

Higher Airy structures and topological recursion for singular spectral curves

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Abstract. We give elements towards the classification of quantum Airy structures based on the $W(\mathfrak{gl}_r)$ -algebras at self-dual level based on twisted modules of the Heisenberg VOA of \mathfrak{gl}_r for twists by arbitrary elements of the Weyl group \mathfrak{S}_r . In particular, we construct a large class of such quantum Airy structures. We show that the system of linear ODEs forming the quantum Airy structure and determining uniquely its partition function is equivalent to a topological recursion à la Chekhov–Eynard–Orantin on singular spectral curves. In particular, our work extends the definition of the Bouchard–Eynard topological recursion (valid for smooth curves) to a large class of singular curves and indicates impossibilities to extend naively the definition to other types of singularities. We also discuss relations to intersection theory on moduli spaces of curves, giving a general ELSV-type representation for the topological recursion amplitudes on smooth curves, and formulate precise conjectures for application in open r -spin intersection theory.

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1. Introduction

Introduced by Kontsevich and Soibelman in [44], Airy structures consist of a system of linear PDEs depending on a parameter \hbar and satisfying a compatibility condition, so that they admit a simultaneous solution which is unique when properly normalised. This solution is called “partition function” and is encoded in Taylor coefficients $F_{g,n}$ indexed by integers $n \geq 1$ and (half-)integers $g \geq 0$, which are determined by a topological recursion, i.e., a recursion on $2g - 2 + n$. The interest in Airy structures comes from the numerous applications of the topological recursion in enumerative geometry, see, e.g., [34].

The purpose of Part I of this work is to construct new Airy structures from representations of the $\mathcal{W}(\mathfrak{gl}_r)$ -algebra. The latter is realised as a sub-VOA of the Heisenberg VOA \mathcal{F}_{Cr} . Twisting the free field representation of \mathcal{F}_{Cr} by an arbitrary element of the Weyl group $\sigma \in \mathfrak{S}_r$ gives rise to an (untwisted) representation of $\mathcal{W}(\mathfrak{gl}_r)$. Applying a “dilaton shift”, [9] constructed all the Airy structures that can arise when σ is an r -cycle or an $(r - 1)$ -cycle. Our work explores the possibility of constructing Airy structures from arbitrary $\sigma \in \mathfrak{S}_r$: we will obtain in Theorem 2.11 conditions on σ and the dilaton shifts that are sufficient for the success of this construction. This gives rise to many new Airy structures and thus partition functions, for which it would be desirable to find enumerative interpretations. If one contents oneself with the existence of a partition function only solving the PDEs to leading order in \hbar (i.e., existence of $F_{g=0,n}$), we give less restrictive conditions for this in Theorem 2.13. Our approach and the results are presented in Section 2 while Section 3 is devoted to the proof of the main Theorem 2.11.

In Part II, we show that the topological recursion for all these Airy structures can be equivalently formulated via residue (hence, period) computations on possibly singular spectral curves. Roughly speaking, a spectral curve is a branched cover of complex curves $x: C \rightarrow C_0$ equipped with a meromorphic function y and a bidifferential $\omega_{0,2}$ on C^2 . The original formulation of the topological recursion, with a spectral curve as input, was developed by Chekhov, Eynard and Orantin [36, 37] in the case of smooth curves with simple ramifications. The output of this CEO recursion is a family of multidifferentials $\omega_{g,n}$ on C^n that have poles at ramification points of x , are symmetric under permutation of the n copies of C , and are obtained recursively by residue computations on C . As observed already in [37] and revisited in [4, 44], the corresponding Airy structure is based on the Virasoro algebra and related to the $\mathcal{W}(\mathfrak{gl}_2)$ -algebra; for each g and n , $F_{g,n}$ and $\omega_{g,n}$ contain the same information packaged in a different way. The definition of the CEO topological recursion was extended in [14, 16] to smooth curves with higher-order ramification points, and its correspondence with $\mathcal{W}(\mathfrak{gl}_r)$ -Airy structures when σ is an r -cycle was established in [9]. An important application of this correspondence is a conceptual proof of symmetry of

the $\omega_{g,n}$ based on the representation theory of \mathcal{W} -algebras. This led to the discovery of a non-trivial criterion on the order of y at ramification points for the symmetry to hold. From [16], it is easy to propose a definition of the CEO recursion that is also valid for singular spectral curves (see equation (5.7)), i.e., if C has several irreducible components intersecting at ramification points of arbitrary order. It is however unclear (and in fact not always true) that this definition leads to symmetric $\omega_{g,n}$.

In Proposition 5.18 and Theorem 5.23, we show that this definition naturally arises from the $\mathcal{W}(\mathfrak{gl}_r)$ -Airy structures with arbitrary permutation σ encoded the ramification profile over a branchpoint in a normalisation of C , and dilaton shifts specifying the order of y at the ramification points. Smooth spectral curves correspond to σ having a single cycle. Besides, the basic properties of Airy structures guarantee that the corresponding $\omega_{g,n}$ are symmetric. The results of Part I therefore give sufficient conditions on the ramification type and the order of y at ramification points for the symmetry to hold, see Definitions 5.19–5.22 and 5.25. The central result of Part II is then Theorem 5.23 giving the correspondence between those Airy structures and our extension of the CEO topological recursion.

In terms of spectral curves, the CEO-like topological recursion provides the (unique when properly normalised) solution to the so-called “abstract loop equations”. The latter express that certain polynomial combinations of the $\omega_{g,n}$ are holomorphic at the ramification points. This in fact provides tools that have been used to establish applications of the topological recursion, e.g., in matrix models [10], in Hurwitz theory [12, 29], and for the reconstruction of WKB expansions [6, 7, 41]. The setup of abstract loop equations was developed in [10, 13] for smooth curves with simple ramifications and extended in [45, Section 7.6] to higher-order ramifications. In Section 5.3, we define a notion of abstract loop equations for arbitrary spectral curves (Definition 5.12) and prove they admit at most one normalised solution (Proposition 5.15), which must then be given by our extension of the CEO recursion, see (5.7). The question of existence of a solution is then reduced to proving that this formula yields symmetric $\omega_{g,n}$. As we establish the equivalence of the abstract loop equations with the differential constraints built from $\mathcal{W}(\mathfrak{gl}_r)$ -algebra representations (Proposition 5.18), it is sufficient to check that the latter form an Airy structure to establish symmetry. If the associated PDEs admit a partition function solving them to leading order in \hbar , this is sufficient to prove symmetry of the $\omega_{0,n}$ (Theorem 5.26).

Our results give a definition of topological recursion for many spectral curves that could not be treated before, for instance,

$$\begin{aligned} y(x - y^r) &= 0, & r \geq 1, \\ (1 - xy^3) &= 0, \\ x^2 - y^2 &= 0, \\ (1 - xy^3)(x - y + 1) &= 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned}
(1 - xy^3)(x - y + 1)(x - y + t) &= 0, & t \in \mathbb{C} \setminus \{1\}, \\
(1 - xy^3)(x - y + 1)(y - t) &= 0, & t \in \mathbb{C}, \\
(1 - xy^2)(x - y + 1)(x - ty) &= 0, & t \in \mathbb{C} \setminus \{1\}.
\end{aligned}$$

In general, nodal points such that y tends to distinct non-zero values on both sides are admissible. If $C_i: P_i(x, y) = 0$ are admissible for $i \in \{1, 2\}$ and do not intersect in \mathbb{C}^2 at zeroes of dx or singular points of C_i , then $C: P_1(x, y)P_2(x, y) = 0$ is admissible. The complete list of conditions defining admissible spectral curves according to our work can be found in Section 5.4, and they only regulate the behavior of x and y at the points where the cardinality of the fibers of x jump (zeroes of dx and singular points). Examples of non-admissible curves are:

$$\begin{aligned}
1 - x^2y^5 &= 0, \\
(x - y^2)(1 - xy^2) &= 0, \\
(x - y)(x - y^r) &= 0 \quad r \in \mathbb{Z}, \\
x^p - y^q &= 0 \quad p, q \text{ coprime and } p > 1, q > 0, \\
(x - y^2)^2 &= 0, \\
(x - y^2)(x - y^3) &= 0.
\end{aligned} \tag{1.2}$$

In general, the following cases are not admissible for us: non-reduced curves, curves where dy and dx have a common zero, reducible curves where there is a point at which at least two irreducible components C_1 and C_2 meet and where dx has a zero and y is regular on C_1 and y is not identically zero on C_2 . It would be desirable to understand if the admissibility conditions can be weakened even more with a suitable modification of the topological recursion residue formula.

We expect all the partition functions of the Airy structures present in this article to admit an enumerative interpretation, i.e., that $\omega_{g,n}$ or $F_{g,n}$ can be computed via intersection theory on a certain moduli space of curves. This was achieved by Eynard in [32, 33] in the case of simple ramification points on smooth curves and y holomorphic, in a form that has a structure similar to the ELSV formula [31], and found applications in Gromov–Witten theory [38] and Hurwitz theory [46, 55]. The case of y having a simple pole was later treated by Chekhov and Norbury [24, 49]. Part III explores the generalisations of this link to other spectral curves that can directly be reached by combining known results with the results of Parts I and II. This stresses the role of Laplace-type integrals on the spectral curve. Although it is not essential in the theory, in the case of global spectral curves the Laplace transform of $\omega_{0,2}$ enjoys a factorisation property reminiscent of the use of R -matrices in the theory of Frobenius manifolds. This was known by [33, Appendix B] for smooth spectral curves with simple ramifications, and we show in Corollary 7.12 that it extends to singular spectral curves in a slightly different form.

In Sections 7.2.3–7.5.4, building on [9] and Theorem 5.23 we generalise Eynard’s formula from [32, 33] to all smooth spectral curves with arbitrary ramification and y of order 1 at the ramification points – this involves Witten spin classes. In the case of global smooth spectral curves (i.e., obeying the precise conditions Definition 7.27), it leads in Corollary 7.29 to an intersection-theoretic formula for decomposition of the $\omega_{g,n}$ on the descendants of the primary differentials. We deduce in Corollary 7.32 that the coefficients of expansion of $\omega_{g,n}$ near simple poles of dx are given by an ELSV-like formula and displaying a quasi-polynomiality property. This answers a question of Shadrin to the first-named author. In Section 8, we apply our general results to the $\mathcal{W}(\mathfrak{gl}_3)$ -constraints of Alexandrov [1] for the open intersection theory developed in [18, 21, 51, 56]. The open intersection numbers can be packaged in a generating series $\omega_{g,n}^{\text{open}}$. As Alexandrov’s $\mathcal{W}(\mathfrak{gl}_3)$ -constraints have been identified with an Airy structure in [9] for $\sigma = (12)(3)$, we deduce from the results of Part II that $\omega_{g,n}^{\text{open}}$ satisfies our extension of the CEO topological recursion applied to the curve $y(y^2 - 2x) = 0$ (Corollary 8.8). Using modified \mathcal{W} -constraints, Safnuk had derived in [53] a residue formula associated to this curve, but its structure is different and not easily generalisable. On the other hand, we can easily conjecture a residue formula for the open r -spin theory. This conjecture is equivalent to the $\mathcal{W}(\mathfrak{gl}_r)$ constraints first mentioned in [9, Section 6.3]: we expose it in more detail, in particular specifying the normalisations necessary for the comparison, and give some support in its favor by comparison with [20].

Remark 1.1. At the time of writing, there are several foundational conjectures in open intersection theory. In Sections 8.1 and 8.5, we formulate the ones that are directly relevant for us and explain the logical dependence of our statements on these conjectures.

In fact, one of the initial motivations of our work was to generalise the structure of Safnuk’s residue formula [53] to higher r and seek along this line for a definition of topological recursion for curves with many irreducible components. Our conclusion is that, although we do not know how to generalise the structure of Safnuk’s recursion, there is a simpler and general definition of the recursion, which retrieves open intersection theory when applied to the reducible curve $y(y^2 - 2x) = 0$.

The $\mathcal{W}(\mathfrak{gl}_r)$ -representations that we consider have an explicit though lengthy expression. We extract from them a few concrete calculations:

- General formulas for $F_{0,3}$, $F_{\frac{1}{2},2}$ and $F_{1,1}$, and partial computations for $F_{0,4}$ in the undeformed case. Their symmetry gives necessary constraints for the twist σ and the dilaton shifts. Remarkably, the symmetry conditions governing $F_{0,3}$ and $F_{0,4}$ exactly match those found in Theorem 2.13 which are sufficient for all $(F_{0,n})_{n \geq 0}$ to be symmetric. Taking the additional symmetry conditions of $F_{\frac{1}{2},2}$ for a generic

choice of $F_{\frac{1}{2},1}$ into account, we find that these are only slightly weaker than the sufficient conditions under which we have an Airy structure according to Theorem 2.11. Hence, we are inclined to think that also this result is generically optimal. This could be implied by the analysis of the symmetry of $F_{\frac{1}{2},n}$ for higher $n \geq 3$.

- If $\sigma = (1 \cdots r - 1)(r)$ or $(1 \cdots r)$, we derive in Section 2.4 from the \mathcal{W} -constraints a homogeneity property, the dilaton equation, and the string equation when it applies.

While this work was in the final writing stage, we learned that results similar to our Theorem 2.11 but restricted to the case of cycles of equal lengths (perhaps with a fixed point) are obtained in an independent work of Bouchard and Mastel [17]. The question of defining a Chekhov–Eynard–Orantin topological recursion on singular curves mentioned in their work is solved by Part II of our work.

Notation

As usual, \mathbb{N} is the set of nonnegative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For $n \in \mathbb{N}$, we denote $[n] = \{1, \dots, n\}$, and in particular $[0] = \emptyset$. More generally, if $a \leq b$ are integers, we denote by $[a, b]$, $[a, b)$, $(a, b]$, (a, b) the integer segments, where the bracket (resp. parenthesis) means we include (resp. exclude) the corresponding endpoint. If $a > b$, we set $[a, b] = \emptyset$. Furthermore, $z_{[n]} = \{z_1, \dots, z_n\}$.

A partition of $r \in \mathbb{N}$, denoted $\lambda \vdash r$, is $\lambda = (\lambda_1, \dots, \lambda_\ell)$ such that $\lambda_1 + \dots + \lambda_\ell = r$. We will sometimes (but not always) require $\lambda_j \geq \lambda_{j+1}$, in which case we say that λ is a *descending partition*. Often it is convenient to express equal blocks $\lambda_j = \lambda_{j+1} = \dots = \lambda_{j+n}$ as λ_j^n . Moreover, to any descending partition λ one can associate a Young diagram \mathbb{Y}_λ in a bijective way. For example, all notations

$$\lambda = (4, 3, 3, 1) \leftrightarrow \lambda = (4, 3^2, 1) \leftrightarrow \mathbb{Y}_\lambda = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \\ \hline \square & \square & \square & \\ \hline \square & & & \\ \hline \end{array}$$

characterise the same descending partition $\lambda \vdash 11$. The *size* of λ is $|\lambda| = \sum_i \lambda_i$ and its *length* is $\ell(\lambda) = \max\{i \mid \lambda_i > 0\}$. Given partitions $\lambda^1, \dots, \lambda^n$ of integers r_1, \dots, r_n , we will write $(\lambda^1, \dots, \lambda^n) := (\lambda_1^1, \dots, \lambda_{\ell(\lambda^1)}^1, \lambda_1^2, \dots, \lambda_{\ell(\lambda^n)}^n)$ for their concatenation, which we see as a partition of $\sum_{j=1}^n r_j$.

If A is a finite set, we write $\mathbf{L} \vdash A$ to say that \mathbf{L} is a partition of A , that is, an unordered tuple of pairwise disjoint non-empty subsets of A whose union is A . We denote by $\|\mathbf{L}\|$ the number of sets in the partition \mathbf{L} .

All of our vector spaces or algebraic spaces are over \mathbb{C} . We denote by $\mathbb{C}\langle y \rangle$ a one-dimensional complex vector space equipped with a non-zero linear form y .

Part I

Classification of $\mathcal{W}(\mathfrak{gl}_r)$ -Airy structures

2. Constructing Airy structures

In this section, we recall the definition of Airy structures and their partition function and its adaptation to the infinite-dimensional setting for which it will be used. We present the main results of Part I, i.e., we exhibit the new Airy structures that can be constructed from $\mathcal{W}(\mathfrak{gl}_r)$ -algebra modules, while the proofs are carried out in Section 3.

2.1. Preliminaries on Airy structures

2.1.1. Finite dimension. We first present the definition when E is a finite-dimensional \mathbb{C} -vector space. For convenience, let us fix a basis $(e_a)_{a \in A}$ of E and $(x_a)_{a \in A}$ be the dual basis of E^* . We consider the graded algebra of differential operators \mathcal{D}_E^{\hbar} , also called *Weyl algebra*. It is the quotient of the free algebra generated by $\hbar^{\frac{1}{2}}$, $(x_a)_{a \in A}$ and $(\hbar \partial_{x_a})_{a \in A}$, modulo the relations generated by

$$[\hbar \partial_{x_a}, x_b] = \hbar \delta_{a,b}, \quad [x_a, x_b] = [\hbar \partial_{x_a}, \hbar \partial_{x_b}] = 0, \quad a, b \in I, \quad \hbar \text{ central},$$

and equipped with the grading

$$\deg x_a = \deg \hbar \partial_{x_a} = \deg \hbar^{\frac{1}{2}} = 1.$$

We will write $P = Q + \mathcal{O}(n)$ for two elements $P, Q \in \mathcal{D}_E^{\hbar}$ if they agree up to at least degree $n - 1$ and use the notation

$$P = Q + o(\hbar^n) \quad \text{if } P - Q \in \hbar^n \mathcal{D}_E^{\hbar}.$$

Definition 2.1. A family $(H_i)_{i \in I}$ of elements of \mathcal{D}_E^{\hbar} is an *Airy structure on E in normal form* with respect to the basis $(x_a)_{a \in A}$ if $I = A$ and it satisfies the following conditions:

- *The degree 1 condition:* for all $i \in I$, we have

$$H_i = \hbar \partial_{x_i} + \mathcal{O}(2). \tag{2.1}$$

- *The subalgebra condition:* there exist $f_{i,j}^k \in \mathcal{D}_E^{\hbar}$ such that for all $i, j \in I$

$$[H_i, H_j] = \hbar \sum_{k \in A} f_{i,j}^k H_k. \tag{2.2}$$

A family $(H_i)_{i \in I}$ of elements of \mathcal{D}_E^{\hbar} is an *Airy structure* if there exist two matrices $\mathcal{N} \in \mathbb{C}^{A \times I}$ and $\mathcal{M} \in \mathbb{C}^{I \times A}$ such that

$$\begin{aligned} \forall a, b \in A, \quad \sum_{i \in I} \mathcal{N}_{a,i} \mathcal{M}_{i,b} &= \delta_{a,b}, \\ \forall i, j \in I, \quad \sum_{a \in A} \mathcal{M}_{i,a} \mathcal{N}_{a,j} &= \delta_{i,j} \end{aligned} \quad (2.3)$$

and the family $\tilde{H}_a = \sum_{i \in I} \mathcal{N}_{a,i} H_i$ indexed by $a \in A$ is an *Airy structure in normal form*.

Being an Airy structure does not depend on a choice of basis, but being an Airy structure in normal form does. Airy structures in Definition 2.1 would be called in [9] “crosscapped higher quantum Airy structure”. “Crosscapped” refers to the presence of half-integer powers of \hbar , and we comment on it in Section 2.3.5. “Higher” means that, compared to the definition in [4, 44], it can contain terms of degree higher than 2. “Quantum” is used to distinguish it in [4, 44] from the classical Airy structures where the Weyl algebra is replaced by the Poisson algebra of polynomial functions on T^*E . We simplified the terminology as the restriction to maximum degree 2 and the classical Airy structures will not play any role in this article and handling half-integer powers of \hbar does not lead to any complication in the theory.

The essential property of an Airy structure is that it specifies uniquely a formal function on E .

Theorem 2.2 ([44, Theorem 2.4.2], [9, Proposition 2.20]). *If $(H_i)_{i \in I}$ is an Airy structure on E , then the system of linear differential equations*

$$\forall a \in A, \quad Z^{-1} H_a Z \cdot 1 = 0 \quad (2.4)$$

admits a unique solution Z of the form $Z = \exp(F)$ with

$$F = \sum_{\substack{g \in \frac{1}{2}\mathbb{Z}_{\geq 0} \\ n \in \mathbb{Z}_{> 0} \\ 2g + 2 - n > 0}} \frac{\hbar^{g-1}}{n!} F_{g,n}, \quad F_{g,n} \in \text{Sym}^n(E^*). \quad (2.5)$$

The function Z (resp. $F_{g,n}$) is called the partition function (resp. free energies). Equation (2.4) implies a recursive formula for $F_{g,n}$ on $2g - 2 + n > 0$. We will typically be interested in the coefficients of decomposition of the free energies on a given basis $(x_a)_{a \in A}$ of linear coordinates of E , for which we use the notation

$$F_{g,n} = \sum_{a_1, \dots, a_n \in A} F_{g,n}[a_1, \dots, a_n] x_{a_1} \cdots x_{a_n}. \quad (2.6)$$

When the Airy structure has normal form with respect to this basis, the explicit recursion to obtain the $F_{g,n}[a_1, \dots, a_n]$ appears in [9, Sections 2.2 and 2.3]. We reproduce it in equation (2.32) at the only place where it is used in the article. We will be led to work with Airy structures that are not given in normal form (cf. Section 3.1), but for which there is an equivalent formulation of the recursion in terms of spectral curves (Part II), that is often more efficient for calculations (cf. Section 6).

2.1.2. Infinite dimension. In this article, we need to handle Airy structures for certain infinite-dimensional vector spaces. This requires some amendments to the previous definitions which we now explain.

A filtered vector space is a vector space E together with a collection of subspaces

$$0 \subseteq \mathcal{F}_1 E \subseteq \mathcal{F}_2 E \subseteq \dots \subseteq E,$$

called the filtration. Throughout this paper, we will assume that for any filtered vector space E , the $\mathcal{F}_p E$ are finite-dimensional, and that $E = \bigcup_{p>0} \mathcal{F}_p E$. Two filtrations $\mathcal{F}, \mathcal{F}'$ on a vector space E are equivalent if for any $p > 0$ there exists $p' > 0$ such that $\mathcal{F}_p E \subseteq \mathcal{F}'_{p'} E$ and for any $q' > 0$ there exists $q > 0$ such that $\mathcal{F}'_{q'} E \subseteq \mathcal{F}_q E$. In particular, all filtrations on a given vector space satisfying our extra assumptions are equivalent.

A filtered set is a set A together with a collection of subsets

$$\emptyset \subseteq f_1 A \subseteq f_2 A \subseteq \dots \subseteq A.$$

Again, we assume all $f_p A$ are finite and $A = \bigcup_{p>0} f_p A$. A filtered basis of a filtered vector space (E, \mathcal{F}) is the data of a filtered set (A, f) and a family $(e_a)_{a \in A}$ of elements of E such that $(e_a)_{a \in f_p A}$ is a basis of $\mathcal{F}_p E$.

Let (E, \mathcal{F}) be a filtered vector space, and for convenience choose a filtered basis $(e_a)_{a \in A}$ and the corresponding linear coordinates $(x_a)_{a \in A}$. We consider the completed Weyl algebra with respect to this filtration, $\widehat{\mathcal{D}}_E^{\hbar}$. It consists of elements of the form

$$\sum_{\substack{m, n \in \mathbb{Z}_{\geq 0} \\ j \in \frac{1}{2}\mathbb{Z}_{\geq 0}}} \sum_{\substack{a_1, \dots, a_n \in A \\ \bar{a}_1, \dots, \bar{a}_m \in A}} \frac{\hbar^j}{n!m!} C^{(j)}[\bar{a}_1, \dots, \bar{a}_m; a_1, \dots, a_n] x_{\bar{a}_1} \cdots x_{\bar{a}_m} \hbar \partial_{x_{a_1}} \cdots \hbar \partial_{x_{a_n}}, \quad (2.7)$$

where for any $p > 0$, the coefficients $C^{(j)}[\bar{a}_1, \dots, \bar{a}_m; a_1, \dots, a_n]$ vanish for all but finitely many $a_1, \dots, a_n \in f_p A$ and $j, \bar{a}_1, \dots, \bar{a}_m$. One can check this does form a (graded) algebra.

If (I, f') is a filtered set (which is unrelated to (A, f)), we say that a family $(D_i)_{i \in I}$ of elements of $\widehat{\mathcal{D}}_E^{\hbar}$ is filtered if for any $p > 0$, the coefficients

$$C_i^{(j)}[\bar{a}_1, \dots, \bar{a}_m; a_1, \dots, a_n]$$

in decomposition (2.7) of D_b vanish for all but finitely many $a_1, \dots, a_n \in f_p A$ and $i, j, \bar{a}_1, \dots, \bar{a}_n$.

Definition 2.3. A filtered family $(H_i)_{i \in I}$ of elements of $\widehat{\mathcal{D}}_E^{\hbar}$ is an Airy structure on E in normal form with respect to $(x_a)_{a \in A}$ if $(I, f') = (A, f)$, and

- the degree one condition holds;
- the subalgebra condition holds for some filtered family $(f_{i,j}^k)_{i,j,k \in I}$ of elements of $\widehat{\mathcal{D}}_E^{\hbar}$.

A filtered family $(H_i)_{i \in I}$ of elements of $\widehat{\mathcal{D}}_E^{\hbar}$ is an Airy structure on E if there exist $\mathcal{N} \in \mathbb{C}^{A \times I}$ and $\mathcal{M} \in \mathbb{C}^{I \times A}$ such that

- for each $i \in I$, $\mathcal{N}_{a,i}$ vanish for all but finitely many $a \in A$;
- for each $a \in A$, $\mathcal{M}_{i,a}$ vanish for all but finitely many $i \in I$;
- \mathcal{N} and \mathcal{M} are inverse to each other in the sense of (2.3), where the sums are finite due to the previous two points;
- the family $\tilde{H}_a = \sum_{i \in I} \mathcal{N}_{a,i} H_i$ indexed by $a \in A$ (which is a filtered family of elements of $\widehat{\mathcal{D}}_E^{\hbar}$ by the first point) is an Airy structure on E in normal form with respect to $(x_a)_{a \in A}$.

The notion of Airy structure does not depend on the choice of filtered basis, and only depends on the equivalence class of the filtration of E . We will sometimes omit to specify the filtration when it is evident. The existence and uniqueness of the partition function (Theorem 2.2) extends to this infinite-dimensional setting, and for each g, n , and p , the summation of (2.6) with a_1 restricted to $f_p A$ is finite, that is, $F_{g,n} \in \overline{\text{Sym}}^n(E^*)$.

2.2. Preliminaries on the $\mathcal{W}(\mathfrak{gl}_r)$ -algebra and its twisted modules

The Airy structures constructed in this paper are obtained by considering twisted modules of Heisenberg vertex operator algebras (VOAs), taking subalgebras of the associated algebras of modes, and using a dilaton shift to break homogeneity. The idea of this construction dates back to [48]. It was developed more systematically in [9, Sections 3 and 4], and we refer to that paper for details. Here we summarise the main points of the construction.

2.2.1. The Heisenberg vertex operator algebra and $\mathcal{W}(\mathfrak{gl}_r)$ -algebra.

Definition 2.4. Let \mathfrak{h} be a finite-dimensional vector space with non-degenerate inner product $\langle \cdot, \cdot \rangle$. The Heisenberg Lie algebra associated to \mathfrak{h} is given by

$$\widehat{\mathfrak{h}} = (\mathfrak{h}[t^{\pm 1}] \oplus \mathbb{C}K) \otimes \mathbb{C}_{\hbar}, \quad [\xi_l + aK, \eta_m + bK] = \hbar \langle \xi, \eta \rangle l \delta_{l+m,0} K,$$

where we write $\xi_l := \xi \otimes t^l$, and $\mathbb{C}_{\hbar} := \mathbb{C}[\hbar^{\frac{1}{2}}]$. The associated *Weyl algebra* is defined as a quotient of its universal enveloping algebra: $\mathcal{H}_{\hbar} := \mathfrak{U}(\widehat{\mathfrak{h}})/(K-1)$. The *Fock space* \mathcal{F}_{\hbar} is the representation of \mathcal{H}_{\hbar} generated by a vector $|0\rangle$ and relations $\xi_l|0\rangle = 0$ for $\xi \in \mathfrak{h}$, $l \geq 0$. The *Heisenberg VOA* is the vertex operator algebra with underlying vector space \mathcal{F}_{\hbar} equipped with vacuum $|0\rangle$, state-field correspondence $Y: \mathcal{F}_{\hbar} \rightarrow \text{End } \mathcal{F}_{\hbar}[[\tilde{x}^{\pm 1}]]$ given by

$$\begin{aligned} Y(|0\rangle, \tilde{x}) &= \text{id}_{\mathcal{F}_{\hbar}}, \\ Y(\xi_{-1}|0\rangle, \tilde{x}) &= \sum_{l \in \mathbb{Z}} \xi_l \tilde{x}^{-l-1}, \\ Y(\xi_{-k_1}^1 \cdots \xi_{-k_n}^n |0\rangle, \tilde{x}) &= : \frac{1}{(k_1-1)!} \frac{d^{k_1-1}}{d\tilde{x}^{k_1-1}} Y(\xi_{-1}^1 |0\rangle, \tilde{x}) \cdots \\ &\quad \times \frac{1}{(k_n-1)!} \frac{d^{k_n-1}}{d\tilde{x}^{k_n-1}} Y(\xi_{-1}^n |0\rangle, \tilde{x}) :, \end{aligned} \quad (2.8)$$

and conformal vector $\varpi = \frac{1}{2} \sum_j \chi_{-1}^j \chi_{-1}^j |0\rangle$ for an orthonormal basis $(\chi^j)_j$ of \mathfrak{h} .

The \mathcal{W} -algebra associated to the general linear Lie algebra \mathfrak{gl}_r at the self-dual level is denoted by $\mathcal{W}(\mathfrak{gl}_r)$. It can be constructed as a sub-VOA of the Heisenberg VOA \mathcal{F}_{\hbar} attached to its Cartan subalgebra $\mathfrak{h} \cong \mathbb{C}^r \subset \mathfrak{gl}_r$. We identify \mathfrak{h} with its dual using the Killing form and take $\chi^j \in \mathfrak{h}$ to correspond to the roots under this identification. Note that they are orthonormal.

Theorem 2.5 ([5, 47]). *$\mathcal{W}(\mathfrak{gl}_r)$ is the sub-VOA of \mathcal{F}_{\hbar} freely and strongly generated by the elements*

$$w_i = e_i(\chi_{-1}^1, \dots, \chi_{-1}^r) |0\rangle, \quad i \in [r], \quad (2.9)$$

where e_i is the i -th elementary symmetric polynomial in r variables.

We introduce the modes $W_{i,k}$ and their (i -form valued) generating series $W_i(\tilde{x})$ with the formulas

$$W_i(\tilde{x}) := \sum_{k \in \mathbb{Z}} \frac{W_{i,k} (d\tilde{x})^i}{\tilde{x}^{i+k}} := Y(w_i, \tilde{x})(d\tilde{x})^i. \quad (2.10)$$

Remark 2.6. Contrarily to [9], we do not include a factor of r^{i-1} in the definition of w_i . Our convention is that $W_{i,k}$ is the coefficient of $\tilde{x}^{-(k+i)}$. This coincides with the convention taken in [9, Section 3.3.4] but differs from the convention $\tilde{x}^{-(k+1)}$ used in the rest of [9]. We also find it convenient to consider the generating series of modes of conformal weight i to be i -differential forms. The variable \tilde{x} is often denoted by z in the VOA literature. However, in Part II, we will see that this variable can be interpreted as the pullback under the normalisation morphism of a function

usually denoted x for a spectral curve. For consistency, we therefore chose to use the letter \tilde{x} , and use the letter z for local coordinates on (the normalisation of) the spectral curve.

2.2.2. The mode algebra and its subalgebras. Let \mathcal{A} be the associative algebra of modes of $\mathcal{W}(\mathfrak{gl}_r)$, see [40, Section 4.3]. Furthermore, let $L(\mathcal{A})$ denote the set of possibly infinite sums of ordered monomials in \mathcal{A} whose degree and conformal weight is bounded; we equip it with the bracket $\hbar^{-1}[\cdot, \cdot]$, making it a Lie algebra.

Definition 2.7. We say that a subset $S \subset [r] \times \mathbb{Z}$ of modes generates a Lie subalgebra of $L(\mathcal{A})$ if the left \mathcal{A} -ideal generated by $\bigoplus_{(i,k) \in S} W_{i,k}$ is a Lie subalgebra of $L(\mathcal{A})$. An equivalent condition is the existence of $f_{(i,k),(i',k')}^{(j,l)} \in L(\mathcal{A})$ such that

$$\forall (i,k), (i',k') \in S, \quad [W_{i,k}, W_{i',k'}] = \hbar \sum_{(j,l) \in I_\lambda} f_{(i,k),(i',k')}^{(j,l)} W_{j,l}.$$

More precisely, it is first required that the right-hand side defines an element of $L(\mathcal{A})$, and then that it coincides with the left-hand side.

Given a partition $\lambda \vdash r$, we set

$$\lambda(i) := \min \left\{ m \geq 0 \mid \sum_{j=1}^m \lambda_j \geq i \right\},$$

and we define the index set

$$I_\lambda := \{(i,k) \in [r] \times \mathbb{Z} \mid \lambda(i) + k > 0\}. \quad (2.11)$$

Theorem 2.8 ([9, Theorem 3.16]). *For any descending partition $\lambda \vdash r$, I_λ generates a Lie subalgebra of $L(\mathcal{A})$.*

2.2.3. Twisted modules. Let $\sigma \in \mathfrak{S}_r$ be an arbitrary element of the Weyl group of \mathfrak{gl}_r . It is a permutation of the elements χ^i and can thus be decomposed into $d \geq 1$ cycles $\sigma = \sigma_1 \cdots \sigma_d$ with each cycle σ_μ of length $r_\mu \geq 1$ such that $r_1 + \cdots + r_d = r$. If $d = 1$, then σ is a transitive element. After relabelling of the elements of the basis of the Cartan, we can assume that σ_μ acts as

$$\sigma_\mu: \chi^{1+r[\mu-1]} \rightarrow \chi^{2+r[\mu-1]} \rightarrow \cdots \rightarrow \chi^{r_\mu+r[\mu-1]} \rightarrow \chi^{1+r[\mu-1]},$$

while keeping all other χ^i fixed, where we introduced the notation $\mathbf{r}_{[\mu]} := \sum_{\nu=1}^{\mu} r_\nu$. It is then easy to check that

$$v^{\mu,a} := \sum_{j=1}^{r_\mu} \vartheta_{r_\mu}^{-aj} \chi^{j+\mathbf{r}_{[\mu-1]}}, \quad \mu \in [d], \quad a \in [r_\mu], \quad \vartheta_{r_\mu} := e^{2i\pi/r_\mu}$$

is an eigenvector of the action of σ on \mathfrak{h} with eigenvalue $\vartheta_{r_\mu}^a$. We define the *currents* via the state field-correspondence

$$J^{\mu,a}(\tilde{x}) := {}^\sigma Y(v_{(-1)}^{\mu,a} |0\rangle, \tilde{x}) d\tilde{x} := \sum_{k \in \frac{a}{r_\mu} + \mathbb{Z}} J_{r_\mu k}^\mu \frac{d\tilde{x}}{\tilde{x}^{k+1}} \quad (2.12)$$

with fractional mode expansion on these eigenvectors, after introducing the differential operators

$$J_k^\mu = \begin{cases} \hbar \partial_{x_k^\mu} & \text{if } k > 0, \\ \hbar^{\frac{1}{2}} Q_\mu & \text{if } k = 0, \\ -k, x_{-k}^\mu & \text{if } k < 0 \end{cases} \quad (2.13)$$

with $Q_\mu \in \mathbb{C}$, acting on the space

$$\mathcal{T} := \mathbb{C}_{\hbar^{\frac{1}{2}}}[(x_a^\mu)_{\mu \in [d], a > 0}].$$

Note that the formal variables \tilde{x} and x_i^μ are unrelated. The state-field correspondence ${}^\sigma Y$ can be extended to the whole space $\mathcal{F}_{\mathfrak{h}}$ by using formula (2.8). This turns $(\mathcal{T}, {}^\sigma Y)$ into a twisted representation of $\mathcal{F}_{\mathfrak{h}}$, whose restriction to $\mathcal{W}(\mathfrak{gl}_r)$ becomes an (untwisted) representation of $\mathcal{W}(\mathfrak{gl}_r)$. For details see [9, 28]. We define the *twist modes* ${}^\sigma W_{i,k}$ by

$${}^\sigma W_i(\tilde{x}) := {}^\sigma Y(w_i, \tilde{x}) (d\tilde{x})^i = \sum_{k \in \mathbb{Z}} {}^\sigma W_{i,k} \frac{(d\tilde{x})^i}{\tilde{x}^{k+i}},$$

where w_i are the strong generators of $\mathcal{W}(\mathfrak{gl}_r)$ from (2.9). They are differential operators acting on \mathcal{T} .

Lemma 2.9 ([9, Proposition 4.5 and Lemma 4.15]). *For an automorphism σ with cycles of respective lengths r_μ , the σ -twisted modes read*

$${}^\sigma W_{i,k} = \sum_{M \subseteq [d]} \sum_{\substack{i_\mu \in [r_\mu], \mu \in M \\ \sum_\mu i_\mu = i}} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^M \\ \sum_\mu k_\mu = k}} \prod_{\mu \in M} \frac{1}{r_\mu^{i_\mu - 1}} W_{i_\mu, k_\mu}^\mu, \quad (2.14)$$

where the W_{i_μ, k_μ}^μ are defined by

$$W_{i_\mu, k_\mu}^\mu = \frac{1}{r_\mu} \sum_{j_\mu=0}^{\lfloor i_\mu/2 \rfloor} \frac{i_\mu! \hbar^{j_\mu}}{2^{j_\mu} j_\mu! (i_\mu - 2j_\mu)!} \times \sum_{\substack{p_{2j_\mu+1}^\mu, \dots, p_{i_\mu}^\mu \in \mathbb{Z} \\ \sum_l p_l^\mu = r_\mu k_\mu}} \Psi_{r_\mu}^{(j_\mu)}(p_{2j_\mu+1}^\mu, p_{2j_\mu+2}^\mu, \dots, p_{i_\mu}^\mu) : \prod_{l=2j_\mu+1}^{i_\mu} J_{p_l^\mu}^\mu :, \quad (2.15)$$

with coefficients $\Psi_{r\mu}^{(j\mu)}(\dots) \in \mathbb{Q}$ admitting a representation in terms of sums over r -th roots of unity

$$\begin{aligned} & \Psi_r^{(j)}(a_{2j+1}, \dots, a_i) \\ & := \frac{1}{i!} \sum_{\substack{m_1, \dots, m_i=0 \\ m_l \neq m_{l'}}}^{r-1} \left(\prod_{l'=1}^j \frac{\vartheta^{m_{2l'-1} + m_{2l'}}}{(\vartheta^{m_{2l'}} - \vartheta^{m_{2l'-1}})^2} \prod_{l=2j+1}^i \vartheta^{-m_l a_l} \right), \end{aligned} \quad (2.16)$$

where $\vartheta = e^{2i\pi/r}$.

In terms of generating series, this lemma can be restated as

$$\begin{aligned} {}^\sigma W_i(\tilde{x}) &= \sum_{M \subseteq [d]} \sum_{\substack{i_\mu \in [r_\mu], \mu \in M \\ \sum_\mu i_\mu = i}} \prod_{\mu \in M} \frac{W_{i_\mu}^\mu(\tilde{x})}{r_\mu^{i_\mu - 1}}, \\ W_i^\mu(\tilde{x}) &:= \frac{1}{r_\mu} \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{i! \hbar^j}{2^j j! (i - 2j)!} \left(\frac{d\tilde{x}}{\tilde{x}} \right)^{2j} \\ & \times \left(\sum_{a_{2j+1}, \dots, a_i=0}^{r_\mu - 1} \Psi_{r_\mu}^{(j)}(a_{2j+1}, \dots, a_i) : \prod_{l=2j+1}^i J^{\mu, a_l}(\tilde{x}) : \right). \end{aligned} \quad (2.17)$$

If we also take a generating series in i by defining

$$\begin{aligned} \mathcal{W}^\mu(\tilde{x}, u) &:= \frac{u^{r_\mu}}{r_\mu} + \sum_{i=1}^{r_\mu} W_i^\mu(\tilde{x}) u^{r_\mu - i}, \\ \mathcal{W}(\tilde{x}, u) &:= u^r + \sum_{i=1}^r {}^\sigma W_i(\tilde{x}) u^{r-i}, \end{aligned}$$

this can be written compactly as

$$\mathcal{W}(\tilde{x}, u) = \prod_{\mu=1}^d r_\mu \mathcal{W}^\mu(\tilde{x}, u).$$

Introducing the filtered vector space

$$E = \bigoplus_{k>0} \bigoplus_{\mu=1}^d \mathbb{C} \langle x_k^\mu \rangle, \quad E_p := \bigoplus_{k \leq p} \bigoplus_{\mu=1}^d \mathbb{C} \langle x_k^\mu \rangle, \quad (2.18)$$

we see from the condition of summations in (2.14)–(2.15) that each $W_{i,k}$ belongs to the completed Weyl algebra $\widehat{\mathcal{D}}_E^\hbar$ according to the definitions in Section 2.1.2. Even

more, any family $(W_{i,k})_{(i,k) \in I}$ for which $\min_I k > -\infty$ is a filtered family of elements of $\widehat{\mathcal{D}}_E^{\hbar}$. Our goal is to construct Airy structures from these operators. The degree one condition requires the operators to have the form $\hbar \partial + \mathcal{O}(2)$. Unfortunately, ${}^\sigma W_{i,k}$ does not have this form as it is homogeneous of degree $\deg {}^\sigma W_{i,k} = i$ in view of (2.14). Thus in order to construct an Airy structure, we first have to break this homogeneity.

2.3. Airy structures from twisted $\mathcal{W}(\mathfrak{gl}_r)$ modules

We can now summarise the main findings of Part I. Before presenting the new basic Airy structures we have found in Section 2.3.2, we review in Section 2.3.1 the ones known from [9]. Section 2.3.3 gives the general form of the Airy structures that can be obtained by deformation of those basic ones through conjugation in the Weyl algebra, which will be important for the correspondence with spectral curves in Part II. This provides so far *sufficient conditions* on the parameters of our construction so that it does produce Airy structures. It is natural to wonder if these conditions are also necessary (classification problem). In Section 2.3.4, we summarise the conditions we have found *necessary* to get Airy structures.

2.3.1. From a twist with a transitive element. First, let σ be a transitive element, i.e., $d = 1$. Then

$${}^\sigma W_{i,k} = r^{1-i} W_{i,k}^1,$$

and to simplify the notation we can omit the superscript 1. One can break up the homogeneity of the differential operators ${}^\sigma W_{i,k}$ by performing a *dilaton shift*

$$J_{-s} \rightarrow J_{-s} - 1 \tag{2.19}$$

for a fixed $s > 0$ while keeping all other J_a for $a \neq -s$ unchanged. Formally, one defines

$${}^\sigma H_{i,k} := \widehat{T} \cdot {}^\sigma W_{i,k} \cdot \widehat{T}^{-1}, \quad \widehat{T} := \exp\left(-\frac{J_s}{\hbar s}\right).$$

It follows from the Baker–Campbell–Hausdorff formula that conjugating with \widehat{T} means shifting the J_s as in (2.19). The action on the completed Weyl algebra $\widehat{\mathcal{D}}_E^{\hbar}$ is then well-defined. Then, certain subsets of the modes ${}^\sigma H_{i,k}$ yield Airy structures.

Theorem 2.10 ([9, Theorem 4.9]). *Let $\sigma \in \mathfrak{S}_r$ be transitive and $s \in [r + 1]$ with $r = \pm 1 \pmod s$ and $J_0 = 0$. Let us define $\lambda^{r,s} \vdash r$ to be the descending partition*

$$\lambda^{r,s} = \begin{cases} (r) & \text{if } s = 1, \\ (1^r) & \text{if } s = r + 1, \\ ((r' + 1)^{r''}, (r')^{s-r''}) & \text{if } r = r's + r'' \text{ with } r'' \in \{1, s - 1\}. \end{cases} \tag{2.20}$$

Then, the family of operators

$$r^{i-1} {}^\sigma H_{i,k}, \quad i \in [r], \quad k \geq 1 - \lambda^{r,s}(i) + \delta_{i,1}$$

forms an Airy structure in normal form on $E = \bigoplus_{k>0} \mathbb{C}\langle x_k \rangle$, seen as a filtered vector space when equipped with the filtration $E_p = \bigoplus_{k \leq p} \mathbb{C}\langle x_k \rangle$.

The partition $\lambda^{r,s}$ chosen in (2.20) to define the mode set determines the sub-algebra associated to this Airy structure by using Theorem 2.8. The corresponding Young diagrams are depicted in Table 1.

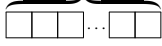
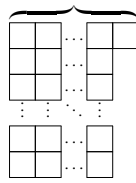
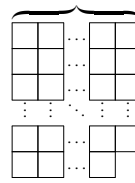
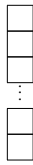
r, s	$s = 1$	$r = r's + 1$	$r = r's + s - 1$	$s = r + 1$
$\lambda^{r,s}$	r boxes 	$(r'+1)$ columns 	$(r'+1)$ columns 	

Table 1. The partitions $\lambda^{r,s}$ associated to different values s .

2.3.2. A generalisation to arbitrary twists. Let $\sigma \in \mathfrak{S}_r$ be a permutation with d cycles of respective lengths r_1, \dots, r_d , so that $r_1 + \dots + r_d = r$. Then the differential operators ${}^\sigma W_{i,k}$ act on the space

$$\mathbb{C}_{\hbar^{\frac{1}{2}}} \left[(x_a^\mu)_{\mu \in [d], a > 0} \right].$$

Again, we will break up the homogeneity of ${}^\sigma W_{i,k}$ by performing a dilaton shift. There are d independent families of variables $(x_a^\mu)_{a>0}$ labelled by $\mu \in [d]$ in which we can perform the shift. Two types of shifts will in fact lead to Airy structures:

- simultaneous shifts in each of the d sets of variables;
- simultaneous shifts in all but one set of variables, and the label μ of the unshifted set of variables corresponds to a fixed point $r_\mu = 1$.

Let us thus choose $s_\mu \in \mathbb{N}^* \cup \{\infty\}$ and, for each μ such that $s_\mu \neq \infty$, $t_\mu \in \mathbb{C}^*$, for each $\mu \in [d]$ and define

$${}^\sigma H_{i,k} := \hat{T} \cdot {}^\sigma W_{i,k} \cdot \hat{T}^{-1}, \quad \hat{T} := \prod_{\mu \in [d]: s_\mu \neq \infty} \exp\left(-r_\mu t_\mu \frac{J_{s_\mu}^\mu}{\hbar s_\mu}\right). \quad (2.21)$$

We chose to include a normalisation factor r_μ to simplify later computations. Remember that conjugating with \hat{T} means nothing but shifting

$$\forall \mu \in [d], \quad k \in \mathbb{Z}, \quad J_{-k}^\mu \rightarrow J_{-k}^\mu - r_\mu t_\mu \delta_{k, s_\mu}$$

and its action on the completed Weyl algebra $\widehat{\mathcal{D}}_E^\hbar$ is well-defined. It turns out that, with the right choice of parameters, certain subsets of these operators ${}^\sigma H_{i,k}$ indeed form an Airy structure.

Theorem 2.11. *Let $d \geq 2$, $r_1, \dots, r_d \geq 1$ and $s_1, \dots, s_d \in \mathbb{N}^* \cup \{\infty\}$ be such that*

$$\frac{r_1}{s_1} \geq \frac{r_2}{s_2} \geq \dots \geq \frac{r_d}{s_d}. \quad (2.22)$$

Let $Q_1, \dots, Q_d \in \mathbb{C}$, $t_1, \dots, t_{d-1} \in \mathbb{C}^$, and $t_d \in \mathbb{C}^*$ if $s_d \neq \infty$. Assume that*

$$\sum_{\mu=1}^d Q_\mu = 0, \quad t_\mu^{r_\mu} \neq t_\nu^{r_\nu} \quad \text{whenever } \mu \neq \nu \text{ and } (r_\mu, s_\mu) = (r_\nu, s_\nu).$$

Define $r = \sum_{\mu=1}^d r_\mu$ and let $\sigma \in \mathfrak{S}_r$ be a permutation made of disjoint cycles of respective lengths (r_1, \dots, r_d) . Assume that

- $r_1 = -1 \pmod{s_1}$;
- $s_\mu = 1$ for any $\mu \notin \{1, d\}$;
- $r_d = 1 \pmod{s_d}$,

and define $\lambda \vdash r$ to be the descending partition

$$\lambda = \begin{cases} ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}, r_d^{s_d}) & \text{if } r_d \neq 1, \\ ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}) & \text{if } r_d = 1, \end{cases} \quad (2.23)$$

where we set $r'_\mu := \lfloor \frac{r_\mu}{s_\mu} \rfloor$. Then, the family

$${}^\sigma H_{i,k}, \quad i \in [r], \quad k \geq 1 - \lambda(i) + \delta_{i,1}$$

is an Airy structure (not necessarily in normal form) on the filtered vector space E given in (2.18).

We call the case $s_d = \infty$ the *exceptional case* and the other case the *standard case*. The proof of the theorem will be presented in Section 3. The exceptional case $d = 2$ was already obtained in [9, Theorem 4.16].

Remark 2.12. Note that the conditions imply that $s_1 \in [r_1 + 1]$ and if $r_d > 1$, then $s_d \in [r_d - 1]$.

- If there are pairwise distinct $\mu_1, \mu_2, \mu_3 \in [d]$ with $\lfloor \frac{r\mu_1}{s_{\mu_1}} \rfloor = \lfloor \frac{r\mu_2}{s_{\mu_2}} \rfloor = \lfloor \frac{r\mu_3}{s_{\mu_3}} \rfloor$, then there is an $m \in \{1, 2, 3\}$ for which $s_{\mu_m} = 1$.

Then the family

$${}^\sigma H_{i,k}, \quad i \in [r], k \geq 1 - \lambda(i) + \delta_{i,1}$$

satisfies the degree one condition of an Airy structure and there is a unique solution $Z = \exp \hbar^{-1} F_0$ to

$$\forall i \in [r], k \geq 1 - \lambda(i) + \delta_{i,1}, \quad Z^{-1} {}^\sigma H_{i,k} Z \cdot 1 = o(\hbar^{\frac{1}{2}}) \quad (2.25)$$

with $F_0 = \sum_{n \geq 3} \frac{1}{n!} F_{0,n}$, where $F_{0,n} \in \text{Sym}^n(E^*)$.

Here, the partition λ specifying the mode set is obtained from the concatenation $(\lambda^{r_1, s_1}, \dots, \lambda^{r_d, s_d})$, which is built from the partitions λ^{r_μ, s_μ} specified in Table 1, by shifting a box from the first row of $\lambda^{r_{\mu+1}, s_{\mu+1}}$ to the last row of λ^{r_μ, s_μ} for each $\mu \in [d-1]$. In this construction, we are assuming that we chose a labelling satisfying (2.22). See also equation (3.21) for an explicit description of this partition. The proof of Theorem 2.13 can be found in Section 3.4.

2.3.3. Arbitrary dilaton shifts and changes of polarisation. In order to connect with the theory of the Chekhov–Eynard–Orantin topological recursion, we ought to be able to conjugate the ${}^\sigma W_{i,k}$ with more general operators, inducing dilaton shifts in several of the variables and also making a *change in polarisation*. This section is completely parallel to [9, Section 4.1.5].

First, let us consider a general dilaton shift

$$\hat{T} := \exp \left(\sum_{\substack{\mu \in [d] \\ k > 0}} (\hbar^{-1} F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] + \hbar^{-\frac{1}{2}} F_{\frac{1}{2},1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right]) \frac{J_k^\mu}{k} \right),$$

$$s_\mu := \min \{k > 0 \mid F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] \neq 0\}$$

with arbitrary scalars $F_{h,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right]$ for $h \in \{0, \frac{1}{2}\}$. The seemingly complicated way to denote these scalars will become natural in Part II, see, e.g., equation (5.14). Effectively, this shifts

$$J_{-k}^\mu \rightarrow J_{-k}^\mu + F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] + \hbar^{\frac{1}{2}} F_{\frac{1}{2},1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right]$$

and by construction of the completed Weyl algebra, its action on $\widehat{\mathcal{D}}_E^{\hbar}$ is well-defined. It should be interpreted as a deformation of the case where there is a single non-zero shift

$$F_{0,1} \left[\begin{smallmatrix} \mu \\ -s_\mu \end{smallmatrix} \right] = -r_\mu t_\mu. \quad (2.26)$$

The $F_{\frac{1}{2},1}$ give an extra possible deformation as we have allowed half-integer powers of \hbar .

Another deformation we would like to consider is the *change of polarisation*, given by conjugation with the operator

$$\widehat{\Phi} = \exp\left(\frac{1}{2\hbar} \sum_{\substack{\mu, \nu \in [d] \\ k, l > 0}} F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right] \frac{J_k^\mu J_l^\nu}{kl}\right),$$

where $F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right]$ are arbitrary scalars. Effectively, it shifts

$$J_{-k}^\mu \rightarrow J_{-k}^\mu + \sum_{\substack{\nu \in [d] \\ l > 0}} F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right] \frac{J_l^\nu}{l}. \quad (2.27)$$

Once again, the action on $\widehat{\mathcal{D}}_E^\hbar$ is well-defined. We want to consider the conjugated operators

$${}^\sigma H_{i,k} := \widehat{\Phi} \widehat{T} \cdot {}^\sigma W_{i,k} \cdot \widehat{T}^{-1} \widehat{\Phi}^{-1}. \quad (2.28)$$

Theorem 2.14. *Defining t_μ by formula (2.26) and with the same conditions for d , $(r_\mu, s_\mu, t_\mu, Q_\mu)_{\mu=1}^d$ and the same range for (i, k) as in Theorem 2.11, the operators ${}^\sigma H_{i,k}$ in equation (2.28) form an Airy structure on E .*

Remark 2.15. In the same vein, the result of Theorem 2.13 with the exact same conditions on $(r_\mu, s_\mu, t_\mu)_{\mu=1}^d$ holds for the operators (2.28) as well.

These results are proved in Section 3 and will be reformulated in terms of spectral curves in Section 5.

2.3.4. Necessary conditions. It turns out that for a family of differential operators $({}^\sigma H_{i,k})$ of the type considered in Section 2.3.2, i.e., $F_{0,1} \left[\begin{smallmatrix} \mu \\ k \end{smallmatrix} \right] = -r_\mu t_\mu \delta_{k, -s_\mu}$, the implication in Theorem 2.13 can be reversed.

Proposition 2.16. *For a family $({}^\sigma H_{i,k})_{k > -\lambda(i) + \delta_{i,1}}$ of differential operators as defined in formula (2.21), there exists a solution Z to equation (2.25) if and only if $(r_\mu, s_\mu, t_\mu)_{\mu=1}^d$ satisfy the conditions in Theorem 2.13.*

To prove this proposition, we will exploit a reinterpretation of the results of this section in terms of spectral curves which will be presented in Section 5. The proof of this proposition can be found in Section 6.3.

Motivated by Proposition 2.16, it is natural to ask whether the sufficient conditions stated in Theorem 2.11 for the family of differential operators to be an Airy structure are also necessary to impose in order to guarantee an all order solution to the associated system of differential equations. We will investigate the analogous question phrased in the setting of spectral curves in Section 6 by analysing the symmetry of the multidifferentials $\omega_{0,3}$, $\omega_{0,4}$, and $\omega_{\frac{1}{2},2}$ which are the counterparts of the free energies $F_{0,3}$, $F_{0,4}$ and $F_{\frac{1}{2},2}$. Doing so we can prove that for generic values of $(t_\mu, Q_\mu)_{\mu=1}^d$

most of the conditions stated in Theorem 2.11 are also necessary, and we believe that a more thorough analysis would actually lead to the conclusion that they are all indeed necessary.

Proposition 2.17. *Let $d \geq 2$, $r_1, \dots, r_d \geq 1$ and $s_1, \dots, s_d \in \mathbb{N}^* \cup \{\infty\}$ be such that $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$ and choose λ as in (2.23). Assume that for all $Q_1, \dots, Q_d \in \mathbb{C}$, $t_1, \dots, t_{d-1} \in \mathbb{C}^*$, and $t_d \in \mathbb{C}^*$ if $s_d \neq \infty$ such that*

$$\sum_{\mu=1}^d Q_\mu = 0, \quad \text{and} \quad t_\mu^{r_\mu} \neq t_\nu^{r_\nu} \quad \text{whenever} \quad \mu \neq \nu \quad \text{and} \quad (r_\mu, s_\mu) = (r_\nu, s_\nu),$$

the operators $({}^\sigma H_{i,k})_{k > -\lambda(i) + \delta_{i,1}}$ defined in formula (2.21) form an Airy structure. Then necessarily

- $r_1 = -1 \bmod s_1$;
- $s_\mu \in \{1, 2\}$ for all $\mu \notin \{1, d\}$;
- $r_d = 1 \bmod s_d$.

Moreover, for $d > 2$, if $(r_\mu, s_\mu) = (r_\nu, s_\nu)$ for $\mu \neq \nu$, then necessarily $s_\mu = s_\nu = 1$.

Again we need to postpone the proof of this proposition to Section 6.3 since we first have to establish the required tools. Besides that, in the course of the proof of Theorem 2.11 in Section 3, we will see that coprimality of r_μ and s_μ and non-resonance condition for the t_μ 's (Remark 3.4), as well as non-vanishing of all but maybe one dilaton shift (Remark 3.16) are obvious necessary conditions to obtain Airy structures with our method. The assumption $\sum_{\mu=1}^d Q_\mu = 0$ may not always be necessary, and we obtain finer information on this in Proposition 6.7, but we adopted it here to simplify the statement of Proposition 2.17.

2.3.5. Half-integer or integer powers of \hbar ? There are several reasons to allow half-integer powers of \hbar in Airy structures instead of just integer power. Our construction admits natural extra degrees of freedom when σ has at least two cycles, namely the parameters Q in Theorem 2.11. This is relevant for applications to open intersection theory, where we have to allow indices g to be both integer or *half-integer*, see Section 8.1 and Theorem 8.7. As it does not lead to any complications, we write the whole article allowing $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. The possibility of half-integer genus was already addressed in [9], which called such Airy structures *crosscapped*. If all monomials in H_i only feature integer powers of \hbar then $F_{g,n}$ in (2.5) vanishes for half-integers g . It is therefore straightforward to specialise our results to allow only integer g , as is more common in topological recursions. Let us however note that half-integer g already made their appearance in certain other applications of the topological recursion, such as non-hermitian matrix models [22], enumeration of non-

orientable discretised surfaces [23], Chern–Simons theory with gauge groups $\mathrm{SO}(N)$ or $\mathrm{Sp}(2N)$ [11], etc.

2.4. String, dilaton and homogeneity equations

In this section, we give more explicit forms of some of the lowest order constraints of the Airy structures found in Theorems 2.10 and 2.11. In the particular case of a transitive twist or its exceptional analogue, we obtain the dilaton equation. With a transitive twist and $s_1 = r_1 + 1$ or its exceptional analogue (i.e., for the Airy structures already known from [9]), we also get a string equation. The string equation does not occur in any of the other (new) Airy structures.

Lemma 2.18. *Let us consider one of the Airy structures from Theorem 2.10 or Theorem 2.11. For any $g, n \geq 0$ such that $2g - 2 + (n + 1) > 0$, any $\mu, \mu_1, \dots, \mu_n \in [d]$ and $p_1, \dots, p_n > 0$, we have*

$$\begin{aligned} & \sum_{\mu=1}^d t_\mu F_{g,n+1} \left[\begin{smallmatrix} \mu & \mu_1 & \dots & \mu_n \\ s_\mu & p_1 & \dots & p_n \end{smallmatrix} \right] \\ &= \sum_{m=1}^n \frac{p_m}{r_{\mu_m}} F_{g,n} \left[\begin{smallmatrix} \mu_1 & \dots & \mu_n \\ p_1 & \dots & p_n \end{smallmatrix} \right] + \delta_{g,1} \delta_{n,0} \left(\frac{r_\mu^2 - 1}{24r_\mu} + \frac{Q_\mu^2}{2r_\mu} \right). \end{aligned} \quad (2.29)$$

Proof. We express the constraint ${}^\sigma H_{i=2,k=0} \cdot Z = 0$, as it is always part of the Airy structure. From (2.14)–(2.15) and taking into account $J_0^\mu = \hbar^{\frac{1}{2}} Q_\mu$ and the evaluations

$$\begin{aligned} \Psi_r^{(0)}(q_1, q_2) &= \frac{1}{2} (r^2 \delta_{r|q_1} \delta_{r|q_2} - r \delta_{r|q_1+q_2}), \\ \Psi_r^{(1)}(\emptyset) &= -\frac{r(r^2 - 1)}{24}, \end{aligned} \quad (2.30)$$

we obtain

$$\begin{aligned} {}^\sigma W_{i=2,k=0} &= \sum_{\mu=1}^d \frac{1}{r_\mu} W_{2,0}^\mu + \sum_{\substack{\mu \neq \nu \\ k > 0}} W_{1,-k}^\mu W_{1,k}^\nu + \sum_{\mu < \nu} W_{1,0}^\mu W_{1,0}^\nu \\ &= \sum_{\mu=1}^d \left(\sum_{k > 0} \frac{r_\mu \delta_{r_\mu | k} - 1}{r_\mu} J_{-k}^\mu J_k^\mu + \frac{(r_\mu - 1) Q_\mu^2 \hbar}{2r_\mu} - \frac{(r_\mu^2 - 1) \hbar}{24r_\mu} \right) \\ &\quad + \sum_{\mu \neq \nu} \left(\sum_{k > 0} J_{-r_\mu k}^\mu J_{r_\nu k}^\nu + \frac{\hbar Q_\mu Q_\nu}{2} \right). \end{aligned} \quad (2.31)$$

To get ${}^\sigma H_{i=2,k=0}$, we have to apply the dilaton shifts $J_{-s_\mu}^\mu \rightarrow J_{-s_\mu}^\mu - r_\mu t_\mu$, which simply results in adding a term $\sum_{\mu=1}^d t_\mu J_{s_\mu}^\mu$ to (2.31). Expressing the constraint

$$\forall k > 0, \quad {}^\sigma H_{i=1,k} \cdot Z = \left(\sum_{\mu=1}^d J_{r_\mu k}^\mu \right) \cdot Z = 0,$$

and using the assumption $\sum_{\mu=1}^d Q_\mu = 0$, we see that ${}^\sigma H_{i=2,k=0} \cdot Z = 0$ implies that Z is annihilated by the operator

$$\sum_{\mu=1}^d \left\{ t_\mu J_{s_\mu}^\mu - \sum_{k>0} \frac{1}{r_\mu} J_{-k}^\mu J_k^\mu - \hbar \left(\frac{r_\mu^2 - 1}{24r_\mu} + \frac{Q_\mu^2}{2r_\mu} \right) \right\}.$$

By the representation (2.13) of the J s, this yields (2.29) for the coefficients (2.5) of the partition function. ■

The partition functions of these Airy structures for $\sigma = (1 \cdots r)$ or $(1 \cdots r - 1)(r)$ (already constructed in [9]) enjoy an extra property of homogeneity, which turn (2.29) into an analogue of the dilaton equation.

Corollary 2.19. *Assume $d = 1$ and $(r_1, s_1) = (r, s)$ with $r = \pm 1 \pmod s$. Then, the coefficients (2.5) of the partition function of the Airy structure from Theorem 2.10 satisfy $F_{g,n}[p_1, \dots, p_n] = 0$ whenever $\sum_{m=1}^n p_m \neq s(2g - 2 + n)$, and the dilaton equation*

$$F_{g,n+1}[s, p_1, \dots, p_n] = s(2g - 2 + n) F_{g,n}[p_1, \dots, p_n] + \frac{r_1^2 - 1}{24} \delta_{g,1} \delta_{n,0}.$$

Corollary 2.20. *Assume $d = 2$, $r_1 = -1 \pmod{s_1}$, $(r_2, s_2) = (1, \infty)$ and $t_1 = \frac{1}{r_1}$. Then, the coefficients of the partition function of the Airy structure described in Theorem 2.11 (also appearing in [9, Theorem 4.16]) satisfy*

$$F_{g,n} \left[\frac{1}{p_1} \cdots \frac{1}{p_n} \right] = 0$$

whenever $\sum_{m=1}^n p_m \neq s_1(2g - 2 + n)$, and the dilaton equation

$$\begin{aligned} F_{g,n+1} \left[\frac{1}{s_1} \frac{1}{p_1} \cdots \frac{1}{p_n} \right] &= s_1(2g - 2 + n) F_{g,n} \left[\frac{1}{p_1} \cdots \frac{1}{p_n} \right] \\ &+ \left(\frac{r_1^2 - 1}{24} + \frac{(r_1 + 1)Q_1^2}{2} \right) \delta_{g,1} \delta_{n,0}. \end{aligned}$$

The proof of these two corollaries goes by induction on $2g - 2 + n > 0$, analysing the Airy structure constraints.

Proof of Corollary 2.19. In this case, we only need to consider $g \in \mathbb{Z}_{\geq 0}$. The Airy structure is in normal form up to an overall normalisation, and we can decompose for $i \in (r)$ and $k \geq 1 - \lambda(i)$,

$$r^{i-1} \sigma H_{i,k} = J_{\Pi(i,k)} - \sum_{\substack{\ell, j \in \mathbb{Z}_{\geq 0} \\ 2 \leq \ell + 2j \leq r}} \frac{\hbar^j}{\ell!} \sum_{\mathbf{q} \in (\mathbb{Z}^*)^\ell} C^{(j)}[\Pi(i,k)|q_1, \dots, q_\ell] : J_{q_1} \cdots J_{q_\ell} :,$$

where $\Pi(i,k) := rk + s(i-1)$. Setting $F_{0,2}[p_1, p_2] := |p_1| \delta_{p_1+p_2,0}$, [9, Corollary 2.16] gives the following formula for the coefficients of the partition function:

$$F_{g,n}[p_1, \dots, p_n] = \sum_{\substack{\ell, j \in \mathbb{Z}_{\geq 0} \\ 2 \leq \ell + 2j \leq r \\ \mathbf{q} \in (\mathbb{Z}^*)^\ell}} \frac{C^{(j)}[p_1|q_1, \dots, q_\ell]}{\ell!} \sum_{\rho \vdash [\ell]} \sum_{\substack{\mu \vdash_{\rho} (n) \\ h: \rho \rightarrow \mathbb{Z}_{\geq 0} \\ \ell + j + \sum_{\rho \in \rho} (h_{\rho} - 1) = g}} \prod_{\rho \in \rho} F_{h_{\rho}, |\rho| + |\mu_{\rho}|}[\mathbf{q}_{\rho}, \mathbf{p}_{\mu_{\rho}}]. \quad (2.32)$$

Here, $\rho \vdash [\ell]$ means that ρ is a set of non-empty subsets of $[\ell]$ which are pairwise disjoint and whose union is ℓ , and for $\rho \in \rho$ we denote $\mathbf{q}_{\rho} := (q_m)_{m \in \rho}$. Then, $\mu \vdash_{\rho} (n)$ is a family of (possibly empty) pairwise disjoint subsets μ_{ρ} of (n) indexed by $\rho \in \rho$, whose union is (n) . The double prime over the summation means that the terms involving $F_{0,1}[q_m]$ or $F_{0,2}[q_m, q_{m'}]$ are excluded from the sum. We note that the summation condition is equivalent to

$$2g - 2 + n + (1 - \ell - 2j) = \sum_{\rho \in \rho} (2h_{\rho} - 2 + |\rho| + |\mu_{\rho}|).$$

Since $\ell + 2j \geq 2$, this is indeed a recursion on $2g - 2 + n \geq 0$ to compute $F_{g,n}$ starting from the value of $F_{0,2}$.

The coefficient $F_{0,2}$ obviously satisfies the homogeneity property. Assume the $F_{g',n'}$ for $0 \leq 2g' - 2 + n' < 2g - 2 + n$ satisfy homogeneity. So, the summands that can contribute to (2.32) are such that

$$\begin{aligned} s(2g - 2 + n + (1 - \ell - 2j)) &= \sum_{\rho \in \rho} s(2h_{\rho} - 2 + |\rho| + |\mu_{\rho}|) \\ &= \sum_{l=1}^{\ell} q_l + \sum_{m=2}^n p_m. \end{aligned} \quad (2.33)$$

Writing $p_1 = \Pi(i,k)$ and applying the dilaton shift $J_{-s} \rightarrow J_{-s} - 1$ to (2.15), we know that $C^{(j)}[p_1|\mathbf{q}]$ is a linear combination of terms inside which $\sum_{l=1}^{\ell} q_l - s\ell' = rk$,

where $\ell + \ell' = i - 2j$. Hence

$$\sum_{l=1}^{\ell} q_l = p_1 + s(1 - 2j - \ell). \quad (2.34)$$

Together with (2.33), this proves homogeneity of $F_{g,n}$. By induction, homogeneity is established for all g, n .

We then apply Lemma 2.18. In our case, there is a single μ , $t_\mu = \frac{1}{r_\mu}$ and $Q_\mu = 0$. Using homogeneity to simplify the right-hand side of (2.29), we obtain

$$F_{g,n+1}[s, p_1, \dots, p_n] = s(2g - 2 + n)F_{g,n}[p_1, \dots, p_n] + \delta_{g,0}\delta_{n,1} \frac{r^2 - 1}{24}. \quad \blacksquare$$

Proof of Corollary 2.20. The argument is similar and we only point the minor differences that must be taken into consideration. Although half-integer g and j are now allowed, this does not spoil the sum constraints appearing in the recursive formula for $F_{g,n}$ and which were used in the argument. According to (2.14) and (2.21), we have for $i \in (r_1 + 1]$

$${}^\sigma H_{i,k} = r_1^{-(i-1)} \left(W_{i,k}^1 + \sum_{k' \in \mathbb{Z}} r_1 W_{i-1,k'}^1 J_{k-1-k'}^2 \right) \Big|_{J_{-s_1}^1 \rightarrow J_{-s_1-1}^1}, \quad (2.35)$$

where by convention $W_{r_1+1,k}^1 = 0$. The analysis of the previous proof applies to the term $W_{i,k}^1$. Due to the equation ${}^\sigma H_{1,k} \cdot Z = (J_{r_1 k}^1 + J_k^2) = 0$, we can obtain a recursion in normal form involving only $F_{g,n} \left[\begin{smallmatrix} 1 & \dots & 1 \\ * & \dots & * \end{smallmatrix} \right]$ by substituting

$$J_{k_2}^2 \rightarrow \begin{cases} 0 & \text{if } k_2 < 0, \\ -\hbar^{\frac{1}{2}} Q_1 & \text{if } k_2 = 0, \\ -J_{r_1 k_2}^1 & \text{if } k_2 > 0 \end{cases} \quad (2.36)$$

in (2.35). This converts $W_{i-1,k'}^1 J_{k-k'}^2$ into $-W_{i-1,k'}^1 J_{r_1(k-k')}^1$ or 0, and makes it contribute to a coefficient $C^{(j)}[p_1|q_1, \dots, q_\ell]$ where now $q_\ell = r_1(k - k')$ and

$$\left(\sum_{l=1}^{\ell-1} q_l \right) - s_1 \ell' = r_1 k'$$

with $(\ell - 1) + \ell' = i - 1 - 2j$. Hence

$$\sum_{l=1}^{\ell} q_l = r_1 k - s_1 \ell' = p_1 + s_1(1 - 2j - \ell),$$

which is the same as (2.34), and this is all what we need to repeat the previous proof and establish homogeneity.

We then specialise Lemma 2.18 to our case, that is, $t_1 = \frac{1}{r_1}$ and $t_2 = 0$, while $Q_2 = -Q_1$. Setting $\mu_i = 1$ for all $i \in [n]$ in (2.29) and using homogeneity to simplify the right-hand side, we deduce

$$F_{g,n+1}\left[\begin{matrix} 1 & 1 & \dots & 1 \\ s_1 & p_1 & \dots & p_n \end{matrix}\right] = s_1(2g - 2 + n)F_{g,n}\left[\begin{matrix} 1 & \dots & 1 \\ p_1 & \dots & p_n \end{matrix}\right] \\ + \left(\frac{r_1^2 - 1}{24} + \frac{(r_1 + 1)Q_1^2}{2}\right)\delta_{g,1}\delta_{n,1}. \quad \blacksquare$$

For the two above cases, we also have a string equation when $s_1 = r_1 + 1$.

Lemma 2.21. *Assume $d = 1$ and $(r_1, s_1) = (r, r + 1)$. Then, the coefficients of the partition function of the Airy structure described in Theorem 2.10 satisfy*

$$F_{g,1+n}[1, p_1, \dots, p_n] = \sum_{m=1}^n p_m F_{g,n}[p_1, \dots, p_{m-1}, p_m - r, p_{m+1}, \dots, p_n] \\ + \delta_{g,0}\delta_{n,2}\delta_{p_1+p_2,r}.$$

Lemma 2.22. *Assume $d = 2$, $s_1 = r_1 + 1$ and $(r_2, s_2) = (1, \infty)$. Then, the coefficients of the partition function of the Airy structure described in Theorem 2.11 satisfy*

$$F_{g,1+n}\left[\begin{matrix} 1 & 1 & \dots & 1 \\ 1 & p_1 & \dots & p_n \end{matrix}\right] = \sum_{m=1}^n p_m F_{g,n}\left[\begin{matrix} 1 & \dots & 1 & 1 & \dots & 1 \\ p_1 & \dots & p_{m-1} & p_m - r & \dots & p_n \end{matrix}\right] \\ + \delta_{g,0}\delta_{n,2}\delta_{p_1+p_2,r} + \delta_{g,\frac{1}{2}}\delta_{n,1}\delta_{p_1,r}Q_1.$$

Proof. The string equation corresponds to the operator $H_{i=2,k=-1}$, which is only present in the Airy structure for $d = 1$ when $s = r + 1$, and for $d = 2$ if $s_1 = r_1 + 1$. The proof of the above relations is then similar to the one in Lemma 2.18. \blacksquare

3. Proof of Theorems 2.11, 2.14, and 2.13

The first part of this section is devoted to the proof of Theorems 2.11 and 2.14. These theorems state that certain collections of operators $(\sigma H_{i,k})_{(i,k) \in I}$ defined in equations (2.21) and (2.28) are Airy structures. In fact, we only prove the second theorem, and note that the first is a special case. We first prove this in the standard case, proceeding as follows:

- (I) The operators of an Airy structure must be of the form $J_a^\mu + \mathcal{O}(2)$. It is thus necessary to first identify the degree zero and degree one term of $\sigma H_{i,k}$ in order to check this condition.

(II) In general, one will find that

$${}^\sigma H_{i,k} = c_{i,k} + \sum_{\substack{\mu \in [d] \\ a \in \mathbb{Z}}} \mathcal{M}_{(i,k),(\mu,a)} J_a^\mu + \mathcal{O}(2) \quad (3.1)$$

for some matrix \mathcal{M} and constants $c_{i,k}$. This means that for generic i, k , we may expect terms proportional to $J_a^\mu = a x_a^\mu$. We will thus construct an index set $I \subset [r] \times \mathbb{Z}$ such that for all $(i, k) \in I$, we have

$$c_{i,k} = 0 \quad \text{and} \quad \mathcal{M}_{(i,k),(\mu,a)} = 0 \quad \text{if } a \leq 0.$$

(III) Nevertheless, even restricted to I the degree one term (3.1) is in general a linear combination of many J_a^μ s for $a > 0$. In order to bring the operators into the normal form of an Airy structure $J_a^\mu + \mathcal{O}(2)$ where each (μ, a) appears in a unique operator, we will show that the matrix \mathcal{M} restricted to I is invertible under certain constraints on the dilaton shifts. One can then obtain the operators

$${}^\sigma \tilde{H}_a^\mu := \sum_{(i,k) \in I} (\mathcal{M}^{-1})_{(\mu,a),(i,k)} {}^\sigma H_{i,k} = \hbar \partial_{x_a^\mu} + \mathcal{O}(2),$$

which are of the desired form.

(IV) The modes $(\tilde{H}_a^\mu)_{(\mu \in [d], a > 0)}$ satisfy the subalgebra condition if and only if the $({}^\sigma W_{i,k})_{(i,k) \in I}$ do. The latter is satisfied when I is induced by a descending partition of r as specified in Theorem 2.8. This criterion thus allows for an easy check of whether the mode set constructed in (III) satisfies the subalgebra condition (2.2) of a higher quantum Airy structure.

Together the results of (III) and (IV) will directly imply Theorem 2.14 and hence Theorem 2.11 in the standard case. The steps (I) to (III) will be carried out in Section 3.1, and step (IV) is performed in Section 3.2. In Section 3.3, we treat in less detail the exceptional case, as it resembles the standard case in many ways.

The purpose of Section 3.4 is to prove Theorem 2.13 that addresses the question in which case the operators ${}^\sigma H_{i,k}$ admit a partition function solving the associated system of differential equations to leading order in $\hbar^{\frac{1}{2}}$. We will start in Section 3.4.1 by deriving sufficient conditions for this question to have a positive answer for general families of differential operators satisfying the degree one condition. Then we apply this result to our case at hand and finally prove Theorem 2.13 in Section 3.4.2.

First, let us recall some notation. Let $\sigma \in \mathfrak{S}_r$ be a permutation with d cycles of respective length r_μ such that $r = r_1 + \dots + r_d$. We can then define the dilaton shifted

modes

$${}^\sigma H_{i,k} := \hat{T} \cdot {}^\sigma W_{i,k} \cdot \hat{T}^{-1}, \quad \hat{T} := \exp \left(\sum_{\substack{\mu \in [d] \\ k > 0}} \hbar^{-1} F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] \frac{J_k^\mu}{k} \right),$$

$$s_\mu := \min \{k > 0 \mid F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] \neq 0\},$$

which in the following are the central objects of study. Here ${}^\sigma W_{i,k}$ are the σ -twisted modes from Lemma 2.9. Compared to equation (2.28), we do not introduce the coefficients $F_{\frac{1}{2},1}$ and $F_{0,2}$ here yet, as it turns out these are not important for most of the proof. As in equation (2.10), it will be useful to gather these in a generating function,

$${}^\sigma H_i(\tilde{x}) = \sum_{k \in \mathbb{Z}} \frac{{}^\sigma H_{i,k}}{\tilde{x}^k} \left(\frac{d\tilde{x}}{\tilde{x}} \right)^i = \hat{T} \cdot {}^\sigma W_i(\tilde{x}) \cdot \hat{T}^{-1}.$$

We will also recall from equation (2.12)

$$J^{\mu,a}(\tilde{x}) := \sum_{k \in \frac{a}{r_\mu} + \mathbb{Z}} J_{r_\mu k}^\mu \frac{d\tilde{x}}{\tilde{x}^{k+1}}$$

and define $J^{\mu,\text{tot}}(\tilde{x}) = \sum_{a=0}^{r_\mu-1} J^{\mu,a}(\tilde{x})$. To treat these currents more uniformly, we introduce $\tilde{C} = \bigsqcup_{\mu=1}^d \tilde{C}_\mu$, the union of d copies of a formal neighbourhood \tilde{C}_μ of the origin in the complex plane. We use the notation $\left(\frac{\mu}{z}\right)$ for the coordinate in the \tilde{C}_μ . We define a function $\tilde{x}: \tilde{C} \rightarrow \mathbb{C}$ by $\tilde{x}\left(\frac{\mu}{z}\right) = z^{r_\mu}$. See Part II for more on this viewpoint. We then define a unified current

$$J\left(\frac{\mu}{z}\right) := \frac{J^{\mu,\text{tot}}(\tilde{x}(z))}{r_\mu} = \sum_{k \in \mathbb{Z}} J_k^\mu \frac{d\tilde{x}}{r_\mu \tilde{x}^{k/r_\mu+1}} = \sum_{k \in \mathbb{Z}} J_k^\mu \frac{dz}{z^{k+1}}.$$

The factor of r_μ in the denominator is a convention making (3.2) simpler. We sometimes omit μ from the notation and simply denote $z \in \tilde{C}$. With these notations, we see that the dilaton shift induces

$$J(z) \rightarrow J(z) + \omega_{0,1}(z), \quad \omega_{0,1}\left(\frac{\mu}{z}\right) := \sum_{k > 0} F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] z^{k-1} dz.$$

We also use the shorthand notation $t_\mu = -\frac{1}{r_\mu} F_{0,1} \left[\begin{smallmatrix} \mu \\ -s_\mu \end{smallmatrix} \right]$ for the leading coefficient.

3.1. The degree one condition

In this subsection and the next one, we assume that all s_μ are finite. Let us begin by identifying the degree one component of ${}^\sigma H_i$. Let π_1 be the projection to degree one.

Lemma 3.1. For any $i \in [r]$,

$$\pi_1(\sigma H_i(\xi)) = \sum_{z \in \tilde{x}^{-1}(\xi)} \sum_{\substack{Z \subseteq \tilde{x}^{-1}(\xi) \setminus \{z\} \\ |Z|=i-1}} J(z) \prod_{z' \in Z} \omega_{0,1}(z'). \quad (3.2)$$

So we have

$$\pi_1(\sigma H_{i,k}) = \sum_{\substack{\mu \in [d] \\ a \in \mathbb{Z}}} \mathcal{M}_{(i,k),(\mu,a)} J_a^\mu \quad (3.3)$$

with a matrix $\mathcal{M}_{(i,k),(\mu,a)}$ having the property that, for each (μ, a) , there exists $K_{\mu,a}$ such that for any $i \in [r]$ and $k \geq K_{\mu,a}$ we have $\mathcal{M}_{(i,k),(\mu,a)} = 0$.

Note that the right-hand side is a symmetric function in the elements of $\tilde{x}^{-1}(\xi)$, and therefore it contains only integral powers of ξ .

Proof. From equation (2.17), we see that $\sigma W_i(\xi)$ is a linear combination of currents

$$\hbar^{\sum_{\mu} j_{\mu}} \prod_{\mu \in M} \prod_{l=2j_{\mu}+1}^{i_{\mu}} J^{\mu, a_l^{\mu}}(\xi) :.$$

After the dilaton shift, these will only contribute to the degree one component of $\sigma H_i(\xi)$ if $j_{\mu} = 0$ for all $\mu \in M$. We have

$$\begin{aligned} \pi_1(\sigma W_i(\xi)) &= \pi_1 \left(\sum_{M \subseteq [d]} \sum_{\substack{i_{\mu} \in [r_{\mu}], \mu \in M \\ \sum_{\mu} i_{\mu} = i}} \prod_{\mu \in M} \frac{1}{r_{\mu}^{i_{\mu}}} \left(\sum_{a_1^{\mu}, \dots, a_{i_{\mu}}^{\mu} \in [0, r_{\mu}]} \Psi_{r_{\mu}}^{(0)}(a_1^{\mu}, \dots, a_{i_{\mu}}^{\mu}) \right. \right. \\ &\quad \left. \left. \times : \prod_{\mu \in M} \prod_{l=1}^{i_{\mu}} J^{\mu, a_l^{\mu}}(\xi) : \right) \right), \end{aligned} \quad (3.4)$$

and therefore on the right-hand side we need to take the contribution of the shifts in all factors but one. Definition (2.16) of the $\Psi^{(0)}$ is nothing but a sum over subsets of Galois conjugates of the function \tilde{x} , so we obtain

$$\sum_{a_1^{\mu}, \dots, a_{i_{\mu}}^{\mu} \in [0, r_{\mu}]} \Psi_{r_{\mu}}^{(0)}(a_1^{\mu}, \dots, a_{i_{\mu}}^{\mu}) : \prod_{l=1}^{i_{\mu}} J^{\mu, a_l^{\mu}}(\xi) : = \sum_{\substack{Z \subseteq \tilde{x}^{-1}(\xi) \cap \tilde{C}_{\mu} \\ |Z|=i_{\mu}}} : \prod_{z' \in Z} J\left(\frac{\mu}{z'}\right) :.$$

The sum over $M \subseteq [d]$ in (3.4) “globalises” this sum of subsets from one component \tilde{C}_{μ} to all of \tilde{C} . As stated before, the degree one projection extracts the dilaton shifts of all but one (the choice of z) of these factors, which proves (3.2). We obtain the matrix \mathcal{M} by expanding this equation in ξ and collecting the contributions of J_a^{μ} . The vanishing property comes from the fact that $\omega_{0,1}$ contains only nonnegative powers of z . ■

We will restrict the range of indices on both sides and show that the matrix \mathcal{M} is invertible in order to bring the differential operators into the normal form of an Airy structure. But first, we would like to see that this matrix is invertible without restricting it to any subspace yet. Define

$$\sigma H(\xi, u) := \sum_{i=1}^r \sigma H_i(\xi) u^{r-i}.$$

Then

$$\begin{aligned} \pi_1(\sigma H(\xi, u)) &= \sum_{i=1}^r \sum_{z \in \tilde{x}^{-1}(\xi)} \sum_{\substack{Z \subseteq \tilde{x}^{-1}(\xi) \setminus \{z\} \\ |Z|=i-1}} J(z) \prod_{z' \in Z} \omega_{0,1}(z') u^{r-i} \\ &= \sum_{z \in \tilde{x}^{-1}(\xi)} J(z) \prod_{z' \in \tilde{x}^{-1}(\xi) \setminus \{z\}} (u + \omega_{0,1}(z')). \end{aligned} \quad (3.5)$$

We are looking for the inverse to this operation. Remember from (2.26) that for any $\mu \in [d]$ we write $t_\mu := -\frac{1}{r_\mu} F_{0,1} \left[\begin{smallmatrix} \mu \\ -s_\mu \end{smallmatrix} \right]$.

Lemma 3.2. *Assume $\gcd(r_\mu, s_\mu) = 1$ for all $\mu \in [d]$ and $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$ for any distinct μ, ν such that $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$. Then the currents can be recovered from $\pi_1(\sigma H(\xi, u))$ as follows:*

$$J(z) = \operatorname{Res}_{u=-\omega_{0,1}(z)} \frac{\pi_1(\sigma H(\tilde{x}(z), u)) du}{\prod_{z' \in \tilde{x}^{-1}(\tilde{x}(z))} (u + \omega_{0,1}(z'))}. \quad (3.6)$$

Proof. If we plug equation (3.5) back in equation (3.6), we get

$$\begin{aligned} &\operatorname{Res}_{u=-\omega_{0,1}(z)} \sum_{\zeta \in \tilde{x}^{-1}(\tilde{x}(z))} J(\zeta) \prod_{\zeta' \in \tilde{x}^{-1}(\tilde{x}(z)) \setminus \{\zeta\}} (u + \omega_{0,1}(\zeta')) \\ &\times \prod_{z' \in \tilde{x}^{-1}(\tilde{x}(z))} (u + \omega_{0,1}(z'))^{-1} du = \operatorname{Res}_{u=-\omega_{0,1}(z)} \sum_{\zeta \in \tilde{x}^{-1}(\tilde{x}(z))} \frac{J(\zeta) du}{u + \omega_{0,1}(\zeta)}. \end{aligned}$$

Because we took z in a small neighbourhood of zero (but of course not zero itself), the conditions of the lemma ensure that all $\omega_{0,1}(\zeta)$ for ζ in the same fiber have different values. Therefore, the only contribution to the residue comes from $\zeta = z$. ■

Remark 3.3. The other inverse relationship,

$$\begin{aligned} \pi_1(\sigma H(\xi, u)) &= \sum_{z \in \tilde{x}^{-1}(\xi)} \prod_{z' \in \tilde{x}^{-1}(\xi) \setminus \{z\}} (u + \omega_{0,1}(z')) \\ &\times \operatorname{Res}_{u'=-\omega_{0,1}(z)} \frac{\pi_1(\sigma H(\tilde{x}(z), u')) du'}{\prod_{z' \in \tilde{x}^{-1}(\tilde{x}(z))} (u' + \omega_{0,1}(z'))}, \end{aligned}$$

follows immediately from Lagrange interpolation since $\pi_1(\sigma H(\tilde{x}(z), u))$ is polynomial in u .

Remark 3.4. For Lemma 3.2 and Remark 3.3 to work, all we really need is that $\omega_{0,1}$ takes distinct values on all elements of the fiber of \tilde{x} near the ramification point $\xi = 0$, i.e., the map $(\tilde{x}, \omega_{0,1}): \tilde{C} \rightarrow T^*\mathbb{P}^1$ is an embedding into a punctured neighbourhood of the ramification point. The conditions $\gcd(r_\mu, s_\mu) = 1$ and $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$ for $\mu \neq \nu$ such that $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ ensure this. Though in general not necessary at this point, it will become crucial to impose these stricter conditions in Lemma 3.11. See Remark 3.12 for this.

In the undeformed case, i.e., if $\omega_{0,1}$ is monomial, this is in fact necessary as well as sufficient. Indeed, suppose we are in the setting of Theorem 2.11 and there is a μ for which $\gcd(r_\mu, s_\mu) = d_\mu > 1$, let $z \in \tilde{C}_\mu$, and let ϑ be a primitive r_μ -th root of unity. Then

$$x(z) = x(\vartheta^{r_\mu/d_\mu} z) \quad \text{and} \quad \omega_{0,1}(z) = \omega_{0,1}(\vartheta^{r_\mu/d_\mu} z).$$

So in equation (3.5), the dependence on J_*^μ is given by

$$\begin{aligned} & \prod_{z' \in \tilde{x}^{-1}(\xi)} (u + \omega_{0,1}(z')) \sum_{i=1}^{r_\mu} \frac{J(\frac{\mu}{\vartheta^i z})}{u + \omega_{0,1}(\vartheta^i z)} \\ &= \prod_{z' \in \tilde{x}^{-1}(\xi)} (u + \omega_{0,1}(z')) \sum_{i=1}^{r_\mu/d_\mu} \frac{\sum_{j=1}^{d_\mu} J(\frac{\mu}{\vartheta^{i+jr_\mu/d_\mu} z})}{u + \omega_{0,1}(\vartheta^i z)}. \end{aligned}$$

Performing the sum over j , we see that $\pi_1(\sigma H(\xi, u))$ only depends on J_a^μ for $d_\mu \mid a$. Therefore, it is impossible to retrieve all J_a^μ from $\pi_1(\sigma H)$.

In the case $\mu \neq \nu$ such that $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ and $t_\mu^{r_\mu} = t_\nu^{r_\nu}$, the situation is similar. First note that by redefining the local coordinate on \tilde{C}_μ , we can actually assume $t_\mu = t_\nu$. Writing the local coordinates as $z \in \tilde{C}_\mu$ and $z' \in \tilde{C}_\nu$, we then have $\omega_{0,1}(\vartheta^i z) = \omega_{0,1}(\vartheta^i z')$ for all $i \in [r_\mu]$. By the same argument as above, $\pi_1(H)$ then depends on J_a^μ and J_b^ν only in the combination $J_a^\mu + J_a^\nu$, so again we can never retrieve the individual J_a^μ and J_a^ν .

Since the $\omega_{0,1}$ are power series with exponents bounded below (by s_μ on component \tilde{C}_μ), we can see that for any i , the set of k such that $\sigma H_{i,k}$ gets a contribution from J_a^μ with $a \leq 0$ in equation (3.2) is bounded from above. Therefore, the following definition makes sense.

Definition 3.5. For $i \in [r]$, we define $k_{\min}^{\geq}(i)$ to be the smallest K such that for all $k \geq K$, $\pi_1(\sigma H_{i,k})$ given by equation (3.2) lies in the linear span of J_a^μ with $\mu \in [d]$ and $a > 0$ solely.

Analogously, for $i \in [r]$ we choose $k_{\min}^{\geq}(i)$ to be the lowest bound for which $\pi_1(\sigma H_{i,k})$ with $k \geq k_{\min}^{\geq}(i)$ only features J_a^μ with $\mu \in [d]$ and $a \geq 0$.

It turns out that $k_{\min}^>(i)$ can be approximated as follows.

Lemma 3.6. *If $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$ and $\gcd(r_\mu, s_\mu) = 1$ for all $\mu \in [d]$, we have $k_{\min}^>(i) \leq \delta_{\mathbf{r}, \mathbf{s}}^>(i)$, where*

$$\delta_{\mathbf{r}, \mathbf{s}}^>(i) := \begin{cases} -\lfloor \frac{s_1(i-1)}{r_1} \rfloor + \delta_{i,1}, & 1 \leq i \leq r_1, \\ -\lfloor \frac{s_2(i-r_1-1)}{r_2} \rfloor - s_1 + \delta_{i,1+r_1}, & r_1 < i \leq \mathbf{r}[2], \\ \vdots & \vdots \\ -\lfloor \frac{s_d(i-\mathbf{r}[d-1]-1)}{r_d} \rfloor - s_{[d-1]} + \delta_{i,1+\mathbf{r}[\mu-1]}, & \mathbf{r}[d-1] < i \leq \mathbf{r}, \end{cases} \quad (3.7)$$

where for a subset $M \subseteq [d]$ we denoted $\mathbf{r}_M := \sum_{\mu \in M} r_\mu$ and $\mathbf{s}_M := \sum_{\mu \in M} s_\mu$.

Proof. Recall that k is the exponent in

$$\frac{\pi_1(\sigma H_{i,k})}{\xi^k} \left(\frac{d\xi}{\xi} \right)^i. \quad (3.8)$$

As we also get i factors of $\frac{d\xi}{\xi}$ on the right-hand side of equation (3.2) (one from J and $i-1$ from the $\omega_{0,1}$), we will concentrate on the remaining powers of ξ .

From Lemma 3.1, we see that in order to determine an upper bound $k_{\max} \in \mathbb{Q}$ for the exponents k in (3.8) for which we get a non-vanishing contribution from J_a^μ with $a \leq 0$ it suffices to inspect $a = 0$ and to take only the leading order of all $\omega_{0,1}$ s into account. More specifically, to make k in (3.8) as large as possible, we need to choose the $\omega_{0,1}$ s efficiently: an $\omega_{0,1}$ on branch ν has leading order $(z')^{s_\nu} \frac{dz'}{z'} = \xi^{\frac{s_\nu}{r_\nu}} \frac{d\xi}{\xi}$. So, to minimise the power of ξ , we need to take $\frac{s_\nu}{r_\nu}$ minimal, i.e., $\frac{r_\nu}{s_\nu}$ maximal, i.e., ν minimal. From this it follows that the maximal k such that J_a^* , $a \leq 0$ can contribute is found by first taking all r_1 factors $\omega_{0,1}$ with arguments on component $\nu = 1$, then the r_2 on component $\nu = 2$, up until we get to $i-1$ factors.

If $i \in [r_1]$, we get $-k_{\max} = \frac{s_1}{r_1}(i-1)$ from this, while for $i \in (\mathbf{r}[\mu-1], \mathbf{r}[\mu]]$, we similarly get

$$-k_{\max} = (i - \mathbf{r}[\mu-1] - 1) \frac{s_\mu}{r_\mu} + \sum_{\nu=1}^{\mu-1} \frac{s_\nu}{r_\nu} r_\nu.$$

Of course, it might still happen that the coefficient of the power $\xi^{-k_{\max}}$ vanishes due to cancelling contributions of different combinations of $\omega_{0,1}$ s. However, in any case $k_{\max} \in \mathbb{Q}$ is an upper bound for $k \in \mathbb{Q}$ for which J_a^* with $a \leq 0$ can still contribute. Therefore,

$$k_{\min}^>(i) \leq \min \left\{ K \in \mathbb{Z} \mid K > -\left(s_{[\mu-1]} + (i - \mathbf{r}[\mu-1] - 1) \frac{s_\mu}{r_\mu} \right) \right\}, \quad (3.9)$$

where $i \in (\mathbf{r}_{[\mu-1]}, \mathbf{r}_{[\mu]})$. Finally, let us argue that the right-hand side of the above equation is nothing but $\mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$ as defined in (3.7). If $i \in (\mathbf{r}_{[\mu-1]} + 1, \mathbf{r}_{[\mu]})$, this is clearly true, as then $(i - \mathbf{r}_{[\mu-1]}) \frac{s_\mu}{r_\mu} \notin \mathbb{Z}$, and taking the integral part implies strict inequality. If $i = \mathbf{r}_{[\mu-1]} + 1$, we need to add 1 to obtain strict inequality, explaining the Kronecker delta in (3.7). ■

Remark 3.7. For generic $\omega_{0,1}$, i.e., generic values for dilaton shifts, we even have $k_{\min}^>(i) = \mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$ for all i . Indeed, it should be clear from the preceding discussion that the case $k_{\min}^>(i) < \mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$ can only occur if the leading order inspected in the proof of Lemma 3.6 vanishes. This can only happen if $\xi^{-k_{\max}}$ gets contributions from several combinations of $\omega_{0,1}$ s, which however is only possible if in the case $i \in (\mathbf{r}_{[\mu-1]}, \mathbf{r}_{[\mu]})$ we have a $\nu \neq \mu$ with $\frac{r_\nu}{s_\nu} = \frac{r_\mu}{s_\mu}$. This can be compared in Part II to the results of Lemma 5.4 and Proposition 5.13.

The lower bound k_{\min}^{\geq} can be approximated in the same way.

Lemma 3.8. *If $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$ and $\gcd(r_\mu, s_\mu) = 1$ for all $\mu \in [d]$, then we have $k_{\min}^{\geq}(i) \leq \mathfrak{d}_{\mathbf{r},\mathbf{s}}^{\geq}(i)$, where*

$$\mathfrak{d}_{\mathbf{r},\mathbf{s}}^{\geq}(i) := - \left\lfloor \frac{s_\mu(i - \mathbf{r}_{[\mu-1]} - 1)}{r_\mu} \right\rfloor - s_{[\mu-1]}, \quad i \in (\mathbf{r}_{[\mu-1]}, \mathbf{r}_{[\mu]}]. \quad (3.10)$$

Proof. By the same arguments as in the proof of Lemma 3.6, we have

$$k_{\min}^{\geq}(i) \leq \min \left\{ K \in \mathbb{Z} \mid K \geq - \left(s_{[\mu-1]} + (i - \mathbf{r}_{[\mu-1]} - 1) \frac{s_\mu}{r_\mu} \right) \right\}, \quad i \in (\mathbf{r}_{[\mu-1]}, \mathbf{r}_{[\mu]}].$$

The right-hand side of this equation is of course nothing but (3.10). ■

From now on, we will always assume that $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$ if not stated otherwise. The modes selected with the help of $\mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$ via

$$\sigma H_{i,k}, \quad i \in [r], \quad k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$$

shall be shown to be an Airy structure for certain $(r_\mu, s_\mu)_{\mu=1}^d$. For future reference, let us therefore define the following index set.

Definition 3.9. We define the index set $I_{\mathbf{r},\mathbf{s}}^>$ to be

$$I_{\mathbf{r},\mathbf{s}}^> := \{(i, k) \in [r] \times \mathbb{Z} \mid k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)\} \quad (3.11)$$

with $\mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$ as in (3.7). Analogously, we set $I_{\mathbf{r},\mathbf{s}}^{\geq} := \{(i, k) \in [r] \times \mathbb{Z} \mid k \geq \mathfrak{d}_{\mathbf{r},\mathbf{s}}^{\geq}(i)\}$.

With the fact that $k_{\min}^>(i) \leq \mathfrak{d}_{\mathbf{r},\mathbf{s}}^>(i)$ and equation (3.7), we have two different characterisations of $I_{\mathbf{r},\mathbf{s}}^>$. The first property tells us that for $(i, k) \in I_{\mathbf{r},\mathbf{s}}^>$ the degree one

projection of ${}^\sigma H_{i,k}$ is a linear combination of J_a^μ with $a > 0$ only while the characterisation in (3.7) will be important later in order to check whether the modes satisfy the subalgebra condition. In the following, we will need yet another characterisation of $I_{r,s}^>$.

Lemma 3.10. *For $(i, k) \in [r] \times \mathbb{Z}$, we have $(i, k) \notin I_{r,s}^>$ if and only if for all $\mu \in [d]$,*

$$r_\mu(k + s_{[\mu-1]}) + s_\mu(i - r_{[\mu-1]} - 1) \leq 0. \quad (3.12)$$

Proof. First, let us take $(i, k) \in [r] \times \mathbb{Z}$ for which equation (3.12) holds for all $\mu \in [d]$. Then especially it holds for $\lambda \in [d]$ chosen so that $i \in (r_{[\lambda-1]}, r_{[\lambda]})$. In this case, (3.12) can be rewritten as

$$k \leq -\frac{s_\lambda}{r_\lambda}(i - r_{[\lambda-1]} - 1) - s_{[\lambda-1]}, \quad (3.13)$$

which in comparison with (3.9) means nothing but $k < \delta_{r,s}^>(i)$ and thus $(i, k) \notin I_{r,s}^>$.

For the other direction, suppose $(i, k) \notin I_{r,s}^>$. Then again if we choose λ so that $i \in (r_{[\lambda-1]}, r_{[\lambda]})$, the statement that $k < \delta_{r,s}^>(i)$ translates into (3.13). Hence, for arbitrary $\mu \in [d]$ we can use the inequality in order to obtain

$$\begin{aligned} & r_\mu(k + s_{[\mu-1]}) + s_\mu(i - r_{[\mu-1]} - 1) \\ & \leq \left(s_\mu - \frac{r_\mu s_\lambda}{r_\lambda}\right)(i - r_{[\lambda-1]} - 1) + s_\mu(r_{[\lambda-1]} - r_{[\mu-1]}) - r_\mu(s_{[\lambda-1]} - s_{[\mu-1]}). \end{aligned}$$

Now suppose $\mu \leq \lambda$. Then by assumption, $s_\mu - \frac{r_\mu s_\lambda}{r_\lambda} \leq 0$ and therefore

$$r_\mu(k + s_{[\mu-1]}) + s_\mu(i - r_{[\mu-1]} - 1) \leq s_\mu r_{[\mu,\lambda]} - r_\mu s_{[\mu,\lambda]} \leq 0,$$

where the last inequality follows from our assumption that $\frac{r_\mu}{s_\mu} \geq \frac{r_\nu}{s_\nu}$ for all $\nu \geq \mu$. The case $\mu > \lambda$ can be treated similarly. Note that in this case $s_\mu - \frac{r_\mu s_\lambda}{r_\lambda} \geq 0$ and thus

$$r_\mu(k + s_{[\mu-1]}) + s_\mu(i - r_{[\mu-1]} - 1) \leq \left(s_\mu - \frac{r_\mu s_\lambda}{r_\lambda}\right)r_\lambda + r_\mu s_{[\lambda,\mu]} - s_\mu r_{[\lambda,\mu]} \leq 0,$$

where the last inequality follows from our assumption that $\frac{r_\mu}{s_\mu} \leq \frac{r_\nu}{s_\nu}$ for all $\nu \leq \mu$. ■

Remember that we defined the matrix $\mathcal{M}_{(i,k),(\mu,a)}$ to be the collection of coefficients

$$\pi_1({}^\sigma H_{i,k}) = \sum_{\mu \in [d], a \in \mathbb{Z}} \mathcal{M}_{(i,k),(\mu,a)} J_a^\mu$$

of the projection to degree one. It is given abstractly in Lemma 3.1. So far we found out that \mathcal{M} admits a two-sided inverse, and moreover we characterised those modes featuring only derivatives in degree one. The next step is now to combine both results in order to bring the operators $({}^\sigma H_{i,k})_{(i,k) \in I_{r,s}^>}$ into the normal form of an Airy structure.

Lemma 3.11. *Assume $\gcd(r_\mu, s_\mu) = 1$ for all $\mu \in [d]$ and $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$ for any distinct μ, ν with $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$. Then, there exist matrices $\mathcal{N}_{(\mu,a),(i,k)}$ and $\mathcal{M}_{(i,k),(\mu,a)}$ indexed by $(\mu, a) \in [d] \times \mathbb{Z}_{>0}$ and $(i, k) \in I_{r,s}^>$ (see (3.11)) obeying the vanishing properties in Definition 2.3, that are inverse to each other, and such that*

$$\sigma \tilde{H}_{\mu,a} = \sum_{(i,k) \in I_{r,s}^>} \mathcal{N}_{(\mu,a),(i,k)} \sigma H_{i,k}, \quad (\mu, a) \in [d] \times \mathbb{Z}_{>0}$$

satisfy the degree one condition.

Proof. From Lemma 3.2 and Remark 3.3, we know that the matrix \mathcal{M} encoding the coefficients of the degree one projection of the modes $\sigma H_{i,k}$ admits a two-sided inverse if we keep the full range of indices $(i, k) \in [r] \times \mathbb{Z}$ and $(\mu, a) \in [d] \times \mathbb{Z}$. However, we actually need such a relation between the semi-infinite column vectors

$$\mathbf{H}_- := (\pi_1(\sigma H_{i,k}))_{(i,k) \in I_{r,s}^>} \quad \text{and} \quad \mathbf{J}_- := (J_a^\mu)_{(\mu,a) \in [d] \times \mathbb{Z}_{>0}}.$$

For this, let us also introduce the semi-infinite column vectors

$$\mathbf{H}_+ := (\pi_1(\sigma H_{i,k}))_{(i,k) \notin I_{r,s}^>} \quad \text{and} \quad \mathbf{J}_+ := (J_a^\mu)_{(\mu,a) \in [d] \times \mathbb{Z}_{\leq 0}},$$

and the infinite column vectors

$$\mathbf{H} := \begin{pmatrix} \mathbf{H}_- \\ \mathbf{H}_+ \end{pmatrix} \quad \text{and} \quad \mathbf{J} := \begin{pmatrix} \mathbf{J}_- \\ \mathbf{J}_+ \end{pmatrix}.$$

We can then write equation (3.5) symbolically as $\mathbf{H} = \mathcal{M} \cdot \mathbf{J}$. The vanishing properties of the matrix \mathcal{M} guarantee that this product is well-defined (i.e., the evaluation of each entry involves only finite sums). By definition of $I_{r,s}$, this splits as

$$\begin{pmatrix} \mathbf{H}_- \\ \mathbf{H}_+ \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{--} & 0 \\ \mathcal{M}_{-+} & \mathcal{M}_{++} \end{pmatrix} \begin{pmatrix} \mathbf{J}_- \\ \mathbf{J}_+ \end{pmatrix}.$$

Writing \mathcal{N} for the inverse of \mathcal{M} , the relation $\mathcal{M} \cdot \mathcal{N} = \text{id}$ implies that $\mathcal{M}_{--} \cdot \mathcal{N}_{--} = \text{id}_-$ (with obvious notation). As these are semi-infinite matrices, we cannot conclude yet that $\mathcal{N}_{--} \cdot \mathcal{M}_{--} = \text{id}_-$. However, using the same arguments as above to prove this, it is sufficient to show that $\mathcal{N}_{+-} = 0$. For this let us inspect \mathcal{N} further. According to Lemma 3.2, seen \mathcal{N} as a linear operator we have

$$\mathcal{N}: \Upsilon(\xi, u) \mapsto \text{Res}_{u=-\omega_{0,1}(z)} \frac{\Upsilon(\tilde{x}(z), u) du}{\prod_{z' \in \tilde{x}^{-1}(\tilde{x}(z))} (u + \omega_{0,1}(z'))}. \quad (3.14)$$

By a straightforward calculation, we see that if $z \in \tilde{C}_\mu$ and $z' \in \tilde{C}_\nu$, then as $z \rightarrow 0$,

$$\omega_{0,1}(z') - \omega_{0,1}(z) = \begin{cases} \mathcal{O}(z^{s_\mu-1} dz), & \nu \geq \mu, \\ \mathcal{O}(z^{\frac{r_\mu s_\nu}{r_\nu}-1} dz), & \nu < \mu. \end{cases}$$

More precisely, a thorough analysis analogous to the one in Lemma 5.4 shows that

$$\prod_{z' \in \tilde{\mathcal{X}}^{-1}(\tilde{\mathcal{X}}(z)) \setminus \{z\}} (-\omega_{0,1}(z) + \omega_{0,1}(z')) = t_\mu z^{u_\mu} (dz)^{r-1} + \mathcal{O}(z^{u_\mu+1} (dz)^{r-1}) \quad (3.15)$$

for some non-vanishing t_μ , where

$$u_\mu := -r - s_\mu + \sum_{v \leq \mu} r_\mu s_v + \sum_{v > \mu} r_v s_\mu + 1.$$

Thus, considering the term related to ${}^\sigma H_{i,k}$, we see after some elementary addition of exponents that

$$\mathcal{N}\left(u^{r-i} \frac{1}{\xi^k} \left(\frac{d\xi}{\xi}\right)^i\right) \in \mathcal{O}\left(z^{-r_\mu(k+s_{[\mu-1]})-s_\mu(i-r_{[\mu-1]})-1} \left(\frac{dz}{z}\right)\right).$$

Since by Lemma 3.10, for $(i, k) \notin I_{\mathbf{r},\mathbf{s}}^>$ we have

$$r_\mu(k + s_{[\mu-1]}) + s_\mu(i - r_{[\mu-1]} - 1) \leq 0,$$

this transformation maps the vector \mathbf{H}_+ to \mathbf{J}_+ and therefore indeed $\mathcal{N}_{+-} = 0$ implying the sought-after relation $\mathcal{N}_{--} \cdot \mathcal{M}_{--} = \text{id}_-$.

From (3.14), one can check the desired vanishing property for the entries of \mathcal{N}_{--} , and therefore applying the matrix \mathcal{N}_{--} to the semi-infinite column vector with entries

$$\pi_1({}^\sigma H_{i,k}) = \sum_{\substack{\mu \in [d] \\ a > 0}} \mathcal{M}_{(i,k),(\mu,a)} \hbar \partial_{x_a^\mu}, \quad (i, k) \in I_{\mathbf{r},\mathbf{s}}^>$$

is well-defined and gives the semi-infinite column vector \mathbf{J}_- .

To completely prove the degree one property, we need to check that for any $(i, k) \in I_{\mathbf{r},\mathbf{s}}^>$, the degree zero component of ${}^\sigma H_{i,k}$ is vanishing. Following the proof of Lemma 3.1, one finds that the projection to degree zero of ${}^\sigma H_{i,k}$ for arbitrary i and k is

$$\pi_0({}^\sigma H_i(\xi)) = \sum_{\substack{Z \subseteq \tilde{\mathcal{X}}^{-1}(\xi) \\ |Z|=i}} \prod_{z \in Z} \omega_{0,1}(z). \quad (3.16)$$

Similarly to Lemma 3.6, we then see that in order to get a non-vanishing contribution to $\pi_0({}^\sigma H_{i,k})$, we need

$$k \leq -\left(s_{[\mu-1]} + (i - r_{[\mu-1]}) \frac{s_\mu}{r_\mu}\right),$$

where the difference with that lemma is the substitution of $i - 1$ by i . By the proof of that lemma,

$$\delta_{\mathbf{r},\mathbf{s}}^>(i) > -\left(s_{[\mu-1]} + (i - r_{[\mu-1]} - 1) \frac{s_\mu}{r_\mu}\right) > -\left(s_{[\mu-1]} + (i - r_{[\mu-1]}) \frac{s_\mu}{r_\mu}\right). \quad (3.17)$$

Hence $\pi_0({}^\sigma H_{i,k}) = 0$ for $(i, k) \in I_{\mathbf{r},\mathbf{s}}^>$, and this concludes the proof. \blacksquare

Remark 3.12. In comparison with Lemma 5.4, we can see that the assumptions $\gcd(r_\mu, s_\mu) = 1$ for all $\mu \in [d]$ and $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$ for any distinct μ, ν with $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ are crucial for the vanishing of \mathcal{N}_{+-} as they ensure that (3.15) starts with the correct order in z . So, in the case these conditions are not satisfied, we cannot expect the differential operators to be an Airy structure in general. A case in which this is indeed failing is provided by the example

$$x\left(\frac{\mu}{z}\right) = \frac{z^2}{2}, \quad y\left(\frac{1}{z}\right) = \frac{1}{z}, \quad y\left(\frac{2}{z}\right) = \frac{1}{z} + z$$

with $\omega_{0,1} = ydx$ presented in the same way as in Section 6. If we tried to compute the free energies associated to this input data recursively, we would find a non-symmetric $F_{0,3}$. Indeed, using a correspondence which will be established in Section 4, we can associate a multidifferential $\omega_{0,3}$ to $F_{0,3}$ which can be computed explicitly and shown to be non-symmetric as

$$\omega_{0,3}\left(\frac{2}{z_1} \frac{1}{z_2} \frac{1}{z_3}\right) = 0 \neq -\frac{dz_1 dz_2 dz_3}{z_1^2 z_2^2 z_3^2} = \omega_{0,3}\left(\frac{1}{z_3} \frac{1}{z_2} \frac{2}{z_1}\right).$$

Hence, the differential operators associated with the above input data cannot form an Airy structure.

3.2. The subalgebra condition

Having proven the degree one condition for the modes

$${}^\sigma H_{i,k}, \quad (i,k) \in I_{r,s}^>$$

with index set $I_{r,s}^>$ as defined in (3.11), we need to check whether these modes form a graded Lie subalgebra as required for Airy structures. In order to do so, we use Theorem 2.8 stating that if I is induced by a descending partition, then $({}^\sigma H_{i,k})_{(i,k) \in I}$ generate a graded Lie subalgebra. By this we mean that there exist $\lambda_1 \geq \dots \geq \lambda_\ell$ with $\sum_{j=1}^\ell \lambda_j = r$ such that $I = I_\lambda$, where

$$I_\lambda := \{(i,k) \in [r] \times \mathbb{Z} \mid \forall i \in [r], \lambda(i) + k > 0\},$$

and we set

$$\lambda(i) := \min \left\{ m \mid \sum_{j=1}^m \lambda_j \geq i \right\}.$$

In the case at hand, we want to check whether $I_{r,s}^> \cup \{(1,0)\}$ is induced by a descending partition. We will later then exclude ${}^\sigma H_{1,0}$ from the associated mode set by setting this mode to zero. Explicitly, this means that we want to classify the cases in which there exists a descending partition $\lambda \vdash r$ such that $\lambda(i) = 1 - \delta_{r,s}^>(i) + \delta_{i,1}$. Writing

$i = i' + r_{[v-1]}$ for $i' \in [r_v]$ again assuming $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$, we can write out $\delta_{\mathbf{r},s}^>(i)$ using Lemma 3.6 and obtain

$$\lambda(i) = 1 + \left\lfloor \frac{s_v(i' - 1)}{r_v} \right\rfloor + s_{[v-1]} - \delta_{i',1} \delta_{v>1}. \quad (3.18)$$

The case $d = 1$ was (up to a small addition we will need later) studied in [9], resulting in the following correspondence.

Lemma 3.13. *Let $r, s \geq 1$ be coprime. Then there exists a descending partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ such that*

$$\lambda(i') = 1 + \left\lfloor \frac{s(i' - 1)}{r} \right\rfloor, \quad i' \in [r] \quad (3.19)$$

if and only if $r = \pm 1 \pmod s$. In this case, λ is given by

$$\lambda_1 = \dots = \lambda_{r''} = r' + 1, \quad \lambda_{r''+1} = \dots = \lambda_\ell = r', \quad \ell = s - \delta_{s,r+1}, \quad (3.20)$$

writing $r = r's + r''$ with $r'' \in \{1, s - 1\}$. In particular, we have $\lambda = (1)$ for $r = 1$, $\lambda = (r)$ for $s = 1$, and $\lambda = (1^r)$ for $s = r + 1$.

Proof. In the case $r = 1$, the statement trivially holds and is independent of the choice of s . So let $r > 1$ now. The case where $s \in [r + 1]$ was already discussed in [9, Proposition B.1] and leads to the above classification of cases in which we can find a descending partition satisfying (3.19).

So what is left to prove is that in the case $s > r + 1$, there is no partition λ for which (3.19) holds. Suppose the opposite is true. Then since $\lambda(i' + 1) - \lambda(i') \in \{0, 1\}$, it follows that

$$\left\lfloor \frac{si'}{r} \right\rfloor - \left\lfloor \frac{s(i' - 1)}{r} \right\rfloor \leq 1$$

for all $i' \in [r - 1]$. Now writing $s = s'r + s''$ for some $s'' \in [0, r - 1]$, we see that the above is equivalent to

$$s' + \left\lfloor \frac{s''i'}{r} \right\rfloor - \left\lfloor \frac{s''(i' - 1)}{r} \right\rfloor \leq 1.$$

This inequality can only be satisfied if $s' = 1$, and

$$\left\lfloor \frac{s''i'}{r} \right\rfloor - \left\lfloor \frac{s''(i' - 1)}{r} \right\rfloor \leq 0$$

for all $i' \in [r - 1]$. Remember that we assumed $s > r + 1$ which implies $s'' > 1$. However, in this case we can always find an $i' \in [r - 1]$ for which $\left\lfloor \frac{s''i'}{r} \right\rfloor - \left\lfloor \frac{s''(i'-1)}{r} \right\rfloor = 1$, which is a contradiction. ■

For later reference, if r and s satisfy the properties of Lemma 3.13, let us write $\lambda^{r,s}$ for the associated partition λ and $\ell^{r,s} := s - \delta_{s,r+1}$ for its length.

To extend the statement of the last lemma to the case $d > 1$, we need to make a few observations. First, notice that for a partition $\lambda \vdash r$ we have

$$\lambda(i+1) - \lambda(i) = 1 \quad \text{if and only if} \quad i = \sum_{j=1}^m \lambda_j$$

for some $m > 0$ and $\lambda(i+1) - \lambda(i) = 0$ otherwise. Therefore, a descending partition λ satisfying (3.18) encodes the length of the intervals for which the right-hand side of the equation stays constant. Now notice that for $i \in (r_{[v-1]} + 1, r_{[v]})$ the right-hand side of (3.18) is exactly of the form as in (3.19) up to a constant shift, i.e.,

$$\lambda(i) = \lambda^{r_v, s_v}(i') + s_{[v-1]}$$

for $i = i' + r_{[v-1]}$ with $i' \in (1, r_v]$. Hence for λ to be descending, r_v and s_v need to satisfy the properties from Lemma 3.13. Moreover, if we analyse the jumping behaviour of $\lambda(i)$ at the transition $i \in [r_{[v-1]}, r_{[v-1]} + 2]$, we obtain a full description of λ . For this observe that because of the Kronecker delta

$$\begin{aligned} \lambda(r_{[v-1]} + 1) - \lambda(r_{[v-1]}) &= 0, \\ \lambda(r_{[v-1]} + 2) - \lambda(r_{[v-1]} + 1) &= 1 \end{aligned}$$

assuming $r_v > 1$ in the second line. Therefore, a partition λ describing the right-hand side of (3.18) must be of the form

$$\lambda = \begin{pmatrix} \lambda_1^{r_1, s_1} & \lambda_2^{r_1, s_1} & \cdots & \lambda_{\ell^{r_1, s_1}-1}^{r_1, s_1} & \lambda_{\ell^{r_1, s_1}}^{r_1, s_1} + 1 \\ \lambda_1^{r_2, s_2} - 1 & \lambda_2^{r_2, s_2} & \cdots & \lambda_{\ell^{r_2, s_2}-1}^{r_2, s_2} & \lambda_{\ell^{r_2, s_2}}^{r_2, s_2} + 1 \\ \lambda_1^{r_3, s_3} - 1 & \lambda_2^{r_3, s_3} & \cdots & \lambda_{\ell^{r_3, s_3}-1}^{r_3, s_3} & \lambda_{\ell^{r_3, s_3}}^{r_3, s_3} + 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{r_d, s_d} - 1 & \lambda_2^{r_d, s_d} & \cdots & \lambda_{\ell^{r_d, s_d}-1}^{r_d, s_d} & \lambda_{\ell^{r_d, s_d}}^{r_d, s_d} \end{pmatrix}, \quad (3.21)$$

where in the case $s_\mu = 1$ the μ -th line

$$(\dots, \lambda_1^{r_\mu, s_\mu} - 1, \lambda_2^{r_\mu, s_\mu}, \dots, \lambda_{\ell^{r_\mu, s_\mu}-1}^{r_\mu, s_\mu}, \lambda_{\ell^{r_\mu, s_\mu}}^{r_\mu, s_\mu} + 1, \dots)$$

must be replaced by just (\dots, r_μ, \dots) . This partition is almost just the concatenation $(\lambda^{r_1, s_1}, \dots, \lambda^{r_d, s_d})$ but it has one box moved from the first row of $\lambda^{r_{\mu+1}, s_{\mu+1}}$ to the last row of λ^{r_μ, s_μ} for each $\mu \in [d-1]$. A classification of the cases in which λ is descending is now given as follows.

Lemma 3.14. *Let $d \geq 2$. Given $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$ with r_μ and s_μ coprime for all μ , there exists a descending partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of $r = r_1 + \dots + r_d$ such that (3.18) is satisfied if and only if the following holds:*

- (i) $r_1 = -1 \bmod s_1$;
- (ii) $s_\mu = 1$ for all $\mu \in (1, d)$;
- (iii) $r_d = +1 \bmod s_d$.

In this case, λ is given by

$$\lambda = \begin{cases} ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}, r'_d{}^{s_d}), & r_d \neq 1, \\ ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}), & r_d = 1, \end{cases} \quad (3.22)$$

where $r'_\mu := \lfloor r_\mu / s_\mu \rfloor$.

Proof. Suppose properties (i)–(iii) are satisfied. Then inserting the explicit expressions for the partitions λ^{r_μ, s_μ} into formula (3.21) immediately tells us that λ is of the form (3.22) which is a descending partition as claimed.

Conversely, in order to argue that (i)–(iii) are necessary for λ to be a descending partition, first note that every row in (3.21) has to be descending individually which forces λ^{r_μ, s_μ} to be descending for all μ . Hence, Lemma 3.13 tells us that necessarily $r_\mu = \pm 1 \bmod s_\mu$. To further constrain the choice for r_μ and s_μ , note that in the case $s_\mu \geq 3$ the requirement that $\lambda_1^{r_\mu, s_\mu} - 1 \geq \lambda_2^{r_\mu, s_\mu}$ implies that $r_\mu = 1 \bmod s_\mu$ and $\lambda_{\ell^{r_\mu, s_\mu} - 1}^{r_\mu, s_\mu} \geq \lambda_{\ell^{r_\mu, s_\mu}}^{r_\mu, s_\mu} + 1$ forces $r_\mu = -1 \bmod s_\mu$. This explains (i) and (iii) and tells us that $s_\mu \leq 2$ for $\mu \in (1, d)$. Suppose now $s_\mu = 2$ for some $\mu \in (1, d)$. Then since we assume λ to be descending, we find that $\lceil \frac{r_\mu}{2} \rceil - 1 = \lambda_1^{r_\mu, 2} - 1 \geq \lambda_1^{r_\mu, 2} + 1 = \lfloor \frac{r_\mu}{2} \rfloor + 1$ which is a contradiction. Thus, the only possibility we are left with is indeed $s_\mu = 1$ for all $\mu \in (1, d)$. ■

We now have everything at hand to prove the standard case of Theorem 2.14.

Proof of Theorem 2.14 (standard case). Recall that the situation of the theorem is as follows:

$$\begin{aligned} {}^\sigma H_{i,k} &:= \widehat{\Phi} \widehat{T} \cdot {}^\sigma W_{i,k} \cdot \widehat{T}^{-1} \widehat{\Phi}^{-1}, \\ \widehat{T} &:= \exp \left(\sum_{\mu \in [d]} \sum_{k > 0} (\hbar^{-1} F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] + \hbar^{-\frac{1}{2}} F_{\frac{1}{2},1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right]) \frac{J_k^\mu}{k} \right) = \widehat{T}_2 \widehat{T}_1, \\ \widehat{T}_1 &:= \exp \left(\frac{1}{\hbar} \sum_{\mu \in [d]} \sum_{k \geq s_\mu} F_{0,1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] \frac{J_k^\mu}{k} \right), \\ \widehat{T}_2 &:= \exp \left(\frac{1}{\hbar^{\frac{1}{2}}} \sum_{\mu \in [d]} \sum_{k > 0} F_{\frac{1}{2},1} \left[\begin{smallmatrix} \mu \\ -k \end{smallmatrix} \right] \frac{J_k^\mu}{k} \right), \\ \widehat{\Phi} &:= \exp \left(\frac{1}{2\hbar} \sum_{\substack{\mu, \nu \in [d] \\ k, l > 0}} F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right] \frac{J_k^\mu J_l^\nu}{kl} \right). \end{aligned}$$

Up to now, we have only considered the conjugation with \widehat{T}_1 , so let us finish the argument for that case first.

The selected modes

$${}^\sigma H_{i,k} := \widehat{T}_1 \cdot {}^\sigma W_{i,k} \cdot \widehat{T}_1^{-1}, \quad i \in [r], \quad k \geq i - \lambda(i) + \delta_{i,1} \quad (3.23)$$

with the partition

$$\lambda = \begin{cases} ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}, r_d'^{s_d}), & r_d \neq 1, \\ ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}), & r_d = 1 \end{cases}$$

exactly correspond to the modes $({}^\sigma H_{i,k})_{(i,k) \in I_{r,s}^>}$, where $I_{r,s}^>$ is the index set defined in (3.11) by performing the identification of index sets via Lemma 3.14. Thus, Lemma 3.11 tells us that after a change of basis, modes (3.23) satisfy the degree one condition. Since by assumption

$${}^\sigma H_{1,0} = {}^\sigma W_{1,0} = J_0^1 + \dots + J_0^d = \hbar^{\frac{1}{2}} (Q_1 + \dots + Q_d) = 0,$$

modes (3.23) satisfy the subalgebra condition if the modes $({}^\sigma H_{i,k})_{(i,k) \in I_\lambda}$ do. Here I_λ is defined as in (2.11). Now using that ${}^\sigma H_{i,k}$ is obtained from ${}^\sigma W_{i,k}$ via conjugation, the claim immediately follows from Theorem 2.8.

For the general case, conjugating also with \widehat{T}_2 and $\widehat{\Phi}$, note first of all that conjugation preserves commutation relations, so the subalgebra condition still holds. For the degree one condition, note that conjugation by \widehat{T}_2 gives the shifts

$$J_{-k}^\mu \rightarrow J_{-k}^\mu + \hbar^{\frac{1}{2}} F_{\frac{1}{2},1}^\mu[-k],$$

which preserves degrees, and only acts on J_k^μ with $k < 0$, which do not occur in $\pi_1(\widehat{T}_1 \cdot {}^\sigma W_{i,k} \cdot \widehat{T}_1^{-1})$ by the previous parts of the computation. Likewise, conjugation by $\widehat{\Phi}$ acts as in equation (2.27), which again preserves degrees and only affects J_k^μ with $k < 0$, so it also preserves the degree one condition. ■

3.3. The exceptional case

Contrary to the case considered before, let us now allow $s_\mu = \infty$ for $\mu \in [d]$. Let us write

$$\widetilde{\mathcal{C}}_+ := \bigsqcup_{\mu \in [d], s_\mu \neq \infty} \widetilde{\mathcal{C}}_\mu,$$

and $\widetilde{\mathcal{C}}_-$ for the collection of all components $\widetilde{\mathcal{C}}_\mu$ on which $s_\mu = \infty$.

Lemma 3.15. *For any $i \in [r]$,*

$$\pi_1({}^\sigma H_i(\xi)) = \sum_{z \in \widetilde{x}^{-1}(\xi)} \sum_{\substack{Z \subseteq \widetilde{x}^{-1}(\xi) \setminus \{z\} \cap \widetilde{\mathcal{C}}_+ \\ |Z|=i-1}} J(z) \prod_{z' \in Z} \omega_{0,1}(z'). \quad (3.24)$$

Proof. The proof of this lemma is verbatim to the one of Lemma 3.1 taking into account that $\omega_{0,1}(\frac{\mu}{z}) = 0$ for all $\mu \in [d]$ with $s_\mu = \infty$. ■

Remark 3.16. Let us make two important observations. First notice that if we write $r_+ := \sum_{\mu \in [d], s_\mu \neq \infty} r_\mu$, then

$$\pi_1(\sigma H_{r_++1}(\xi)) = \sum_{z \in \tilde{x}^{-1}(\xi) \cap \tilde{C}_-} J(z) \prod_{z' \in \tilde{C}_+} \omega_{0,1}(z'),$$

and moreover that for all $i > r_+ + 1$, we have

$$\pi_1(\sigma H_i(\xi)) = 0.$$

Especially, from the last identity we deduce that, in order to end up with an Airy structure, it is necessary to have at most one $\mu \in [d]$ for which $s_\mu = \infty$. Moreover, necessarily for this μ we need $r_\mu = 1$. Otherwise, there is no hope to obtain an Airy structure.

Motivated by Remark 3.16 in the following, we will assume that only $(r_d, s_d) = (1, \infty)$ while for all other $\mu \in [d-1]$ we have $s_\mu \neq \infty$, what we call the exceptional case in Theorem 2.11. Moreover, let us assume that as before $\frac{r_1}{s_1} \geq \dots \geq \frac{r_{d-1}}{s_{d-1}}$. Rather than working with expression (3.24), we will mainly use that by (2.14) we have

$$\sigma H_{i,k} = \sigma H'_{i,k} + \sum_{a \in \mathbb{Z}} \sigma H'_{i-1,k-a} J_a^d, \quad i \in [r], k \in \mathbb{Z},$$

where $\sigma H'_{i,k}$ is obtained from $\sigma H_{i,k}$ by formally setting J_*^d equal to zero. Of course, $\sigma H'_{i,k}$ may be computed via (3.2) replacing d by $d-1$, i.e., these are modes of the standard case. Therefore, as the modes $\sigma H_{i,k}$ are build up from modes considered in equation (3.2) and an additional factor J_*^d , we can use the analysis of the standard case from the previous section in order to prove the degree one condition for these operators.

Let us select the following modes. For $i < r$, we define $\delta_{r,s}^>(i)$ exactly as in (3.7) and set $\delta_{r,s}^>(r) := 1 - s_{[d-1]}$. Again, we define $I^>$ as in (3.11) to be the index set associated to this choice of $\delta_{r,s}^>$.

Lemma 3.17. *Assume $\gcd(r_\mu, s_\mu) = 1$ for all $\mu \in [d-1]$ and $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$ for any distinct μ, ν with $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$. Then there exists an invertible matrix \mathcal{N} such that*

$$\sigma \tilde{H}_{\mu,a} = \sum_{(i,k) \in I_{r,s}^>} \mathcal{N}_{(\mu,a),(i,k)} \sigma H_{i,k}, \quad (\mu, a) \in [d] \times \mathbb{Z}_{>0},$$

satisfy the degree one condition.

Proof. First, let us argue that

$$\pi_1(\sigma H_{i,k}) = \pi_1(\sigma H'_{i,k}) + \sum_{a \in \mathbb{Z}} \pi_0(\sigma H'_{i-1,k-a}) J_a^d \quad (3.25)$$

for $k \geq \delta_{r,s}^>(i)$ is a linear combination of J_a^d 's with $a > 0$ only. Since for $i < r$ we defined $\delta_{r,s}^>(i)$ exactly as in (3.7), it follows from Lemma 3.6 that the first term $\pi_1(\sigma H'_{i,k})$ is a linear combination of J_a^d 's with $a > 0$ only. That the second term

$$\sum_{a \in \mathbb{Z}} \pi_0(\sigma H'_{i-1,k-a}) J_a^d$$

is a linear combination of $(J_a^d)_{a>0}$ only follows from the fact that $\pi_0(\sigma H'_{i-1,k-a}) = 0$ unless $k - a < \delta_{r,s}^>(i)$ as observed in (3.17). Hence,

$$\pi_1(\sigma H_{i,k}) = \pi_1(\sigma H'_{i,k}) + \sum_{a > k - \delta_{r,s}^>(i)} \pi_0(\sigma H'_{i-1,k-a}) J_a^d, \quad i \in [r], k \geq \delta_{r,s}^>(i)$$

indeed lies in the linear span of $(J_a^d)_{(\mu,a) \in [d] \times \mathbb{Z}_{>0}}$.

In order to bring the operators into normal form, let us make use of our observation made earlier in Remark 3.16 that

$$\pi_1(\sigma H_{r,k}) = \sum_{a > k - \delta_{r,s}^>(r)} \pi_0(\sigma H'_{r-1,k-a}) J_a^d.$$

This can be rephrased in the sense that

$$\pi_1(\sigma H_{r,k}) = \sum_{a > 0} \mathcal{A}_{k - \delta_{r,s}^>(r) + 1, a} J_a^d,$$

where \mathcal{A} is an upper triangular matrix whose diagonal entries

$$\mathcal{A}_{a,a} = \pi_0(\sigma H'_{r-1, \delta_{r,s}^>(r) - 1}) = (-1)^{r+d} \prod_{\mu=1}^{d-1} (F_{0,1}[-s_\mu^\mu])^{r\mu} \neq 0$$

can be read off from (3.16) by taking only the leading order contributions of the $\omega_{0,1}$'s into account. Thus, one can find a two-sided inverse of \mathcal{A} , and applying it to the semi-infinite vector $(\sigma H_{r,k})_{k \geq \delta_{r,s}^>(r)}$, we get $(\sigma \tilde{H}_{r,k})_{k \geq \delta_{r,s}^>(r)}$ for which

$$\pi_1(\sigma \tilde{H}_{r,k}) = J_{k - \delta_{r,s}^>(r) + 1}^d.$$

By taking again suitable linear combinations, we can use the above modes in order to eliminate all J_a^d from $\pi_1(\sigma H_{i,k})$ for $i < r$, i.e., get operators $(\bar{H}_{i,k})_{(i,k) \in I_{r,s}^>, i < r}$ that satisfy

$$\forall i \in [r], k \geq \delta_{r,s}^>(i), \quad \pi_1(\sigma \bar{H}_{i,k}) = \pi_1(\sigma H'_{i,k}).$$

This expression is exactly the degree one projection of the operators considered in Lemma 3.1, where we shifted in all cycles. From Lemma 3.11, we know that they can be brought in normal form, provided that we can argue at last that the degree zero projection of these modes is vanishing. This is indeed the case: since $\pi_0(\sigma H_{i,k}) = \pi_0(\sigma H'_{i,k})$ and we know by (3.17) that the degree zero projection of $H'_{i,k}$ vanishes as long as $k \geq \delta_{r,s}^>(i)$, we see the same holds for $\sigma H_{i,k}$. ■

This provides us with everything we need to prove Theorem 2.11 in the exceptional case.

Proof of Theorem 2.14 (exceptional case). As in the standard case, we know that conjugation with

$$\begin{aligned}\hat{T}_2 &:= \exp\left(\frac{1}{\hbar^{\frac{1}{2}}} \sum_{\mu \in [d]} \sum_{k > 0} F_{\frac{1}{2},1} \left[\begin{matrix} \mu \\ -k \end{matrix} \right] \frac{J_k^\mu}{k}\right), \\ \hat{\Phi} &:= \exp\left(\frac{1}{2\hbar} \sum_{\substack{\mu, \nu \in [d] \\ k, l > 0}} F_{0,2} \left[\begin{matrix} \mu & \nu \\ -k & -l \end{matrix} \right] \frac{J_k^\mu J_l^\nu}{kl}\right)\end{aligned}$$

preserves the Airy structure conditions. It is hence sufficient to prove that

$$\begin{aligned}\sigma H_{i,k} &:= \hat{T}_1 \cdot \sigma W_{i,k} \cdot \hat{T}_1^{-1}, \quad i \in [r], k \geq 1 - \lambda(i) + \delta_{i,1}, \\ \hat{T}_1 &:= \exp\left(\frac{1}{\hbar} \sum_{\substack{\mu \in [d] \\ s_\mu \neq \infty}} \sum_{k \geq s_\mu} F_{0,1} \left[\begin{matrix} \mu \\ -k \end{matrix} \right] \frac{J_k^\mu}{k}\right),\end{aligned}$$

where we chose the partition

$$\lambda = ((r'_1 + 1)^{s_1}, r_2, r_3, \dots, r_{d-1}),$$

form an Airy structure. Indeed, as we have

$$\sigma H_{1,0} = \hbar^{\frac{1}{2}}(Q_1 + \dots + Q_d) = 0,$$

which vanishes by assumption, the selected modes already satisfy the subalgebra condition using Theorem 2.8. On the other hand, since λ is chosen so that

$$1 - \lambda(i) + \delta_{i,1} = \delta_{r,s}^>(i)$$

for all $i \in [r]$, the selected modes also satisfy the degree one condition by (3.17) as required. ■

3.4. Approximate solution to the partial differential equations

Having analysed the cases in which $({}^\sigma H_{i,k})_{(i,k) \in I_{r,s}^>}$ is an Airy structure and hence giving rise to a partition function Z solving the associated system of differential equations to all orders in \hbar , we now turn our attention to the question in which case the associated differential equations are solved at leading order in the \hbar expansion. For this we will use a technical lemma that holds for arbitrary families of differential operators satisfying the degree one condition.

3.4.1. The existence of a partition function. Suppose we are in the setting of Section 2.1.1, i.e., E is a finite-dimensional vector space over \mathbb{C} with $(x_a)_{a \in A}$ a basis of E^* . Moreover, let π_k denote the projection to the k -graded piece of \mathcal{D}_E^\hbar . This section is devoted to the following question: In which case does a family $(H_a)_{a \in A}$ of differential operators satisfying the degree one condition admits a solution to

$$\forall a \in A, \quad e^{-\hbar^{-1}F_0} H_a e^{\hbar^{-1}F_0} \cdot 1 = o(\hbar^{\frac{1}{2}}) \quad (3.26)$$

of the form $F_0 = \sum_{n \geq 3} \frac{1}{n!} F_{0,n}$, where $F_{0,n} \in \text{Sym}^n(E^*)$? We remark that, due to the degree one condition, such an F_0 is unique if it exists, and the free energies obey a recursion in n . Regarding the question of existence, we have the following lemma.

Lemma 3.18. *Suppose there is a family $(H_a)_{a \in A}$ of elements in \mathcal{D}_E^\hbar that (after normalisation) satisfies the degree one condition (2.1) of an Airy structure and there is a second family $(H_j)_{j \in J}$ whose elements satisfy*

$$\pi_0(H_j) = 0, \quad \pi_1(H_j) \in \mathbb{C} \langle \hbar \partial_{x_*}, \hbar^{\frac{1}{2}} \rangle$$

so that the combined family $(H_i)_{i \in A \cup J}$ satisfies the subalgebra condition (2.2) up to order $\hbar^{\frac{3}{2}}$, i.e., there exist $f_{i_1, i_2}^{i_3} \in \mathcal{D}_E^\hbar$ so that for all $i_1, i_2 \in A \cup J$

$$[H_{i_1}, H_{i_2}] - \hbar \sum_{i_3 \in A \cup J} f_{i_1, i_2}^{i_3} H_{i_3} = o(\hbar^{\frac{3}{2}}).$$

Then $(H_a)_{a \in A}$ admits a solution F_0 to the associated system of differential equations (3.26).

Proof. After a change of basis, we can assume that the family $(H_a)_{a \in A}$ is normalised, which means the operators take the form

$$\begin{aligned} H_a &= \hbar \partial_{x_a} + x^{\geq 2} + x^{\geq 1} (\hbar \partial)^{\geq 1} + (\hbar \partial)^{\geq 2} + o(\hbar^{\frac{1}{2}}), \quad a \in A, \\ H_j &= \sum_{b \in A} \mathcal{M}_{j,b} \hbar \partial_{x_b} + x^{\geq 2} + x^{\geq 1} (\hbar \partial)^{\geq 1} + (\hbar \partial)^{\geq 2} + o(\hbar^{\frac{1}{2}}), \quad j \in J \end{aligned}$$

for some matrix \mathcal{M} . Here we used the same notation as in [44, Section 2.4] to represent terms in which certain symbols may appear with a fixed order. In the following, we will construct free energies $F_{0,n}$ inductively in $n \geq 3$ so that

$$e^{-\hbar^{-1} \sum_{k=3}^n F_{0,k}} H_i e^{\hbar^{-1} \sum_{k=3}^n F_{0,k}} (1) = x^{\geq n} + \hbar^{\geq \frac{1}{2}} x^{\geq 0} \quad (3.27)$$

for all $i \in A \cup J$. Then taking $F := \lim_{n \rightarrow \infty} \sum_{k=3}^n F_{0,k}$, we obtain a free energy that satisfies (3.26) as claimed.

Note that since $H_i \cdot 1 = x^{\geq 2} + \hbar^{\geq \frac{1}{2}} x^{\geq 0}$, the first non-trivial case is $n = 3$. For this let us write $H_i = (\hbar\partial)^{-1} - H_{i;0,2} + x^{\geq 3} + x^{\geq 1}(\hbar\partial)^{\geq 1} + (\hbar\partial)^{\geq 2} + o(\hbar^{\frac{1}{2}})$, i.e., we write $-H_{i;0,2}$ for the terms $\hbar^0 x^{\geq 2} (\hbar\partial)^{=0}$. Then for $a_1, a_2 \in A$, we have

$$[H_{a_1}, H_{a_2}] = \hbar(\partial_{a_2} H_{a_1;0,2} - \partial_{a_1} H_{a_2;0,2} + x^{\geq 2} + x^{\geq 0}(\hbar\partial)^{\geq 1} + o(\hbar^{\frac{1}{2}}))$$

and since by assumption the commutator has to lie in the left ideal generated by $\{H_i\}_{i \in A \cup J}$, we get $\partial_{a_2} H_{a_1;0,2} - \partial_{a_1} H_{a_2;0,2} = 0$. As the last equality holds for all $a_1, a_2 \in A$, then there exists a unique homogeneous polynomial $F_{0,3}$ of degree three in x_* such that $\partial_a F_{0,3} = H_{a;0,2}$. With this choice, (3.27) is indeed satisfied for all $i \in A$ with $n = 3$. So far we only used the exact same arguments as in [44, Theorem 2.4.2]. Now we need to prove the compatibility of our choice for $F_{0,3}$ with $(H_j)_{j \in J}$. For this, notice that from

$$[H_j, H_a] = \hbar \left(\partial_a H_{j;0,2} - \sum_{b \in A} \mathcal{M}_{j,b} \partial_b H_{a;0,2} + x^{\geq 2} + x^{\geq 0}(\hbar\partial)^{\geq 1} + o(\hbar^{\frac{1}{2}}) \right)$$

which holds for all $j \in J$ and $a \in A$, we can deduce that

$$\partial_a \left(H_{j;0,2} - \sum_{b \in A} \mathcal{M}_{j,b} \partial_b F_{0,3} \right) = 0,$$

which in turn implies that $H_{j;0,2} = \sum_{b \in A} \mathcal{M}_{j,b} \partial_b F_{0,3}$ since the term in brackets is a homogeneous polynomial of order two. Therefore, we see that for $n = 3$ equation (3.27) is indeed satisfied for all $i \in J$ as well.

Now for the induction step, suppose we have already constructed $\sum_{k=3}^n F_{0,k}$ solving (3.27) for all $i \in A \cup J$. Then after conjugation

$$\bar{H}_i := e^{-\hbar^{-1} \sum_{k=3}^n F_{0,k}} H_i e^{\hbar^{-1} \sum_{k=3}^n F_{0,k}},$$

the operators take the form

$$\begin{aligned} \bar{H}_a &= \hbar \partial_{x_a} - \bar{H}_{a;0,n} + x^{\geq n+1} + x^{\geq 1}(\hbar\partial)^{\geq 1} + (\hbar\partial)^{\geq 2} + o(\hbar^{\frac{1}{2}}), \quad a \in A, \\ \bar{H}_j &= \sum_{b \in A} \mathcal{M}_{j,b} \hbar \partial_{x_b} - \bar{H}_{j;0,n} + x^{\geq n+1} + x^{\geq 1}(\hbar\partial)^{\geq 1} + (\hbar\partial)^{\geq 2} + o(\hbar^{\frac{1}{2}}), \quad j \in J, \end{aligned}$$

where we wrote $-\bar{H}_{i;0,n}$ for the piece $\hbar^0 x^{-n} (\hbar \partial)^{=0}$ in \bar{H}_i . We use the exact same argument as for the base case and deduce from

$$[H_{a_1}, H_{a_2}] = \hbar(\partial_{a_2} H_{a_1;0,n} - \partial_{a_1} H_{a_2;0,n} + x^{\geq 2} + x^{\geq 0} (\hbar \partial)^{\geq 1} + o(\hbar^{\frac{1}{2}}))$$

that $\partial_{a_2} H_{a_1;0,n} - \partial_{a_1} H_{a_2;0,n} = 0$. Hence, there is a homogeneous polynomial $F_{0,n+1}$ of degree $n+1$ satisfying $\partial_a F_{n+1} = \bar{H}_{a;0,n}$ as required. The compatibility with the operators $(\bar{H}_j)_{j \in J}$ similarly follows from inspecting

$$[H_j, H_a] = \hbar\left(\partial_a H_{j;0,n} - \sum_{b \in A} \mathcal{M}_{j,b} \partial_b H_{a;0,n} + x^{\geq 2} + x^{\geq 0} (\hbar \partial)^{\geq 1} + o(\hbar^{\frac{1}{2}})\right)$$

with $j \in J, a \in A$, from which we deduce that

$$\partial_a \left(H_{j;0,n} - \sum_{b \in A} \mathcal{M}_{j,b} \partial_b F_{0,n+1} \right) = 0.$$

For this to be satisfied, $H_{j;0,n} - \sum_{b \in A} \mathcal{M}_{j,b} \partial_b F_{0,n+1}$ must itself vanish. So we see that with our choice for $F_{0,n+1}$, we indeed have

$$\begin{aligned} e^{-\hbar^{-1} \sum_{k=3}^{n+1} F_{0,k}} H_i e^{\hbar^{-1} \sum_{k=3}^{n+1} F_{0,k}} \cdot 1 &= e^{-\hbar^{-1} F_{0,n+1}} \bar{H}_i e^{\hbar^{-1} F_{0,n+1}} \cdot 1 \\ &= x^{\geq n+1} + \hbar^{\geq \frac{1}{2}} x^{\geq 0} \end{aligned}$$

for all $i \in A \cup J$ which proves the induction step. ■

Remark 3.19. For expositional reasons, we chose to prove Lemma 3.18 in the setting of finite-dimensional vector spaces but we stress that the statement of the lemma carries over to the case where E is a filtered vector space and $(H_a)_{a \in A}$ is a filtered family of elements in $\hat{\mathcal{D}}_E^{\hbar}$. We leave it to the reader to check the details.

Remark 3.20. In the language of [9, 44], Lemma 3.18 is only a statement about the *classical limit* of the family $(H_a)_{a \in A}$ since all higher-order terms in $\hbar^{\frac{1}{2}}$ essentially play no role. We chose to present the statement in the *quantum* setting nevertheless since all differential operators considered in this paper come with a natural \hbar -refinement.

3.4.2. The proof of Theorem 2.13. Let us now apply Lemma 3.18 to the family of modes $({}^\sigma H_{i,k})_{(i,k) \in I_{r,s}^{\geq}}$ from Section 2.3.3 in order to work out sufficient conditions under which the operators admit a solution to their associated system of differential equations up to corrections in $\hbar^{\frac{1}{2}}$. In Lemmas 3.11 and 3.17, we already analysed the cases in which this family satisfies the degree one condition. So in order to apply Lemma 3.18, we need to find a set $J \subseteq [r] \times \mathbb{Z}$ so that the combined family $({}^\sigma H_{i,k})_{(i,k) \in I_{r,s}^{\geq} \cup J}$ satisfies the subalgebra condition and the operators $({}^\sigma H_{i,k})_{(i,k) \in J}$ feature no terms involving x_*^* in degree one. As shown in Lemma 3.8,

the latter requirement forces that we choose $J \subseteq I_{\mathbf{r},\mathbf{s}}^{\geq} \setminus I_{\mathbf{r},\mathbf{s}}^{>}$. Therefore, again by making use of Theorem 2.8, the question of when the operators $(\sigma H_{i,k})_{(i,k) \in I_{\mathbf{r},\mathbf{s}}^{\geq}}$ admit a solution to the associated system of differential equations to leading order in $\hbar^{\frac{1}{2}}$ is reduced to the question whether we can find a descending partition $\lambda \vdash r$ for which $I_{\mathbf{r},\mathbf{s}}^{>} \subseteq I_{\lambda} \subseteq I_{\mathbf{r},\mathbf{s}}^{\geq}$ because if such a partition exists, we can simply take $J = I_{\lambda} \setminus I_{\mathbf{r},\mathbf{s}}^{>}$ and apply Lemma 3.18. A characterisation when this is possible is presented in the following.

Lemma 3.21. *Given $\frac{r_1}{s_1} \geq \dots \geq \frac{r_d}{s_d}$ with r_{μ} and s_{μ} coprime integers for all μ , there exists a descending partition λ such that*

$$I_{\mathbf{r},\mathbf{s}}^{>} \subseteq I_{\lambda} \subseteq I_{\mathbf{r},\mathbf{s}}^{\geq} \quad (3.28)$$

if the following points are satisfied:

- (i) $r_{\mu} = \pm 1 \pmod{s_{\mu}}$ for all $\mu \in [d]$.
- (ii) For all $\mu_1 \neq \mu_2$ with $s_{\mu_i} > 2$ such that either

$$r_{\mu_1} = 1 \pmod{s_{\mu_1}} \quad \text{and} \quad r_{\mu_2} = 1 \pmod{s_{\mu_2}}$$

or

$$r_{\mu_1} = -1 \pmod{s_{\mu_1}} \quad \text{and} \quad r_{\mu_2} = -1 \pmod{s_{\mu_2}},$$

one has $\lfloor \frac{r_{\mu_1}}{s_{\mu_1}} \rfloor \neq \lfloor \frac{r_{\mu_2}}{s_{\mu_2}} \rfloor$.

- (iii) If there are pairwise distinct $\mu_1, \mu_2, \mu_3 \in [d]$ with $\lfloor \frac{r_{\mu_1}}{s_{\mu_1}} \rfloor = \lfloor \frac{r_{\mu_2}}{s_{\mu_2}} \rfloor = \lfloor \frac{r_{\mu_3}}{s_{\mu_3}} \rfloor$, then there is an $m \in \{1, 2, 3\}$ for which $s_{\mu_m} = 1$.

Proof. First, let us describe what a possibly non-descending partition λ' looks like that satisfies $I_{\lambda'} = I_{\mathbf{r},\mathbf{s}}^{\geq}$. On the intervals $i = i' + \mathbf{r}_{[v-1]} \in (\mathbf{r}_{[v-1]}, \mathbf{r}_{[v]})$, we have

$$1 - \delta_{\mathbf{r},\mathbf{s}}(i) = \lambda^{r_v, s_v}(i') + s_{[v-1]},$$

where λ^{r_v, s_v} is the partition described in Lemma 3.13. To analyse the transitions between these intervals, let us for a moment assume that $r_v > 1$ and $s_v < r_v$ for all $v \in d$. These special cases will be treated separately later. Under these assumptions,

$$(1 - \delta_{\mathbf{r},\mathbf{s}}^{\geq}(\mathbf{r}_{[v]} + 1)) - (1 - \delta_{\mathbf{r},\mathbf{s}}^{\geq}(\mathbf{r}_{[v-1]})) = (1 + s_{[v]}) - s_{[v]} = 1,$$

and thus the partition $\lambda' = (\lambda^{r_1, s_1}, \dots, \lambda^{r_d, s_d})$ satisfies $I_{\lambda'} = I_{\mathbf{r},\mathbf{s}}^{\geq}$. Since by assumption (i) each block λ^{r_v, s_v} in λ' is descending, we only need to check whether it is descending at the transitions, i.e., whether $\lambda_{\ell^{r_v, s_v}}^{r_v, s_v} \geq \lambda_1^{r_{v+1}, s_{v+1}}$ for all $v \in [d-1]$, and if not, whether we can modify λ' into a descending partition λ satisfying (3.28). For this note that $\lambda_{\ell^{r_v, s_v}}^{r_v, s_v} \geq \lambda_1^{r_{v+1}, s_{v+1}}$ holds if and only if

$$\left\lfloor \frac{r_v}{s_v} \right\rfloor \geq \left\lceil \frac{r_{v+1}}{s_{v+1}} \right\rceil.$$

This means that in the case $\lfloor \frac{r_\nu}{s_\nu} \rfloor > \lfloor \frac{r_{\nu+1}}{s_{\nu+1}} \rfloor$ for all $\nu \in [d-1]$, the partition λ' is indeed descending and so we can choose $\lambda = \lambda'$. If however there is a $\mu \in [d]$ for which $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_{\mu+1}}{s_{\mu+1}} \rfloor$, we need to be more careful. The case where at least one of s_μ or $s_{\mu+1}$ equals 1 is unproblematic since then $\lfloor \frac{r_\mu}{s_\mu} \rfloor \geq \lceil \frac{r_{\mu+1}}{s_{\mu+1}} \rceil$ is automatically satisfied. However, if $s_\mu, s_{\mu+1} > 1$, we have $\lfloor \frac{r_\mu}{s_\mu} \rfloor < \lceil \frac{r_{\mu+1}}{s_{\mu+1}} \rceil$. The lack of λ' being descending at this transition can be cured as follows. First, notice that (ii) forces that $r_\mu = -1 \bmod s_\mu$ and $r_{\mu+1} = 1 \bmod s_{\mu+1}$. One should remark here that the second case $r_\mu = 1 \bmod s_\mu$ and $r_{\mu+1} = -1 \bmod s_{\mu+1}$ allowed by (ii) can only occur if $s_\mu = s_{\mu+1} = 2$ and is hence contained in the first case. Pictorially, we are in the following setting:

$$\lambda' = \left. \begin{array}{c} \vdots \\ \left. \begin{array}{c} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \vdots \end{array} \right\} = \lambda^{r_\mu, s_\mu}, \\ \left. \begin{array}{c} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \dots \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\ \vdots \end{array} \right\} = \lambda^{r_{\mu+1}, s_{\mu+1}}. \end{array} \quad (3.29)$$

We will now make use of the fact that there is a certain ambiguity in choosing λ , so that I_λ lies between $I_{\mathbf{r}, \mathbf{s}}^>$ and $I_{\mathbf{r}, \mathbf{s}}^\geq$. So far we focused on the partition λ' satisfying $I_{\lambda'} = I_{\mathbf{r}, \mathbf{s}}^\geq$. Now notice that deleting an element

$$(\mathbf{r}_{[\mu]} + 1, -s_{[\mu]}) \in I_{\mathbf{r}, \mathbf{s}}^\geq \setminus I_{\mathbf{r}, \mathbf{s}}^>$$

affects the partition describing the index set in the sense that one box from the first row of $\lambda^{r_{\mu+1}, s_{\mu+1}}$ gets shifted to the last row of λ^{r_μ, s_μ} , i.e., the index sets associated to the partitions

$$\lambda' = (\lambda_1^{r_1, s_1}, \dots, \lambda_{\ell^{r_\mu, s_\mu} - 1}^{r_\mu, s_\mu}, \lambda_{\ell^{r_\mu, s_\mu}}^{r_\mu, s_\mu}, \lambda_1^{r_{\mu+1}, s_{\mu+1}}, \lambda_2^{r_{\mu+1}, s_{\mu+1}}, \dots, \lambda_{\ell^{r_d, s_d}}^{r_d, s_d}), \\ \lambda'' = (\lambda_1^{r_1, s_1}, \dots, \lambda_{\ell^{r_\mu, s_\mu} - 1}^{r_\mu, s_\mu}, \lambda_{\ell^{r_\mu, s_\mu} + 1}^{r_\mu, s_\mu} + 1, \lambda_1^{r_{\mu+1}, s_{\mu+1}} - 1, \lambda_2^{r_{\mu+1}, s_{\mu+1}}, \dots, \lambda_{\ell^{r_d, s_d}}^{r_d, s_d})$$

are related by $I_{\lambda''} = I_{\lambda'} \setminus \{(\mathbf{r}_{[\mu]} + 1, -s_{[\mu]})\}$. The index set $I_{\lambda''}$ then still satisfies $I_{\mathbf{r}, \mathbf{s}}^\geq \subseteq I_{\lambda''} \subseteq I_{\mathbf{r}, \mathbf{s}}^\geq$ as required. We notice now that if we perform such a box shift in the case displayed in (3.29), we indeed obtain a partition λ'' that is descending at the transition between the μ -th and $(\mu+1)$ -th blocks. This solves the problem of having two subsequent blocks with $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_{\mu+1}}{s_{\mu+1}} \rfloor$. However, if there is a sequence $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \dots = \lfloor \frac{r_{\mu+a}}{s_{\mu+a}} \rfloor$ with $a > 1$, then by (iii) we must already have $s_{\mu+2} = \dots = s_{\mu+a} = 1$.

This means that if also $s_{\mu+1} = 1$, there is nothing to worry about, and in the case $s_{\mu+1} > 1$ we have

$$\lambda' = \left. \begin{array}{c} \dots \\ \left. \begin{array}{c} \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \end{array} \right\} = \lambda^{r_\mu, s_\mu}, \\ \left. \begin{array}{c} \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \end{array} \right\} = \lambda^{r_{\mu+1}, s_{\mu+1}}, \\ \left. \begin{array}{c} \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \end{array} \right\} = \lambda^{r_{\mu+2}, s_{\mu+2}}, \\ \left. \begin{array}{c} \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \\ \begin{array}{ccc} \square & \dots & \square \\ \square & & \square \\ \square & & \square \end{array} \\ \vdots \end{array} \right\} = \lambda^{r_{\mu+a}, s_{\mu+a}}
 \end{array}$$

and so we can perform a box shift in order to construct a descending partition λ from λ' . This finishes the construction of a partition λ satisfying (3.28) if $s_\nu < r_\nu$ and $r_\nu > 1$ for all $\nu \in [d]$.

Hence, we are left with the cases in which possibly $r_\mu = 1$ or $s_\mu = r_\mu + 1$ for $\mu \in [d]$. So suppose there is a μ with $s_\mu = r_\mu = 1$ and assume that $\mu \in [d]$ is minimal with this property. Then from our assumption that $\frac{r_{v_1}}{s_{v_1}} \geq \frac{r_{v_2}}{s_{v_2}}$ for $v_1 \leq v_2$, we deduce that necessarily $s_\nu \leq r_\nu$ and $r_\nu > 1$ for all $\nu < \mu$. Hence, we know from our previous analysis that we can find a partition $\lambda \vdash \mathbf{r}_{[\mu-1]}$ whose associated index set I_λ satisfies

$$I_{\mathbf{r}, \mathbf{s}}^\geq \cap \{(i, k) \mid i \leq \mathbf{r}_{[\mu-1]}\} \subseteq I_\lambda \subseteq I_{\mathbf{r}, \mathbf{s}}^\geq \cap \{(i, k) \mid i \leq \mathbf{r}_{[\mu-1]}\}.$$

Now let $\delta \geq \mu$ be maximal with $r_\nu = s_\nu = 1$ for all $\nu \in [\mu, \delta]$. Then we compute

$$(1 - \delta_{\mathbf{r}, \mathbf{s}}^\geq(\mathbf{r}_{[v-1]} + 1)) - (1 - \delta_{\mathbf{r}, \mathbf{s}}^\geq(\mathbf{r}_{[v-1]})) = 1$$

for all $v \in [\mu, \delta]$ and thus

$$I_{\mathbf{r}, \mathbf{s}}^\geq \cap \{(i, k) \mid i \leq \mathbf{r}_{[\delta]}\} \subseteq I_{\lambda'} \subseteq I_{\mathbf{r}, \mathbf{s}}^\geq \cap \{(i, k) \mid i \leq \mathbf{r}_{[\delta]}\}$$

if we choose $\lambda' = (\lambda, 1^{\delta - \mu + 1})$ which is descending. Since we have chosen δ maximal, we must have $s_\nu > r_\nu$ for all $\nu > \delta$ which however implies that $\delta \in \{d - 2, d - 1\}$ by (iii). First, let us assume $\delta = d - 1$. Then

$$-\delta_{\mathbf{r}, \mathbf{s}}^\geq(\mathbf{r}_{[d-1]} + 1) + \delta_{\mathbf{r}, \mathbf{s}}^\geq(\mathbf{r}_{[d-1]}) = 1,$$

and thus

$$I_{\mathbf{r}, \mathbf{s}}^\geq \subseteq I_{(\lambda', 1^{r_d})} \subseteq I_{\mathbf{r}, \mathbf{s}}^\geq.$$

Now suppose $\delta = d - 2$. In this case, we need to be more careful. First, let us assume $r_{d-1} > 1$ which due to (i) forces $s_{d-1} = r_{d-1} + 1$ and thus also $r_d = 1$ due to (ii). We find that between the two transition points

$$\begin{aligned} -\delta_{r,s}^{\geq}(\mathbf{r}_{[d-2]} + 1) + \delta_{r,s}^{\geq}(\mathbf{r}_{[d-2]}) &= 1, \\ -\delta_{r,s}^{\geq}(\mathbf{r}_{[d-1]} + 1) + \delta_{r,s}^{\geq}(\mathbf{r}_{[d-1]}) &= 2. \end{aligned}$$

The fact that $\delta_{r,s}^{\geq}$ jumps by two between $\mathbf{r}_{[d-1]} \rightarrow \mathbf{r}_{[d-1]} + 1 = \mathbf{r}$ is only a mild problem that can be cured by not including $(\mathbf{r}_{[d]}, -s_{[d-1]})$ into our index set. Indeed, if we choose $\lambda'' = (\lambda', 1^{r_{d-1}}, 1)$, then the associated index set indeed has the desired property that $I_{r,s}^{\geq} \subseteq I_{\lambda''} \subseteq I_{r,s}^{\geq}$.

Lastly, let us assume that $r_{d-1} = 1$. In this case, (i) and (ii) only leave us with $s_{d-1} = 2$ and $r_d = 1$. Recall here that we treated the case $s_{d-1} = 1$ earlier. This time we find

$$\begin{aligned} -\delta_{r,s}^{\geq}(\mathbf{r}_{[d-2]} + 1) + \delta_{r,s}^{\geq}(\mathbf{r}_{[d-2]}) &= 1, \\ -\delta_{r,s}^{\geq}(\mathbf{r}_{[d-1]} + 1) + \delta_{r,s}^{\geq}(\mathbf{r}_{[d-1]}) &= 2. \end{aligned}$$

Again we remedy the jump of 2 in $\delta_{r,s}^{\geq}$ by deleting $(\mathbf{r}_{[d]}, -s_{[d-1]})$ from the index set and finally as in the case before we find that $(\lambda', 1, 1)$ satisfies the desired property $I_{r,s}^{\geq} \subseteq I_{(\lambda', 1, 1)} \subseteq I_{r,s}^{\geq}$. ■

Remark 3.22. The statement of Lemma 3.21 can easily be extended to the exceptional case. If $r_d = 1$, then $\delta_{r,s}^{\geq}$ and $\delta_{r,s}^{\geq}$ actually do not depend on the choice of s_d for finite s_d . Therefore, we can extend the definition of $\delta_{r,s}^{\geq}$ and $\delta_{r,s}^{\geq}$ to the case $s_d = \infty$ by choosing $\delta_{r,s}^{\geq}(r) = \delta_{r,s}^{\geq}(r) + 1 = -s_{[d-1]} + 1$ as in the finite setting. Thus, all the results of Lemma 3.21 easily translate to the exceptional case if we adopt the convention that we say $r_d = 1 \bmod s_d$ if $(r_d, s_d) = (1, \infty)$ and interpret $\frac{r_d}{s_d}$ as being zero.

As motivated in the introduction of this subsection, we can now easily use Lemma 3.21 to prove Theorem 2.13.

Proof of Theorem 2.13. The partition λ in the statement of the theorem is chosen so that $1 - \lambda(i) + \delta_{i,1} = \delta_{r,s}^{\geq}(i)$ for all $i \in [r]$. Thus, we already know from Lemmas 3.11 and 3.17 that the selected modes satisfy the degree one condition. Moreover, Lemma 3.21 tells us that we can choose a partition λ' so that the modes $({}^\sigma H_{i,k})_{(i,k) \in J}$ associated to the index set $J = I_{\lambda'} \setminus I_{r,s}^{\geq}$ satisfy the properties required in Lemma 3.18. Therefore, the lemma ensures the sought-after existence of a leading order solution to the system of differential equations associated to $({}^\sigma H_{i,k})_{(i,k) \in I_\lambda}$. ■

Part II

Spectral curve descriptions

In this part, we translate the differential constraints coming from the \mathcal{W} -algebra representations from Section 2 into constraints on the order of poles of certain combinations of multidifferentials $\omega_{g,n}$ on a spectral curve built from the coefficients $F_{g,n}$. The latter constraints are called “abstract loop equations”. In a second step, we show that the unique solution to the abstract loop equations is provided by an adaptation of Bouchard–Eynard topological recursion to the setting of singular spectral curves. In fact, this provides us with the right definition of the topological recursion à la Chekhov–Eynard–Orantin in this setting, together with the proof that it is well-defined.

4. From Airy structures to local spectral curves

4.1. Fields for a single cycle

We will start by reconsidering Section 2.2.3 in the case of σ consisting of a single cycle of length r . In this case, we can omit all μ -indices, and consider

$$J_k = \begin{cases} \hbar \partial_{x_k} & \text{if } k > 0, \\ \hbar^{\frac{1}{2}} Q & \text{if } k = 0, \\ -k x_{-k} & \text{if } k < 0 \end{cases}$$

the standard representation of the Heisenberg algebra of \mathfrak{gl}_r . It is useful to write $\tilde{x} = z^r$. We split the current as follows:

$$\begin{aligned} J(z) &= \sum_{k \in \mathbb{Z}} \frac{J_k dz}{z^{k+1}} = J_+(z) + J_-(z) + \hbar^{\frac{1}{2}} \frac{Q dz}{z}, \\ J_+(z) &= \sum_{k > 0} J_{-k} z^{k-1} dz, \\ J_-(z) &= \sum_{k > 0} \frac{J_k dz}{z^{k+1}}. \end{aligned}$$

Choose a primitive r -th root of unity ϑ and let $\mathfrak{f}(z) = \{z, \vartheta z, \dots, \vartheta^{r-1} z\}$. Set

$$\omega_{0,2}^{\text{std}}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

We can rewrite (2.17) as

$$\begin{aligned}
 W_i(\tilde{x}) &= \frac{1}{r} \sum_{Z:[i] \leftrightarrow f(z)} \sum_{\substack{0 \leq j \leq [i/2] \\ A_0 \sqcup A_+ \sqcup A_- = [2j+1, i]}} \frac{\hbar^j}{2^j j! (i-2j)!} \prod_{l=1}^j \omega_{0,2}^{\text{std}}(Z_{2l-1}, Z_{2l}) \\
 &\times \prod_{l \in A_0} \frac{Q dz}{z} \prod_{l \in A_+} J_+(Z_l) \prod_{l \in A_-} J_-(Z_l). \tag{4.1}
 \end{aligned}$$

As in Section 2.3.3, let us apply a general dilaton shift and change of polarisation to these operators. We take

$$\begin{aligned}
 \hat{T} &= \exp\left(\sum_{k>0} (\hbar^{-1} F_{0,1}[-k] + \hbar^{-\frac{1}{2}} F_{\frac{1}{2},1}[-k]) \frac{J_k}{k}\right), \\
 \hat{\Phi} &= \exp\left(\frac{1}{2\hbar} \sum_{k_1, k_2 > 0} F_{0,2}[-k_1, -k_2] \frac{J_{k_1} J_{k_2}}{k_1 k_2}\right),
 \end{aligned}$$

in which we can always assume that $F_{0,2}[-k_1, -k_2] = F_{0,2}[-k_2, -k_1]$, and introduce

$$H_i(\tilde{x}) := \hat{\Phi} \hat{T} \cdot W_i(\tilde{x}) \cdot \hat{T}^{-1} \hat{\Phi}^{-1}.$$

The effect of the dilaton shift \hat{T} in formula (4.1) is to replace $J_+(z)$ by $J_+(z) + \hbar^{\frac{1}{2}}(\omega_{\frac{1}{2},1}(z) - Q \frac{dz}{z}) + \omega_{0,1}(z)$, where

$$\omega_{0,1}(z) := \sum_{k>0} F_{0,1}[-k] z^{k-1} dz, \quad \omega_{\frac{1}{2},1}(z) := Q \frac{dz}{z} + \sum_{k>0} F_{\frac{1}{2},1}[-k] z^{k-1} dz.$$

Using the Baker–Campbell–Hausdorff formula, it is easy to see that the net effect of the change of polarisation $\hat{\Phi}$ is to replace $\omega_{0,2}^{\text{std}}$ by

$$\omega_{0,2}(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \sum_{k_1, k_2 > 0} F_{0,2}[-k_1, -k_2] z_1^{k_1-1} z_2^{k_2-1} dz_1 dz_2$$

and to replace $J_-(z)$ by¹

$$\mathcal{J}_-(z) := \sum_{k>0} J_k d\xi_{-k}(z), \tag{4.2}$$

where for $k > 0$,

$$d\xi_{-k}(z) := \frac{dz}{z^{k+1}} + \sum_{l>0} \frac{F_{0,2}[-k, -l]}{k} d\xi_l(z) = \text{Res}_{z'=0} \left(\int_0^{z'} \omega_{0,2}(\cdot, z) \right) \frac{dz'}{(z')^{k+1}}.$$

¹More precisely, the conjugation by $\hat{\Phi}$ leaves $J_-(z)$ invariant and adds extra terms with positive J s in $J_+(z)$. We collect all terms with negative (resp. positive) J s in \mathcal{J}_+ (resp. \mathcal{J}_-), thus leading to (4.2).

For uniformity, we also define for $k \geq 0$,

$$d\xi_k(z) := z^{k-1} dz.$$

So, we can rewrite

$$\begin{aligned}\omega_{0,1}(z) &= \sum_{k>0} F_{0,1}[-k] d\xi_k(z), \\ \omega_{\frac{1}{2},1}(z) &= Q d\xi_0(z) + \sum_{k>0} F_{\frac{1}{2},1}[-k] d\xi_k(z), \\ \omega_{0,2}(z_1, z_2) &= \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \sum_{k_1, k_2 > 0} F_{0,2}[-k_1, -k_2] d\xi_{k_1}(z_1) d\xi_{k_2}(z_2),\end{aligned}$$

so that we have

$$\mathcal{J}_+(z) := J_+(z) = \sum_{k>0} k x_k d\xi_k(z).$$

We then obtain

$$\begin{aligned}rH_i(\tilde{x}) &= \sum_{\substack{Z: [i] \rightarrow \mathfrak{f}(z) \\ 0 \leq j \leq \lfloor i/2 \rfloor \\ A_0 \sqcup A_{\frac{1}{2}} \sqcup A_+ \sqcup A_- = [2j+1, i]}} \frac{\hbar^{j + \frac{1}{2}|A_{\frac{1}{2}}|}}{2^j j! (i - 2j)!} \prod_{l=1}^j \omega_{0,2}(Z_{2l-1}, Z_{2l}) \\ &\quad \times \prod_{l \in A_0} \omega_{0,1}(Z_l) \prod_{l \in A_{\frac{1}{2}}} \omega_{\frac{1}{2},1}(Z_l) \prod_{z' \in A_+} \mathcal{J}_+(z') \prod_{z' \in A_-} \mathcal{J}_-(z'). \quad (4.3)\end{aligned}$$

We prefer to convert this expression into a sum over subsets $Z \subseteq \mathfrak{f}(z)$ of cardinality i . Then, we have to sum over partitions $B_1 \sqcup \dots \sqcup B_j \sqcup A_0 \sqcup A_{\frac{1}{2}} \sqcup A_+ \sqcup A_- = Z$, where $|B_l| = 2$ for any $l \in [j]$, and it can arise in exactly $2^j (i - 2j)!$ terms in (4.3) corresponding to the choice of an order within each pair B_j , and the choice of a labelling by $[2j + 1, i]$ for the elements in $A_0 \sqcup A_{\frac{1}{2}} \sqcup A_+ \sqcup A_-$. So only the factor $1/j!$ remains. It can also be erased by forgetting the ordering of B_1, \dots, B_j . More precisely, introducing the set $\mathcal{P}(\mathfrak{f}(z))$ whose elements are sets of disjoint pairs in $\mathfrak{f}(z)$, and writing $\sqcup \mathbf{P} := \bigsqcup_{P \in \mathbf{P}} P$ if $\mathbf{P} \in \mathcal{P}(\mathfrak{f}(z))$, we obtain

$$\begin{aligned}rH_i(\tilde{x}) &= \sum_{\substack{Z \subseteq \mathfrak{f}(z) \\ |Z|=i}} \sum_{\substack{(\sqcup \mathbf{P}) \sqcup A_0 \sqcup A_{\frac{1}{2}} \sqcup A_+ \sqcup A_- = Z \\ \mathbf{P} \in \mathcal{P}(\mathfrak{f}(z))}} \hbar^{|\mathbf{P}| + \frac{1}{2}|A_{\frac{1}{2}}|} \prod_{\{z', z''\} \in \mathbf{P}} \omega_{0,2}(z', z'') \\ &\quad \times \prod_{z' \in A_0} \omega_{0,1}(z') \prod_{z' \in A_{\frac{1}{2}}} \omega_{\frac{1}{2},1}(z') \prod_{z' \in A_+} \mathcal{J}_+(z') \prod_{z' \in A_-} \mathcal{J}_-(z').\end{aligned}$$

4.2. Fields for an arbitrary twist

We now return to the general situation of Section 2.2.3. Let σ be a permutation of $[r]$ with cycles of lengths r_μ labelled by $\mu \in [d]$. For each $\mu \in [d]$, we have generators J_k^μ of the Heisenberg algebra of \mathfrak{gl}_{r_μ} :

$$J_k^\mu = \begin{cases} \hbar \partial_{x_k^\mu} & \text{if } k > 0, \\ \hbar^{\frac{1}{2}} Q_\mu & \text{if } k = 0, \\ -k x_{-k}^\mu & \text{if } k < 0, \end{cases}$$

whose currents we split as $J_+^\mu(z)$ and $J_-^\mu(z)$ in the same way as in Section 4.1. We obtain modes W_{i_μ, k_μ}^μ indexed by $i_\mu \in [r_\mu]$ and $k_\mu \in \mathbb{Z}$ for a representation of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra given by (2.15). To match Section 4.1, we introduce for each $\mu \in [d]$ formal variables z such that $\tilde{x} = z^{r_\mu}$. These z thus depend on μ , but they will appear in generating series with superscript μ so that one can infer directly from the formula which power r_μ one should use to relate it to the global variable \tilde{x} .

At this stage, we are naturally led to introduce a curve which is the union of copies of a formal disk for each $\mu \in [d]$:

$$\tilde{C} = \bigsqcup_{\mu=1}^d \tilde{C}_\mu, \quad \tilde{C}_\mu := \text{Spec } \mathbb{C}[[z]].$$

When necessary to avoid confusion, points in \tilde{C} will be denoted by $(\frac{\mu}{z})$ to indicate in which copy of the formal disk we consider them. One can consider \tilde{x} as a branched cover

$$\tilde{C} \rightarrow V := \text{Spec } \mathbb{C}[[X]]$$

given by $z \mapsto z^{r_\mu}$ on the μ -th copy of the z -formal disk. The smooth (but reducible) curve \tilde{C} is in fact the normalisation $\pi: \tilde{C} \rightarrow C$ of the singular curve

$$C = \text{Spec } \mathbb{C}[[x, z]] / \left(\prod_{\mu=1}^d (x - z^{r_\mu}) \right).$$

The branched cover $\tilde{x}: \tilde{C} \rightarrow V$ factors through $x: C \rightarrow V$. This is the local picture we will globalise later in Section 5 by considering more general branched covers

$$\tilde{C} \xrightarrow{\pi} C \xrightarrow{x} V, \quad \tilde{x} = x \circ \pi,$$

where V, \tilde{C} are regular curves and C is a possibly singular curve whose normalisation is \tilde{C} . For the moment, we stick to the local setting.

Let us again consider a general dilaton shift and change of polarisation

$$\hat{T} = \exp \left(\sum_{\substack{\mu \in [d] \\ k > 0}} (\hbar^{-1} F_{0,1}[\begin{smallmatrix} \mu \\ -k \end{smallmatrix}] + \hbar^{-\frac{1}{2}} F_{\frac{1}{2},1}[\begin{smallmatrix} \mu \\ -k \end{smallmatrix}]) \frac{J_k^\mu}{k} \right),$$

$$\hat{\Phi} = \exp \left(\frac{1}{2\hbar} \sum_{\substack{\mu, \nu \in [d] \\ k, l > 0}} F_{0,2}[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix}] \frac{J_k^\mu J_l^\nu}{kl} \right)$$

and the conjugated operator

$$H_i(\tilde{x}) = \hat{\Phi} \hat{T} \cdot W_i(\tilde{x}) \cdot \hat{T}^{-1} \hat{\Phi}^{-1} = \sum_{k \in \mathbb{Z}} \frac{H_{i,k}(\mathrm{d}\tilde{x})^i}{\tilde{x}^{k+i}}.$$

To express $H_i(\tilde{x})$, we introduce the basis of meromorphic 1-forms $\mathrm{d}\xi_k^\mu$ on \tilde{C} , indexed by $\mu \in [d]$ and $k \in \mathbb{Z}$. It is defined by

$$k \geq 0: \mathrm{d}\xi_k^\mu \left(\frac{\nu}{z} \right) = \delta_{\mu,\nu} z^{k-1} \mathrm{d}z, \quad (4.4)$$

$$k > 0: \mathrm{d}\xi_{-k}^\mu \left(\frac{\nu}{z} \right) = \delta_{\mu,\nu} \frac{\mathrm{d}z}{z^{k+1}} + \sum_{l > 0} \frac{F_{0,2}[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix}]}{k} z^{l-1} \mathrm{d}z. \quad (4.5)$$

We also introduce the meromorphic forms $\omega_{0,1}$, $\omega_{\frac{1}{2},1}$ and bidifferential $\omega_{0,2}$ on \tilde{C} :

$$\omega_{0,1} = \sum_{\substack{\mu \in [d] \\ k > 0}} F_{0,1}[\begin{smallmatrix} \mu \\ -k \end{smallmatrix}] \mathrm{d}\xi_k^\mu,$$

$$\omega_{\frac{1}{2},1} = \sum_{\mu \in [d]} Q_\mu \mathrm{d}\xi_0^\mu + \sum_{k > 0} F_{\frac{1}{2},1}[\begin{smallmatrix} \mu \\ -k \end{smallmatrix}] \mathrm{d}\xi_k^\mu,$$

$$\omega_{0,2} = \omega_{0,2}^{\mathrm{std}} + \sum_{\substack{\mu_1, \mu_2 \in [d] \\ k_1, k_2 > 0}} F_{0,2}[\begin{smallmatrix} \mu_1 & \mu_2 \\ -k_1 & -k_2 \end{smallmatrix}] \mathrm{d}\xi_{k_1}^{\mu_1} \mathrm{d}\xi_{k_2}^{\mu_2},$$

where

$$\omega_{0,2}^{\mathrm{std}} \left(\frac{\nu_1}{z_1} \frac{\nu_2}{z_2} \right) = \frac{\delta_{\nu_1, \nu_2} \mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

For $k \in \mathbb{Z}$, we introduce the 1-form on \tilde{C}

$$\mathrm{d}\xi_k^* \left(\frac{\mu}{z} \right) = \mathrm{d}\xi_k^\mu \left(\frac{\mu}{z} \right).$$

We recall that the index $\mu \in [d]$ of the component to which a point $z' \in \tilde{C}$ belongs is implicit in the data of z' .

Similarly to Section 4.1, the effect of the dilaton shift is to replace $J_+^\mu(z)$ by

$$J_+^\mu(z) + \hbar^{\frac{1}{2}} \left(\omega_{\frac{1}{2},1} \left(\frac{\mu}{z} \right) - Q_\mu \frac{\mathrm{d}z}{z} \right) + \omega_{0,1} \left(\frac{\mu}{z} \right),$$

while the effect of the change of polarisation is to replace $\omega_{0,2}^{\text{std}}(z_1, z_2)$ by $\omega_{0,2}(\frac{\mu_1}{z_1}, \frac{\mu_2}{z_2})$ and $J_-^\mu(z)$ by

$$\mathcal{J}_-(\frac{\mu}{z}) := \sum_{k>0} J_k^\mu d\xi_{-k}^*(\frac{\mu}{z}).$$

For uniformity, we also set

$$\mathcal{J}_+(\frac{\mu}{z}) := J_+^\mu(z) = \sum_{k>0} k x_k^\mu d\xi_k^*(\frac{\mu}{z}).$$

We can repeat the argument of Section 4.1 with several μ s, defining the fiber over \tilde{x} in \tilde{C}

$$\tilde{f}(z) := \bigsqcup_{\mu=1}^d \tilde{f}_\mu(z)$$

and getting

$$\begin{aligned} H_i(\tilde{x}) &= \sum_{\substack{Z \subseteq \tilde{f}(z) \\ |\tilde{Z}|=i}} \sum_{\substack{(\sqcup \mathbf{P}) \sqcup A_0 \sqcup A_{\frac{1}{2}} \sqcup A_+ \sqcup A_- = Z \\ \mathbf{P} \in \tilde{\mathcal{P}}(\tilde{f}(z))}} \hbar^{|\mathbf{P}| + \frac{1}{2}|A_{\frac{1}{2}}|} \prod_{\{z', z''\} \in \mathbf{P}} \omega_{0,2}(z', z'') \\ &\times \prod_{z' \in A_0} \omega_{0,1}(z') \prod_{z' \in A_{\frac{1}{2}}} \omega_{\frac{1}{2},1}(z') \prod_{z' \in A_+} \mathcal{J}_+(z') \prod_{z' \in A_-} \mathcal{J}_-(z'). \end{aligned}$$

4.3. Action of the fields on the partition function

Given a formal function

$$F = \sum_{\substack{g \geq 0, n \geq 1 \\ 2g-2+n > 0}} \sum_{\substack{\mu_1, \dots, \mu_n \in [d] \\ k_1, \dots, k_n > 0}} \frac{\hbar^{g-1}}{n!} F_{g,n} \left[\begin{smallmatrix} \mu_1 & \dots & \mu_n \\ k_1 & \dots & k_n \end{smallmatrix} \right] \prod_{i=1}^n x_{k_i}^{\mu_i}, \quad (4.6)$$

let us compute

$$G_i(\tilde{x}) = e^{-F} H_i(\tilde{x}) e^F \cdot 1.$$

The partition function e^F is annihilated by the differential operators above a certain index in the \mathcal{W} if and only if the G_i satisfy certain bounds on their pole orders as $\tilde{x} \rightarrow 0$. Because F is a function (i.e., it does not contain a differential part), it commutes with $\omega_{0,2}$, $\omega_{0,1}$, $\omega_{\frac{1}{2},1}$, and \mathcal{J}_+ . The only non-trivial computation is

$$e^{-F} \mathcal{J}_-(z') e^F = \mathcal{J}_-(z') + [\mathcal{J}_-(z'), F],$$

where each term $\mathcal{J}_-(z')$ obtained like this has to act on a later $[\mathcal{J}_-(z''), F]$, as it annihilates 1. The \mathcal{J}_- commute among each other, so we get a partition of A_- into

sets of operators acting on a single copy of F . We obtain

$$\begin{aligned}
 G_i(\tilde{x}) &= \sum_{\substack{Z \subseteq \mathfrak{f}(z) \\ |Z|=i}} \sum_{\substack{(\sqcup \mathbf{P}) \sqcup A_0 \sqcup A_1 \sqcup A_+ \sqcup A_- = Z \\ \mathbf{P} \in \mathcal{P}(\mathfrak{f}(z))}} \hbar^{|\mathbf{P}| + \frac{1}{2}|A_{\frac{1}{2}}|} \prod_{\{z', z''\} \in \mathbf{P}} \omega_{0,2}(z', z'') \\
 &\times \prod_{z' \in A_0} \omega_{0,1}(z') \prod_{z' \in A_{\frac{1}{2}}} \omega_{\frac{1}{2},1}(z') \prod_{z' \in A_+} \left(\sum_{\tilde{k}_{z'} \geq 1} \tilde{k}_{z'} x_{\tilde{k}_{z'}}^{\mu_{z'}} d\xi_{\tilde{k}_{z'}}^*(z') \right) \\
 &\times \sum_{\substack{\mathbf{L} \vdash A_- \\ g_L, m_L \geq 0, L \in \mathbf{L} \\ 2g_L - 2 + m_L + |L| > 0}} \prod_{\substack{L \in \mathbf{L} \\ \nu_L: [m_L] \rightarrow [d] \\ \ell_L: [m_L] \rightarrow \mathbb{N}^* \\ k_L: L \rightarrow \mathbb{N}^*}} \left(\frac{\hbar^{g_L - 1 + |L|}}{m_L!} \right. \\
 &\times F_{g_L, |L| + m_L} \left[\begin{matrix} \mu_{|L} & \nu_{L,1} & \dots & \nu_{L,m_L} \\ k_L & \ell_{L,1} & \dots & \ell_{L,m_L} \end{matrix} \right] \prod_{l \in [m_L]} x_{\ell_{L,l}}^{\nu_{L,l}} \prod_{z' \in L} d\xi_{-k_{z'}}^*(z') \Big), \quad (4.7)
 \end{aligned}$$

where $\mu: A_+ \sqcup A_- \rightarrow [d]$ associates to z' the index $\mu_{z'} \in [d]$ such that $z' \in \mathfrak{f}_{\mu_{z'}}(z)$, and we identify $\mu_{|L}$ and \mathbf{k}_L with the tuples $(\mu_{z'})_{z' \in L}$ and $(k_{z'})_{z' \in L}$, respectively.

We decompose G_i in homogeneous terms with respect to the exponent of \hbar and the number of x_k^μ :

$$G_i(\tilde{x}) = \sum_{g, n \geq 0} \frac{\hbar^g}{n!} G_{i;g,n}(\tilde{x}).$$

In order to completely rephrase this in terms of spectral curves, we need to get rid of the x_k^μ and replace them with $d\xi$ s. For every n , prepare a tuple $w_{[n]} = (w_j)_{j=1}^n$ of points on \tilde{C} and define

$$\mathcal{E}_{g,n}^{(i)}(\tilde{x}; w_{[n]}) := \prod_{j=1}^n \text{ad}_{\mathfrak{f}_-(w_j)} G_{i;g,n}(\tilde{x}).$$

To compute it, we introduce the multidifferential forms for $g \geq 0$ and $n \geq 1$ such that $2g - 2 + n > 0$,

$$\omega_{g,n}(z_1, \dots, z_n) := \sum_{\substack{\mu_1, \dots, \mu_n \in [d] \\ k_1, \dots, k_n > 0}} F_{g,n} \left[\begin{matrix} \mu_1 & \dots & \mu_n \\ k_1 & \dots & k_n \end{matrix} \right] \prod_{j=1}^n d\xi_{-k_j}^{\mu_j}(z_j). \quad (4.8)$$

Besides, under this action, we get

$$\text{ad}_{\mathfrak{f}_-(w)} \left(\sum_{k>0} k x_k^\nu d\xi_k^\mu(z) \right) = \sum_{k>0} k d\xi_{-k}^\nu(w) d\xi_k^\mu(z),$$

which is the series expansion of $\omega_{0,2}(\frac{\nu}{w}, \frac{\mu}{z})$ with $|z| < |w|$.

We then notice that the sums over $\tilde{k}_{z'}$, k_L , ν_L and ℓ_L in (4.7) recombine into

$$\begin{aligned} \mathcal{E}_{g,n}^{(i)}(\tilde{x}, w_{[n]}) &= \sum_{\substack{Z \subseteq \mathfrak{f}(z) \\ |\tilde{Z}|=i}} \sum_{\substack{(\sqcup \mathbf{P}) \sqcup A_0 \sqcup A_{\frac{1}{2}} \sqcup A_+ \sqcup A_- = Z \\ \mathbf{P} \in \mathcal{P}(\mathfrak{f}(z))}} \sum_{\iota: A_+ \hookrightarrow [n]} \sum_{\substack{\mathbf{L} \vdash A_- \\ \mathbf{M} \vdash \mathbf{L}[n] \setminus \iota(A_+)}} \\ &\quad \sum_{\substack{g_L \geq 0, L \in \mathbf{L} \\ 2g_L - 2 + |L| + |\mathbf{M}_L| > 0 \\ \frac{1}{2}|A_{\frac{1}{2}}| + |\mathbf{P}| + |A_-| + \sum_L (g_L - 1) = g}} \prod_{\{z', z''\} \in \mathbf{P}} \omega_{0,2}(z', z'') \prod_{z' \in A_0} \omega_{0,1}(z') \\ &\quad \times \prod_{z' \in A_{\frac{1}{2}}} \omega_{\frac{1}{2},1}(z') \prod_{z' \in A_+} \omega_{0,2}(w_{\iota(z')}, z') \prod_{L \in \mathbf{L}} \omega_{g,|L|+|\mathbf{M}_L|}(L, w_{\mathbf{M}_L}). \end{aligned}$$

We now observe that the factors $\omega_{0,1}$, $\omega_{\frac{1}{2},1}$, $\omega_{0,2}$ can be treated uniformly by summing over partitions $\mathbf{L} \vdash Z$ and allowing $(g_L, |L| + m_L) = (0, 1), (\frac{1}{2}, 1), (0, 2)$, which were exactly the terms for which $2g_L - 2 + |L| + m_L \leq 0$. We get

$$\mathcal{E}_{g,n}^{(i)}(\tilde{x}; w_{[n]}) = \sum_{\substack{Z \subseteq \mathfrak{f}(z) \\ |\tilde{Z}|=i}} \sum_{\mathbf{L} \vdash Z} \sum_{\substack{g_L \geq 0, L \in \mathbf{L} \\ g = i + \sum_L (g_L - 1)}} \prod_{L \in \mathbf{L}} \omega_{g_L, |L| + |\mathbf{N}_L|}(L, w_{\mathbf{N}_L}).$$

4.4. From partial differential equations to abstract loop equations

Theorem 2.14 gives sufficient conditions on the values of $(r_\mu)_{\mu=1}^d$, of positive integers $(s_\mu)_{\mu=1}^d$, of scalars $(t_\mu)_{\mu=1}^d$ and $(Q_\mu)_{\mu=1}^d$ to get a unique F of the form (4.6) such that for any $i \in [r]$ and $k \geq \delta_{r,s}(i)$,

$$e^{-F} H_{i,k} e^F \cdot 1 = 0.$$

The translation of these differential constraints in terms of the *correlators*

$$\omega = (\omega_{g,n})_{g,n}$$

defined in (4.8) is called “abstract loop equations”. It says that for any $n \geq 0$, we have

$$\mathcal{E}_{g,n}^{(i)}(\tilde{x}; w_{[n]}) \in o(\tilde{x}^{-\delta_{r,s}(i)}) \cdot \left(\frac{d\tilde{x}}{\tilde{x}}\right)^i, \quad \tilde{x} \rightarrow 0.$$

In other words, $\mathcal{E}_{g,n}^{(i)}(\tilde{x}, w_{[n]})$ is meromorphic and has a pole of order strictly less than $\delta_{r,s}(i) + i$ at the point $\tilde{x} = 0$ in V . If we let $\tilde{\mathcal{E}}_{g,n}^{(i)}(z, w_{[n]})$ be its pullback to a meromorphic i -differential on \tilde{C} , this is tantamount to requiring that for any $\mu \in [d]$,

$$\tilde{\mathcal{E}}_{g,n}^{(i)}\left(\frac{\mu}{z}; w_{[n]}\right) \in o(z^{-r\mu \delta_{r,s}(i)}) \cdot \left(\frac{dz}{z}\right)^i, \quad z \rightarrow 0.$$

5. Topological recursion on global spectral curves

We are going to formalise what we have found in the context of global, possibly singular spectral curves. This will lead us to define the appropriate notion of abstract loop equations in Section 5.2, and to show in Section 5.4 that its unique solution is given by an appropriate topological recursion à la Chekhov–Eynard–Orantin, that is, by computing residues on the normalisation of the singular curve.

5.1. Spectral curves

Definition 5.1. A *spectral curve* is a triple $\mathcal{C} = (C, x, y)$, where C is a reduced analytic curve over \mathbb{C} , and x, y are meromorphic functions on C such that all fibers of x are finite and $\omega_{0,1} := y dx$ has no poles at ramifications of x .

Note that C is not necessarily connected, compact, or irreducible. We will work with its normalisation $\pi: \tilde{C} \rightarrow C$, which is a smooth curve. We have meromorphic functions $\tilde{x} = x \circ \pi$ and $\tilde{y} = y \circ \pi$ defined on \tilde{C} . Let $\mathfrak{b} \subset \mathbb{C}$ be the set of points b that have a neighbourhood U_b such that the cardinality of the fiber of x is constant on $U_b \setminus \{b\}$ and strictly smaller at b itself. It is the collection of branchpoints of \tilde{x} and images of locally reducible points away from ∞ . We also denote $\alpha = x^{-1}(b)$ and $\tilde{\alpha} = \tilde{x}^{-1}(b)$. We assume that \mathfrak{b} is finite. As a result, $\tilde{\alpha}$ and α are also finite. Note that, since we assumed that all fibers of x are finite, the same is true of \tilde{x} and there cannot be an irreducible component of \tilde{C} where \tilde{x} is constant.

If $\alpha \in \alpha$, we let $U_\alpha \subset C$ be a small neighbourhood of α that is invariant under local Galois transformations and

$$\tilde{U}_\alpha := \pi^{-1}(U_\alpha), \quad \tilde{U}'_\alpha = \tilde{U}_\alpha \setminus \pi^{-1}(\alpha), \quad V_\alpha = x(U_\alpha), \quad V'_\alpha = V_\alpha \setminus \{x(\alpha)\}.$$

Without loss of generality, we can assume that $V_\alpha \subset \mathbb{C}$. If $z \in \tilde{U}_\alpha$, we define

$$\mathfrak{f}_\alpha(z) = \tilde{x}^{-1}(\tilde{x}(z)) \cap \tilde{U}_\alpha, \quad \mathfrak{f}'_\alpha(z) = \mathfrak{f}_\alpha(z) \setminus \{z\}, \quad \tilde{\alpha}_\alpha = \pi^{-1}(\alpha).$$

Note that $\tilde{\alpha}_\alpha$ is in bijection with the set of branches in \tilde{C} above α , and we denote $d_\alpha := |\tilde{\alpha}_\alpha|$. For each $\mu \in \tilde{\alpha}_\alpha$, we introduce a small neighbourhood \tilde{C}_μ of μ in \tilde{C} such that $\pi(\tilde{C}_\mu) = U_\alpha$, as well as $\tilde{C}'_\mu = \tilde{C}_\mu \setminus \{\mu\}$. We have of course

$$\tilde{U}_\alpha = \bigsqcup_{\mu \in \tilde{\alpha}_\alpha} \tilde{C}_\mu.$$

By taking a smaller neighbourhood, we can always assume that the $(\tilde{C}_\mu)_{\mu \in \tilde{\alpha}}$ are pairwise disjoint. As anticipated in Section 4.2, if we want to insist that a point $z \in \tilde{U}_\alpha$

belongs to \tilde{C}_μ , we will denote it by $(\frac{\mu}{z})$. The fibers can be decomposed

$$f_\alpha(z) = \bigsqcup_{\mu \in \tilde{\alpha}_\alpha} f_\mu(z), \quad f_\mu(z) := f_\alpha(z) \cap \tilde{C}_\mu.$$

We denote $|f_\mu(z)|$, which is independent of $z \in \tilde{C}'_\mu$, by r_μ , and $|f_\alpha(z)|$, which is independent of $z \in \tilde{U}'_\alpha$, by r_α . In particular,

$$r_\alpha = \sum_{\mu \in \tilde{\alpha}_\alpha} r_\mu.$$

If γ is a small loop in V_α around $x(\alpha)$, it induces a Galois transformation in the cover $\tilde{x}|_{\tilde{U}'_\alpha}$, that is, a permutation σ_α of $f_\alpha(z)$ for each $z \in \tilde{U}'_\alpha$ that on $\tilde{f}_\mu(z)$ restricts to a cyclic transformation of order r_μ . This integer represents the order of ramification at $\mu \in \tilde{\alpha}_\alpha$ of $\tilde{x}|_{\tilde{C}_\mu}$.

Remark 5.2. If $|\tilde{\alpha}_\alpha| = 1$, C is irreducible locally at α , hence smooth at α . We can then use the same symbol to denote the point $\alpha \in C$ and the unique point above it in \tilde{C} . If $|\tilde{\alpha}_\alpha| > 1$, C is reducible locally at α , hence singular at α . If $|\tilde{\alpha}_\alpha| = 2$, α is a node. For $\mu \in \tilde{\alpha}$, we have $r_\mu = 1$ if and only if μ is not a ramification point of \tilde{x} . We say that the spectral curve is smooth if all ramification points in C are smooth.

For each $\alpha \in \alpha$ and $\mu \in \tilde{\alpha}_\alpha$, there exists a local coordinate ζ on \tilde{C}_μ such that

$$\tilde{x}(\frac{\mu}{z}) = x(\alpha) + \zeta(z)^{r_\mu}.$$

As in Section 4.2, when working with local coordinates, it should be clear from the context which \tilde{C}_μ is involved. Specifying such coordinates requires the choice of a r_μ -th root of unity for $\tilde{x} - x(\alpha)$. We assume such a choice is fixed. We also choose a primitive r_μ -th root of unity, denoted by ϑ_μ . If $z \in \tilde{C}'_\mu$, the set of coordinates of the points in $f_\alpha(z)$ is

$$\{\vartheta_\mu^j \zeta^{r_\mu/r_\nu} \mid \nu \in \tilde{\alpha}_\alpha, j \in [r_\nu]\}.$$

Let us write locally at $\mu \in \alpha$ the Laurent series expansion of the function \tilde{y}

$$\tilde{y}(\frac{\mu}{z}) \sim \sum_{k>0} \frac{1}{r_\mu} F_{0,1}[\frac{\mu}{-k}] \zeta^{k-r_\mu}$$

and define

$$s_\mu := \min \{k \in \mathbb{Z} \mid F_{0,1}[\frac{\mu}{-k}] \neq 0\} \in \mathbb{Z} \cup \{+\infty\}.$$

In particular, $s_\mu = +\infty$ if y vanishes identically in the connected component of μ in \tilde{C} . If s_μ is finite, we introduce

$$t_\mu := -\frac{1}{r_\mu} F_{0,1}[\frac{\mu}{-s_\mu}].$$

We equip $\tilde{\alpha}_\alpha$ with a total order \preceq satisfying

$$\mu \preceq \nu \Rightarrow \frac{r_\mu}{s_\mu} \geq \frac{r_\nu}{s_\nu}$$

and denote by $<$ the corresponding strict order (such orders exist). Note that the inequality still makes sense for μs such that $s_\mu = +\infty$. Then, in agreement with Section 3, for $\lambda, \mu \in \tilde{\alpha}_\alpha$ we let

$$\begin{aligned} [\mu] &:= \{\nu \in \tilde{\alpha}_\alpha \mid \nu \preceq \mu\}, \\ [\mu] &:= \{\nu \in \tilde{\alpha}_\alpha \mid \nu < \mu\}, \\ [\lambda, \mu] &:= \{\nu \in \alpha_\alpha \mid \lambda \preceq \nu \preceq \mu\}, \end{aligned}$$

and likewise for the open segments (λ, μ) , $(\lambda, \mu]$, etc. For instance, $[\mu] = [\min \tilde{\alpha}_\alpha, \mu]$. If $M \subseteq \tilde{\alpha}_\alpha$, we let

$$\mathbf{r}_M := \sum_{\mu \in M} r_\mu, \quad \mathbf{s}_M := \sum_{\mu \in M} s_\mu.$$

For $\mu \in \tilde{\alpha}_\alpha$, we define

$$\Delta_\mu := \mathbf{r}_{[\mu]s_\mu} - \mathbf{s}_{[\mu]}r_\mu.$$

Definition 5.3. For $\alpha \in \alpha$, we define a function of $z \in \tilde{U}_\alpha$ by

$$Y_\alpha(z) := \prod_{z' \in \tilde{f}'_\alpha(z)} (\tilde{y}(z') - \tilde{y}(z)).$$

In the next subsection, we will need to study the order of vanishing of these functions at $\tilde{\alpha}$. This is given by the following lemma.

Lemma 5.4. *If one of the following conditions is satisfied:*

- (i) *there exist distinct $\mu, \nu \in \tilde{\alpha}_\alpha$ such that $s_\mu = s_\nu = +\infty$; or*
- (ii) *there exists at least one $\mu \in \tilde{\alpha}_\alpha$ such that $s_\mu = +\infty$ and $r_\mu > 1$,*

then $Y_\alpha(z)$ vanishes identically on \tilde{C}_μ for the μ involved in these conditions. Otherwise, for any $\mu \in \tilde{\alpha}_\alpha$, we have $Y_\alpha(z) \in \mathcal{O}(\zeta^{v_\mu})$ when $z \in \tilde{C}_\mu$ approaches μ , where

$$v_\mu = (s_\mu - r_\mu)(r_\alpha - 1) - \Delta_\mu. \quad (5.1)$$

If furthermore either

- (iii) *there exist distinct $\mu, \nu \in \tilde{\alpha}_\alpha$ such that s_μ, s_ν are finite, $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ and $t_\mu^{r_\nu} = t_\nu^{r_\mu}$; or*
- (iv) *there exists $\mu \in \tilde{\alpha}_\alpha$ such that s_μ is finite and $\gcd(r_\mu, s_\mu) > 1$,*

then $Y_\alpha(z) \in \mathcal{O}(\zeta^{v_\mu+1})$. If none of the above conditions are satisfied, then there exists a non-zero scalar $t_{\alpha,\mu}$ such that $Y_\alpha(z) - t_{\alpha,\mu}\zeta^{v_\mu} \in \mathcal{O}(\zeta^{v_\mu+1})$.

Proof. If $s_\mu = +\infty$, \tilde{y} is identically zero for $z \in \tilde{C}_\mu$. Conditions (i) and (ii) both imply there is a $z' \in \tilde{f}'_\alpha(z)$ such that $\tilde{y}(z') \equiv 0$ as well, so one of the factors in $Y_\alpha(z)$ vanishes identically.

We now assume that (i) and (ii) are not satisfied. Let us add for the moment the assumption that all s_μ are finite. We compute

$$\prod_{z' \in \tilde{f}'_\mu(z)} (\tilde{y}(z') - \tilde{y}(z)) = \left(\prod_{j=1}^{r_\mu-1} (\vartheta_\mu^{s_\mu j} - 1) \right) (-t_\mu)^{r_\mu-1} \zeta^{(s_\mu-r_\mu)(r_\mu-1)} + \mathcal{O}(\zeta^{(s_\mu-r_\mu)(r_\mu-1)+1}), \quad (5.2)$$

and observe that the scalar prefactor in the first term is non-zero if and only if r_μ and s_μ are coprime – in that case it is equal to $r_\mu t_\mu^{r_\mu-1}$. For $\nu \in \tilde{\alpha}_\alpha$ distinct from μ , we have

$$\begin{aligned} \prod_{z' \in \tilde{f}'_\nu(z)} (\tilde{y}(z') - \tilde{y}(z)) &= \prod_{j=0}^{r_\nu-1} (-t_\nu \vartheta_\nu^j \zeta^{r_\mu(s_\nu-r_\nu)/r_\nu} + t_\mu \zeta^{s_\mu-r_\mu} + \dots) \\ &= t_{\mu,\nu} \zeta^{\min(r_\mu s_\nu, r_\nu s_\mu) - r_\mu r_\nu} + \mathcal{O}(\zeta^{\min(r_\mu s_\nu, r_\nu s_\mu) - r_\mu r_\nu + 1}), \end{aligned} \quad (5.3)$$

where \dots are higher-order terms, and

$$t_{\mu,\nu} = \begin{cases} -t_\nu^{r_\nu} & \text{if } r_\mu s_\nu < r_\nu s_\mu, \\ t_\mu^{r_\nu} & \text{if } r_\mu s_\nu > r_\nu s_\mu, \\ t_\mu^{r_\nu} - t_\nu^{r_\nu} & \text{if } r_\mu s_\nu = r_\nu s_\mu. \end{cases}$$

We have $t_{\mu,\nu} = 0$ if and only if (μ, ν) obeys condition (iii). Multiplying (5.2) with the product of (5.3) over all $\nu \neq \mu$, we deduce that $Y_\alpha(z) = t_{\alpha,\mu} \zeta^{v_\mu} + \mathcal{O}(\zeta^{v_\mu+1})$ and $t_{\alpha,\mu} = 0$ if and only if conditions (iii) and (iv) are satisfied, with the exponent

$$\begin{aligned} v_\mu &= (s_\mu - r_\mu)(r_\mu - 1) + \sum_{\nu \neq \mu} (\min(r_\mu s_\nu, r_\nu s_\mu) - r_\mu r_\nu) \\ &= r_\mu - s_\mu + \sum_{\nu \in \tilde{\alpha}_\alpha} (\min(r_\mu s_\nu, r_\nu s_\mu) - r_\mu r_\nu) \\ &= -r_\mu(r_\alpha - 1) - s_\mu + \sum_{\nu < \mu} r_\mu s_\nu + \sum_{\nu \geq \mu} r_\nu s_\mu \\ &= (s_\mu - r_\mu)(r_\alpha - 1) - \Delta_\mu \end{aligned} \quad (5.4)$$

as claimed. This concludes the proof in absence of an infinite s .

Now let us assume there exists a unique $\mu_- \in \tilde{\alpha}_\alpha$ such that $s_{\mu_-} = +\infty$. As we assume that (i) and (ii) are not satisfied, we must have $r_{\mu_-} = 1$. If $\mu \neq \mu_-$ and we take $z \in \tilde{C}_\mu$, we only need to pay attention to factor (5.3) for $\nu = \mu_-$, and in

fact equation (5.3) remains valid, hence $Y_\alpha(z) = t_{\alpha,\mu} \zeta^{v_\mu} + \mathcal{O}(\zeta^{v_\mu+1})$ with the same expression for v_μ and the same discussion for the (non-)vanishing of $t_{\alpha,\mu}$. Notice that by definition of the order, we must have $\mu_- = \max(\tilde{\alpha}_\alpha)$, so μ_- does not appear in Δ_μ . If $z \in \tilde{C}_{\mu_-}$, factor (5.2) is absent in $Y_\alpha(z)$ and the other factors for $v \neq \mu_-$ are as in (5.3). But, as μ_- only appears in v_μ via the first term of the first line of (5.4), which can be consistently set to 0 since $r_{\mu_-} = 1$, the formula for v_μ remains valid. ■

Remark 5.5. Notice that if (iv) does not hold, i.e., any finite s_μ is coprime to r_μ , the condition $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ is equivalent to $(r_\mu, s_\mu) = (r_\nu, s_\nu)$. In that case, condition (iii) can be replaced by a more symmetric one:

- (iii') there exist distinct $\mu, \nu \in \tilde{\alpha}_\alpha$ such that s_μ, s_ν are finite, $(r_\mu, s_\mu) = (r_\nu, s_\nu)$ and $t_\mu^{r_\mu} = t_\nu^{r_\nu}$.

5.2. Correlators, master loop equations and topological recursion

Let $\mathcal{C} = (C, x, y)$ be a spectral curve with normalisation $\pi: \tilde{C} \rightarrow C$.

Definition 5.6. A *fundamental bidifferential of the second kind* on \mathcal{C} is an element

$$B \in H^0(\tilde{C} \times \tilde{C}; K_{\tilde{C}}^{\boxtimes 2}(2\Delta))^{\mathfrak{S}_2}$$

with biresidue 1 on the diagonal $\Delta \subset \tilde{C} \times \tilde{C}$, where $K_{\tilde{C}}$ is the sheaf of differentials on \tilde{C} .

A *crosscap differential* on \mathcal{C} is the data of a (possibly empty) divisor D on $\tilde{C} \setminus \tilde{\alpha}$ and

$$q \in H^0(\tilde{C}; K_{\tilde{C}}(D + \tilde{\alpha}))$$

such that

$$\forall \alpha \in \alpha, \quad \sum_{\mu \in \tilde{\alpha}_\alpha} \operatorname{Res}_{z=\mu} q(z) = 0.$$

Definition 5.7. A *family of correlators* is a family of multidifferentials

$$\omega = (\omega_{g,n})_{g \in \frac{1}{2}\mathbb{N}, n > 0}$$

on \tilde{C} such that $\omega_{0,1} = \tilde{y} d\tilde{x}$, $\omega_{0,2}$ is a fundamental bidifferential of the second kind on \mathcal{C} , $\omega_{\frac{1}{2},1}$ is a crosscap differential, and for $2g - 2 + n > 0$,

$$\omega_{g,n} \in H^0(\tilde{C}^n; (K_{\tilde{C}}(*\tilde{\alpha}))^{\boxtimes n})^{\mathfrak{S}_n}.$$

It satisfies the *projection property* if for $2g - 2 + n > 0$,

$$\omega_{g,n}(z_1, z_{[2,n]}) = \sum_{\mu \in \tilde{\alpha}} \operatorname{Res}_{z=\mu} \left(\int_{\mu}^z \omega_{0,2}(\cdot, z_1) \right) \omega_{g,n}(z, z_{[2,n]}). \quad (5.5)$$

Note that (5.5) is automatically satisfied for $(g, n) = (0, 2)$. Differentials satisfying the projection property cannot have residues, and if they are holomorphic, they must vanish. We can always assume by taking smaller neighbourhoods that the divisor D of the crosscap differential is supported outside $\bigsqcup_{\alpha \in \mathfrak{a}} \tilde{U}_\alpha$.

Definition 5.8. Let ω be a family of correlators, and $i \geq 1$, $g \in \frac{1}{2}\mathbb{N}$ and $n \geq 0$. The genus g , i -disconnected, n -connected correlator is defined by

$$\mathcal{W}_{g,i,n}(z_{[i]}; w_{[n]}) := \sum_{\substack{\mathbf{L} \vdash [i] \\ \bigsqcup_{L \in \mathbf{L}} N_L = [n] \\ i + \sum_L (g_L - 1) = g}} \prod_{L \in \mathbf{L}} \omega_{g_L, |L| + |N_L|}(z_L, w_{N_L}).$$

We define $\mathcal{W}'_{g,i,n}$ by the same formula, but omitting any summand containing some $\omega_{0,1}$.

If $i \in [r_\alpha]$, we let $\varkappa_i: \tilde{U}_\alpha^{(i)} \rightarrow V_\alpha$ be the smooth curve obtained by taking the fibered product of i copies of $\tilde{x}: \tilde{U}'_\alpha \rightarrow \mathbb{C}$, deleting the big diagonal $\Delta(i)$, and quotienting by the (free) action of \mathfrak{S}_i . Points in $\tilde{U}_\alpha^{(i)}$ are exactly subsets of cardinality i of $\mathfrak{f}_\alpha(z)$ for some $z \in \tilde{U}'_\alpha$. We have natural holomorphic maps

$$(\tilde{U}'_\alpha)^i \setminus \Delta(i) \xrightarrow{q_i} \tilde{U}_\alpha^{(i)} \xrightarrow{\varkappa_i} V'_\alpha,$$

where q_i forgets the order of elements of an i -tuple and

$$\varkappa_i(\{z_1, \dots, z_n\}) = \tilde{x}(z_1) = \dots = \tilde{x}(z_i).$$

Let $\iota_i: \tilde{U}_\alpha^i \setminus \Delta^{(i)} \rightarrow \tilde{C}^i$ be the natural inclusion. We introduce

$$\begin{aligned} \mathcal{E}_{\alpha;g,n}^{(i)} &:= \frac{(x_i q_i)^*}{i!} \iota_i^* (\mathcal{W}_{g,i,n}) \in H^0(V'_\alpha \times \tilde{C}^n; K_{V'_\alpha}^{\otimes i} \boxtimes K_{\tilde{C}}(*\tilde{\mathfrak{a}})^{\boxtimes n})^{\mathfrak{S}_n}, \\ \tilde{\mathcal{E}}_{\alpha;g,n}^{(i)} &:= \tilde{x}^* \mathcal{E}_{\alpha;g,n}^{(i)}, \end{aligned}$$

where all operations do not concern the last n variables. More concretely,

$$\begin{aligned} \mathcal{E}_{\alpha;g,n}^{(i)}(x_0; z_{[n]}) &= \sum_{\substack{Z \subseteq \tilde{x}^{-1}(x_0) \cap \tilde{U}_\alpha \\ |Z|=i}} \mathcal{W}_{g,i,n}(Z; z_{[n]}), \\ \tilde{\mathcal{E}}_{\alpha;g,n}^{(i)}(z_0; z_{[n]}) &= \sum_{\substack{Z \subseteq \mathfrak{f}_\alpha(z_0) \\ |Z|=i}} \mathcal{W}_{g,i,n}(Z; z_{[n]}). \end{aligned} \tag{5.6}$$

The symmetry factor $i!$ disappeared since $\mathcal{W}_{g,i,n}$ is symmetric in its i first variables. Note that reading (5.6) in the local coordinate ζ_0 of $z_0 \in \tilde{U}'_\mu$ each term may be multivalued (i.e., fractional powers of $\zeta(z_0)$ could appear), however the sum is single-valued as it is the pullback along \tilde{x} of a 1-form on V'_α .

Definition 5.9. We say that a family of correlators satisfies the *master loop equations* if for any $g \in \frac{1}{2}\mathbb{N}$ and $n \geq 0$ such that $2g - 2 + (n + 1) > 0$, for any $\alpha \in \alpha$ and $i \in [r_\alpha]$, any $\mu \in \tilde{\alpha}'_\alpha$, when $z_0 \in \tilde{C}'_\mu$ approaches μ , we have

$$\sum_{i=1}^{r_\alpha} \tilde{\mathcal{E}}_{\alpha;g,n}^{(i)}(z_0; z_{[n]})(-\omega_{0,1}(z_0))^{r_\alpha-i} = \mathcal{O}(\zeta^{-1+v_\mu+(r_\mu-1)(r_\alpha-1)}(d\zeta)^{r_\alpha}).$$

The relevance of this notion comes from the fact that the master loop equation can be solved by the topological recursion.

Proposition 5.10. *Assume that none of conditions (i), (ii), (iii), and (iv) appearing in Lemma 5.4 are satisfied. Then, if ω is a family of correlators satisfying the master loop equation (Definition 5.9) and the projection property (Definition 5.7), we must have for any $g \in \frac{1}{2}\mathbb{N}$ and $n \geq 0$ such that $2g - 2 + (n + 1) > 0$,*

$$\begin{aligned} & \omega_{g,n+1}(z_0, z_{[n]}) \\ &= \sum_{\alpha \in \alpha} \sum_{\mu \in \tilde{\alpha}'_\alpha} \text{Res}_{z=\mu} \left(\sum_{Z \subseteq \tilde{\alpha}'_\alpha(z)} K_\mu^{(1+|Z|)}(z_0; z, Z) \mathcal{W}'_{g,1+|Z|,n}(z, Z; z_{[n]}) \right), \end{aligned} \quad (5.7)$$

where for $i \geq 2$ we have introduced the i -th recursion kernel for $|Z| = i - 1$

$$K_\mu^{(i)}(z_0; z, Z) := - \frac{\int_\mu^z \omega_{0,2}(\cdot, z_0)}{\prod_{z' \in Z} ((\tilde{y}(z') - \tilde{y}(z)) d\tilde{x}(z))}. \quad (5.8)$$

Proof. The proof is similar to [45, Theorem 7.6.5], the only difference being the order of the pole in the master loop equation. For completeness, we include the argument here. By definition,

$$\mathcal{W}'_{g,1,n} = \omega_{g,n+1}.$$

By the projection property and Definition 5.3,

$$\begin{aligned} \omega_{g,n+1}(z_0, z_{[n]}) &= \sum_{\alpha \in \alpha} \sum_{\mu \in \tilde{\alpha}'_\alpha} \text{Res}_{z=\mu} \left(\int_\mu^z \omega_{0,2}(\cdot, z_0) \right) \mathcal{W}'_{g,1,n}(z; z_{[n]}) \\ &= - \sum_{\alpha \in \alpha} \sum_{\mu \in \tilde{\alpha}'_\alpha} \text{Res}_{z=\mu} K_\mu^{(r_\alpha)}(z_0; \tilde{\alpha}'_\alpha(z)) \mathcal{W}'_{g,1,n}(z_0; z_{[n]}) \\ &\quad \times Y_\alpha(z) \cdot (d\tilde{x}(z))^{(r_\alpha-1)}, \end{aligned} \quad (5.9)$$

where we noticed that

$$K_\mu^{(r_\alpha)}(z_0; \tilde{\alpha}'_\alpha(z)) = - \frac{\int_\mu^z \omega_{0,2}(\cdot, z_0)}{Y_\alpha(z)(d\tilde{x}(z))^{(r_\alpha-1)}}$$

always considering $z \in \mathfrak{f}_\alpha(z)$ as the first element of the set. Let $\alpha \in \mathfrak{a}$, and use the combinatorial identity [45, Lemma 7.6.4], which states that

$$\begin{aligned} & \sum_{\{z\} \subseteq Z \subseteq \mathfrak{f}_\alpha(z)} \mathcal{W}'_{g,|Z|,n}(Z; z[n]) \prod_{z' \in \mathfrak{f}_\alpha(z) \setminus Z} ((\tilde{y}(z') - \tilde{y}(z)) d\tilde{x}(z)) \\ &= \sum_{i=1}^{r_\alpha} \tilde{\mathcal{E}}_{\alpha;g,n}^{(i)}(z; z[n]) (-\omega_{0,1}(z))^{r_\alpha - i}. \end{aligned}$$

Isolating the term $Z = \{z\}$ and substituting in (5.9), we obtain

$$\begin{aligned} & \omega_{g,n+1}(z_0, z[n]) \\ &= \sum_{\alpha \in \mathfrak{a}} \sum_{\mu \in \tilde{\mathfrak{a}}_\alpha} \operatorname{Res}_{z=\mu} K_\mu^{(r_\alpha)}(z_0; \mathfrak{f}_\alpha(z)) \left(- \sum_{i=1}^{r_\alpha} \tilde{\mathcal{E}}_{\alpha;g,n}^{(i)}(z; z[n]) (-\omega_{0,1}(z))^{r_\alpha - i} \right. \\ & \quad \left. + \sum_{\{z\} \subset Z \subseteq \mathfrak{f}_\alpha(z)} \mathcal{W}'_{g,|Z|,n}(Z; z[n]) \prod_{z' \in \mathfrak{f}_\alpha(z) \setminus Z} ((\tilde{y}(z') - \tilde{y}(z)) d\tilde{x}(z)) \right) \\ &= \sum_{\alpha \in \mathfrak{a}} \sum_{\mu \in \tilde{\mathfrak{a}}_\alpha} \operatorname{Res}_{z=\mu} \left(\sum_{\{z\} \subset Z \subseteq \mathfrak{f}_\alpha(z)} K_\mu^{(|Z|)}(z_0; Z) \mathcal{W}'_{g,|Z|,n}(Z; z[n]) \right). \quad (5.10) \end{aligned}$$

By Lemma 5.4 and the assumption, we know that for $z \in \tilde{\mathcal{C}}'_\mu$ approaching μ

$$Y_\alpha(z) (d\tilde{x}(z))^{r_\alpha - 1} \sim t_{\alpha,\mu} \zeta^{\nu_\mu + (r_\mu - 1)(r_\alpha - 1)} (d\zeta)^{r_\alpha - 1}$$

for some non-zero scalar $t_{\alpha,\mu}$. Since the numerator of the recursion kernel vanishes at order 1 at $z = \mu$, the master loop equation implies that the first term inside the bracket of (5.10) is $\mathcal{O}(d\zeta)$ hence does not contribute to the residue. Besides, the contribution of the second sum can be simplified by observing that

$$K_\mu^{(|Z|)}(z_0; Z) = K_\mu^{(r_\alpha)}(z_0; \mathfrak{f}_\alpha(z)) \prod_{z' \in \mathfrak{f}_\alpha(z) \setminus Z} ((\tilde{y}(z') - \tilde{y}(z)) d\tilde{x}(z)).$$

Redefining Z by removing z from it, we obtain the desired formulas. \blacksquare

Remark 5.11. From the proof, we see that if one of conditions (i) and (ii) appearing in Lemma 5.4 is satisfied, the recursion kernel is ill-defined as the denominator vanishes identically in the neighbourhood of some μ . Besides, if one of conditions (iii) or (iv) is satisfied, the same thing could occur or at least the order of vanishing of the denominator is finite but higher than the one specified by Definition 5.9. In the latter case, one can still ask for the analogue of Proposition 5.10 simply by modifying the master loop equation to require that the first sum in (5.10) is $\mathcal{O}(d\zeta)$.

We note that the right-hand side of (5.7) involves only $\omega_{g',n'}$ with $2g' - 2 + n' < 2g - 2 + (n + 1)$. For a fixed $\omega^{\text{un}} = (\omega_{0,1}, \omega_{\frac{1}{2},1}, \omega_{0,2})$, there exists at most one way to complete it into a system of correlators satisfying the master loop equation and the projection property: the $\omega_{g,n}$ are then determined by (5.7) inductively on $2g - 2 + n > 0$. However, such a system of correlators may actually fail to exist at all. Indeed, (5.7) gives a non-symmetric role to z_0 compared to z_1, \dots, z_n , therefore the $\omega_{g,n+1}(z_0, \dots, z_n)$ that (5.7) compute may fail to be symmetric, and so would not respect Definition 5.7.

5.3. Abstract loop equations

We now address the aforementioned problem of existence of the solution to the master loop equations, thanks to the results obtained in Section 4. We first introduce a seemingly different notion of “abstract loop equations” valid in the setting of Section 5.1. It will turn out that they give the right generalisation of “abstract loop equations” proposed in [13] for smooth spectral curves. We will show that, under admissibility conditions on the spectral curves that pertain to our constructions of Airy structures in Section 2, the abstract loop equations have a solution satisfying the projection properties, and imply the master loop equation. Therefore, this solution must be given by the topological recursion formula (5.7), and this proves a posteriori that this definition is well-posed, i.e., it produces inductively only multidifferentials that are symmetric under permutations of all their variables. A direct proof of symmetry by residue computations on \tilde{C} seems rather elusive.

Let \mathcal{C} be a spectral curve as in Section 5.1. We introduce integers $\delta_\alpha(i)$ for each $\alpha \in \alpha$ and $i \in [r_\alpha]$, matching Lemma 3.6. If $i \in [r_\alpha]$, we first decompose it into $i = \mathbf{r}[\lambda] + i'$ for the unique $\lambda \in \tilde{\alpha}_\alpha$ such that $\mathbf{r}[\lambda] < i \leq \mathbf{r}[\lambda]$. Then, $i' \in [r_\lambda]$ and we have

$$\delta_\alpha(i) := - \left\lfloor \frac{s_\lambda(i' - 1)}{r_\lambda} \right\rfloor - \mathbf{s}[\lambda] + \delta_{i',1}. \tag{5.11}$$

Definition 5.12. We say that a family of correlators satisfies the *abstract loop equations* if for any $g \in \frac{1}{2}\mathbb{N}$ and $n \geq 0$ such that $2g - 2 + (n + 1) > 0$, for any $\alpha \in \alpha$ and $i \in [r_\alpha]$ when $x_0 \rightarrow x(\alpha)$, we have

$$\mathcal{E}_{\alpha;g,n}^{(i)}(x_0; z_{[n]}) = \mathcal{O} \left(x_0^{-(\delta_\alpha(i) - 1 - \delta_{i,1})} \left(\frac{dx_0}{x_0} \right)^i \right). \tag{5.12}$$

This condition is equivalent to the property that, for any $\mu \in \tilde{\alpha}_\alpha$, we have when $z_0 \in \tilde{C}'_\mu$ approaches μ

$$\tilde{\mathcal{E}}_{\alpha;g,n}^{(i)}(z_0; z_{[n]}) = \mathcal{O} \left(\zeta_0^{-r_\mu(\delta_\alpha(i) - 1 - \delta_{i,1})} \left(\frac{d\zeta_0}{\zeta_0} \right)^i \right).$$

Proposition 5.13. *Assume that none of conditions (i), (ii), (iii), (iv) appearing in Lemma 5.4 are satisfied. Then, the abstract loop equations imply the master loop equations.*

Proof. We treat the case where s_μ is finite for all $\mu \in \alpha$. The case where there could exist $\mu_{\alpha,-} \in \tilde{\alpha}_\alpha$ (which is then unique) such that $s_{\mu_{\alpha,-}} = +\infty$ is left as exercise to the reader.

For each $\alpha \in \alpha$ and $i \in [r_\alpha]$, the abstract loop equations imply that for any $\mu \in \tilde{\alpha}_\alpha$, we have when $z_0 \in \tilde{U}'_\mu$ approaches μ

$$\tilde{\mathcal{E}}_{\alpha;g,n}^{(i)}(z_0; z_{[n]})(-\omega_{0,1}(z))r_\alpha^{-i} = O(\zeta^{-r_\mu(\mathfrak{d}_\alpha(i)-1-\delta_{i,1})-i+(s_\mu-1)(r_\alpha-i)}(d\zeta)r_\alpha).$$

Comparing with Definition 5.9, the result will be proved after we justify that

$$\begin{aligned} \mathfrak{p}_\mu(i) &:= -r_\mu(\mathfrak{d}_\alpha(i) - 1 - \delta_{i,1}) - i + (s_\mu - 1)(r_\alpha - i) \\ &\quad - (-1 + \mathfrak{v}_\mu + (r_\mu - 1)(r_\alpha - 1)) \end{aligned}$$

is always nonnegative. We recall the definition of \mathfrak{v}_μ in (5.1):

$$\mathfrak{v}_\mu = (s_\mu - r_\mu)(r_\alpha - 1) - \mathbf{r}_{[\mu]}s_\mu + \mathbf{s}_{[\mu]}r_\mu.$$

We decompose $i = \mathbf{r}_{[\lambda]} + i'$ with the unique $\lambda \in \tilde{\alpha}_\alpha$ such that $\mathbf{r}_{[\lambda]} < i \leq \mathbf{r}_{[\lambda]}$ and $i' \in [\mathbf{r}_\lambda]$, and we denote $\lambda_\alpha := \min \tilde{\alpha}_\alpha$. Inserting the definition of $\mathfrak{d}_\alpha(i)$ from (5.11), we obtain

$$\mathfrak{p}_\mu(i) = s_\mu(1 - i + \mathbf{r}_{[\mu]}) + r_\mu \left(1 + \left\lfloor \frac{s_\lambda(i' - 1)}{r_\lambda} \right\rfloor + \mathbf{s}_{[\lambda]} - \mathbf{s}_{[\mu]} - \delta_{\lambda > \lambda_\alpha} \delta_{i',1} \right).$$

We are going to use often the inequality

$$\lfloor x \rfloor > x - 1. \quad (5.13)$$

Checking nonnegativity of $\mathfrak{p}_\mu(i)$ is done by a case discussion.

- If $\mu = \lambda$, this becomes

$$\mathfrak{p}_\mu(i) = s_\mu(1 - i') + r_\mu \left(1 + \left\lfloor \frac{s_\mu(i' - 1)}{r_\mu} \right\rfloor - \delta_{\mu > \lambda_\alpha} \delta_{i',1} \right).$$

For $i' = 1$ and $\mu > \lambda_\alpha$, we get $\mathfrak{p}_\mu(i) = 0$. For $i' = 1$ and $\mu = \lambda_\alpha$, we get $\mathfrak{p}_\mu(i) = r_{\lambda_\alpha} > 0$. For $i' \geq 2$, using (5.13) yields directly $\mathfrak{p}_\mu(i) > 0$.

- If $\mu < \lambda$, we have

$$\mathfrak{p}_\mu(i) = s_\mu(1 - \mathbf{r}_{[\mu,\lambda]} - i') + r_\mu \left(1 + \left\lfloor \frac{s_\lambda(i' - 1)}{r_\lambda} \right\rfloor + \mathbf{s}_{[\mu,\lambda]} - \delta_{i',1} \right).$$

For $i' = 1$, this simplifies into

$$\mathfrak{p}_\mu(i) = -s_\mu \mathbf{r}_{[\mu, \lambda]} + r_\mu \mathbf{s}_{[\mu, \lambda]}.$$

By definition of the order, for all $\nu \in [\mu, \lambda)$, we have $\frac{r_\nu}{s_\nu} \geq \frac{r_\mu}{s_\mu}$, therefore $\mathfrak{p}_\mu(i) \geq 0$.

For $i' \geq 2$, we can use $\frac{s_\lambda}{r_\lambda} \geq \frac{s_\mu}{r_\mu}$ and (5.13) and obtain $\mathfrak{p}_\mu(i) > 0$.

- If $\mu \succ \lambda$, we rather have

$$\mathfrak{p}_\mu(i) = s_\mu(1 + \mathbf{r}_{[\lambda, \mu]} - i') + r_\mu \left(1 + \left\lfloor \frac{s_\lambda(i' - 1)}{r_\lambda} \right\rfloor - \mathbf{s}_{[\lambda, \mu]} - \delta_{\lambda \succ \lambda_\alpha} \delta_{i', 1} \right).$$

For $i' = 1$ and $\lambda = \lambda_\alpha$, this simplifies to

$$\mathfrak{p}_\mu(i) = s_\mu \mathbf{r}_{[\mu]} - r_\mu \mathbf{s}_{[\mu]} + r_\mu,$$

and thanks to the inequality $\frac{r_\nu}{s_\nu} \geq \frac{r_\mu}{s_\mu}$ for all $\nu \in [\mu)$, we deduce $\mathfrak{p}_\mu(i) \geq r_\mu > 0$.

For $i' = 1$ and $\lambda \succ \lambda_\alpha$, we have

$$\mathfrak{p}_\mu(i) = s_\mu \mathbf{r}_{[\lambda, \mu]} - r_\mu \mathbf{s}_{[\lambda, \mu]}.$$

Due to the inequality $\frac{r_\nu}{s_\nu} \geq \frac{r_\mu}{s_\mu}$ for all $\nu \in [\lambda, \mu)$, we have again $\mathfrak{p}_\mu(i) \geq 0$. For $i' \geq 2$, we use inequality (5.13) to write

$$\begin{aligned} \mathfrak{p}_\mu(i) &> s_\mu(1 + \mathbf{r}_{[\lambda, \mu]} - i') + r_\mu \left(\frac{s_\lambda(i' - 1)}{r_\lambda} - \mathbf{s}_{[\lambda, \mu]} \right) \\ &> r_\mu(i' - 1) \left(\frac{s_\lambda}{r_\lambda} - \frac{s_\mu}{r_\mu} \right) + \mathbf{r}_{[\lambda, \mu]} s_\mu - \mathbf{s}_{[\lambda, \mu]} r_\mu, \end{aligned}$$

and due to the ordering we find again $\mathfrak{p}_\mu(i) > 0$. ■

Remark 5.14. In the proof we see that for any $\alpha \in \alpha$, if $|\tilde{\alpha}_\alpha| > 1$, there exists $\mu \in \tilde{\alpha}_\alpha$ and $i \in [r_\alpha]$ such that $\mathfrak{p}_\mu(i) = 0$. Therefore, we do use all the vanishing provided by the abstract loop equations to derive the master loop equations.

Combining with Proposition 5.10, we obtain the following result.

Proposition 5.15. *Assume that none of conditions (i), (ii), (iii), (iv) appearing in Lemma 5.4 are satisfied. For a fixed $(\omega_{0,1}, \omega_{0,2}, \omega_{1, \frac{1}{2}})$, the topological recursion (5.7) gives the unique – if it exists, i.e., if the result is symmetric in all variables – solution to the abstract loop equations.*

The notion of abstract loop equation was first introduced [10, 13] for smooth curves with simple ramifications and was shown therein to be a mechanism implying directly the topological recursion. This was extended to higher-order ramifications on smooth curves having \tilde{y} holomorphic near α in [9, 15, 45], and to the more general case where $y dx$ is holomorphic near α in [9]. The novelty of Propositions 5.10 and 5.13 here is the treatment of possibly singular curves.

5.4. Topological recursion for admissible spectral curves

In this subsection, we express the abstract loop equations in a more algebraic way, which will make the bridge to Airy structures. The converse route was anticipated in Section 4.

Let \mathcal{C} be a spectral curve. We can attach to it a local spectral curve matching the definitions in Section 4.2. Namely, we let

$$\tilde{\mathcal{C}}^{\text{loc}} = \bigsqcup_{\mu \in \tilde{\alpha}} \tilde{\mathcal{C}}_{\mu}^{\text{loc}}, \quad \tilde{\mathcal{C}}_{\mu}^{\text{loc}} := \text{Spec } \mathbb{C}[[\zeta]].$$

We recall for each $\alpha \in \alpha$ and ramification point $\mu \in \tilde{\alpha}_{\alpha}$ above α , we have a local coordinate ζ such that

$$\tilde{x}\left(\frac{\mu}{z}\right) = x(\alpha) + \zeta(z)^{r_{\mu}}.$$

For each $\mu \in \tilde{\alpha}$, define $\tilde{\mathcal{C}}_{\mu}^{\text{loc}'}$:= $\text{Spec } \mathbb{C}((\zeta))$ and let

$$\mathcal{L}_{\mu} := H^0(\tilde{\mathcal{C}}_{\mu}^{\text{loc}'}; K_{\tilde{\mathcal{C}}_{\mu}^{\text{loc}'}}) \cong \mathbb{C}((\zeta))d\zeta$$

be a copy of the space of formal Laurent series, and

$$\mathcal{L} = H^0(\tilde{\mathcal{C}}^{\text{loc}'}; K_{\tilde{\mathcal{C}}^{\text{loc}'}}) \cong \bigoplus_{\mu \in \tilde{\alpha}} \mathcal{L}_{\mu}.$$

We denote by $\text{Loc}_{\mu}: H^0(\tilde{U}; K_{\tilde{U}}(*\tilde{\alpha})) \rightarrow \mathcal{L}_{\mu}$ the linear map associating to a meromorphic differential its all-order Laurent series expansion near μ using the local coordinate ζ in \tilde{U}_{μ} , and $\text{Loc} = \bigoplus_{\mu \in \tilde{\alpha}} \text{Loc}_{\mu}$. We define elements $d\xi_k^{\mu} \in \mathcal{L}$, indexed by $\mu \in \tilde{\alpha}$ and $k \geq 0$

$$d\xi_k^{\mu}\left(\frac{\nu}{\zeta}\right) = \delta_{\mu,\nu} \zeta^{k-1} d\zeta.$$

We introduce the standard bidifferential of the second kind on \tilde{U} , that is

$$\omega_{0,2}^{\text{std}}\left(\begin{matrix} \mu_1 & \mu_2 \\ z_1 & z_2 \end{matrix}\right) := \frac{\delta_{\mu_1, \mu_2} d\zeta(z_1) d\zeta(z_2)}{(\zeta(z_1) - \zeta(z_2))^2}.$$

Let now ω be a family of correlators on \mathcal{C} . We can encode the correlators $\omega_{g,n}$ with $2g - 2 + n \geq 0$ by the following Laurent series expansion:

$$\begin{aligned} \text{Loc}(\omega_{0,1}) &= \sum_{\substack{\mu \in \tilde{\alpha} \\ k > 0}} F_{0,1}\left[\begin{matrix} \mu \\ -k \end{matrix}\right] d\xi_k^{\mu}, \\ \text{Loc}(\omega_{\frac{1}{2},1}) &= \sum_{\mu \in \tilde{\alpha}} \left(Q_{\mu} d\xi_0^{\mu} + \sum_{k > 0} F_{\frac{1}{2},1}\left[\begin{matrix} \mu \\ -k \end{matrix}\right] d\xi_k^{\mu} \right), \\ \text{Loc}^{\otimes 2}(\omega_{0,2} - \omega_{0,2}^{\text{std}}) &= \sum_{\substack{\mu_1, \mu_2 \in \tilde{\alpha} \\ k_1, k_2 > 0}} F_{0,2}\left[\begin{matrix} \mu_1 & \mu_2 \\ -k_1 & -k_2 \end{matrix}\right] d\xi_{k_1}^{\mu_1} d\xi_{k_2}^{\mu_2}. \end{aligned} \tag{5.14}$$

Using the fundamental bidifferential of the second kind, we introduce another family of differentials $d\xi_{-k}^\mu$, now globally defined on \tilde{C} and indexed by $\mu \in \tilde{\alpha}$ and $k > 0$

$$d\xi_{-k}^\mu(z) = \text{Res}_{z'=\mu} \left(\int_\mu^{z'} \omega_{0,2}(\cdot, z) \right) \frac{d\xi(z')}{(\xi(z'))^{k+1}}. \quad (5.15)$$

Notice that it is such that for any $\mu, \nu \in \tilde{\alpha}$,

$$\text{Loc}_\nu(d\xi_{-k}^\mu) = \frac{\delta_{\mu,\nu} d\xi}{\xi^{k+1}} + \sum_{l>0} \frac{F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right]}{k} d\xi_l^\nu.$$

Assuming that ω satisfies the projection property, by symmetry we can apply this property to each variable to obtain the existence of a finite decomposition for $2g - 2 + n > 0$

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{\substack{\mu_1, \dots, \mu_n \in \tilde{\alpha} \\ k_1, \dots, k_n > 0}} F_{g,n} \left[\begin{smallmatrix} \mu_1 & \dots & \mu_n \\ k_1 & \dots & k_n \end{smallmatrix} \right] \prod_{i=1}^n d\xi_{-k_i}^{\mu_i}(z_i), \quad (5.16)$$

where $F_{g,n} \left[\begin{smallmatrix} \mu \\ k \end{smallmatrix} \right]$ are scalars.

Definition 5.16. The *partition function* Z associated to ω and satisfying the projection property is defined as

$$Z := e^F, \quad F = \sum_{\substack{g \in \frac{1}{2}\mathbb{N}, n \geq 1 \\ 2g - 2 + n > 0}} \sum_{\substack{\mu_1, \dots, \mu_n \in \tilde{\alpha} \\ k_1, \dots, k_n > 0}} \frac{\hbar^{g-1}}{n!} F_{g,n} \left[\begin{smallmatrix} \mu_1 & \dots & \mu_n \\ k_1 & \dots & k_n \end{smallmatrix} \right] \prod_{i=1}^n x_{k_i}^{\mu_i}.$$

We now would like to translate the abstract loop equations on ω into constraints for its partition function. For this purpose, we introduce for each $\alpha \in \alpha$ a copy $W_{\alpha; i, k}$ of the differential operators in equation (2.14) indexed by $i \in [r_\alpha]$ and $k \in \mathbb{Z}$ forming a representation of the $\mathcal{W}(\mathfrak{gl}_{r_\alpha})$ -VOA using as twists permutations σ_α which is a product of disjoint cycles of respective orders $(r_\mu)_{\mu \in \tilde{\alpha}_\alpha}$. They are described in terms of the Heisenberg generators indexed by $\mu \in \tilde{\alpha}$ and $k \in \mathbb{Z}$

$$J_k^\mu = \begin{cases} \hbar \partial_{x_k^\mu} & \text{if } k > 0, \\ \hbar^{\frac{1}{2}} Q_\mu & \text{if } k = 0, \\ -k x_{-k}^\mu & \text{if } k < 0, \end{cases}$$

where we use

$$Q_\mu = \text{Res}_{z=\mu} \omega_{\frac{1}{2}, 1}(z)$$

coming from the crosscap differential. Then, we construct the dilaton shift and the change of polarisation

$$\begin{aligned}\widehat{T} &= \exp\left(\sum_{\substack{\mu \in \tilde{\alpha} \\ k > 0}} (\hbar^{-1} F_{0,1}[\begin{smallmatrix} \mu \\ -k \end{smallmatrix}]) + \hbar^{-\frac{1}{2}} F_{\frac{1}{2},1}[\begin{smallmatrix} \mu \\ -k \end{smallmatrix}]) \frac{J_k^\mu}{k}\right), \\ \widehat{\Phi} &= \exp\left(\frac{1}{2\hbar} \sum_{\substack{\mu, \nu \in \tilde{\alpha} \\ k, l > 0}} F_{0,2}[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix}] \frac{J_k^\mu J_l^\nu}{kl}\right).\end{aligned}$$

Definition 5.17. To a spectral curve \mathcal{C} equipped with a crosscap differential $\omega_{\frac{1}{2},1}$ and a fundamental bidifferential of the second kind $\omega_{0,2}$, we associate the system of differential operators indexed by $\alpha \in \alpha$, $i \in [r_\alpha]$ and $k \in \mathbb{Z}$

$$H_{\alpha;i,k} := \widehat{\Phi} \widehat{T} \cdot \sigma_\alpha W_{i,k} \cdot \widehat{T}^{-1} \widehat{\Phi}^{-1},$$

where $W_{i,k}^\sigma$ is as in equation (2.14), and σ_α is the monodromy permutation at α . We also introduce the set

$$\mathcal{I} := \{(\alpha, i, k) \mid \alpha \in \alpha, i \in [r_\alpha], k \geq \mathfrak{d}_\alpha(i) - \delta_{i,1}\}.$$

Proposition 5.18. *Assume that none of conditions (i), (ii), (iii), (iv) appearing in Lemma 5.4 are satisfied, and let ω be a system of correlators satisfying the projection property. Then, the abstract loop equations for ω are equivalent to the following system of differential equations for its partition function:*

$$\forall (\alpha, i, k) \in \mathcal{I}, \quad e^{-F} H_{\alpha;i,k} e^F \cdot 1 = 0. \quad (5.17)$$

Proof. If $|\alpha| = 1$, this is the computation done in Section 4.4. Given the formalism that we introduced, it is straightforward to adapt it to handle several α s, where $H_{\alpha;i,k}$ now form a representation of the direct sum over $\alpha \in \alpha$ of the $\mathcal{W}(\mathfrak{gl}_{r_\alpha})$ -VOAs. ■

It is now easy to combine the construction of Airy structures in Theorem 2.11 with Propositions 5.10 and 5.13 to obtain our second main result. We recall that we have defined

$$t_\mu = -\frac{1}{r_\mu} F_{0,1}[\begin{smallmatrix} \mu \\ -s_\mu \end{smallmatrix}].$$

Definition 5.19. We say $\alpha \in \alpha$ is *regularly admissible* if

- C is irreducible locally at α , that is, $|\tilde{\alpha}_\alpha| = 1$.
- \tilde{y} is holomorphic near α and $d\tilde{y}(\alpha) \neq 0$.

In that case, in all the previous definitions and constructions in the neighbourhood \tilde{U}_α , we replace $\tilde{y}(z)$ by $\tilde{y}(z) - \tilde{y}(\alpha)$. In particular, we take $s_\alpha = r_\alpha + 1$, and the value of $\tilde{y}(\alpha)$ plays absolutely no role in all the results we have mentioned.

Definition 5.20. We say $\alpha \in \alpha$ is *irregularly admissible* if

- for any $\mu \in \tilde{\alpha}_\alpha$ such that $r_\mu > 1$, \tilde{y} has a pole at μ but $\tilde{y}d\tilde{x}$ is regular at μ . In particular, this imposes $s_\mu \in [1, r_\mu)$;
- for any distinct $\mu, \nu \in \tilde{\alpha}_\alpha$ such that $(r_\mu, s_\mu) = (r_\nu, s_\nu)$, we have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$;
- if $|\tilde{\alpha}_\alpha| > 1$, there exist distinct $\mu_+, \mu_- \in \tilde{\alpha}_\alpha$ such that $r_{\mu_\pm} = \mp 1 \pmod{s_{\mu_\pm}}$ and $\frac{r_{\mu_+}}{s_{\mu_+}} \geq \frac{r_{\mu_-}}{s_{\mu_-}}$, and for any $\mu \in \tilde{\alpha}_\alpha \setminus \{\mu_-, \mu_+\}$, we have $s_\mu = 1$ and $\frac{r_{\mu_+}}{s_{\mu_+}} \geq r_\mu \geq \frac{r_{\mu_-}}{s_{\mu_-}}$;
- if $|\tilde{\alpha}_\alpha| = 1$, then $r_\mu = \pm 1 \pmod{s_\mu}$ for $\mu \in \tilde{\alpha}_\alpha$.

These conditions always imply that for any μ , we have $\gcd(r_\mu, s_\mu) = 1$; in other words, the plane curve $(\tilde{C}, \tilde{x}, \tilde{y})$ is locally irreducible at μ . Here, the second condition avoids pathology (iii) in Lemma 5.4 and the next results. The third condition is then equivalent to avoiding pathology (iv) in Lemma 5.4, because $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ and (r_μ, s_μ) coprime, (r_ν, s_ν) coprime imply that $(r_\mu, s_\mu) = (r_\nu, s_\nu)$. The third and fourth conditions match those in Theorem 2.11 if $d > 1$, and the case $d = 1$ corresponds to Theorem 2.10.

Definition 5.21. We say $\alpha \in \alpha$ is *exceptionally admissible* if

- there exists a unique $\mu_- \in \tilde{\alpha}_\alpha$ such that $s_{\mu_-} = +\infty$, and it has $r_{\mu_-} = 1$;
- the three first properties in Definition 5.20 that do not involve μ_- are satisfied;
- there exists $\mu_+ \in \tilde{\alpha}_\alpha \setminus \{\mu_-\}$ such that $r_{\mu_+} = -1 \pmod{s_{\mu_+}}$;
- for any $\mu \in \tilde{\alpha}_\alpha \setminus \{\mu_-, \mu_+\}$, we have $s_\mu = 1$ and $\frac{r_{\mu_+}}{s_{\mu_+}} \geq r_\mu$.

Allowing infinite s , the first condition guarantees that we avoid pathologies (i) and (ii) in Lemma 5.4, which make the denominator of the recursion kernel be identically zero in some open set. The last two conditions match those in Theorem 2.11.

Definition 5.22. A spectral curve $\mathcal{C} = (C, x, y)$ is *admissible* if all $\alpha \in \alpha$ are either regularly, or irregularly, or exceptionally admissible. The tuple $(r_\mu, s_\mu)_{\mu \in \tilde{\alpha}_\alpha}$ is called the type of the ramification point $\alpha \in \alpha$.

Theorem 5.23. Let \mathcal{C} be an admissible spectral curve equipped with a fundamental bidifferential of the second kind $\omega_{0,2}$ and with a crosscap differential $\omega_{\frac{1}{2},1}$. Then there exists a unique way to complete $(\omega_{0,1}, \omega_{\frac{1}{2},1}, \omega_{0,2})$ into a system of correlators ω satisfying the projection property and the abstract loop equations (or the master loop equations). Moreover, $\omega_{g,n}$ is computed by the topological recursion (5.7) by induction on $2g - 2 + n > 0$, and the result of this formula is symmetric in all its variables. The coefficients $F_{g,n}$ determined by its decomposition are the coefficients of expansion of the partition function of the Airy structure introduced in Definition 5.17.

For smooth curves with simple ramifications – i.e., $\tilde{C} = C$, $|\tilde{\alpha}_\alpha| = 1$ and $r_\alpha = 2$ for all $\alpha \in \alpha$ – the symmetry is proved in [36, Theorem 4.6]. For admissible smooth curves, Theorem 5.23 is proved in [9, Theorem 5.32]. For singular curves, the admissibility condition we have adopted is not far from being optimal for this formulation of the abstract loop equation/formulas like (5.7). It may not be impossible to define a topological recursion for more general spectral curves, but either formula (5.7) will have to be different or the exponents in the master loop equations/abstract loop equations should be increased, in a consistent way so that there still exist a unique symmetric solution.

We obtained Theorem 5.23 by exploiting the dictionary established in Section 4 and using Theorem 2.14. It is therefore natural to ask what the analogous statement of Theorem 2.13 (with the addition of Theorem 2.15) in the setting of spectral curves should be. For this we make the following observation.

Proposition 5.24. *Again assume that none of conditions (i), (ii), (iii), (iv) appearing in Lemma 5.4 are satisfied, and let ω be a system of correlators satisfying the projection property. Then, ω satisfies the abstract loop equations for $g = 0^2$ if and only if the associated partition function satisfies*

$$\forall(\alpha, i, k) \in \mathcal{I}, \quad e^{-F} H_{\alpha; i, k} e^F \cdot 1 = o(\hbar^{\frac{1}{2}}). \quad (5.18)$$

From this it becomes clear how to translate Theorem 2.13.

Definition 5.25. We say $\alpha \in \alpha$ is *admissible in genus 0* if

- there is at most one $\mu_- \in \tilde{\alpha}_\alpha$ for which $s_{\mu_-} = +\infty$. If such a μ_- exists, then $r_{\mu_-} = 1$;
- for any distinct $\mu, \nu \in \tilde{\alpha}_\alpha$ such that $(r_\mu, s_\mu) = (r_\nu, s_\nu)$, we have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$;
- $r_\mu = \pm 1 \pmod{s_\mu}$ for all $\mu \in \tilde{\alpha}_\alpha$;
- for all $\mu_1 \neq \mu_2$ with $s_{\mu_i} > 2$ such that either

$$r_{\mu_1} = 1 \pmod{s_{\mu_1}} \quad \text{and} \quad r_{\mu_2} = 1 \pmod{s_{\mu_2}}$$

or

$$r_{\mu_1} = -1 \pmod{s_{\mu_1}} \quad \text{and} \quad r_{\mu_2} = -1 \pmod{s_{\mu_2}}$$

one has $\lfloor \frac{r_{\mu_1}}{s_{\mu_1}} \rfloor \neq \lfloor \frac{r_{\mu_2}}{s_{\mu_2}} \rfloor$;

- if there are pairwise distinct $\mu_1, \mu_2, \mu_3 \in \tilde{\alpha}_\alpha$ with $\lfloor \frac{r_{\mu_1}}{s_{\mu_1}} \rfloor = \lfloor \frac{r_{\mu_2}}{s_{\mu_2}} \rfloor = \lfloor \frac{r_{\mu_3}}{s_{\mu_3}} \rfloor$, then there is an $m \in \{1, 2, 3\}$ for which $s_{\mu_m} = 1$.

A spectral curve $\mathcal{C} = (C, x, y)$ is *admissible in genus 0* if all $\alpha \in \alpha$ are.

²With this we mean that for any $n \geq 2$, $\alpha \in \alpha$ and $i \in [r_\alpha]$, equation (5.12) is satisfied for $g = 0$ as $x_0 \rightarrow x(\alpha)$.

Theorem 5.26. *If \mathcal{C} is a spectral curve admissible in genus 0, equipped with a fundamental bidifferential of the second kind $\omega_{0,2}$, then there exists a unique way to complete $(\omega_{0,1}, \omega_{0,2})$ into a system of correlators $(\omega_{0,n})_{n \geq 1}$ satisfying the projection property and the abstract loop equations (or the master loop equations) for $g = 0$. Moreover, $\omega_{0,n}$ is computed by the topological recursion (5.7) by induction on $n > 0$, and the result of this formula is symmetric in all its variables. The $F_{0,n}$ determined by its decomposition are the free energies solving (5.18).*

Clearly, not every curve admissible in genus 0 is admissible, and there is indeed a good reason for this definition. Suppose we are in the setting of Theorem 5.26. Then given a crosscap differential $\omega_{\frac{1}{2},1}$, one could ask whether one can extend the family $(\omega_{0,1}, \omega_{0,2}, \omega_{0,3}, \dots, \omega_{\frac{1}{2},1})$ to a system of correlators ω satisfying the abstract loop equations for all $g \geq 0$. Doing explicit calculations, we will observe in Section 6.1.3 that this is not always possible without at least imposing certain conditions on the choice of $\omega_{\frac{1}{2},1}$.

5.5. Decoupling of exceptional components

If we erase some or all of the components of an exceptionally admissible local spectral curve \mathcal{C} indexed by the $\mu_- \in \alpha$ such that $s_{\mu_-} = \infty$, we still obtain an admissible local spectral curve \mathcal{C}' . We prove below a decoupling result if $\omega_{0,2}$ has no cross-terms with these components and $\omega_{\frac{1}{2},1}$ vanishes on these components. This decoupling means that the computing $\omega_{g,n}$ on \mathcal{C} and restricting to \mathcal{C}' gives the same result as restricting $(\omega_{0,1}, \omega_{0,2}, \omega_{\frac{1}{2},1})$ to \mathcal{C}' and then computing $\omega_{g,n}$ by the topological recursion on \mathcal{C}' .

Proposition 5.27. *Let (C, x, y) be an exceptionally admissible spectral curve equipped with a fundamental bidifferential of the second kind $\omega_{0,2}$ and with a crosscap differential $\omega_{\frac{1}{2},1}$.*

Let α' be a non-empty subset of exceptionally admissible ramification points, and denote $\tilde{\alpha}'$ the set of $\mu_- \in \tilde{\alpha}$ such that $\mu_- \in \tilde{\alpha}_\alpha$ for some $\alpha \in \alpha'$ and $s_{\mu_-} = \infty$. Assume that for any $\mu_- \in \tilde{\alpha}'$ and $\nu \in \tilde{\alpha} \setminus \tilde{\alpha}'$, we have

$$(\text{Loc}_{\mu_-} \otimes \text{Loc}_\nu)(\omega_{0,2}) = 0, \quad \text{Loc}_{\mu_-}(\omega_{\frac{1}{2},1}) = 0.$$

Then, if we denote by ω the outcome of topological recursion and

$$\text{Loc}' = \bigsqcup_{\mu \in \tilde{\alpha} \setminus \tilde{\alpha}'} \text{Loc}_\mu,$$

the Loc' -projection of the system of correlators ω obtained from $(C, x, y, \omega_{0,2}, \omega_{\frac{1}{2},1})$ by the topological recursion (5.7), satisfies the topological recursion on the local

spectral curve

$$(\tilde{C}^{\text{loc}})' = \bigsqcup_{\mu \in \tilde{\alpha} \setminus \tilde{\alpha}'} \tilde{C}_\mu$$

equipped with the restriction of x , y , $\omega_{0,2}$, $\omega_{\frac{1}{2},1}$ onto $(\tilde{C}^{\text{loc}})'$.

Proof. We detail the proof in the case of a single ramification point with $s_{\mu_-} = \infty$. The general case follows because topological recursion is local. It suffices to work from the start with the normalised local spectral curve attached to (C, x, y) . We write \tilde{C}_-^{loc} for the connected component of \tilde{C}^{loc} associated to μ_- and \tilde{C}_+^{loc} for all other components.

First, let us prove all $\omega_{g,n}$ with exactly one argument in \tilde{C}_-^{loc} are zero. We will prove this by induction on the Euler characteristic. The base cases hold, as $\omega_{0,1}$ and $\omega_{\frac{1}{2},1}$ vanish on \tilde{C}_-^{loc} and $\omega_{0,2}$ does not have cross-terms. For the induction step, let us recall the topological recursion formula (5.7):

$$\omega_{g,n+1}(z_0, z_{[n]}) = \sum_{\alpha \in \tilde{\alpha}} \sum_{\mu \in \tilde{\alpha}_\alpha} \text{Res}_{z=\mu} \left(\sum_{\{z\} \subset Z \subseteq \mathfrak{f}_\alpha(z)} K_\mu^{(|Z|)}(z_0; Z) \mathcal{W}'_{g,|Z|,n}(Z; z_{[n]}) \right),$$

$$K_\mu^{(m)}(z_0; z_{[m]}) := - \frac{\int_\mu^z \omega_{0,2}(\cdot, z_0)}{\prod_{l=2}^m ((\tilde{y}(z_l) - \tilde{y}(z_1)) d\tilde{x}(z_1))}.$$

In this case, $\alpha = \{0\}$, and $\tilde{\alpha}_0 = \{0 \in \tilde{C}_+^{\text{loc}}, 0 \in \tilde{C}_-^{\text{loc}}\}$.

Let us analyse this formula with $z_0 \in \tilde{C}_-^{\text{loc}}$ and all $z_i \in \tilde{C}_+^{\text{loc}}$ for $i \in [n]$. First, because the $\omega_{0,2}$ does not contain cross-terms, we must have $z \in C_-$, so also $\mu = 0 \in \tilde{C}_-^{\text{loc}}$. Because $\tilde{x}|_{\tilde{C}_-^{\text{loc}}}$ is injective, $\mathfrak{f}_0(z)$ contains no other point in \tilde{C}_-^{loc} . Hence for all terms in the sum, $Z \cap \tilde{C}_-^{\text{loc}} = \{z\}$, and by the induction hypothesis, all $\mathcal{W}'(Z; z_{[n]})$ are zero (they contain a factor $\omega_{g',n'}$ with $2g' - 2 + n' < 2g - 2 + n$ and one argument in \tilde{C}_-^{loc}).

Now, we will prove the proposition using a similar induction. The base cases, $\omega_{0,1}$ and $\omega_{0,2}$ (and the trivial $\omega_{\frac{1}{2},1}$), do indeed not mix several components.

For the induction step, we again look at the topological recursion formula. Let us look at the terms contributing in the case z_0 and all $z_{[n]}$ are in \tilde{C}_+^{loc} . As before, $z \in \tilde{C}_+^{\text{loc}}$. Furthermore, $\mathfrak{f}_0(z)$ contains exactly one element, say ζ , in \tilde{C}_-^{loc} , so any Z contains at most one such element. Any term in the sum not containing ζ also contributes to the topological recursion on \tilde{C}_+^{loc} , and all terms including ζ must vanish by the first part of this proof, as they have a factor $\omega_{g',n'}$ with exactly one argument, namely ζ , in \tilde{C}_-^{loc} . ■

6. Calculations for low $2g - 2 + n$

In this section, we calculate some of the first correlators in the unique way of Theorem 5.23, but for not necessarily admissible spectral curves. In this generality, there is no guarantee for the correlators to be symmetric, and we will find that indeed they are not for certain choices of parameters. As correlators coming from Airy structures are symmetric by construction, these calculations give necessary conditions for our collections of differential operators to form Airy structures. These conditions are summarised in Section 2.3.4.

6.1. The standard case

Let us consider the spectral curve with a unique ramification point α at which d irreducible components labelled by $\tilde{\alpha}_\alpha = \tilde{\alpha}$ intersect, and defined for $\mu \in \tilde{\alpha}$ and z on the μ -th component of the normalisation by the formulas

$$x\left(\frac{\mu}{z}\right) = z^{r_\mu}, \quad y\left(\frac{\mu}{z}\right) = -t_\mu z^{s_\mu - r_\mu}. \quad (6.1)$$

We equip it with the bidifferential and the crosscap differential

$$\omega_{0,2}\left(\frac{\mu_1}{z_1} \frac{\mu_2}{z_2}\right) = \delta_{\mu_1, \mu_2} \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad \omega_{\frac{1}{2},1}\left(\frac{\mu}{z}\right) = \frac{Q_\mu dz}{z}.$$

We assume that $\gcd(r_\mu, s_\mu) = 1$, and that for $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ and $\mu \neq \nu$, we must have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$. The correlators for $\chi = 2 - 2g - n = -1$ are

$$\begin{aligned} \omega_{0,3}(z_1, z_2, z_3) &= \sum_{\mu \in \tilde{\alpha}} \operatorname{Res}_{z=\mu} \sum_{z' \in \tilde{f}'(z)} K_\mu(z_1, z, z') \\ &\quad \times (\omega_{0,2}(z, z_2) \omega_{0,2}(z', z_3) + \omega_{0,2}(z, z_3) \omega_{0,2}(z', z_2)), \\ \omega_{\frac{1}{2},2}(z_1, z_2) &= \sum_{\mu \in \tilde{\alpha}} \operatorname{Res}_{z=\mu} \sum_{z' \in \tilde{f}'(z)} K_\mu(z_1, z, z') \\ &\quad \times (\omega_{0,2}(z, z_2) \omega_{\frac{1}{2},1}(z') + \omega_{0,2}(z', z_2) \omega_{\frac{1}{2},1}(z)), \\ \omega_{1,1}(z_1) &= \sum_{\mu \in \tilde{\alpha}} \operatorname{Res}_{z=\mu} \sum_{z' \in \tilde{f}'(z)} K_\mu(z_1, z, z') \\ &\quad \times (\omega_{0,2}(z, z') + \omega_{\frac{1}{2},1}(z) \omega_{\frac{1}{2},1}(z')), \end{aligned} \quad (6.2)$$

where

$$K_\mu(z_1, z, z') = \frac{\int_\mu^z \omega_{0,2}(\cdot, z_1)}{(y(z) - y(z')) dx(z)}.$$

In this section, we compute these correlators and show that the symmetry of $\omega_{0,3}$ and $\omega_{\frac{1}{2},2}$ poses constraints on the parameters $(r_\mu, s_\mu, t_\mu, Q_\mu)_{\mu \in \tilde{\alpha}}$. We also obtain

similar constraints from partial calculation of $\omega_{0,4}$. In the light of Proposition 5.24 and Theorem 5.26, the symmetry of $\omega_{0,3}$ and $\omega_{0,4}$ is necessary so that the family of differential operators admits a partition function solving the associated system of differential equations to leading order in $\hbar^{\frac{1}{2}}$. In the same way, Proposition 5.18 and Theorem 5.23 tell us that additionally we need to take into account the symmetry conditions for $\omega_{\frac{1}{2},2}$ which are necessary constraints to obtain Airy structures from the construction presented in Section 2. These ideas will be used in Section 6.3 to prove Propositions 2.16 and 2.17.

6.1.1. Genus zero. In this section, we calculate $\omega_{0,3}$ and obtain constraints on the parameters of the spectral curve (6.1) that are necessary for the symmetry of the correlators $\omega_{0,3}$ and $\omega_{0,4}$.

Proposition 6.1. *Assume r_μ and s_μ are coprime for all $\mu \in \tilde{\alpha}$. Then $\omega_{0,3}$ is symmetric if and only if the following holds:*

- (i) $r_\mu = \pm 1 \pmod{s_\mu}$ for all $\mu \in \tilde{\alpha}$.
- (ii) For all $\mu_1 \neq \mu_2$ with $s_{\mu_i} > 2$ such that either

$$r_{\mu_1} = 1 \pmod{s_{\mu_1}} \quad \text{and} \quad r_{\mu_2} = 1 \pmod{s_{\mu_2}}$$

or

$$r_{\mu_1} = -1 \pmod{s_{\mu_1}} \quad \text{and} \quad r_{\mu_2} = -1 \pmod{s_{\mu_2}},$$

one has $\lfloor \frac{r_{\mu_1}}{s_{\mu_1}} \rfloor \neq \lfloor \frac{r_{\mu_2}}{s_{\mu_2}} \rfloor$.

When these conditions are satisfied, then $\omega_{0,3}$ is given by

$$\begin{aligned} \omega_{0,3} \left(\begin{matrix} \mu_1 & \mu_2 & \mu_3 \\ z_1 & z_2 & z_3 \end{matrix} \right) \\ = \sum_{\mu} \frac{c_{\mu} \delta_{\mu_1, \mu_2, \mu_3, \mu}}{t_{\mu} r_{\mu}} \sum_{k_1, k_2, k_3 > 0} \frac{k_1 k_2 k_3 dz_1 dz_2 dz_3}{z_1^{k_1+1} z_2^{k_2+1} z_3^{k_3+1}} \delta_{k_1+k_2+k_3, s_{\mu}}, \end{aligned} \quad (6.3)$$

where

$$c_{\mu} := \begin{cases} -r'_{\mu}, & r_{\mu} = r'_{\mu} s_{\mu} + 1, \\ r'_{\mu} + 1, & r_{\mu} = r'_{\mu} s_{\mu} + s_{\mu} - 1. \end{cases} \quad (6.4)$$

Proposition 6.2. *Assume conditions (i) and (ii) of Proposition 6.1 hold. Then $\omega_{0,4}$ can only be symmetric if*

- (iii) For any distinct μ_1, μ_2, μ_3 with $\lfloor \frac{r_{\mu_1}}{s_{\mu_1}} \rfloor = \lfloor \frac{r_{\mu_2}}{s_{\mu_2}} \rfloor = \lfloor \frac{r_{\mu_3}}{s_{\mu_3}} \rfloor$, there is an $m \in \{1, 2, 3\}$ for which $s_{\mu_m} = 1$.

Notice that the conditions from Propositions 6.1 and 6.2 are exactly the defining properties of a curve admissible in genus 0, i.e., those curves for which we already know that all $(\omega_{0,n})_{n \geq 1}$ computed by the topological recursion will be symmetric.

Corollary 6.3. *Let \mathcal{C} be a spectral curve of type (6.1) equipped with the standard bidifferential. Then the topological recursion formula applied to $(\omega_{0,1}, \omega_{0,2})$ yields a family of symmetric multidifferentials $(\omega_{0,n})_{n \geq 1}$ if and only if \mathcal{C} is admissible in genus 0.*

Before we prove Propositions 6.1 and 6.2, we first give some more general considerations for calculating the genus zero correlators.

Recalling equations (5.7) and (5.8), we see that the recursion kernel can be split into factors coming from different irreducible components of the spectral curve. First, note that in $K(z_1; z, Z)$, we always need z_1 and z to lie in the same irreducible component, and then we have

$$\int_{\mu}^z \omega_{0,2}(\cdot, z_1) = \frac{z dz_1}{z_1(z_1 - z)}.$$

Because $\omega_{0,2}$ in our current situation does not mix irreducible components, this shows immediately that to get symmetric correlators, we need

$$\omega_{0,3}\left(\begin{matrix} \mu & \nu & \lambda \\ z_1 & z_2 & z_3 \end{matrix}\right) = 0$$

unless $\mu = \nu = \lambda$: if one (say μ) is different from the other two, we can use the recursion with respect to its variable and get $z \in \tilde{\mathcal{C}}_{\mu}$, so both terms in (6.2) would involve an $\omega_{0,2}$ between two different irreducible components.

This vanishing can then be used to calculate $\omega_{0,4}$ with arguments in exactly three different irreducible components. All terms involving $K^{(2)}$ would also involve a vanishing $\omega_{0,3}$, while the same argument as for $\omega_{0,3}$ above shows that the contribution of $K^{(3)}$ to such $\omega_{0,4}$ should vanish – it also involves only $\omega_{0,2}$. In fact, this same argument could be applied to $\omega_{0,4}\left(\begin{matrix} \mu & \nu & \nu & \nu \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right)$ (we only need one irreducible component to be different from all others), but this turns out not to give a new constraint, so we omit it here.

Remark 6.4. This argument can be used inductively to show that all $\omega_{0,n}$ with exactly one argument on a given irreducible component must vanish. This is analogous to the proof of Proposition 5.27, but also uses that all arguments of the recursion kernel must couple to a different correlator, as we restrict to genus zero. However, in general the recursive computation of these correlators does require $K^{(m)}$ of order m up to the degree of \tilde{x} , and therefore becomes quite complicated.

For the remainder of this section, we restrict to recursion kernels coupled to $\omega_{0,2}$'s, which is sufficient for our calculations. We also assume $z_1 \in \tilde{\mathcal{C}}_{\mu}$. From the shape of the recursion kernel, we obtain several possible contributions (combining factors from the kernel and the correlators):

(1) There will always be one term

$$\frac{zdz_1}{z_1(z_1 - z)} \frac{dzdz_m}{(z - z_m)^2}$$

with $z_m \in \tilde{C}_\mu$.

(2) For any other $z_m \in \tilde{C}_\mu$, we get a contribution

$$\frac{\vartheta_\mu^{a_m}}{r_\mu t_\mu (\vartheta_\mu^{a_m s_\mu} - 1) z^{s_\mu - 1}} \frac{dz_m}{(\vartheta_\mu^{a_m} z - z_m)^2},$$

where ϑ_μ is a primitive r_μ -th root of unity and we need to sum over all subsets $\{a_m\} \subset [r_\mu - 1]$ of size determined by the number of $z_m \in \tilde{C}_\mu$.

(3) For any $v \neq \mu$ and $z_m \in \tilde{C}_v$, we get a contribution

$$\frac{\vartheta_v^{b_m} z^{r_\mu/r_v - 1}}{r_v (t_v \vartheta_v^{b_m s_v} z^{r_\mu s_v/r_v - 1} - t_\mu z^{s_\mu - 1})} \frac{dz_m}{(\vartheta_v^{b_m} z^{r_\mu/r_v} - z_m)^2}$$

and we need to sum over all subsets $\{b_m\} \subset [r_v]$ of size determined by the number of $z_m \in \tilde{C}_v$.

We need to take the series expansion of each of these near $z = 0$. For (1), this is

$$\frac{zdz_1}{z_1(z_1 - z)} \frac{dzdz_m}{(z - z_m)^2} = dzdz_1 dz_m \sum_{\ell, k \geq 1} k z_1^{-\ell - 1} z_m^{-k - 1} z^{\ell + k - 1}.$$

The summations for cases (2) and (3) get quite complicated for general sizes of the subset, but for size one, they are computable. For case (2), we get

$$\begin{aligned} & \sum_{a=1}^{r_\mu - 1} \frac{\vartheta_\mu^a}{r_\mu t_\mu (\vartheta_\mu^{a s_\mu} - 1) z^{s_\mu - 1}} \frac{dz_m}{(\vartheta_\mu^a z - z_m)^2} \\ &= \frac{dz_m}{r_\mu^2 t_\mu} \sum_{a=1}^{r_\mu - 1} \sum_{\ell=0}^{r_\mu - 1} \ell \vartheta_\mu^{a s_\mu \ell} \sum_{k \geq 1} k \vartheta_\mu^{a k} z_m^{-k - 1} z^{k - s_\mu} \\ &= \frac{dz_m}{r_\mu^2 t_\mu} \sum_{k \geq 1} \sum_{\ell=0}^{r_\mu - 1} \ell (r_\mu \delta_{r_\mu | s_\mu \ell + k} - 1) k z_m^{-k - 1} z^{k - s_\mu} \\ &= \frac{dz_m}{r_\mu t_\mu} \sum_{k \geq 1} \left(\ell_\mu(k) - \frac{r_\mu - 1}{2} \right) k z_m^{-k - 1} z^{k - s_\mu}, \end{aligned}$$

where $\ell_\mu(k)$ is the unique $\ell \in [0, r_\mu)$ such that $r_\mu \mid s_\mu \ell + k$.

For case (3), there are three different subcases, depending on the sign of $\frac{r_\mu}{s_\mu} - \frac{r_\nu}{s_\nu}$:

(3a) If $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$, we get

$$\begin{aligned} & \sum_{b=1}^{r_\nu} \frac{\vartheta_\nu^{b_m} z^{r_\mu/r_\nu-1}}{r_\nu (t_\nu \vartheta_\nu^{b_m s_\nu} z^{r_\mu s_\nu/r_\nu-1} - t_\mu z^{s_\mu-1})} \frac{dz_m}{(\vartheta_\nu^{b_m} z^{r_\mu/r_\nu} - z_m)^2} \\ &= dz_m \sum_{k \geq 1} \frac{t_\mu^{r_\mu - \ell_\mu(k) - 1} t_\nu^{\ell_\mu(k)}}{t_\nu^{r_\mu} - t_\mu^{r_\mu}} k z_m^{-k-1} z^{k-s_\mu}. \end{aligned}$$

(3b) If $\frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}$, we get

$$\begin{aligned} & \sum_{b=1}^{r_\nu} \frac{\vartheta_\nu^{b_m} z^{r_\mu/r_\nu-1}}{r_\nu (t_\nu \vartheta_\nu^{b_m s_\nu} z^{r_\mu s_\nu/r_\nu-1} - t_\mu z^{s_\mu-1})} \frac{dz_m}{(\vartheta_\nu^{b_m} z^{r_\mu/r_\nu} - z_m)^2} \\ &= -\frac{dz_m}{t_\mu} \sum_{\substack{k, \ell \geq 1 \\ r_\nu | s_\nu(\ell-1) + k}} \left(\frac{t_\nu}{t_\mu}\right)^{\ell-1} k z_m^{-k-1} z^{(s_\nu(\ell-1)+k)\frac{r_\mu}{r_\nu} - \ell s_\mu}. \end{aligned}$$

(3c) If $\frac{r_\mu}{s_\mu} < \frac{r_\nu}{s_\nu}$, we get

$$\begin{aligned} & \sum_{b=1}^{r_\nu} \frac{\vartheta_\nu^{b_m} z^{r_\mu/r_\nu-1}}{r_\nu (t_\nu \vartheta_\nu^{b_m s_\nu} z^{r_\mu s_\nu/r_\nu-1} - t_\mu z^{s_\mu-1})} \frac{dz_m}{(\vartheta_\nu^{b_m} z^{r_\mu/r_\nu} - z_m)^2} \\ &= \frac{dz_m}{t_\nu} \sum_{\substack{k, \ell \geq 1 \\ r_\nu | s_\nu \ell - k}} \left(\frac{t_\mu}{t_\nu}\right)^{\ell-1} k z_m^{-k-1} z^{(k-\ell s_\nu)\frac{r_\mu}{r_\nu} + (\ell-1)s_\mu}. \end{aligned}$$

Proof of Proposition 6.1. We have already argued that in the symmetric case $\omega_{0,3}$ vanishes unless all arguments are on the same component. Let us therefore first calculate the value of $\omega_{0,3}$ in exactly this case. If all arguments lie on \tilde{C}_μ , we get a contribution from case (1) and the one-argument version of case (2). Then taking the residue, we obtain

$$\begin{aligned} \frac{\omega_{0,3}\left(\frac{\mu}{z_1} \frac{\mu}{z_2} \frac{\mu}{z_3}\right)}{dz_1 dz_2 dz_3} &= \text{Res}_{z=0} \left(dz \sum_{\ell_2, k_2 \geq 1} k_2 z_1^{-\ell_2-1} z_2^{-k_2-1} z^{\ell_2+k_2-1} \right. \\ &\quad \times \frac{1}{r_\mu t_\mu} \sum_{k_3 \geq 1} \left(\ell_\mu(k_3) - \frac{r_\mu-1}{2} \right) k_3 z_3^{-k_3-1} z^{k_3-s_\mu} \\ &\quad + dz \sum_{\ell_3, k_3 \geq 1} k_3 z_1^{-\ell_3-1} z_3^{-k_3-1} z^{\ell_3+k_3-1} \\ &\quad \times \frac{1}{r_\mu t_\mu} \sum_{k_2 \geq 1} \left(\ell_\mu(k_2) - \frac{r_\mu-1}{2} \right) k_2 z_2^{-k_2-1} z^{k_2-s_\mu} \left. \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{r_\mu t_\mu} \operatorname{Res}_{z=0} dz \sum_{\ell, k_2, k_3 \geq 1} k_2 k_3 (\ell_\mu(k_2) + \ell_\mu(k_3) - r_\mu + 1) \\
 &\quad \times z_1^{-\ell-1} z_2^{-k_2-1} z_3^{-k_3-1} z^{\ell+k_2+k_3-s_\mu-1} \\
 &= \frac{1}{r_\mu t_\mu} \sum_{k_1, k_2, k_3 \geq 1} k_2 k_3 (\ell_\mu(k_2) + \ell_\mu(k_3) - r_\mu + 1) \\
 &\quad \times z_1^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1} \delta_{k_1+k_2+k_3, s_\mu}.
 \end{aligned}$$

For $r_\mu = 1$, this expression vanishes and hence is symmetric. In the case $r_\mu > 1$, we know from [9, Proposition B.2] that for $s_\mu \in [r_\mu + 1]$, this is symmetric if and only if $r_\mu = \pm 1 \pmod{s_\mu}$. If however, $r_\mu > 1$ and $s_\mu > r_\mu + 1$, it is easy to see that the correlators can never be symmetric. Let us assume $\omega_{0,3}$ is symmetric. Then for all $k_1, k_2, k_3 > 0$ satisfying $k_1 + k_2 + k_3 = s_\mu$, we must have

$$k_2 k_3 (\ell_\mu(k_2) + \ell_\mu(k_3) - r_\mu + 1) = k_1 k_3 (\ell_\mu(k_1) + \ell_\mu(k_3) - r_\mu + 1).$$

After plugging $k_1 = 1$, $k_2 = r_\mu$, and $k_3 = s_\mu - r_\mu - 1$ into the above equation, the right-hand side vanishes and we obtain

$$r_\mu (\ell_\mu(s_\mu - r_\mu - 1) - r_\mu + 1) = 0.$$

From this we deduce that $\ell_\mu(s_\mu - r_\mu - 1) = r_\mu - 1$. This means that there must exist such m that $m r_\mu = s_\mu - r_\mu - 1 + s_\mu (r_\mu - 1)$. This in turn implies that $r_\mu = 1$ which contradicts our starting assumption. We therefore conclude that the symmetry condition for $\omega_{0,3} \left(\begin{smallmatrix} \mu & \mu & \mu \\ z_1 & z_2 & z_3 \end{smallmatrix} \right)$ is exactly captured by (i).

To show equation (6.3), recall that $\ell_\mu(k)$ is the unique $\ell \in [0, r_\mu)$ such that $r_\mu \mid s_\mu \ell + k$. In other words, there is an m such that $m(r'_\mu s_\mu + \varepsilon_\mu) = s_\mu \ell_\mu(k) + k$. Viewing this formula modulo s_μ shows that for $k < s_\mu$

$$m = \begin{cases} k & \text{if } \varepsilon_\mu = 1, \\ s_\mu - k & \text{if } \varepsilon_\mu = s_\mu - 1. \end{cases}$$

From this, it follows easily that $(\ell_\mu(k_2) + \ell_\mu(k_3) - r_\mu + 1) = c_\mu$ as $k_2, k_3 < s_\mu$.

Now let us turn our attention to the case where the arguments $\omega_{0,3}$ lie on two different components. We will see that in this case the symmetry is controlled by condition (ii). Let us first show that this condition is indeed necessary for the symmetry of $\omega_{0,3}$. So suppose there are μ and ν violating (ii), i.e., $s_\mu, s_\nu > 2$, $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_\nu}{s_\nu} \rfloor$ and either $r_\mu = 1 \pmod{s_\mu}$ and $r_\nu = 1 \pmod{s_\nu}$ or $r_\mu = -1 \pmod{s_\mu}$ and $r_\nu = -1 \pmod{s_\nu}$. After relabelling, we may also assume that $\frac{r_\mu}{s_\mu} \geq \frac{r_\nu}{s_\nu}$. In the following, we will show by explicit computation that in this case $\omega_{0,3} \left(\begin{smallmatrix} \mu & \mu & \nu \\ z_1 & z_2 & z_3 \end{smallmatrix} \right) \neq 0$. This however means that $\omega_{0,3}$ cannot be symmetric since $\omega_{0,3} \left(\begin{smallmatrix} \nu & \mu & \mu \\ z_3 & z_2 & z_1 \end{smallmatrix} \right)$ is always vanishing.

In the case $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$, $\omega_{0,3}(\frac{\mu}{z_1} \frac{\mu}{z_2} \frac{\nu}{z_3})$ gets a contribution from case (1) and the one-argument version of case (3a). Then taking the residue gives

$$\begin{aligned} \frac{\omega_{0,3}(\frac{\mu}{z_1} \frac{\mu}{z_2} \frac{\nu}{z_3})}{dz_1 dz_2 dz_3} &= \operatorname{Res}_{z=0} dz \sum_{k_1, k_2, k_3 \geq 1} k_2 k_3 z_1^{-k_1-1} z_2^{-k_2-1} z^{k_1+k_2-1} \\ &\quad \times \frac{t_\mu^{r_\mu - \ell_\mu(k) - 1} t_\nu^{\ell_\mu(k)}}{t_\nu^{r_\mu} - t_\mu^{r_\mu}} z_m^{-k_3-1} z^{k_3 - s_\mu} \\ &= \sum_{k_1, k_2, k_3 \geq 1} \frac{t_\mu^{r_\mu - \ell_\mu(k_3) - 1} t_\nu^{\ell_\mu(k_3)}}{t_\nu^{r_\mu} - t_\mu^{r_\mu}} k_2 k_3 \delta_{k_1+k_2+k_3, s_\mu} \\ &\quad \times z_1^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1}. \end{aligned}$$

This is clearly non-vanishing since we assumed that $s_\mu, s_\nu > 2$.

If now $\frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}$, the correlator gets a contribution from (1) and (3b) leading to

$$\begin{aligned} \frac{\omega_{0,3}(\frac{\mu}{z_1} \frac{\mu}{z_2} \frac{\nu}{z_3})}{dz_1 dz_2 dz_3} &= -\frac{1}{t_\mu} \operatorname{Res}_{z=0} dz \sum_{\substack{k_1, k_2, k_3, \ell \geq 1 \\ r_\nu | s_\nu(\ell-1) + k_3}} k_2 k_3 z_1^{-k_1-1} z_2^{-k_2-1} z^{k_1+k_2-1} \left(\frac{t_\nu}{t_\mu}\right)^{\ell-1} \\ &\quad \times z_3^{-k_3-1} z^{(s_\nu(\ell-1)+k_3)\frac{r_\mu}{r_\nu} - \ell s_\mu} \\ &= -\frac{1}{t_\mu} \sum_{\substack{k_1, k_2, k_3, \ell \geq 1 \\ r_\nu | s_\nu(\ell-1) + k_3}} \left(\frac{t_\nu}{t_\mu}\right)^{\ell-1} k_2 k_3 z_1^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1} \\ &\quad \times \delta_{k_1+k_2+(s_\nu(\ell-1)+k_3)\frac{r_\mu}{r_\nu} - \ell s_\mu, 0} \\ &= -\frac{1}{t_\mu} \sum_{k_1, k_2, k_3 \geq 1} \sum_{\ell' \geq 0} \left(\frac{t_\nu}{t_\mu}\right)^{\ell_\nu(k_3) + r_\nu \ell'} k_2 k_3 z_1^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1} \\ &\quad \times \delta_{k_1+k_2 - a_{\mu, \nu}(k_3) + \ell'(r_\mu s_\nu - r_\nu s_\mu), 0}, \end{aligned} \quad (6.5)$$

where we set

$$a_{\mu, \nu}(k) := -(s_\nu \ell_\nu(k) + k) \frac{r_\mu}{r_\nu} + (\ell_\nu(k) + 1) s_\mu. \quad (6.6)$$

The key observation to make now is that $a_{\mu, \nu}(1) > 1$ under our assumptions on μ and ν . Indeed, in the case $r_\mu = 1 \pmod{s_\mu}$ and $r_\nu = 1 \pmod{s_\nu}$, we find that

$$a_{\mu, \nu}(1) = -r_\mu + (r_\mu - 1 + s_\mu) = s_\mu - 1 > 1, \quad (6.7)$$

where we used that $\ell_\nu(1) = \lfloor \frac{r_\nu}{s_\nu} \rfloor = \lfloor \frac{r_\mu}{s_\mu} \rfloor$. Similarly, if $r_\mu = -1 \pmod{s_\mu}$ and $r_\nu = -1 \pmod{s_\nu}$, an analogous calculation gives $a_{\mu, \nu}(1) = s_\nu - 1 > 1$. Hence, we find that

the residue

$$\begin{aligned} & \operatorname{Res}_{z_1=0} \operatorname{Res}_{z_2=0} \operatorname{Res}_{z_3=0} z_1^{a_{\mu,v}(1)-1} z_2 z_3 \omega_{0,3} \left(\begin{matrix} \mu & \mu & v \\ z_1 & z_2 & z_3 \end{matrix} \right) \\ &= -\frac{1}{t_\mu} \sum_{\ell' \geq 0} \left(\frac{t_v}{t_\mu} \right)^{\ell_v(1)+r_v \ell'} \delta_{\ell'(r_\mu s_v - r_v s_\mu), 0} = -\frac{1}{t_\mu} \left(\frac{t_v}{t_\mu} \right)^{\ell_v(1)} \end{aligned}$$

is non-vanishing, telling us that $\omega_{0,3} \left(\begin{matrix} \mu & \mu & v \\ z_1 & z_2 & z_3 \end{matrix} \right) \neq 0$. This shows that condition (ii) is indeed necessary to ensure the symmetry of $\omega_{0,3}$.

Let us now finally argue why (i) and (ii) are also sufficient to ensure symmetry. Indeed, our spectral curve is admissible in genus 0 if additionally condition (iii) of Proposition 6.2 is satisfied. In this case, $\omega_{0,3}$ computed by the topological recursion must be symmetric by Theorem 5.26. Note now that condition (iii) is trivially satisfied in the case $d \leq 2$. In the case $d > 2$, since $\omega_{0,3} \left(\begin{matrix} \mu & \mu & \mu \\ z_1 & z_2 & z_3 \end{matrix} \right)$ can as well be computed after restricting x , y , and $\omega_{0,2}$ to the component \tilde{C}_μ (a one-component spectral curve for which (iii) is trivially satisfied), the correlator must be symmetric under permutation of the arguments. Similarly, for $\mu \neq v$ we can compute $\omega_{0,3} \left(\begin{matrix} \mu & v & \mu \\ z_1 & z_2 & z_3 \end{matrix} \right)$, $\omega_{0,3} \left(\begin{matrix} \mu & \mu & v \\ z_1 & z_2 & z_3 \end{matrix} \right)$, etc. after restricting the input data to the two-component curve $\tilde{C}_\mu \sqcup \tilde{C}_v$ for which (iii) holds as well. Finally, if all arguments lie on different components μ, v, λ , we always have $\omega_{0,3} \left(\begin{matrix} \mu & v & \lambda \\ z_1 & z_2 & z_3 \end{matrix} \right) = 0$ which of course is symmetric. Thus, although we are not imposing condition (iii) of Proposition 6.2 the correlator $\omega_{0,3}$ is already symmetric given that (i) and (ii) hold. \blacksquare

Proof of Proposition 6.2. Suppose $\omega_{0,4}$ is symmetric and there are distinct μ, v, λ with $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_v}{s_v} \rfloor = \lfloor \frac{r_\lambda}{s_\lambda} \rfloor$ and $s_\mu, s_v, s_\lambda > 1$. After relabelling, we can assume that $\frac{r_\mu}{s_\mu} \geq \frac{r_v}{s_v} \geq \frac{r_\lambda}{s_\lambda}$. As argued before,

$$\omega_{0,4} \left(\begin{matrix} \mu & v & \mu & \lambda \\ z_1 & z_3 & z_2 & z_4 \end{matrix} \right) = 0$$

because $\mu \neq v$. So it suffices to show that $\omega_{0,4} \left(\begin{matrix} \mu & \mu & v & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \neq 0$ in order to get a contradiction.

In the case $\frac{r_\mu}{s_\mu} > \frac{r_v}{s_v} \geq \frac{r_\lambda}{s_\lambda}$, the correlator $\omega_{0,4} \left(\begin{matrix} \mu & \mu & v & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right)$ gets a contribution from (1) and twice from (3b) giving us

$$\begin{aligned} & \frac{\omega_{0,4} \left(\begin{matrix} \mu & \mu & v & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right)}{dz_1 dz_2 dz_3 dz_4} \\ &= -\frac{1}{t_\mu^2} \operatorname{Res}_{z=0} dz \sum_{\substack{k_1, k_2, k_3, k_4, \ell_3, \ell_4 \geq 1 \\ r_v | s_v (\ell_3 - 1) + k_3 \\ r_\lambda | s_\lambda (\ell_4 - 1) + k_4}} k_2 k_3 k_4 z_1^{-k_1-1} z_2^{-k_2-1} z^{k_1+k_2-1} \\ & \quad \times \left(\frac{t_v}{t_\mu} \right)^{\ell_3-1} z_3^{-k_3-1} z^{(s_v(\ell_3-1)+k_3) \frac{r_\mu}{r_v} - \ell_3 s_\mu} \end{aligned}$$

$$\begin{aligned}
 & \times \left(\frac{t_\lambda}{t_\mu} \right)^{\ell_4-1} z_4^{-k_4-1} z^{(s_\lambda(\ell_4-1)+k_4) \frac{r_\mu}{r_\lambda} - \ell_4 s_\mu} \\
 & = -\frac{1}{t_\mu^2} \sum_{\substack{k_1, k_2, k_3, k_4 \geq 1 \\ \ell'_3, \ell'_4 \geq 0}} k_2 k_3 k_4 \left(\frac{t_\nu}{t_\mu} \right)^{\ell_\nu(k_3) + \ell'_3 r_\nu} \left(\frac{t_\lambda}{t_\mu} \right)^{\ell_\lambda(k_4) + \ell'_4 r_\lambda} \left(\prod_{m=1}^4 z_m^{-k_m-1} \right) \\
 & \quad \times \delta_{k_1+k_2-a_{\mu,\nu}(k_3) + \ell'_3(r_\mu s_\nu - r_\nu s_\mu) - a_{\mu,\lambda}(k_4) + \ell'_4(r_\mu s_\lambda - r_\lambda s_\mu), 0},
 \end{aligned}$$

where we defined $a_{\mu,\nu}(k_3)$ and $a_{\mu,\lambda}(k_4)$ as in (6.6). Again, the key observation to make is that $a_{\mu,\nu}(1) > 0$. We have already shown this in (6.7) for two cases. In the other two remaining cases, a straightforward calculation shows that $a_{\mu,\nu}(1) = 1$ in case $r_\mu = -1 \bmod s_\mu$ and $r_\nu = 1 \bmod s_\nu$, and $a_{\mu,\nu}(1) = (s_\mu - 1)(s_\nu - 1)$ if $r_\mu = 1 \bmod s_\mu$ and $r_\nu = -1 \bmod s_\nu$ confirming that indeed $a_{\mu,\nu}(1)$ is a strictly positive integer. Thus, the residue

$$\begin{aligned}
 & \text{Res}_{z_1=0} \cdots \text{Res}_{z_4=0} z_1^{a_{\mu,\nu}(1)} z_2^{a_{\mu,\lambda}(1)} z_3 z_4 \omega_{0,4} \left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \\
 & = -\frac{a_{\mu,\lambda}(1)}{t_\mu^2} \sum_{\ell'_3, \ell'_4 \geq 0} \left(\frac{t_\nu}{t_\mu} \right)^{\ell_\nu(1) + \ell'_3 r_\nu} \left(\frac{t_\lambda}{t_\mu} \right)^{\ell_\lambda(1) + \ell'_4 r_\lambda} \delta_{\ell'_3(r_\mu s_\nu - r_\nu s_\mu) + \ell'_4(r_\mu s_\lambda - r_\lambda s_\mu), 0} \\
 & = -\frac{a_{\mu,\lambda}(1)}{t_\mu^2} \left(\frac{t_\nu}{t_\mu} \right)^{\ell_\nu(1)} \left(\frac{t_\lambda}{t_\mu} \right)^{\ell_\lambda(1)}
 \end{aligned}$$

is non-vanishing and hence $\omega_{0,4} \left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \neq 0$ giving us the desired contradiction.

Now if $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu} > \frac{r_\lambda}{s_\lambda}$, we can proceed similarly. This time the correlator receives contributions from (1), (3a), and (3b) leading to

$$\begin{aligned}
 \frac{\omega_{0,4} \left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right)}{dz_1 dz_2 dz_3 dz_4} & = -\frac{1}{t_\mu} \sum_{\substack{k_1, k_2, k_3, k_4 \geq 1 \\ \ell'_4 \geq 0}} k_2 k_3 k_4 \frac{t_\mu^{r_\mu - \ell_\mu(k_3) - 1} t_\nu^{\ell_\mu(k_3)}}{t_\mu^{r_\mu} - t_\nu^{r_\mu}} \left(\frac{t_\lambda}{t_\mu} \right)^{\ell_\lambda(k_4) + \ell'_4 r_\lambda} \\
 & \quad \times \left(\prod_{m=1}^4 z_m^{-k_m-1} \right) \delta_{k_1+k_2+k_3-s_\mu - a_{\mu,\lambda}(k_4) + \ell'_4(r_\mu s_\lambda - r_\lambda s_\mu), 0}.
 \end{aligned}$$

Therefore, also in this case we find that $\omega_{0,4} \left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \neq 0$ as

$$\begin{aligned}
 & \text{Res}_{z_1=0} \cdots \text{Res}_{z_4=0} z_1^{s_\mu-1} z_2^{a_{\mu,\lambda}(1)} z_3 z_4 \omega_{0,4} \left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \\
 & = \frac{a_{\mu,\lambda}(1)}{t_\mu} \frac{t_\mu^{r_\mu - \ell_\mu(1) - 1} t_\nu^{\ell_\mu(1)}}{t_\nu^{r_\mu} - t_\mu^{r_\mu}} \left(\frac{t_\lambda}{t_\mu} \right)^{\ell_\lambda(1)} \neq 0.
 \end{aligned}$$

Similarly, if $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu} = \frac{r_\lambda}{s_\lambda}$, an analogous calculation gives

$$\text{Res}_{z_1=0} \cdots \text{Res}_{z_4=0} z_1^{s_\mu-1} z_2^{s_\mu-1} z_3 z_4 \omega_{0,4} \left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right) \neq 0. \quad \blacksquare$$

6.1.2. The Bouchard–Eynard formula for (0, 3) does not hold in general. Let us compare $\omega_{0,3}$ with

$$\begin{aligned}\check{\omega}_{0,3}\left(\begin{matrix} \mu_1 & \mu_2 & \mu_3 \\ z_1 & z_2 & z_3 \end{matrix}\right) &= \sum_{\mu} \operatorname{Res}_{z=\mu} \frac{\omega_{0,2}(z, z_1)\omega_{0,2}(z, z_2)\omega_{0,2}(z, z_3)}{dx(z)dy(z)} \\ &= - \sum_{\mu} \delta_{\mu_1, \mu_2, \mu_3, \mu} \sum_{k_1, k_2, k_3 > 0} \frac{k_1 k_2 k_3 dz_1 dz_2 dz_3}{z_1^{k_1+1} z_2^{k_2+1} z_3^{k_3+1}} \frac{\delta_{k_1+k_2+k_3, s_{\mu}}}{t_{\mu} r_{\mu} (s_{\mu} - r_{\mu})}.\end{aligned}$$

According to [14, Proposition 11], $\check{\omega}_{0,3} = \omega_{0,3}$ for regularly admissible spectral curves. More generally, if the spectral curve is admissible, comparing with Proposition 6.1, we see that $\omega_{0,3} = \check{\omega}_{0,3}$ if and only if $c_{\mu} = \frac{1}{s_{\mu} - r_{\mu}}$ or $s_{\mu} \leq 2$ for each $\mu \in \tilde{\alpha}$. The condition $c_{\mu} = \frac{1}{s_{\mu} - r_{\mu}}$ implies that $s_{\mu} = r_{\mu} + 1$, which by property (ii) of Proposition 6.1 can only hold for one μ . Comparing with Theorem 2.11, we see that, e.g., $(r_1, s_1, r_2, s_2) = (5, 3, 3, 2)$ gives an Airy structure for which $\omega_{0,3} \neq \check{\omega}_{0,3}$. We conclude that [14, Proposition 11] does not always hold in our general setup.

6.1.3. $(g, n) = (\frac{1}{2}, 2)$. In Section 5.4, we already asked the question whether topological recursion applied to any curve admissible in genus 0, which in this section we will continue to assume to be of the form (6.1), together with a choice of $\omega_{0,2}$ and $\omega_{\frac{1}{2},1}$ will yield symmetric $\omega_{g,n}$ not just for $g = 0$ but also for $g > 0$. By studying the symmetry of $\omega_{\frac{1}{2},2}$, we will see that this is not always possible unless the parameters Q_{μ} in $\omega_{\frac{1}{2},1}\left(\begin{matrix} \mu \\ z \end{matrix}\right) = \frac{Q_{\mu} dz}{z}$ satisfy certain compatibility conditions. Let us first look at two examples.

Example 6.5. Let us consider the two-component curve of type $(r_1, s_1) = (r_2, s_2) = (3, 2)$ and choose $t_1 = -t_2 = 1$. Then the topological recursion gives us

$$\begin{aligned}\omega_{\frac{1}{2},2}\left(\begin{matrix} 1 & 1 \\ z_1 & z_2 \end{matrix}\right) &= -\left(\frac{Q_1}{3} + \frac{Q_2}{2}\right) \frac{dz_1 dz_2}{z_1^2 z_2^2}, \\ \omega_{\frac{1}{2},2}\left(\begin{matrix} 2 & 2 \\ z_1 & z_2 \end{matrix}\right) &= \left(\frac{Q_1}{2} + \frac{Q_2}{3}\right) \frac{dz_1 dz_2}{z_1^2 z_2^2},\end{aligned}$$

which are always symmetric under the exchange of arguments, while for arguments lying on different components, we find that

$$\omega_{\frac{1}{2},2}\left(\begin{matrix} 1 & 2 \\ z_1 & z_2 \end{matrix}\right) = \frac{Q_1}{2} \frac{dz_1 dz_2}{z_1^2 z_2^2}, \quad \omega_{\frac{1}{2},2}\left(\begin{matrix} 2 & 1 \\ z_1 & z_2 \end{matrix}\right) = -\frac{Q_2}{2} \frac{dz_1 dz_2}{z_1^2 z_2^2},$$

which means that $\omega_{\frac{1}{2},2}$ can only be symmetric if $Q_1 + Q_2 = 0$.

Example 6.6. Let us look at the two-component curve with $(r_1, s_1) = (8, 3)$ and $(r_2, s_2) = (5, 3)$. Furthermore, we choose $t_1 = t_2 = 1$. In this case, $\omega_{\frac{1}{2},2}$ vanishes for

arguments lying on different components and

$$\begin{aligned}\omega_{\frac{1}{2},2}\left(\begin{matrix} 1 & 1 \\ z_1 & z_2 \end{matrix}\right) &= -\frac{5Q_1 + 8Q_2}{4} \frac{dz_1 dz_2}{z_1^2 z_2^3} - \frac{Q_1 + 4Q_2}{4} \frac{dz_1 dz_2}{z_1^3 z_2^2}, \\ \omega_{\frac{1}{2},2}\left(\begin{matrix} 2 & 2 \\ z_1 & z_2 \end{matrix}\right) &= -\frac{6Q_2}{5} \frac{dz_1 dz_2}{z_1^2 z_2^3} - \frac{Q_2}{5} \frac{dz_1 dz_2}{z_1^3 z_2^2}.\end{aligned}$$

For the above to be symmetric under the exchange of z_1 and z_2 , we need $Q_1 = Q_2 = 0$ which means in this case there is no ambiguity in choosing the Q_μ .

These two examples are special instances of the following classification.

Proposition 6.7. *Assume $(r_\mu, s_\mu)_{\mu \in \tilde{\alpha}}$ are integers satisfying condition (i) and (ii) from Proposition 6.1. Then $\omega_{\frac{1}{2},2}$ is symmetric if and only if*

(iv) *If $s_\mu > 2$ and $r_\mu = 1 \pmod{s_\mu}$ for $\mu \in \tilde{\alpha}$, then*

$$\sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} Q_v = 0 \quad \text{and} \quad \forall \ell \in [s_\mu - 3], \quad \sum_{\substack{v \neq \mu \\ s_v = 1, \lfloor \frac{r_\mu}{s_\mu} \rfloor = r_v}} Q_v t_v^{r_v \ell} = 0.$$

(v) *If $s_\mu > 2$ and $r_\mu = -1 \pmod{s_\mu}$ for $\mu \in \tilde{\alpha}$, then*

$$Q_\mu + \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} Q_v = 0 \quad \text{and} \quad \forall \ell \in [s_\mu - 3], \quad \sum_{\substack{v \neq \mu \\ s_v = 1, \lceil \frac{r_\mu}{s_\mu} \rceil = r_v}} Q_v t_v^{-r_v \ell} = 0.$$

(vi) *For all $\mu \neq v$ with $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_v}{s_v} \rfloor$, $r_\mu = -1 \pmod{s_\mu}$, $r_v = 1 \pmod{s_v}$ and $s_\mu, s_v > 1$, we have*

$$Q_\mu = -Q_v.$$

If these conditions are satisfied, we have if $r_\mu = r'_\mu s_\mu + 1$,

$$\begin{aligned}\omega_{\frac{1}{2},2}\left(\begin{matrix} \mu & \mu \\ z_1 & z_2 \end{matrix}\right) &= \sum_{k_1, k_2 > 0} \frac{k_1 k_2 dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} \left(-\frac{Q_\mu r'_\mu}{t_\mu r_\mu} \delta_{k_1+k_2, s_\mu} \right. \\ &+ \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} = \frac{r_v}{s_v}}} \frac{t_\mu^{r_\mu-1}}{t_v^{r_\mu} - t_\mu^{r_\mu}} Q_v \delta_{k_1+k_2, s_\mu} \delta_{s_\mu, 2} \\ &- \sum_{\substack{v \neq \mu \\ s_v = 1, \lfloor \frac{r_\mu}{s_\mu} \rfloor = r_v}} \frac{Q_v}{t_\mu} \left(\frac{t_v}{t_\mu} \right)^{r_v (s_\mu-2)} \delta_{k_1+k_2, 2} \delta_{s_\mu > 2} \\ &+ \left. \sum_{\substack{v \neq \mu: s_v = 2, \\ \frac{r_\mu}{s_\mu} < \frac{r_v}{s_v}, \lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_v}{s_v} \rfloor}} \frac{Q_v}{t_\mu} (t_\mu t_v)^{r_v} \delta_{k_1+k_2, 2} \delta_{s_\mu > 2} \right), \quad (6.8)\end{aligned}$$

and if $r_\mu = r'_\mu s_\mu + s_\mu - 1$ and $s_\mu > 2$, we have

$$\begin{aligned} \omega_{\frac{1}{2},2}\left(\begin{matrix} \mu & \mu \\ z_1 & z_2 \end{matrix}\right) &= \sum_{k_1, k_2 > 0} \frac{k_1 k_2 dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} \left(\frac{Q_\mu(r'_\mu + 1)}{t_\mu r_\mu} \delta_{k_1+k_2, s_\mu} \right. \\ &+ \sum_{\substack{v \neq \mu \\ s_v=1, \lfloor \frac{r'_\mu}{s_\mu} \rfloor = r_v}} \frac{Q_v(t_\mu)}{t_\mu} \left(\frac{t_\mu}{t_v} \right)^{r_v(s_\mu-2)} \delta_{k_1+k_2, 2} \delta_{s_\mu > 2} \\ &- \sum_{\substack{v \neq \mu: s_v=2, \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}, \lfloor \frac{r'_\mu}{s_\mu} \rfloor = \lfloor \frac{r_v}{s_v} \rfloor}} \frac{Q_v(t_v)}{t_\mu} \left(\frac{t_v}{t_\mu} \right)^{r_v} \delta_{k_1+k_2, 2} \delta_{s_\mu > 2} \Big). \end{aligned} \quad (6.9)$$

Moreover, for $\mu \neq v$ we have

$$\begin{aligned} \omega_{\frac{1}{2},2}\left(\begin{matrix} \mu & v \\ z_1 & z_2 \end{matrix}\right) &= \begin{cases} -\frac{dz_1 dz_2}{z_1^2 z_2^2} \frac{Q_\mu(t_\mu)}{t_\mu} \left(\frac{t_\mu}{t_v} \right)^{r'_\mu} & \text{if } (r_\mu, s_\mu), (r_v, s_v) \text{ satisfy (vi) and } \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}, \\ -\frac{dz_1 dz_2}{z_1^2 z_2^2} \frac{Q_\mu t'_\mu r'_v}{t'_\mu r'_\mu - t'_v r'_\mu} & \text{if } r_\mu = r_v \text{ and } s_\mu = s_v = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (6.10)$$

Before proving Proposition 6.7, we need the following technical fact.

Lemma 6.8. *Let $r_1, r_2, s_1, s_2 > 0$ with r_μ and s_μ coprime and $\frac{r_1}{s_1} > \frac{r_2}{s_2}$. Further assume the integers satisfy all constraints from Proposition 6.1. Then*

$$\Delta := r_1 s_2 - r_2 s_1$$

takes the following values:

- $\Delta = 1$ if $s_1 = 1$, $r_1 = \lfloor \frac{r_2}{s_2} \rfloor + 1$, and $r_2 = -1 \pmod{s_2}$ or we have $s_2 = 1$, $r_2 = \lfloor \frac{r_1}{s_1} \rfloor$, and $r_1 = 1 \pmod{s_1}$.
- $\Delta = \max\{s_1, s_2\} - 2$ if $s_1 = 2$, $\lfloor \frac{r_1}{s_1} \rfloor = \lfloor \frac{r_2}{s_2} \rfloor$, and $r_2 = 1 \pmod{s_2}$ or we have $s_2 = 2$, $\lfloor \frac{r_1}{s_1} \rfloor = \lfloor \frac{r_2}{s_2} \rfloor$, and $r_1 = -1 \pmod{s_1}$.
- Otherwise $\Delta \geq \max\{s_1, s_2\} - 1$.

Proof. Let us write $r_\mu := r'_\mu s_\mu + r''_\mu$ with $r''_\mu \in [0, s_\mu)$. Then due to (i), we know that $r''_\mu \in \{1, s_\mu - 1\}$. We can thus rewrite

$$\Delta = s_1 s_2 (r'_1 - r'_2) + r''_1 s_2 - r''_2 s_1. \quad (6.11)$$

By simply plugging in the indicated values for r_μ, s_μ , it is easy to see that they indeed produce $\Delta \in \{1, \max\{s_1, s_2\} - 2\}$. For instance, setting $s_1 = 1$, $r_1 = r'_2 + 1$, and $r''_2 = s_2 - 1$, we directly obtain $\Delta = 1$ as was claimed.

It is therefore only left to prove that in all other cases $\Delta \geq \max\{s_1, s_2\} - 1$. To do so, let us first assume $r'_1 = r'_2$, in which case (6.11) reduces to

$$\Delta = r''_1 s_2 - r''_2 s_1 = \begin{cases} s_1 - s_2 & \text{if } r''_1 = s_1 - 1, \quad r''_2 = s_2 - 1, \\ s_1 s_2 - s_1 - s_2 & \text{if } r''_1 = s_1 - 1, \quad r''_2 = 1, \\ s_1 + s_2 - s_1 s_2 & \text{if } r''_1 = 1, \quad r''_2 = s_2 - 1, \\ s_2 - s_1 & \text{if } r''_1 = 1, \quad r''_2 = 1. \end{cases}$$

Let us go through the cases separately. First assume $r''_1 = s_1 - 1$ and $r''_2 = s_2 - 1$. In this case, (ii) forbids $s_1, s_2 > 2$. Notice now that since we assume that $\frac{r_1}{s_1} > \frac{r_2}{s_2}$, necessarily $\Delta > 0$ and therefore the only allowed cases are $s_2 = 1$ with $s_1 > 1$ or $s_2 = 2$ with $s_1 > 2$. While $s_2 = 1$ and $s_1 > 1$ gives $\Delta = s_2 - 1 = \max\{s_1, s_2\} - 1$, considering the second case yielded $\Delta = s_2 - 2 = \max\{s_1, s_2\} - 2$.

For our further analysis of the case $r'_1 = r'_2$, we can assume that $s_1, s_2 > 2$. Then if $r''_1 = s_1 - 1$ and $r''_2 = 1$, one has

$$\Delta = s_1(s_2 - 1) - s_2 \geq 3(s_2 - 1) - s_2 = 2s_2 - 3 \geq s_2,$$

and similarly also $\Delta \geq s_1$. Now assume $r''_1 = 1$ and $r''_2 = s_2 - 1$. Then due to $s_1, s_2 > 2$, one has

$$\Delta = s_2 - s_1(s_2 - 1) < s_2 - 2(s_2 - 1) = 2 - s_2 < 0,$$

which contradicts the assumption that $\frac{r_1}{s_1} > \frac{r_2}{s_2}$. Finally, notice that the case $r''_1 = 1$ and $r''_2 = 1$ for $s_1, s_2 > 2$ is forbidden by (ii).

Now assume that $r'_1 \neq r'_2$. Due to $\frac{r_1}{s_1} > \frac{r_2}{s_2}$, we know that $r'_1 > r'_2$, which taking into account (6.11) implies that

$$\Delta \geq s_1 s_2 + r''_1 s_2 - r''_2 s_1 = s_1(s_2 - r''_2) + r''_1 s_2 \geq s_1 + r''_1 s_2,$$

which is always larger than $\max\{s_1, s_2\} - 1$ unless $r''_1 = 0$. However, if $r''_1 = 0$, then necessarily $s_1 = 1$. Then either $r'_1 = r'_2 + 1$ implying that

$$\Delta = s_2 - r''_2 = \begin{cases} 1 & \text{if } r''_2 = s_2 - 1, \\ s_2 - 1 & \text{if } r''_2 = 1 \end{cases}$$

or we have $r'_1 > r'_2 + 1$ yielding

$$\Delta \geq 2s_2 - r''_2 \geq s_2 > \max\{s_1, s_2\} - 1.$$

Note that the cases above, in which $\Delta = 1$, are exactly those considered in the beginning of the proof. ■

Proof of Proposition 6.7. Let us start by computing $\omega_{\frac{1}{2},2}(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix})$. Inspecting equation (6.2), it should be clear that one may proceed as in the computation of $\omega_{0,n}$. While the second term in the bracket in (6.2) gives a contribution

$$\begin{aligned} & \omega_{\frac{1}{2},2}(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}) \\ &= \dots + \operatorname{Res}_{z=0} \sum_{a=1}^{r_\mu-1} \frac{z dz_1}{z_1(z_1-z)} \frac{\vartheta_\mu^a}{r_\mu t_\mu (\vartheta_\mu^{a s_\mu} - 1) z^{s_\mu-1}} \frac{dz_2}{(\vartheta_\mu^a z - z_2)^2} \frac{Q_\mu dz}{z} \\ &= \dots + \sum_{k_1, k_2 > 0} \frac{dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} k_2 Q_\mu \delta_{k_1+k_2, s_\mu} \frac{2\ell_\mu(k_2) - r_\mu + 1}{2r_\mu t_\mu}, \end{aligned} \quad (6.12)$$

the first one gets a contribution of type (1) and additionally a factor

$$\frac{1}{r_\nu (t_\nu \vartheta_\nu^{b s_\nu} z^{r_\mu s_\nu / r_\nu - 1} - t_\mu z^{s_\mu - 1})} \frac{Q_\nu dz}{z},$$

where one sums over $b \in [r_\nu]$ if $\nu \neq \mu$ and over $b \in [r_\mu - 1]$ if $\nu = \mu$. This gives

$$\begin{aligned} & \omega_{\frac{1}{2},2}(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}) \\ &= \operatorname{Res}_{z=0} \sum_{\nu \in \tilde{\alpha}} \sum_{b=1}^{r_\mu - \delta_{\mu, \nu}} \frac{z dz_1}{z_1(z_1-z)} \frac{dz dz_2}{(z-z_2)^2} \frac{Q_\nu}{r_\nu (t_\nu \vartheta_\nu^{b s_\nu} z^{r_\mu \frac{s_\nu}{r_\nu} - 1} - t_\mu z^{s_\mu - 1})} \frac{dz}{z} + \dots \\ &= \sum_{k_1, k_2 > 0} \frac{dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} \left\{ k_2 Q_\mu \delta_{k_1+k_2, s_\mu} \frac{\ell_\mu(k_2) - r_\mu + 1}{r_\mu t_\mu} \right. \\ & \quad + k_2 \sum_{\substack{\nu \neq \mu \\ \frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}}} \frac{t_\mu^{r_\mu-1}}{t_\nu^{r_\mu} - t_\mu^{r_\mu}} Q_\nu \delta_{k_1+k_2, s_\mu} \\ & \quad - k_2 \sum_{\substack{\nu \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}}} \sum_{\ell' \geq 0} \frac{t_\nu^{r_\nu \ell'}}{t_\mu^{r_\nu \ell' + 1}} Q_\nu \delta_{k_1+k_2+\ell'(r_\mu s_\nu - r_\nu s_\mu), s_\mu} \\ & \quad \left. + k_2 \sum_{\substack{\nu \neq \mu \\ \frac{r_\mu}{s_\mu} < \frac{r_\nu}{s_\nu}}} \sum_{\ell' > 0} \frac{t_\mu^{r_\nu \ell' - 1}}{t_\nu^{r_\nu \ell'}} Q_\nu \delta_{k_1+k_2-\ell'(r_\mu s_\nu - r_\nu s_\mu), s_\mu} \right\}, \end{aligned} \quad (6.13)$$

where in the second line we also included the contribution from (6.12). Let us further on use the notation

$$\omega_{\frac{1}{2},2}(\begin{smallmatrix} \mu_1 & \mu_2 \\ z_1 & z_2 \end{smallmatrix}) = \sum_{k_1, k_2 > 0} \frac{dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} F_{\frac{1}{2},2}[\begin{smallmatrix} \mu_1 & \mu_2 \\ k_1 & k_2 \end{smallmatrix}].$$

Inspecting (6.13), we notice that for $s_\mu \leq 2$ the components $F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \mu \\ k_1 & k_2 \end{smallmatrix}]$ are always symmetric under the exchange of arguments. To be more precise, for $s_\mu = 1$ the components all vanish and for $s_\mu = 2$ they get a contribution from the first and second line of (6.13) solely. Hence, using that $\ell_\mu(1) = r'_\mu$ for $s_\mu = 2$, we see that in this case $F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \mu \\ k_1 & k_2 \end{smallmatrix}]$ is indeed given by (6.8).

Now let us assume that $s_\mu > 2$. Then due to property (ii) of Proposition 6.1, the contribution $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ in (6.13) has to vanish and one ends up with

$$\begin{aligned}
F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \mu \\ k_1 & k_2 \end{smallmatrix}] &= -\delta_{k_1+k_2, s_\mu} k_2 \left(\frac{Q_\mu(r_\mu - 1 - \ell_\mu(k_2))}{r_\mu t_\mu} + \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}}} \frac{Q_\nu}{t_\mu} \right) \\
&+ \frac{k_2}{t_\mu} \sum_{\ell' > 0} \left(- \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}}} \left(\frac{t_\nu}{t_\mu} \right)^{r_\nu \ell'} Q_\nu \delta_{k_1+k_2+\ell'(r_\mu s_\nu - r_\nu s_\mu), s_\mu} \right. \\
&+ \left. \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} < \frac{r_\nu}{s_\nu}}} \left(\frac{t_\mu}{t_\nu} \right)^{r_\nu \ell'} Q_\nu \delta_{k_1+k_2-\ell'(r_\mu s_\nu - r_\nu s_\mu), s_\mu} \right). \tag{6.14}
\end{aligned}$$

Since $r_\mu s_\nu - r_\nu s_\mu \neq 0$ for $\frac{r_\mu}{s_\mu} \neq \frac{r_\nu}{s_\nu}$, we can analyse the symmetry constraints coming from the first and second line of (6.14) individually.

First, assume $r_\mu = r'_\mu s_\mu + 1$. Then if $k_1 + k_2 = s_\mu$, we can use that

$$r_\mu - 1 - \ell_\mu(k_2) = r_\mu - 1 - r'_\mu k_2 = r'_\mu k_1$$

to find that

$$F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \mu \\ k_1 & k_2 \end{smallmatrix}] = -k_1 k_2 \frac{Q_\mu r'_\mu}{r_\mu t_\mu} - k_2 \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}}} \frac{Q_\nu}{t_\mu}. \tag{6.15}$$

This expression is symmetric under the exchange of k_1 and k_2 if and only if

$$\sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}}} Q_\nu = 0. \tag{6.16}$$

Let us proceed by analysing the symmetry constraints coming from coefficients with $k_1 + k_2 < s_\mu$, i.e., we consider the second line of (6.14). Clearly, the case $s_\mu = 3$ is symmetric. Therefore, now assume that $s_\mu > 3$. In this case, we can only expect a non-vanishing $F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \mu \\ k_1 & k_2 \end{smallmatrix}]$ if $|r_\mu s_\nu - r_\nu s_\mu| \in [s_\mu - 3]$ for some $\nu \neq \mu$. Due to Lemma 6.8, we know that for all ν either $|r_\mu s_\nu - r_\nu s_\mu| = 1$ or $|r_\mu s_\nu - r_\nu s_\mu| \geq s_\mu - 2$, which limits the cases where $F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \mu \\ k_1 & k_2 \end{smallmatrix}] \neq 0$ for $k_1 + k_2 < s_\mu$ severely. To be more

precise, Lemma 6.8 tells us that $|r_\mu s_\nu - r_\nu s_\mu| = 1$ if and only if $s_\nu = 1$ and $r_\nu = \lfloor \frac{r_\mu}{s_\mu} \rfloor$. Therefore, for $2 < k_1 + k_2 < s_\mu$, we have

$$\begin{aligned} F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right] &= -\frac{k_2}{t_\mu} \sum_{\ell' > 0} \sum_{\substack{v \neq \mu \\ s_v = 1, r_v = \lfloor \frac{r_\mu}{s_\mu} \rfloor}} \left(\frac{t_v}{t_\mu} \right)^{r_v \ell'} Q_v \delta_{k_1 + k_2 + \ell', s_\mu} \\ &= -\frac{k_2}{t_\mu} \sum_{\substack{v \neq \mu \\ s_v = 1, r_v = \lfloor \frac{r_\mu}{s_\mu} \rfloor}} \left(\frac{t_v}{t_\mu} \right)^{r_v (s_\mu - k_1 - k_2)} Q_v, \end{aligned} \quad (6.17)$$

which is symmetric if and only if for all $\ell \in [s_\mu - 3]$, we have

$$\sum_{\substack{v \neq \mu \\ s_v = 1, r_v = \lfloor \frac{r_\mu}{s_\mu} \rfloor}} \left(\frac{t_v}{t_\mu} \right)^{r_v \ell} Q_v = 0.$$

Together with (6.16) this explains symmetry condition (iv). Regarding the formula for $F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right]$ stated in (6.8) notice that for $s_\mu = 2$, we have

$$F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right] = -\frac{Q_\mu r'_\mu}{r_\mu t_\mu} \delta_{k_1 + k_2, 2} + \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} = \frac{r_v}{s_v}}} \frac{t_\mu^{r_\mu - 1}}{t_v^{r_\mu} - t_\mu^{r_\mu}} Q_v \delta_{k_1 + k_2, s_\mu}$$

while for $s_\mu > 2$ one has

$$\begin{aligned} F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right] &= -\frac{Q_\mu r'_\mu}{r_\mu t_\mu} \delta_{k_1 + k_2, s_\mu} - \sum_{\substack{v \neq \mu \\ s_v = 1, \lfloor \frac{r_\mu}{s_\mu} \rfloor = r_v}} \frac{Q_v}{t_\mu} \left(\frac{t_v}{t_\mu} \right)^{r_v (s_\mu - 2)} \delta_{k_1 + k_2, 2} \\ &+ \sum_{\substack{v \neq \mu: s_v = 2, \\ \frac{r_\mu}{s_\mu} < \frac{r_v}{s_v}, \lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_v}{s_v} \rfloor}} \frac{Q_v}{t_\mu} \left(\frac{t_\mu}{t_v} \right)^{r_v} \delta_{k_1 + k_2, 2}. \end{aligned}$$

In order to obtain the first two terms in the above expression, one applies the symmetry constraint (iv) on (6.15) and (6.17). The third term is due to contributions $v \neq \mu$ in (6.14), for which $|r_\mu s_\nu - r_\nu s_\mu| = s_\mu - 2$. Lemma 6.8 tells us that this is the case for exactly those $v \neq \mu$ with $s_v = 2$, $\frac{r_\mu}{s_\mu} < \frac{r_v}{s_v}$, and $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_v}{s_v} \rfloor$. This now explains the origin of all terms occurring in (6.8), which closes the analysis of the case $r_\mu = r'_\mu s_\mu + 1$.

Now assume that $r_\mu = r'_\mu s_\mu + s_\mu - 1$ and $s_\mu > 2$. First, let us inspect $F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right]$ given by (6.14) for $k_1 + k_2 = s_\mu$. In this case, we can use that

$$r_\mu - 1 - \ell_\mu(k_2) = r_\mu - k_1(r'_\mu + 1)$$

in order to find that

$$F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right] = k_1 k_2 \frac{Q_\mu (r'_\mu + 1)}{r_\mu t_\mu} - k_2 \left(\frac{Q_\mu}{t_\mu} + \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} \frac{Q_v}{t_\mu} \right)$$

for $k_1 + k_2 = s_\mu$. This is symmetric if and only if

$$Q_\mu + \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} Q_v = 0. \quad (6.18)$$

Now let us turn to the case $k_1 + k_2 < s_\mu$. Again, we may only expect a contribution from $v \neq \mu$ in the second line of (6.14) possibly leading to a non-symmetric term if $|r_\mu s_v - r_v s_\mu| = 1$. Now Lemma 6.8 says that $|r_\mu s_v - r_v s_\mu| = 1$ if and only if $s_v = 1$ and $r_v = \lceil \frac{r_\mu}{s_\mu} \rceil$. Thus, for $2 < k_1 + k_2 < s_\mu$, we find that

$$F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \mu \\ k_1 & k_2 \end{matrix} \right] = \frac{k_2}{t_\mu} \sum_{\substack{v \neq \mu \\ s_v=1, r_v=\lceil \frac{r_\mu}{s_\mu} \rceil}} \left(\frac{t_\mu}{t_v} \right)^{r_v (s_\mu - k_1 - k_2)} Q_v,$$

which is symmetric if and only if

$$\sum_{\substack{v \neq \mu \\ s_v=1, r_v=\lceil \frac{r_\mu}{s_\mu} \rceil}} \left(\frac{t_\mu}{t_v} \right)^{r_v \ell} Q_v = 0$$

for all $\ell \in [s_\mu - 3]$. This condition together with (6.18) are of course nothing but (v). Since the derivation of formula (6.9) is in line with the one of (6.8), we omit a further discussion.

Now let us consider $F_{\frac{1}{2},2} \left[\begin{matrix} \mu & v \\ k_1 & k_2 \end{matrix} \right]$ for $v \neq \mu$. First, assume $\frac{r_\mu}{s_\mu} = \frac{r_v}{s_v}$. Note that in this case, property (ii) of Proposition 6.1 forces $s_\mu = s_v \leq 2$. Using the same approach as in the derivation of (6.13), one finds that

$$F_{\frac{1}{2},2} \left[\begin{matrix} \mu & v \\ k_1 & k_2 \end{matrix} \right] = -k_2 \frac{Q_\mu t_\mu^{\ell_\mu(k_1)} t_v^{\ell_\mu(k_2)}}{t_\mu^{r_\mu} - t_v^{r_v}} \delta_{k_1+k_2, s_\mu}. \quad (6.19)$$

For $s_\mu = s_v = 1$, this expression is always vanishing and hence symmetric. Conversely, for $s_\mu = s_v = 2$, we have

$$F_{\frac{1}{2},2} \left[\begin{matrix} \mu & v \\ k_1 & k_2 \end{matrix} \right] = F_{\frac{1}{2},2} \left[\begin{matrix} v & \mu \\ k_2 & k_1 \end{matrix} \right] \quad \text{if and only if} \quad Q_\mu = -Q_v,$$

which is captured in condition (vi).

Now let us consider the case $\frac{r_\mu}{s_\mu} \neq \frac{r_\nu}{s_\nu}$. Without loss of generality, we can assume that $\frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}$. In this case, we have

$$F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \nu \\ k_1 & k_2 \end{matrix} \right] = -k_2 \frac{Q_\mu}{t_\mu} \sum_{\ell' \geq 0} \left(\frac{t_\nu}{t_\mu} \right)^{r_\nu \ell' + \ell_\nu(k_2)} \delta_{k_1 + (k_2 + s_\nu \ell_\nu(k_2)) \frac{r_\mu}{r_\nu} - s_\mu \ell_\nu(k_2) + \ell'(r_\mu s_\nu - r_\nu s_\mu), s_\mu}. \quad (6.20)$$

Let us first simplify this expression before turning to $F_{\frac{1}{2},2} \left[\begin{matrix} \nu & \mu \\ k_2 & k_1 \end{matrix} \right]$. Comparing formula (6.20) with (6.5), we can deduce that property (ii) of Proposition 6.1 ensures that $F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \nu \\ k_1 & k_2 \end{matrix} \right] = 0$ unless $k_1 = 1$. And moreover, also $F_{\frac{1}{2},2} \left[\begin{matrix} \mu & \nu \\ k_1 & k_2 \end{matrix} \right] = 0$ unless

$$1 + (k_2 + s_\nu \ell_\nu(k_2)) \frac{r_\mu}{r_\nu} - s_\mu \ell_\nu(k_2) = s_\mu. \quad (6.21)$$

This relation cannot be satisfied for $k_2 \geq s_\nu$. Indeed, if $r_\nu = 1 \pmod{s_\nu}$, then

$$\begin{aligned} (k_2 + s_\nu \ell_\nu(k_2)) \frac{r_\mu}{r_\nu} - s_\mu \ell_\nu(k_2) &= k_2 (r_\mu - r'_\nu s_\mu) - \left\lfloor \frac{k_2 r'_\nu}{r_\nu} \right\rfloor (r_\mu s_\nu - r_\nu s_\mu) \\ &= \frac{s_\mu}{s_\nu} k_2 + \left(\frac{k_2}{s_\nu} - \left\lfloor \frac{k_2 r'_\nu}{r_\nu} \right\rfloor \right) (r_\mu s_\nu - r_\nu s_\mu) \\ &\geq \frac{s_\mu}{s_\nu} k_2 \geq s_\mu \end{aligned}$$

for all $k_2 \geq s_\nu$ where we used that

$$\ell_\nu(k) = \begin{cases} k r'_\nu - \left\lfloor \frac{k r'_\nu}{r_\nu} \right\rfloor r_\nu & \text{if } r_\nu = r'_\nu s_\nu + 1, \\ -k(r'_\nu + 1) + \left\lceil \frac{k(r'_\nu + 1)}{r_\nu} \right\rceil r_\nu & \text{if } r_\nu = r'_\nu s_\nu + s_\nu - 1 \end{cases}$$

and that $\left\lfloor \frac{k r'_\nu}{r_\nu} \right\rfloor \leq \frac{k}{s_\nu}$. The case $r_\nu = -1 \pmod{s_\nu}$ can be covered using similar arguments. On the other hand, for $1 \leq k_2 < s_\nu$, plugging in the explicit expression for $\ell_\nu(k_2)$, we find that (6.21) is equivalent to

$$s_\mu - 1 = \begin{cases} k_2 (r_\mu - s_\mu r'_\nu) & \text{if } r_\nu = r'_\nu s_\nu + 1, \\ s_\mu + (s_\nu - k_2) (r_\mu - s_\mu r'_\nu) & \text{if } r_\nu = r'_\nu s_\nu + s_\nu - 1. \end{cases} \quad (6.22)$$

Let us first analyse the case $r_\nu = r'_\nu s_\nu + 1$. Since

$$r_\mu - s_\mu r'_\nu = r''_\mu + s_\mu \left(\left\lfloor \frac{r_\mu}{s_\mu} \right\rfloor - \left\lfloor \frac{r_\nu}{s_\nu} \right\rfloor + \delta_{s_\nu, 1} \right), \quad (6.23)$$

equation (6.22) can only be satisfied if $s_\nu > 1$ and $\left\lfloor \frac{r_\mu}{s_\mu} \right\rfloor = \left\lfloor \frac{r_\nu}{s_\nu} \right\rfloor$. Suppose the latter is true, then since we assumed that $\frac{r_\mu}{s_\mu} > \frac{r_\nu}{s_\nu}$, we have $s_\nu r''_\mu > s_\mu$. Therefore, if we had $r''_\mu = 1$ and $s_\mu > 2$, this would violate property (ii) of Proposition 6.1, as in this

case $s_\nu, s_\mu > 2$. Hence, we can assume that $r''_\mu = s_\mu - 1$. Using (6.23), this means that (6.22) is satisfied if and only if $k_2 = 1$.

The case where $r_\nu = r'_\nu s_\nu + s_\nu - 1$ can be treated similarly. We can assume that $s_\nu > 2$ since $s_\nu \leq 2$ is a special case of the one considered before. Now since $r_\mu - s_\mu r'_\nu \geq 0$, we always have

$$s_\mu + (s_\nu - k_2)(r_\mu - s_\mu r'_\nu) \geq s_\mu,$$

implying that (6.22) cannot be satisfied for any $k_2 > 0$. Thus, $F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \nu \\ k_1 & k_2 \end{smallmatrix}] = 0$ for all $k_i > 0$ if $s_\nu > 2$ and $r_\nu = r'_\nu s_\nu + s_\nu - 1$. To sum up, assuming that (i) and (ii) from Proposition 6.1 hold, we have

$$F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \nu \\ k_1 & k_2 \end{smallmatrix}] = \begin{cases} -\frac{Q_\mu}{t_\mu} \left(\frac{t_\nu}{t_\mu}\right)^{\ell_\nu(1)} & \text{if } k_1 = k_2 = 1, s_\nu > 1, \lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_\nu}{s_\nu} \rfloor, \\ & r_\mu = -1 \pmod{s_\mu}, r_\nu = 1 \pmod{s_\nu}, \\ 0 & \text{otherwise.} \end{cases} \quad (6.24)$$

One can obtain a similar expression for $F_{\frac{1}{2},2}[\begin{smallmatrix} \nu & \mu \\ k_2 & k_1 \end{smallmatrix}]$. First, one finds that

$$F_{\frac{1}{2},2}[\begin{smallmatrix} \nu & \mu \\ k_2 & k_1 \end{smallmatrix}] = k_1 \sum_{\ell \geq 0} \frac{t_\nu^\ell}{t_\mu^{\ell+1}} Q_\nu \delta_{r_\mu | k_1 - (\ell+1)s_\mu} \delta_{k_2 + (k_1 - (\ell+1)s_\mu) \frac{r_\nu}{r_\mu} + \ell s_\nu, 0}.$$

Then comparing the above expression with (6.5), one can simplify the expression for $F_{\frac{1}{2},2}[\begin{smallmatrix} \nu & \mu \\ k_2 & k_1 \end{smallmatrix}]$ as we did with $F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \nu \\ k_1 & k_2 \end{smallmatrix}]$ before. One finds that

$$F_{\frac{1}{2},2}[\begin{smallmatrix} \nu & \mu \\ k_2 & k_1 \end{smallmatrix}] = \begin{cases} \frac{Q_\nu}{t_\nu} \left(\frac{t_\nu}{t_\mu}\right)^{r_\mu - \ell_\mu(1)} & \text{if } k_1 = k_2 = 1, s_\nu > 1, \lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_\nu}{s_\nu} \rfloor, \\ & r_\mu = -1 \pmod{s_\mu}, r_\nu = 1 \pmod{s_\nu}, \\ 0 & \text{otherwise.} \end{cases} \quad (6.25)$$

Now comparing (6.24) with (6.25), it is clear that

$$F_{\frac{1}{2},2}[\begin{smallmatrix} \mu & \nu \\ k_1 & k_2 \end{smallmatrix}] = F_{\frac{1}{2},2}[\begin{smallmatrix} \nu & \mu \\ k_2 & k_1 \end{smallmatrix}]$$

if and only if

$$\frac{Q_\mu}{t_\mu} \left(\frac{t_\nu}{t_\mu}\right)^{\ell_\nu(1)} = -\frac{Q_\nu}{t_\nu} \left(\frac{t_\nu}{t_\mu}\right)^{r_\mu - \ell_\mu(1)} \quad (6.26)$$

in the case $r_\mu = r'_\mu s_\mu + s_\mu - 1$, $r_\nu = r'_\nu s_\nu + 1$, $s_\nu > 1$, and $r'_\mu = r'_\nu$. Notice now that due to $\ell_\nu(1) = r'_\nu$ and $\ell_\mu(1) = r'_\mu - 1$, equation (6.26) is equivalent to $Q_\mu = -Q_\nu$, which is nothing but condition (vi). Finally, notice that equations (6.19) and (6.24) directly imply the formula for $\omega_{\frac{1}{2},2}(\begin{smallmatrix} \mu & \nu \\ z_1 & z_2 \end{smallmatrix})$ stated in (6.10). This finishes the proof of Proposition 6.7. \blacksquare

6.1.4. $(g, n) = (1, 1)$. We study separately the two types of terms in (6.2):

$$\omega_{1,1}(\mu_1) = \omega_{1,1}^I(\mu_1) + \omega_{1,1}^{II}(\mu_1),$$

where $\omega_{1,1}^I$ is the contribution from $\omega_{0,2}$, and $\omega_{1,1}^{II}$ is the one from $\omega_{\frac{1}{2},1} \cdot \omega_{\frac{1}{2},1}$.

Lemma 6.9. *Let $\mu \in \tilde{\alpha}$. Then*

$$\omega_{1,1}^I(\mu) = \frac{r_\mu^2 - 1}{24r_\mu t_\mu} \frac{dz}{z^{s_\mu+1}}.$$

Proof. We have

$$\omega_{1,1}^I(\mu) = \text{Res}_{z=\mu} \sum_{a=1}^{r_\mu-1} K_\mu \left(\begin{matrix} \mu & \mu & \mu \\ z_1 & z & \vartheta_\mu^a z \end{matrix} \right) \frac{\vartheta_\mu^a}{(1 - \vartheta_\mu^a)^2} \frac{(dz)^2}{z^2}, \quad (6.27)$$

where the presence of $\omega_{0,2}(z, z')$ forces the three variables in the recursion kernel to belong to the same component \tilde{C}_μ of \tilde{C} . To handle the sum, we use the following identity for a not divisible by r_μ :

$$\frac{\vartheta_\mu^a}{(1 - \vartheta_\mu^a)^2} = \sum_{m=0}^{r_\mu-1} \frac{m(r_\mu - m)}{2r_\mu} \vartheta_\mu^{am}. \quad (6.28)$$

The contribution of each term of the right-hand side of (6.28) to (6.27) can be computed similarly to (3a). After computation of the residue, we find

$$\omega_{1,1}^I(\mu) = - \sum_{m=0}^{r_\mu-1} \frac{m(r_\mu - m)(r_\mu - 1 - 2\ell_\mu(m))}{4r_\mu^2 t_\mu} \frac{dz_1}{z_1^{s_\mu+1}}, \quad (6.29)$$

and it is enough to sum over $m \in [r_\mu - 1]$. Recall that $\ell_\mu(m)$ is defined as the unique integer in $[0, r_\mu)$ such that $\ell_\mu(m) = -mc_\mu \pmod{r_\mu}$, where c_μ is given by equation (6.4).

In the case $r_\mu = r'_\mu s_\mu + 1$, we can decompose $m = m' s_\mu + m''$ with $m'' \in [s_\mu]$. It puts $m \in [r_\mu - 1]$ into bijection with $(m', m'') \in [0, r'_\mu) \times [s_\mu]$, and we get $\ell_\mu(m) = r'_\mu m - r_\mu m' = r'_\mu m'' - m'$. Then, we have

$$\begin{aligned} \sum_{m=1}^{r_\mu-1} m(r_\mu - 1)(r_\mu - 1 - 2\ell_\mu(m)) &= \sum_{m'=0}^{r_\mu-1} \sum_{m''=1}^{s_\mu} (m' s_\mu + m'')(r_\mu - m' s_\mu - m'') \\ &\quad \times (r_\mu - 1 - 2r'_\mu m'' + 2m') = -\frac{r_\mu(r_\mu^2 - 1)}{6}, \end{aligned}$$

whence

$$\omega_{1,1}^I(\mu) = \frac{r_\mu^2 - 1}{24r_\mu t_\mu} \frac{dz_1}{z_1^{s_\mu+1}}.$$

In the case $r_\mu = r'_\mu s_\mu + s_\mu - 1$, we decompose $m = m' s_\mu + m''$ with $m'' \in [0, s_\mu)$. It puts $m \in [r_\mu]$ into bijection with $(m', m'') \in [0, r'_\mu] \times [0, s_\mu)$ and gives

$$\ell_\mu(m) = -(r'_\mu + 1)m + (m' + 1)r_\mu = r_\mu - m' - (r'_\mu + 1)m''.$$

Then, we have

$$\begin{aligned} \sum_{m=0}^{r_\mu} m(r_\mu - 1)(r_\mu - 1 - 2\ell_\mu) &= \sum_{m'=0}^{r'_\mu} \sum_{m''=1}^{s_\mu-1} (m' s_\mu + m'')(r_\mu - m' s_\mu - m'') \\ &\quad \times (2m' + 2(r'_\mu + 1)m'' - r_\mu - 1) = -\frac{r_\mu(r_\mu^2 - 1)}{6}. \end{aligned}$$

So, we obtain the same formula for $\omega_{1,1}^I$ in both cases. \blacksquare

Remark 6.10. This computation can also be done directly from the Airy structure, as

$$\omega_{1,1}^I\left(\frac{\mu}{z}\right) = \sum_{k>0} F_{1,1}[k]|_{\mathcal{Q}=0} \frac{dz}{z^{k+1}}$$

and $F_{1,1}[k]$ is the constant term of order \hbar in the unique operator of the Airy structure of the form $\hbar \partial_{x_k} + \mathcal{O}(2)$. In fact, as (6.27) coincides with the topological recursion on the sole component \tilde{C}_μ of the spectral curve, the value of $F_{1,1}[k]|_{\mathcal{Q}=0}$ must coincide at $t_\mu = \frac{1}{r_\mu}$ with the one computed in [9, Lemma B.3]. This is indeed the case, but we note the calculation by this other method involves the sum

$$\Psi^{(1)}(\emptyset) = -\frac{1}{2} \sum_{m=0}^{r_\mu-1} \frac{m(r_\mu - m)}{2r_\mu} = -\frac{r_\mu(r_\mu^2 - 1)}{24},$$

which is much simpler than (6.29), although it leads to the same result.

To obtain $\omega_{1,1}^{\text{II}}$, we can split the contribution of the various v as we did in the derivation of equation (6.13). The details are omitted and we only give the outcome:

$$\begin{aligned} \omega_{1,1}^{\text{II}}\left(\frac{\mu}{z_1}\right) &= dz_1 \mathcal{Q}_\mu \left\{ -z_1^{-(s_\mu+1)} \mathcal{Q}_\mu \frac{r_\mu - 1}{2r_\mu t_\mu} - \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} = \frac{r_v}{s_v}}} \frac{t_\mu^{r_\mu-1}}{t_\mu^{r_\mu} - t_v^{r_\mu}} \frac{\mathcal{Q}_v}{z_1^{s_\mu+1}} \right. \\ &\quad - \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} \sum_{\substack{k_1 > 0 \\ \ell' \geq 0}} \frac{t_v^{r_v \ell'}}{t_\mu^{r_v \ell'+1}} \frac{\mathcal{Q}_v}{z_1^{k_1+1}} \delta_{k_1+\ell'(s_v r_\mu - s_\mu r_v), s_\mu} \\ &\quad \left. + \sum_{\substack{v \neq \mu \\ \frac{r_\mu}{s_\mu} < \frac{r_v}{s_v}}} \sum_{\substack{k_1 > 0 \\ \ell' > 0}} \frac{t_\mu^{r_\mu \ell'-1}}{t_v^{r_v \ell'}} \frac{\mathcal{Q}_v}{z_1^{k_1+1}} \delta_{k_1+\ell'(s_\mu r_v - s_v r_\mu), s_\mu} \right\}. \end{aligned}$$

6.2. The exceptional case

We proceed considering a spectral curve with only one branchpoint. Let us only slightly change the setting and add a component – the exceptional component – indexed by $\mu_- \in \tilde{\alpha}$ on which

$$x\left(\frac{\mu_-}{z}\right) = z, \quad y\left(\frac{\mu_-}{z}\right) = 0,$$

i.e., $r_{\mu_-} = 1$ and $s_{\mu_-} = \infty$. On all other components $\mu \in \tilde{\alpha} \setminus \{\mu_-\}$ we still take

$$x\left(\frac{\mu}{z}\right) = z^{r_\mu}, \quad y\left(\frac{\mu}{z}\right) = -t_\mu z^{s_\mu - r_\mu},$$

and we equip the curve with the standard bidifferential and the crosscap differential

$$\omega_{\frac{1}{2},1}\left(\frac{\mu}{z}\right) = \frac{Q_\mu dz}{z}.$$

As before, we only require that $\gcd(r_\mu, s_\mu) = 1$, and that for $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu}$ and $\mu \neq \nu$ we must have $t_\mu^{r_\mu} \neq t_\nu^{r_\nu}$. In the following, we will consider and partially compute the correlators $\omega_{0,3}$, $\omega_{0,4}$ and $\omega_{\frac{1}{2},2}$ and characterise the cases in which they are symmetric. These correlators are still given by formula (6.2), and we will learn that in the exceptional case they mostly behave as in the standard case which was discussed in the preceding sections.

Lemma 6.11. *The correlators $(\omega_{0,n})_{n \geq 1}$ are symmetric if and only if conditions (i), (ii) of Proposition 6.1 and condition (iii) of Proposition 6.2 are satisfied. Assuming the above conditions, $\omega_{\frac{1}{2},2}$ is symmetric if and only if (iv), (v), and (vi) of Proposition 6.7 hold. In this case, the correlators $\omega_{0,3}$ and $\omega_{\frac{1}{2},2}$ are still given by the formulas stated in Propositions 6.1 and 6.7, where any Kronecker delta involving $s_{\mu_-} = \infty$ evaluates to 0.*

Remark 6.12. One should remark that all expressions occurring in (i) to (vi) make sense even with $s_{\mu_-} = \infty$ for a single $\mu_- \in \tilde{\alpha}$ if we understand $r_{\mu_-} = 1 \bmod s_{\mu_-}$ and interpret $\frac{r_{\mu_-}}{s_{\mu_-}}$ as being zero.

Proof of Lemma 6.11. Let us first focus on the correlators $(\omega_{0,n})_{n \geq 1}$. Regarding those, Theorem 5.26 already tells us that (i), (ii), and (iii) are sufficient conditions for these correlators to be symmetric. Conversely, we can argue that these conditions are also necessary for the symmetry by looking at $\omega_{0,3}$ and $\omega_{0,4}$.

First of all, we notice that in the case μ_1, μ_2, μ_3 satisfy $\mu_i \neq \mu_-$, the correlator $\omega_{0,3}\left(\frac{\mu_1}{z_1}, \frac{\mu_2}{z_2}, \frac{\mu_3}{z_3}\right)$ is computed as in the standard case considered in Proposition 6.1 since the bidifferential does not mix the components. So first, we can deduce that restriction of $\omega_{0,3}$ to the non-exceptional components is symmetric if and only if $(r_\mu, s_\mu)_{\mu \neq \mu_-}$ satisfy (i) and (ii) and second, we know that in this case $\omega_{0,3}$ is given

by (6.3). The same argument applied to $\omega_{0,4}$ tells us that $(r_\mu, s_\mu)_{\mu \neq \mu_-}$ have to satisfy (iii) for the correlator to be symmetric.

Hence, we only need to argue that (i)–(iii) involving the component \tilde{C}_{μ_-} are necessary to impose. However, condition (i) is satisfied by our convention that $r_{\mu_-} = 1 \bmod s_{\mu_-}$. Regarding property (ii), suppose there is a $\mu \neq \mu_-$ for which $s_\mu > 2$, $r_\mu = 1 \bmod s_\mu$, and $\lfloor \frac{r_\mu}{s_\mu} \rfloor = 0$. Then of course we must already have $r_{\mu_2} = 1$. In this case, the correlator

$$\begin{aligned} \frac{\omega_{0,3}\left(\frac{\mu}{z_1} \frac{\mu}{z_2} \frac{\mu_-}{z_3}\right)}{dz_1 dz_2 dz_3} &= \frac{1}{dz_1} \operatorname{Res}_{z=\mu} K_\mu\left(\frac{\mu}{z_1} \frac{\mu}{z} \frac{\mu_-}{z}\right) \sum_{k_2, k_3 \geq 1} k_2 k_3 z^{k_2+k_3-2} z_2^{-k_2-1} z_3^{-k_3-1} (dz)^2 \\ &= -\frac{1}{t_\mu} \sum_{k_1, k_2, k_3 \geq 1} k_2 k_3 z_1^{-k_1-1} z_2^{-k_2-1} z_3^{-k_3-1} \delta_{k_1+k_2+k_3, s_\mu}, \quad (6.30) \end{aligned}$$

where we used that

$$K_\mu\left(\frac{\mu}{z_1} \frac{\mu}{z} \frac{\mu_-}{z}\right) = -\frac{dz_1}{t_\mu dz} \sum_{k_1 \geq 1} z^{k_1+1-s_\mu} z_1^{-k_1-1}.$$

This is clearly non-zero, unlike $\omega_{0,3}\left(\frac{\mu}{z_1} \frac{\mu_-}{z_3} \frac{\mu}{z_2}\right) = 0$, and therefore we see that (ii) must indeed be satisfied for all $(r_\mu, s_\mu)_{\mu \in \tilde{\alpha}}$ for $\omega_{0,3}$ to be symmetric.

Regarding (iii) we proceed similarly. Suppose there are $\mu \neq \nu$ distinct from μ_- with $s_\mu, s_\nu > 1$ and $\lfloor \frac{r_\mu}{s_\mu} \rfloor = \lfloor \frac{r_\nu}{s_\nu} \rfloor = 0$. Then (i) and (ii) only leave us with $r_\mu = r_\nu = 1$ and $s_\mu = s_\nu = 2$. We compute

$$\begin{aligned} \frac{\omega_{0,4}\left(\frac{\mu}{z_1} \frac{\mu}{z_2} \frac{\nu}{z_3} \frac{\mu_-}{z_4}\right)}{dz_1 dz_2 dz_3 dz_4} &= \frac{1}{dz_1} \operatorname{Res}_{z=\mu} K_\mu\left(\frac{\mu}{z_1} \frac{\mu}{z} \frac{\nu}{z} \frac{\mu_-}{z}\right) \\ &\quad \times \sum_{k_2, k_3, k_4 \geq 1} k_2 k_3 k_4 z^{k_2+k_3+k_4-3} z_2^{-k_2-1} z_3^{-k_3-1} z_4^{-k_4-1} (dz)^3 \\ &= -\frac{1}{t_\mu(t_\mu - t_\nu)} \frac{1}{z_1^2 z_2^2 z_3^2 z_4^2}, \end{aligned}$$

which is non-vanishing, where

$$K_\mu\left(\frac{\mu}{z_1} \frac{\mu}{z} \frac{\nu}{z} \frac{\mu_-}{z}\right) = -\frac{dz_1}{(t_\mu - t_\nu)t_\mu (dz)^2} \sum_{k_1 \geq 1} z^{k_1-2} z_1^{-k_1-1}.$$

Using that always $\omega_{0,4}\left(\frac{\mu}{z_1} \frac{\nu}{z_3} \frac{\mu}{z_2} \frac{\mu_-}{z_4}\right) = 0$, this shows that (ii) is indeed necessary to impose.

Now let us turn to $\omega_{\frac{1}{2}, 2}$. In the following, set $\omega'_{\frac{1}{2}, 2} := \omega_{\frac{1}{2}, 2}|_{\mathcal{Q}_{\mu_- = 0}}$, i.e., by $\omega'_{\frac{1}{2}, 2}$ we denote the restriction of $\omega_{\frac{1}{2}, 2}$ to the non-exceptional components. Then for distinct $\mu, \nu \in \tilde{\alpha} \setminus \{\mu_-\}$, we have

$$\omega_{\frac{1}{2}, 2}\left(\frac{\mu}{z_1} \frac{\nu}{z_2}\right) = \omega'_{\frac{1}{2}, 2}\left(\frac{\mu}{z_1} \frac{\nu}{z_2}\right)$$

as the bidifferential is not mixing the components. Thus, from the analysis done in Proposition 6.7, we deduce that the condition ensuring the symmetry of this correlator is (vi) and in the symmetric case it can be evaluated using formula (6.10). However, if the arguments lie on the same component $\mu \neq \mu_-$, we obtain an additional contribution

$$\begin{aligned} \omega_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}\right)} &= \omega'_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}\right)} + r_\mu Q_{\mu_-} \operatorname{Res}_{z=\mu} K_\mu \left(\begin{smallmatrix} \mu & \mu & \mu_- \\ z_1 & z & z^{r_\mu} \end{smallmatrix}\right) \sum_{k_2 \geq 1} k_2 z^{k_2-2} z_2^{-k_2-1} (dz)^2 dz_2 \\ &= \omega'_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}\right)} - \sum_{k_1, k_2 \geq 1} \frac{dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} k_2 \frac{Q_{\mu_-}}{t_\mu} \delta_{k_1+k_2, s_\mu} \end{aligned}$$

coming from the first term in the bracket in (6.2). This is symmetric for $s_\mu \leq 2$, while for $s_\mu > 3$ we can use that $\omega'_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}\right)}$ is computed via (6.14) in order to get

$$\begin{aligned} \omega_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}\right)} &= \sum_{k_1, k_2 \geq 1} \frac{dz_1 dz_2}{z_1^{k_1+1} z_2^{k_2+1}} \left\{ -\delta_{k_1+k_2, s_\mu} k_2 \left(\frac{Q_\mu(r_\mu - 1 - \ell_\mu(k_2))}{r_\mu t_\mu} + \sum_{\substack{v \in \tilde{\alpha} \setminus \{\mu\} \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} \frac{Q_v}{t_\mu} \right) \right. \\ &\quad + \frac{k_2}{t_\mu} \sum_{\ell' > 0} \left(- \sum_{\substack{v \in \tilde{\alpha} \setminus \{\mu, \mu_-\} \\ \frac{r_\mu}{s_\mu} > \frac{r_v}{s_v}}} \left(\frac{t_v}{t_\mu}\right)^{r_v \ell'} Q_v \delta_{k_1+k_2+\ell'(r_\mu s_v - r_v s_\mu), s_\mu} \right. \\ &\quad \left. \left. + \sum_{\substack{v \in \tilde{\alpha} \setminus \{\mu, \mu_-\} \\ \frac{r_\mu}{s_\mu} < \frac{r_v}{s_v}}} \left(\frac{t_\mu}{t_v}\right)^{r_v \ell'} Q_v \delta_{k_1+k_2-\ell'(r_\mu s_v - r_v s_\mu), s_\mu} \right) \right\}. \end{aligned}$$

Comparing formula (6.14) with the above equation and following the characterisation of the symmetry of (6.14), it should be clear that the condition for the symmetry of $\omega_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu & \mu \\ z_1 & z_2 \end{smallmatrix}\right)}$ above is encoded in (iv) and (v).

Now let us turn to the computation of correlators with arguments lying on the exceptional component. It is a straightforward calculation to find that

$$\omega_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu_- & \mu_- \\ z_1 & z_2 \end{smallmatrix}\right)} = \frac{dz_1 dz_2}{z_1^2 z_2^2} \sum_{\substack{v \neq \mu_- \\ (r_v, s_v) = (1, 2)}} \frac{Q_\mu}{t_\mu}, \quad (6.31)$$

which is always symmetric. Notice at this point that this is in accordance with (iv) and (v) as the two conditions do not imply any constraints for μ_- . Now let us briefly argue that formula (6.8) exactly produces the result for $\omega_{\frac{1}{2},2}^{\left(\begin{smallmatrix} \mu_- & \mu_- \\ z_1 & z_2 \end{smallmatrix}\right)}$ we obtained above. Since $r'_{\mu_-} = \lfloor \frac{r_{\mu_-}}{s_{\mu_-}} \rfloor = 0$, the first term in (6.8) vanishes. The two terms after

are empty sums, hence vanishing. Thus the last term in (6.8) is the only one potentially leading to a non-zero contribution, and indeed one finds that the sum condition in this sum coincides with the one in (6.31). We therefore deduce that formula (6.8) even applies for the exceptional case.

It is also straightforward to compute

$$\begin{aligned} \omega_{\frac{1}{2},2}\left(\begin{matrix} \mu & \mu_- \\ z_1 & z_2 \end{matrix}\right) &= -\delta_{r_\mu+1,s_\mu} \frac{Q_\mu}{t_\mu} \frac{dz_1 dz_2}{z_1^2 z_2^2}, \\ \omega_{\frac{1}{2},2}\left(\begin{matrix} \mu & \mu_- \\ z_1 & z_2 \end{matrix}\right) &= \delta_{r_\mu+1,s_\mu} \frac{Q_{\mu_-}}{t_\mu} \frac{dz_1 dz_2}{z_1^2 z_2^2} \end{aligned} \tag{6.32}$$

for $\mu \neq \mu_-$. We therefore deduce the symmetry condition that for all $\mu \neq \mu_-$ with $s_\mu = r_\mu + 1$ necessarily $Q_\mu = -Q_{\mu_-}$. This condition is however covered in (vi). Notice moreover that (6.10) applied to the case at hand indeed produces (6.32).

At last, let us show that the statement of Proposition 6.2 is true in the exceptional case as well. For this notice that if we choose $\mu, \nu, \lambda \in \tilde{\alpha}$ pairwise distinct with $\frac{r_\mu}{s_\mu} = \frac{r_\nu}{s_\nu} = \frac{r_\lambda}{s_\lambda}$, then already $s_\mu < \infty$. Thus, $\omega_{0,4}\left(\begin{matrix} \mu & \mu & \nu & \lambda \\ z_1 & z_2 & z_3 & z_4 \end{matrix}\right)$ gets the same contributions as in the standard case which means that the further discussion of the symmetry of this correlator must be in line with the proof of Proposition 6.2. ■

6.3. Necessary conditions for symmetry

In Corollary 6.3, we showed that the topological recursion formula outputs symmetric $(\omega_{0,n})_{n \geq 1}$ if and only if the considered spectral curve is admissible in genus 0. Using Proposition 5.24, this means that the associated differential operators $H_{\alpha;i,k}$, which we introduced in Definition 5.17, admit a partition function solving the associated system of differential equations to leading order in $\hbar^{\frac{1}{2}}$ in the sense of equation (2.25). This, on the other hand, implies that for the differential operators of the type considered in Section 2.3.2 – which are exactly those corresponding to spectral curves of the form (6.1) – the reverse direction of Theorem 2.13 is true as well. This is exactly the statement of Proposition 2.16 which we have hereby proven. Note that by Lemma 6.11 the statement of Corollary 6.3 holds in the exceptional case too.

The next natural step is to investigate the necessary conditions for the operators $H_{\alpha;i,k}$ to be an Airy structure. Suppose they are, then we know from Theorem 5.23 that all $\omega_{g,n}$ have to be symmetric. So especially the symmetry conditions for $\omega_{\frac{1}{2},2}$ will impose further necessary conditions on the $H_{\alpha;i,k}$ to be an Airy structure. We will analyse how they restrict the choice of $(r_\mu, s_\mu)_{\mu \in \mu \in \tilde{\alpha}}$ in case we ask $\omega_{\frac{1}{2},2}$ to be symmetric for all $Q_\mu \in \mathbb{C}$, $\mu \in \tilde{\alpha}$ only being subject to the constraint

$$\sum_{\mu \in \tilde{\alpha}} Q_\mu = 0. \tag{6.33}$$

Before we begin our analysis, let us remark that during our calculation in Sections 6.1 and 6.2 we were always assuming that $\gcd(r_\mu, s_\mu) = 1$,

$$t_\mu^{r_\mu} \neq t_\nu^{r_\nu} \quad \text{whenever } (r_\mu, s_\mu) = (r_\nu, s_\nu) \text{ and } \mu \neq \nu, \quad (6.34)$$

and that $s_\mu = \infty$ for at most one $\mu \in \tilde{\alpha}$. However, we already argued in Remarks 3.4 and 3.16 that these assumptions are indeed necessary so that the differential operators are an Airy structure. In order to be closer to the notation of Section 2.3.2, let us choose a lexicographic ordering $\mu: [d] \rightarrow \tilde{\alpha}$ satisfying $\mu_j < \mu_{j+1}$.

Proof of Proposition 2.17. Due to (i) of Proposition 6.1, we know that necessarily $r_\nu = \pm 1 \pmod{s_\nu}$ for all $\nu \in \tilde{\alpha}$. If $s_{\mu_1} > 2$, then (ii) tells us that we must have $\frac{r_{\mu_1}}{s_{\mu_1}} > \frac{r_{\mu_2}}{s_{\mu_2}}$. Therefore,

$$\sum_{\substack{v \neq \mu_1 \\ \frac{r_{\mu_1}}{s_{\mu_1}} > \frac{r_\nu}{s_\nu}}} Q_\nu = \sum_{j=2}^d Q_{\mu_j}$$

and hence – since we assume $\omega_{\frac{1}{2}, 2}$ to be symmetric for all choices of Q_μ satisfying (6.33) – condition (iv) of Proposition 6.7 forbids that $r_{\mu_1} = 1 \pmod{s_{\mu_1}}$ and $s_{\mu_1} > 2$ as we can always choose the Q s such that

$$\sum_{j \in [d]} Q_{\mu_j} \neq 0.$$

Thus, we are left with $r_{\mu_1} = -1 \pmod{s_{\mu_1}}$.

Now let us see which values the symmetry conditions allow (r_{μ_j}, s_{μ_j}) to take in the case $j \notin \{1, d\}$. First, let us assume $\frac{r_{\mu_1}}{s_{\mu_1}} = \frac{r_{\mu_j}}{s_{\mu_j}}$. Then by (ii), necessarily $s_{\mu_j} \in \{1, 2\}$. If however $\frac{r_{\mu_1}}{s_{\mu_1}} > \frac{r_{\mu_j}}{s_{\mu_j}}$, then conditions (iv) and (v) also force $s_{\mu_j} \in \{1, 2\}$, using that we can choose the Q s arbitrary except that they must satisfy (6.33).

As one can use similar arguments in order to show that also $r_{\mu_d} = 1 \pmod{s_{\mu_d}}$, we omit the further discussion of this case.

Now let us analyse the implications coming from condition (vi) in the case of generic Q s. If $d = 2$, condition (vi) is always satisfied since the requirement that $Q_\mu = -Q_\nu$ for $\mu \neq \nu$ is nothing but property (6.33). Therefore, let us assume that $d > 2$. In this case, (vi) exactly forbids that $r_\mu = r_\nu$ and $s_\mu = s_\nu = 2$ for $\mu \neq \nu$. Hence, taking into account the other constraints we have already derived for $(r_\mu, s_\mu)_{\mu \in \tilde{\alpha}}$ we deduce that for $d > 2$ necessarily $s_\mu = s_\nu = 1$ whenever $(r_\mu, s_\mu) = (r_\nu, s_\nu)$ for some $\mu \neq \nu$.

Note that due to Lemma 6.11, the above discussion covers both the standard and the exceptional case. ■

Part III

Applications to intersection theory

We expect that the coefficients $F_{g,n}$ of the partition function for all basic Airy structures (those of Sections 2.3.1 and 2.3.2) and the $\omega_{g,n}$ of the corresponding topological recursion can be represented as integrals of distinguished cohomology classes on $\bar{\mathcal{M}}_{g,n}$ or its relatives. As soon as such a representation is known for one spectral curve admitting a single ramification point and whose type belongs to a certain set, it is relatively easy to extend it to any spectral curve having ramification points whose type belong to this set. The reason is that the corresponding Airy structure is built from the basic Airy structures by direct sums, change of polarisations and further dilaton shifts, cf. Section 2.3.3. In terms of partition functions, this is sometimes called “Givental decomposition”.

We develop this idea for two types for which the link to $\bar{\mathcal{M}}_{g,n}$ is already known:

- the type $(r, s) = (r, r + 1)$ is related to Witten r -spin theory. The $r = 2$ subcase is Eynard’s formula [32, 33], and by generalising it to any r , we answer a question of Shadrin to the first-named author;
- the type $(r_1, s_1, r_2, s_2) = (r, r + 1, 1, \infty)$ is related to open r -spin intersection theory as discussed in Section 8.

The type (r, s) for other s is discussed in Section 7.5.4 assuming the existence of a special class on $\bar{\mathcal{M}}_{g,n}$ which has only been constructed for $(r, s) = (2, 1)$ so far [24, 49]. It turns out that Laplace-type integrals play an important role in such representations, and we first study them in the preliminary Section 7.1. The method is general: if in the future an enumerative interpretation is found for a larger set of types, it is rather automatic to follow the strategy at work in these two examples and extend our representations to any spectral curve having ramifications whose types belong to this larger set.

7. Representation of correlators via intersection theory

7.1. The Laplace isomorphisms

Let $r \geq 1$. If $m \geq 1 - r$ is an integer, we introduce the r -fold factorial either by induction

$$m!^{(r)} = \begin{cases} 1 & \text{if } 1 - r \leq m \leq 0, \\ m \cdot (m - r)!^{(r)} & \text{if } m > 0 \end{cases} \quad (7.1)$$

or equivalently in terms of the Gamma function

$$m!^{(r)} = \frac{r^{\frac{m}{r}+1} \Gamma(\frac{m}{r} + 1)}{r^{\binom{m}{r}} \Gamma(\binom{m}{r})},$$

where we have defined $\binom{dr+a}{r} = \frac{a}{r}$ if $d \in \mathbb{Z}$ and $a \in [r]$.

Definition 7.1. We introduce two isomorphisms:

$$\begin{aligned} \mathfrak{L}^+ : \mathbb{C}[[\zeta]]d\zeta &\rightarrow u^{\frac{1}{r}} \mathbb{C}[[u^{\frac{1}{r}}]], \\ k\zeta^{k-1}d\zeta &\mapsto k!^{(r)} u^{\frac{k}{r}}, \\ \mathfrak{L}^- : \mathbb{C}[\zeta^{-1}]\zeta^{-2}d\zeta &\rightarrow \epsilon^{\frac{1}{r}} \mathbb{C}[\epsilon^{\frac{1}{r}}], \\ k!^{(r)}\zeta^{-(k+1)}d\zeta &\mapsto \epsilon^{\frac{k}{r}}. \end{aligned}$$

Abusing notation slightly, we also write \mathfrak{L}^\pm for the maps extended to the domain $\mathbb{C}((\zeta))d\zeta$ by defining them to be zero on any monomial not in the original domain of definition.

The first map can be realised by integrating over paths from 0 to ∞ in the x -plane.

Lemma 7.2. We have

$$\mathfrak{L}^+ = \sum_{j=0}^{r-1} \mathfrak{J}_j \int_{\mathbb{R}_+} e^{-\frac{\zeta^r}{ur}},$$

where the constants are

$$\mathfrak{J}_j := \sum_{a=1}^r \frac{e^{-\frac{2i\pi ja}{r}}}{r^{\frac{a}{r}} \Gamma(\frac{a}{r})}. \tag{7.2}$$

Proof. Let β_j be from 0 to ∞ in the angular direction $e^{\frac{2i\pi j}{r}}$. Consider integration along a formal combination of paths $\beta = \sum_{j=0}^{r-1} \mathfrak{J}_j \beta_j$ for some $\mathfrak{J}_j \in \mathbb{C}$. For $k = dr + a$ with $d \geq 0$ and $a \in [r]$, we compute

$$\begin{aligned} \int_{\beta} e^{-\frac{\zeta^r}{ur}} k\zeta^{k-1}d\zeta &= \sum_{j=0}^{r-1} \mathfrak{J}_j e^{\frac{2i\pi ja}{r}} (ur)^{\frac{k}{r}} \frac{k}{r} \int_{\mathbb{R}_+} e^{-\tilde{x}} \tilde{x}^{\frac{k}{r}-1} d\tilde{x} \\ &= r \hat{\mathfrak{J}}_a u^{\frac{k}{r}} r^{\frac{k}{r}} \Gamma\left(\frac{k}{r} + 1\right) \\ &= \hat{\mathfrak{J}}_a u^{\frac{k}{r}} r^{\frac{a}{r}} \Gamma\left(\frac{a}{r}\right) k!^{(r)} \end{aligned}$$

with the change of variable $\tilde{x} = \zeta^r / ur$ and the discrete Fourier transform for $a \in \mathbb{Z}$

$$\hat{\mathfrak{J}}_a = \frac{1}{r} \sum_{j=0}^{r-1} e^{\frac{2i\pi ja}{r}} \mathfrak{J}_j.$$

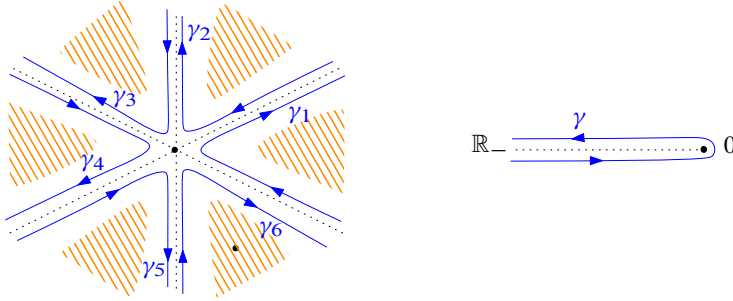


Figure 1. Left panel: contours in the ζ -plane for $r = 6$. The striped regions correspond to $\text{Re } x > M$. Right panel: Hankel contour in the x plane.

We get

$$\int_{\rho} e^{-\zeta^r/ur} k \zeta^{k-1} d\zeta = \mathfrak{L}_+[\zeta^{k-1} d\zeta]$$

provided we choose

$$\hat{\mathfrak{A}}_a = \frac{1}{r^{\frac{a}{r}} \Gamma\left(\frac{a}{r}\right)}.$$

This entails the result by inverse discrete Fourier transform. ■

The second map can be realised by contour integration. To this end, let γ be the Hankel contour giving the Gamma function representation (Figure 1)

$$\frac{1}{\Gamma(\alpha)} = \int_{\gamma} e^x x^{-\alpha} dx, \quad \alpha \in \mathbb{C},$$

that is, γ goes from $-\infty - i0$ to $-i0$, then round the origin to $+i0$ and ends in $-\infty + i0$. Under the branched covering $\zeta \mapsto x(\zeta) = \zeta^r$, this contour has r different lifts $(\gamma_j)_{j=1}^r$, which we label so that γ_j comes from the asymptotic direction $e^{-\frac{i\pi}{r}(2j+1)}(+\infty + i0)$, approaches the origin and then ends in the asymptotic direction $e^{-\frac{i\pi}{r}(2j-1)}(+\infty - i0)$. These contours belong to the lattice of rank $(r - 1)$ of Lefschetz thimbles

$$V := H_1(\mathbb{C}, \mathbb{S}_M^-; \mathbb{Z}),$$

where $\mathbb{S}_M^- = \{z \in \mathbb{C} \mid \text{Re } x < -M\}$ for some large $M > 0$. The homology class $\sum_{j=1}^r \gamma_j$ is trivial, and omitting one γ_j we get a basis of V .

Lemma 7.3. *We have*

$$\mathfrak{L}^- = \sum_{j=1}^r \frac{\mathfrak{A}_j}{2i\pi r} \int_{\gamma_j} e^{\frac{\zeta^r}{r}}$$

with the constants already appearing in (7.2).

Proof. Let $\epsilon > 0$, $j \in [r]$. Consider the integral

$$\mathcal{I}_j(\alpha) := \frac{1}{2i\pi} \int_{\gamma_j} e^{\frac{\epsilon \zeta^r}{r}} \zeta^\alpha d\zeta \tag{7.3}$$

for complex $\alpha \in \mathbb{C}$. Here, $\zeta \mapsto \zeta^\alpha$ is defined in the usual way as the analytic function on $\mathbb{C} \setminus \mathbb{R}_-$ such that for $\zeta \in \mathbb{R}_^*$, we have $\lim_{\epsilon \rightarrow 0^+} (\zeta \pm i\epsilon)^\alpha = e^{\pm i\pi\alpha} |\zeta|^\alpha$.

Let us first consider $\text{Re } \alpha > 0$. In this case, as the integrand is regular, we can squeeze the Hankel contour to the half-axes of angles $e^{-\frac{i\pi(2k+1)}{r}}$ for $k \in [r]$. After a change of variable $\tilde{x} = -\frac{\epsilon \zeta^r}{r}$, we find

$$\begin{aligned} \mathcal{I}_j(\alpha) &= \left(\frac{\epsilon}{r}\right)^{-\frac{\alpha+1}{r}} \left(-e^{-\frac{i\pi(2j+1)(\alpha+1)}{r}} + e^{-\frac{i\pi(2j-1)(\alpha+1)}{r}}\right) \cdot \frac{1}{r} \int_{\mathbb{R}_+} e^{-\tilde{x}} \tilde{x}^{\frac{\alpha+1}{r}-1} d\tilde{x} \\ &= \left(\frac{\epsilon}{r}\right)^{-\frac{\alpha+1}{r}} e^{-\frac{2i\pi j(\alpha+1)}{r}} \cdot \frac{2i}{r} \sin\left(\frac{\pi(\alpha+1)}{r}\right) \Gamma\left(\frac{\alpha+1}{r}\right). \end{aligned} \tag{7.4}$$

We note that by definition in (7.3), $\mathcal{I}_j(\alpha)$ is an entire function of $\alpha \in \mathbb{C}$. This is also true for the right-hand side of (7.4): the Gamma function has simple poles when $\frac{\alpha+1}{r} \in 2\mathbb{Z}_{\leq 0}$, which are compensated by a zero in the prefactor. Then, by analytic continuation, identity (7.4) remains true for all $\alpha \in \mathbb{C}$.

We apply it to $\alpha = -(k+1)$, where k is a positive integer that we decompose as $k = rd + a$ with $a \in [r]$ and $d \geq 0$. Then

$$\begin{aligned} \mathcal{I}_j(-(k+1)) &= \left(\frac{\epsilon}{r}\right)^{\frac{k}{r}} e^{\frac{2i\pi jk}{r}} \frac{2i}{r} \sin\left(-\frac{\pi jk}{r}\right) \Gamma\left(-\frac{k}{r}\right) \\ &= \epsilon^{\frac{k}{r}} \frac{2i\pi}{r} e^{\frac{2i\pi ja}{r}} \frac{1}{r^{\frac{k}{r}} \Gamma\left(\frac{k}{r} + 1\right)} \\ &= \epsilon^{\frac{k}{r}} e^{\frac{2i\pi ja}{r}} \frac{2i\pi}{r^{\frac{a}{r}} \Gamma\left(\frac{a}{r}\right) k!(r)}. \end{aligned}$$

Coming back to the definition (7.2) of \mathfrak{A}_j , we get

$$\sum_{j=1}^r \frac{\mathfrak{A}_j}{2i\pi r} \int_{\gamma_j} e^{\frac{\epsilon x}{r}} \frac{k!(r) d\zeta}{\zeta^{k+1}} = \mathfrak{L}^{-}\left[\frac{k!(r) d\zeta}{\zeta^{k+1}}\right]. \quad \blacksquare$$

7.2. Laplace transform on curves

7.2.1. Total Laplace transform. Let C be a curve with normalisation $\pi: \tilde{C} \rightarrow C$, equipped with a meromorphic 1-form dx . We shall rely on the notations introduced in Section 5.4. We do not require that \tilde{C} comes from a spectral curve in the sense of Definition 5.1, neither that $d\tilde{x}$ is the differential of a meromorphic function on \tilde{C} . In particular, $d\tilde{x}$ could have simple poles, in which case a local primitive \tilde{x} has logarithmic singularities and would be multivalued on \tilde{C} .

Recall that we write $d\tilde{x} = \pi^* dx$, that α is the set of zeroes of dx (including singular points) and $\tilde{\alpha} = \pi^{-1}(\alpha)$. Denote the order of a zero of dx at $\mu \in \tilde{\alpha}$ by $r_\mu - 1$ (this could be zero if $\pi(a)$ is singular). Around each $\mu \in \tilde{\alpha}$, we have a local coordinate such that

$$\int_\mu d\tilde{x} = \zeta^{r_\mu}.$$

We define the vector spaces

$$V = \bigoplus_{\mu \in \tilde{\alpha}} V_\mu, \quad V_\mu := \bigoplus_{l=1}^{r_\mu} \mathbb{C} \cdot e_{\mu,l}$$

and equip V with the pairing

$$\eta(e_{\mu,l} \otimes e_{\nu,m}) = \frac{\delta_{\mu,\nu} \delta_{l+m,r_\mu}}{r_\mu}. \tag{7.5}$$

Remark 7.4. The fact that e_{μ,r_μ} are null vectors for η is a convention: it simplifies the formulas in Section 7.2.3 but has no effect elsewhere. The factor r_μ is a choice that will simplify formulas in Sections 7.2.3 and 7.5.4 but has no effect elsewhere.

When needed, we shall decompose integers $k \in \mathbb{Z}$ as

$$k = \bar{k} r_\mu + \hat{k}, \quad \hat{k} \in [r_\mu]$$

and the index $\mu \in \tilde{\alpha}$ that one should use will be clear from the context. In order to get rid of fractional powers of the Laplace variable, we introduce the isomorphism

$$E_\mu^*: \epsilon^{\frac{1}{r_\mu}} \mathbb{C}[\epsilon^{\frac{1}{r_\mu}}] \rightarrow V_\mu^*[\epsilon], \quad \epsilon^{\frac{k}{r}} \mapsto e_{\mu,\hat{k}}^* \epsilon^{\bar{k}},$$

where $e_{\mu,l}^*$ is the dual basis with respect to the pairing η . Importing Definition 7.1, for each $\mu \in \tilde{\alpha}$, we consider the local Laplace map

$$\mathfrak{L}_\mu^- = E_\mu \circ \mathfrak{L}^- \circ \text{Loc}_\mu: H^0(\tilde{C}, K_{\tilde{C}}(*\tilde{\alpha})) \rightarrow V_\mu^*[\epsilon]$$

and the total map

$$\mathfrak{L}_{\text{tot}}^- = \left(\bigoplus_{\mu \in \tilde{\alpha}} \mathfrak{L}_\mu^- \right): H^0(\tilde{C}, K_{\tilde{C}}(*\tilde{\alpha})) \rightarrow V^*[\epsilon].$$

We define in a similar way

$$\begin{aligned} E_\mu: u^{\frac{1}{r_\mu}} \mathbb{C}[[u^{\frac{1}{r_\mu}}]] &\rightarrow V_\mu[[u]], \\ \mathfrak{L}_\mu^+: H^0(\tilde{C}, K_{\tilde{C}}(*\tilde{\alpha})) &\rightarrow V_\mu[[u]], \\ \mathfrak{L}_{\text{tot}}^+: H^0(\tilde{C}, K_{\tilde{C}}(*\tilde{\alpha})) &\rightarrow V[[u]], \end{aligned}$$

where the role of $e_{\mu,k}^*$ is now played by $e_{\mu,k}$.

Remark 7.5. From Lemmas 7.2 and 7.3, we see that the natural variables Laplace dual to $x = \zeta^r$ are not u and ϵ^{-1} , but ru and $r\epsilon^{-1}$. Moreover, by a different choice of constants (replacing multiple factorials by values of the Γ function at rational numbers), the transforms could have been given by a single integral. We give the Laplace transform in this way to conform to conventions in the literature using r -fold factorials for the case of a smooth spectral curve and its relation to the Witten r -spin class.

7.2.2. Two generating series. Assume that we are given a holomorphic 1-form $\omega_{0,1}$ in a neighbourhood of $\tilde{\alpha}$ in \tilde{C} .

Definition 7.6. We introduce $\mathsf{T}(u) \in V[[u]]$ given by the formula

$$\begin{aligned} \mathsf{T}(u) &:= \sum_{\mu \in \tilde{\alpha}} \mathsf{T}_{\mu,k} e_{\mu,\hat{k}} u^{\bar{k}} := \left(\sum_{\substack{\mu \in \tilde{\alpha} \\ s_{\mu} \neq \infty}} e_{\mu,\hat{s}_{\mu}} u^{\bar{s}_{\mu}} \right) + \mathfrak{L}_{\text{tot}}^+[\omega_{0,1}](u) \\ &= \sum_{\substack{\mu \in \tilde{\alpha} \\ s_{\mu} \neq \infty}} \left(e_{\mu,\hat{s}_{\mu}} u^{\bar{s}_{\mu}} + \sum_{k>0} (k - r_{\mu})!^{(r_{\mu})} F_{0,1} \left[\begin{matrix} \mu \\ -k \end{matrix} \right] e_{\mu,\hat{k}} u^{\bar{k}} \right). \end{aligned}$$

Assume that we are given a fundamental bidifferential of the second kind $\omega_{0,2}$ on the smooth curve \tilde{C} .

Definition 7.7. We introduce $\mathsf{B}(u, v) \in V^{\otimes 2}[[u, v]]$ given by the formula

$$\begin{aligned} \mathsf{B}(u, v) &:= (\mathfrak{L}_{\text{tot}}^+)^{\otimes 2}(\omega_{0,2} - \omega_{0,2}^{\text{std}}) \\ &= \sum_{\substack{\mu, v \in \alpha \\ k, l > 0}} (k - r_{\mu})!^{(r_{\mu})} (l - r_{\nu})!^{(r_{\nu})} F_{0,2} \left[\begin{matrix} \mu & \nu \\ -k & -l \end{matrix} \right] e_{\mu,\hat{k}} \otimes e_{\nu,\hat{l}} u^{\bar{k}} v^{\bar{l}}, \quad (7.6) \end{aligned}$$

where the second line follows from the decomposition in (5.14).

These definitions in particular apply to admissible spectral curves equipped with a fundamental bidifferential of the second kind, but make sense in this greater generality. Their relevance will become clear in Section 7.5.

7.2.3. Factorisation property for B. We prove in this section a factorisation property for B when \tilde{C} is smooth compact connected curve and $d\tilde{x}$ is a meromorphic 1-form. Here we only need the data of a smooth compact curve \tilde{C} , of a meromorphic 1-form that we denote $d\tilde{x}$, and of a fundamental bidifferential of the second kind on \tilde{C} , which defines $\mathsf{B}(u, v)$ in Definition 7.7. We do not need the full data of a spectral curve, nor \tilde{C} to be the normalisation of a curve. In light of Remark 7.5, and for this section only, the “right” Laplace variables ru and rv (where r depends on the branch point) will play a role, so that the result will be best formulated for the following object.

Definition 7.8. Let $B_{\mu,v}(u, v)$ be the projection of $B(u, v)$ into $(V_\mu \otimes V_\nu)[[u, v]]$ and define

$$\bar{B}(u, v) := \sum_{\mu, v \in \tilde{\alpha}} B_{\mu,v}(u/r_\mu, v/r_\nu).$$

Proposition 7.9. Assume that \tilde{C} is a smooth compact connected curve, $d\tilde{x}$ is a meromorphic 1-form, and $\omega_{0,2}$ is a fundamental bidifferential of the second kind on \tilde{C} . Then

$$\bar{B}(u, v) = \frac{1}{u+v} (u\bar{B}(u, 0) + v\bar{B}(0, v) - uv\bar{B}(u, 0) \star \bar{B}(0, v)), \quad (7.7)$$

where $A \star B = (\text{id} \otimes \eta \otimes \text{id})(A \otimes B)$. Besides, we have the compatibility relation

$$\bar{B}(u, 0) - \bar{B}(0, -u) + u\bar{B}(u, 0) \star \bar{B}(0, -u) = 0. \quad (7.8)$$

Remark 7.10. Such a property appeared in the case where C is smooth and dx has simple zeroes in [33, Appendix B].

Proof. We have introduced in (5.15) the family of meromorphic 1-forms

$$d\xi_{-k}^\mu(z) = \text{Res}_{z'=\mu} \left(\int_{\mu}^{z'} \omega_{0,2}(z, \cdot) \right) \frac{d\zeta(z')}{\zeta(z')^{k+1}} \in H^0(\tilde{C}, K_{\tilde{C}}((k+1)\mu)) \quad (7.9)$$

indexed by $\mu \in \tilde{\alpha}$ and $k > 0$. We have for $\nu \in \tilde{\alpha}$,

$$\text{Loc}_\nu(d\xi_{-k}^\mu) = \frac{\delta_{\mu,\nu} d\zeta}{\zeta^{k+1}} + \sum_{l>0} \frac{F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right]}{k} \zeta^{l-1} d\zeta. \quad (7.10)$$

The idea of the proof is to derive a recursion for these forms using the action of $d(\frac{\cdot}{d\tilde{x}})$, see equation (7.12) below. We will first prove it for the polar part near the ramification points, and use that \tilde{C} is smooth and compact and $d\tilde{x}$ meromorphic on \tilde{C} to get an equality of globally defined meromorphic forms. This implies a recursion for the regular part of expansion (7.10), i.e., the coefficients $F_{0,2}$, and this will imply the desired relation for $B(u_1, u_2)$.

Since $d\tilde{x} = r_\nu \zeta^{r_\nu-1} d\zeta$ near the ramification point ν , we have

$$\text{Loc}_\nu \left[-d \left(\frac{d\xi_{-k}^\mu}{d\tilde{x}} \right) \right] = \frac{k+r_\nu}{r_\nu} \frac{\delta_{\mu,\nu} d\zeta}{\zeta^{k+r_\nu+1}} + \sum_{l=1}^{r_\nu-1} \frac{F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right]}{k} \frac{r_\nu-l}{r_\nu} \frac{d\zeta}{\zeta^{r_\nu-l+1}} + \mathcal{O}(d\zeta).$$

As $d\tilde{x}$ is meromorphic, the 1-form

$$d \left(\frac{d\xi_{-k}^\mu}{d\tilde{x}} \right) + \frac{k+r_\mu}{r_\mu} d\xi_{-(k+r_\mu)}^\mu + \sum_{\nu \in \tilde{\alpha}} \sum_{l=1}^{r_\nu-1} \frac{F_{0,2} \left[\begin{smallmatrix} \mu & \nu \\ -k & -l \end{smallmatrix} \right]}{k} \frac{r_\nu-l}{r_\nu} d\xi_{l-r_\nu}^{\nu} \quad (7.11)$$

is holomorphic on $\tilde{\mathcal{C}}$. Since $\tilde{\mathcal{C}}$ is compact and smooth, $H_1(\tilde{\mathcal{C}}, \mathbb{C})$ is a finite-dimensional symplectic space equipped with the intersection pairing, and

$$\mathbb{K} := \left\{ \gamma \in H_1(\tilde{\mathcal{C}}, \mathbb{C}) \mid \int_{\gamma} \omega_{0,2}(\cdot, z) = 0 \right\}$$

is a Lagrangian subspace. From definition (7.9), we see that integrating any $d\xi_{-k}^{\mu}$ for $k > 0$ and $\mu \in \tilde{\alpha}$ along a cycle in \mathbb{K} gives zero. The same is true for the first term in (7.11) since it is an exact form. As the period map induces a non-degenerate pairing $H^0(\tilde{\mathcal{C}}, K_{\tilde{\mathcal{C}}}) \otimes \mathbb{K} \rightarrow \mathbb{C}$ and (7.11) is sent to 0, we deduce the identity between meromorphic forms

$$d\left(\frac{d\xi_{-k}^{\mu}}{d\tilde{x}}\right) + \frac{k+r_{\mu}}{r_{\mu}} d\xi_{-(k+r_{\mu})}^{\mu} + \sum_{v \in \tilde{\alpha}} \sum_{l=1}^{r_v-1} \frac{F_{0,2} \left[\begin{smallmatrix} \mu & v \\ -k & -l \end{smallmatrix} \right]}{k} \frac{r_v-l}{r_v} d\xi_{l-r_v}^v = 0. \quad (7.12)$$

We now would like to apply the local Laplace transform $\mathfrak{L}_v^+[\cdot](v)$ to this relation. Recall that \mathfrak{L}^+ by definition kills the polar part. So, by direct computation on the basis elements with Definition 7.1, we have

$$\forall \varphi \in H^0(\tilde{\mathcal{C}}, K_{\tilde{\mathcal{C}}}(*\tilde{\alpha})), \quad \mathfrak{L}_{\rho}^+ \left[d\left(\frac{\varphi}{d\tilde{x}}\right) \right](v) = \frac{1}{r_{\rho}} [v^{-1} \mathfrak{L}_{\rho}^+[\varphi](v)]_+,$$

where $[\cdot]_+$ keeps only the nonnegative powers of v . Expansion (7.10) implies for any meromorphic form

$$\mathfrak{L}_v^+[d\xi_{-k}^{\mu}](v) = \sum_{l>0} \frac{F_{0,2} \left[\begin{smallmatrix} \mu & v \\ -k & -l \end{smallmatrix} \right]}{k} (l-r_v)!^{(r_v)} e_{v,\hat{l}} v^{\bar{l}}.$$

By comparison with (7.6), $B_{\mu,\rho}(u, v)$ can be obtained by the generating series

$$B_{\mu,\rho}(u, v) = \sum_{k>0} k!^{(r_{\mu})} e_{\mu,\hat{k}} u^{\bar{k}} \otimes \mathfrak{L}_{\rho}^+[d\xi_{-k}^{\mu}](v).$$

Applying $\mathfrak{L}_{\rho}^+[\cdot](v)$ to (7.12), multiplying by $k!^{(r_{\mu})} e_{\mu,\hat{k}} u^{\bar{k}}$, and summing over $k > 0$ then yields

$$\begin{aligned} & (r_{\rho}v)^{-1} (B_{\mu,\rho}(u, v) - B_{\mu,\rho}(u, 0)) + (r_{\mu}u)^{-1} (B_{\mu,\rho}(u, v) - B_{\mu,\rho}(0, v)) \\ & + \sum_{v \in \tilde{\alpha}} \frac{1}{r_v} \sum_{l=1}^{r_v-1} (\text{id} \otimes e_{v,l}^*) [B_{\mu,v}(u, 0)] (e_{v,r_v-l}^* \otimes \text{id}) [B_{v,\rho}(0, v)] = 0. \end{aligned}$$

Noticing that $\eta(e_{v,l} \otimes e_{v,r_v-m}) = r_v^{-1} \delta_{l,m}$ for $l, m \in [r_v - 1]$ and $\eta(e_{\mu,r_{\mu}}, -) = 0$, this can be rewritten as

$$\begin{aligned} & (r_{\mu}u + r_{\rho}v) B_{\mu,\rho}(u, v) - r_{\mu}u B_{\mu,\rho}(u, 0) - r_{\rho}v B_{\mu,\rho}(0, v) \\ & + r_{\mu}r_{\rho}uv \sum_{v \in \tilde{\alpha}} (\text{id} \otimes \eta \otimes \text{id}) [B_{\mu,v}(u, 0) \otimes B_{v,\rho}(0, v)] = 0. \end{aligned}$$

Replacing u and v by u/r_μ and v/r_ρ , respectively, and summing over $\mu, \rho \in \tilde{\alpha}$, this implies the desired formula (7.7). Since the left-hand side is a formal power series in u, v , it can be specialised at $v = -u$. This is possible on the right-hand side if and only if (7.8) is satisfied. ■

For each $\mu \in \tilde{\alpha}$, we have an orthogonal direct sum $V_\mu \cong {}^s V_\mu \oplus {}^0 V_\mu$ with

$${}^s V_\mu = \bigoplus_{l=1}^{r_\mu-1} \mathbb{C}.e_{\mu,l}, \quad {}^0 V_\mu = \mathbb{C}.e_{\mu,r_\mu},$$

and we decompose $V \cong {}^0 V \oplus {}^s V$ with

$${}^s V := \bigoplus_{\mu \in \tilde{\alpha}} {}^s V_\mu, \quad {}^0 V := \bigoplus_{\mu \in \tilde{\alpha}} {}^0 V_\mu.$$

Definition 7.11. We introduce two generating series

$$\begin{aligned} R(u) &:= \eta^\perp - u \bar{B}(u, 0) \in (V^s)^{\otimes 2}[[u]], \\ R^\perp(u) &:= \eta^\perp - u \bar{B}(0, u) \in (V^s)^{\otimes 2}[[u]], \end{aligned}$$

where $\eta^\perp \in ({}^s V)^{\otimes 2} \subset V^{\otimes 2}$ is induced by the pairing η , i.e.,

$$\eta^\perp = \sum_{\mu \in \tilde{\alpha}} \left(\sum_{l=1}^{r_\mu-1} r_\mu e_{\mu,l} \otimes e_{\mu,r_\mu-l} \right), \quad (7.13)$$

We write ${}^{00}B$, ${}^{0s}B$, ${}^{s0}B$, and ${}^{ss}B$ for the projections of \bar{B} on the adequate subspaces in both arguments, and likewise for R .

Corollary 7.12. *Under the assumptions of Proposition 7.9, we have*

$$\begin{aligned} {}^{ss}\bar{B}(u, v) &= \frac{1}{u+v} (\eta^\perp - {}^{ss}R(u) \star {}^{ss}R^\perp(v)), \\ {}^{s0}\bar{B}(u, v) &= \frac{-1}{u+v} ({}^{s0}R(u) + {}^{ss}R(u) \star {}^{s0}R^\perp(v)), \\ {}^{0s}\bar{B}(u, v) &= \frac{-1}{u+v} ({}^{0s}R^\perp(v) + {}^{0s}R(u) \star {}^{ss}R^\perp(v)), \\ {}^{00}\bar{B}(u, v) &= \frac{-1}{u+v} ({}^{00}R(u) + {}^{00}R^\perp(v) + {}^{0s}R(u) \star {}^{s0}R^\perp(v)), \end{aligned}$$

and the following compatibility relations hold:

$$\begin{aligned} {}^{ss}R(u) \star {}^{ss}R^\perp(-u) &= \eta^\perp, \\ {}^{ss}R(u) \star {}^{s0}R^\perp(-u) &= -{}^{s0}R(u), \\ {}^{0s}R(u) \star {}^{ss}R^\perp(-u) &= -{}^{0s}R^\perp(-u), \\ {}^{0s}R(u) \star {}^{s0}R^\perp(-u) &= -{}^{00}R(u) - {}^{00}R^\perp(-u). \end{aligned}$$

The structure of the first line is familiar from Givental formalism and from [33, Appendix B], but the other three are new.

7.2.4. Primary differentials and their descendants.

Definition 7.13. We consider a new family of meromorphic differentials on \tilde{C} , indexed by $\mu \in \tilde{\alpha}$ and $k > 0$:

$$d\hat{\xi}_{-k}^\mu := \mathcal{D}^{\bar{k}}(d\xi_{-\hat{k}}^\mu), \quad \text{where } \mathcal{D}(\varphi) := -d(\varphi/d\tilde{x}). \tag{7.14}$$

For $k \in [r_\mu]$, they are equal to $d\xi_{-k}^\mu$, while the $k > r_\mu$ ones are called descendants. Borrowing the language of Frobenius manifolds, we call $(d\xi_{-k}^\mu)_{\mu,k}$ the canonical basis and $(d\hat{\xi}_{-k}^\mu)_{\mu,k}$ the flat basis.

The aim of this section is to compute the change of basis, from the canonical basis to the flat basis of differentials, when \tilde{C} is smooth compact connected as in the previous section. This generalises computations done in [33] for simple ramifications. As we shall see in Section 7.5.4, the correspondence between topological recursion and intersection theory takes a particularly nice form in the $\hat{\xi}$ -basis.

By an easy recursion from (7.12), passing from ξ to $\hat{\xi}$ is change of basis with triangular structure. Thus, there exist scalars $A \begin{bmatrix} \mu & \nu \\ -k & -l \end{bmatrix}$ indexed by $\mu, \nu \in \tilde{\alpha}$ and $k, m > 0$, such that

$$d\xi_{-k}^\mu = \sum_{\substack{\rho \in \tilde{\alpha} \\ m > 0}} A \begin{bmatrix} \mu & \rho \\ -k & -m \end{bmatrix} d\hat{\xi}_{-m}^\rho, \tag{7.15}$$

where the sum on the right-hand side is finite.

Definition 7.14. We introduce

$$A_{\mu,\nu}(u, v) = \left(\sum_{k,l > 0} A \begin{bmatrix} \mu & \nu \\ -k & -l \end{bmatrix} k!^{(r_\mu)} e_{\mu,\hat{k}} \otimes e_{\nu,\hat{l}} u^{\bar{k}} v^{\bar{l}} \right) \in V_\mu \otimes V_\nu \llbracket u, v \rrbracket, \quad \mu, \nu \in \tilde{\alpha},$$

$$\bar{A}(u, v) = \left(\sum_{\mu,\nu \in \tilde{\alpha}} A_{\mu,\nu}(u/r_\mu, v) \right) \in V^{\otimes 2} \llbracket u, v \rrbracket,$$

$$\eta^\vee = \sum_{\mu \in \tilde{\alpha}} \sum_{k=1}^{r_\mu} k e_{\mu,k} \otimes e_{\mu,k}.$$

Note that in \bar{A} we do not rescale the variable v , unlike for \bar{B} . Due to the definition of the primary differentials, we have $\bar{A}(0, v) = \bar{A}(0, 0) = \eta^\vee$. We use the notation ss for the projection to $(V^s)^{\otimes 2}$ as in Definition 7.11.

Proposition 7.15. Assume that \tilde{C} is a smooth compact connected curve, $d\tilde{x}$ is a meromorphic 1-form on \tilde{C} and $\omega_{0,2}$ is a fundamental bidifferential of the second kind on \tilde{C} .

We have

$$\bar{A}(u, v) = \frac{\eta^\vee - u\bar{B}(u, 0) \star \eta^\vee}{1 - uv}.$$

In particular,

$${}^{ss}\bar{A}(u, v) = \frac{R(u) \star \eta^\vee}{1 - uv}.$$

Proof. Inserting the change of basis (7.15) into relation (7.12) and identifying the coefficient of $d\hat{\xi}_{-m}^\rho$ yields, for any $\mu, \rho \in \tilde{\alpha}$ and $k, m > 0$,

$$A\left[\begin{smallmatrix} \mu & \rho \\ -k & -m+r_\rho \end{smallmatrix}\right] = \frac{k+r_\mu}{r_\mu} A\left[\begin{smallmatrix} \mu & \rho \\ -(k+r_\mu) & -m \end{smallmatrix}\right] + F_{0,2}\left[\begin{smallmatrix} \mu & \rho \\ -k & m-r_\rho \end{smallmatrix}\right] \frac{m}{kr_\rho} \delta_{m < r_\rho},$$

where by convention A with nonnegative lower indices vanish. Multiplying this equality by $k!(r_\mu)e_{\mu, \hat{k}} \otimes e_{\rho, \hat{m}} u^{\bar{k}} v^{\bar{m}}$ and summing over $k, m > 0$, we obtain

$$\begin{aligned} vA_{\mu, \rho}(u, v) &= (r_\mu u)^{-1}(A_{\mu, \rho}(u, v) - A_{\mu, \rho}(0, v)) \\ &\quad + \sum_{k>0} \sum_{m=1}^{r_\rho-1} F_{0,2}\left[\begin{smallmatrix} \mu & \rho \\ -k & m-r_\rho \end{smallmatrix}\right] (k-r_\mu)! \binom{r_\mu}{r_\rho} e_{\mu, \hat{k}} \otimes e_{\rho, m} u^{\bar{k}} \\ &= (r_\mu u)^{-1}(A_{\mu, \rho}(u, v) - A_{\mu, \rho}(0, v)) \\ &\quad + \left(\sum_{k, l > 0} F_{0,2}\left[\begin{smallmatrix} \mu & \rho \\ -k & -l \end{smallmatrix}\right] (k-r_\mu)! (l-r_\rho)! \binom{r_\rho}{r_\rho} e_{\nu, \hat{k}} \otimes e_{\rho, l} u^{\bar{k}} v^{\bar{l}} \right) \star \eta^\vee \\ &= (r_\mu u)^{-1}(A_{\mu, \rho}(u, v) - A_{\mu, \rho}(0, v)) + B_{\mu, \rho}(u, 0) \star \eta^\vee, \end{aligned}$$

where we have used $\eta(e_{\rho, l}, e_{\nu, m}) = r_\rho^{-1} \delta_{\nu, \rho} \delta_{l+m, r_\rho}$, the fact that $(l-r_\rho)! \binom{r_\rho}{r_\rho} = 1$ when $l < r_\rho$, which are the only terms contributing to the sum, and recognised $B_{\mu, \rho}(u, 0)$ from Definition 7.7. We now make the substitution $u \mapsto u/r_\mu$, sum over $\mu, \rho \in \tilde{\alpha}$ and use $\bar{A}(0, v) = \eta^\vee$. This results in

$$v\bar{A}(u, v) = u^{-1}(\bar{A}(u, v) - \eta^\vee) + \bar{B}(u, 0) \star \eta^\vee,$$

whose solution is

$$\bar{A}(u, v) = \frac{\eta^\vee - u\bar{B}(u, 0) \star \eta^\vee}{1 - uv}.$$

As the projection of η^\vee onto $(V^s)^{\otimes 2}$ coincides with $\eta^\perp \star \eta^\vee$, we deduce

$${}^{ss}\bar{A}(u, v) = \frac{(\eta^\perp - u\bar{B}(u, 0)) \star \eta^\vee}{1 - uv},$$

where we recognise $R(u)$ from Definition 7.11. ■

7.3. Review of Witten r -spin classes

For $r \geq 2$, we denote by $w_{g,n}^{r\text{spin}}(k_1, \dots, k_n)$ the Witten r -spin class, first imagined by [60] and constructed in [26, 52], cf. also [50]. For all integers $g, n \geq 0$ such that $2g - 2 + n > 0$ and $k_1, \dots, k_n \in \mathbb{Z}$, this is a Chow class

$$w_{g,n}^{r\text{spin}}(k_1, \dots, k_n) \in CH^*(\bar{\mathcal{M}}_{g,n}).$$

It is defined via the moduli space of r -spin structures $\bar{\mathcal{M}}_{g;k_1, \dots, k_n}^{r\text{spin}}$ parametrizing pointed curves (C, p_1, \dots, p_n) with a line bundle $L \rightarrow C$ together with an isomorphism of $L^{\otimes r}$ to $K_C^{\log}(-\sum_{i=1}^n k_i p_i)$. Denoting by \mathcal{C} the universal curve and by \mathcal{L} the universal line bundle, and considering the projection

$$\mathcal{L} \rightarrow \mathcal{C} \xrightarrow{\pi} \bar{\mathcal{M}}_{g;k_1, \dots, k_n}^{r\text{spin}} \xrightarrow{p} \bar{\mathcal{M}}_{g,n},$$

the naive definition is

$$w_{g,n}^{r\text{spin}}(k_1, \dots, k_n) = r^{-g} p_* c_{\text{top}}((R^1 \pi_* \mathcal{L})^\vee).$$

This works in genus 0, but if $g > 0$, then $R^1 \pi_* \mathcal{L}$ is not a vector bundle as $R^0 \pi_* \mathcal{L}$ is non-zero, so the general construction is more involved. For positive k_i , Witten’s class vanishes if one of the k_i is divisible by r . It is concentrated in codimension³

$$D = \frac{(r - 2)(g - 1) - n + \sum_{i=1}^n k_i}{r}. \tag{7.16}$$

In particular, the class vanishes unless the right-hand side of (7.16) is an integer, and its integration on $\bar{\mathcal{M}}_{g,n}$ vanishes unless $D = \dim \bar{\mathcal{M}}_{g,n} = 3g - 3 + n$.

We are primarily interested in indices ranging over $[r]$, or $[r - 1]$ since the class vanishes for index equal to r , but the following property, conjectured by Jarvis–Kimura–Vaintrob [42, Descent Axiom 1.9], explains the appearance of the r -fold factorial in all our formulas, see also [27, Lemma 4.2.8].

Lemma 7.16 ([52, Proposition 5.1]). *Let $g, n, k_1, \dots, k_n \geq 0$ be integers such that $2g - 2 + n > 0$, and decompose $k_i = d_i r + a_i$ with $a_i \in [r]$ and $d_i \geq 0$. We have*

$$w_{g,n}^{r\text{spin}}(k_1, \dots, k_n) = w_{g,n}^{r\text{spin}}(a_1, \dots, a_n) \prod_{i=1}^n r^{-d_i} (k - r)!^{(r)} \psi_i^{d_i}.$$

This formula is consistent with the case $k \in [r]$ due to the initial condition in definition (7.1) of the r -fold factorial.

³The notion of codimension for Chow classes refers to homology. Therefore, when the Chow class can be realised by a cohomology class, this codimension is twice the cohomological degree.

Remark 7.17. Witten’s class can be interpreted as a cohomological field theory in the following way: its vector space $V = V^{r\text{spin}}$ has a basis $(e_i)_{i=1}^{r-1}$ and we define for $a_1, \dots, a_n \in [r - 1]$

$$W_{g,n}^{r\text{spin}}(e_{a_1} \otimes \dots \otimes e_{a_n}) := w_{g,n}^{r\text{spin}}(a_1, \dots, a_n) \in CH^*(\bar{\mathcal{M}}_{g,n}).$$

The vector space $V^{r\text{spin}}$ is equipped with product and multiplication

$$(e_a | e_b) = \frac{\delta_{a+b,r}}{r}, \quad (e_{a_1} \bullet e_{a_2} | e_{a_3}) = W_{0,3}(e_{a_1} \otimes e_{a_2} \otimes e_{a_3}) = \delta_{a_1+a_2+a_3,r+1}.$$

We included a factor r^{-1} to align with (7.5). It could be removed by rescaling the product without changing the class $W_{g,n}^{r\text{spin}}$ and without effect on the formulas we are going to write.

We define the partition function of the r -spin theory by

$$\begin{aligned} Z^{r\text{spin}}[(x_k)_{k>0}] &= \exp\left(\sum_{\substack{g,n \in \mathbb{Z}_{\geq 0} \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{k_1, \dots, k_n > 0} \left(\int_{\bar{\mathcal{M}}_{g,n}} w_{g,n}^{r\text{spin}}(k_1, \dots, k_n)\right) \prod_{i=1}^n r^{\lfloor \frac{k_i}{r} \rfloor} k_i x_{k_i}\right) \\ &= \exp\left(\sum_{\substack{g,n \in \mathbb{Z}_{\geq 0} \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\substack{a_1, \dots, a_n \in [r-1] \\ d_1, \dots, d_n \geq 0}} \left(\int_{\bar{\mathcal{M}}_{g,n}} w_{g,n}^{r\text{spin}}(a_1, \dots, a_n) \prod_{i=1}^n \psi_i^{d_i}\right) \right. \\ &\quad \left. \times \prod_{i=1}^n (d_i r + a_i)!^{(r)} x_{d_i r + a_i}\right), \end{aligned}$$

where we took into account the dimension constraint (7.16) to get the second line. Due to the aforementioned vanishing, it is independent of the times with indices divisible by r . The function $Z^{r\text{spin}}$ was identified in [39] with the tau function for the r -KdV hierarchy. It is also known that $Z^{r\text{spin}}$ satisfies $\mathcal{W}(\mathfrak{gl}_r)$ -constraints; see [9] for the history of this result. From there, Milanov proved in [48] that $Z^{r\text{spin}}$ is the partition function of the Airy structure of Theorem 2.10 with $(r, s) = (r, r + 1)$, as well as the topological recursion à la Bouchard–Eynard [14] for the associated correlators

$$\begin{aligned} \omega_{g,n}^{r\text{spin}}(\zeta_1, \dots, \zeta_n) &= \sum_{\substack{a_1, \dots, a_n \in [r-1] \\ d_1, \dots, d_n \geq 0}} \left(\int_{\bar{\mathcal{M}}_{g,n}} w_{g,n}(a_1, \dots, a_n) \prod_{i=1}^n \psi_i^{d_i}\right) \\ &\quad \times \prod_{i=1}^n \frac{(d_i r + a_i)!^{(r)} d\zeta_i}{\zeta_i^{d_i r + a_i + 1}}. \end{aligned} \tag{7.17}$$

This result was apparently also obtained by Bouchard and Eynard in an unpublished draft, and appeared in [30] in a form closer to the one we state here.

Theorem 7.18. *The correlators $\omega^{r\text{spin}}$ are computed by the topological recursion for the spectral curve (without crosscap)*

$$x(\zeta) = \zeta^r, \quad y(\zeta) = -\frac{\zeta}{r}, \quad \omega_{0,2}(\zeta_1, \zeta_2) = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2}. \quad (7.18)$$

Proof. We start from [30, Theorem 7.3], which shows that the topological recursion for the spectral curve

$$x(z) = z^r, \quad y(z) = z, \quad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \quad (7.19)$$

yields

$$\begin{aligned} \sum_{\substack{a_1, \dots, a_n \in [r-1] \\ d_1, \dots, d_n \geq 0}} (-r)^{2-2g-n} \left(\int_{\bar{\mathcal{M}}_{g,n}} w_{g,n}^{r\text{spin}}(a_1, \dots, a_n) \prod_{i=1}^n \psi_i^{d_i} \right) \\ \times \prod_{i=1}^n \frac{(d_i r + a_i)!^{(r)} dz_i}{z_i^{dr_i + a_i + 1}}. \end{aligned} \quad (7.20)$$

The spectral curve (7.18) can be obtained from (7.19) by multiplying y by $-\frac{1}{r}$. This multiplies (7.20) by $(-r)^{2g-2+n}$. So the factors of r cancel, and we indeed obtain the correlators $\omega_{g,n}^{r\text{spin}}(\zeta_1, \dots, \zeta_n)$. ■

7.4. Deformations on Witten r -spin classes

We now recall well-known actions on family of classes, originating from the work of Givental. As our focus is not on cohomological field theories, some of the actions we allow may not preserve this property and do not belong *stricto sensu* to the Givental group. See, e.g., [50, 54, 57] for more background.

7.4.1. Translations. Given a formal series $\mathbb{T}(u) \in uV^{r\text{spin}}[[u]]$, we can define a new family of Chow classes

$$[\widehat{\mathbb{T}} \cdot W^{r\text{spin}}]_{g,n}: (V^{r\text{spin}})^{\otimes n} \rightarrow CH^*(\bar{\mathcal{M}}_{g,n}).$$

We first decompose

$$\mathbb{T}(u) = \sum_{\substack{d \geq 1 \\ a \in [r-1]}} \mathbb{T}_{rd+a} e_a u^d$$

and assume that $\mathbb{T}_{r+1} \neq 1$. Then we introduce

$$\tilde{\mathbb{T}}(u) := \frac{\mathbb{T}(u) - \mathbb{T}_{r+1} e_1 u}{1 - \mathbb{T}_{r+1}} = \sum_{\substack{d \geq 1 \\ a \in [r-1] \\ (d,a) \neq (1,1)}} \frac{\mathbb{T}_{rd+a}}{1 - \mathbb{T}_{r+1}} e_a u^d$$

and set

$$[\widehat{\mathbb{T}} \cdot W_{g,n}^{r\text{spin}}](-) = \sum_{m \geq 0} \frac{1}{m!} (\pi_m)_* W_{g,n+m}^{r\text{spin}}(- \otimes \widetilde{\mathbb{T}}(\psi_{n+1}) \otimes \cdots \otimes \widetilde{\mathbb{T}}(\psi_{n+m})),$$

where $\pi_m: \overline{\mathcal{M}}_{g,n+m} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the forgetful morphism. This definition is well-posed, i.e., the sum has finitely many non-zero terms. Indeed, if we evaluate on $\bigotimes_{i=1}^n e_{a_i}$, the codimension (after pushforward) of the summand proportional to $\prod_{j=1}^m \tau_{d_j r + b_j}$ with $d_j r + b_j \geq r + 2$ is

$$\begin{aligned} & \frac{1}{r} \left((g-1)(r-2) - (n+m) + \sum_{i=1}^n a_i + \sum_{j=1}^m (d_j r + b_j) \right) - m \\ & \geq \frac{1}{r} \left((g-1)(r-2) - n + \sum_{i=1}^n a_i \right) + \frac{m}{r}, \end{aligned}$$

which for fixed g, n, a_1, \dots, a_n becomes larger than $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$ for m large enough, forcing this summand to vanish. The coefficient τ_{r+1} plays a special role, which reflects the dilaton equation

$$(p_1)_*(w_{g,n+1}^{r\text{spin}}(a_1, \dots, a_n, 1) \cdot \psi_{n+1}) = (2g - 2 + n) w_{g,n}^{r\text{spin}}(a_1, \dots, a_n).$$

The change from \mathbb{T} to $\widetilde{\mathbb{T}}$ reflects this.

7.4.2. Sums over stable graphs. Let now V be a finite-dimensional vector space and consider a family of classes

$$\Omega_{g,n}: V^{\otimes n} \rightarrow CH^*(\overline{\mathcal{M}}_{g,n}), \quad g, n \in \mathbb{Z}_{\geq 0}, 2g - 2 + n > 0.$$

Given a symmetric formal power series $B(u_1, u_2) \in V^{\otimes 2}[[u_1, u_2]]$, we can define a new such family $[\widehat{\mathbb{B}} \cdot \Omega_{g,n}]_{g,n}$ by sums over stable graphs.

Let $G_{g,n}$ be the set of stable graphs of type (g, n) . For a vertex v in a stable graph, we denote by $h(v)$ the genus and by $k(v)$ the valency. Then

$$[\widehat{\mathbb{B}} \cdot \Omega]_{g,n} = \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut } \Gamma|} \eta_{\Gamma} (\iota_{\Gamma})_* \left[\prod_{v \in \text{Vert}(\Gamma)} \Omega_{h(v), k(v)} \prod_{\{e, e'\} \in \text{Edge}(\Gamma)} B(\psi_e, \psi_{e'}) \right],$$

where $\iota_{\Gamma}: \prod_{v \in \text{Vert}(\Gamma)} \overline{\mathcal{M}}_{h(v), k(v)} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the natural inclusion of the boundary stratum associated to Γ . To read this formula, half-edges label the punctures on the curves whose moduli spaces sit at the vertices. So, we have ψ -classes $\psi_e, \psi_{e'}$ associated to an edge $\{e, e'\}$, and for each half-edge there is a copy of V^* coming from the vertex it starts from, and a copy of V coming from the contribution of the edge it belongs to. The symbol η_{Γ} indicates that we pair them in the natural way. This

definition is well-posed because the dimension of the moduli spaces at the vertices is smaller than the one of $\bar{\mathcal{M}}_{g,n}$, so that only finitely many powers of ψ -classes can contribute in the sum.

Remark 7.19. This differs slightly from the so-called R -action in Givental formalism, by the fact that we do not decorate leaves of the stable graph, and we do not assume that B has a factorisation property in terms of an R -matrix in the style of (7.7).

7.5. Intersection theory for regularly admissible spectral curves

Let $\mathcal{S} = (C, x, y, \omega_{0,2})$ be a regularly admissible spectral curve equipped with a fundamental bidifferential of the second kind. In particular, C must be smooth, and we can write C and α instead of \tilde{C} and $\tilde{\alpha}$, respectively. In this context, as in Definition 7.11, we rather take

$$V = \bigoplus_{\alpha \in \alpha} V^{r_\alpha \text{ spin}}.$$

This amounts to set $e_{\alpha, r_\alpha} = 0$ in all subsequent formulas.

Following Definition 7.6, we have a generating series

$$\begin{aligned} \mathbb{T}(u) &:= \left(\sum_{\alpha \in \alpha} e_{\alpha,1} u \right) + \mathfrak{L}_{\text{tot}}^+(\omega_{0,1}) \\ &= \sum_{\alpha \in \alpha} \left(e_{\alpha,1} u + \sum_{k > 0} (k - r_\alpha)!^{(r_\alpha)} F_{0,1} \left[\begin{smallmatrix} \alpha \\ -k \end{smallmatrix} \right] e_{\alpha, \hat{k}} u^{\bar{k}} \right) \\ &=: \sum_{\alpha \in \alpha} \sum_{k > 0} \mathbb{T}_{\alpha,k} e_{\alpha, \hat{k}} u^{\bar{k}}. \end{aligned} \tag{7.21}$$

Equivalently, the definition of \mathbb{T} means that we have the expansion

$$\text{Loc}_\alpha(y - y(\alpha)) = -\frac{\zeta}{r_\alpha} + \sum_{k > 0} \frac{k!^{(r_\alpha)} \mathbb{T}_{\alpha,k+r_\alpha}}{r_\alpha} \zeta^k d\zeta$$

when $z \rightarrow \alpha$ in the local coordinate such that $x = x(\alpha) + \zeta^{r_\alpha}$. Due to the regularly admissible condition, we indeed have $\mathbb{T}(u) \in \mathcal{O}(u)$ and

$$\mathbb{T}_{\alpha, r_\alpha+1} = 1 + F_{0,1} \left[\begin{smallmatrix} \alpha \\ -(r_\alpha+1) \end{smallmatrix} \right] \neq 1,$$

so we can use it to act on Witten r_α -spin class. Recall that we have a second generating series $B(u, v)$ from Definition 7.7.

Definition 7.20. We let these generating series act on the direct sum of Witten classes to define

$$\Omega_{g,n}^{\mathcal{S}} := \left[\widehat{B\hat{T}} \cdot \left(\bigoplus_{\alpha \in \alpha} W^{r_\alpha \text{ spin}} \right) \right]_{g,n} : V^{\otimes n} \rightarrow CH^*(\bar{\mathcal{M}}_{g,n}).$$

Theorem 7.21. *If $\mathcal{S} = (C, x, y, \omega_{0,2})$ is a regularly admissible spectral curve equipped with a fundamental bidifferential of the second kind $\omega_{0,2}$ and zero crosscap form, then for $2g - 2 + n > 0$,*

$$(\mathfrak{L}_{\text{tot}}^-)^{\otimes n}(\omega_{g,n}) = \int_{\bar{\mathcal{M}}_{g,n}} \frac{\Omega_{g,n}^{\mathcal{S}}}{\prod_{i=1}^n (1 - \epsilon_i \psi_i)},$$

where ϵ_i is the variable in the i -th Laplace transform.

Proof. The spectral curve

$$x_0\left(\frac{\alpha}{z}\right) = z^{r\alpha}, \quad y_0\left(\frac{\alpha}{z}\right) = -\frac{z}{r}, \quad \omega_{0,2}\left(\frac{\alpha_1}{z_1} \frac{\alpha_2}{z_2}\right) = \frac{\delta_{\alpha_1, \alpha_2} dz_1 dz_2}{(z_1 - z_2)^2}$$

has $T = 0$ and $B = 0$, hence $\Omega^{\mathcal{S}} = \bigoplus_{\alpha \in \mathfrak{a}} W_{g,n}^{r\alpha \text{ spin}}$. It is obtained by taking independent copies of (7.18) indexed by $\alpha \in \mathfrak{a}$. So, its correlators are equal to (7.17), that is,

$$\begin{aligned} \omega_{g,n}^0\left(\frac{\alpha_1}{z_1} \dots \frac{\alpha_n}{z_n}\right) &= \sum_{\substack{d_1, \dots, d_n \geq 0 \\ l_i \in [r\alpha_i]}} \left(\int_{\bar{\mathcal{M}}_{g,n}} W_{g,n}^0 \left(\bigotimes_{i=1}^n e_{\alpha_i, l_i} \right) \prod_{i=1}^n \psi_i^{d_i} \right) \\ &\quad \times \prod_{i=1}^n \frac{(d_i r\alpha_i + l_i)!^{(r\alpha_i)} dz_i}{z_i^{d_i r\alpha_i + l_i + 1}}, \end{aligned} \tag{7.22}$$

where $W_{g,n}^0 = \bigoplus_{\alpha \in \mathfrak{a}} W^{r\alpha \text{ spin}}$. Applying $\bigotimes_{i=1}^n \mathfrak{L}_{\alpha_i}^-$ to equation (7.22) cancels the factorials and replaces $z^{-(d_i r\alpha_i + l_i + 1)} dz$ with $e_{\alpha_i, l_i}^* \in^{d_i}$. The sum over d_i can then be performed, and this entails the claim in this special case.

Applying Theorem 5.23 to the special case, let Z_0 be the partition function of the Airy structure corresponding to this special case. Its coefficients are

$$F_{g,n}^0\left[\frac{\alpha_1}{k_1} \dots \frac{\alpha_n}{k_n}\right] := \left(\int_{\bar{\mathcal{M}}_{g,n}} W_{g,n} \left(\bigotimes_{i=1}^n e_{\alpha_i, \hat{k}_i} \right) \prod_{i=1}^n \psi_i^{\hat{k}_i} \right) \prod_{i=1}^n k_i!^{(r\alpha_i)}. \tag{7.23}$$

It corresponds to $F_{0,1}^0[-(r\alpha+1)] = -1$.

Applying now Theorem 5.23 to a general regularly admissible spectral curve, the partition function of the corresponding Airy structure is

$$\begin{aligned} Z &= \exp\left(\frac{1}{2\hbar} \sum_{\substack{\alpha, \beta \in \mathfrak{a} \\ k, l > 0}} F_{0,2}\left[\begin{matrix} \alpha & \beta \\ -k & -l \end{matrix}\right] \frac{J_k^\alpha J_l^\beta}{k l}\right) \cdot Z_1, \\ Z_1 &= \exp\left(\frac{1}{\hbar} \sum_{\substack{\alpha \in \mathfrak{a} \\ k > 0}} (F_{0,1}\left[\begin{matrix} \alpha \\ -k \end{matrix}\right] + \delta_{k, r\alpha+1}) \frac{J_k^\alpha}{k}\right) \cdot Z_0. \end{aligned}$$

The operation taking Z_0 to Z_1 is the shift of times

$$x_{\alpha,k} \rightarrow x_{\alpha,k} + \frac{F_{0,1}[\frac{\alpha}{-k}] + \delta_{k,r_\alpha+1}}{k} = x_{\alpha,k} + \frac{\mathbb{T}_{\alpha,k}}{k!(r_\alpha)},$$

taking into account (7.21). In terms of the coefficients $F_{g,n}^1$ of Z_1 , we get

$$\begin{aligned} F_{g,n}^1[\frac{\alpha_1 \dots \alpha_n}{k_1 \dots k_n}] &= \sum_{m \geq 0} \frac{1}{m!} \sum_{\substack{\beta_1, \dots, \beta_n \in \alpha \\ l_1, \dots, l_m > 0}} F_{g,n+m}^0[\frac{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_m}{k_1 \dots k_n l_1 \dots l_m}] \prod_{i=1}^m \frac{\mathbb{T}_{\beta_i, l_i}}{l_i!(r_{\beta_i})} \\ &= \left(\int_{\bar{\mathcal{M}}_{g,n}} [\hat{\mathbb{T}} \cdot W^0]_{g,n} \left(\bigotimes_{i=1}^n e_{\alpha_i, \hat{k}_i} \right) \prod_{i=1}^n \psi_i^{\bar{k}_i} \right) \prod_{i=1}^n k_i!(r_{\alpha_i}) \end{aligned}$$

by using (7.23) and comparing with Section 7.4.1. Note the cancellation of the factorials that was the motivation for our definition of \mathbb{T} in (7.21). Applying $\mathcal{L}_{\text{tot}}^-$ to the corresponding correlators kills the remaining factorials, we would prove the desired formula in the case $B = 0$.

For general B , we should still take Z_1 to Z , and this amounts at the level of coefficients to summing over stable graphs. Comparing with Section 7.4.2, one can check in a similar way that the factorials completely disappear, so that the sum over d_i becomes the geometric series in the Laplace variable ϵ_i . ■

7.5.1. The conjectural (r, s) classes. Consider the basic case of irregularly admissible smooth spectral curves with one ramification point:

$$x = z^r, \quad y = -\frac{z^{s-r}}{r}, \quad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

with $r \geq 2$, $s \in [r - 1]$ and $r = \pm 1 \pmod s$. It corresponds to the Airy structures of Theorem 2.10, already obtained in [9]. The coefficients of its partition function have the following basic properties from Corollary 2.19 and Proposition 6.1:

- (i) Homogeneity: $F_{g,n}[p_1, \dots, p_n] = 0$ unless we have $\sum_{m=1}^n p_m = s(2g - 2 + n)$.
- (ii) Dilaton equation: $F_{g,n+1}[s, p_1, \dots, p_n] = s(2g - 2 + n)F_{g,n}[p_1, \dots, p_n]$ for $2g - 2 + n > 0$.
- (iii) Special values:

$$\begin{aligned} F_{1,1}[p] &= \frac{r^2 - 1}{24} \delta_{p,s}, \\ F_{0,3}[p_1, p_2, p_3] &= c p_1 p_2 p_3 \delta_{p_1+p_2+p_3,s}, \end{aligned}$$

where c is as in equation (6.4).

- (iv) If $s = 1$, then $F_{0,n} = 0$ for any $n \geq 3$. Indeed, as $c = 0$ in this case, we have $F_{0,3} = 0$, and as it is the only initial data needed for topological recursion (2.32) in genus 0, all the genus 0 sector vanishes.
- (v) $F_{g,n}[p_1, \dots, p_n] = 0$ whenever there exists $i \in [n]$ such that $r \mid p_i$.

Remark 7.22. There is no string equation, as the operator $H_{i=2,k=-1}$ is not part of the Airy structure.

Mimicking Theorem 7.18 and taking into account these properties, we are led to propose the following conjecture, in a slightly more precise form than [9, Section 6.2].

Conjecture 7.23. *For each $r \geq 2$ and $s \in [r - 1]$ such that $r = \pm 1 \pmod s$, there exist cohomology classes $w_{g,n}^{(r,s)}(a_1, \dots, a_n) \in CH^*(\bar{\mathcal{M}}_{g,n})$ indexed by $a_i \in [r]$ and $g, n \in \mathbb{Z}_{\geq 0}$ such that $2g - 2 + n > 0$, so that*

- (o) for any $d_i \geq 0$,

$$\begin{aligned}
 & F_{g,n}[d_1 r + a_1, \dots, d_n r + a_n] \\
 &= \prod_{i=1}^n (d_i r + a_i)! \int_{\bar{\mathcal{M}}_{g,n}} w_{g,n}^{(r,s)}(a_1, \dots, a_n) \prod_{i=1}^n \psi_i^{d_i}; \quad (7.24)
 \end{aligned}$$

- (i) $w_{g,n}^{(r,s)}(\mathbf{a})$ have pure Chow codimension

$$2 \left(\frac{\sum_{i=1}^n a_i - s(2g - 2 + n)}{r} + (3g - 3 + n) \right);$$

- (ii) denoting by $\pi: \bar{\mathcal{M}}_{g,n+1} \rightarrow \bar{\mathcal{M}}_{g,n}$ the forgetful morphism, we have the dilaton equation

$$\psi_{n+1} \pi^*(w_{g,n}^{(r,s)}(\mathbf{a})) = w_{g,n+1}^{(r,s)}(s, \mathbf{a}); \quad (7.25)$$

- (iii) we have the special values

$$\begin{aligned}
 w_{0,3}^{(r,s)}(a_1, a_2, a_3) &= c \delta_{a_1+a_2+a_3,s} \mathbf{1} \in H^0(\bar{\mathcal{M}}_{0,3}), \\
 w_{1,1}^{(r,s)}(a) &= \delta_{a,s} \frac{r^2 - 1}{s} \psi_1 \in H^2(\bar{\mathcal{M}}_{1,1});
 \end{aligned}$$

- (iv) $w_{0,n}^{(r,s=1)} = 0$ for all n ;
- (v) $w_{g,n}^{(r,s)}(a_1, \dots, a_n) = 0$ whenever there exists $i \in [n]$ such that $a_i = r$.

The conjecture is proved in the case $(r, s) = (2, 1)$ by Norbury [49] and $s = r - 1$ for general $r \geq 2$ by Chidambaram, Garcia-Failde and Giacchetto [25]. They construct $w_{g,n}^{(r,r-1)}$ as a cohomology class obtained by pushforward from the moduli space of r -spin curves; in their work, it is denoted by $\Theta_{g,n}^r$, or simply by $\Theta_{g,n}$ if $r = 2$.

Assuming Conjecture 7.23 holds, we will deform formula (7.24) to generalise Theorem 7.21 for any smooth admissible spectral curve. Before this, we need to discuss action of translation and sums over stable graphs on the (r, s) class.

7.5.2. Deformation of the (r, s) classes. We introduce

$$W_{g,n}^{(r,s)}: (V^{r\text{spin}})^{\otimes n} \rightarrow H^*(\bar{\mathcal{M}}_{g,n}),$$

$$\bigotimes_{i=1}^n e_{a_i} \mapsto w_{g,n}^{(r,s)}(a_1, \dots, a_n).$$

If we have a formal series $\mathbb{T}(u) \in V^{r\text{spin}}[[u]]$, we will see that under certain conditions, we can define analogously to Section 7.4.1 a new family of cohomology classes

$$[\widehat{\mathbb{T}}W^{(r,s)}]_{g,n}: (V^{r\text{spin}})^{\otimes n} \rightarrow H^*(\bar{\mathcal{M}}_{g,n}).$$

Assuming $\mathbb{T}_s \neq 1$, this is done in terms of the modified generating series

$$\tilde{\mathbb{T}}(u) := \frac{\mathbb{T}(u) - \mathbb{T}_s e_s}{1 - \mathbb{T}_s} = \sum_{\substack{d \geq 0 \\ a \in [r-1] \\ rd+a > s}} \frac{\mathbb{T}_{rd+a} e_a u^d}{1 - \mathbb{T}_s}$$

via the formula

$$[\widehat{\mathbb{T}}W^{(r,s)}]_{g,n}(-) = \sum_{m \geq 0} \frac{1}{m!} (\pi_m)_* W_{g,n+m}^{(r,s)}(- \otimes \tilde{\mathbb{T}}(\psi_{n+1}) \otimes \dots \otimes \tilde{\mathbb{T}}(\psi_{n+m})), \quad (7.26)$$

where $\pi_m: \bar{\mathcal{M}}_{g,n+m} \rightarrow \bar{\mathcal{M}}_{g,n}$ is the forgetful morphism. As in (7.4.1), the handling of \mathbb{T}_s is tailored to be compatible with the dilaton equation (7.25).

Lemma 7.24. *Assume that $\mathbb{T}_s \neq 1$ and $\mathbb{T}_a = 0$ for $a < s$. Then equation (7.26) is well-defined, i.e., for any evaluation on an element of $(V^{r\text{spin}})^{\otimes n}$, the sum on the right-hand side is finite.*

Proof. The argument is similar to Section 7.4.1. If we evaluate on $\bigotimes_{i=1}^n e_{a_i}$, due to (i) the complex codimension of the summand proportional to $\prod_{j=1}^m \mathbb{T}_{d_j r + b_j}$ with $d_j r + b_j \geq s + 1$ is

$$\frac{1}{r} \left(s(2g - 2 + n + m) - \sum_{i=1}^n a_i - \sum_{j=1}^m (d_j r + b_j) \right) \leq \frac{1}{r} \left(s(2g - 2 + n) - \sum_{i=1}^n a_i - m \right),$$

which for fixed a_1, \dots, a_n is negative for m is large enough, forcing this summand to vanish. ■

On the other hand, the action of any $B \in V^{r\text{spin}}[[u, v]]$ via sums over stable graphs is well-defined since it always involves finite sums.

7.5.3. Intersection theory for admissible smooth local spectral curves. Let $\mathcal{S} = (C, x, y, \omega_{0,2})$ be an admissible smooth spectral curve equipped with a fundamental bidifferential of the second kind. Recall from Section 7.2 the definition of the vector space V and the generating series T and B that can be associated to \mathcal{S} .

Definition 7.25. We let them act on the direct sum of (r_α, s_α) -classes to define

$$\Omega_{g,n}^{\mathcal{S}} = \left[\widehat{B\hat{T}} \left(\bigoplus_{\alpha \in \alpha} W^{(r_\alpha, s_\alpha)} \right) \right]_{g,n} : V^{\otimes n} \rightarrow CH^*(\overline{\mathcal{M}}_{g,n}).$$

Here we suppose the (r, s) class is a Chow class to put them on the same footing as the Witten r -spin classes, and denote $W^{(r, r+1)} := W^{r\text{spin}}$ for uniformity. The admissibility condition for irregular ramification points matches the condition of Lemma 7.24, so the definition is well-posed.

Theorem 7.26. Assume Conjecture 7.23 holds, and let $\mathcal{S} = (C, x, y, \omega_{0,2})$ be an admissible smooth spectral curve equipped with a fundamental bidifferential of the second kind $\omega_{0,2}$ and zero crosscap form. Then, for $2g - 2 + n > 0$

$$(\mathcal{L}_{\text{tot}}^-)^{\otimes n}(\omega_{g,n}) = \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Omega_{g,n}^{\mathcal{S}}}{\prod_{i=1}^n (1 - \epsilon_i \psi_i)},$$

where ϵ_i is the variable of the i -th Laplace transform.

Proof. The proof, which relies on the correspondence of Theorem 5.23 – already proved in [9] – is similar to that of Theorem 7.21, so we omit the details: the regular ramification points are treated by Theorem 7.21 itself, and the irregular ramification points using Section 7.5.2. ■

7.5.4. Intersection numbers on the flat basis.

Definition 7.27. Let $\mathcal{S} = (C, x, y, \omega_{0,2})$ be a spectral curve equipped with a fundamental bidifferential of the second kind. We say that \mathcal{S} is nearly compact if there exists a smooth compact connected curve \overline{C} containing \tilde{C} such that $d\tilde{x}$ admits an analytic continuation to a meromorphic 1-form on \overline{C} without zeroes on $\overline{C} \setminus \tilde{C}$, and $\omega_{0,2}$ admits an analytic continuation as a fundamental bidifferential of the second kind on \overline{C} .

This definition is tailored to allow \tilde{x} to be multivalued (for instance, due to logarithmic singularities) on \overline{C} . Many examples of non-compact but nearly compact spectral curves can be found in mirror symmetry and in Hurwitz theory. In the nearly compact case, the results of Sections 7.2.3 and 7.2.4 apply to \overline{C} : we have the factorisation property for \overline{B} via the R-matrix (Corollary 7.12), and we can consider the basis of primary differentials and their descendants on \overline{C} (Proposition 7.15).

Definition 7.28. We introduce the covector of primary differentials:

$$d\Xi(z) = \left(\sum_{\substack{\mu \in \tilde{\alpha} \\ a \in [r_\mu]}} d\xi_{-a}^\mu(z) e_{\mu,a}^* \right) \in H^0(\tilde{\mathcal{C}}, K_{\tilde{\mathcal{C}}}(*\tilde{\alpha})) \otimes V^*.$$

Corollary 7.29. Consider the situation described in Theorem 7.21 or Theorem 7.26, and assume that the spectral curve \mathcal{S} is nearly compact. We have the following decomposition on the basis of primary differentials and their descendants:

$$\omega_{g,n}(z_1, \dots, z_n) = \int_{\bar{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\mathcal{S}} \otimes \left(\bigotimes_{i=1}^n d\Xi(z_i) \right) \cdot \left(\bigotimes_{i=1}^n \frac{\mathbb{R}(\psi_i) \star \eta^\vee}{1 - \mathcal{D}_i \psi_i} \right).$$

Here, \mathcal{D}_i is the operator $\mathcal{D}: \varphi \mapsto -d(\varphi/d\tilde{x})$ acting from the right on the 1-forms in $d\Xi(z_i)$, and “ \cdot ” means that the first tensor factor from the i -th factor on the right must be inserted by the i -th position in the multilinear map $\Omega_{g,n}^{\mathcal{S}}$, while the second tensor factor must be inserted in the linear form $d\Xi_i(z_i)$.

Proof. Under the assumptions, we already know that the coefficients of decomposition of the $\omega_{g,n}$ on the ξ -basis are

$$F_{g,n}[\mu_1 \dots \mu_n; k_1 \dots k_n] = \int_{\bar{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\mathcal{S}}((e_{\mu_i, \hat{k}_i})_{i=1}^n) \prod_{i=1}^n k_i!^{(r_{\mu_i})} \psi_i^{\bar{k}_i}. \quad (7.27)$$

We rather want to express them on the $\hat{\xi}$ -basis. With the change of basis (7.14) this yields:

$$\begin{aligned} & \omega_{g,n}(z_1, \dots, z_n) \\ &= \sum_{\substack{\mu_1, \dots, \mu_n \in \tilde{\alpha} \\ k_1, \dots, k_n > 0}} \sum_{\substack{\rho_1, \dots, \rho_n \in \tilde{\alpha} \\ l_1, \dots, l_n > 0}} \left(\int_{\bar{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\mathcal{S}}(e_{\mu_1, \hat{k}_1} \otimes \dots \otimes e_{\mu_n, \hat{k}_n}) \prod_{i=1}^n k_i!^{(r_{\mu_i})} \psi_i^{\bar{k}_i} \right) \\ & \quad \times A \left[\begin{array}{cc} \mu_i & \rho_i \\ -k_i & -l_i \end{array} \right] d\hat{\xi}_{-l_i}^{\rho_i}(z_i) \\ &= \int_{\bar{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\mathcal{S}} \otimes \left(\bigotimes_{i=1}^n d\Xi(z_i) \right) \\ & \quad \times \bigotimes_{i=1}^n \left(\sum_{\substack{\mu_i, \rho_i \in \tilde{\alpha} \\ k_i, l_i > 0}} A \left[\begin{array}{cc} \mu_i & \rho_i \\ -k_i & -l_i \end{array} \right] k_i!^{(r_{\mu_i})} e_{\mu_i, \hat{k}_i} \otimes e_{\rho_i, l_i} \psi_i^{\bar{k}_i} \mathcal{D}_i^{\bar{l}_i} \right) \\ &= \int_{\bar{\mathcal{M}}_{g,n}} \Omega_{g,n}^{\mathcal{S}} \otimes \left(\bigotimes_{i=1}^n d\Xi(z_i) \right) \cdot \bigotimes_{i=1}^n \bar{A}(\psi_i, \mathcal{D}_i), \end{aligned}$$

where “ \cdot ” is the contraction of multilinear maps with tensors in the order prescribed by the statement of the claim, and \mathcal{D}_i acts from the right on the 1-forms in $d\Xi(z_i)$. The

situations of Theorem 7.21 or Theorem 7.26 correspond to smooth spectral curves, for which (7.27) vanishes whenever one of the k_i is divisible by r_{μ_i} (see item (v) of Section 7.5.1). Subsequently, we can replace $\bar{A}(\psi_i, \mathcal{D}_i)$ by its projection ${}^{ss}\bar{A}(\psi_i, \mathcal{D}_i)$. We conclude by using the second formula of Proposition 7.15. ■

7.5.5. TR-ELSV formula and quasi-polynomiality. In the situation of Corollary 7.29, let $P \subset \bar{C}$ be a set of points which are not zeroes of $d\tilde{x}$. Given $p \in P$, we introduce

$$d_p := (-\text{ord}_p d\tilde{x}) \in \mathbb{Z}_{\geq 0}, \quad c_p = \text{Res}_p d\tilde{x},$$

and we choose a local coordinate X near p such that $X(p) = 0$ and

$$d\tilde{x} = \begin{cases} c_p \frac{dX}{X} & \text{if } d_p = 1, \\ X^{-d_p} dX & \text{if } d_p \neq 1. \end{cases}$$

Definition 7.30. We introduce for $(p, \ell) \in P \times \mathbb{Z}_{>0}$ and $\mu \in \tilde{\alpha}, k \in [r_\mu]$ the quantities

$$S \left[\begin{smallmatrix} p & \mu \\ \ell & -k \end{smallmatrix} \right] = \text{Res}_{z=p} \frac{d\xi_{-k}^\mu(z)}{X(z)^\ell} = \text{Res}_{z=p} \text{Res}_{z'=\mu} \frac{k \omega_{0,2}(z, z')}{X(z)^\ell \zeta(z')^k},$$

$$\mathcal{S} \left[\begin{smallmatrix} p \\ \ell \end{smallmatrix} \right] = \sum_{\substack{\mu \in \tilde{\alpha} \\ k \in [r_\mu]}} S \left[\begin{smallmatrix} p & \mu \\ \ell & -k \end{smallmatrix} \right] e_{\mu,k}^*.$$

They give the all-order series expansion (indicated with the symbol \approx) of the primary differentials near p :

$$d\Xi(z) \underset{z \rightarrow p}{\approx} d \left(\sum_{\ell > 0} \mathcal{S} \left[\begin{smallmatrix} p \\ \ell \end{smallmatrix} \right] \frac{X^\ell}{\ell} \right)$$

and are closely related to the S -matrix in Givental formalism.

Lemma 7.31. *With the convention that \mathcal{D} acts from the right on primary differentials, we have for $p \in P$*

$$d\Xi(z) \frac{1}{1 - \mathcal{D}\psi} \underset{z \rightarrow p}{\approx} \begin{cases} d \left(\sum_{\ell > 0} \frac{\mathcal{S} \left[\begin{smallmatrix} p \\ \ell \end{smallmatrix} \right]}{1 + c_p^{-1} \ell \psi} \frac{X^\ell}{\ell} \right) & \text{if } d_p = 1, \\ d \left(\sum_{\ell > 0} \mathcal{S} \left[\begin{smallmatrix} p \\ \ell \end{smallmatrix} \right] G_{\frac{\ell}{d_p-1}} \left((d_p - 1) X^{d_p-1} \psi \right) \frac{X^\ell}{\ell} \right) & \text{otherwise,} \end{cases}$$

where in the second line we introduced

$$G_a(x) := \sum_{m \geq 0} (-1)^m \prod_{j=0}^{m-1} (j + a) x^m = \int_0^\infty \frac{e^{-t/x}}{(1+t)^a} \frac{dt}{x} = \frac{e^{1/x} \Gamma(1 - a, \frac{1}{x})}{x^a},$$

and $\Gamma(1 - a, x)$ is the incomplete Gamma function.

Proof. Let us write for uniformity $d\tilde{x} = \kappa_p X^{-d_p} dX$ with $\kappa_p = c_p \neq 0$ if $d_p = 1$ and $\kappa_p = 1$ otherwise. The operator \mathcal{D} can be expressed in terms of X :

$$\mathcal{D}\varphi = -d\left(\frac{X^{d_p}\varphi}{\kappa_p dX}\right).$$

In particular, for $\ell > 0$

$$\mathcal{D}(X^{\ell-1}dX) = -\frac{(d_p - 1 + \ell)}{\kappa_p} X^{d_p-1+\ell-1}dX,$$

and by induction for $m \geq 0$

$$\begin{aligned} \mathcal{D}^m(X^{\ell-1}dX) &= \begin{cases} (-1)^m(\ell/c_p)^m X^{\ell-1}dX & \text{if } d_p = 1, \\ (-1)^m(d_p - 1)^m \prod_{j=1}^m (j + \frac{\ell}{d_p-1}) X^{m(d_p-1)+\ell-1}dX & \text{otherwise.} \end{cases} \end{aligned}$$

We then multiply by ψ^m and sum over $m \geq 0$. ■

Since p is not a zero of $d\tilde{x}$, for $2g - 2 + n > 0$ the form $\omega_{g,n}(z_1, \dots, z_n)$ is holomorphic at $z_i = p \in P$, and so we have an all-order series expansion of the form

$$\omega_{g,n}(z_1, \dots, z_n) \approx d_1 \cdots d_n \left(\sum_{\ell_1, \dots, \ell_n > 0} H_{g,n} \left[\begin{smallmatrix} p_1 & \dots & p_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix} \right] \prod_{i=1}^n \frac{X_i^{\ell_i}}{\ell_i} \right).$$

Equivalently,

$$H_{g,n} \left[\begin{smallmatrix} p_1 & \dots & p_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix} \right] = \operatorname{Res}_{z_1=p_1} \cdots \operatorname{Res}_{z_n=p_n} \frac{\omega_{g,n}(z_1, \dots, z_n)}{\prod_{i=1}^n X(z_i)^{\ell_i}}.$$

When we only expand near simple poles of $d\tilde{x}$, the formula for $H_{g,n}$ becomes particularly simple due to Lemma 7.31.

Corollary 7.32 (TR-ELSV formula). *Under the assumptions of Corollary 7.29 and assuming that $P \subset \bar{C}$ is a set of simple poles of $d\tilde{x}$, we have for $2g - 2 + n > 0$, $p_1, \dots, p_n \in P$ and $\ell_1, \dots, \ell_n > 0$*

$$H_{g,n} \left[\begin{smallmatrix} p_1 & \dots & p_n \\ \ell_1 & \dots & \ell_n \end{smallmatrix} \right] = \int_{\bar{\mathcal{M}}_{g,n}} \left(\Omega_{g,n}^S \otimes \bigotimes_{i=1}^n \frac{S \left[\begin{smallmatrix} p_i \\ \ell_i \end{smallmatrix} \right]}{1 + c_{p_i}^{-1} \ell_i \psi_i} \right) \cdot \left(\bigotimes_{i=1}^n R(\psi_i) \star \eta^\vee \right),$$

where $c_p = \operatorname{Res}_p d\tilde{x}$.

Remark 7.33. Apart from the non-trivial dependence in ℓ_i contained in S , the dependence in the ℓ_i is polynomial, coming from the geometric series expansion of

$$\frac{1}{1 + c_{p_i}^{-1} \ell_i \psi_i},$$

from which only finitely many terms contribute. This type of behavior is called “quasi-polynomiality”. The assumptions of Corollary 7.32 in particular apply for spectral curves of Hurwitz theory and Gromov–Witten theory, where the $H_{g,n}$ store the desired enumerative invariants. Lemma 7.31 shows that, in the situation where enumerative invariants are stored in the expansion of $\omega_{g,n}$ near points which are not simple poles of $d\tilde{x}$, the quasi-polynomiality is a priori destroyed and the structure is more complicated: the coefficient of $d(X^{\ell_i}(z_i)/\ell_i)$ will receive contributions from $\mathcal{S} \left[\begin{smallmatrix} p_i \\ \ell'_i \end{smallmatrix} \right]$ for all $\ell'_i \leq \ell_i$ such that $(d_p - 1)$ divides $(\ell_i - \ell'_i)$.

8. Open intersection numbers

In this section, we shall propose precise conjectures about open r -spin intersection numbers using the partition function of the Airy structure with twist σ of cycle type $(r - 1, 1)$ with no dilaton shift attached to the fixed point, as obtained in Theorem 2.11. Before this, we review the relation between this partition function for $r = 2$ and the open intersection theory, and Safnuk’s topological recursion [53] for it. The latter has peculiar features related to the reducibility of the spectral curve, our general approach shines a new light on this. These relations depend on some foundational conjectures in open intersection theory scattered in the literature and that we make explicit.

8.1. Review of open intersection theory

The enumerative geometry of open Riemann surfaces was developed by Pandharipande–Solomon–Tessler in genus 0 in [51], and its extension to all genera was announced by Solomon and Tessler. The upshot is that for $g, n, b, m \geq 0$ such that

$$-\bar{\chi} := 2\bar{g} - 2 + 2n + m > 0, \quad \bar{g} := 2g + b - 1, \tag{8.1}$$

there is a moduli space $\mathcal{M}_{g,n;b,m}$ parametrizing Riemann surfaces of genus g with b (unlabelled) boundary components, n labelled interior marked points and m labelled boundary marked points, equipped with spin structure and a “grading”. It is a real orbifold of dimension

$$D := 6g - 6 + 3b + 2n + m,$$

and admits several connected components indexed by the distribution of the boundary marked points on the boundary components. Note that \bar{g} and $\bar{\chi}$ are respectively the genus and the Euler characteristic of the surface doubled along its boundary. This moduli space admits a compactification $\bar{\mathcal{M}}_{g,n;b,m}$, on which one can seek to define and calculate intersection numbers. Denoting by \mathbb{L}_i the cotangent line bundle at the

i -th interior marked point, and according to the statement of [3, Theorem 1.1], whose proof by Solomon and Tessler has not yet appeared, it is possible to define

$$\langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ (\tau_0^\partial)^m \rangle_{g,n;b,m} = 2^{1-g-\frac{b+m}{2}} \int_{\bar{\mathcal{M}}_{g,n;b,m}} e\left(\bigoplus_{i=1}^n \mathbb{L}_i^{\oplus d_i}, s\right) \in \mathbb{Q}, \quad (8.2)$$

whenever $d_1, \dots, d_n \geq 0$ are such that

$$D = \sum_{i=1}^n 2d_i.$$

Here, e is the Euler class relative to some boundary condition s , and [51, 56] give suitable boundary conditions so that the number is unambiguously defined.

Remark 8.1. For $b = k = 0$, the moduli space $\bar{\mathcal{M}}_{g,n;0,0}$ coincides with the moduli space of spin structures, which is a \mathbb{Z}_2 -orbifold cover of the Deligne–Mumford moduli space of pointed Riemann surfaces $\bar{\mathcal{M}}_{g,n}$. In fact, $\bar{\mathcal{M}}_{g,n;0,0}$ has two connected components, distinguished by the parity of the corresponding spin structures. On the component containing the even (resp. odd) spin structures, the degree of the cover is $2^{g-1}(2^g + 1)$ (resp. $2^{g-1}(2^g - 1)$), and the virtual fundamental class (= the Witten 2-spin class) is the fundamental class of the even component minus the one of the odd components. After pushforward, this yields a multiple of the fundamental class of $\bar{\mathcal{M}}_{g,n}$ by a factor of

$$\frac{1}{2}(2^{g-1}(2^g + 1) - 2^{g-1}(2^g - 1)) = 2^{g-1},$$

where the extra $\frac{1}{2}$ comes from the \mathbb{Z}_2 -orbifold structure of $\bar{\mathcal{M}}_{g,n;0,0}$. So, the conventional factor of 2 in (8.2) is such that we retrieve for $b = k = 0$ the usual ψ -class intersection

$$\langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \rangle_{g,n;0,0} = \int_{\bar{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}, \quad \psi_i = c_1(\mathbb{L}_i).$$

We thank Ran Tessler for a remark on this point, see also [58, Lemma 6.20 and Remark 6.21].

It is expected that one can also define geometrically boundary descendants, which we would like to denote

$$\langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m} \in \mathbb{Q},$$

where now $d_1, \dots, d_n, k_1, \dots, k_m \geq 0$ satisfy the dimension constraint

$$D = \sum_{i=1}^n 2d_i + \sum_{j=1}^m 2k_j, \quad (8.3)$$

or equivalently

$$3(2g - 2 + n + m + b) = \sum_{i=1}^n (2d_i + 1) + \sum_{j=1}^m (2k_j + 2). \quad (8.4)$$

One could then consider the generating series

$$Z^{\text{open}}[Q; t^\circ; t^\partial] = \exp \left(\sum_{\substack{g, b, m, n \geq 0 \\ \bar{\chi} < 0}} \frac{\hbar^{g-1+\frac{b}{2}} Q^b}{m!n!} \langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g, n; b, m} \right. \\ \left. \times \prod_{i=1}^n (2d_i + 1)!! t_{d_i}^\circ \prod_{j=1}^m (2k_j + 2)!! t_{k_j}^\partial \right),$$

where b is the number determined from g, n, m, d_i, k_i via (8.3).

A combinatorial model for the intersection numbers (8.2) – i.e., without boundary descendants – has been proposed in [3], refining [58], where only their sum over all possible b s was obtained. As a consequence, their generating series has a matrix integral description. It turns out this matrix integral allows naturally for the insertion of extra parameters, thus defining a generating series of the form

$$Z^{\text{ABT}}[Q; t^\circ; t^\partial] = \exp \left(\sum_{\substack{g, b, m, n \geq 0 \\ \bar{\chi} < 0}} \frac{\hbar^{g-1+\frac{b}{2}} Q^b}{m!n!} \langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g, n; b, m}^{\text{ABT}} \right. \\ \left. \times \prod_{i=1}^n (2d_i + 1)!! t_{d_i}^\circ \prod_{j=1}^m (2k_j + 2)!! t_{k_j}^\partial \right).$$

We will not need its precise definition, which can be found in [3, (3.14) and Lemma 3.2].

The Kontsevich–Penner matrix model is another important character of the story. It is defined by

$$\mathcal{Z}^{\text{KP}, N}(\Lambda) = \int_{\mathcal{H}_N} \frac{dH}{c_{\Lambda, N, \hbar}} \exp \left\{ \hbar^{-\frac{1}{2}} \text{Tr} \left(\frac{H^3}{6} - \frac{H^2 \Lambda}{2} \right) \right\} \left(\frac{\det(\Lambda)}{\det(\Lambda - H)} \right)^Q,$$

where \mathcal{H}_N is the space of hermitian matrices of size N , and $c_{\Lambda, N, \hbar}$ is some normalizing constant. It determines a unique generating series of the form

$$Z^{\text{KP}}[Q; t^\circ; t^\partial] = \exp \left(\sum_{\substack{g, b, m, n \geq 0 \\ \bar{\chi} < 0}} \frac{\hbar^{g-1+\frac{b}{2}} Q^b}{m!n!} \langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g, n; b, m}^{\text{KP}} \right. \\ \left. \times \prod_{i=1}^n (2d_i + 1)!! t_{d_i}^\circ \prod_{j=1}^m (2k_j + 2)!! t_{k_j}^\partial \right)$$

such that for any $N \geq 1$,

$$\mathcal{Z}_N^{\text{KP}}(\Lambda) = Z^{\text{KP}}\left[Q; \left(t_d^\circ = \frac{\hbar^{\frac{1}{2}} \text{Tr} \Lambda^{-(2d+1)}}{2d+1}\right)_{d \geq 0}; \left(t_k^\partial = \frac{\hbar^{\frac{1}{2}} \text{Tr} \Lambda^{-(2k+2)}}{2k+2}\right)_{k \geq 0}\right].$$

Remark 8.2. In contrast with the aforementioned works, we have included in our definition of the generating series a variable \hbar which is redundant because of the dimension constraint (8.3), and we have not written separately the contribution of the closed Riemann surfaces ($b = 0$). Our definition can be obtained from [3] by the following substitutions:

$$\begin{aligned} N &\rightarrow Q, & H &\rightarrow \hbar^{-\frac{1}{6}} H, & \Lambda &\rightarrow \hbar^{-\frac{1}{6}} \Lambda, \\ t_d &\rightarrow \hbar^{\frac{d-1}{3}} (2d+1)!! t_d^\circ, & s_k &\rightarrow \hbar^{\frac{k}{3}-\frac{1}{6}} (2k+2)!! t_k^\partial, \end{aligned}$$

under which $\tau_N^{\text{open}} \rightarrow Z^{\text{ABT}}$ and $\tau_N \rightarrow Z^{\text{KP}}$. Our definition can be obtained from [1] by substituting there

$$N \rightarrow Q, \quad T_{2d+1} \rightarrow \hbar^{\frac{d-1}{3}} t_{2d+1}^\circ, \quad T_{2k+2} \rightarrow \hbar^{\frac{k}{3}-\frac{1}{6}} t_k^\partial.$$

Finally, note that the definition of the Kontsevich–Penner matrix model of [3] can be obtained from the one in [1] by the substitution $\Phi \rightarrow \Lambda - H$.

It is expected that these three collections of numbers coincide.

Conjecture 8.3. *There exists a geometric definition of the open intersection numbers with boundary descendants, and it is such that $Z^{\text{open}} = Z^{\text{ABT}}$, that is*

$$\langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m} = \langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m}^{\text{ABT}}.$$

Conjecture 8.4. *There exists a geometric definition of the open intersection numbers with boundary descendants, and it is such that $Z^{\text{open}} = Z^{\text{KP}}$, that is*

$$\langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m} = \langle \tau_{d_1}^\circ \cdots \tau_{d_n}^\circ \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m}^{\text{KP}}.$$

Conjecture 8.5. *We have $Z^{\text{KP}} = Z^{\text{ABT}}$.*

Obviously, any two of the conjectures imply the third one. They are supported by partial results:

- The specialisation of Conjecture 8.3 to $t_k^\partial = 0$ for $k > 0$ is proved in [3] conditionally to [3, Theorem 1.1], whose proof was announced by Solomon and Tessler but has not appeared yet. The same specialisation in Conjecture 8.4 was anticipated in [53] and proved there for $g = 0, \frac{1}{2}, 1$.
- The specialisation of Conjecture 8.5 to $Q = 1$ is proved in [1] via integrability techniques, and the specialisation to $Q = \pm 1$ in [3, Section 4.2] by matrix integral techniques.

- The string and dilaton equations satisfied by Z^{ABT} and Z^{KP} are the same [3, Section 4.3].

Alexandrov has proposed various collections of differential operators relevant to the study of the Kontsevich–Penner model – and therefore to open intersection theory in light of Conjecture 8.4. We summarise what is relevant for our exposition:

- (a) In [1], Alexandrov used a representation of the \mathfrak{gl}_1 -Heisenberg algebra to construct operators $(\hat{L}_k^o)_{k \geq 0}$ and $(\hat{M}_k^o)_{k \geq -2}$ (see equations (7.4) and (7.14) therein) annihilating the $Q = 1$ specialization of Z^{KP} .
- (b) In [53], Safnuk introduced a modification of these differential operators, denoted by $(\hat{L}_k)_{k \geq -1}$ and $(\hat{M}_k)_{k \geq -2}$ (see equations (2.9) and (2.10) therein), which still annihilate the $Q = 1$ specialization of Z^{KP} .
- (c) In [2], Alexandrov used a twisted representation of the Heisenberg algebra of \mathfrak{gl}_3 to construct a free field representation of the algebra $\mathcal{W}(\mathfrak{sl}_3)$ and operators $(\hat{\mathcal{L}}_k^Q)_{k \geq -1}$ and $(\hat{\mathcal{M}}_k^Q)_{k \geq -2}$ (see equation (72) therein) annihilating Z^{KP} .

All those operators are related by taking (possibly infinite) linear combinations, but it turns out that choosing one or the other set of operators affects the structure of the recursion one deduces for $\langle \dots \rangle^{\text{KP}}$. For sake of comparison and completeness, we review in Section 8.2 the definition of the operators in (a) and (b) and the topological recursion with strange features that Safnuk derived from the operators in (b). In Section 8.3, we explain that the operators in (c) directly compare to Airy structures for $\sigma = (12)(3)$ and thus provide a CEO-like topological recursion, whose structure is more transparent and more general than [53].

8.2. Review of Safnuk’s recursion

Consider the following representation of the Heisenberg VOA for \mathfrak{gl}_1 :

$$J(z) = \sum_{k \in \mathbb{Z}} \frac{J_k}{z^{k+1}}, \quad J_k := \begin{cases} \partial_{t_k} & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -k t_{-k} & \text{if } k < 0 \end{cases}$$

and introduce the normal ordered products

$$L(z) = \frac{1}{2} : J(z)^2 : = \sum_{k \in \mathbb{Z}} \frac{L_k}{z^{k+2}},$$

$$M(z) = \frac{1}{3} : J(z)^3 : = \sum_{k \in \mathbb{Z}} \frac{M_k}{z^{k+3}}.$$

These operators form a representation of the $\mathcal{W}(\mathfrak{sl}_3)$ -algebra. The collection of operators mentioned in (a) is

$$\begin{aligned} \widehat{L}_k^o &:= L_{2k} + (k + 2)J_{2k} - J_{2k+3} + \delta_{k,0} \left(\frac{1}{8} + \frac{3}{2} \right), \\ \widehat{M}_k^o &:= M_{2k} + 2(k + 3)L_{2k} - 2L_{2k+3} - 2(k + 3)J_{2k+3} + J_{2k+6} \\ &\quad + \left(\frac{95}{12} + 6k + \frac{4}{3}k^2 \right) J_{2k} + \frac{23}{4} \delta_{k,0}. \end{aligned}$$

Then Alexandrov proved in [1] that $(\widehat{L}_k^o)_{k \geq 0}$ and $(\widehat{M}_k^o)_{k \geq -2}$ annihilate the specialization of Z^{KP}

$$\begin{cases} \widehat{L}_k^o \tau_1 = 0, & k \geq 0, \\ \widehat{M}_k^o \tau_1 = 0, & k \geq -2. \end{cases}$$

The collection of modified operators mentioned in (b) reads

$$\widehat{L}_k := \widehat{L}_k^o, \quad \widehat{M}_k := -\widehat{M}_k^o + 2(k + 2)\widehat{L}_k^o.$$

Let us sketch the strategy of Safnuk in [53] – for a better comparison, we set $\hbar = 1$ till the end of this subsection. We first rewrite these operators. To this end, we change the currents to include as zero-mode $Q = 1$ and a dilaton shift (Safnuk does this later in the computation):

$$\widetilde{\mathcal{J}}(z) := \sum_{k \in \mathbb{Z}} \frac{\widetilde{\mathcal{J}}_k}{z^{k+1}}, \quad \widetilde{\mathcal{J}}_k := J_k - \delta_{k,-3} + \delta_{k,0}.$$

Let us also define, following Safnuk, the differential operators

$$\mathcal{D}_1 := dz \left(-\frac{d}{dz} + \frac{1}{z} \right), \quad \mathcal{D}_2 := \frac{(dz)^2}{2} \left(\frac{d^2}{dz^2} - \frac{3}{z} \frac{d}{dz} + \frac{3}{z^2} \right),$$

the 1-form $\eta = -z^2 dz$, and the projection operators

$$\mathcal{P}^{(i)}: \mathbb{C}[[z^{\pm 1}]]dz \rightarrow \mathbb{C}[[z^{-2}]]z^{-i}dz, \quad i \in \{2, 3\}.$$

Then, taking only the parts of the generating series that annihilate $Z^{\text{KP}}|_{Q=1}$, we get

$$\begin{aligned} \mathcal{J}(z) &:= \widetilde{\mathcal{J}}(z)dz, \\ \mathcal{L}(z) &:= \sum_{k \geq -1} \frac{dz}{z^{2k+4}} \widehat{L}_k = \mathcal{P}^{(2)} \left(\frac{1}{2\eta} \left(: \mathcal{J}^2 : + \mathcal{D}_1 \mathcal{J} + \frac{(dz)^2}{4z^2} \right) \right), \\ \mathcal{M}(z) &:= \sum_{k \geq -2} \frac{dz}{z^{2k+7}} \widehat{M}_k = \mathcal{P}^{(3)} \left(\frac{1}{3\eta^2} \left(: \mathcal{J}^3 : - \mathcal{D}_2 \mathcal{J} + \frac{3(dz)^2}{4z^2} \mathcal{J} \right) \right). \end{aligned}$$

The last term in the last two lines can be absorbed by *defining*

$$\mathcal{J}^2(z) = : \mathcal{J}^2(z) : + \frac{(dz)^2}{4z^2}, \quad \mathcal{J}^3(z) = \mathcal{J}(z) \cdot \mathcal{J}^2(z).$$

The square of $\mathcal{J}(z)$ itself will not be defined due to infinite sums, and somehow definition (8.2) implements the right operator product expansion from the $\mathcal{W}(\mathfrak{sl}_3)$ -algebra, cf. equation (2.17). We then get the following operators annihilating $Z^{\text{KP}}|_{Q=1}$:

$$\mathcal{L} = \mathcal{P}^{(2)}\left(\frac{1}{2\eta}(\mathcal{J}^2 + \mathcal{D}_1\mathcal{J})\right), \quad \mathcal{M} = \mathcal{P}^{(3)}\left(\frac{1}{3\eta^2}(\mathcal{J}^3 - \mathcal{D}_2\mathcal{J})\right).$$

Now, if we write $Z^{\text{KP}}|_{Q=1} = e^F$ and commute this through these operators, this gives the equations

$$\begin{aligned} 0 &= \mathcal{P}^{(2)}\left(\frac{1}{2\eta}e^{-F}(\mathcal{J}^2 + \mathcal{D}_1\mathcal{J})e^F\right) = \mathcal{P}^{(2)}\left(\frac{1}{2\eta}(U^2 + \mathcal{D}_1U) \cdot 1\right), \\ 0 &= \mathcal{P}^{(3)}\left(\frac{1}{3\eta^2}e^{-F}(\mathcal{J}^3 - \mathcal{D}_2\mathcal{J})e^F\right) = \mathcal{P}^{(3)}\left(\frac{1}{3\eta^2}(U^3 - \mathcal{D}_2U) \cdot 1\right), \end{aligned}$$

where

$$U(z) := e^{-F} \mathcal{J}(z) e^F = \mathcal{J}(z) + [\mathcal{J}(z), F]$$

is the operator appearing in [53]. In order to recover a spectral curve topological recursion from this, one should define

$$\delta_z := \mathcal{J}_-(z) = \sum_{k \geq 1} \frac{dz}{z^{k+1}} \partial_{t_k}, \quad \omega_{g,n}(z_1, \dots, z_n) := \delta_{z_1} \cdots \delta_{z_n} F_{g,n}.$$

If we also introduce the unstable terms

$$\omega_{0,1}(z) := \eta(z), \quad \omega_{0,2} = [\delta_1, \mathcal{J}_2] = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad \omega_{\frac{1}{2},1} := \frac{dz}{z},$$

coming from the dilaton shift, the positive part of $\mathcal{J}(z)$ and the zero mode $J_0 = 1$, respectively, we see that

$$U(z) = \delta_z + \omega_{0,1}(z) + \omega_{\frac{1}{2},1}(z) + \delta^{-1} \omega_{0,2} + \sum_{2g-2+n>0} \delta F_{g,n}.$$

Reinterpreting the projection operators as residues with the recursion kernel results in the following theorem.

Theorem 8.6 ([53, Theorem 5.3]). *The $\omega_{g,n}$ obey a modified topological recursion on the spectral curve with crosscap form*

$$\mathbb{C}, \quad x(z) = \frac{z^2}{2}, \quad y(z) = -z, \quad \omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad \omega_{\frac{1}{2},1} = \frac{dz}{z},$$

given by

$$\omega_{g,n+1}(z_0, z_{[n]}) = \operatorname{Res}_{w=0} \left(K^{(2)}(z_0, w) (\mathcal{W}'_{g,2,n}(w, w; z_{[n]}) + \mathcal{D}_1 \omega_{g-\frac{1}{2},n+1}(w, z_{[n]})) \right. \\ \left. + K^{(3)}(z_0, w) (\mathcal{W}'_{g,3,n}(w, w, w; z_{[n]}) + \mathcal{D}_2 \omega_{g-1,n+1}(w, z_{[n]})) \right),$$

where \mathcal{W}' is as in Definition 5.8 and

$$K^{(j)}(z, w) = \left((-1)^j \int_{\xi=0}^{-w} \omega_{0,2}(z, \xi) - \int_{\xi=0}^w \omega_{0,2}(z, \xi) \right) \frac{1}{2^j (-w^2 dw)^{j-1}}.$$

This form of the topological recursion has some odd features. For one, it is a recursion of order 3, even though the degree of x is only 2. This is also noted by Safnuk, who computes the quantum curve in [53, Theorem 7.1] and obtains the semi-classical limit $y(y^2 - 2x) = 0$. Moreover, the formula does not include summation over fibers of x – rather, the variable w is inserted several times. Finally, the operators \mathcal{D}_j introduce uncommon derivatives. Safnuk posits that these quirks come from the reducibility of the curve, but in the next section, we will see this relation is not straightforward.

8.3. Relation to topological recursion

In a previous work, it was shown that the operators in (c), i.e., in [2], coincide, up to a change of variables, with the reduction to $\mathcal{W}(\cong \mathfrak{l}_3)$ of an $\mathcal{W}(\mathfrak{gl}_3)$ -Airy structure, leading to the following result.

Theorem 8.7 ([9, Proposition 6.3]). *The function Z^{KP} is the partition function of the Airy structure of Theorem 2.11 with parameters $(r_1, s_1, r_2, s_2) = (2, 3, 1, \infty)$, $t_1 = \frac{1}{2}$, and $Q_1 = -Q_2 = Q$, specialised to*

$$x_{2d+1}^1 = t_d^\circ, \quad x_{2d+2}^1 = t_d^\partial, \quad x_{d+1}^2 = 0 \quad (d \geq 0). \tag{8.5}$$

Thanks to Propositions 5.18 and 5.23, we can now convert this into a CEO-like topological recursion on the reducible spectral curve consisting of the union of two components intersecting at $z = 0$

$$C = C_1 \cup C_2, \quad C_1: \begin{cases} x(z) = z^2, \\ y(z) = -\frac{z}{2}, \end{cases} \quad C_2: \begin{cases} x(z) = z, \\ y(z) = 0. \end{cases} \tag{8.6}$$

Due to the constraint $H_{i=1,k=d+1} \cdot Z = 0$ in the Airy structure, the partition function only depends on the variable $x_{2d+2}^1 - x_{d+1}^2$. Accordingly, no information is lost after the specialisation to (8.5). More precisely, each occurrence of x_{2d+1}^1 in a monomial is associated with the insertion of $(2d + 1)!! \tau_d^\circ$, each occurrence of x_{d+1}^2 with the insertion of $(2d + 2)!! \tau_d^\partial$, and each occurrence of x_{2d+2}^1 with $-(d + 1)!! \tau_d^\partial$. Taking

this into account and using (5.16), the suitable definition for the correlators in terms of open intersection numbers is

$$\begin{aligned} & \omega_{h,n+m}^{\text{KP}} \left(\frac{1}{z_1} \cdots \frac{1}{z_n} \frac{2}{z_{n+1}} \cdots \frac{2}{z_{n+m}} \right) \\ &= \sum_{\substack{g, b \in \mathbb{Z}_{\geq 0} \\ g + \frac{b}{2} = h}} \sum_{N_0 \sqcup N_\partial = [n]} \sum_{\substack{d_1, \dots, d_n \geq 0 \\ k_1, \dots, k_m \geq 0}} (-1)^m Q^b \left\langle \prod_{i \in N_0} \tau_{d_i}^\circ \prod_{i \in N_\partial} \tau_{d_i}^\partial \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{g, |N_0|; b, m + |N_\partial|}^{\text{KP}} \\ & \times \prod_{i \in N_0} \frac{(2d_i + 1)!! dz_i}{z_i^{2d_i+2}} \prod_{i \in N_\partial} \frac{(d_i + 1)! dz_i}{z_i^{2d_i+3}} \prod_{j=1}^m \frac{(k_j + 1)! dz_{n+j}}{z_{n+j}^{k_j+2}}, \end{aligned}$$

where the number of boundaries b is determined in terms of g, n, d, k, m via (8.3).

Corollary 8.8. *For $g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $n \geq 1$ such that $2g - 2 + n > 0$, $\omega_{g,n}^{\text{KP}}$ is computed by the topological recursion on the spectral curve (8.6) equipped with bidifferential and crosscap form*

$$\omega_{0,2} \left(\frac{\mu_1}{z_1} \frac{\mu_2}{z_2} \right) = \delta_{\mu_1, \mu_2} \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad \omega_{\frac{1}{2}, 1} \left(\frac{\mu}{z} \right) = (-1)^{\mu+1} Q \frac{dz}{z}$$

for $\mu, \mu_1, \mu_2 \in \{1, 2\}$.

Several sanity checks of this corollary can be proposed. At $Q = 0$, the variables t^∂ become irrelevant and Z^{KP} specialises to the Witten–Kontsevich partition function [43, 59]

$$Z^{\text{KP}}[Q = 0, t^\circ, t^\partial] = \sum_{\substack{g \geq 0, n \geq 1 \\ 2g - 2 + n > 0}} \frac{\hbar^{g-1}}{n!} \left(\int_{\bar{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} \right) \prod_{i=1}^n (2d_i + 1)!! t_{d_i}^\circ. \quad (8.7)$$

This can also be checked on the topological recursion side. Indeed, Theorem 7.18 for $r = 2$ states that the topological recursion for the spectral curve

$$x(\zeta) = \zeta^2, \quad y(\zeta) = -\frac{\zeta}{2}, \quad \omega_{0,2}(\zeta_1, \zeta_2) = \frac{d\zeta_1 d\zeta_2}{(\zeta_1 - \zeta_2)^2} \quad (8.8)$$

produces the (g, n) -correlator for $2g - 2 + n > 0$:

$$\omega_{g,n}(\zeta_1, \dots, \zeta_n) = \sum_{d_1, \dots, d_n \geq 0} \left(\int_{\bar{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i} \right) \prod_{i=1}^n \frac{(2d_i + 1)!! d\zeta_i}{\zeta_i^{2d_i+2}}.$$

Note that this case was proved before with a slightly different normalisation by Eynard–Orantin [36], see also [35].

At $Q = 0$, we are in position to apply Proposition 5.27, showing that the second component decouples, i.e., $\omega_{g,n}^{\text{KP}} \left(\frac{1}{z_1} \cdots \frac{1}{z_n} \right) |_{Q=0}$ coincides with the correlators of (8.8). So, the correlators of the spectral curve (8.6) at $Q = 0$ agree with the correlators associated to (8.7), as predicted by the corollary.

8.4. Review of open r -spin theory

It is expected that there exists an open analogue of Witten r -spin theory, related to a space $\overline{\mathcal{M}}_{g,n;b,m}^{r\text{spin}}$, which specialises to the open theory of Section 8.1 for $r = 2$. It would give for each (g, n, b, m) such that $-\bar{\chi} > 0$ a collection of numbers

$$\langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m}^{r\text{spin}} \in \mathbb{Q}$$

indexed by $d_i, k_i \geq 0$ and $a_i \in [r]$, which we can collect in a generating series

$$\begin{aligned} & Z^{\text{open } r\text{spin}}[Q; t^\circ; t^\partial] \\ &= \exp \left(\sum_{\substack{g,b,n,m \geq 0 \\ -\bar{\chi} > 0}} \frac{\hbar^{g-1+\frac{b}{2}} Q^b}{m!n!} \sum_{\substack{d_1, \dots, d_n \geq 0 \\ a_1, \dots, a_n \in [r] \\ k_1, \dots, k_m \geq 0}} \left\langle \prod_{i=1}^n \tau_{d_i}^\circ(a_i) \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{g,n;b,m}^{r\text{spin}} \right. \\ & \quad \left. \times \prod_{i=1}^n (d_i r + a_i)! t_{a_i, d_i}^\circ \prod_{j=1}^m (r k_j + r)! t_{k_j}^\partial \right). \end{aligned} \quad (8.9)$$

These numbers should vanish unless

$$(r+1)(2g-2+b+n+m) = \sum_{i=1}^n (r d_i + a_i) + \sum_{j=1}^m (r k_j + r). \quad (8.10)$$

Besides, for $m \geq 1$, each insertion⁴ of $\tau_d^\circ(r)$ should amount to inserting $(-1/r^{d+1})\tau_d^\partial$.

There are several possible choices of conventions (in particular, for orientations) that could affect these numbers by a prefactor depending only on the topology. We fix them by the normalisation of the consistency relations with the intersection numbers that are already defined. For $r = 2$ and $m \geq 1$, we want to retrieve the open intersection numbers of Section 8.1

$$\left\langle \prod_{i=1}^n \tau_{d_i}^\circ(1) \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{g,n;b,m}^{2\text{spin}} = \left\langle \prod_{i=1}^n \tau_{d_i}^\circ \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{g,n;b,m}. \quad (8.11)$$

Notice that the dimension constraint (8.4) forces $b + m$ to be even, so this is indeed an identity in \mathbb{Q} . In absence of boundaries $b = m = 0$, we want to retrieve the Witten

⁴For $d = 0$, this matches the normalisation [20, Theorem 1.5 and Section 6.1], namely $\tau_0^\circ(r) \leftrightarrow (-1/r)\tau_0^\partial$. The τ_d^∂ correspond to boundary descendent insertion: as they have not been defined geometrically yet, we do not have a natural normalisation to compare to. The factor r^{d+1} is natural from the numerical perspective in combination with the identity $(dr+r)!^{(r)} = r^{d+1}(d+1)!$.

r -spin class intersections of Section 7.3

$$\langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \rangle_{g,n;0,0}^{r\text{spin}} = \int_{\bar{\mathcal{M}}_{g,n}} w_{g,n}^{r\text{spin}}(a_1, \dots, a_n) \prod_{i=1}^n \psi_i^{d_i}.$$

For disks without boundary descendants – that is, $(g, b) = (0, 1)$ and $k_j = 0$ – the open r -spin intersection numbers have been defined in [19] in the form

$$\langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) (\tau_0^\partial)^m \rangle_{0,n;1,m} = \int_{\mathcal{P}\bar{\mathcal{M}}_{0,n;1,m}^{r\text{spin}}} e\left(\mathbb{W} \oplus \bigoplus_{i=1}^n \mathbb{L}_i^{\oplus d_i}, s\right), \quad (8.12)$$

where $\mathcal{P}\bar{\mathcal{M}}$ is a partial compactification of the moduli space of r -spin disks and \mathbb{W} is a bundle which is the open analogue of $R^1\pi_*\mathcal{L}$. We stress that a_i in [19, 20] corresponds to our $a_i - 1$. These numbers are computed explicitly for $d_i = 0$ in [20, Theorem 1.2], and in particular

$$\langle \tau_0^\circ(a) \tau_0^\partial \rangle_{0,1;1,1} = \delta_{a,1}.$$

The dimension constraint (8.10) is the natural generalisation of [20, Section 6.2.1] allowing boundary descendants, and coincides with (8.4) for $r = 2$.

Bertola and Yang have constructed in [8] a particular solution of the extended r -KdV hierarchy, generalising the $r = 2$ construction from [18]. Up to a change of normalisation, this solution is mentioned in [20] under the name Φ and depends on a redundant parameter ε and times $(T_k)_{k>0}$. We shall use the latter normalisation, and for uniformity denote it by $Z^{r\text{BY}}[\varepsilon; (T_k)_{k>0}]$. It gives, for each (\bar{g}, n, m) such that $\bar{\chi} > 0$ (see (8.1)), a collection of numbers

$$\langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{\bar{g};n;m}^{r\text{BY}}$$

by writing down the following expansion:

$$\begin{aligned} & Z^{r\text{BY}}[\varepsilon = (-r\hbar)^{\frac{1}{2}}; (T_{dr+a} = (-r)^{d+\frac{1}{2}-\frac{3(dr+a)}{2(r+1)}} (t_{a,d}^\circ - \delta_{a,r} r^{d+1} t_d^\partial))_{a \in [r]}]_{d \geq 0} \\ &= \exp\left(\sum_{\substack{\bar{g}, n, m \geq 0 \\ -\bar{\chi} > 0}} \frac{\hbar^{\frac{\bar{g}-1}{2}}}{m!n!} \sum_{\substack{d_1, \dots, d_n \geq 0 \\ a_1, \dots, a_n \in [r] \\ k_1, \dots, k_m \geq 0}} \left\langle \prod_{i=1}^n \tau_{d_i}^\circ(a_i) \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{\bar{g};n;m}^{r\text{BY}} \right. \\ & \quad \left. \times \prod_{i=1}^n (rd_i + a_i)! t_{a_i, d_i}^\circ \prod_{j=1}^m (rk_j + r)! t_{k_j}^\partial \right). \end{aligned}$$

In absence of an extra variable in $Z^{r\text{BY}}$ playing the role that Q has in (8.9), one cannot define numbers depending individually on (g, b) , but only on the doubled genus $\bar{g} = 2g + b - 1$.

Conjecture 8.9. *There exists a geometric definition of the open r -spin intersection numbers, and it satisfies*

$$\begin{aligned} & \langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{\bar{g};n;m}^{r\text{BY}} \\ &= \sum_{\substack{g,b \in \mathbb{Z}_{\geq 0} \\ 2g+b-1=\bar{g}}} \langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g;n;b,m}^{r\text{spin}}. \end{aligned} \tag{8.13}$$

This conjecture is formulated in the restricted case $t_d^\partial = 0$ for $d > 0$ in [20], perhaps because no geometric construction of boundary descendants in the open r -spin theory is available yet. Under this restriction, it is supported by the following results:

- The $r = 2$ case was proved in [18], and in agreement with (8.11) we have

$$\begin{aligned} & Z^{\text{ABT}}[Q = 1; (t_d^\circ)_{d \geq 0}; (t_k^\partial)_{k \geq 0}] \\ &= Z^{\text{open } 2\text{spin}}[Q = 1, (t_{1,d}^\circ = t_d^\circ, t_{2,d}^\circ = 0)_{d \geq 0}; (t_k^\partial)_{k \geq 0}]. \end{aligned}$$

- The conjecture is proved in [20] for $\bar{g} = 0$ for general r . In that case, there is a single term $(g, b) = (0, 1)$ in the right-hand side of (8.13).

8.5. Conjectural relation to topological recursion

We now propose a direct generalisation of Section 8.3. We consider the Airy structure of Theorem 2.11 with

$$d = 2, \quad (r_1, s_1; r_2, s_2) = (r, r + 1; 1, \infty), \quad Q_1 = -Q_2 = Q, \quad t_1 = \frac{1}{r},$$

which is also given in [9, Theorem 4.16]. We denote by $Z^{r\star}$ its partition function and we decompose its coefficients as

$$\begin{aligned} & F_{h,n+m}^{r\star} \left[d_1 \overset{1}{r+a_1} \cdots d_n \overset{1}{r+a_n} k_1 \overset{2}{+1} \cdots k_m \overset{2}{+1} \right] \\ &= \sum_{\substack{g,b \in \mathbb{Z}_{\geq 0} \\ g+\frac{b}{2}=h}} (-1)^m Q^b \prod_{i=1}^n (d_i r + a_i)^{(r)} \\ &\quad \times \prod_{j=1}^m (k_j + 1)! \langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g;n;b,m}^{r\star}. \end{aligned} \tag{8.14}$$

According to Section 5.4, the corresponding correlators should be defined as

$$\begin{aligned}
 \omega_{h,n+m}^{r\star} & \left(\begin{array}{c} 1 \cdots 1 \\ z_1 \cdots z_n \end{array} \begin{array}{c} 2 \cdots 2 \\ z_{n+1} \cdots z_{n+m} \end{array} \right) \\
 & := \sum_{\substack{g,b \in \mathbb{Z}_{\geq 0} \\ g + \frac{b}{2} = h}} \sum_{N_{\circ} \sqcup N_{\partial} = [n]} \sum_{\substack{a: N_{\circ} \rightarrow [r-1] \\ d_1, \dots, d_n \geq 0 \\ k_1, \dots, k_m \geq 0}} (-1)^m Q^b \\
 & \times \left\langle \prod_{i \in N_{\circ}} \tau_{d_i}^{\circ}(a_i) \prod_{i \in N_{\partial}} \tau_{d_i}^{\partial} \prod_{j=1}^m \tau_{k_j}^{\partial} \right\rangle_{g, |N_{\circ}|; b, m + |N_{\partial}|}^{r\star} \\
 & \times \prod_{i \in N_{\circ}} \frac{(d_i r + a_i)!^{(r)} dz_i}{z_i^{d_i r + a_i + 1}} \prod_{i \in N_{\partial}} \frac{(d_i + 1)! dz_i}{z_i^{d_i r + r + 1}} \prod_{j=1}^m \frac{(k_j + 1)! dz_{n+j}}{z_{n+j}^{k_j + 2}}. \quad (8.15)
 \end{aligned}$$

Note that we have converted all $\tau_d^{\circ}(r)$ into $-r^{-(d+1)}\tau_d^{\partial}$, which turned the r -fold factorial $(rd + r)!^{(r)} = r^{d+1}(d + 1)!$ into a usual factorial in the corresponding factors. By Theorem 5.23, they satisfy the topological recursion on the reducible spectral curve with two components intersecting at $z = 0$:

$$C = C_1 \cup C_2, \quad C_1: \begin{cases} x(z) = z^r, \\ y(z) = -\frac{z}{r}, \end{cases} \quad C_2: \begin{cases} x(z) = z, \\ y(z) = 0 \end{cases} \quad (8.16)$$

equipped with

$$\omega_{0,2}(\begin{smallmatrix} \mu_1 & \mu_2 \\ z_1 & z_2 \end{smallmatrix}) = \delta_{\mu_1, \mu_2} \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad \omega_{\frac{1}{2}, 1}(\begin{smallmatrix} \mu \\ z \end{smallmatrix}) = (-1)^{\mu+1} Q \frac{dz}{z}.$$

For comparison, let us examine the basic properties of $\langle \dots \rangle^{r\star}$. Firstly, due to the constraint $(J_{k_r}^1 + J_k^2)Z^{r\star} = 0$ for $k > 0$ and definition (8.14), each insertion of $\tau_d^{\circ}(r)$ amounts to the insertion of $-r^{-(d+1)}\tau_d^{\partial}$ while incrementing the number m by 1. Secondly, the constraint $H_{i=2, k=0}Z^{r\star} = 0$ gives, by computations similar to those of Section 2.4, the string equation

$$\begin{aligned}
 \left\langle \tau_0^{\circ}(1) \prod_{i=1}^n \tau_{d_i}^{\circ}(a_i) \prod_{j=1}^m \tau_{k_j}^{\partial} \right\rangle_{g, 1+n; b, m}^{r\star} & = \sum_{l=1}^n \left\langle \tau_{d_l-1}^{\circ}(a_l) \prod_{i \neq l} \tau_{d_i}^{\circ}(a_i) \prod_{j=1}^m \tau_{k_j}^{\partial} \right\rangle_{g, n; b, m}^{r\star} \\
 & + \sum_{l=1}^m \left\langle \tau_{k_l-1}^{\partial} \prod_{i=1}^n \tau_{d_i}^{\circ}(a_i) \prod_{j \neq l} \tau_{k_j}^{\partial} \right\rangle_{g, n; b, m}^{r\star} \\
 & + \delta_{g, b, m, 0} \delta_{n, 2} \delta_{d_1, d_2, 0} \delta_{a_1 + a_2, r} \\
 & + \delta_{g, m, 0} \delta_{b, n, 1} \delta_{d_1, 0} \delta_{a_1, r} \\
 & + \delta_{g, n, 0} \delta_{b, m, 1} \delta_{k_1, 0}, \quad (8.17)
 \end{aligned}$$

with obvious vanishing conventions for insertion of negative indices. This last term gives the special value

$$\langle \tau_0^\circ(1) \tau_0^\partial \rangle_{0,1;1,1}^{r\star} = 1. \tag{8.18}$$

Note this is compatible with the computation of (6.32) with the specialisation $t_1 = \frac{1}{r}$. Indeed, the latter yields

$$\omega_{\frac{1}{2},2} \left(\begin{matrix} 1 & 2 \\ z_1 & z_2 \end{matrix} \right) = -rQ \frac{dz_1 dz_2}{z_1^2 z_2^2} = -Q \frac{dz_1 r!(r) dz_2}{z_1^2 z_2^2},$$

and thus after taking (8.15) into account, (8.18) describes the only non-vanishing intersection number for $(g, n; b, m) = (0, 1; 1, 1)$. Thirdly, we have from Corollary 2.20 the dilaton equation

$$\begin{aligned} & \left\langle \tau_1^\circ(1) \prod_{i=1}^n \tau_{d_i}^\circ(a_i) \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{g,1+n;b,m}^{r\star} \\ &= (2g - 2 + n + m) \left\langle \prod_{i=1}^n \tau_{d_i}^\circ(a_i) \prod_{j=1}^m \tau_{k_j}^\partial \right\rangle_{g,n;b,m}^{r\star} \\ &+ \frac{r-1}{24} \delta_{g,1} \delta_{n,b,m,0} + \frac{1}{2} \delta_{g,n,m,0} \delta_{b,2} \end{aligned} \tag{8.19}$$

and the homogeneity property which says that $\langle \prod_{i=1}^n \tau_{d_i}^\circ(a_i) \prod_{j=1}^m \tau_{k_j}^\partial \rangle_{g,n;b,m}^{r\star}$ vanishes unless the dimension constraint (8.10) holds.

We predict that the partition function $Z^{r\star}$ describes the full open r -spin intersection theory in any genera and with arbitrary descendants.

Conjecture 8.10. *There is a geometric definition of the open r -spin intersection numbers, and it satisfies*

$$\langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m}^{r\text{spin}} = \langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m}^{r\star}.$$

A weaker prediction, involving only quantities whose definition is available at the time of writing, is that the Bertola–Yang $Z^{r\text{BY}}$ partition function satisfies $\mathcal{W}(\text{gl}_r)$ -constraints with zero mode values $Q_1 = -Q_2 = 1$. Including the expected normalisations, this would translate into the following.

Conjecture 8.11. *We have for $b, m > 0$*

$$\langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{\bar{g};n;m}^{r\text{BY}} = \sum_{\substack{g,b \in \mathbb{Z}_{\geq 0} \\ g + \frac{b}{2} = \bar{g}}} \langle \tau_{d_1}^\circ(a_1) \cdots \tau_{d_n}^\circ(a_n) \tau_{k_1}^\partial \cdots \tau_{k_m}^\partial \rangle_{g,n;b,m}^{r\star}.$$

In support of the conjectures, we see that the basic properties listed for $\langle \cdots \rangle^{r\star}$ match the ones listed for $\langle \cdots \rangle$ in the range of parameters in which the comparison

is possible. The dilaton equation from [20, Proposition 5.3] matches the restriction of (8.19) to $(g, b) = (0, 1)$ and $k_j = 0$. The string equation from [20, Proposition 5.2] matches the same restriction of (8.17), and observe that in absence of boundary descendants, the second sum in the right-hand side of (8.17) is absent. The identification of $\tau_d^\circ(r)$ with $-r^{-(d+1)}\tau_d^\partial$ for $d = 0$ is manifest in [20, Theorem 1.1]. Their extension to $g > 0$ expected in [20, Section 6.2] also matches our proposal.

Topological recursion relations (TTR) involving in a linear way the open r -spin intersection numbers mentioned in (8.12) and the closed r -spin intersection numbers are given in [20, Theorem 4.1]. The information of (8.12) should be encoded for us in $F_{\frac{1}{2},n}^{r,*}$, as it is easy to see that the W -constraints indeed imply some quadratic relation with a similar structure involving $F_{0,n}^{r,*}$ in a non-linear way and $F_{\frac{1}{2},n}^{r,*}$ in a linear way. As $F_{0,n}^{r,*}$ contains only information from the closed sector and satisfies W -constraints on its own, the structure somehow resembles TRR, but establishing an exact match is left to future work.

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