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The Structure of the Quasi-invariant Set of a Linear Measure

By

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Abstract

Let μ be a probability measure on a locally convex Hausdorff space E and $A(\mu)$ be the quasi-invariant set of μ. If $\mu^*(A(\mu))>0$, then there exist a finite-dimensional subspace L, a thick subgroup G of L and a countable subgroup $\{x_i\}$ such that $A(\mu)=\bigcup_{i=1}^{\infty}(G+x_i)$. If E is Souslin, then $A(\mu)$ is a Borel subset. If E is Souslin and if $\mu(A(\mu))>0$, then $A(\mu)=\bigcup_{i=1}^{\infty}(L+x_i)$.

§1. Introduction

Let E be a locally convex Hausdorff space and E' be the topological dual of E. Denote by $C(E, E')$ the cylindrical σ-algebra on E, the minimal σ-algebra which makes each $\langle\ ,x'\rangle$, $x'\in E'$, measurable. Denote by $B(E)$ the Borel σ-algebra on E generated by all open subsets. Then it holds that $C(E, E')\subset B(E)$ and these σ-algebras are translation invariant, that is, for every $x\in E$ and $A\in C(E, E')$ (resp. $B(E)$) it follows that $A-x\in C(E, E')$ (resp. $B(E)$). Let μ be a probability measure on $C(E, E')$ or on $B(E)$. For $x\in E$ we set $\mu_x(A)=\mu(A-x)$ and $A(\mu)=\{x\in E: \mu_x\sim\mu \text{ (equivalent)}\}$. The set $A(\mu)$ is called the quasi-invariant set of μ. It is well-known that $A(\mu)$ is an additive subgroup of E.

Skorohod [9] stated the following assertions concerning the structure of the quasi-invariant set $A(\mu)$ (however there are some gaps in the proof).

Let E be a separable Hilbert space. Then

(1) *$A(\mu)$ is a Borel subset of E,*

(2) *if $\mu(F)=0$ for every finite-dimensional subspace $F\subset E$, then $\mu(A(\mu))=0$, and*

(3) *if $\mu(A(\mu))=1$, then there exists a sequence L_i of finite-dimensional subspaces such that $\mu\left(\bigcup_{i=1}^{\infty}L_i\right)=1$,*

see [9], §19. On the other hand, Okazaki [7] proved the following results.

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Let G be a complete separable metrizable abelian topological group and μ be a probability measure on G. Then

(1) *$A(\mu)$ is a Borel subset of G, and*

(2) *if $\mu(A(\mu))>0$, then $A(\mu)$ is a locally compact σ-compact topological group with respect to the induced topology from G.*

There are similarities between these two results. In fact, each locally compact locally convex Hausdorff space is finite-dimensional and each locally compact σ-compact subgroup of a locally convex Hausdorff space is of the form R^n+ (countable subgroup) (the structure theorem of the locally compact σ-compact abelian topological group).

In this paper, we generalize the above results of Skorohod as follows. Firstly, suppose that E is a general locally convex Hausdorff space and $\mu^*(A(\mu))>0$, then there exist a finite-dimensional subspace L, a thick subgroup G of L and a countable subgroup $\{x_i\}$ of E such that $A(\mu)=\bigcup_{i=1}^{\infty}(G+x_i)$ (Theorem 1), where μ^* denotes the outer measure. Secondly, suppose that E is a Souslin locally convex Hausdorff space, then $A(\mu)$ is a Borel subset of E and if $\mu(A(\mu))>0$, $A(\mu)$ can be written as $A(\mu)=\bigcup_{i=1}^{\infty}(L+x_i)$, where L is a finite-dimensional subspace and $\{x_i\}$ is a countable subgroup of E (Theorem 2).

§ 2. Preliminaries

Let (G, B) be a measurable group, that is, G is a group with a σ-algebra B satisfying

(1) $x \longrightarrow x^{-1}$ is B-measurable, and

(2) $(x, y) \longrightarrow xy$ is $B\otimes B-B$-measurable,

where $B\otimes B$ is the product σ-algebra on $G\times G$, see Halmos [4], § 59 and Yamasaki [12], Part B, Chapter 1, § 1.

Let (G, B) be a measurable group and μ be a measure on (G, B). Then (G, B, μ) is called separated if for every $g\neq e$ (e is the unit of G), there exists $A\in B$ such that $\mu(A)>0$ and $\mu(A\cap Ag)=0$, see Halmos [4], § 62 and Yamasaki [12], Part B, Chapter 1, § 4.

A subset A of a topological group G is called precompact (or bounded) if for every neighborhood U of e, there exists a finite sequence $g_1, g_2, \cdots, g_n$ in G such that $A\subset\bigcup_{i=1}^{n}(Ug_i)$. If a topological group G has a precompact neighborhood of e, then G is called locally precompact. If a topological group G is written as $G=\bigcup_{n=1}^{\infty}A_n$, where each A_n is precompact, then G is called σ-precompact, see Halmos [4], § 0 and Yamasaki [12], Part B, Chapter 1, § 3.

We shall use the following facts in later.

Fact 1. *A topological group G is locally precompact and σ-precompact if and only if G can be imbedded densely and isomorphically (algebraically and topologically isomorphic) into a locally compact σ-compact topological group $\bar{G}$.*

The group $\bar{G}$ is uniquely determined within an isomorphism and $\bar{G}$ is called the completion of G, see Bourbaki [1], Chapter 2, § 3, Halmos [4], § 0 and Yamasaki [12], Part B, Chapter 1, § 3, Theorem 3.1. As for the Borel structure of G and $\bar{G}$, the following result is known, see Yamasaki [12], Part B, Chapter 1, § 3, Theorem 3.2.

Fact 2. *Let G be a locally precompact σ-precompact topological group and Bu be the σ-algebra on G generated by all uniformly continuous functions on G. Let $\overline{Ba}$ be the Baire σ-algebra on the completion $\bar{G}$, the σ-algebra generated by all continuous functions on $\bar{G}$. Then it holds that $Bu=\overline{Ba}\cap G$. Moreover, (G, Bu) is a measurable group.*

A topological group G is called a thick group if

(1) G is locally precompact and σ-precompact, and

(2) for the right Haar measure $\bar{\lambda}$ on $(\bar{G}, \overline{Ba})$, G is thick with respect to $\bar{\lambda}$, that is, for every $B\in\overline{Ba}$ with $B\cap G=\phi$ it follows that $\bar{\lambda}(B)=0$,

see Halmos [4], § 62 and Yamasaki [12], Part B, Chapter 1, § 3.

Let G be a thick group. Then there is a right invariant measure λ on (G, Bu), that is, $\lambda(Ag)=\lambda(A)$ for every $A\in Bu$ and every $g\in G$. In fact, λ is the restriction of $\bar{\lambda}$ to G, see Yamasaki [12], Part B, Chapter 1, § 3, Theorem 3.3. This right invariant measure λ on (G, Bu) is called the Haar measure of G. It is known that, in general, a right invariant measure on (G, Bu) is unique up to a constant factor.

In the sequel, every group which we consider is abelian. Hence we denote by $x+y$ the group operation (instead of xy).

§ 3. Quasi-invariant Set for Cylindrical Measure

Let E be a locally convex Hausdorff space and μ be a probability measure on $C(E, E')$. Then $(E, C(E, E'))$ is a measurable group and the quasi-invariant set $A(\mu)$ is an additive subgroup of E, see Yamasaki [12], Part B, Chapter 1, § 5, Theorem 5.1. Remark also that $C(E, E')\otimes C(E, E')=C(E\times E, (E\times E)')$. The next lemma is an easy consequence of this fact.

Lemma 1. *$(A(\mu), C(E, E')\cap A(\mu))$ is a measurable group.*

Proof. We show that $\Psi:(x, y)\longrightarrow x-y$ is $\beta\otimes\beta-\beta$-measurable, where $\beta=C(E, E')\cap A(\mu)$. For every $B\in\beta$, we can find C so that $B=C\cap A(\mu)$ and $C\in C(E, E')$. Then we have $\Psi^{-1}(B)=\Psi^{-1}(C)\cap A(\mu)\times A(\mu)\in C(E, E')\otimes C(E, E')\cap$

$A(\mu)\times A(\mu)=\beta\otimes\beta$.

Suppose that $\mu^*(A(\mu))>0$ where μ^* is the outer measure. Let ν be the restriction of μ^* to $(A(\mu), C(E, E')\cap A(\mu))$. ν is defined as follows. Take $C\in C(E, E')$ such that $A(\mu)\subset C$ and $\mu(C)=\mu^*(A(\mu))$. Then for every $A\in C(E, E')\cap A(\mu)$ with $A=D\cap A(\mu)$, $D\in C(E, E')$, $\nu(A)$ is given by $\nu(A)=\mu(C\cap D)$.

Lemma 2. *Suppose that $\mu^*(A(\mu))>0$ and ν be the restriction of μ^* to $(A(\mu), C(E, E')\cap A(\mu))$. Then ν is a quasi-invariant measure on $A(\mu)$, that is, ν and ν_x are equivalent for every $x\in A(\mu)$.*

Proof. Let $C\in C(E, E')$ be $A(\mu)\subset C$ and $\mu^*(A(\mu))=\mu(C)$. For every $A\in C(E, E')\cap A(\mu)$, we write $A=D\cap A(\mu)$, where $D\in C(E, E')$. Then we have for each $x\in A(\mu)$, $A-x=(D-x)\cap A(\mu)$ and $A(\mu)\subset(C-x)\cap C\subset C$. Hence it follows that $\mu(C-x)=\mu(C)=\mu^*(A(\mu))$ and $\nu_x(A)=\nu(A-x)=\nu((D-x)\cap A(\mu))=\mu(C\cap(D-x))$ $=\mu((C-x)\cap(D-x))=\mu(C\cap D-x)=\mu_x(C\cap D)$ for every $x\in A(\mu)$. Since $\nu(A)=\mu(C\cap D)$, for every $x\in A(\mu)$, $\nu_x(A)=0$ if and only if $\nu(A)=0$.

Lemma 3. *Suppose that $\mu^*(A(\mu))>0$ and ν be the restriction of μ^* to $(A(\mu), C(E, E')\cap A(\mu))$. Then $(A(\mu), C(E, E')\cap A(\mu), \nu)$ is separated.*

Proof. For every $x\neq 0$ in $A(\mu)$, we shall show the existence of A in $C(E, E')\cap A(\mu)$ such that $\nu(A)>0$ and that $A\cap(A+x)=\phi$. Let $x'\in E'$ be $\langle x, x'\rangle=1$ (Hahn-Banach theorem) and set $A=\{y\in A(\mu): 0\leqq\langle y, x'\rangle<1\}$. Then it holds that $A\cap(A+x)=\phi$ and $A(\mu)=\bigcup_{n=1}^{\infty}(A+nx)$. By Lemma 2, it must be $\nu(A)>0$.

Lemma 4. *Suppose that $\mu^*(A(\mu))>0$ and ν be the restriction of μ^* to $(A(\mu), C(E, E')\cap A(\mu))$. Then there exists a ($\sigma$-finite by (1) below) measure λ on $(A(\mu), C(E, E')\cap A(\mu))$ such that*

(1) *$\lambda\sim\nu$ (equivalent), and*

(2) *$\lambda_x=\lambda$ for each x in $A(\mu)$ ($A(\mu)$-invariant).*

Proof. The assertions follow by Mackey [6], Lemma 7, Umemura [10], Proposition 6.2 and Yamasaki [12], Part B, Chapter 1, § 1, Theorems 1.4 and 1.1.

Suppose that $\mu^*(A(\mu))>0$ and ν be the restriction of μ^* to $(A(\mu), C(E, E')\cap A(\mu))$. Let λ be a σ-finite invariant measure on $A(\mu)$ equivalent to ν (Lemma 4). We remark that $(A(\mu), C(E, E')\cap A(\mu), \lambda)$ is also separated since so is $(A(\mu), C(E, E')\cap A(\mu), \nu)$ and λ is equivalent to ν.

Now let τ be the Weil topology of $A(\mu)$ derived by the invariant measure $(A(\mu), C(E, E')\cap A(\mu), \lambda)$, that is, the basis of neighborhoods of 0 in τ is given by the family

$$U_{A,\varepsilon}=\{x\in A(\mu): \lambda(A\ominus(A+x))<\varepsilon\},$$

where $A\in C(E, E')\cap A(\mu)$ be $0<\lambda(A)<\infty$, $\varepsilon>0$ and $\ominus$ is the symmetric difference, see Halmos [4], § 62, Yamasaki [12], Part B, Chapter 1, § 4 and Weil [11], Appendice 1. Then the following result is known. For the proof, we refer to Halmos [4], § 62 and Yamasaki [12], Part B, Chapter 1, § 4, Theorem 4.1.

Lemma 5. *$(A(\mu), \tau)$ is a Hausdorff topological group and thick. Moreover, $Bu \subset C(E, E') \cap A(\mu)$ and the restriction $\lambda | Bu$ is the Haar measure on $(A(\mu), Bu)$.*

Lemma 6. *The Weil topology τ is finer than the weak topology $\sigma(E, E')$ on $A(\mu)$. In particular it holds that $Bu = C(E, E') \cap A(\mu)$ in Lemma 5.*

Proof. We show that each x' is τ-continuous. Let N be $\lambda(\{x \in A(\mu): |\langle x, x'\rangle| \leqq N\}) > 0$ and let $A \subset \{x \in A(\mu): |\langle x, x'\rangle| \leqq N\}$ be $0 < \lambda(A) < \infty$. Then for every $x \in U_{A, \lambda(A)} = \{x \in A(\mu): \lambda(A \ominus (A+x)) < \lambda(A)\}$, it follows that $|\langle x, x'\rangle| \leqq 2N$ since $U_{A, \lambda(A)} \subset A - A \subset \{x \in A(\mu): |\langle x, x'\rangle| \leqq 2N\}$. Remark that if $\lambda(A \ominus (A+x)) < \lambda(A)$ then $A \cap (A+x) \neq \phi$. Thus the additive functional x' is τ-continuous.

Theorem 1. *Let E be a locally convex Hausdorff space and μ be a probability measure on $C(E, E')$. Suppose that $\mu^*(A(\mu)) > 0$. Then*

(1) *there exists a topology τ on $A(\mu)$ such that $(A(\mu), \tau)$ is a Hausdorff topological group and a thick group,*

(2) *the restriction ν of μ^* to $A(\mu)$ is equivalent to the Haar measure on the thick group $(A(\mu), \tau)$, and*

(3) *there exist a finite-dimensional subspace $L \subset E$, a thick subgroup G of L (with respect to the Euclidean topology) and a countable subgroup $\{x_\iota\} \subset E$ such that $A(\mu) = \bigcup_{\iota=1}^{\infty} (G + x_\iota) \subset \bigcup_{\iota=1}^{\infty} (L + x_\iota)$.*

Proof. (1) and (2) follow from Lemmas 5 and 6. We shall prove (3). Consider the natural injection $\iota: (A(\mu), \tau) \to (E, \sigma(E, E'))$. By Lemma 6, ι is continuous. ι can be extended to the completion $(A(\mu), \tau)^-$ into $(E')^a$ (the algebraic dual of E' which is the completion of $(E, \sigma(E, E'))$, see Bourbaki [1], Chapter 2, § 3, Theorem 3.1. Let $\bar{\iota}$ be the extension. Then $\bar{\iota}$ is a continuous homomorphism on the locally compact σ-compact topological group $(A(\mu), \tau)^-$ into $((E')^a, \sigma((E')^a, E'))$. The image $\bar{\iota}((A(\mu), \tau)^-)$ is algebraically isomorphic with $(A(\mu), \tau)^- / \ker \bar{\iota}$. We put the topology T on $\bar{\iota}((A(\mu), \tau)^-)$ induced by the quotient topology of $(A(\mu), \tau)^- / \ker \bar{\iota}$. Then $(\bar{\iota}((A(\mu), \tau)^-), T)$ is again a locally compact σ-compact abelian topological group and T is finer than the weak topology $\sigma((E')^a, E')$. By the structure theorem of locally compact σ-compact abelian topological group, there exist natural numbers n, d (possibly 0) and a compact abelian group K such that $(\bar{\iota}((A(\mu), \tau)^-), T)$ is isomorphic (algebraically and topologically) with the direct sum $R^n \oplus Z^d \oplus K$, where R (resp. Z) denotes the real numbers (resp. integers), see Hewitt and Ross [5], Theorem (9.8) and Weil [11], § 29. Since K is isomorphic to a compact subgroup of the vector space $((E')^a, \sigma((E')^a, E'))$, it must be $K=\{0\}$. Let $\phi: R^n \oplus Z^d \to (\bar{\iota}((A(\mu), \tau)^-), T) \subset (E')^a$ be an isomorphism. Then ϕ is in fact linear on R^n, hence $L=\phi(R^n)$ is a finite-dimensional subspace of $(E')^a$ contained in $\bar{\iota}((A(\mu), \tau)^-)$. $D=\phi(Z^n)$ is a discrete countable subgroup of $(\bar{\iota}((A(\mu), \tau)^-), T)$. We have proved that $(\bar{\iota}((A(\mu), \tau)^-), T)=L \oplus D$. Remark that L is open and closed subgroup of $(\bar{\iota}((A(\mu), \tau)^-), T)$. Since $\iota(A(\mu))=A(\mu)$ is dense in $(\bar{\iota}((A(\mu), \tau)^-), T)=L \oplus D$, it follows that $A(\mu) \cap L$ is a dense subgroup of L with respect to the Euclidean topology of L. By $A(\mu) \cap L \subset E \cap L \subset L$, we obtain $L=\overline{A(\mu) \cap L} \subset \overline{E \cap L} \subset E$ since the finite-dimensional subspace $E \cap L$ is complete with respect to $\sigma((E')^a, E')$. We denote $\{x_i\}=A(\mu) \cap D$. Then it holds that $A(\mu)=A(\mu) \cap L+\{x_i\}$. We prove that the subgroup $G=A(\mu) \cap L$ is thick in L with respect to the Euclidean topology of L. In fact, for every Baire subset C in L satisfying $G \cap C=\phi$, $(\bar{\iota})^{-1}(C)$ is a Baire subset of $(A(\mu), \tau)^-$ and $(\bar{\iota})^{-1}(C) \cap A(\mu)=\phi$. By the thickness of $(A(\mu), \tau)$, it holds that $\bar{\lambda}((\bar{\iota})^{-1}(C))=0$, where $\bar{\lambda}$ is the Haar measure on $(A(\mu), \tau)^-$. Remark that the Haar measure on $(\bar{\iota}((A(\mu), \tau)^-), T)=L \oplus D$ coincides with the image measure $\bar{\iota}(\bar{\lambda})$ up to a constant factor. Furthermore, the Haar measure on L is the restriction $\bar{\iota}(\bar{\lambda})|L$ since L is an open and closed subgroup in $(\bar{\iota}((A(\mu), \tau)^-), T)$. Thus we have proved that for every Baire subset C in L with $G \cap C=\phi$, the Haar measure of C is zero, which shows the thickness of G in L. This completes the proof.

§ 4. Quasi-invariant Set for Borel Measure

Let E be a Souslin locally convex Hausdorff space and μ be a probability measure on the Borel σ-algebra $B(E)$. $(E, B(E))$ is a measurable group since the product σ-algebra $B(E) \otimes B(E)$ coincides with the Borel σ-algebra on the product space $E \times E$, see Schwartz [8], Part I, Chapter II. Moreover, μ is a Radon measure, that is, for every Borel subset $A \in B(E)$, $\mu(A)=\sup\{\mu(K): K$ is compact and $K \subset A\}$, see Schwartz [8], Part I, Chapter II, § 3.

Lemma 7. *$A(\mu)$ is a Borel subset.*

Proof. We shall give a sketch of the proof. For details, see Okazaki [7], Proposition 12. For every Borel subset $B \in B(E)$, the function $\mu(B-x)$ in x is Borel measurable since E is Souslin and μ is Radon. Let $M(E)$ be the set of all probability measures on E. We consider the σ-algebra $\mathcal{M}$ on $M(E)$ generated by $\nu \to \nu(B)$, $B \in B(E)$, that is, $\mathcal{M}$ is the minimal σ-algebra on $M(E)$ which makes each $\nu \to \nu(B)$, $B \in B(E)$, measurable. Then the mapping $\Psi: (E, B(E)) \to (M(E) \times M(E), \mathcal{M} \otimes \mathcal{M})$, $\Psi(x)=(\mu_x, \mu)$, is measurable. By Dubins and Freedman [2], 2.11, the set $D=\{(\xi, \eta) \in M(E) \times M(E): \xi \sim \eta\}$ belongs to $\mathcal{M} \otimes \mathcal{M}$ (here we use the fact that $B(E)$ is countably generated, see Schwartz [8], Part I, Chapter II, § 1, Corollary of Lemma 18). Hence we obtain $A(\mu)=\Psi^{-1}(D)$ belongs to $B(E)$.

Theorem 2. *Let E be a Souslin locally convex Hausdorff space and μ be a* *probability measure on* $(E, B(E))$. *Suppose that* $\mu(A(\mu))>0$. *Then*

(1) $A(\mu)$ *is a locally compact* σ*-compact topological subgroup of* E *with respect to the induced topology from* E,

(2) *the restriction* $\nu=\mu|A(\mu)$ *is equivalent to the Haar measure of* $A(\mu)$, *and*

(3) *there exist a finite-dimensional subspace* $L\subset E$ *and a countable subgroup* $\{x_i\}$ *of* E *such that* $A(\mu)=\bigcup_{i=1}^{\infty}(L+x_i)$.

Proof. By Lemma 7, $A(\mu)$ is a Souslin topological group with respect to the induced topology from E, see Schwartz [8], Part I, Chapter II, §1, Theorem 3. The restriction $\nu=\mu|A(\mu)$ is a quasi-invariant Radon measure on $A(\mu)$ with respect to the induced topology from E. Furthermore, $(A(\mu), B(A(\mu)), \nu)$ is a separated measurable group, see Lemma 3. Thus by Mackey [6], Lemma 7, Umemura [10], Proposition 6.2 and Yamasaki [12], Part B, Chapter 1, §1, Theorems 1.4 and 1.1, there exists a σ-finite invariant Radon measure λ which is equivalent to ν, see Lemma 4. Hence by Gowrisankaran [3], $A(\mu)$ is a locally compact σ-compact topological group with respect to the induced topology from E and λ is the Haar measure of $A(\mu)$ up to a constant factor. The σ-compactness follows by the σ-finiteness of the Haar measure. By the structure theorem of a locally compact σ-compact abelian topological group, $A(\mu)$ is isomorphic with $R^n\oplus Z^d\oplus K$ by Hewitt and Ross [5], Theorem (9.8) and Weil [11], §29, see the proof of Theorem 1. Since K is a compact subgroup of the vector space E, it follows that $K=\{0\}$. Let ϕ: $R^n\oplus Z^d\to A(\mu)$ be an isomorphism. Then ϕ is linear on R^n. We put $L=\phi(R^n)$ and $\{x_i\}=\phi(Z^d)$. Then L is a finite-dimensional subspace and $A(\mu)=\bigcup_{i=1}^{\infty}(L+x_i)$. This completes the proof.

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