we put*

$$q_j^{\sim}:=\deg(s_2^{\sim})=\min\{q_j, 1+\deg_t P\}.$$

Proof. Let $f_\nu^\sim, g_\nu^\sim \in \mathcal{K}[t]$ be the polynomials in Lemma 10.2. We put

$$h_\nu^\sim := t^{q_j^\sim} f_\nu^\sim + g_\nu^\sim \in \mathcal{K}[t].$$

Note that, by (10.5) in Lemma 10.2 and $\deg(s_{\nu-1}^\sim) \geqq 1$, we have

$$\deg(g_\nu^\sim) \leqq q_j^\sim - 1. \tag{10.9}$$

Since $\mathcal{O}$ is a unique factorization domain, this $h_\nu^\sim$ can be written as the form

$$h_\nu^\sim(t) = \sum_{i=0}^{N} (b_{i\nu}/a_{i\nu}) t^i \qquad (N := q_j^\sim + \deg(f_\nu^\sim))$$

where $a_{i\nu}, b_{i\nu} \in \mathcal{O}$ are taken such as $a_{i\nu}$ and $b_{i\nu}$ have no non-unit common divisor, that is, they are coprime for $0 \leqq i \leqq N$. We take

$$c_\nu := \mathrm{GCD}\{b_{0\nu}, b_{1\nu}, \cdots, b_{N\nu}\}$$

where the notation GCD denotes the greatest common divisor (Note that such c_ν is uniquely determined up to unit elements, since $\mathcal{O}$ is a unique factorization domain.). Then we have

$$h_\nu^\sim / c_\nu = \sum_{i=0}^{N} (b'_{i\nu}/a_{i\nu}) t^i$$

where $b'_{i\nu}$ and $a_{i\nu}$ are coprime for $0 \leqq i \leqq N$, and where

$$\mathrm{GCD}\{b'_{0\nu}, b'_{1\nu}, \cdots, b'_{N\nu}\} = 1 \quad \text{up to unit elements.}$$

Now we take

$$d_\nu := \mathrm{LCM}\{a_{0\nu}, a_{1\nu}, \cdots, a_{N\nu}\}$$

where LCM denotes the least common multiplier. Then it follows that

$$(d_\nu/c_\nu) h_\nu^\sim \in \mathcal{O}[t] \quad \text{and is a primitive polynomial.} \tag{10.10}$$

Thus, if we set

$$f_\nu := (d_\nu/c_\nu) f_\nu^\sim, \quad g_\nu := (d_\nu/c_\nu) g_\nu^\sim$$

then (10.9) and (10.10) yield $f_\nu, g_\nu \in \mathcal{O}[t]$. Such constructions of c_ν, d_ν and of f_ν, g_ν easily imply the desired conditions (10.6)-(10.8).

The proof of Lemma 10.3 is complete. Q. E. D.

Definition 10.4. We define $s_\nu(t) \in \mathcal{O}[t]$ for $1 \leqq \nu \leqq k$ as follows:

1) For $\nu = 1, 2$, we set

$$s_1 := s_1^\sim, \quad s_2 := s_2^\sim.$$

2) For $3 \leqq \nu \leqq k$, we define $s_\nu = (d_\nu/c_\nu) s_\nu^\sim$ by Lemma 10.3, where k and $s_\nu^\sim$ are defined by Definition 10.1.

Proposition 10.5. *For the finite sequence* $\{s_\nu(t); 1 \leqq \nu \leqq k\} \subset \mathcal{O}[t]$ *given by Definition* 10.4, *it follows that*

$$\deg(s_{k-1}) = 1. \tag{10.11}$$

Proof. We prove (10.11) by contradiction. Note that if $q_j^{\sim}=\deg(s_2)=1$ then $k=3$ follows, from $\deg(s_3)=\deg(s_3^{\sim})<\deg(s_2^{\sim})=q_j^{\sim}=1$. Thus (10.11) holds if $q_j^{\sim}=1$. Hence we may assume that $q_j^{\sim}=\deg(s_2^{\sim})\geqq 2$.

We assume that the conclusion (10.11) is not true. Then (10.7) in Lemma 10.3 yields

$$\begin{cases} \deg(f_k)=\deg(s_1)-\deg(s_{k-1})\leqq\deg(s_1)-2 \quad \text{and} \\ \deg(g_k)=\deg(s_2)-\deg(s_{k-1})\leqq\deg(s_2)-2. \end{cases} \tag{10.12}$$

Note that the following inequalities hold:

$$\deg(s_1)=\deg(s_1^{\sim})=\begin{cases} \deg_t(tP-x_1)\leqq q & \text{if } s_1^{\sim}=tP-x_1. \\ \deg_t(Q_j)\leqq q_j & \text{if } s_1^{\sim}=Q_j. \end{cases} \tag{10.13}$$

$$\deg(s_2)=\deg(s_2^{\sim})=\begin{cases} \deg_t(Q_j)\leqq q_j & \text{if } s_1^{\sim}=tP-x_1. \\ \deg_t(tP-x_1)\leqq q & \text{if } s_1^{\sim}=Q_j. \end{cases} \tag{10.14}$$

By virtue of (10.12)-(10.14), we can write $f_k(t)s_2(t)+g_k(t)s_1(t)$ as the following form:

$$\begin{aligned} f_k s_2+g_k s_1 &= \begin{cases} f_k Q_j+g_k(tP-x_1) & \text{if } s_1=tP-x_1,\ s_2=Q_j \\ f_k(tP-x_1)+g_k Q_j & \text{if } s_1=Q_j,\ s_2=tP-x_1 \end{cases} \\ &=\Big(\sum_{i=0}^{q-2} e_i t^i\Big)Q_j+\Big(\sum_{i=0}^{q_j-2} e_i' t^i\Big)(tP-x_1) \quad (\text{where } e_i,\ e_i'\in\mathcal{O}). \end{aligned} \tag{10.15}$$

Let us recall the $(q+q_j)$-square matrix $D_j=D(Q_j,\ tP-x_1)$ defined by Notation 8.8 and Definition 8.5. We note the

Remark 10.6. Let $a_i\in\mathcal{O}$ $(0\leqq i\leqq q-1)$ and $b_i\in\mathcal{O}$ $(0\leqq i\leqq q_j-1)$. Then the following 1) and 2) are equivalent:

1) $\Big(\sum_{i=0}^{q-1} a_i t^i\Big)Q_j+\Big(\sum_{i=0}^{q_j-1} b_i t^i\Big)(tP-x_1)=\sum_{i=0}^{q+q_j-1} c_i t^i.$

2) $(c_{q+q_j-1},\ c_{q+q_j-2},\ \cdots,\ c_0)=(a_{q-1},\ \cdots,\ a_0;\ b_{q_j-1},\ \cdots,\ b_0)D_j.$

Remark 10.6 and (10.15) yield that the relation

$$s_k=f_k s_2+g_k s_1\in\mathcal{O}\cap(s_1,\ s_2)\mathcal{O}[t],$$

which is (10.6) for $\nu=k$, can be written as the following form:

$$(0,\ \cdots,\ 0,\ s_k)=(0,\ e_{q-2},\ \cdots,\ e_0;\ 0,\ e'_{q_j-2},\ \cdots,\ e'_0)D_j. \tag{10.16}$$

Let δ_i be the $(i,\ q+q_j)$-cofactor of D_j and let D_j^c be the cofactor matrix of D_j. Operating D_j^c to (10.16) we have

$$(0,\ \cdots,\ 0,\ s_k)D_j^c=(0,\ e_{q-2},\ \cdots,\ e_0;\ 0,\ e'_{q_j-2},\ \cdots,\ e'_0)R_j I_{q+q_j} \tag{10.16$'$}$$

since $D_jD_j^c=(\det D_j)I_{q+q_j}=R_jI_{q+q_j}$.

On the other hand, $D_j^cD_j=R_jI_{q+q_j}$ yields

$$\delta_q Q_{j0}-\delta_{q+q_j}x_1=R_j,$$

where $Q_{j0}(x', \xi_1)$ denotes the coefficient of degree 0 in t of $Q_j(t, x', \xi_1)$. Hence we have

$$\delta_q\notin(R_j) \quad \text{or} \quad \delta_{q+q_j}\notin(R_j). \tag{10.17}$$

Indeed, if we assume that (10.17) is false then we can find $\delta_q^{\sim}$, $\delta_{q+q_j}^{\sim}$ such that $\delta_q=R_j\delta_q^{\sim}$, $\delta_{q+q_j}=R_j\delta_{q+q_j}^{\sim}$. Hence we get

$$(1-\delta_q^{\sim}Q_{j0}+\delta_{q+q_j}^{\sim}x_1)R_j=0.$$

Since $R_j\neq 0$, it follows $1-\delta_q^{\sim}Q_{j0}+\delta_{q+q_j}^{\sim}x_1=0$. This is a contradiction because $Q_{j0}(0, 0)=0$. Thus (10.17) is true.

Note that (10.16)$'$ is equivalent to

$$s_k(\delta_1, \cdots, \delta_{q+q_j})=R_j(0, e_{q-2}, \cdots, e_0; 0, e'_{q_j-2}, \cdots, e'_0). \tag{10.16$''$}$$

Thus (10.17) yields

$$s_k\in(R_j) \tag{10.18}$$

since the ideal (R_j) is a prime ideal of $\mathcal{O}$ (Theorem 9.1).

We return to the equality (10.16). Let D'_j be the $(q+q_j-2)$-square matrix which is obtained by excluding from D_j the first column and row and the $(q+1)$-th column and row. Then (10.16) can be written as the form

$$(0, \cdots, 0, s_k)=(e_{q-2}, \cdots, e_0; e'_{q_j-2}, \cdots, e'_0)D'_j. \tag{10.19}$$

Restricting (10.19) on $\{R_j=0\}$, and using (10.18), we have

$$0=(e_{q-2}, \cdots, e_0; e'_{q_j-2}, \cdots, e'_0)D'_j|_{\{R_j=0\}}. \tag{10.20}$$

Since

$$h_k=t^{q_j}f_k+g_k=\begin{cases} \sum_{i=0}^{q-2} e_it^{i+q_j}+\sum_{i=0}^{q_j-2} e'_it^i & (\text{if } s_1=tP-x_1) \\ \sum_{i=0}^{q_j-2} e'_it^{i+1+\deg(P)}+\sum_{i=0}^{\deg(P)-1} e_it^i & (\text{if } s_1=Q_j) \end{cases}$$

is a primitive polynomial (Lemma 10.3), it follows that

$$(e_{q-2}, \cdots, e_0; e'_{q_j-2}, \cdots, e'_0)|_{\{R_j=0\}}\not\equiv 0. \tag{10.21}$$

By virtue of (10.20) and (10.21), we get

$$\det(D'_j)|_{\{R_j=0\}}\equiv 0.$$

Hence the Rückert's Nullstellensatz yields $\det(D'_j)\in\mathrm{Rad}[(R_j)]$. Since (R_j) is a prime ideal, we get

(10.22) $$\det(D_j') \in (R_j).$$

But (10.22) is a contradiction. Indeed, by the definition of the matrix D_j', it follows that

$$\mathrm{ord}[(\det D_j')(x_1, x', \xi_1)|_{(x', \xi_1)=0}] = q_j - 1$$

since $Q_j(t, x', \xi_1)$ is a Weierstrass polynomial in t. On the other hand, we know

$$\mathrm{ord}[R_j(x_1, 0, 0)] = q_j.$$

Thus we have

$$(\det D_j')(x_1, 0, 0) \notin (R_j(x_1, 0, 0))$$

which contradicts (10.22). This contradiction comes from our assumption $\deg(s_{k-1}) \geqq 2$. Hence it follows that $\deg(s_{k-1}) = 1$ as desired.

The proof of Proposition 10.5 is complete. Q. E. D.

By virtue of Proposition 10.5, the polynomial $s_{k-1}(t)$ can be written as

(10.23) $$s_{k-1}(t, x, \xi_1) = a(x, \xi_1)t + b(x, \xi_1).$$

In this situation we have the

Proposition 10.7. *It follows*

(10.24) $$a(x, \xi_1)|_{\{R_j=0\}} \not\equiv 0.$$

Proof. We first show the assertion (10.24) in the case $k=3$.
If $k=3$ then Proposition 10.5 yields

$$\deg(s_2) = 1.$$

By Definitions 10.4 and 10.1, we have

$$s_2 = \begin{cases} Q_j & \text{if } q_j \leqq 1 + \deg_t P. \\ tP - x_1 & \text{if } q_j > 1 + \deg_t P. \end{cases}$$

Thus, in the case $q_j \leqq 1 + \deg_t P$, we conclude that the Weierstrass polynomial Q_j has degree one. Hence $a(x, \xi_1) \equiv 1$ holds. On the other hand, in the case $q_j > 1 + \deg_t P$, we get $\deg_t P = 0$ which derives $a(x, \xi_1) = P$. Since the Claim 2) in the proof of Proposition 8.9 shows that the leading coefficient of P does not vanish identically on $\{R_j = 0\}$, (10.24) also holds in the case $q_j > 1 + \deg_t P$. Hence Proposition 10.7 holds if $k=3$.

Now we prove Proposition 10.7 in the case $k \geqq 4$. We assume that the conclusion (10.24) is not true. Then, since (R_j) is a prime ideal of $\mathcal{O} := \mathcal{O}_{M \times C, (0,0)}$, the Rückert's Nullstellensatz yields $a(x, \xi_1) \in (R_j)$. Hence we can find a germ $\tilde{a} \in \mathcal{O}$ such that

(10.25) $$a = R_j \tilde{a}.$$

By virtue of (10.25) with the assumption $k-1\geqq 3$, Lemma 10.3 yields

$$R_j\tilde{a}t+b=s_{k-1}=f_{k-1}s_2+g_{k-1}s_1\in(Q_j,\ tP-x_1)\mathcal{O}[t].$$

Since $R_j\in(Q_j,\ tP-x_1)\mathcal{O}[t]$ (Corollary 8.7), we get

$$b\in\mathcal{O}\cap(Q_j,\ tP-x_1)\mathcal{O}[t]. \tag{10.26}$$

By Proposition 8.6, (10.26) yields that there exist $e_i\in\mathcal{O}$ $(1\leqq i\leqq q+q_j)$, $e\in\mathcal{O}$ and $d\in\mathcal{O}-\{0\}$ such that

$$b=Q_{j0}e_q-x_1e_{q+q_j} \quad \text{and} \tag{10.27}$$

$$e_i=(e/d)\delta_i \qquad \text{for } 1\leqq i\leqq q+q_j \tag{10.28}$$

where δ_i denotes the $(i,\ q+q_j)$-cofactor of the matrix D_j in Notation 8.8.

Recall the relation

$$R_j=\delta_qQ_{j0}-\delta_{q+q_j}x_1 \tag{10.29}$$

which is a consequence of $D_j^cD_j=R_jI_{q+q_j}$ (D^c denotes the cofactor matrix of D_j). By (10.27)-(10.29) we have

$$db=Q_{j0}de_q-x_1de_{q+q_j}=Q_{j0}e\delta_q-x_1e\delta_{q+q_j}=eR_j. \tag{10.30}$$

Claim 10.8. *It follows that $b\in(R_j)$.*

Proof. Since (R_j) is a prime ideal of $\mathcal{O}$, if we assume that Claim 10.8 is false then (10.30) yields $d\in(R_j)$, that is, there exists a germ $d^{\sim}\in\mathcal{O}$ such that $d=R_jd^{\sim}$. Thus (10.28) can be written as

$$R_jd^{\sim}e_i=de_i=e\delta_i \qquad \text{for } 1\leqq i\leqq q+q_j.$$

Hence, by (10.17), we get $e\in(R_j)$. But we can choose $d,\ e\in\mathcal{O}$ in (10.28) such as d and e are coprime since $\mathcal{O}$ is a unique factorization domain. Thus it is a contradiction that both d and e lie in (R_j). Hence Claim 10.8 follows.

Q. E. D.

We continue the proof of Proposition 10.7. Recall the relation

$$s_{k-1}=f_{k-1}s_2+g_{k-1}s_1 \tag{10.31}$$

which is a consequence of Lemma 10.3. Since the assumption $k\geqq 4$ yields $\deg(s_{k-2})>\deg(s_{k-1})=1$, we have

$$\begin{cases} \deg(f_{k-1})=\deg(s_1)-\deg(s_{k-2})\leqq\deg(s_1)-2 \quad \text{and} \\ \deg(g_{k-1})=\deg(s_2)-\deg(s_{k-2})\leqq\deg(s_2)-2. \end{cases}$$

Thus the inequalities (10.13, 14) imply that there exist $c_i,\ c_i'\in\mathcal{O}$ such that

$$f_{k-1}s_2+g_{k-1}s_1=\Big(\sum_{i=0}^{q-2}c_it^i\Big)Q_j+\Big(\sum_{i=0}^{q_j-2}c_i't^i\Big)(tP-x_1).$$

Then Remark 10.6 yields that (10.31) can be written as

$$(10.32) \qquad (0, \cdots, 0, a, b)=(0, c_{q-2}, \cdots, c_0; 0, c'_{q_j-2}, \cdots, c'_0)D_j.$$

Recall the $(q+q_j-2)$-square matrix D'_j obtained by excluding the first column and row, and the $(q+1)$-th column and row of D_j, which is used in the proof of Proposition 10.5. Then the relation (10.32) can be written as the form

$$(10.32)' \qquad (0, \cdots, 0, a, b)=(c_{q-2}, \cdots, c_0; c'_{q_j-2}, \cdots, c'_0)D'_j.$$

Restricting (10.32)′ on $\{R_j=0\}$ and using $a, b\in(R_j)$ we have

$$0=(c_{q-2}, \cdots, c_0; c'_{q_j-2}, \cdots, c'_0)D'_j|_{\{R_j=0\}}$$

which yields

$$(10.33) \qquad (\det D'_j)|_{\{R_j=0\}}=0$$

since the polynomial $h_{k-1}=t^{q_j}f_{k-1}+g_{k-1}$ is a primitive polynomial. Then Rückert's Nullstellensatz and the primeness of (R_j) imply that (10.33) is equivalent to

$$(10.34) \qquad \det D'_j\in(R_j).$$

But (10.34) is a contradiction since

$$\det D'_j(x_1, 0, 0)\in(x_1)^{q_j-1}, \notin(R_j(x_1, 0, 0))=(x_1)^{q_j}$$

as like as in the proof of Proposition 10.5. This contradiction comes from our assumption that Proposition 10.7 is not true. Hence Proposition 10.7 follows.

The proof of Proposition 10.7 is complete. Q.E.D.

As a corollary of Propositions 10.5 and 10.7, we get Theorem 6.11:

Proof of Theorem 6.11. For the polynomial $s_{k-1}(t)=a(x, \xi_1)t+b(x, \xi_1)$, we define a map germ $\rho:(M\times \boldsymbol{C}, (0, 0))\to(\boldsymbol{C}\times M\times \boldsymbol{C}, (0, 0, 0))$ by setting as

$$(10.35) \qquad \rho(x, \xi_1):=(-\{b(x, \xi_1)/a(x, \xi_1)\}, x, \xi_1).$$

We show that ρ induces a meromorphic inverse $V_{j2}\to V_{j1}$ of the map germ $\pi_{j2}: V_{j1}=\{tP-x_1=Q_j=0\}\to V_{j2}=\{R_j=0\}$.

Note that $V_{j2}-\{a(x, \xi_1)=0\}\neq\emptyset$ (Proposition 10.7) implies that the intersection germ $\Sigma_j:=V_{j2}\cap\{a=0\}$ is a germ of nowhere dense analytic subset of V_{j2}. Let $(x^0, \xi_1^0)\in V_{j2}-\Sigma_j$. Since π_{j2} is an open map germ, that is, π_{j2} is surjective to $(V_{j2}, (0, 0))$, there exists $t^0\in(\boldsymbol{C}, 0)$ such that $(t^0, x^0, \xi_1^0)\in V_{j1}$. Then we have

$$(10.36) \qquad a(x^0, \xi_1^0)t^0+b(x^0, \xi_1^0)=s_{k-1}(t^0, x^0, \xi_1^0)=0$$

by virtue of $s_{k-1}=f_{k-1}s_2+g_{k-1}s_1\in(tP-x_1, Q_j)$. Since $a(x^0, \xi_1^0)\neq0$, the equation (10.36) yields

$$(t^0, x^0, \xi_1^0)=\rho(x^0, \xi_1^0)$$

which shows that the induced map germ

$$\pi'_{j2}: V_{j1}-\pi_{j2}^{-1}(\Sigma_j) \longrightarrow V_{j2}-\Sigma_j$$

is a biholomorphic map germ.

It only remains to show that

(10.37) $\pi_{j2}^{-1}(\Sigma_j)$ is a germ of a nowhere dense analytic subset of V_{j1}.

But this is easy: Since π_{j2} is an open map germ at (0, 0, 0), we have

$$\pi_{j2}^{-1}(\Sigma_j) \neq V_{j1}.$$

Then the irreducibility of V_{j1} at (0, 0, 0) yields (10.37).

The proof of Theorem 6.11 is complete. Q. E. D.

Chapter IV. Appendices

§ 11. Generalities of Newton Polygons

In this section we summarize basic facts on Newton polygons. The aim of this section is to give proofs of Propositions 2.11 and 2.12.

Let S be a domain in $C^{n-1}(n \geqq 2)$ which contains the origin throughout this section. For a holomorphic germ

$$f(y, \tau)=\sum_{\nu=0}^{\infty} c_\nu(y)\tau^\nu \in \mathcal{O}_{S\times C,(0,0)}(c_\nu \in \mathcal{O}_{S,0})$$

we define its Newton polygon N(f), the strict boundary ∂^0N(f) of N(f), and segments and vertices of N(f), by Definition 2.3. We also use Notation 2.4.

Definition 11.1. Let N be a Newton polygon. For a vertex $A \in$ Ver N:

1) We define the *left* [or *right*, resp.] *segment* $L(A)$ [$R(A)$] of A as follows: We arrange vertices of N as

$$\text{Ver N} =: \{A(\mu)=(a(\mu),\ p-b(\mu)) : 0 \leqq \mu \leqq m\}$$

where finite sequenses $\{a(\mu)\}$, $\{b(\mu)\}$ are monotonely increasing in μ. We set

$$L(A(\mu)):=\begin{cases} \{tA(\mu)+(1-t)A(\mu-1)\,;\ 0 \leqq t \leqq 1\} & \text{if } \mu \geqq 1 \\ A(0)+0\times \bar{\boldsymbol{R}}_+ & \text{if } \mu=0. \end{cases}$$

$$R(A(\mu)):=\begin{cases} \{tA(\mu)+(1-t)A(\mu+1))\,;\ 0 \leqq t \leqq 1\} & \text{if } \mu<m \\ A(m)+\bar{\boldsymbol{R}}_+\times 0 & \text{if } \mu=m. \end{cases}$$

2) We set $\kappa(L(A))$ [or, $\kappa(R(A))$ resp.] $\in \bar{\boldsymbol{Q}}_+ \cup \{\infty\}$ by

$$\kappa(L(A(\mu))):=\begin{cases} \kappa(\mu) & \text{if } \mu \geqq 1 \\ \infty & \text{if } \mu=0 \end{cases} \quad \left[\kappa(R(A(\mu))):=\begin{cases} \kappa(\mu) & \text{if } \mu<m \\ 0 & \text{if } \mu=m \end{cases}\right]$$

where $\kappa(\mu):=p(\mu)/q(\mu)\in\boldsymbol{Q}_+$ is defined by (2.8) in Notation 2.4.

First we show the

Proposition 11.2. *Let f, g be germs on $(S\times\boldsymbol{C}, (0, 0))$ and let $A\in\mathrm{N}(f)$, $B\in\mathrm{N}(g)$. Then the following statements are equivalent:*

1) $A+B\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))$.
2) $A\in\mathrm{Ver}\,\mathrm{N}(f)$, $B\in\mathrm{Ver}\,\mathrm{N}(g)$ *such that*

$$\min\{\kappa(L(A)), \kappa(L(B))\}>\max\{\kappa(R(A)), \kappa(R(B))\}\,. \tag{11.1}$$

Proof. Note that the vector sum $\mathrm{N}(f)+\mathrm{N}(g)$ is a closed, $\bar{\boldsymbol{R}}_+^2$-invariant, convex set, thus we can define $\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))$.

We first show 1)⇒2). Since 1) yields

$$A+B\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))\subset\partial^{\circ}(\mathrm{N}(f)+\mathrm{N}(g))$$

we easily have $A\in\partial^{\circ}\mathrm{N}(f)$ and $B\in\partial^{\circ}\mathrm{N}(g)$.

To show $A\in\mathrm{Ver}\,\mathrm{N}(f)$, we derive the following implication (11.2):

$$\begin{cases} A=tA'+(1-t)A''(A',\ A''\in\mathrm{N}(f)\quad\text{with}\quad A'\neq A''\quad\text{and}\quad t\in[0,\ 1]) \\ \Rightarrow t=0\quad\text{or}\quad t=1. \end{cases} \tag{11.2}$$

Indeed, $A+B\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))$ yields the implication

$$A+B=t(A'+B)+(1-t)(A''+B)\Rightarrow t=0\quad\text{or}\quad t=1$$

since $A'+B$, $A''+B\in\mathrm{N}(f)+\mathrm{N}(g)$ with $A'+B\neq A''+B$ and $t\in[0, 1]$. Hence (11.2) follows, that is, $A\in\mathrm{Ver}\,\mathrm{N}(f)$. We similarly have $B\in\mathrm{Ver}\,\mathrm{N}(g)$.

Now we show the inequality (11.1) under the assumptions

$$A\in\mathrm{Ver}\,\mathrm{N}(f),\ B\in\mathrm{Ver}\,\mathrm{N}(g)\quad\text{and}\quad A+B\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))\,. \tag{11.3}$$

Since $\kappa(L(A))>\kappa(R(A))$ and $\kappa(L(B))>\kappa(R(B))$ are trivial, (11.1) is equivalent to

$$\kappa(L(A))>\kappa(R(B))\quad\text{and}\quad\kappa(L(B))>\kappa(R(A))\,. \tag{11.4}$$

By the symmetricity of A and B, we only have to check the first inequality of (11.4). Note that in the case $\kappa(L(A))=\infty$ or $\kappa(R(B))=0$ the assertion is trivial, hence we may assume $\kappa(L(A))<\infty$ and $\kappa(R(B))>0$, that is, both $L(A)$ and $R(B)$ are segments in the sense of Definition 2.3.

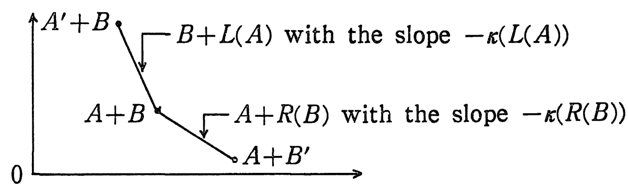
We can therefore write $L(A)$ and $R(B)$ as

$$L(A)=\{tA+(1-t)A'\,;\ t\in[0,\ 1]\}$$

$$R(B)=\{tB+(1-t)B'\,;\ t\in[0,\ 1]\}$$

where $A'\in\mathrm{Ver}\,\mathrm{N}(f)$, $B'\in\mathrm{Ver}\,\mathrm{N}(g)$ with $A\neq A'$, $B\neq B'$. Note that the first inequality of (11.4) is equivalent to the following (11.5) (see figure):

(11.5) $A+B \notin$ the closed half plane given by the closure of the upper side of the line joining $A'+B$ and $A+B'$.



We note that $A+B \in \mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))$ implies (11.5), since $A'+B$, $A+B'$ both belong to $\mathrm{N}(f)+\mathrm{N}(g)$. Hence we have the first inequality of (11.4) as desired. Thus 1) implies 2).

Conversely we assume 2). It suffices for 1) to show that there exist two lines σ, σ' passing through $A+B$ with non-positive distinct slopes such that

(11.6) $\mathrm{N}(f)+\mathrm{N}(g) \subset$ the closure of

(the upper side of σ)$\cap$(the upper side of σ').

We construct σ, σ' as the following forms:

$$\begin{cases} \sigma := A+B+\{(x, y);\ y=-cx\} \\ \sigma' := A+B+\{(x, y);\ y=-c'x\} \\ c := \min\{\kappa(L(A)), \kappa(L(B))\} > c' := \max\{\kappa(R(A)), \kappa(R(B))\} \end{cases}$$

where, in the case $c=\infty$, the line "$y=-cx$" denotes the vertical line $x=0$, and "the upper side of σ" denotes the right side of σ.

Let v[or v' resp.] be a linear functional on $\boldsymbol{R}^2$ which takes positive values on the upper side of the line $y=-cx$[$y=-c'x$]. Then the inequality (11.1) yields the following inclusions:

$$\mathrm{N}(f) \subset \{A';\ v(A') \geqq v(A) \ \text{ and } \ v'(A') \geqq v'(A)\}$$

$$\mathrm{N}(g) \subset \{B';\ v(B') \geqq v(B) \ \text{ and } \ v'(B') \geqq v'(B)\}$$

which imply

$$\mathrm{N}(f)+\mathrm{N}(g) \subset \{A'+B';\ v(A'+B') \geqq v(A+B) \ \text{ and } \ v'(A'+B') \geqq v'(A+B)\}.$$

Hence we get (11.6) as desired.

The proof of Proposition 11.2 is complete. Q.E.D.

Proposition 11.3 (*the additivity property of Newton polygons*). *For any holomorphic germs f and g on $(S \times \boldsymbol{C}, (0, 0))$, it follows that*

(11.7) $$\mathrm{N}(fg)=\mathrm{N}(f)+\mathrm{N}(g).$$

Proof. We take the Taylor expansions

$$f(y,\tau)=\sum_{\nu=0}^{\infty} c_\nu(y)\tau^\nu, \qquad g(y,\tau)=\sum_{\nu=0}^{\infty} d_\nu(y)\tau^\nu.$$

Of course we have $(fg)(y,\tau)=\sum_{\nu=0}^{\infty}\Big(\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\Big)\tau^\nu$.

We first show

$$\text{N}(fg)\subset \text{N}(f)+\text{N}(g). \tag{11.8}$$

It suffices for (11.8) to prove

$$\Big(\text{ord}\Big[\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\Big], \nu\Big)\in \text{N}(f)+\text{N}(g) \quad \text{if} \quad \sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\neq 0 \tag{11.8$'$}$$

since $\text{N}(f)+\text{N}(g)$ is a convex, $\bar{\boldsymbol{R}}_+^2$-invariant set.

Assume $\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\neq 0$. Then we have

$$\infty>\text{ord}\Big[\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\Big]\geqq \min_{0\leqq\lambda\leqq\nu}\{\text{ord}[c_\lambda]+\text{ord}[d_{\nu-\lambda}]\}. \tag{11.9}$$

Choosing $\lambda'(0\leqq\lambda'\leqq\nu)$ to attain the right hand side of (11.9), we have

$$\begin{aligned}\Big(\text{ord}\Big[\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\Big], \nu\Big)&\in(\text{ord}[c_{\lambda'}], \lambda')+(\text{ord}[d_{\nu-\lambda'}], \nu-\lambda')+\bar{\boldsymbol{R}}_+^2\\ &\subset \text{N}(f)+\text{N}(g).\end{aligned}$$

Hence the inclusion (11.8) follows.

Now we prove the converse inclusion of (11.8). It suffices to show

$$\text{Ver}(\text{N}(f)+\text{N}(g))\subset \text{N}(fg). \tag{11.10}$$

Let $A\in\text{N}(f)$, $B\in\text{N}(g)$ satisfy $A+B\in\text{Ver}(\text{N}(f)+\text{N}(g))$. By virtue of Proposition 11.2, we may assume there exist $\lambda'(0\leqq\lambda'\leqq\nu)$ such that

$$A=(\text{ord}[c_{\lambda'}], \lambda')\in\text{Ver N}(f), \quad B=(\text{ord}[d_{\nu-\lambda'}], \nu-\lambda')\in\text{Ver N}(g).$$

We must show

$$\text{ord}\Big[\sum_{\lambda=0}^{\nu} c_\lambda d_{\nu-\lambda}\Big]=\text{ord}[c_{\lambda'}]+\text{ord}[d_{\nu-\lambda'}]. \tag{11.11}$$

Note that it suffices for (11.11) to verify

$$\text{ord}[c_\lambda]+\text{ord}[d_{\nu-\lambda}]>\text{ord}[c_{\lambda'}]+\text{ord}[d_{\nu-\lambda'}] \quad \text{for all} \quad \lambda(\neq\lambda'),\ 0\leqq\lambda\leqq\nu. \tag{11.12}$$

Since $(\text{ord}[c_\lambda], \lambda)\in\text{N}(f)$, $(\text{ord}[d_{\nu-\lambda}], \nu-\lambda)\in\text{N}(g)$, we have the following inequalities:

$$\left\{\begin{array}{l}\lambda\geqq-\kappa(L(A))(\text{ord}[c_\lambda]-\text{ord}[c_{\lambda'}])+\lambda' \text{ and}\\ \nu-\lambda\geqq-\kappa(R(B))(\text{ord}[d_{\nu-\lambda}]-\text{ord}[d_{\nu-\lambda'}])+\nu-\lambda'\end{array}\right\} \quad \text{for } \lambda'\leqq\forall\lambda\leqq\nu. \tag{11.13}$$

$$(11.14)\quad \left\{\begin{array}{l} \lambda\geqq-\kappa(R(A))(\mathrm{ord}[c_{\lambda}]-\mathrm{ord}[c_{\lambda'}])+\lambda' \text{ and} \\ \nu-\lambda\geqq-\kappa(L(B))(\mathrm{ord}[d_{\nu-\lambda}]-\mathrm{ord}[d_{\nu-\lambda'}])+\nu-\lambda' \end{array}\right\} \text{ for } 0\leqq\forall\lambda\leqq\lambda'.$$

Consider the case $\lambda>\lambda'$. The inequalities (11.13) yield

$$\begin{aligned} &\mathrm{ord}[c_{\lambda}]+\mathrm{ord}[d_{\nu-\lambda}] \\ &\quad\geqq\mathrm{ord}[c_{\lambda'}]+\mathrm{ord}[d_{\nu-\lambda'}]+(\lambda-\lambda')\{1/\kappa(R(B))-1/\kappa(L(A))\} \\ &\quad>\mathrm{ord}[c_{\lambda'}]+\mathrm{ord}[d_{\nu-\lambda'}] \end{aligned}$$

since $\kappa(L(A))>\kappa(R(B))$. In the case $\lambda<\lambda'$, the inequalities (11.14) imply (11.12) as similar as the case $\lambda>\lambda'$. Hence (11.11) follows. Thus we have (11.10) as desired. The proof of Proposition 11.3 is complete. Q. E. D.

Corollary 11.4. *Let* $A\in\mathrm{N}(f)$, $B\in\mathrm{N}(g)$ *such that*

$$A+B\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))=\mathrm{Ver}\,\mathrm{N}(fg).$$

Then it follows that the left segment $L(A+B)$ *of* $A+B$ *is given by*

$$(11.15)\qquad L(A+B)=\left\{\begin{array}{lll} A+L(B) & \textit{if} & \kappa(L(A))>\kappa(L(B)). \\ B+L(A) & \textit{if} & \kappa(L(A))<\kappa(L(B)). \\ L(A)+L(B) & \textit{if} & \kappa(L(A))=\kappa(L(B)). \end{array}\right.$$

Proof. We classify the proof in the following three cases:

$$\left\{\begin{array}{l} \textit{Case } 1\quad \kappa(L(A))\neq\kappa(L(B)),\quad \textit{Case } 2\quad \kappa(L(A))=\kappa(L(B))<\infty \text{ and} \\ \textit{Case } 3\quad \kappa(L(A))=\kappa(L(B))=\infty. \end{array}\right.$$

First we consider the *case* 1. By the symmetricity of the roles f and g, we may assume $\kappa(L(A))>\kappa(L(B))$. Since $\kappa(L(B))<\infty$, $L(B)$ is a segment of $\mathrm{N}(g)$. Hence we can find $B'(\neq B)\in\mathrm{Ver}\,\mathrm{N}(g)$ such that $L(B)$ can be written as

$$L(B)=\{tB+(1-t)B'\,;\ t\in[0,1]\}.$$

Since the proof of 2)⇒1) in Proposition 11.2 shows that

$$\mathrm{N}(f)+\mathrm{N}(g)\subset A+B+\{(x,y)\,;\ y\geqq-\kappa(L(B))x\}$$

it suffices for $L(A+B)=A+L(B)$ to show

$$(11.16)\qquad A+B'\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g)).$$

The relation $R(B')=L(B)$ and the inequality (11.1) yield

$$\kappa(L(A))>\kappa(L(B))=\kappa(R(B'))$$

and

$$\kappa(L(B'))>\kappa(R(B'))=\kappa(L(B))>\kappa(R(A)).$$

Hence, by Proposition 11.2, we get (11.16) as desired.

Next we consider the *case* 2. Since both $L(A)$, $L(B)$ are segments, we can write them as

$$L(A)=\{tA+(1-t)A'\,;\ t\in[0,1]\}\quad\text{and}\quad L(B)=\{tB+(1-t)B'\,;\ t\in[0,1]\}$$

where $A'(\neq A)\in\mathrm{Ver}\,\mathrm{N}(f)$, $B'(\neq B)\in\mathrm{Ver}\,\mathrm{N}(g)$. Then it follows

$$\kappa(L(A'))>\kappa(L(A))=\kappa(L(B))=\kappa(R(B'))$$

and

$$\kappa(L(B'))>\kappa(L(B))=\kappa(L(A))=\kappa(R(A')).$$

Hence Proposition 11.2 yields that $A'+B'\in\mathrm{Ver}(\mathrm{N}(f)+\mathrm{N}(g))$ as desired.

In the *case* 3, it is trivial that

$$\mathrm{N}(f)+\mathrm{N}(g)\subset A+B+\bar{\boldsymbol{R}}_+\times\boldsymbol{R}$$

which shows $L(A+B)=A+B+0\times\bar{\boldsymbol{R}}_+=L(A)+L(B)$.

The proof of Corollary 11.4 is complete. Q.E.D.

Remark 11.5. Proposition 2.11 follows from Proposition 11.3 and Lemma 0.2.

Proof. The additivity property of Newton polygons immediately implies

$$\mathrm{N}\Big(\prod_{j=1}^{r}f_j(y,\tau)^{\nu(j)}\Big)=\sum_{j=1}^{r}\nu(j)\mathrm{N}(f_j).$$

Thus the left equality of Proposition 2.11 is a direct consequence of Proposition 11.3. Hence it suffices for Proposition 2.11 to verify

$$\mathrm{N}(f^{\Phi})=\sum_{\mu=1}^{m}\mathrm{N}_{q(\mu),p(\mu)}. \tag{11.17}$$

But this immediately follows from Lemma 0.2. Indeed, since we assume $p(1)/q(1)>\cdots>p(m)/q(m)$ in Notation 2.4, we get

$$\mathrm{Ver}\Big(\sum_{\mu=1}^{m}\mathrm{N}_{q(\mu),p(\mu)}\Big)=\mathrm{Ver}\,\mathrm{N}(f^{\Phi})$$

which shows (11.17). Q.E.D.

Now we recall the coprimeness condition (Definition 2.5).

Proposition 11.6. *Let f, g be holomorphic germs on $(S\times\boldsymbol{C},(0,0))$ satisfying $f(0,\tau)g(0,\tau)\not\equiv0$. Then the following statements are equivalent:*

1) *$\mathrm{N}(fg)$ satisfies the coprimeness condition.*

2) *Both $\mathrm{N}(f)$, $\mathrm{N}(g)$ satisfy the coprimeness condition, and the following condition holds:*

$$\begin{cases}\kappa(L(A))\neq\kappa(L(B))\\ \qquad\textit{for all}\quad A\in\mathrm{Ver}\,\mathrm{N}(f),\ B\in\mathrm{Ver}\,\mathrm{N}(g)\ \textit{satisfying}\\ \qquad\kappa(L(A))<\infty,\quad \kappa(L(B))<\infty.\end{cases}\tag{11.18}$$

Proof. Since $(fg)(y, 0)=f(y, 0)g(y, 0)$, we only have to consider the second condition in Definition 2.5.

We assume that $\mathrm{N}(fg)$ satisfies the coprimeness condition. We first show the condition (11.18) by contradiction. We assume that there exist $A\in\mathrm{N}(f)$, $B\in\mathrm{N}(g)$ such that

$$\kappa(L(A))=\kappa(L(B))<\infty. \tag{11.19}$$

Then it follows

$$\kappa(L(A))=\kappa(L(B))>\kappa(R(B))$$

and

$$\kappa(L(B))=\kappa(L(A))>\kappa(R(A)).$$

Hence $A+B\in\mathrm{Ver}\,\mathrm{N}(fg)$ by Propositions 11.2 and 11.3. Then Corollary 11.4 yields $L(A+B)=L(A)+L(B)$, which shows that $\mathrm{N}(fg)$ does not satisfy the coprimeness condition. This contradicts the assumption. Hence we have the implication "1)$\Rightarrow$(11.18)" in Proposition 11.6.

Next we prove, under (11.18), that $\mathrm{N}(f)$ and $\mathrm{N}(g)$ both satisfy the coprimeness condition if and only if $\mathrm{N}(fg)$ satisfies the same condition. Note that $\mathrm{N}(f)$ and $\mathrm{N}(g)$ can be written as the form (11.20), since $f(y, 0)g(y, 0)\not\equiv 0$ and $f(0, \tau)g(0, \tau)\not\equiv 0$:

$$\begin{cases} \mathrm{N}(f)=\sum_{\mu=1}^{m_1}\mathrm{N}_{q_1(\mu),\,p_1(\mu)}, \quad \mathrm{N}(g)=\sum_{\nu=1}^{m_2}\mathrm{N}_{q_2(\nu),\,p_2(\nu)}. \\ p_1(1)/q_1(1)>\cdots>p_1(m_1)/q_1(m_1)>0. \\ p_2(1)/q_2(1)>\cdots>p_2(m_2)/q_2(m_2)>0. \end{cases} \tag{11.20}$$

Since the condition (11.18) is equivalent to

$$p_1(\mu)/q_1(\mu)\neq p_2(\nu)/q_2(\nu) \quad \text{for all} \quad 1\leqq\mu\leqq m_1,\ 1\leqq\nu\leqq m_2, \tag{11.21}$$

Proposition 11.3 yields that

$$\mathrm{N}(fg)=\mathrm{N}(f)+\mathrm{N}(g)=\sum_{\mu=1}^{m_1}\mathrm{N}_{q_1(\mu),\,p_1(\mu)}+\sum_{\nu=1}^{m_2}\mathrm{N}_{q_2(\nu),\,p_2(\nu)} \tag{11.22}$$

with the *distinct* ratios $\{p_1(\mu)/q_1(\mu)\}_{1\leqq\mu\leqq m_1}\cup\{p_2(\nu)/q_2(\nu)\}_{1\leqq\nu\leqq m_2}$. Hence we get the following equivalence under the condition (11.18):

$$\begin{aligned} &\text{Both } \mathrm{N}(f) \text{ and } \mathrm{N}(g) \text{ satisfy the coprimeness condition.} \\ \Leftrightarrow &\begin{cases} p_1(\mu) \text{ and } q_1(\mu) \text{ are coprime for } 1\leqq\mu\leqq m_1, \text{ and} \\ p_2(\nu) \text{ and } q_2(\nu) \text{ are coprime for } 1\leqq\nu\leqq m_2. \end{cases} \\ \Leftrightarrow\ &\mathrm{N}(fg)=\mathrm{N}(f)+\mathrm{N}(g) \text{ satisfies the coprimeness condition.} \end{aligned} \tag{11.23}$$

By virtue of (11.23) and the implication "the condition 1)$\Rightarrow$(11.18)" we get the desired equivalence between 1) and 2) in Proposition 11.6.

The proof of Proposition 11.6 is complete. Q.E.D.

Proof of Proposition 2.12. We may assume our situation be as follows: Let S be a domain in $\boldsymbol{C}^{n-1}$ containing the origin. Let $f(y, \tau)\in\mathcal{O}_{S\times C,(0,0)}$ be a holomorphic germ which has a finite order $p\in[1, \infty)$ with resprct to τ.

Let us denote the irreducible decomposition of f locally at (0, 0) by

$$f=\prod_{j=1}^{r} f_j^{\nu(j)}. \tag{11.24}$$

We assume that the Newton polygon $\mathrm{N}(f)$ satisfies the coprimeness condition, that is, the following two conditions hold:

$$\mathrm{N}(f)\cap(\boldsymbol{R}\times 0)\neq\emptyset. \tag{11.25}$$

(11.26) $p(\mu)$ and $q(\mu)$ are coprime for $1\leqq\mu\leqq m:=\#\mathrm{Seg}\,\mathrm{N}(f)$.

Recall that the integers $p(\mu)$ and $q(\mu)$ in (11.26) are defined by

(11.27)
$$\begin{cases} \text{Write } \mathrm{Ver}\,\mathrm{N}(f)=:\{(a(\mu),\ p-b(\mu));\ 0\leqq\mu\leqq m\} \text{ with} \\ \qquad 0=a(0)<a(1)<\cdots<a(m)=q:=\mathrm{ord}_0[f(y, 0)] \\ \qquad 0=b(0)<b(1)<\cdots<b(m)=p=\mathrm{ord}_0[f(0, \tau)] \\ \text{and put } p(\mu):=b(\mu)-b(\mu-1) \text{ and } q(\mu):=a(\mu)-a(\mu-1). \end{cases}$$

It suffices for Proposition 2.12, to show the following

(11.28)
$$\begin{cases} 1)\ \ \nu(j)=1 \quad \text{for } 1\leqq j\leqq r. \\ 2)\ \ \mathrm{N}(f_j) \text{ satisfies the coprimeness condition for } 1\leqq j\leqq r. \\ 3)\ \ \text{There exist subsets } \mathrm{M}_j \text{ of } \{1, 2, \cdots, m\}(1\leqq j\leqq r) \text{ such that} \\ \quad \text{(a)}\ \ \mathrm{M}_j\cap\mathrm{M}_k=\emptyset \ \text{ if } \ j\neq k. \\ \quad \text{(b)}\ \ \{1, 2, \cdots, m\}=\bigcup_{j=1}^{r}\mathrm{M}_j \\ \quad \text{(c)}\ \ \mathrm{N}(f_j)=\sum_{\mu\in\mathrm{M}_j}\mathrm{N}_{q(\mu), p(\mu)} \quad \text{for all } 1\leqq j\leqq r. \end{cases}$$

We first show the assertion 2) in (11.28). We fix j, and put

$$g_j:=f_j^{\nu(j)-1}\prod_{i\neq j} f_i^{\nu(i)}.$$

Then, applying Proposition 11.6 to $f=f_j g_j$, we get the coprimeness of $\mathrm{N}(f_j)$.

Next we show the assertion 1). Assume that 1) is not true, then there is a number $j(1\leqq j\leqq r)$ such that $\nu(j)\geqq 2$. For such a number j, we put

$$\tilde{g_j}:=f_j^{\nu(j)-2}\prod_{i\neq j} f_i^{\nu(i)}.$$

Then, applying Proposition 11.6 to $f=f_j^2\tilde{g_j}$, we have

(11.29) $\mathrm{N}(f_j^2)$ satisfies the coprimeness condition.

Then, applying Proposition 11.6 to f_j^2, we get

$$(11.30)\qquad \begin{cases} \kappa(L(A))\neq\kappa(L(B)) \\ \qquad \text{for all } A,\ B\in\mathrm{Ver}\,\mathrm{N}(f_j) \text{ satisfying } \kappa(L(A)),\ \kappa(L(B))<\infty\,. \end{cases}$$

But (11.30) is impossible for $A=B$ satisfying $\kappa(L(A))<\infty$ (note that the existence of such a vertex A is a consequence of $\#\mathrm{Ver}\,\mathrm{N}(f_j)\geqq 2$). This contradiction comes from the assumption $\nu(j)\geqq 2$. Thus we get the assertion 1) in (11.28).

Now we prove the assertion 3) in (11.28). By virtue of 1) in (11.28), we have the following irreducible decomposition of f locally at (0, 0):

$$(11.24)'\qquad f=\prod_{j=1}^{r} f_j\,.$$

Since each $\mathrm{N}(f_j)$ satisfies the coprimeness condition, we can write $\mathrm{N}(f_j)$ as the following form for $1\leqq j\leqq r$:

$$(11.31)\qquad \begin{cases} \mathrm{N}(f_j)=\sum_{\nu=1}^{m_j} \mathrm{N}_{q_j(\nu),\,p_j(\nu)} \\ p_j(1)/q_j(1)>p_j(2)/q_j(2)>\cdots>p_j(m_j)/q_j(m_j)>0\,. \end{cases}$$

Claim 11.7. *If $j\neq k$ then it follows that*

$$(11.32)\qquad \{p_j(\nu)/q_j(\nu)\}_{1\leq\nu\leq m_j}\cap\{p_k(\nu')/q_k(\nu')\}_{1\leq\nu'\leq m_k}=\emptyset\,.$$

Proof. If contrary, there exist numbers j and $k(j\neq k)$ and $\nu,\ \nu'$ with $1\leqq \nu\leqq m_j$, $1\leqq\nu'\leqq m_k$ such that

$$(11.33)\qquad p_j(\nu)/q_j(\nu)=p_k(\nu')/q_k(\nu')\,.$$

Then Proposition 11.6 yields that $\mathrm{N}(f_jf_k)$ does not satisfy the coprimeness condition. Hence, regarding f as $f=(f_jf_k)(\prod_{i\neq j,k} f_i)$, Proposition 11.6 leads us to $\mathrm{N}(f)$ also does not satisfy the coprimeness condition, which contradicts the assumption of Proposition 2.12. Thus Claim 11.7 follows. Q.E.D.

By virtue of Proposition 11.3 and Claim 11.7, we get

$$(11.34)\qquad \mathrm{N}(f)=\sum_{j=1}^{r}\mathrm{N}(f_j)=\sum_{j=1}^{r}\sum_{\nu=1}^{m_j}\mathrm{N}_{q_j(\nu),\,p_j(\nu)}$$

with the *distinct* ratios $\bigcup_{j=1}^{r}\{p_j(\nu)/q_j(\nu)\}_{1\leq\nu\leq m_j}$.

Comparing (11.34) with

$$\begin{cases} \mathrm{N}(f)=\sum_{\mu=1}^{m}\mathrm{N}_{q(\mu),\,p(\mu)} \\ p(1)/q(1)>p(2)/q(\ )>\cdots>p(m)/q(m)\,, \end{cases}$$

we get the following disjoint decomposition of the set of ratios:

$$(11.35)\qquad \{p(\mu)/q(\mu)\}_{1\leq\mu\leq m}=\bigcup_{j=1}^{r}\{p_j(\nu)/q_j(\nu)\}_{1\leq\nu\leq m_j}\,.$$

Note that the coprimeness conditions of $\mathrm{N}(f)$ and $\mathrm{N}(f_j)$ yield the following equivalence:

$$p(\mu)/q(\mu)=p_j(\nu)/q_j(\nu)\Leftrightarrow\begin{cases} p(\mu)=p_j(\nu) \quad \text{and} \\ q(\mu)=q_j(\nu). \end{cases} \tag{11.36}$$

Thus, putting $\mathrm{M}_j\subset\{1, 2, \cdots, m\}$ for $1\leqq j\leqq r$ as

$$\mathrm{M}_j:=\{\mu\,;\ \exists\nu\ (1\leqq\nu\leqq m_j) \text{ such that } p(\mu)/q(\mu)=p_j(\nu)/q_j(\nu)\},$$

we conclude the assertion 3) in (11.28) as follows: Indeed, the assertions (a) and (b) in 3) follow from the decomposition (11.35). Then the assertion (c) follows from (11.31) and (11.36).

The proof of Proposition 2.12 is complete. Q.E.D.

§ 12. Proof of Lemma 7.5

In this section we prove Lemma 7.5.

Recall the notation: let $t\mapsto\Psi^{\sim}(t, y', \eta_1)=(X\,;\Xi, Z)(t, y', \eta_1)$ be the characteristic curve of $F(x\,;\xi, z)=G(x\,;\xi_1, \xi'', z)-\xi_n$, such that

$$\Psi^{\sim}(0, y', \eta_1)=(0, y'\,;\eta_1 dx_1, 0)\in E=T_S^*M\times\{0\}.$$

We assume the assumptions [B.1]-[B.4] of Theorem 5.1. For simplicity of notations, we *denote the variables* $(y', \eta_1)\in T_S^*M\cong\boldsymbol{C}^{n-1}\times\boldsymbol{C}$ *by* (y, η).

Lemma 12.1. *We consider the following commutative diagram of holomorphic map germs*:

$$\begin{array}{ccc} (\boldsymbol{C}, t^0) & \xrightarrow{\ Y\ } & (\boldsymbol{C}^N, y^0) \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle H} \\ & & (\boldsymbol{C}, z^0) \end{array} \tag{12.1}$$

Then, for $\forall i\geqq 1$ *it follows*

$$(1/i\,!)\partial_t^i h(t)=\sum_{1\leqq|\alpha|\leqq i}(1/\alpha\,!)(\partial_y^\alpha H)(Y(t)) \times\sum_{\{i(j,\lambda)\}}\prod_{j=1}^{N}\prod_{\lambda=1}^{\alpha_j}(1/i(j, \lambda)\,!)\partial_t^{i(j,\lambda)}Y_j(t) \tag{12.2}$$

where $\{i(j, \lambda)\,;\ 1\leqq j\leqq N,\ 1\leqq\lambda\leqq\alpha_j\}$ *runs through the following set*:

$$\begin{cases} i(j, \lambda)\geqq 1 \quad \textit{for any } (j, \lambda), \textit{ and} \\ \sum_{j=1}^{N}\sum_{\lambda=1}^{\alpha_j} i(j, \lambda)=i. \end{cases} \tag{12.3}$$

Proof. Let $t^\wedge\in(\boldsymbol{C}, t^0)$, $y^\wedge=Y(t^\wedge)\in(\boldsymbol{C}^N, y^0)$. We take the Taylor expansions of H [or Y resp.] at $y^\wedge$ [at $t^\wedge$]:

$$\begin{cases} H(y)=H(y^{\wedge})+\sum_{|\alpha|\geqq 1}(1/\alpha !)\partial_y^\alpha H(y^{\wedge})(y-y^{\wedge})^\alpha \\ Y_j(t)=Y_j(t^{\wedge})+\sum_{i=1}^{\infty}(1/i!)\partial_t^i Y_j(t^{\wedge})(t-t^{\wedge})^i \qquad \text{for } 1\leqq j\leqq N. \end{cases}$$

Substituting $y_j=Y_j(t)$, $y_j^{\wedge}=Y_j(t^{\wedge})$, we have the following expression of $h(t)$:

$$\begin{aligned} & h(t)(=H(Y(t))) \\ & =H(Y(t^{\wedge}))+\sum_{|\alpha|\geqq 1}(1/\alpha !)\partial_y^\alpha H(Y(t^{\wedge}))\{Y(t)-Y(t^{\wedge})\}^\alpha \\ & =h(t^{\wedge})+\sum_{|\alpha|\geqq 1}(1/\alpha !)\partial_y^\alpha H(Y(t^{\wedge}))\prod_{j=1}^{N}\{\sum_{i(j)\geqq 1}(1/i(j)!)\partial_t^{i(j)}Y_j(t^{\wedge})(t-t^{\wedge})^{i(j)}\}^{\alpha_j}. \end{aligned}$$

Thus the coefficient of $(t-t^{\wedge})^i$ in the expansion of $h(t)$ is given by

$$\sum_{|\alpha|\geqq 1}(1/\alpha !)\partial_y^\alpha H(Y(t^{\wedge}))\sum_{\{i(j,\lambda)\}}\prod_{j=1}^{N}\prod_{\lambda=1}^{\alpha_j}(1/i(j,\lambda)!)\partial_t^{i(j,\lambda)}Y_j(t^{\wedge})$$

where $\{i(j,\lambda)\}$ satisfies (12.3). Note that $i(j,\lambda)\geqq 1$ yields

$$i=\sum_{j=1}^{N}\sum_{\lambda=1}^{\alpha_j}i(j,\lambda)\geqq\sum_{j=1}^{N}\alpha_j=|\alpha|.$$

Hence α runs through $1\leqq|\alpha|\leqq i$.

The proof of Lemma 12.1 is complete. Q.E.D.

Now we prove Lemma 7.5. Since the assertion 6) in Lemma 7.5 is trivial by the fact $\partial_{\xi_n}F\equiv -1$, we only have to prove the assertions 1)-5).

We use induction on $i\geqq 1$. Note that

$$\mathrm{N}((y)^q 1)\subset\mathrm{N}(f) \quad\text{and}\quad \mathrm{N}((y)^q(\eta)1)\Subset\mathrm{N}(f)$$

which are direct consequences of the definitions

$$q:=\mathrm{ord}[f(y,0)] \quad\text{and}\quad f(y,\eta):=F(0,y;\eta dx_1,0).$$

Hence it suffices to prove Lemma 7.5 for $1\leqq i\leqq q$.

Step 0. *When* $i=1$.

1) Since $\partial_t X_1(0,y,\eta)=\partial_{\xi_1}F(o,y;\eta dx_1,0)=\partial_\eta f(y,\eta)$, it follows

$$\mathrm{N}[(\eta)\{\partial_t X_1(0,y,\eta)-\partial_\eta f(y,\eta)\}]=\mathrm{N}(0)=\emptyset\Subset\mathrm{N}(f).$$

2) For $2\leqq j\leqq n$, we have $\Xi_j(0,y,\eta)=0$ which yields

$$\begin{aligned} \partial_t\Xi_j(0,y,\eta)&=-\Xi_j(0,y,\eta)\partial_z F(0,y;\eta dx_1,0)-\partial_{x_j}F(0,y;\eta dx_1,0) \\ &=-\partial_{y_j}f(y,\eta). \end{aligned}$$

Hence we get

$$\mathrm{N}[(y)\partial_t\Xi_j(0,y,\eta)]=\mathrm{N}[(y)\partial_{y_j}f(y,\eta)]\subset\mathrm{N}(f).$$

3) Since $\Xi_j(0,y,\eta)=\delta_{1j}\eta$ (δ_{1j} is a Kronecker's delta) for $1\leqq j\leqq n$, we get

$$\partial_t Z(0, y, \eta) = \eta \partial_{\xi_1} F(0, y; \eta dx_1, 0) = \eta \partial_\eta f(y, \eta).$$

Thus we have

$$\mathrm{N}[\partial_t Z(0, y, \eta)] \subset \mathrm{N}(f).$$

4) For $2 \leqq j \leqq n-1$, the assumption [B.1] yields $\partial_{\xi_j} F(0, y; \eta dx_1, 0) \in (y, \eta)$ hence we get

$$\partial_t X_j(0, y, \eta) = \partial_{\xi_j} F(0, y; \eta dx_1, 0) \in (y, \eta).$$

5) Since $\partial_t \Xi_1(0, y, \eta) = -\eta \partial_z F(0, y; \eta dx_1, 0) - \partial_{x_1} F(0, y; \eta dx_1, 0)$

$$\equiv -\partial_{x_1} F(0, y; 0, 0) \quad \mathrm{mod}(\eta)$$

the assumption [B.4], that is, $\mathrm{ord}[F(x; 0, 0)] = q$ implies

$$\partial_t \Xi_1(0, y, \eta) \in (y)^{q-1} + (\eta).$$

The assertions 1)-5) in Lemma 7.5 have been proved for $i=1$.

Next let $i \geqq 2$. We assume that Lemma 7.5 is true for i' $(1 \leqq i' \leqq i-1)$.

Step 1. *Proof of the assertion* 1).
We use the following

Notation 12.2. Set $\boldsymbol{Z}_+ := \boldsymbol{N} \cup \{0\}$. For a multi-index

$$(\alpha; \beta, k) = (\alpha_1, \alpha'', \alpha_n; \beta_1, \beta'; k) \in \boldsymbol{Z}_+^n \times \boldsymbol{Z}_+^n \times \boldsymbol{Z}_+$$

we set

$$F^{(\alpha_1, \beta', k)}_{(\alpha'', \alpha_n, \beta_1)}(y, \eta) := \partial_x^\alpha \partial_\xi^\beta \partial_z^k F(x; \xi, z)|_{(x; \xi, z) = (0, y; \eta dx_1, 0)} \tag{12.4}$$

Note that the sub-index $(\alpha'', \alpha_n, \beta_1)$ is associate with the tangential variables (x'', x_n, ξ_1) of $E = T^*_S M \times \{0\} \cong T^*_S M$.

According to Lemma 12.1, it follows

$$\begin{aligned}
(12.5) \qquad & (1/i!)\partial_t^i X_1(0, y, \eta) \\
&= (1/i)[(1/(i-1)!)\partial_t^{i-1} \partial_{\xi_1} F(X; \Xi, Z)]|_{t=0} \\
&= (1/i) \sum_{1 \leq |\alpha| + |\beta| + k \leq i-1} (\alpha! \beta! k!)^{-1} F^{(\alpha_1, \beta', k)}_{(\alpha'', \alpha_n, \beta_1 + 1)} \\
&\quad \times \sum_{\{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})
\end{aligned}$$

where $K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})$ is given by

(12.6) $K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})$

$$:= \prod_{\lambda=1}^{\alpha_1} (1/u(1, \lambda)!)\partial_t^{u(1,\lambda)} X_1(0, y, \eta) \prod_{j=2}^{n} \prod_{\lambda=1}^{\beta_j} (1/u(j, \lambda)!)\partial_t^{u(j,\lambda)} \Xi_j(0, y, \eta)$$

$$\times \prod_{\lambda=1}^{k} (1/u(n+1, \lambda)!)\partial_t^{u(n+1,\lambda)} Z(0, y, \eta) \prod_{j=2}^{n} \prod_{\lambda=1}^{\alpha_j} (1/d(j, \lambda)!)\partial_t^{d(j,\lambda)} X_j(0, y, \eta)$$

$$\times \prod_{\lambda=1}^{\beta_1} (1/d(1, \lambda)!)\partial_t^{d(1,\lambda)} \Xi_1(0, y, \eta)$$

and where $\{u(j, \lambda), d(j, \lambda)\}$ runs through the following set:

$$\text{(12.7)} \qquad \begin{cases} u(j, \lambda) \geqq 1, \quad d(j, \lambda) \geqq 1 \quad \text{and} \\ \sum_{j=1}^{n+1} \sum_{\lambda} u(j, \lambda) + \sum_{j=1}^{n} \sum_{\lambda} d(j, \lambda) = i-1. \end{cases}$$

We classify the proof of the step 1 into the following four cases:

$$\begin{cases} \text{Case 1.1.} \quad \alpha_1 + |\beta'| + k \geqq 1. \\ \text{Case 1.2.} \quad \alpha_1 + |\beta'| + k = 0, \quad \beta_1 \geqq 1. \\ \text{Case 1.3.} \quad \alpha_1 + |\beta'| + k + \beta_1 = 0, \quad |\alpha''| \geqq 1. \\ \text{Case 1.4.} \quad \alpha_1 + |\beta'| + k + \beta_1 + |\alpha''| = 0. \end{cases}$$

Case 1.1. In this case we note that $K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})$ belongs to the ideal

$$(\partial_t^{u(1,\lambda)} X_1) + \sum_{j=2}^{n} (\partial_t^{u(j,\lambda)} \Xi_j) + (\partial_t^{u(n+1,\lambda)} Z)|_{t=0}.$$

Thus it follows

$$\text{(12.8)} \quad (y)^{i-1}(\eta)(K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}))$$

$$\subset (y)^{i-1}(\eta)\left[(\partial_t^{u(1,\lambda)} X_1) + \sum_{j=2}^{n} (\partial_t^{u(j,\lambda)} \Xi_j) + (\partial_t^{u(n+1,\lambda)} Z)\right]\Big|_{t=0}$$

$$\subset (y)^{i-u(1,\lambda)}[(y)^{u(1,\lambda)-1}(\eta)(\partial_t^{u(1,\lambda)} X_1|_{t=0})]_{(1)}$$

$$+ \sum_{j=2}^{n} (y)^{i-1-u(j,\lambda)}(\eta)[(y)^{u(j,\lambda)}(\partial_t^{u(j,\lambda)} \Xi_j|_{t=0})]_{(2)}$$

$$+ (y)^{i-u(n+1,\lambda)}(\eta)[(y)^{u(n+1,\lambda)-1}(\partial_t^{u(n+1,\lambda)} Z|_{t=0})]_{(3)}.$$

Note that (12.7) yields

$$i-1 \geqq \sum_{j=1}^{n+1} u(j, \lambda) \geqq u(j, \lambda) \qquad \text{for } 1 \leqq j \leqq n+1$$

and that the three blackets $[\ ]_{(\nu)}$ $(\nu=1, 2, 3)$ in the rightest hand side of (12.8) satisfy

$$\mathrm{N}([\ ]_{(\nu)}) \subset \mathrm{N}(f)$$

which is a consequence of the inductive assumption. Hence we get

$$\begin{aligned}&\mathrm{N}[(y)^{i-1}(\eta)K(\alpha,\ \beta,\ k\ ;\ \{u(j,\ \lambda),\ d(j,\ \lambda)\})]\\&\qquad\subset\text{convex hull}[\mathrm{N}((y)1)\cup\mathrm{N}((\eta)1)]+\mathrm{N}(f)\\&\qquad\Subset\mathrm{N}(f).\end{aligned}$$

Case 1.2. In this case, note that

$$K(\alpha,\ \beta,\ k\ ;\ \{u(j,\ \lambda),\ d(j,\ \lambda)\})\in\prod_{\lambda=1}^{\beta_1}(\partial_t^{d(1,\lambda)}\Xi_1|_{t=0}).$$

Hence it follows

$$\begin{aligned}&(y)^{i-1}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})(K(\alpha,\ \beta,\ k\ ;\ \{u(j,\ \lambda),\ d(j,\ \lambda)\})\\&\quad\subset(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})\prod_{\lambda=1}^{\beta_1}(y)^{d(1,\lambda)-1}(\partial_t^{d(1,\lambda)}\Xi_1|_{t=0})\\&\quad\subset(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})\prod_{\lambda=1}^{\beta_1}[(y)^{q-1}+(\eta)]\end{aligned}$$

by the inductive assumption. Note that

$$\prod_{\lambda=1}^{\beta_1}[(y)^{q-1}+(\eta)]\subset(y)^{q-1}+(\eta)^{\beta_1}$$

which yields

$$\begin{aligned}(12.9)\qquad&(y)^{i-1}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})(K(\alpha,\ \beta,\ k\ ;\ \{u(j,\ \lambda),\ d(j,\ \lambda)\})\\&\quad\subset(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})[(y)^{q-1}+(\eta)^{\beta_1}].\end{aligned}$$

We first observe

$$(12.10)\qquad \mathrm{N}[(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})(\eta)^{\beta_1}]\Subset\mathrm{N}(f).$$

Indeed, we can write

$$\begin{aligned}(12.11)\qquad&\mathrm{N}[(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}(\eta)(F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)})(\eta)^{\beta_1}]\\&\quad=\mathrm{N}[(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}1]+\mathrm{N}[(\eta)^{\beta_1+1}F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)}]\\&\quad=\mathrm{N}[(y)^{i-1-\Sigma\{d(1,\lambda)-1\}-|\alpha''|-\alpha_n}1]+\mathrm{N}[(y'')^{\alpha''}(y_n)^{\alpha_n}(\eta)^{\beta_1+1}F_{(\alpha'',\alpha_n,\beta_1+1)}^{(0,0,0)}]\end{aligned}$$

by virtue of the additivity of Newton polygons. Note that

$$i-1-\sum_{\lambda}\{d(1,\ \lambda)-1\}-|\alpha''|-\alpha_n\geqq i-1-\sum_{j,\lambda}d(j,\ \lambda)+\beta_1\geqq\beta_1>0$$

which yields the first term in the rightest hand side of (12.11) is contained in $\mathrm{N}((y)1)$. On the other hand, we have

$$\mathrm{N}[(y'')^{\alpha''}(y_n)^{\alpha_n}(\eta)^{\beta_1+1}F_{(\alpha''\alpha_n,\beta_1+1)}^{(0,0,0)}]\subset\mathrm{N}(f).$$

Hence (12.11) implies (12.10).

Next we observe

$$\mathrm{N}[(y)^{i-1-\Sigma\{d(1,\lambda)-1\}}(\eta)F^{(0,0,0)}_{(\alpha'',\alpha_n,\beta_1+1)})(y)^{q-1}]\in\mathrm{N}(f). \tag{12.12}$$

Indeed, we have

$$(y)^{i-1-\Sigma\{d(i,\lambda)-1\}}(\eta)(y)^{q-1}\subset(y)^q(\eta)$$

which yields (12.12).

By virtue of (12.10), (12.12), the inclusion (12.9) implies

$$\mathrm{N}[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha'',\alpha_n,\beta_1+1)})K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\})]\in\mathrm{N}(f)$$

in the case 1.2 as desired.

Case 1.3. In this case we have

$$K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\})\in\prod_{j=2}^{n-1}\prod_{\lambda=1}^{\alpha_j}(\partial_t^{d(j,\lambda)}X_j|_{t=0}).$$

Thus it follows

$$\begin{aligned}&\mathrm{N}[(y)^{i-1}(\eta)(F^{(0,0,0)}_{(\alpha'',\alpha_n,1)})(K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\})]\\ &\subset\mathrm{N}[(\eta)(F^{(0,0,0)}_{(\alpha'',\alpha_n,1)})\prod_{j=2}^{n-1}\prod_{\lambda=1}^{\alpha_j}(y)^{d(j,\lambda)-1}(\partial_t^{d(j,\lambda)}X_j|_{t=0})]\\ &\quad+\mathrm{N}[(y)^{i-1-\Sigma''\{d(j,\lambda)-1\}}1]\quad\Big(\text{we denote }\Sigma'':=\sum_{j=2}^{n-1}\Big)\\ &\subset\mathrm{N}[(y,\eta)^{|\alpha''|}(\eta)F^{(0,0,0)}_{(\alpha'',\alpha_n,1)}]+\mathrm{N}[(y)^{i-1-\Sigma''\{d(j,\lambda)-1\}}1]\end{aligned}\tag{12.13}$$

which is a consequence of the inductive assumption. Since (12.7) yields

$$\begin{aligned}i-1-\sum_{j=2}^{n-1}\sum_{\lambda=1}^{\alpha_j}\{d(j,\lambda)-1\}&=|\alpha''|+i-1-\sum_{j=2}^{n-1}\sum_{\lambda=1}^{\alpha_j}d(j,\lambda)\\ &\geqq|\alpha''|+\alpha_n\end{aligned}$$

we have

$$\begin{aligned}&\mathrm{N}[(\eta)F^{(0,0,0)}_{(\alpha'',\alpha_n,1)}]+\mathrm{N}[(y)^{i-1-\Sigma''\{d(j,\lambda)-1\}}1]\\ &\quad\subset\mathrm{N}[(y)^{|\alpha''|+\alpha_n}(\eta)F^{(0,0,0)}_{(\alpha'',\alpha_n,1)}]\\ &\quad\subset\mathrm{N}(f).\end{aligned}$$

Hence (12.13) implies

$$\begin{aligned}&\mathrm{N}[(y)^{i-1}(\eta)F^{(0,0,0)}_{(\alpha'',\alpha_n,1)}K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\})]\\ &\quad\subset\mathrm{N}[(y,\eta)^{|\alpha''|}1]+\mathrm{N}(f)\\ &\quad\in\mathrm{N}(f)\end{aligned}$$

as desired, since $|\alpha''|\geqq 1$ in the case 1.3.

Case 1.4. This case contributes to (12.5) as the following sum:

$$(1/i)\sum_{\alpha_n=1}^{i-1}(1/\alpha_n!)F^{(0,0,0)}_{(0,\alpha_n,1)}\sum_{\{d(n,\lambda)\}}\prod_{\lambda=1}^{\alpha_n}(1/d(n,\lambda)!)\partial_t^{d(n,\lambda)}X_n|_{t=0}\tag{12.14}$$

where $\{d(n, \lambda); 1 \leqq \lambda \leqq \alpha_n\}$ runs through

(12.15) $$d(n, \lambda) \geqq 1 \quad \text{and} \quad \sum_{\lambda=1}^{\alpha_n} d(n, \lambda) = i-1.$$

If $\alpha_n < i-1$, then it follows

$$\begin{aligned} \mathrm{N}[(y)^{i-1}(\eta) F_{(0, \alpha_n, 1)}^{(0, 0, 0)}] &\subset \mathrm{N}[(y)^{i-1-\alpha_n} 1] + \mathrm{N}[(y)^{\alpha_n}(\eta) F_{(0, \alpha_n, 1)}^{(0, 0, 0)}] \\ &\subset \mathrm{N}[(y)1] + \mathrm{N}(f) \\ &\in \mathrm{N}(f). \end{aligned}$$

It remains the terms in the case $\alpha_n = i-1$. Note that (12.15) yields

$$d(n, \lambda) = 1 \qquad \text{for all } 1 \leqq \lambda \leqq \alpha_n = i-1$$

in this case. Hence it suffices to observe the following term:

(12.16) $$(1/i)(1/(i-1)!) F_{(0, i-1, 1)}^{(0, 0, 0)} \{\partial_t X_n(0, y, \eta)\}^{i-1}.$$

Since

$$\partial_t X_n(0, y, \eta) \equiv -1 \quad \text{and} \quad F_{(0, i-1, 1)}^{(0, 0, 0)} = \partial_{y_n}^{i-1} \partial_\eta f(y, \eta)$$

the term (12.16) is nothing but

$$(1/i!)(-1)^{i-1} \partial_{y_n}^{i-1} \partial_\eta f(y, \eta).$$

Thus the proof in the cases 1.1-1.4 leads us to

$$\mathrm{N}[(y)^{i-1}(\eta)\{(1/i!)\partial_t^i X_1(0, y, \eta) - (1/i!)(-1)^{i-1} \partial_{y_n}^{i-1} \partial_\eta f(y, \eta)\}] \in \mathrm{N}(f)$$

which is the assertion 1) of Lemma 7.5.

Step 2. *Proof of the assertion* 2).

We use the following simpler notation than Notation 12.2:

Notation 12.3. For a multi-index $(\alpha; \beta, k) = (\alpha_1, \alpha'; \beta_1, \beta', k)$, we set

(12.4)′ $$F_{(\alpha', \beta_1)}^{(\alpha_1, \beta', k)}(y, \eta) := \partial_x^\alpha \partial_\xi^\beta \partial_z^k F(x; \xi, z)|_{(x; \xi, z) = (0, y; \eta dx_1, 0)}$$

that is, we denote (α_1, α'') in Notation 12.2 by α'.

By virtue of Lemma 12.1, for $2 \leqq j \leqq n$, we have the following expression:

$$\begin{aligned} &(1/i!)\partial_t^i \Xi_j(0, y, \eta) \\ &\quad = (1/i)[(1/(i-1)!)\partial_t^{i-1}\{-\Xi_j \partial_z F(X; \Xi, Z) - \partial_{x_j} F(X; \Xi, Z)\}|_{t=0}] \\ &\quad = (1/i) \sum_{1 \leq |\alpha| + |\beta| + k \leq i-1} (\alpha! \beta! k!)^{-1} \partial_{(\alpha', \beta_1)}^{(\alpha_1, \beta', k)} \{-\xi_j \partial_z F - \partial_{x_j} F\}|_{t=0} \\ &\qquad \times \sum_{\{u(j, \lambda), d(j, \lambda)\}} K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \end{aligned}$$

where $K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})$ is defined by (12.6), and where the index $\{u(j, \lambda), d(j, \lambda)\}$ runs through the set determined by (12.7).

We classify the proof in the step 2 into the following five cases:

$$\begin{cases} \text{Case 2.1.} & |\beta'|+k\geqq 1. \\ \text{Case 2.2.} & |\beta'|+k=0, \quad \alpha_1\geqq 1, \quad \text{and} \quad \beta_1\geqq 1. \\ \text{Case 2.3.} & |\beta'|+k+\beta_1=0, \quad \alpha_1\geqq 1. \\ \text{Case 2.4.} & |\beta'|+k+\alpha_1=0, \quad \beta_1\geqq 1. \\ \text{Case 2.5.} & |\beta'|+k+\alpha_1+\beta_1=0. \end{cases}$$

Case 2.1. Note that in this case we have

$$K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\} \in \sum_{j=2}^{n}(\partial_t^{u(j,\lambda)}\Xi_j)+(\partial_t^{u(n+1,\lambda)}Z)|_{t=0}.$$

Hence it follows

$$\begin{aligned}&(y)^i(K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}))\\ &\subset \sum_{j=2}^{n}(y)^{i-u(j,\lambda)}[(y)^{u(j,\lambda)}\partial_t^{u(j,\lambda)}\Xi_j|_{t=0}]_{(1)}\\ &\quad +(y)^{i+1-u(n+1,\lambda)}[(y)^{u(n+1,\lambda)-1}\partial_t^{u(n+1,\lambda)}Z|_{t=0}]_{(2)}.\end{aligned}$$

Since the inductive assumption yields the blackets $[\ ]_{(\nu)}$ $(\nu=1, 2)$ satisfy

$$\mathrm{N}([\cdots]_{(\nu)})\subset \mathrm{N}(f),$$

we get

$$\mathrm{N}[(y)^iK(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}]\subset \mathrm{N}(f)$$

in the case 2.1.

Case 2.2. In this case we note that the inductive assumption yields

$$\begin{aligned}&(y)^i(K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}))\\ &\subset (y)^i(\partial_t^{u(1,\lambda)}X_1(0, y, \eta))(\partial_t^{d(1,\lambda)}\Xi_1(0, y, \eta))\\ &\subset (y)^{i+2-u(1,\lambda)-d(1,\lambda)}[(y)^{u(1,\lambda)-1}(\partial_t^{u(1,\lambda)}X_1|_{t=0})]\\ &\quad \times[(y)^{d(1,\lambda)-1}(\partial_t^{d(1,\lambda)}\Xi_1|_{t=0})]\\ &\subset (y)^3[(y)^{u(1,\lambda)-1}(\partial_t^{u(1,\lambda)}X_1|_{t=0})\{(y)^{q-1}+(\eta)\}]\end{aligned}$$

since $i-1\geqq u(1, \lambda)+d(1, \lambda)$. Then the following facts

$$\mathrm{N}[(y)^3(y)^{q-1}]\Subset \mathrm{N}(f) \quad \text{and} \quad \mathrm{N}[(y)^{u(1,\lambda)-1}(\eta)\partial_t^{u(1,\lambda)}X_1(0, y, \eta)]\subset \mathrm{N}(f)$$

yield

$$\mathrm{N}[(y)^iK(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})]\subset \mathrm{N}(f)$$

in the case 2.2.

To consider the remaining Cases 2.3-2.5, we need the

Remark 12.4. If $|\beta'|+k=0$ and $2\leqq j\leqq n$, then it follows

$$\partial^{(\alpha_1,0,0)}_{(\alpha',\beta_1)}\{-\xi_j\partial_z F-\partial_{x_j}F\}(0, y; \eta dx_1, 0)=-F^{(\alpha_1,0,0)}_{(\alpha'+e_j,\beta_1)}. \tag{12.17}$$

Proof. Since the derivation does not involve that of in the ξ_j-direction we have

$$\partial^{(\alpha_1,0,0)}_{(\alpha',\beta_1)}\{-\xi_j\partial_z F-\partial_{x_j}F\}=-\xi_j\partial^{(\alpha_1,0,1)}_{(\alpha',\beta_1)}F-\partial^{(\alpha_1,0,0)}_{(\alpha'+e_j,\beta_1)}F.$$

Thus, restricting this on $E=T^*_S M\times\{0\}$ and using $\Xi_j(0, y, \eta)=0$, we get (12.17) as desired. Q.E.D.

Case 2.3. In this case, Remark 12.4 yields

$$([\partial^{(\alpha_1,0,0)}_{(\alpha',0)}\{-\xi_j\partial_z F-\partial_{x_j}F\}(0, y; \eta dx_1, 0)]K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}))$$
$$\subset(F^{(\alpha_1,0,0)}_{(\alpha'+e_j,0)})(\partial_t^{u(1,\lambda)}X_1(0, y, \eta)).$$

Thus it suffices to show

$$\mathrm{N}[(y)^i F^{(\alpha_1,0,0)}_{(\alpha'+e_j,0)}\partial_t^{u(1,\lambda)}X_1(0, y, \eta)]\subset\mathrm{N}(f). \tag{12.18}$$

We first note

$$F^{(\alpha_1,0,0)}_{(\alpha'+e_j,0)}\in(y)^{q-1-|\alpha'|-\alpha_1}+(\eta). \tag{12.19}$$

Indeed, the assumption [B.4] implies

$$\partial^{(\alpha_1,0,0)}_{(\alpha'+e_j,0)}F(x; 0, 0)\in(x)^{q-\alpha_1-|\alpha'|-1}.$$

Hence we get

$$F^{(\alpha_1,0,0)}_{(\alpha'+e_j,0)}(y, \eta)|_{\eta=0}\in(y)^{q-1-|\alpha'|-\alpha_1}.$$

Thus (12.19) follows.

By virtue of (12.19) we have the inclusion

$$\begin{aligned}(y)^i(F^{(\alpha_1,0,0)}_{(\alpha'+e_j,0)})(\partial_t^{u(1,\lambda)}X_1(0, y, \eta))\\ \subset(y)^{i+q-1-|\alpha'|-\alpha_1}(\partial_t^{u(1,\lambda)}X_1|_{t=0})+(y)^i(\eta)(\partial_t^{u(1,\lambda)}X_1|_{t=0}).\end{aligned} \tag{12.20}$$

Note that

$$i+q-1-|\alpha'|-\alpha_1\geqq q+i-1-\sum_{j=2}^{n}\sum_{\lambda=1}^{\alpha_j}d(j, \lambda)-\sum_{\lambda=1}^{\alpha_1}u(1, \lambda)=q$$

since $|\beta'|+k+\beta_1=0$ in the case 2.3. Hence $q:=\mathrm{ord}[f(y, \eta)]$ yields

$$\mathrm{N}[(y)^{i+q-1-|\alpha'|-\alpha_1}1]\subset\mathrm{N}[(y)^q]\subset\mathrm{N}(f).$$

On the other hand, it follows

$$\begin{aligned}&\mathrm{N}[(y)^i(\eta)(\partial_t^{u(1,\lambda)}X_1|_{t=0})\\ &=\mathrm{N}[(y)^{i+1-u(1,\lambda)}1]+\mathrm{N}[(y)^{u(1,\lambda)-1}(\eta)\partial_t^{u(1,\lambda)}X_1|_{t=0}]\end{aligned}$$

$$\subset \mathrm{N}(f)$$

by the inductive assumption. Thus (12.20) implies (12.18) as desired.

Case 2.4. In this case we have

$$K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \in \prod_{\lambda=1}^{\beta_1} (\partial_t^{d(1,\lambda)} \Xi_1|_{t=0}).$$

Thus, by Remark 12.4, it suffices for the proof in the case 2.4 to show

$$\mathrm{N}[(y)^i (F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)}) \prod_{j=1}^{\beta_1} \partial_t^{d(1,\lambda)} \Xi_1|_{t=0}] \subset \mathrm{N}(f). \tag{12.21}$$

The left hand side of (12.21) is contained in

$$\begin{aligned} &\mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}} (F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)}) \prod_{\lambda=1}^{\beta_1} (y)^{d(1,\lambda)-1} \partial_t^{d(1,\lambda)} \Xi_1|_{t=0}] \\ &\subset \mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}} F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)} \prod_{\lambda=1}^{\beta_1} \{(y)^{q-1}+(\eta)\}] \\ &\subset \mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}} F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)} \{(y)^{q-1}+(\eta)^{\beta_1}\}]. \end{aligned} \tag{12.22}$$

Note that the inequality

$$i - \sum_{\lambda=1}^{\beta_1} \{d(1,\lambda)-1\} \geqq \beta_1+1+i-1-\sum_{\lambda=1}^{\beta_1} d(1,\lambda) \geqq \beta_1+1$$

yields

$$\mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}} (y)^{q-1} 1] \subset \mathrm{N}[(y)^q] \subset \mathrm{N}(f).$$

On the other hand, we have

$$\begin{aligned} &\mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}} (\eta)^{\beta_1} F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)}] \\ &\quad \subset \mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}-|\alpha'|-1} 1] + \mathrm{N}[(y)^{|\alpha'|+1} (\eta)^{\beta_1} F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)}] \end{aligned}$$

where we note

$$\begin{aligned} i - \sum_{\lambda=1}^{\beta_1} \{d(1,\lambda)-1\} - |\alpha'| - 1 &\geqq \beta_1 + i - 1 - \sum_{\lambda=1}^{\beta_1} d(1,\lambda) - \sum_{j=2}^{n} \sum_{\lambda=1}^{\alpha_j} d(j,\lambda) \\ &\geqq \beta_1 \\ &> 0. \end{aligned}$$

Thus we get

$$\begin{aligned} \mathrm{N}[(y)^{i-\Sigma\{d(1,\lambda)-1\}} (\eta)^{\beta_1} F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)}] &\subset \mathrm{N}[(y)^{|\alpha'|+1} (\eta)^{\beta_1} F^{(0,0,0)}_{(\alpha'+e_j, \beta_1)}] \\ &= \mathrm{N}[(y)^{|\alpha'|+1} (\eta)^{\beta_1} \partial_y^{\alpha'+e_j} \partial_\eta^{\beta_1} f(y, \eta)] \\ &\subset \mathrm{N}(f). \end{aligned}$$

Hence (12.22) implies (12.21) as desired.

Case 2.5. Since $|\beta'|+k+\alpha_1+\beta_1=0$, Remark 12.4 yields that it suffices to show

$$\mathrm{N}[(y)^i F^{(0,0,0)}_{(\alpha'+e_j,0)}]\subset \mathrm{N}(f). \tag{12.23}$$

The left hand side is contained in

$$\begin{aligned}\mathrm{N}[(y)^{i-1-|\alpha'|}(y)^{|\alpha'|+1}F^{(0,0,0)}_{(\alpha'+e_j,0)}]&\subset \mathrm{N}[(y)^{|\alpha'|+1}F^{(0,0,0)}_{(\alpha'+e_j,0)}]\\ &=\mathrm{N}[(y)^{|\alpha'|+1}\partial_y^{\alpha'+e_j}f(y,\eta)]\\ &\subset \mathrm{N}(f).\end{aligned}$$

Indeed, we have

$$i-1-|\alpha'|\geqq i-1-\sum_{j=2}^{n}\sum_{\lambda=1}^{\alpha_j} d(j,\lambda)=0$$

which yields the first inclusion, and it is trivial that the second inclusion holds.

By the cases 2.1–2.5, we get

$$\mathrm{N}[(y)^i\partial_t^i\Xi_j(0,y,\eta)]\subset \mathrm{N}(f) \qquad \text{for } 2\leqq j\leqq n$$

as desired.

Step 3. *Proof of the assertion* 3).

We use the Leibniz's rule which makes our proof of 3) reduce to another assertions of Lemma 7.5.

We first note that the Leibniz's rule yields

$$\begin{aligned}&(1/i!)\partial_t^i Z(0,y,\eta)\\ &=(1/i!)\partial_t^{i-1}\left\{\sum_{j=1}^{n}\Xi_j\partial_{\xi_j}F(X;\Xi,Z)\right\}\Big|_{t=0}\\ &=(1/i!)\sum_{j=1}^{n}\sum_{s=0}^{i-1}C(i-1,s)\{\partial_t^s\Xi_j|_{t=0}\}\partial_t^{i-1-s}\{\partial_{\xi_j}F(X;\Xi,Z)\}|_{t=0}\end{aligned}$$

where $C(r,s):=(r!)/\{s!(r-s)!\}$. We classify the proof of the assertion 3) into the following four cases:

$$\begin{cases}\text{Case 3.1.} & j\geqq 2,\ s\geqq 1.\\ \text{Case 3.2.} & j\geqq 2,\ s=0.\\ \text{Case 3.3.} & j=1,\ s\geqq 1.\\ \text{Case 3.4.} & j=1,\ s=0.\end{cases}$$

Case 3.1. Since $s\leqq i-1$, the inductive assumption yields

$$\begin{aligned}\mathrm{N}[(y)^{i-1}\partial_t^s\Xi_j(0,y,\eta)]&=\mathrm{N}[(y)^{i-1-s}(y)^s\partial_t^s\Xi_j(0,y,\eta)]\\ &\subset \mathrm{N}[(y)^s\partial_t^s\Xi_j(0,y,\eta)]\\ &\subset \mathrm{N}(f).\end{aligned}$$

Case 3.2. Since $\Xi_j(0, y, \eta)=0$ for $j\geqq 2$,

$$\mathrm{N}[(y)^{i-1}\Xi_j(0, y, \eta)]=\mathrm{N}(0)=\emptyset\subset \mathrm{N}(f).$$

Case 3.3. Note that the inductive assumption yields

$$\begin{aligned}&(y)^{i-1}(\partial_t^s\Xi_1(0, y, \eta))(\partial_t^{i-1-s}\{\partial_{\xi_1}F(X; \Xi, Z)\}|_{t=0})\\&=[(y)^{s-1}(\partial_t^s\Xi_1(0, y, \eta))][(y)^{i-s-1}(\partial_t^{i-s}X_1|_{t=0})](y)\\&\subset[(y)^{q-1}+(\eta)](y)[(y)^{i-s-1}(\partial_t^{i-s}X_1|_{t=0})].\end{aligned}$$

Since $\mathrm{N}((y)^q 1)\subset \mathrm{N}(f)$ and since

$$\begin{aligned}\mathrm{N}[(y)^{i-s-1}(\eta)\partial_t^{i-s}X_1(0, y, \eta)]&=\mathrm{N}[(y)^{i-s-1}(\eta)(-1)^{i-s-1}\partial_{y_n}^{i-s-1}\partial_\eta f]\\&\subset \mathrm{N}(f)\end{aligned}$$

we get

$$\mathrm{N}[(y)^{i-1}\{\partial_t^s\Xi_1(0, y, \eta)\}\partial_t^{i-1-s}\{\partial_{\xi_1}F(X; \Xi, Z)\}|_{t=0}]\subset \mathrm{N}(f)$$

in the case 3.3.

Case 3.4. Note that the assertion 1) of Lemma 7.5 has been already show for i (Step 1). Thus we have

$$\begin{aligned}&\mathrm{N}[(y)^{i-1}\Xi_1(0, y, \eta)\partial_t^{i-1}\{\partial_{\xi_1}F(X; \Xi, Z)\}|_{t=0}]\\&=\mathrm{N}[(y)^{i-1}(\eta)\partial_t^iX_1(0, y, \eta)]\\&\subset \mathrm{N}(f).\end{aligned}$$

By the cases 3.1–3.4, we get the assertion 3) of Lemma 7.5.

Step 4. *Proof of the assertions* 4) *and* 5).

Since the assertion 4) is trivial for $i\geqq 2$, we only have to prove the assertion 5). We may assume $q\geqq 2$, since if $q=1$ then the assertion 5) is trivial.

We use Notation 12.3. By virtue of Lemma 12.1 we have:

$$\begin{aligned}&(1/i!)\partial_t^i\Xi_1(0, y, \eta)\\&=(1/i)[(1/(i-1)!)\partial_t^{i-1}\{-\Xi_1\partial_zF(X; \Xi, Z)-\partial_{x_1}F(X; \Xi, Z)\}|_{t=0}\\&=(1/i)\sum_{1\leqq|\alpha|+|\beta|+k\leqq i-1}(\alpha!\beta!k!)^{-1}\partial_{(\alpha';\beta_1)}^{(\alpha_1,\beta',k)}\{-\xi_1\partial_zF-\partial_{x_1}F\}(0, y, \eta dx_1, 0)\\&\quad\times\sum_{\{u(j,\lambda),d(j,\lambda)\}}K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})\end{aligned}$$

where $\{u(j, \lambda), d(j, \lambda)\}$ and $K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\})$ are defined by (12.7) and (12.6) respectively.

By the Leibniz's rule it follows

$$\begin{aligned}&\partial_{(\alpha',\beta_1)}^{(\alpha_1,\beta',k)}\{-\xi_1\partial_zF-\partial_{x_1}F\}(0, y; \eta dx_1, 0)\\&=-\eta F_{(\alpha',\beta_1)}^{(\alpha_1,\beta',k+1)}-F_{(\alpha',\beta_1-1)}^{(\alpha_1,\beta',k)}-F_{(\alpha',\beta_1)}^{(\alpha_1+1,\beta',k)}.\end{aligned}$$

Remark 12.5. The following inclusion of ideals holds:

$$(y)^{i-1}(\{-\eta F^{(\alpha_1,\beta',k+1)}_{(\alpha',\beta_1)}-F^{(\alpha_1,\beta',k)}_{(\alpha',\beta_1-1)}\}\sum_{\{u(j,\lambda),d(j,\lambda)\}}K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\}))$$
$$\subset(y)^{q-1}+(\eta).$$

Proof. We only have to verify

$$(y)^{i-1}(-F^{(\alpha_1,\beta',k)}_{(\alpha',\beta_1-1)}K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\}))\subset(y)^{q-1}+(\eta). \tag{12.24}$$

Note that this term appears only if $\beta_1\geqq 1$. Hence the left hand side of (12.24) is contained in the following ideal $\mathcal{I}$:

$$\mathcal{I}:=(y)^{i-1}(F^{(\alpha_1,\beta',k)}_{(\alpha',\beta_1-1)}\partial_t^{d(1,\lambda)}\Xi_1(0,y,\eta))$$

Since $d(j,\lambda)\leqq i-1$, the inductive assumption yields

$$\mathcal{I}=(y)^{i-d(1,\lambda)}\{(y)^{d(1,\lambda)-1}(\partial_t^{d(1,\lambda)}\Xi_1(0,y,\eta))\}\subset(y)^{q-1}+(\eta).$$

Thus (12.24) follows. Q.E.D.

By virtue of Remark 12.5, it suffices for the assertion 5) to show

$$(y)^{i-1}(F^{(\alpha_1+1,\beta',k)}_{(\alpha',\beta_1)}K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\}))\subset(y)^{q-1}+(\eta). \tag{12.25}$$

We classify the proof of (12.25) into following three cases:

$$\begin{cases}\text{Case 4.1.} & |\beta'|+k\geqq 1.\\ \text{Case 4.2.} & |\beta'|+k=0,\ \beta_1\geqq 1.\\ \text{Case 4.3.} & |\beta'|+k+\beta_1=0.\end{cases}$$

Case 4.1. In this case we have

$$K(\alpha,\beta,k;\{u(j,\lambda),d(j,\lambda)\})\in\sum_{j=2}^{n}(\partial_t^{u(j,\lambda)}\Xi_j|_{t=0})+(\partial_t^{u(n+1,\lambda)}Z|_{t=0})$$

which implies that the left hand side of (12.25) is contained in

$$(y)^{i-1}\Big\{\sum_{j=2}^{n}(\partial_t^{u(j,\lambda)}\Xi_j(0,y,\eta))+(\partial_t^{u(n+1,\lambda)}Z(0,y,\eta))\Big\}$$
$$=\sum_{j=2}^{n}(y)^{i-1-u(j,\lambda)}[(y)^{u(j,\lambda)}(\partial_t^{u(j,\lambda)}\Xi_j(0,y,\eta))]_{(j)}$$
$$+(y)^{i-u(n+1,\lambda)}[(y)^{u(n+1,\lambda)-1}\partial_t^{u(n+1,\lambda)}Z(0,y,\eta))]_{(1)}.$$

Note that the Newton polygons of the blackets $[\cdots]_{(j)}$ $(1\leqq j\leqq n)$ are contained in $\mathrm{N}(f)$ by the inductive assumption, and that

$$\mathrm{N}(f)\subset\mathrm{N}_{q-1,1}:=\{(s,t);(s/(q-1))+t\geqq 1\}.$$

Hence we get (12.25) in the case 4.1.

Case 4.2. In this case we have

$$K(\alpha, \beta, k; \{u(j, \lambda), d(j, \lambda)\}) \in (\partial_t^{d(1,\lambda)} \Xi_1(0, y, \eta)).$$

Thus the left hand side of (12.25) is contained in

$$(y)^{i-1}(\partial_t^{d(1,\lambda)} \Xi_1(0, y, \eta)) = (y)^{i-d(1,\lambda)}[(y)^{d(1,\lambda)-1}(\partial_t^{d(1,\lambda)} \Xi_1(0, y, \eta))]$$
$$\subset (y)^{q-1} + (\eta)$$

which is a consequence of the inductive assumption.

Case 4.3. In this case we have the left hand side of (12.25) is contained in

$$(y)^{i-1}(F_{(\alpha',0)}^{(\alpha_1+1,0,0)}) \subset (y)^{i-1}\{(y)^{q-1-\alpha_1-|\alpha'|} + (\eta)\}$$

since the assumption [B.4] implies

$$F_{(\alpha',0)}^{(\alpha_1+1,0,0)}|_{\eta=0} \in (y)^{q-1-\alpha_1-|\alpha'|}.$$

Hence the following inequality

$$i-1+q-1-\alpha_1-|\alpha'| \geqq q-1$$

yields (12.25) as desired.

The assertion 5) in Lemma 7.5 is proved.

The proof of Lemma 7.5 is complete. Q.E.D.

§ 13. Summary of Local Dimension Theory

In this section we give a summary of local dimension theory of analytic sets. Our summary starts from a review of a way of regarding an analytic set X of a domain D in $\mathbf{C}^N$ as a reduced complex space $(X, \mathcal{O}_X)$. We only give outlines of this way (for its detail, see [Gr-Re]).

Let X be an analytic set of a domain D, that is, X can be defined locally as a common zero set of finitely many holomorphic germs of functions on D.

Definition 13.1. We define the *ideal sheaf* $\iota(X)$ of X as the sheafication of the following presheaf $\{(U, \iota(U))\}$ of ideals of $\mathcal{O}_D$:

(13.1) $$U: \text{open in } D \longmapsto \iota(U) := \{f \in \mathcal{O}_D(U); f|_{X\cap U} \equiv 0\}.$$

Definition 13.2. We define a *structure sheaf* $\mathcal{O}_X$ on X by

(8.29) $$\mathcal{O}_X := (\mathcal{O}_D/\iota(X))|_X$$

that is, the restriction on $X \subset D$ of the sheaf $\mathcal{O}_D/\iota(X)$ on D, where $\mathcal{O}_D/\iota(X)$ is defined by the sheafication of the presheaf on D which is determined by the following data:

$$U: \text{open in } D \longmapsto \mathcal{O}_D(U)/\imath(X)(U).$$

Note that each stalk ideal $\imath(X)_x := \varinjlim_{U \ni x} \imath(X)(U)$ is a *reduced ideal* of $\mathcal{O}_{D,x}$, that is, if $f^m \in \imath(X)_x (\exists m)$ then $f \in \imath(X)_x$ follows. Thus the stalk $\mathcal{O}_{X,x}$ of the structure sheaf $\mathcal{O}_X$ has no non-trivial nilpotent elements, that is, the ringed space $(X, \mathcal{O}_X)$ is a *reduced* ringed space.

We regard an analytic set X in D as a complex space by this way, where we use the terminology of "complex space" in the sense of [Gr-Re], that is, a ringed space $(X, \mathcal{O})$ is called a complex space if it is a *$\boldsymbol{C}$-ringed space with a coherent structure sheaf $\mathcal{O}$, and with a Hausdorff topological space X.*

Remark 13.3. It needs more work to show that our reduced ringed space $(X, \mathcal{O}_X)$ forms a complex space. This justification is based on the famous *Cartan's coherence theorem* [Gr-Re: Fundamental Theorem 4.2, p. 84], which says that the ideal sheaf $\imath(X)$ is a coherent $\mathcal{O}_D$-sheaf.

Definition 13.4. Let $(X, \mathcal{O}_X)$ be a complex space. We say $(X, \mathcal{O}_X)$ is *locally irreducible* at $x \in X$, if the stalk $\mathcal{O}_{X,x}$ is an integral domain.

Next we explain a way of regarding holomorphic maps in the sense of Definition 3.1 as morphisms in the category of complex spaces.

Let X [or Y, resp.] be an analytic set of a domain $D[D']$, and let $f: X \to Y$ be a holomorphic map in the sense of Definition 3.1. Recall the (0-th) *direct image sheaf* $f_*\mathcal{O}_X$ which is defined as the sheafication of the following presheaf $\{(U, (f_*\mathcal{O}_X)(U))\}$:

(13.2) $$U: \text{open in } Y \longmapsto (f_*\mathcal{O}_X)(U) := \mathcal{O}_X(f^{-1}(U)).$$

We want to construct a morphism of the form

$$(f, f^{\sim}): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

where $f^{\sim}: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a sheaf map on Y, and where $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ are the reduced complex spaces constructed by the way of Definition 13.2.

Since $f: X \to Y$ is a holomorphic map in the sense of Definition 3.1, there exists a holomorphic map $g: D \to D'$ such that f is induced by g:

(13.3) $$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cap\downarrow & g & \cap\downarrow \\ D & \longrightarrow & D' \end{array}$$

We consider the pull-back $g^*: \mathcal{O}_{D'} \to f_*\mathcal{O}_D$ and show the

Lemma 13.5. *There exists a canonical sheaf map $f^{\sim}: \mathcal{O}_Y \to f_*\mathcal{O}_X$ on Y which* *is induced by* g^*.

Proof. Let U be an open set in Y, and let $u \in \mathcal{O}_Y(U)$. Recall that $\mathcal{O}_Y$ is defined by the restriction on $Y \subset D'$ of the sheaf $\mathcal{O}_{D'}/i(Y)$ on D'. Hence we have the following 1) and 2):

1) $$u=(u_y) \in \prod_{y \in U} \mathcal{O}_{Y,y}.$$

2) For any $y \in U$, there exists an open neighborhood E' in D' and a section

$$v \in \mathcal{O}_{D'}(E')/i(Y)(E')$$

such that

$$u_z = s'_{E',z}(v) \qquad \text{for any } z \in E' \cap Y \tag{13.4}$$

where $s'_{E',z}: \mathcal{O}_{D'}(E')/i(Y)(E') \to \mathcal{O}_{Y,z} = \mathcal{O}_{D',z}/i(Y)_z$ is the canonical map.

We need the following simple

Claim 13.6. *If* $v^{\sim} \in i(Y)(E')$ *then* $g^* v^{\sim} \in i(X)(g^{-1}(E'))$. *Hence there exist a canonical map*

$$[g^*]: \mathcal{O}_{D'}(E')/i(Y)(E') \longrightarrow \mathcal{O}_D(g^{-1}(E'))/i(X)(g^{-1}(E')). \tag{13.5}$$

Proof. This claim is a direct consequence of Definition 13.1 and (13.3). Indeed, we have

$$\begin{aligned} v^{\sim} \in i(Y)(E') &\Longleftrightarrow v^{\sim}|_{E' \cap Y} \equiv 0 \\ &\Longrightarrow v^{\sim}(g(x))=0 \quad \forall x \in g^{-1}(E') \cap X \\ &\Longleftrightarrow g^* v^{\sim} \in i(X)(g^{-1}(E')). \end{aligned}$$

Hence Claim 13.6 follows. Q.E.D.

Using the map $[g^*]$ given by (13.5), we define a section

$$w := f^{\sim}(u) \in \mathcal{O}_X(f^{-1}(U)) = (f_* \mathcal{O}_X)(U)$$

as follows:

$$\begin{cases} w=(w_x) \in \prod_{x \in f^{-1}(U)} \mathcal{O}_{X,x} \\ w_x := s_{g^{-1}(E'),x}([g^*]v) \in \mathcal{O}_{D,x}/i(X)_x = \mathcal{O}_{X,x} \end{cases} \tag{13.6}$$

where $s_{g^{-1}(E'),x}: \mathcal{O}_D(g^{-1}(E'))/i(X)(g^{-1}(E')) \to \mathcal{O}_{D,x}/i(X)_x$ is the canonical map.

We must verify that the definition (13.6) is well-defined. Let $y \in U$ $(\subset Y)$, let E'_i $(i=1, 2)$ be open neighborhoods of y in D', and let

$$v_i \in \mathcal{O}_{D'}(E'_i)/i(Y)(E'_i) \qquad (i=1, 2)$$

be sections satisfying

$$u_z = s'_{E'_1,z}(v_1) = s'_{E'_2,z}(v_2) \quad \text{for any } z \in E'_1 \cap E'_2 \cap Y.$$

Putting $E'' := E'_1 \cap E'_2$, we have

$$(v_1 - v_2)|_{E''} = 0 \quad \text{as an element of } \mathcal{O}_{D'}(E'')/i(Y)(E'').$$

Thus we get

$$s_{g^{-1}(E''),x}([g^*](v_1 - v_2)) = 0 \in \mathcal{O}_{X,x} \quad \text{for any } x \in g^{-1}(E'')$$

which shows that the section $w = (w_x) \in \prod_{x \in f^{-1}(U)} \mathcal{O}_{X,x}$ given by (13.6) is determined independently of the choice of v satisfying (13.4). Hence the sheaf map $f^{\sim} : \mathcal{O}_Y \ni u \mapsto w = f^{\sim}(u) \in f_* \mathcal{O}_X$ is well-defined.

The proof of Lemma 13.5 is complete. Q. E. D.

Definition 13.7. Let $(f, f^{\sim}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of complex spaces. We call $(f, f^{\sim})$ is a *finite morphism* if the underlying map $f : X \to Y$ is a finite holomorphic map in the sense of Definition 3.3.

Now we recall a definition of analytic subsets in a complex space, and their local dimensions.

Definition 13.8. Let $(X, \mathcal{O}_X)$ be a complex space. We call a subset Z of X is an *analytic subset* at a point $x \in X$ if there exist finitely many germs $f_1, \cdots, f_k \in \mathcal{O}_{X,x}$ such that the germ (Z, x) of a subset Z at x is given by the common zero set of $f_1, \cdots, f_k$. We denote this by $(Z, x) = \mathrm{Null}(f_1, \cdots, f_k)$.

Definition 13.9. Let $(X, \mathcal{O}_X)$ be a complex space. We define its *local dimension* $\dim_x(X, \mathcal{O}_X)$ *at* a point $x \in X$ by

$$\dim_x(X, \mathcal{O}_X) := \min\{k \,;\, \exists f_1, \cdots, f_k \in \mathcal{O}_{X,x} \text{ such that } \mathrm{Null}(f_1, \cdots, f_k) \cap X = \{x\}\}$$

that is, the minimum integer of such numbers k of germs $f_1, \cdots, f_k \in \mathcal{O}_{X,x}$ which make the point x be isolated in $\mathrm{Null}(f_1, \cdots, f_k)$.

Definition 13.10. Let Z be an analytic set of $(X, \mathcal{O}_X)$ at $x \in Z \subset X$. We define its *local dimension* $\dim_x Z$ at x by

$$\dim_x Z := \dim_x(Z, \mathcal{O}_Z)$$

where $(Z, \mathcal{O}_Z)$ is the closed reduced complex subspace of $(X, \mathcal{O}_X)$ defined by

$$\mathcal{O}_Z := (\mathcal{O}_X / i(Z))|_Z$$

as similar as Definition 13.2.

Now we quote several propositions which are used in §§ 8 and 14.

Propositon 13.11 [Gr-Re: Remark 5.1.1, p. 94]. *Let*

$$(f, f^{\sim}):(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

be a finite morphism of complex spaces. Then

$$\dim_x(X, \mathcal{O}_X) \leqq \dim_{f(x)}(Y, \mathcal{O}_Y).$$

Proposition 13.12 [Gr-Re: Active Lemma 5.2.4, p. 100]. *Let* $(X, \mathcal{O}_X)$ *be a complex space and let* $g \in \mathcal{O}_{X,x}$ *be a germ. If the zero set* Null(g) *of* g *is nowhere dense in* $(X, \mathcal{O}_X)$, *then*

$$\dim_x \mathrm{Null}(g) = \dim_x(X, \mathcal{O}_X) - 1.$$

Proposition 13.13 [Gr-Re: Proposition 5.3.2, p. 103]. *Let* $(X, \mathcal{O}_X)$ *be a complex space which is locally irreducible at* $x \in X$. *Let* Z *be an analytic subset at* x *of* $(X, \mathcal{O}_X)$. *If*

$$\dim_x Z = \dim_x(X, \mathcal{O}_X)$$

then there exists an open neighborhood U *of* x *in* X *such that*

$$Z \cap U = X \cap U.$$

We also use the following theorem in §§ 8 and 14:

Theorem 13.14 [Gr-Re: Finite Mapping Theorem 3.1.3, p. 64]. *Let* $(f, f^{\sim}):(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *be a finite morphism. Then the image* $f(X)$ *is an analytic set in* $(Y, \mathcal{O}_Y)$.

The following famous theorem is used in §§ 4, 8, and 10:

Rückert's Nullstellensatz [Gr-Re: Theorem 4.1.5, p. 82]. *Let* $(X, \mathcal{O}_X)$ *be a complex space, and let* $\mathcal{J} \subset \mathcal{O}_X$ *be a coherent sheaf of ideals with zero set* Null$(\mathcal{J})$. *Let* $\imath(\mathrm{Null}(\mathcal{J}))$ *denote the ideal sheaf of the analytic set* Null$(\mathcal{J})$, *and let* Rad$(\mathcal{J})$ *denote the radical of* $\mathcal{J}$, *that is, the sheafication of the following presheaf* $\{(U, \mathrm{Rad}(\mathcal{J})(U))\}$ *of ideals of* $\mathcal{O}_X$:

$$U: \text{open in } X \mapsto \mathrm{Rad}(\mathcal{J})(U) := \{f \in \mathcal{O}_X(U);\ {}^{\exists} m \in N,\ f^m \in \mathcal{J}(U)\}.$$

Then it follows that

$$\imath(\mathrm{Null}(\mathcal{J})) = \mathrm{Rad}(\mathcal{J}).$$

§ 14. Proof of Lemma 9.5

In this last section we prove the following Theorem 14.1 which contains Lemma 9.5 as a special case.

Let us consider a non-zero germ $f \in \mathcal{O}_{C^2,0}$ of *two independent variables*. For a local coordinate system (x, y) at the origin satisfying

$$f(x, 0)f(0, y)\not\equiv 0, \tag{14.1}$$

we define the Newton polygon N(f) of f with respect to the coordinate system by

$$f(x, y)=\sum_{j,k=0}^{\infty} c_{jk}x^j y^k \longmapsto \mathrm{N}(f):=\mathrm{ch}\Big[\bigcup_{c_{jk}\neq 0}(j, k)+\bar{\boldsymbol{R}}_+^2\Big]$$

where ch[A] denotes the convex hull of $A\subset\boldsymbol{R}^2$, and we set $\bar{\boldsymbol{R}}_+:=\{t\in\boldsymbol{R};\ t\geqq 0\}$.

Since we assume (14.1), we can find positive integers $p(\mu)$, $q(\mu)$ where $1\leqq\mu\leqq m:=\sharp\mathrm{Seg}\,\mathrm{N}(f)$ such that N(f) can be written as

$$\mathrm{N}(f)=\sum_{\mu=1}^{m}\mathrm{N}_{q(\mu),p(\mu)} \tag{14.2}$$

where we set

$$\mathrm{N}_{a,b}:=\{(s, t)\in\bar{\boldsymbol{R}}_+^2;\ (s/a)+(t/b)\geqq 1\}.$$

We may assume that $p(\mu)$, $q(\mu)$ are arranged as

$$p(1)/q(1)>p(2)/q(2)>\cdots>p(m)/q(m). \tag{14.3}$$

In this situation we have the

Theorem 14.1 *There exist positive integers $i(\mu)$, $c(\mu, i)$, $\nu(\mu, i)$ and non-unit germs $f_{\mu i}\in\mathcal{O}_{C^2,0}$ such that the following* 1)–3) *hold*:

1) *The following decomposition*

$$f(x, y)=\prod_{\mu=1}^{m}\prod_{i=1}^{i(\mu)}f_{\mu i}(x, y)^{\nu(\mu,i)} \tag{14.4}$$

is an irreducible decomposition of $f(x, y)$ in the ring $\mathcal{O}_{C^2,0}$.

2) *Each Newton polygon of $f_{\mu i}(x, y)$ is given by*

$$\mathrm{N}(f_{\mu i})=c(\mu, i)\mathrm{N}_{q^{\sim}(\mu),p^{\sim}(\mu)} \quad \text{for } 1\leqq\mu\leqq m,\ 1\leqq i\leqq i(\mu). \tag{14.5}$$

In (14.5), *the integers $p^{\sim}(\mu)$, $q^{\sim}(\mu)$ are defined by*

$$p^{\sim}(\mu):=p(\mu)/(p(\mu), q(\mu)),\quad q^{\sim}(\mu):=q(\mu)/(p(\mu), q(\mu))$$

where (a, b) denotes the greatest common divisor of $a, b\in\boldsymbol{Z}$.

3) *For any μ, $\{c(\mu, i), \nu(\mu, i);\ 1\leqq i\leqq i(\mu)\}$ satisfies*

$$\sum_{i=1}^{i(\mu)}c(\mu, i)\nu(\mu, i)=(p(\mu), q(\mu)). \tag{14.6}$$

Remark 14.2. Theorem 14.1 contains Lemma 9.5 as the special case that the following condition holds:

$$(p(\mu), q(\mu))=1 \quad \text{for all } 1\leqq\mu\leqq m. \tag{14.7}$$

Proof. The conditions (14.6), (14.7) imply

$$i(\mu)=1,\quad c(\mu, 1)=\nu(\mu, 1)=1,\quad \text{and}$$

$$p^{\sim}(\mu)=p(\mu), \quad q^{\sim}(\mu)=q(\mu).$$

Hence Theorem 14.1 yields an irreducible decomposition

$$f(x, y)=\prod_{\mu=1}^{m} f_{\mu 1}(x, y)$$

with the conditions

$$N(f_{\mu 1})=N_{q(\mu), p(\mu)} \qquad \text{for } 1 \leqq \mu \leqq m.$$

Thus Lemma 9.5 follows from Theorem 14.1. Q. E. D.

Now we prove Theorem 14.1. We derive this theorem from the following

Proposition 14.3. *Let* $g \in \mathcal{O}_{C^2,0}-\{0\}$ *be a non-unit germ. Assume that the complex curve* Null(g) *has only one irreducible component locally at the origin. Then, for any local coordinate system* (x, y) *at the origin satisfying*

$$g(x, 0)g(0, y) \not\equiv 0, \tag{14.1'}$$

it follows that N(g) *has only one segment, that is,*

$$\sharp \mathrm{Seg}\, N(g)=1. \tag{14.8}$$

Proof of "Proposition 14.3 $\Rightarrow$ *Theorem* 14.1*".* Let

$$f=\prod_{j=1}^{k} g_j^{\nu^{\sim}(j)} \tag{14.9}$$

be an irreducible decomposition of f in the ring $\mathcal{O}_{C^2,0}$, and let (x, y) be a local coordinate system at the origin satisfying (14.1). Since this coordinate system satisfies (14.1)′ for all g_j, Proposition 14.3 yields

$$N(g_j)=N_{a(j), b(j)} \tag{14.10}$$

for suitable positive integers $a(j)$, $b(j)$. With no loss of generality we may assume

$$b(1)/a(1) \geqq b(2)/a(2) \geqq \cdots \geqq b(k)/a(k).$$

We take integers

$$0=j_0<j_1<\cdots<j_s=k$$

such as

$$\begin{cases} b(1+j_{\lambda-1})/a(1+j_{\lambda-1})=b(j_\lambda)/a(j_\lambda) & \text{and} \\ b(j_{\lambda-1})/a(j_{\lambda-1})>b(j_\lambda)/a(j_\lambda) & \text{for all } 1 \leqq \lambda \leqq s. \end{cases} \tag{14.11}$$

We set a_λ, b_λ $(1 \leqq \lambda \leqq s)$ by the following conditions:

$$\begin{cases} b_\lambda/a_\lambda=b(j_\lambda)/a(j_\lambda) & \text{and} \\ (a_\lambda, b_\lambda)=1. \end{cases} \tag{14.12}$$

Then, for $1\leqq j\leqq k$, we can find an integer $d(j)>0$ such that

$$\begin{cases} a(j)=d(j)a_\lambda \\ b(j)=d(j)b_\lambda \quad \text{for } 1+j_{\lambda-1}\leqq j\leqq j_\lambda . \end{cases}$$

Thus, by the additivity of Newton polygons, (14.9)-(14.12) yield

$$(14.13)\qquad \begin{cases} \mathrm{N}(f)=\sum_{j=1}^{k}\nu^{\sim}(j)\mathrm{N}_{a(j),b(j)}=\sum_{\lambda=1}^{s}\left\{\sum_{j=1+j_{\lambda-1}}^{j_\lambda}\nu^{\sim}(j)d(j)\right\}\mathrm{N}_{a_\lambda,b_\lambda}. \\ b_1/a_1>b_2/a_2>\cdots>b_s/a_s . \end{cases}$$

Now, comparing (14.13) with (14.2), we get

$$\begin{cases} m:=\#\{\mu\}=\#\{\lambda\}=s \quad \text{and} \\ q(\mu)=\left\{\sum_{j=1+j_{\mu-1}}^{j_\mu}\nu^{\sim}(j)d(j)\right\}a_\mu, \quad p(\mu)=\left\{\sum_{j=1+j_{\mu-1}}^{j_\mu}\nu^{\sim}(j)d(j)\right\}b_\mu . \end{cases}$$

Note that the coprimeness (14.12) of a_μ, b_μ implies

$$(14.14)\qquad q^{\sim}(\mu)=a_\mu,\quad p^{\sim}(\mu)=b_\mu,\quad \text{and}\quad (p(\mu),q(\mu))=\sum_{j=1+j_{\mu-1}}^{j_\mu}\nu^{\sim}(j)d(j).$$

We define $i(\mu)$, $c(\mu, i)$, $\nu(\mu, i)$ $(1\leqq i\leqq i(\mu))$ and $f_{\mu i}(x, y)$ as follows:

$$(14.15)\qquad \begin{cases} i(\mu):=j_\mu-j_{\mu-1} \\ c(\mu,i):=d(i+j_{\mu-1}),\quad \nu(\mu,i):=\nu^{\sim}(i+j_{\mu-1}) \\ f_{\mu i}:=g_{i+j_{\mu-1}} . \end{cases}$$

Then, from (14.14), (14.15), we have the irreducible decomposition

$$f=\prod_{j=1}^{k}g_j^{\nu^{\sim}(j)}=\prod_{\mu=1}^{m}\prod_{i=1}^{i(\mu)}g_{i+j_{\mu-1}}^{\nu^{\sim}(i+j_{\mu-1})}=\prod_{\mu=1}^{m}\prod_{i=1}^{i(\mu)}f_{\mu i}^{\nu(\mu,i)}$$

with the conditions

$$\begin{cases} \mathrm{N}(f_{\mu i})=\mathrm{N}(g_{i+j_{\mu-1}})=\mathrm{N}_{a(i+j_{\mu-1}),b(i+j_{\mu-1})} \\ \qquad =d(i+j_{\mu-1})\mathrm{N}_{a_\mu,b_\mu}=c(\mu,i)\mathrm{N}_{q^{\sim}(\mu),p^{\sim}(\mu)} \quad \text{and} \\ \sum_{i=1}^{i(\mu)}c(\mu,i)\nu(\mu,i)=\sum_{i=1}^{i(\mu)}d(i+j_{\mu-1})\nu^{\sim}(i+j_{\mu-1}) \\ \qquad =\sum_{j=1+j_{\mu-1}}^{j_\mu}d(j)\nu^{\sim}(j)=(p(\mu),q(\mu)). \end{cases}$$

Thus Theorem 14.1 follows if Proposition 14.3 is established. Q.E.D.

From now on we prove Proposition 14.3. We shall prove the following contrapositive proposition of Proposition 14.3:

Proposition 14.4. *Let* $g\in\mathcal{O}_{C^2,0}-\{0\}$ *be a non-unit germ. If there exists a* *local coordinate system* (x, y) *at the origin such that*

$$g(x, 0)g(0, y)\not\equiv 0 \quad \textit{and} \tag{14.16}$$

$$m:=\#\mathrm{Seg}\,\mathrm{N}(g)>1 \tag{14.17}$$

then the complex curve $\mathrm{Null}(g)$ *has at least two irreducible components locally at the origin.*

We prove Proposition 14.4 by induction on $\nu:=\mathrm{ord}_0[g]$. Since $\nu=1$ immediately yields $m=1$, Proposition 14.4 is trivial in the case $\nu=1$. Let $\nu\geqq 2$, and assume that Proposition 14.4 holds for any germs with order $<\nu$.

We first show the

Lemma 14.5. *Let* $g(x, y)$ *be a germ with a vanishing order* $\nu\geqq 2$, *and let* (x, y) *be a local coordinate system satisfying* (14.16) *and* (14.17). *Then there exists a local coordinate system* $(x^{\sim}, y^{\sim})$ *at the origin such that*

$$g^{\sim}(x^{\sim}, 0)g^{\sim}(0, y^{\sim})\not\equiv 0 \tag{14.16$^{\sim}$}$$

$$m^{\sim}:=\#\mathrm{Seg}\,\mathrm{N}(g^{\sim})>1 \quad \textit{and} \tag{14.17$^{\sim}$}$$

$$\mathrm{ord}[g^{\sim}(x^{\sim}, y^{\sim})|_{y^{\sim}=0}]=\nu \tag{14.18}$$

where $g^{\sim}(x^{\sim}, y^{\sim})$ *denotes the expression of* g *by the coordinate system* $(x^{\sim}, y^{\sim})$, *and* $\mathrm{N}(g^{\sim})$ *denotes the Newton polygon of* $g^{\sim}$ *with respect to* $(x^{\sim}, y^{\sim})$.

Proof. If either $\mathrm{ord}[g(x, 0)]=\nu$ or $\mathrm{ord}[g(0, y)]=\nu$, then we can take a local coordinate system $(x^{\sim}, y^{\sim})$ as

$$(x^{\sim}, y^{\sim}):=\text{either } (x, y) \text{ or } (y, x).$$

Thus we may assume

$$\nu<\min\{\mathrm{ord}[g(x, 0)], \mathrm{ord}[g(0, y)]\}. \tag{14.19}$$

Using the identification

$$T_0\boldsymbol{C}^2\ni X\partial_x+Y\partial_y \overset{\sim}{\longmapsto} (X, Y)\in\boldsymbol{C}^2$$

we write $g(x, y)$ as the form

$$g(x, y)=\mathrm{Loc}[g](x, y)+h(x, y) \tag{14.20}$$

where $\mathrm{ord}[h]>\nu$, and $\mathrm{Loc}[g]$ is a homogeneous polynomial of degree ν. Note that $\mathrm{Loc}[g]$ can be written as the form

$$\mathrm{Loc}[g](x, y)=cx^iy^j\prod_{k=1}^{n}(x-e_ky)^{\nu(k)} \tag{14.21}$$

where $i, j, n\geqq 0$, $\nu(k)\geqq 1$ satisfy the relation

$$i+j+\sum_{k=1}^{n}\nu(k)=\nu$$

and where $c,\ e_k\in \boldsymbol{C}-\{0\}$. Moreover the condition (14.19) yields

(14.22) $$i>0 \quad \text{and} \quad j>0.$$

Taking a linear coordinate transformation

$$x^{\sim}:=x,\quad y^{\sim}:=x-ey\quad (e\neq 0)$$

we can write Loc[g] as

$$\mathrm{Loc}[g^{\sim}](x^{\sim},\ y^{\sim})=cx^{\sim i}e^{-j}(x^{\sim}-y^{\sim})^{j}\prod_{k=1}^{n}(x^{\sim}-e_k e^{-1}(x^{\sim}-y^{\sim}))^{\nu(k)}$$

$$=ce^{-j}x^{\sim i}(x^{\sim}-y^{\sim})^{j}\prod_{k=1}^{n}\{(1-e_k e^{-1})x^{\sim}+e_k e^{-1}y^{\sim}\}^{\nu(k)}.$$

Hence if we choose e such as $e\in \boldsymbol{C}-\{0,\ e_1,\ \cdots,\ e_n\}$ then we have

(14.23) $$\mathrm{Loc}[g^{\sim}](x^{\sim},\ 0)\not\equiv 0,\quad \text{and} \quad \mathrm{Loc}[g^{\sim}](0,\ y^{\sim})\equiv 0.$$

Since (14.22), (14.23) yield

$$(\nu,\ 0),\ (i,\ \nu-i)\in \mathrm{N}(g^{\sim}) \quad \text{with} \quad (\nu,\ 0)\neq(i,\ \nu-i)$$

$$(0,\ \nu)\notin \mathrm{N}(g^{\sim})$$

we get (14.17)~ and (14.18). Note that $\{x=0\}=\{x^{\sim}=0\}$ which yields

$$\mathrm{ord}[g^{\sim}(0,\ y^{\sim})]=\mathrm{ord}[g(0,\ y)]<\infty.$$

Hence (14.16)~ also follows. The proof of Lemma 14.15 is complete. Q.E.D.

Now we prove Proposition 14.4 for the case $\nu\geqq 2$. By virtue of Lemma 14.5, we may assume

(14.24) $$2\leqq\nu:=\mathrm{ord}[g]=\mathrm{ord}[g(x,\ 0)]<\mathrm{ord}[g(0,\ y)]<\infty.$$

Hence we have the following expression:

(14.25) $$\mathrm{Loc}[g](x,\ y)=cx^{i}\prod_{k=1}^{n}(x-e_k y)^{\nu(k)}$$

where $i\geqq 1,\ n\geqq 0,\ \nu(k)\geqq 1$ with the relation

$$i+\sum_{k=1}^{n}\nu(k)=\nu$$

and where $c,\ e_k\in \boldsymbol{C}-\{0\}$ with $e_k\neq e_{k^{\sim}}$ if $k\neq k^{\sim}$.

In order to prove Proposition 14.4 by induction, we use the notion of *blowing ups* of the complex curve Null(g) with center a point.

Definition 14.6 (see, for example, [Hi: Lecture 1]). Let $Z:=\boldsymbol{C}^2$, let

$$\pi_0: \boldsymbol{C}^2-\{0\} \longrightarrow \boldsymbol{P}^1:=(\boldsymbol{C}^2-\{0\})/(\boldsymbol{C}-\{0\})$$

be the natural map. Let Z' be the closure of

$$\text{graph}(\pi_0)=\{(x, y\,;[\xi:\eta])\in(\boldsymbol{C}^2-\{0\})\times\boldsymbol{P}^1\,;\, x\eta=y\xi\}$$

in $\boldsymbol{C}^2\times\boldsymbol{P}^1$, and let $\pi: Z'\to Z$ be the map induced by the following diagram:

$$\begin{array}{ccc} Z' & \hookrightarrow & \boldsymbol{C}^2\times\boldsymbol{P}^1 \\ \Big\lfloor \pi & & \Big\downarrow \text{projection} \\ & \longrightarrow & Z=\boldsymbol{C}^2 \end{array}$$

This map $\pi: Z'\to Z$ is called the *blowing up* (or the *quadratic transformation*) of *Z with center* $\{0\}$.

Note 14.7. The blowing up $\pi: Z'\to Z$ has the following properties:

$$Z'=\text{graph}(\pi_0)\cup(\{0\}\times\boldsymbol{P}^1) \tag{14.26}$$

$$\pi^{-1}(0)=\{0\}\times\boldsymbol{P}^1 \tag{14.27}$$

and the map $\pi: Z'\to Z$ induces an isomorphism

$$\pi|_{Z'-\pi^{-1}(0)}: Z'-\pi^{-1}(0) \xrightarrow{\sim} Z-\{0\}\,. \tag{14.28}$$

Proof. Since (14.27), (14.28) easily follow from (14.26), we only have to verify (14.26). Let $\{(x_n, y_n\,;[\xi_n:\eta_n])\}_{n=1,2,\dots}$ be a sequence in graph(π_0) which converges to a point $(x, y\,;[\xi:\eta])$ in $\boldsymbol{C}^2\times\boldsymbol{P}^1$. Then the following two cases occur.

$$\begin{cases} \text{Case 1.} \quad \text{When } (x, y)\neq 0. \\ \text{Case 2.} \quad \text{When } (x, y)=0. \end{cases}$$

In the *case* 1, with no loss of generality, we may assume $x\neq 0$. Then we have $x_n\neq 0$ for $n\gg 1$. Hence the condition $(x_n, y_n\,;[\xi_n:\eta_n])\in\text{graph}(\pi_0)$ can be written as

$$[\xi_n:\eta_n]=[1:y_n/x_n] \qquad \text{for } n\gg 1.$$

Thus, taking the limit $n\to\infty$, we have

$$(x, y\,;[\xi:\eta])=(x, y\,;[1:y/x])\in\text{graph}(\pi_0).$$

In the *case* 2, for any $[\xi:\eta]\in\boldsymbol{P}^1$, we can choose a sequence in graph(π_0) which converges to $(0, 0\,;[\xi:\eta])$ as follows: we set

$$(x_n, y_n\,;[\xi_n:\eta_n]):=\begin{cases} (1/n, (\eta/\xi)/n\,;[\xi:\eta]) & \text{if } \xi\neq 0 \\ ((\xi/\eta)/n, 1/n\,;[\xi:\eta]) & \text{if } \eta\neq 0. \end{cases}$$

Then the sequence satisfies the desired property.

Hence we get (14.26) as desired. The proof of Note 14.7 is complete.

Q.E.D.

We remark that Z' is equipped with a structure of complex manifold of dimension 2. Indeed, we define two maps w_x, w_y by

$$\begin{cases} w_x : \boldsymbol{C}^2 \ni (x, y_1) \longmapsto (x, xy_1; [1 : y_1]) \in Z' \\ w_y : \boldsymbol{C}^2 \ni (x_1, y) \longmapsto (yx_1, y; [x_1 : 1]) \in Z' \end{cases}$$

and set

$$\Omega_x := w_x(\boldsymbol{C}^2), \qquad \Omega_y := w_y(\boldsymbol{C}^2).$$

Then they have the following properties (14.29)-(14.31):

(14.29) $$Z' = \Omega_x \cup \Omega_y .$$

(14.30) $$\boldsymbol{C}^2 \underset{w_x}{\overset{\sim}{\longrightarrow}} \Omega_x \subset Z', \qquad \boldsymbol{C}^2 \underset{w_y}{\overset{\sim}{\longrightarrow}} \Omega_y \subset Z'.$$

(14.31) $$\begin{cases} w_x(x, y_1) = w_y(x_1, y) \in \Omega_x \cap \Omega_y \Longleftrightarrow \\ \text{“}xy \neq 0 \text{ and } x_1 = x/y,\ y_1 = y/x\text{” or “}x = y = 0,\ x_1 y_1 = 1\text{”}. \end{cases}$$

Notation 14.8. We denote the variable x_1 [or y_1 resp.] by x/y [y/x].

According with this notation, we have the following 1)-3):

1) The coordinate neighborhoods Ω_x and Ω_y can be written as follows:

(14.32) $$\begin{cases} \Omega_x = \{(x, x(y/x); [1 : y/x]) \in \boldsymbol{C}^2 \times \boldsymbol{P}^1 ; (x, y/x) \in \boldsymbol{C}^2\}. \\ \Omega_y = \{(y(x/y), y; [x/y : 1]) \in \boldsymbol{C}^2 \times \boldsymbol{P}^1 ; (x/y, y) \in \boldsymbol{C}^2\}. \end{cases}$$

2) The blowing up $\pi : Z' \to Z$ can be represented on the coordinate neighborhoods Ω_x, Ω_y as follows:

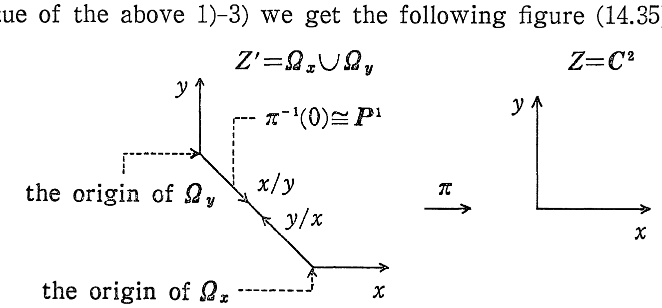
(14.33) $$\begin{cases} \pi|_{\Omega_x} : \Omega_x \cong \boldsymbol{C}^2 \ni (x, y/x) \longmapsto (x, x(y/x)) \in Z. \\ \pi|_{\Omega_y} : \Omega_y \cong \boldsymbol{C}^2 \ni (x/y, y) \longmapsto (y(x/y), y) \in Z. \end{cases}$$

3) In particular it follows that

(14.34) $$\begin{cases} \Omega_x \cap \pi^{-1}(0) \cong \{(x, y/x) ; x = 0\}. \\ \Omega_y \cap \pi^{-1}(0) \cong \{(x/y, y) ; y = 0\}. \end{cases}$$

By virtue of the above 1)-3) we get the following figure (14.35) of Z':

(14.35)



Definition 14.9. Let X be a complex curve defined by $f(x, y)=0$ in a neighborhood of the origin of Z. Let $\nu=\mathrm{ord}_0[f]$ be the (vanishing) order of f at the origin. We can write the pull-back $f\circ\pi$ as

$$(14.36)\qquad \begin{cases} f(x, x(y/x))=x^{\nu}f_1'(x, y/x) & \text{on } (\Omega_x, \Omega_x\cap\pi^{-1}(0)) \\ f(y(x/y), y)=y^{\nu}f_2'(x/y, y) & \text{on } (\Omega_y, \Omega_y\cap\pi^{-1}(0)) \end{cases}$$

where f_1' [or f_2' resp.] is a holomorphic germ defined in a neighborhood of $\Omega_x\cap\pi^{-1}(0)$ [of $\Omega_y\cap\pi^{-1}(0)$]. We set

$$(14.37)\qquad \begin{cases} X_1':=w_x(\{(x, y/x)\,;\, f_1'(x, y/x)=0\})\subset\Omega_x \\ X_2':=w_y(\{(x/y, y)\,;\, f_2'(x/y, y)=0\})\subset\Omega_y\,. \end{cases}$$

Then $X_1'\cup X_2'$ determines a complex curve X' in a neighborhood of $\pi^{-1}(0)$ in Z'. We call X' the *strict transform* of X by the blowing up π. Note that the blowing up $\pi: Z'\to Z$ induces a holomorphic map

$$(14.38)\qquad p: X'\longrightarrow X\,.$$

This map p is called the *strict transformation* of X *with center* $\{0\}$.

Proof of the well-definedness of X'. We have

$$x=y(x/y),\quad y=x(y/x)\quad\text{and}\quad (x/y)(y/x)=1\qquad \text{on } \Omega_x\cap\Omega_y$$

which yield

$$x^{\nu}f_1'(x, y/x)=y^{\nu}(x/y)^{\nu}f_1'(y(x/y), (x/y)^{-1})\,.$$

Thus it follows that

$$(14.39)\qquad f_2'(x/y, y)=(x/y)^{\nu}f_1'(y(x/y), (x/y)^{-1})\,.$$

Since x/y does not vanish on $\Omega_x\cap\Omega_y$, (14.39) implies

$$X_1'=X_2'\qquad \text{on } \Omega_x\cap\Omega_y$$

as desired. Q.E.D.

Now we return the proof of Proposition 14.4. Let $g(x, y)$ be a germ of the order $\nu\geqq 2$, and let (x, y) be the local coordinate system satisfying (14.24) and (14.17). Recall the expression of the localization $\mathrm{Loc}[g]$ at the origin:

$$(14.25)\qquad \begin{cases} \mathrm{Loc}[g](x, y)=cx^{i}\prod_{k=1}^{n}(x-e_k y)^{\nu(k)} \\ \qquad \text{where } i\geqq 1,\ n\geqq 0, \text{ and } c, e_k\in\boldsymbol{C}-\{0\} \text{ with } e_k\neq e_{k^{\sim}}\ (k\neq k^{\sim}). \end{cases}$$

Lemma 14.10. *Set* $X:=\{(x, y)\,;\, g(x, y)=0\}=\mathrm{Null}(g)$, *and let* X' *be the strict transform of* X. *Then*

1) *The pre-image* $p^{-1}(0)=X'\cap\pi^{-1}(0)$ *consists of the following finite points*

$$(14.40)\qquad \{(x/y, y)\in\Omega_y\,;\, y=0,\ x/y=0, e_1, \cdots, e_n\}\,.$$

2) *Let $g_2'(x/y, y)$ be the defining germ of $\Omega_y \cap X'$. Then it follows*

$$\operatorname{ord}_{(0,0)}[g_2'] \leqq i \tag{14.41}$$

$$\operatorname{ord}_{(e_k,0)}[g_2'] \leqq \nu(k) \quad \textit{for } k=1, \cdots, n. \tag{14.42}$$

Proof. We calculate the defining germs $g_1'(x, y/x)$, $g_2'(x/y, y)$ as follows. Since $g(x, y)$ can be written as

$$g=\operatorname{Loc}[g]+h \qquad (\text{where } h(x, y)=\sum_{q+r>\nu} c_{qr}x^q y^r)$$

the definition (14.36) and the expression (14.25) lead us to

$$\begin{aligned} g_2'(x/y, y) &= y^{-\nu}\{\operatorname{Loc}[g](y(x/y), y)+\sum_{q+r>\nu} c_{qr}(x/y)^q y^{q+r}\} \\ &= \operatorname{Loc}[g](x/y, 1)+\sum_{q+r>\nu} c_{qr}(x/y)^q y^{q+r-\nu} \\ &= c(x/y)^i \prod_{k=1}^{n}((x/y)-e_k)^{\nu(k)}+y^3 h_2(x/y, y). \end{aligned} \tag{14.43}$$

Since $\Omega_y \cap \pi^{-1}(0)=\{(x/y, y);\, y=0\}$, we get

$$p^{-1}(0)\cap\Omega_y=\{(x/y, y)=(0, 0),\ (e_k, 0)\in\Omega_y;\ 1\leqq k\leqq n\}. \tag{14.44}$$

We similarly have

$$\begin{aligned} g_1'(x, y/x) &= \operatorname{Loc}[g](1, y/x)+\sum_{q+r>\nu} c_{qr}x^{q+r-\nu}(y/x)^r \\ &= c\prod_{k=1}^{n}(1-e_k(y/x))^{\nu(k)}+x^3 h_1(x, y/x). \end{aligned}$$

Thus $(x, y/x)=(0, 0)\notin\{g_1'=0\}$ which shows that

$$(\Omega_x-\Omega_y)\cap p^{-1}(0)=\emptyset. \tag{14.45}$$

Hence the assertion 1) of Lemma follows from (14.44) and (14.45).

The assertion 2) is a direct consequence of the expression (14.43): Indeed, we have

$$\operatorname{ord}_{(0,0)}[g_2'] \leqq \operatorname{ord}_0[g_2'(x/y, 0)]=i \quad \text{and}$$

$$\operatorname{ord}_{(e_k,0)}[g_2'] \leqq \operatorname{ord}_{e_k}[g_2'(x/y, 0)]=\nu(k).$$

Hence the proof of Lemma 14.10 is complete. Q.E.D.

Corollary 14.11. *The strict transformation $p: X'\to X$ determines the finite holomorphic map germs*

$$p^{(k)}: (X', (e_k, 0)) \longrightarrow (X, (0, 0)) \qquad k=0, 1, \cdots, n \tag{14.46}$$

where we set $e_0:=0$.

We classify the proof of Proposition 14.4 for $\nu\geqq 2$ into the following two cases:

$$\begin{cases} \textit{Case 1.} & \textit{When } n\geqq 1, \textit{ that is, } \sharp p^{-1}(0)=n+1\geqq 2. \\ \textit{Case 2.} & \textit{When } n=0, \textit{ that is, } \sharp p^{-1}(0)=1. \end{cases}$$

Proof of Proposition 14.4 *in the case* 1. Since $p^{(k)}$ $(k=0, 1, \cdots, n)$ are finite map germs, we can regard them as finite morphisms by the way given in § 13. Thus, by virtue of Proposition 13.14, the images $p^{(k)}(X')$ are analytic sets of X, for $0\leqq k\leqq n$. Then Propositions 13.11 and 13.12 yield

$$1=\dim_{(e_k, 0)}X'\leqq\dim_{(0,0)}[p^{(k)}(X')]\leqq\dim_{(0,0)}X=1\,.$$

Hence we conclude that $p^{(k)}(X')$ are complex curves at (0, 0) contained in X. On the other hand the blowing up $\pi: Z'\to Z$ induces the isomorphism (14.28). Thus we get $p^{(k)}(X')\neq p^{(k^{\sim})}(X')$ as germs of curves at (0, 0) if $k\neq k^{\sim}$. Consequently we get $X=\{g=0\}$ has at least two irreducible components locally at the origin in the case 1, as desired. Q.E.D.

It remains the proof of Proposition 14.4 in the case 2.

We observe the effect of the strict transformation $p: X'\to X$ to the Newton polygon $\mathrm{N}(g)$. Since we assume (14.17), (14.24), and the case 2, there exist positive integers m $(\geqq 2)$ and $a(\mu)$, $b(\mu)$ $(1\leqq\mu\leqq m)$ such that

$$\mathrm{N}(g)=\sum_{\mu=1}^{m}\mathrm{N}_{a(\mu),b(\mu)} \qquad \text{where } \mathrm{N}_{a,b}:=\{(s, t); (s/a)+(t/b)\geqq 1\} \tag{14.47}$$

$$b(1)/a(1)>\cdots>b(m-1)/a(m-1)>b(m)/a(m)>1\,. \tag{14.48}$$

The effect of $p: X'\to X$ to $\mathrm{N}(g)$ is given by the

Lemma 14.12. *Let $g_2'(x/y, y)$ be the defining germ of the strict transform X' on the coordinate neighborhood Ω_y. Assume the case 2. Then it follows*

$$\mathrm{N}(g_2')=\sum_{\mu=1}^{m}\mathrm{N}_{a(\mu),b(\mu)-a(\mu)} \tag{14.49}$$

where we denote by $\mathrm{N}(g_2')$ the Newton polygon of g_2' at the unique pre-image $(0, e_0)$ of $(0, 0)\in X$ by p, with respect to the coordinate system $(x/y, y)$.

Proof. Since we assume the case 2, the expression (14.43) yields

$$g_2'(x/y, y)=c(x/y)^{\nu}+\sum_{q+r>\nu}c_{qr}(x/y)^{q}y^{q+r-\nu}\,. \tag{14.50}$$

In particular we have

$$\mathrm{ord}_0[g_2'(x/y, 0)]=\nu, \qquad \mathrm{ord}_0[g_2'(0, y)]=\min\{r-\nu; c_{0r}\neq 0\}.$$

which yield

$$(14.51)\qquad \begin{cases} (\nu,\ 0)\in\partial^{o}\mathrm{N}(g_2')\subset\mathrm{N}(g_2') \quad \text{and} \\ (0,\ b(1)+\cdots+b(m)-a(1)-\cdots-a(m)) \\ =(0,\ b(1)+\cdots+b(m)-\nu)\in\partial^{o}\mathrm{N}(g_2')\subset\mathrm{N}(g_2'). \end{cases}$$

On the other hand, since $(q,\ r)\in\mathrm{N}(g)$ if $c_{qr}\neq 0$, it follows

$$(14.52)\qquad r\geqq-(b(\mu)/a(\mu))[q-\{a(1)+\cdots+a(\mu)\}] \\ +b(\mu+1)+\cdots+b(m) \qquad \text{for } 1\leqq\mu\leqq m.$$

Indeed, Lemma 0.2 leads us to

$$\mathrm{Ver}\ \mathrm{N}(g)=\{(a(1)+\cdots+a(\mu),\ b(\mu+1)+\cdots+b(m))\ ;\ 0\leqq\mu\leqq m\}.$$

By the inequality (14.52), we have

$$q+r-\nu\geqq-\{(b(\mu)/a(\mu))-1\}[q-\{a(1)+\cdots+a(\mu)\}] \\ +a(1)+\cdots+a(\mu)+b(\mu+1)+\cdots+b(m)-\nu.$$

Thus, from $\nu=a(1)+\cdots+a(m)$, we get

$$(14.53)\qquad q+r-\nu\geqq-\{(b(\mu)/a(\mu))-1\}[q-\{a(1)+\cdots+a(\mu)\}] \\ +\{b(\mu+1)-a(\mu+1)\}+\cdots+\{b(m)-a(m)\}.$$

From (14.53) and (14.51) we conclude

$$(14.54)\qquad \mathrm{N}(g_2')\subset\sum_{\mu=1}^{m}\mathrm{N}_{a(\mu),b(\mu)-a(\mu)}.$$

Let us fix $1\leqq\mu<m$. Note that the equalities hold in (14.52) simultaneously for μ and for $\mu+1$ if and only if

$$(q,\ r)=(a(1)+\cdots+a(\mu),\ b(\mu+1)+\cdots+b(m)).$$

Thus we get

$$(14.55)\qquad (a(1)+\cdots+a(\mu),\ b(\mu+1)+\cdots+b(m)-a(\mu+1)-\cdots-a(m))\in\mathrm{N}(g_2') \\ \text{for } 1\leqq\mu<m.$$

By (14.51), (14.54) and (14.55), we conclude the equality (14.49) as desired.
The proof of Lemma 14.12 is complete. Q.E.D.

Proof of Proposition 14.4 *in the case* 2. Let us divide $b(m)$ by $a(m)$:

$$(14.56)\qquad \begin{cases} b(m)=a(m)d+c \\ 0\leqq c<a(m), \quad \text{and} \quad d\geqq 1\ (c,\ d\in\boldsymbol{Z}). \end{cases}$$

We classify the proof as follows:

$$\begin{cases} \text{Case 2a). When } c=0. \\ \text{Case 2b). When } c>0. \end{cases}$$

First we prove Proposition 14.4 in *the case* 2*a*). We consider the following sequence of blowing ups:

$$\begin{array}{ccccccc} Z=\mathbf{C}^2 & \xleftarrow{\pi_1} & Z' & \longleftarrow & \cdots & \xleftarrow{\pi_{d-1}} & Z^{(d-1)} \\ \cup\!\uparrow & & \cup\!\uparrow & & & & \cup\!\uparrow \\ X=g^{-1}(0) & \xleftarrow{p_1} & X' & \longleftarrow & \cdots & \xleftarrow{p_{d-1}} & X^{(d-1)} \end{array} \tag{14.57}$$

where $\pi_j: Z^{(j)} \to Z^{(j-1)}$ is the blowing up of $Z^{(j-1)}$ with center $\{x_{j-1}\}$, and $p_j: X^{(j)} \to X^{(j-1)}$ is the strict transformation of $X^{(j-1)}$ induced by π_j such that $x_j \in X^{(j)}$ satisfies

$$p_j(x_j) = x_{j-1} \qquad \text{for } j \geqq 1 . \tag{14.58}$$

Note that such a sequence (14.57) is determined uniquely if we give x_0 by

$$x_0 := (0, 0) \in X = \mathrm{Null}(g) \tag{14.59}$$

since the germ $(X^{(j-1)}, x_{j-1})$ lies in the case 2, for $1 \leqq j \leqq d-1$.

By virtue of Lemma 14.12, we have

$$\mathrm{N}(g^{(j)}) = \sum_{\mu=1}^{m} \mathrm{N}_{a(\mu),\, b(\mu)-ja(\mu)} \qquad \text{at } x_j \in X^{(j)} \ (0 \leqq j \leqq d-1) \tag{14.60}$$

where $g^{(j)}$ is the defining germ of $X^{(j)}$ at x_j. Then $c=0$ implies that the germ $(X^{(d-1)}, x_{d-1})$ lies in the case 1, since (14.60) yields

$$\mathrm{N}(g^{(d-1)}) = \sum_{\mu=1}^{m-1} \mathrm{N}_{a(\mu),\, b(\mu)-(d-1)a(\mu)} + \mathrm{N}_{a(m),\, a(m)} \tag{14.60$'$}$$

with

$$\begin{cases} m \geqq 2 \quad \text{and} \\ \{b(\mu)-(d-1)a(\mu)\}/a(\mu) = \{b(\mu)/a(\mu)\} - (d-1) \\ \qquad\qquad > \{b(m)/a(m)\} - (d-1) = 1 \qquad \text{for } \mu < m . \end{cases}$$

Thus $X^{(d-1)}$ has at least two irreducible components locally at x_{d-1} (note that the expression (14.60)′ yields $\# \operatorname{Seg} \mathrm{N}(g^{(d-1)}) = \# \operatorname{Seg} \mathrm{N}(g) > 1$).

Since the composite map germ

$$p_1 \circ p_2 \circ \cdots \circ p_{d-1} : (X^{(d-1)}, x_{d-1}) \longrightarrow (X, x_0)$$

is a finite map germ which induces an isomorphic map germ

$$(X^{(d-1)} - \{x_{d-1}\}, x_{d-1}) \xrightarrow{\sim} (X - \{x_0\}, x_0),$$

we conclude that $X = \{g=0\}$ also has at least two irreducible components at $x_0 = (0, 0)$ as desired.

The proof of Proposition 14.4 in the case 2a) is complete.

It only remains *the case* 2*b*). As similar as the case 2a), we consider the following sequence of blowing ups:

(14.57)′
$$\begin{array}{ccccccc} Z=\boldsymbol{C}^2 & \xleftarrow{\pi_1} & Z' \longleftarrow \cdots & \xleftarrow{\pi_{d-1}} & Z^{(d-1)} & \xleftarrow{\pi_d} & Z^{(d)} \\ \bigcup\!\uparrow & & \bigcup\!\uparrow & & \bigcup\!\uparrow & & \bigcup\!\uparrow \\ X=g^{-1}(0) & \xleftarrow{p_1} & X' \longleftarrow \cdots & \xleftarrow{p_{d-1}} & X^{(d-1)} & \xleftarrow{p_d} & X^{(d)} \end{array}$$

where π_j and p_j are as same as in the above proof of the case 2a).

Note that $c>0$ implies

(14.61) $(X^{(j-1)}, x_{j-1})$ lies in the case 2 for $1\leqq j\leqq d$, and

(14.62) $\mathrm{ord}_{x_d}[g^{(d)}]<\nu=\mathrm{ord}_{x_{d-1}}[g^{(d-1)}]=\cdots=\mathrm{ord}_{x_0}[g]$.

Indeed, Lemma 14.12 yields that the Newton polygon $\mathrm{N}(g^{(j)})$ is given by the expression (14.60) with the inequalities

$$\{b(m)-ja(m)\}/a(m)\begin{cases} \geqq d+\{c/a(m)\}-(d-1)>1 & \text{for } j\leqq d-1\,. \\ =d+\{c/a(m)\}-d<1 & \text{for } j=d\,. \end{cases}$$

Thus we have the assertions (14.61) and (14.62).

Note that the expression (14.60) yields $\sharp\,\mathrm{Seg}\,\mathrm{N}[g^{(d)}]=\sharp\,\mathrm{Seg}\,\mathrm{N}(g)>1$. Hence, by virtue of (14.62), we can apply the inductive assumption to the curve $X^{(d)}$, which says that $X^{(d)}$ has at least two irreducible components at x_d.

Thus, the finiteness of the composite map germ

$$p_1\circ p_2\circ\cdots\circ p_d:(X^{(d)}, x_d)\longrightarrow(X, x_0)$$

and the isomorphness of the induced map germ

$$(X^{(d)}-\{x_d\}, x_d)\xrightarrow{\sim}(X-\{x_0\}, x_0)$$

yield that $X=\{g=0\}$ also has at least two irreducible components at $x_0=(0, 0)$ as desired.

The proof of Proposition 14.4 is complete. Q. E. D.

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