



Anita Liebenau · Nick Wormald

Asymptotic enumeration of graphs by degree sequence, and the degree sequence of a random graph

Received November 26, 2018

Abstract. In this paper we relate a fundamental parameter of a random graph, its degree sequence, to a simple model of nearly independent binomial random variables. As a result, many interesting functions of the joint distribution of graph degrees, such as the distribution of the median degree, become amenable to estimation. Our result is established by proving an asymptotic formula conjectured in 1990 for the number of graphs with given degree sequence. In particular, this gives an asymptotic formula for the number of d -regular graphs for all d , as $n \rightarrow \infty$. The key to our results is a new approach to estimating ratios between point probabilities in the space of degree sequences of the random graph, including analysis of fixed points of the associated operators.

Keywords. Asymptotic enumeration, random graph, degree sequence

1. Introduction

We consider the number of graphs with a given degree sequence. In particular, we show that a formula known to hold in the sparse and dense cases as long as the degrees of vertices are somewhat close to each other, also holds for the remaining cases. The main consequence of this is a very simple and consequently useful model for the degree sequence of a random graph. The two most popularly studied models of random graphs are considered here: $\mathcal{G}(n, p)$, in which n vertices have edges included between each pair of them independently with probability p for each pair, and $\mathcal{G}(n, m)$, in which n vertices have m edges included, chosen uniformly at random from the m -subsets of the unordered pairs of vertices. Those classical models of random graphs easily satisfy the required restrictions on degrees with high probability, and it follows that the degree sequence of $\mathcal{G}(n, m)$ is well approximated by a certain sequence of independent binomial variables conditioned on summing to $2m$. A similar connection is provided between the random

Anita Liebenau: School of Mathematics and Statistics, UNSW, Sydney, NSW 2052, Australia; a.liebenau@unsw.edu.au

Nick Wormald: School of Mathematics, Monash University, Melbourne, VIC 3800, Australia; nick.wormald@monash.edu

Mathematics Subject Classification (2020): Primary 05C30; Secondary 05C07, 05C80

graph $\mathcal{G}(n, p)$ and a slightly twisted sequence of independent binomial random variables. This makes a very convenient way of proving results about the degree sequence of $\mathcal{G}(n, p)$. Enumeration formulae for graphs by degree sequence have also led by other routes to a large number of results in random graph theory, some of which we mention in Section 1.2.

The study of the degree sequence of the random graph goes back to the early papers of Erdős and Rényi [15, 16] and has since attracted the attention of many; see, e.g., [1, 3, 9, 22, 33, 34]. For a historical overview, see for example Bollobás' seminal book [10], where the degree sequence is the first major topic. The distribution of the k th largest element d_k of the sequence, for example, was determined quite precisely when k is small. The book [2] by Barbour, Holst and Janson contained much information on the distribution of the number D_k of vertices of degree k . On the other hand, enumerating graphs with given degree sequence has been the interest of various authors over many years. Read [36] found a recursive formula for the number of 3-regular graphs from which he also deduced a simple asymptotic formula. After this, formulae for the number of graphs with given degree sequence $\mathbf{d} = (d_1, \dots, d_n)$ were found by Bender and Canfield [6] and Wormald [39, Theorem 3.3], and for ever-denser ranges of degrees by Bollobás [8] and McKay [28], culminating in papers giving asymptotic formulae for a range of degrees provided the average degree d is $o(\sqrt{n})$ (by McKay and Wormald [31]) or between $cn/\log n$ and $n/2$ for a certain c (by McKay and Wormald [30], also treated more recently by Barvinok and Hartigan [4] for a wider spread of degrees, but with similar density). The complementary ranges of d larger than $n/2$ are automatically covered. Also quite recently, Janson [19, 20] obtained formulae for some sparse degree sequences with maximum degree $\Theta(\sqrt{n})$, and Gao and Wormald [17] for others having slightly larger maximum degree. A case of special interest, which saw no advance since 1990, is the problem of finding the asymptotic number of d -regular graphs for d in the range $c\sqrt{n} \leq d = o(n/\log n)$.

Approximate estimates of the numbers of graphs with given degrees have also been of interest due to their relevance to statistics of real-world networks in areas such as ecology. For instance, Blitzstein and Diaconis [7] used sequential importance sampling (SIS) both to generate such graphs approximately uniformly, and to approximate their numbers. The use of asymptotic formulae in aiding SIS is explained by Chen, Diaconis, Holmes and Liu [13], who use it for sampling contingency tables with given row and column sums. These are equivalent to bipartite graphs with given degrees.

In 1990, McKay and Wormald [30] restated the asymptotic formulae from the sparse and dense cases in a common form, which they conjectured to be valid additionally for all densities in between those two cases and hence for all densities except trivial extremely sparse and dense ones. For a precise statement, see the *Binomial Approximation Conjecture* (Conjecture 1.2) below. It applies to all the typical degree sequences in either model of random graphs defined above.

In this paper, we prove the Binomial Approximation Conjecture. In particular, as a very special case, this implies that the number of d -regular graphs on n vertices is asymp-

totally equal to

$$\frac{\binom{n-1}{d}^n \binom{n}{m}}{\binom{n(n-1)}{2m}} \cdot e^{1/4}$$

for all $1 \leq d \leq n-2$, where $m = dn/2$.

A weakened version of the Binomial Approximation Conjecture was given in 1997 by McKay and Wormald [32] (see Conjecture 1.3) and shown to imply an explicit connection between the degree sequence of a random graph of a given density and a sequence of independent binomial variables. This work opened up a completely new approach to deriving properties of the degree sequence of a random graph, by considering independent binomials. It permits easy access to all the known properties, and a large number that were previously inaccessible. Previously, precise results for order statistics were only known for extreme degrees, but now even the distribution of the median degree can be closely examined for those densities where the conjecture holds.

To present the connection between the degree sequence of $\mathcal{G}(n, m)$ or $\mathcal{G}(n, p)$, and a sequence of independent binomials, let us first make some definitions. We assume that a graph on n vertices has vertex set v_1, \dots, v_n and degree sequence (d_1, \dots, d_n) , so that $d(v_i) = d_i$. If \mathcal{G} is any probability space of random graphs, let $\mathcal{D}(\mathcal{G})$ be the random vector distributed as the degree sequence of a random graph $G \in \mathcal{G}$. Also define $\mathcal{B}_p(n)$ to be the random sequence consisting of n independent binomial variables $\text{Bin}(n-1, p)$.

Let A_n and B_n be two sequences of probability spaces with the same underlying set for each n . Suppose that whenever a sequence of events H_n satisfies $\mathbf{P}(H_n) = n^{-O(1)}$ in either model, it is true that $\mathbf{P}_{A_n}(H_n) \sim \mathbf{P}_{B_n}(H_n)$, where by $f(n) \sim g(n)$ we mean that $f(n)/g(n) \rightarrow 1$ as $n \rightarrow \infty$. We then say that A_n and B_n are *asymptotically quite equivalent* (a.q.e.). Throughout this paper we use ω to be an arbitrary function of n such that $\omega \rightarrow \infty$ as $n \rightarrow \infty$, perhaps different at each occurrence. As we will see later, our main result, combined with existing results, implies the following. Note for part (i) that $B_p(n)|_{\Sigma=2m}$ is independent of p .

Proposition 1.1. *Let n, m be integers and let $0 < p < 1$. Let Σ denote the sum of the components of the random vector $\mathcal{B}_p(n)$ in (i) and $\mathcal{B}_{\hat{p}}(n)$ in (ii).*

- (i) $\mathcal{D}(\mathcal{G}(n, m))$ and $\mathcal{B}_p(n)|_{\Sigma=2m}$ are a.q.e. provided that $\min\{m, \binom{n}{2} - m\} = \omega \log n$.
- (ii) Let \hat{p} be randomly chosen according to the normal distribution with mean p and variance $p(1-p)/n(n-1)$, truncated at 0 and 1. Then $\mathcal{D}(\mathcal{G}(n, p))$ and $\mathcal{B}_{\hat{p}}(n)|_{\Sigma \text{ is even}}$ are a.q.e. provided that $p(1-p) = \omega n^{-2} \log^3 n$.

Note that the assumptions on p and m merely ensure that the graph, or its complement, has number of edges at least a small power of $\log n$; if this fails, the degree sequence is almost trivial and the graph is likely to be uninteresting, either a set of independent edges or its complement.

Proposition 1.1 has significant implications. In particular, it was shown in [32] that a result very similar to Proposition 1.1 (ii) can be used, for whatever range of $p(n)$ it is valid for, to transfer general classes of properties to $\mathcal{D}(\mathcal{G}(n, p))$ from the independent binomial

model $\mathcal{B}_p(n)$. We give details later in this section. It was also observed in [32], using the known asymptotic formulae, that the Binomial Approximation Conjecture holds when $p = o(1/\sqrt{n})$ or $p(1-p) > c/\log n$. However, the rather large gap, where $p(1-p)$ is between roughly $1/\sqrt{n}$ and $1/\log n$, was still open, as the appropriate enumeration results were lacking. This gap has prevented the new approach being fully utilised. Part (i) of the proposition is also appealing as a direct connection between independent binomials and $\mathcal{D}(\mathcal{G}(n, m))$, but the event $\Sigma = 2m$ is fairly “thin”, so some results would require more finesse to be transferred.

In this article we introduce an approach to enumerating graphs by degree sequence that differs significantly from what has previously been applied to this or any similar problems. This new method is versatile enough to be applied to enumeration problems for other discrete structures, as described in Section 8. For this reason, our theoretical results are set in a framework slightly wider than is needed for the enumeration results derived in the present paper. Our method has its origins in some aspects of the enumeration arguments of McKay [28] and Stein [38]. Instead of trying to directly count quantities indexed by some parameter, we obtain estimates of ratios of the quantities with adjacent indices, and then use the ratios combined with our knowledge of the total over all indices to deduce the desired estimates. This arises naturally in the exchangeable pairs approach of Stein (see also [37]). Our estimates of ratios can be improved by iterating the basic relations we derive, so the final estimates can be determined by fixed points of certain operators defined using those relations.

1.1. Conjectures and results

In 1990 McKay and Wormald [30] unified the existing asymptotic formulae for the sparse and the dense case into one form and conjectured this form to hold also for the gap in the range of degrees, as long as the degree sequences are close to regular. Throughout this paper we use the following notation. Given a sequence $\mathbf{d} = (d_1, \dots, d_n)$, let $g(\mathbf{d})$ denote the number of graphs whose degree sequence is \mathbf{d} , let $\mu = \mu(\mathbf{d}) = d/(n-1)$ where $d = \frac{1}{n} \sum_{i=1}^n d_i$, and let $\gamma_2 = \gamma_2(\mathbf{d}) = (n-1)^{-2} \sum_{i=1}^n (d_i - d)^2$. We refer to the following as the *Binomial Approximation Conjecture*.

Conjecture 1.2 (McKay and Wormald [30]). *For some absolute constant $\varepsilon > 0$, if $\mathbf{d} = \mathbf{d}(n)$ satisfies $\max_j |d_j - d| = o(n^\varepsilon \min\{d, n-d-1\}^{1/2})$, $n \min\{d, n-d-1\} \rightarrow \infty$, and $\sum_i d_i$ is even, then*

$$g(\mathbf{d}) \sim \sqrt{2} \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\right) (\mu^\mu (1-\mu)^{1-\mu})^{n(n-1)/2} \prod \binom{n-1}{d_i}. \quad (1.1)$$

Note. The permitted domain of \mathbf{d} is compact for each n , so one can consider the “worst” sequence $\mathbf{d}(n)$ for each n , and arrive at an equivalent way of stating the result: there is a function $\delta(n) \rightarrow 0$ such that the relative error in “ \sim ” is bounded above in absolute value by $\delta(n)$ for all \mathbf{d} under consideration. This was the manner of stating Conjecture 1.3 below in [32].

Recall that $\mathcal{B}_p(n)$ yields a random sequence consisting of n independent binomial variables $\text{Bin}(n-1, p)$. Let $\mathcal{B}_m = \mathcal{B}_m(n)$ denote $\mathcal{B}_p(n)$ conditioned on the sum of the sequence being $2m$ and note that for a given sequence \mathbf{d} with $\sum_{i=1}^n d_i = 2m$ we have

$$\mathbf{P}_{\mathcal{B}_m}(\mathbf{d}) = \binom{n(n-1)}{2m}^{-1} \prod_{i=1}^n \binom{n-1}{d_i},$$

which is, we recall, independent of p . Multiplying by $|\mathcal{G}(n, m)|$ and using Stirling's approximation shows that the formula

$$\mathbf{P}_{\mathcal{D}(\mathcal{G}(n, m))}(\mathbf{d}) \sim \mathbf{P}_{\mathcal{B}_m}(\mathbf{d}) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\right) \quad (1.2)$$

is equivalent to the asymptotic formula in (1.1), as long as m and $n(n-1) - 2m$ both tend to infinity.

Proposition 1.1 can be shown to follow from a conjecture made in 1997, which we present below, that is weaker than Conjecture 1.2. To state it, we define the model \mathcal{E}_p to be $\mathcal{B}_p(n)$, conditioned on *even* sum, and then construct \mathcal{E}'_p from \mathcal{E}_p by weighting each \mathbf{d} with a weight depending only on $2m = \sum d_i$, such that the value m has distribution $\text{Bin}(n(n-1)/2, p)$.

Definition. A probability $p = p(n)$ is *acceptable* if $p(1-p)n^2 = \omega \log n$ and there is a set-valued function $R_p(n)$ of integer sequences of length n with even sum, such that both of the following hold:

- (i) For $\mathbf{d} \in R_p(n)$

$$\mathbf{P}_{\mathcal{D}(\mathcal{G}(n, p))}(\mathbf{d}) \sim \mathbf{P}_{\mathcal{E}'_p}(\mathbf{d}) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\right). \quad (1.3)$$

- (ii) In each of the models \mathcal{E}_p and $\mathcal{D}(\mathcal{G}(n, p))$, we have $\mathbf{P}(R_p(n)) \geq 1 - n^{-\omega}$.

The conjecture in [32] is as follows.

Conjecture 1.3 (McKay and Wormald [32]). *If $p(1-p)n^2 = \omega \log n$ then $p(n)$ is acceptable.*

The fact that the exponential factor in (1.3) is sharply concentrated near 1 was used to prove [32, Theorem 2.6(b)] (see Section 3 of that paper), which we do not state here. The first part of [32, Theorem 2.6(b)] gives bounds on the differences between the expected values of random variables in $\mathcal{D}(\mathcal{G}(n, p))$ and the model $\mathcal{B}_{\hat{p}}(n)|_{\Sigma}$ is even of Proposition 1.1(ii) for any acceptable p . The second part of that theorem similarly bounds the difference between expectations in $\mathcal{D}(\mathcal{G}(n, m))$ and $\mathcal{B}_m(n)$, for any m such that $2m/n(n-1)$ is acceptable. In particular, Conjecture 1.3 implies Proposition 1.1 via [32, Theorem 2.6(b)] with X_n defined as the indicator of the event H_n that appears in the definition of a.q.e. We omit further details of this, since for applications of our main result, one can look at all the consequences of Conjecture 1.3 in [32], of which Proposition 1.1 is just an example. In particular, further results in [32, Section 4] show

that for many properties in $\mathcal{G}(n, p)$, the conditioning on parity in Proposition 1.1 (ii) has negligible effect, so one can consider purely independent binomials. Sometimes the integration required to deal with the distribution of \hat{p} also has negligible effect, and in any case, [32, Theorems 3.7 and 3.8] give some general results on carrying out the integration. To our knowledge, there are so far no detailed applications of the consequence for $\mathcal{G}(n, m)$ given in Proposition 1.1 (i).

It was shown in [32, Theorem 2.5] that p is acceptable if either $\omega(\log n)/n^2 \leq p(1-p) = o(n^{-1/2})$ or $p(1-p) \geq c/\log n$ for some $c > 2/3$, using the known enumeration results in the respective range. The essence of the proof of [32, Theorem 2.5] shows that if (1.1) of Conjecture 1.2 holds for all d near some d_0 , then by known concentration results, $p = d_0/(n-1)$ is acceptable. The same considerations prove that Conjecture 1.3 follows from Conjecture 1.2.

The main result of this paper verifies Conjecture 1.2, and hence Conjecture 1.3 and Proposition 1.1, in the remaining gap range.

Theorem 1.4. *Let $\mu_0 > 0$ be a sufficiently small constant, and let $1/2 < \alpha < 3/5$. Let n and m be integers, and assume that $d = 2m/n$ satisfies $(\log n)^\omega \leq d \leq \mu_0 n$. Let \mathfrak{D} be the set of sequences \mathbf{d} of length n satisfying $\sum_i d_i = 2m$ and $|d_i - d| \leq d^\alpha$ for all $i \in [n]$. Then uniformly for all $\mathbf{d} \in \mathfrak{D}$ we have*

$$\mathbf{P}_{\mathfrak{D}(\mathcal{G}(n,m))}(\mathbf{d}) = \mathbf{P}_{\mathcal{B}_m}(\mathbf{d}) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\right) \left(1 + O\left(\frac{(\log n)^2}{\sqrt{n}} + d^{5\alpha-3}\right)\right),$$

where $\mu = \mu(\mathbf{d}) = d/(n-1)$ and $\gamma_2 = \gamma_2(\mathbf{d})$.

Corollary 1.5. *Conjectures 1.2 and 1.3 and Proposition 1.1 are all true.*

In Section 6 we provide a hand-checkable proof of Theorem 1.4 under the restriction that $\max |d_i - d| = O(\sqrt{d \log n})$ and $\sum_i (d_i - d)^2 \leq 2dn$ (see Theorem 6.3). As a corollary of this and known results, we obtain Conjecture 1.3. Extending the expansions of functions involved to more terms using computer algebra lets us obtain a proof of Conjecture 1.2.

Our method is markedly different from those previously used for this problem, the most successful of which used either the ‘‘configuration model’’ with ‘‘switchings’’, or the estimation of integrals representing coefficients of appropriate generating functions. Instead, we derive a set of three equations relating the numbers of graphs with almost the same degree sequence, and the probability that they contain a given edge or a given pair of incident edges. The equations are derived by examining the operation of moving one end of an edge from one vertex to another, and the operation of deleting an edge. These equations are analysed from the perspective of fixed points of a contraction mapping.

Note that if we build up a random graph by inserting m edges using the method of choosing each endpoint of each edge independently at random, the resulting multigraph has exactly the multinomial degree distribution, that is, $\mathcal{B}_m(n)$ (see for example Cain and Wormald [12]). The only difficulty is that the result can contain loops and multiple edges. For Proposition 1.1, one would need to show that conditioning on the absence of loops

and multiple edges does not significantly affect the degree distribution. This is difficult since the probability of no loop or multiple edges occurring is extremely small even for moderate densities. We believe that our approach to this enumeration problem gives an intuitive explanation of why the distribution of degrees in a random graph is so close to the binomial model, since it shows directly that the distribution of graph counts behaves “locally” in the appropriate way. Such an explanation is absent from the main methods used previously, in [30, 31].

As a by-product of our proof, we obtain asymptotic formulae for the edge probabilities in a random graph with a given degree sequence. For a sequence \mathbf{d} of length n , let $\mathcal{G}(\mathbf{d})$ be a graph chosen uniformly at random from all graphs that have degree sequence \mathbf{d} , and let $\mathbf{P}_{ab}(\mathbf{d})$ be the probability that the edge $v_a v_b$ is present in $\mathcal{G}(\mathbf{d})$.

Theorem 1.6. *Let \mathfrak{D} be as in Theorem 1.4 and let $a, b \in [n]$, $a \neq b$. Then for all $\mathbf{d} \in \mathfrak{D}$,*

$$\mathbf{P}_{ab}(\mathbf{d}) = \frac{d_a d_b}{d(n-1)} \left(1 - \frac{(d_a - d)(d_b - d)}{d(n-1-d)} \right) + O\left(\frac{\sqrt{d \log n}}{n^2} + \frac{(\sqrt{d \log n})^3}{n^3} \right).$$

We also give a more accurate but more complicated formula for $\mathbf{P}_{ab}(\mathbf{d})$ in Lemma 7.1 (ii), and an approximation for sparser degree sequences in (4.4). Our results also provide formulae for the probability that a given path of length 2 occurs in the same graphs.

In Section 2 we provide the notation we use in this paper and some preliminary results. In Section 3 we derive certain recursive formulae that are the core of our method. In Section 4 we re-prove enumeration results for the sparse case to illustrate how the recursive formulae from Section 3 are to be used to obtain explicit formulae. We also take this opportunity to provide a general template on how our proofs in the later parts are structured. In Section 5 the recursive formulae are turned into operators. Section 6 contains the proof of Theorem 1.4 under stricter assumptions on the degree spreads, and gives the proof of Conjecture 1.3, all using easily hand-checkable calculations. As explained above, this is enough to deduce our first major goal, Proposition 1.1. In Section 7 we provide the full proof for Theorem 1.4. We finish the paper with some concluding remarks in Section 8, and the Appendix gives details of some routine calculations.

1.2. Further comments

Studying the degree sequence of a random graph is not the only significant potential use of enumeration formulae for graphs with given degrees. A large number of properties of random regular graphs have been shown using them, in particular, results on subgraphs of random regular graphs or random graphs with given degrees. (McKay [29] refers to many examples.) The asymptotic distribution of the number of edges in the giant component of the random graph, and the size of its 2-core, were first obtained by Pittel and Wormald [35] with heavy use of these formulae. Additionally, Kim and Vu [23, (3.1)] heavily used the asymptotic formula for the number of d -regular graphs, in the known range, to give strong relations between random graphs and random regular graphs. The lack of a formula in the

denser case restricted their work to the sparse case. The results in the present paper could have helped here, and would presumably have simplified the extension of Kim and Vu's work to denser cases, given recently by Dudek, Frieze, Ruciński and Šileikis [14].

Various results on random d -regular graphs [5, 25] use the configuration model and a sophisticated analysis of switchings, which are methods developed for asymptotic enumeration of graphs by degree sequence when the degrees are relatively small. The methods from this paper may well have similar applications, even in cases where the enumeration formulae themselves do not apply.

We note that Isaev and McKay [18] have given a major further development of the methods in [4, 30] to obtain further results useful in the case of very dense graphs. They have also (unpublished) announced progress in pushing this method towards sparser cases. Also after our project was under way, Burstein and Rubinfeld [11] presented an idea of comparing numbers of graphs with given degree sequences that is somewhat related to the techniques in the present paper, but differs in several significant ways. Their approach would give a formula that is valid up to maximum degree $n^{1-\delta}$ for any fixed $\delta > 0$, using a finite amount of computation. However, their general result is not explicit enough to enable the derivation of results as simple as formula (1.1). In particular, we are not aware of any claims of extending the range of validity of the Binomial Approximation Conjecture, apart from the present paper.

2. Preliminaries and notation

For the reader's convenience, we solidify some notation here. Our *graphs* are simple, that is, they have no loops or multiple edges. We write $a \sim b$ to mean that $a/b \rightarrow 1$, $f = O(g)$ if $|f| \leq Cg$ for some constant C , and $f = o(g)$ if $f/g \rightarrow 0$. We use ω to mean a function going to infinity, possibly different in all instances. We use $1 \pm \xi$ to denote a quantity in the interval $[1 - \xi, 1 + \xi]$. Also $\binom{[n]}{2}$ denotes the set of 2-subsets of the set $[n] = \{1, \dots, n\}$. We often consider a vector $\mathbf{d} = (d_1, \dots, d_n)$, and use Δ or $\Delta(\mathbf{d})$ to denote $\max_i d_i$, in line with the notation for maximum degree of a graph. Recall also from the introduction that $g(\mathbf{d})$ denotes the number of graphs whose degree sequence is \mathbf{d} , $\mu = \mu(\mathbf{d}) = d/(n-1)$ where $d = d(\mathbf{d}) = \frac{1}{n} \sum_{i=1}^n d_i$, and $\gamma_2 = (n-1)^{-2} \sum_{i=1}^n (d_i - d)^2$. We write $M_1 = M_1(\mathbf{d}) = \sum_{i=1}^n d_i$. If \mathbf{d} is the degree sequence of a graph, this is the total degree, or twice the number of edges. Parity is an important issue, so we say a vector \mathbf{d} is *even* if $M_1(\mathbf{d})$ is even, and *odd* otherwise. Finally, in this paper multiplication by juxtaposition has precedence over “/”, so for example $j/\mu n^2 = j/(\mu n^2)$.

We first state a simple result by which we leverage an enumeration result from comparisons of related numbers.

Lemma 2.1. *Let \mathcal{S} and \mathcal{S}' be probability spaces with the same underlying set Ω . Let G be a graph with vertex set $\mathbb{X} \subseteq \Omega$ such that $\mathbf{P}_{\mathcal{S}}(v), \mathbf{P}_{\mathcal{S}'}(v) > 0$ for all $v \in \mathbb{X}$. Suppose that $\varepsilon_0, \delta > 0$ are such that $\min\{\mathbf{P}_{\mathcal{S}}(\mathbb{X}), \mathbf{P}_{\mathcal{S}'}(\mathbb{X})\} > 1 - \varepsilon_0 > 1/2$, and such that for every*

edge uv of G ,

$$\frac{\mathbf{P}_{\mathcal{S}'}(u)}{\mathbf{P}_{\mathcal{S}'}(v)} = e^{O(\delta)} \frac{\mathbf{P}_{\mathcal{S}}(u)}{\mathbf{P}_{\mathcal{S}}(v)},$$

where the constant implicit in $O(\cdot)$ is absolute. Let r be an upper bound on the diameter of G and assume $r < \infty$. Then for each $v \in \mathfrak{V}$ we have

$$\mathbf{P}_{\mathcal{S}'}(v) = e^{O(r\delta + \varepsilon_0)} \mathbf{P}_{\mathcal{S}}(v),$$

with again a bound uniform for all v .

Proof. For any $u, v \in \mathfrak{V}$ we may take a telescoping product of ratios along a path joining u to v of length at most r . This gives $\mathbf{P}_{\mathcal{S}'}(u)/\mathbf{P}_{\mathcal{S}'}(v) = e^{O(r\delta)} \mathbf{P}_{\mathcal{S}}(u)/\mathbf{P}_{\mathcal{S}}(v)$. Summing over all $u \in \mathfrak{V}$ gives

$$\frac{\mathbf{P}_{\mathcal{S}'}(\mathfrak{V})}{\mathbf{P}_{\mathcal{S}'}(v)} = \frac{e^{O(r\delta)} \mathbf{P}_{\mathcal{S}}(\mathfrak{V})}{\mathbf{P}_{\mathcal{S}}(v)},$$

and the claim follows using the lower bound $1 - \varepsilon_0$ on $\mathbf{P}_{\mathcal{S}}(\mathfrak{V})$ and $\mathbf{P}_{\mathcal{S}'}(\mathfrak{V})$. \blacksquare

Using the lemma calls for evaluating the ratios of probabilities in an “ideal” probability space, the one by which we are approximating the “true” space. This leads to computing ratios in the conjectured formulae. One we need several times is the following. Let $H(\mathbf{d})$ denote the conjectured formula in the right hand side of (1.2), apart from the error term, that is,

$$H(\mathbf{d}) = \mathbf{P}_{\mathcal{B}_m}(\mathbf{d}) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\right).$$

We use \mathbf{e}_a to denote the elementary unit vector with 1 in its a th coordinate. Noting that $\mu = d/(n-1)$ where $d = M_1/n$ (a function of the degree sequence) has the same value for both cases $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$, and recalling $\gamma_2 = \gamma_2(\mathbf{d}) = (n-1)^{-2} \sum_i (d_i - d)^2$, we have (for all odd \mathbf{d} with $\Delta = \max d_i \leq n/2$ and $\mu > 0$)

$$\begin{aligned} \frac{H(\mathbf{d} - \mathbf{e}_a)}{H(\mathbf{d} - \mathbf{e}_b)} &= \frac{d_a(n-d_b)}{d_b(n-d_a)} \exp\left(\frac{(d_a-d_b)(\gamma_2 + O(\Delta/n^2))}{(n-1)^2(\mu')^2(1-\mu')^2}\right) \\ &= \frac{d_a(n-d_b)}{d_b(n-d_a)} \exp\left(\frac{(d_a-d_b)\gamma_2}{d^2(1-\mu')^2} + O(\Delta^2/(dn)^2)\right), \end{aligned} \quad (2.1)$$

where $\mu' = \mu(\mathbf{d} - \mathbf{e}_a) = \mu(\mathbf{d} - \mathbf{e}_b)$, and of course γ_2 , d , and Δ are defined with respect to \mathbf{d} .

Next we turn to some issues involving existence of graphs with a given degree sequence. For counting purposes we will be considering graphs with vertex set $V = [n]$. Let $\mathcal{A} = \mathcal{A}(n) \subseteq \binom{[n]}{2}$ be a set which we call *allowable pairs*. Note that as usual we regard the edge joining vertices u and v as the unordered pair $\{u, v\}$, and denote this edge by uv following standard graph-theoretic notation. A sequence $\mathbf{d} := (d_1, \dots, d_n)$ is called *\mathcal{A} -realisable* if there is a graph G on vertex set V such that vertex $a \in V$ has degree d_a and all edges of G are allowable pairs. In this case, we say G *realises* \mathbf{d} *over* \mathcal{A} .

In standard terminology, if \mathbf{d} is $\binom{V}{2}$ -realisable, it is *graphical*. Let $\mathcal{G}_{\mathcal{A}}(\mathbf{d})$ be the set of all graphs that realise \mathbf{d} over \mathcal{A} . In this paper, we are particularly interested in the case that $\mathcal{A} = \mathcal{A}^{\text{gr}} = \binom{V}{2}$. Then $\mathcal{G}_{\mathcal{A}}(\mathbf{d})$ is the set of all graphs G on vertex set $[n]$ that have degree sequence \mathbf{d} . By using different definitions of \mathcal{A} , it is also possible to model other enumeration problems. Throughout this paper, all graphs are finite and simple (i.e. have no loops or multiple edges).

Let $E \subseteq \mathcal{A}$, i.e. a subset of the allowable edges. We write $\mathcal{N}_E(\mathbf{d})$ and $\mathcal{N}_E^*(\mathbf{d})$ for the number of graphs $G \in \mathcal{G}_{\mathcal{A}}(\mathbf{d})$ that contain, or do *not* contain, the edge set E , respectively. We abbreviate $\mathcal{N}_E(\mathbf{d})$ to $\mathcal{N}_{ab}(\mathbf{d})$ if $E = \{ab\}$ (i.e. contains the single edge ab), and put $\mathcal{N}(\mathbf{d}) = |\mathcal{G}_{\mathcal{A}}(\mathbf{d})|$. (When \mathcal{N} and similar notation is used, the set \mathcal{A} should be clear by context.)

We pause for a notational comment. In this paper, a subscript ab is always interpreted as an ordered pair (a, b) rather than an edge (and similar for triples). This is irrelevant for $\mathcal{N}_{ab}(\mathbf{d}) = \mathcal{N}_{ba}(\mathbf{d})$ since the two ordered pairs signify the same edge, but the distinction is important with other notation.

Let

$$P_E(\mathbf{d}) = \frac{\mathcal{N}_E(\mathbf{d})}{\mathcal{N}(\mathbf{d})},$$

which is the probability that the edges in E are present in a graph G that is drawn uniformly at random from $\mathcal{G}_{\mathcal{A}}(\mathbf{d})$. Of particular interest are the probability of a single edge av and a path avb , for which we simplify the notation to

$$P_{av}(\mathbf{d}) = P_{\{av\}}(\mathbf{d}), \quad Y_{avb}(\mathbf{d}) = P_{\{av, bv\}}(\mathbf{d}),$$

where a, b and v are all distinct. We will use the following trick several times to switch between degree sequences of differing total degree.

Lemma 2.2. *Let $av \in \mathcal{A}$ and let \mathbf{d} be a sequence of length N . Then*

$$\begin{aligned} \mathcal{N}_{av}(\mathbf{d}) &= \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - \mathcal{N}_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) \\ &= \begin{cases} \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)(1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)) & \text{if } \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. Removing an edge av from a graph in $\mathcal{G}_{\mathcal{A}}(\mathbf{d})$ shows that the number of graphs with that edge is the same as the number of graphs in $\mathcal{G}_{\mathcal{A}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)$ and *no* edge between a and v . (The general form does not apply when $\mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) = 0$ only because $P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)$ is then technically undefined.) \blacksquare

For vertices $a, b \in V$, if \mathbf{d} is a sequence such that $\mathbf{d} - \mathbf{e}_b$ is \mathcal{A} -realisable, we define

$$R_{ab}(\mathbf{d}) = \frac{\mathcal{N}(\mathbf{d} - \mathbf{e}_a)}{\mathcal{N}(\mathbf{d} - \mathbf{e}_b)}.$$

With \mathcal{A} understood we write $\mu = \mu(\mathbf{d}) = \frac{1}{2} M_1 / |\mathcal{A}|$, which is the relative edge density of an \mathcal{A} -realisable graph with degree sequence \mathbf{d} . When \mathcal{A} is undefined, as in the introduction, then the default assumption is that $\mathcal{A} = \binom{[n]}{2}$ (so that this definition of μ extends the

one given in the introduction). Finally, for a vertex $a \in V$, we set $\mathcal{A}(a) = \{v \in V : av \in \mathcal{A}\}$, the “projection” of \mathcal{A} onto the edges incident with vertex a , and, with \mathbf{d} understood, we use $\mathcal{A}^*(a)$ for the set of $v \in \mathcal{A}(a)$ such that $\mathcal{N}_{av}(\mathbf{d}) > 0$.

We can bound the probability of an edge in a simple way using the following switching argument.

Lemma 2.3. *Let \mathbf{d} be a graphical sequence of length n with $\sum d_i = dn$ such that $\Delta = \Delta(\mathbf{d})$ satisfies $2\Delta(\Delta + 1) < dn$. Then for any a and v in $[n]$ we have*

$$P_{av}(\mathbf{d}) \leq \frac{\Delta^2}{dn(1 - \Delta(\Delta + 2)/dn)}.$$

Proof. For each graph G with degree sequence \mathbf{d} and an edge joining a and v , we can perform a switching (of the type used previously in graphical enumeration) by removing both av and another randomly chosen edge $a'v'$, and inserting the edges aa' and vv' , provided that no loops or multiple edges are so formed. The number of such switchings that can be applied to G with the vertices of each edge ordered is at least

$$dn - 2(\Delta + 1)\Delta$$

since there are dn ways to choose a' and v' as the ordered ends of any edge, whereas the number of such choices that are ineligible is at most the number of choices with a' being a or a neighbour of a (which automatically rules out $a' = v$), or similarly for v' . On the other hand, for each graph G' in which av is *not* an edge, the number of ways that it is created by performing such a switching backwards is at most Δ^2 . Counting the set of all possible switchings over all such graphs G and G' two different ways shows that the ratio of the number of graphs with av to the number without av is at most

$$\beta := \frac{\Delta^2}{dn - 2\Delta(\Delta + 1)}.$$

Hence $P_{av}(\mathbf{d}) \leq \beta/(1 + \beta)$, and the lemma follows. ■

We finish the section with some simple sufficient conditions for a sequence to be graphical. We need this since our degree switching arguments require that $\mathcal{N}(\mathbf{d}') > 0$ for several sequences \mathbf{d}' that are related to a *root* sequence \mathbf{d} .

Lemma 2.4. *For $\varepsilon > 0$, the following holds for n sufficiently large. Let $d_i \geq 0$ be integers for all $1 \leq i \leq n$, with $\sum_{1 \leq i \leq n} d_i$ even. Then there exists a graph with degrees d_1, \dots, d_n provided that either of the following hold:*

- (a) *There exists $0 < \mu < 1 - \varepsilon$ with $\max |\mu - d_i/n| < \varepsilon\mu$.*
- (b) *We have $1 \leq d_i \leq 2\sqrt{n} - 2$ for $1 \leq i \leq n$.*

Proof. Koren [24, Section 1] observed that the classical Erdős–Gallai conditions for the existence of a graph with degrees d_1, \dots, d_n are equivalent to the following: $\sum d_i$ is even,

and whenever $S \cap T = \emptyset \neq S \cup T \subseteq [n]$ we have

$$\sum_{i \in S} d_i - \sum_{j \in T} d_j \leq s(n-1-t),$$

where $s = |S|$ and $t = |T|$.

For (a), it suffices to have

$$s(n-1-t) - s\mu n(1+\varepsilon) + t\mu n(1-\varepsilon) \geq 0.$$

We only need to check the extreme values of t , i.e. $t = 0$ and $t = n - s$. In both cases, the function is easily seen to be non-negative for all appropriate s and n sufficiently large, noting that $\mu \leq 1 - \varepsilon$.

For (b), we only need $s\Delta - t \leq s(n-1-t)$ (recall that $1 \leq d_i \leq \Delta$). Again taking the extreme values of t , only the larger one gives a restriction, being $s(\Delta + 2 - s) \leq n$, which is satisfied because $\Delta \leq 2\sqrt{n} - 2$. ■

3. Recursive relations

In this section, we derive certain recursive formulae for the probability and ratio functions $P_{av}(\mathbf{d})$ and $R_{ab}(\mathbf{d})$. These identities will serve as a motivation for operators that we define in the next section. With a view to further applications of this work elsewhere, we consider an arbitrary set \mathcal{A} of allowable pairs.

Our first result expresses the edge probability P_{av} , the ratio R_{ab} , and the path probability Y_{ivj} in terms of each other.

Proposition 3.1. *Let \mathbf{d} be a sequence of length n and let $\mathcal{A} \subseteq \binom{[n]}{2}$.*

(a) *Let $a, v \in V$. If $\mathcal{N}_{av}(\mathbf{d}) > 0$ then*

$$P_{av}(\mathbf{d}) = d_v \left(\sum_{b \in \mathcal{A}^*(v)} R_{ba}(\mathbf{d} - \mathbf{e}_v) \frac{1 - P_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \right)^{-1}.$$

(b) *Let $a, b \in V$. If $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$ are \mathcal{A} -realisable then*

$$R_{ab}(\mathbf{d}) = \frac{d_a}{d_b} \cdot \frac{1 - B(a, b, \mathbf{d} - \mathbf{e}_b)}{1 - B(b, a, \mathbf{d} - \mathbf{e}_a)}, \quad (3.1)$$

where

$$B(i, j, \mathbf{d}') = \frac{1}{d_i} \left(\sum_{v \in \mathcal{A}(i) \setminus \mathcal{A}(j)} P_{iv}(\mathbf{d}') + \sum_{v \in \mathcal{A}(i) \cap \mathcal{A}(j)} Y_{ivj}(\mathbf{d}') \right) \quad (3.2)$$

provided that $B(b, a, \mathbf{d} - \mathbf{e}_a) \neq 1$.

(c) *Let a, v, b be distinct elements of V . If $\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v$ is \mathcal{A} -realisable then*

$$Y_{avb}(\mathbf{d}) = \frac{P_{av}(\mathbf{d})(P_{bv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - Y_{avb}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v))}{1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)}.$$

Remarks. (1) In (a), the summation over all b in $\mathcal{A}^*(v)$, instead of $\mathcal{A}(v)$, is merely for the technicality that $P_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)$ is otherwise undefined.

(2) The condition $\mathcal{N}_{av} > 0$ in part (a) does not reduce the practical usefulness of the proposition since the degree sequences where this condition fails for allowable edges av with $d_a, d_v > 0$ are pathological enough that our method fails on those for other reasons.

(3) For the range of the summations in $B(i, j, \mathbf{d}')$, when $\mathcal{A} = \binom{[n]}{2}$ (as in the applications in this paper) we have $\mathcal{A}(i) \setminus \mathcal{A}(j) = \{j\}$ and $\mathcal{A}(i) \cap \mathcal{A}(j) = [n] \setminus \{i, j\}$.

Proof of Proposition 3.1. To prove part (a) of the proposition, let \mathbf{d} be an \mathcal{A} -realisable sequence. Then every graph $G \in \mathcal{G}_{\mathcal{A}}(\mathbf{d})$ contributes exactly d_v to $\sum_{b \in \mathcal{A}^*(v)} \mathcal{N}_{bv}(\mathbf{d})$. Hence, since $\mathcal{N}_{av}(\mathbf{d}) > 0$ we may write

$$\begin{aligned} d_v &= \sum_{b \in \mathcal{A}^*(v)} \frac{\mathcal{N}_{bv}(\mathbf{d})}{\mathcal{N}(\mathbf{d})} = P_{av}(\mathbf{d}) \sum_{b \in \mathcal{A}^*(v)} \frac{\mathcal{N}_{bv}(\mathbf{d})}{\mathcal{N}_{av}(\mathbf{d})} \\ &= P_{av}(\mathbf{d}) \sum_{b \in \mathcal{A}^*(v)} \frac{\mathcal{N}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)(1 - P_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v))}{\mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)(1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v))}, \end{aligned}$$

by Observation 2.2, noting that $\mathcal{N}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v) \geq \mathcal{N}_{bv}(\mathbf{d}) > 0$ by definition of $\mathcal{A}^*(v)$. Part (a) follows since by definition

$$\frac{\mathcal{N}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{\mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} = R_{ba}(\mathbf{d} - \mathbf{e}_v),$$

and also noting that the summation is non-zero because $a \in \mathcal{A}^*(v)$.

To prove part (b), assume that $\mathbf{d} - \mathbf{e}_b$ is \mathcal{A} -realisable. First note that if $a = b$ then by definition $R_{ab}(\mathbf{d}) = 1$ and so the formula is correct. We can therefore assume henceforth that $a \neq b$. Let J_1 be the set of graphs in $\mathcal{G}_{\mathcal{A}}(\mathbf{d} - \mathbf{e}_b)$ with a distinguished edge incident to a , and J_2 the set of graphs in $\mathcal{G}_{\mathcal{A}}(\mathbf{d} - \mathbf{e}_a)$ with a distinguished edge incident to b . Then $|J_1| = d_a \mathcal{N}(\mathbf{d} - \mathbf{e}_b)$ and $|J_2| = d_b \mathcal{N}(\mathbf{d} - \mathbf{e}_a)$. Applying a *degree switching* to $G \in J_1$ consists in deleting the distinguished edge av and adding a new distinguished edge bv , to produce a graph $G' \in J_2$. The degree switching cannot be performed, i.e. is not valid, if $bv \notin \mathcal{A}$, which includes the case that $v = b$, or bv is an edge of G . Now let $G \in \mathcal{G}_{\mathcal{A}}(\mathbf{d} - \mathbf{e}_b)$ be picked uniformly at random and let v be a random neighbour of a in G . Let E be the event $E_{\mathcal{A}} \cup E_D$ where $E_{\mathcal{A}}$ is the event that $bv \notin \mathcal{A}$ and E_D is the event that bv is an edge of G , and define $B(a, b, \mathbf{d} - \mathbf{e}_b) = \mathbf{P}(E)$ (which we show further below to satisfy (3.2)). Then the number of valid degree switchings is

$$d_a \mathcal{N}(\mathbf{d} - \mathbf{e}_b)(1 - B(a, b, \mathbf{d} - \mathbf{e}_b)).$$

We may count the same switchings from the other direction, i.e. starting with an element of J_2 , using the same argument, and in this case $\mathcal{N}(\mathbf{d} - \mathbf{e}_a) = 0$ is permissible. Equating the two counts gives

$$\frac{\mathcal{N}(\mathbf{d} - \mathbf{e}_a)}{\mathcal{N}(\mathbf{d} - \mathbf{e}_b)} = \frac{d_a}{d_b} \cdot \frac{1 - B(a, b, \mathbf{d} - \mathbf{e}_b)}{1 - B(b, a, \mathbf{d} - \mathbf{e}_a)},$$

where the denominator is non-zero by the hypotheses of (b), noting that $d_b > 0$ because $\mathbf{d} - \mathbf{e}_b$ is \mathcal{A} -realisable. This gives (3.1).

The event E_D can only happen if the vertex v is a neighbour both of a and of b , and hence (as we still assume $a \neq b$)

$$\mathbf{P}(E_D) = \frac{1}{d_a} \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} Y_{avb}(\mathbf{d} - \mathbf{e}_b).$$

On the other hand, the vertex v is always a neighbour of a in a graph $G \in \mathcal{G}_{\mathcal{A}}(\mathbf{d})$, and thus

$$\mathbf{P}(E_{\mathcal{A}}) = \mathbf{P}(\{v \notin \mathcal{A}(b)\}) = \mathbf{P}(\{v \in \mathcal{A}(a) \setminus \mathcal{A}(b)\}) = \frac{1}{d_a} \sum_{v \in \mathcal{A}(a) \setminus \mathcal{A}(b)} P_{av}(\mathbf{d} - \mathbf{e}_b).$$

Noting that $E_{\mathcal{A}} \cap E_D = \emptyset$, so $\mathbf{P}(E) = \mathbf{P}(E_{\mathcal{A}}) + \mathbf{P}(E_D)$, we obtain the stated formula for $B(a, b, \mathbf{d} - \mathbf{e}_b)$. For $B(b, a, \mathbf{d} - \mathbf{e}_a)$ we can just swap a and b .

For (c), the assumptions imply that $\mathcal{N}_{av}(\mathbf{d}) > 0$, and hence by definition,

$$Y_{avb}(\mathbf{d}) = \frac{\mathcal{N}_{\{av,bv\}}(\mathbf{d})}{\mathcal{N}(\mathbf{d})} = P_{av}(\mathbf{d}) \cdot \frac{\mathcal{N}_{\{av,bv\}}(\mathbf{d})}{\mathcal{N}_{av}(\mathbf{d})}. \quad (3.3)$$

Now Lemma 2.2 says that

$$\begin{aligned} \mathcal{N}_{av}(\mathbf{d}) &= \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - \mathcal{N}_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) \\ &= \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)(1 - P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)) \end{aligned}$$

and similarly (cf. the proof of Lemma 2.2) we have

$$\begin{aligned} \mathcal{N}_{\{av,bv\}}(\mathbf{d}) &= \mathcal{N}_{bv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - \mathcal{N}_{\{av,bv\}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) \\ &= \mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)(P_{bv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - P_{abv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)). \end{aligned}$$

Part (c) follows upon substituting these expressions into (3.3). ■

4. The general method and the sparse case for graphs

In this section we present a simple application of the recursive relations found in Section 3. This is a completely new derivation of a known formula for the number of sparse graphs with a given degree sequence. We first give a template of the method, since it is used again in the other proofs in this paper and can be used elsewhere.

Template of the method

Step 1. Obtain an estimate of the ratio between the numbers of graphs of related degree sequences, using Proposition 3.1. This step is the crux of the whole argument.

Step 2. By making suitable definitions, we cause this ratio to appear as the expression $\mathbf{P}_{\mathcal{S}'}(u)/\mathbf{P}_{\mathcal{S}'}(v)$ for some probability space \mathcal{S}' on an underlying set Ω in an application of

Lemma 2.1. Thus, Ω is the set of degree sequences, with probabilities in \mathcal{S}' determined by the random graph under consideration, and the graph G in the lemma has a suitable vertex set \mathfrak{X} of such sequences. Each edge of G is in general a pair of degree sequences $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$ of the form occurring in the definition of $R_{ab}(\mathbf{d})$. Having defined G , we may call any two such degree sequences *adjacent*.

Step 3. Another probability space \mathcal{S} is defined on Ω , by taking a probability space \mathcal{B} directly from a joint binomial distribution, together with a function $\tilde{H}(\mathbf{d})$ that varies quite slowly, and defining probabilities in \mathcal{S} by the equation $\mathbf{P}_{\mathcal{S}}(\mathbf{d}) = \mathbf{P}_{\mathcal{B}}(\mathbf{d})\tilde{H}(\mathbf{d})/\mathbf{E}_{\mathcal{B}}\tilde{H}$.

Step 4. Using sharp concentration results, show that $P(\mathfrak{X}) \approx 1$ in both of the probability spaces \mathcal{S} and \mathcal{S}' (where, by \approx , we mean approximately equal to, with some specific error bound in each case). As part of this, we show that $\mathbf{E}_{\mathcal{B}}\tilde{H} \approx 1$. At this point, we may specify ε_0 for the application of Lemma 2.1.

Step 5. Apply Lemma 2.1 and the conclusions of the previous steps to deduce $P_{\mathcal{S}'}(\mathbf{d}) \approx P_{\mathcal{S}}(\mathbf{d}) \approx \mathbf{P}_{\mathcal{B}}(\mathbf{d})\tilde{H}(\mathbf{d})$. Upon estimating the errors in the approximations, which includes bounding the diameter of the graph G , we obtain an estimate for the probability $P_{\mathcal{S}'}(\mathbf{d})$ of the random graph having degree sequence \mathbf{d} in terms of a known quantity.

Recall that given a sequence \mathbf{d} we write $\Delta(\mathbf{d}) = \max_i d_i$, $M_1(\mathbf{d}) = \sum_i d_i$, $d = d(\mathbf{d}) = M_1/n$, and $\mu = \mu(\mathbf{d}) = d/(n-1)$. Note that in the following result the condition $\Delta(\mathbf{d})^6 + n^\varepsilon = o(nd^2)$ implies in particular that $m/\sqrt{n} \rightarrow \infty$. This restriction is imposed just for simplicity; the technique can still apply in the (less interesting) extremely sparse case. Also recall the probability spaces of random sequences, $\mathcal{G}(n, m)$ and $\mathcal{B}_m(n)$, from Section 1.

Theorem 4.1. *Let $\varepsilon > 0$, and let n and m be integers. Let \mathfrak{D} be any set of sequences of length n with $\sum_i d_i = 2m$ for all $\mathbf{d} \in \mathfrak{D}$ such that $\Delta(\mathbf{d})^6 + n^\varepsilon = o(nd^2)$ uniformly for all $\mathbf{d} \in \mathfrak{D}$. Then uniformly for $\mathbf{d}^* \in \mathfrak{D}$ we have*

$$\mathbf{P}_{\mathcal{G}(n,m)}(\mathbf{d}^*) = \mathbf{P}_{\mathcal{B}_m}(\mathbf{d}^*) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2}\right) \left(1 + O\left(\frac{\Delta(\mathbf{d}^*)^6 + n^\varepsilon}{nd^2} + n^{\varepsilon-1/2}\right)\right).$$

Proof. We can clearly assume that \mathfrak{D} is non-empty and we fix $\mathbf{d}^* \in \mathfrak{D}$. We first estimate ratios of probabilities for adjacent sequences that are typical in the binomial model \mathcal{B}_m , and sequences close to $\mathbf{d}^* \in \mathfrak{D}$ (as per Step 1 in the template above). We make the following definitions. Let

$$\Delta_1 = 2\Delta(\mathbf{d}^*) + n^{\varepsilon/6}$$

and define \mathfrak{D}^+ to be the set of all sequences $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ with $\Delta(\mathbf{d}) \leq \Delta_1$ and $M_1(\mathbf{d}) = 2m$. For an integer $r \geq 0$, denote by Q_r^0 (or Q_r^1) the set of all even (or odd, respectively) sequences in $\mathbb{Z}_{\geq 0}^n$ that have L_1 distance at most r from some sequence in \mathfrak{D}^+ . (Recall that we defined the parity of \mathbf{d} to be the parity of $M_1(\mathbf{d})$.) We will estimate the ratio of the probabilities of adjacent degree sequences in the random graph model using the following. Define R_{ab} as in Section 2 with $\mathcal{A} = \binom{V}{2}$.

Claim 4.2. *Uniformly for all sequences $\mathbf{d} \in Q_1^1$ and for all $a, b \in [n]$,*

$$R_{ab}(\mathbf{d}) = \frac{d_a}{d_b} \left(1 + \frac{(d_a - d_b)(M_1 + M_2)}{M_1^2} \right) \left(1 + O\left(\frac{\Delta_1^6}{d^3 n^2}\right) \right), \quad (4.1)$$

where $M_1 = M_1(\mathbf{d})$ and $M_2 = M_2(\mathbf{d}) = \sum_{v \in [n]} d_v(d_v - 1)$.

Proof. First, let $\mathbf{d} \in \mathfrak{D}^+$, and let n_1 be the number of non-zero coordinates in \mathbf{d} . By summing vertex degrees we find $dn = M_1(\mathbf{d}) \leq n_1 \Delta(\mathbf{d})$. By assumption we therefore have

$$\Delta(\mathbf{d}) \leq \Delta_1 \ll d^{1/3} n^{1/6} = \frac{(dn)^{1/3}}{n^{1/6}} \leq \frac{(\Delta(\mathbf{d})n_1)^{1/3}}{n^{1/6}}, \quad (4.2)$$

which readily implies that $\Delta(\mathbf{d}) = o(n_1^{1/2}/n^{1/4})$. Lemma 2.4(b) applied to the sequence formed by the non-zero coordinates of \mathbf{d} now implies that $\mathcal{N}(\mathbf{d}) > 0$ for n sufficiently large. We can deduce the same conclusion for all $\mathbf{d} \in Q_8^0$, since $\Delta(\mathbf{d})$ and $n_1(\mathbf{d})$ can only change by bounded factors when moving from such \mathbf{d} to the closest member of \mathfrak{D}^+ . Similarly, $M_1(\mathbf{d}) = \sum d_i = dn + O(1)$ for all $\mathbf{d} \in Q_8^0$. It is now clear by Lemma 2.3 and (4.2) that

$$P_{av}(\mathbf{d}) = O(\Delta_1^2/dn) = o(1) \quad \text{for all } \mathbf{d} \in Q_8^0 \text{ and all } a \neq v. \quad (4.3)$$

Next consider any distinct $a, v, b \in [n]$ and $\mathbf{d} \in Q_6^0$, with $d_a > 0$ and $d_v > 0$. Then $\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v \in Q_8^0$ and hence $\mathcal{N}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) > 0$ from above, and also $P_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) = O(\Delta_1^2/dn) < 1$ using (4.3). Thus, for n sufficiently large, $\mathcal{N}_{av}(\mathbf{d}) > 0$ by Lemma 2.2, and we have $\mathcal{N}_{av}(\mathbf{d}) < \mathcal{N}(\mathbf{d})$ since $P_{av}(\mathbf{d}) < 1$ for similar reasons. This establishes the hypotheses for $\mathcal{N}_{av}(\mathbf{d})$ and $\mathcal{N}(\mathbf{d})$ in Proposition 3.1 whenever they are needed below.

It now follows that $Y_{avb}(\mathbf{d}) = O(\Delta_1^4/d^2 n^2)$ for $\mathbf{d} \in Q_6^0$; if d_a or d_v is 0 then this is immediate, and otherwise it follows from Proposition 3.1(c) in view of (4.3), and noting that the numerator is non-negative by definition. Next, definition (3.2) yields $B(a, b, \mathbf{d}) = O(\Delta_1^4/d^2 n)$ for all distinct $a, b \in [n]$ and all $\mathbf{d} \in Q_6^0$ with $d_a > 0$. (In the current setting $\mathcal{A}(i) \setminus \mathcal{A}(j) = \{j\}$ when $i \neq j$, and $d \leq \Delta_1$.) Consequently, (3.1) gives $R_{ab}(\mathbf{d}) = (d_a/d_b)(1 + O(\Delta_1^4/d^2 n))$ for all $\mathbf{d} \in Q_5^1$ and all distinct a, b such that $d_a, d_b > 0$. If now $\mathbf{d} \in Q_4^0$ and $d_v > 0$, we have $\mathcal{N}_{bv}(\mathbf{d}) > 0$ as noted above. Therefore $\sum_{b \in \mathcal{A}^*(v)} d_b = M_1(\mathbf{d}) - d_v$ for such \mathbf{d} . Thus, Proposition 3.1(a) gives

$$P_{av}(\mathbf{d}) = \frac{d_a d_v}{M_1(\mathbf{d})} \left(1 + O\left(\frac{\Delta_1^4}{d^2 n}\right) \right) \quad (4.4)$$

for all $\mathbf{d} \in Q_4^0$ and $a \neq v$. (If d_a and d_v are both non-zero, the proposition applies as mentioned above, and if either is 0, the claim holds trivially.) Using a similar argument, Proposition 3.1(c) gives

$$Y_{avb}(\mathbf{d}) = \frac{d_a [d_v]_2 d_b}{M_1(\mathbf{d})^2} \left(1 + O\left(\frac{\Delta_1^4}{d^2 n}\right) \right)$$

for all $\mathbf{d} \in Q_2^0$ and all distinct $a, v, b \in [n]$. Applying these results to the definition (3.2) of B for $\mathbf{d} \in Q_2^0$ and distinct $a, b \in [n]$, and recalling that $d \leq \Delta(\mathbf{d}) + 2$, now gives the sharper estimate

$$B(a, b, \mathbf{d}) = \left(\frac{d_b}{M_1(\mathbf{d})} + \frac{d_b M_2(\mathbf{d})}{M_1(\mathbf{d})^2} \right) \left(1 + O\left(\frac{\Delta_1^4}{d^2 n} \right) \right),$$

which we note is $O(\Delta_1^2/dn)$ as $M_2(\mathbf{d}) \leq M_1(\mathbf{d})\Delta_1$. Thus, for all $\mathbf{d} \in Q_1^1$ and all a, b , and noting that M_1 and M_2 change by a negligible additive term $O(\Delta_1)$ under bounded perturbations of the elements of the sequence \mathbf{d} ,

$$R_{ab}(\mathbf{d}) = \frac{d_a}{d_b} \cdot \frac{1 - (d_b - 1)/M_1 - (d_b - 1)M_2/M_1^2}{1 - (d_a - 1)/M_1 - (d_a - 1)M_2/M_1^2} \left(1 + O\left(\frac{\Delta_1^6}{d^3 n^2} \right) \right), \quad (4.5)$$

which implies the claim. \blacksquare

We next make the definitions of probability spaces necessary to apply Lemma 2.1 (see Steps 2 and 3 in the template). Let Ω be the underlying set of $\mathcal{B}_m(n)$, $\mathfrak{X} = \mathfrak{D}^+$ and $\mathcal{S}' = \mathcal{D}(\mathcal{G}(n, m))$. Let $H(\mathbf{d}) = \mathbf{P}_{\mathcal{B}_m}(\mathbf{d})\tilde{H}(\mathbf{d})$, where

$$\tilde{H}(\mathbf{d}) = \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2} \right),$$

and define the probability function in \mathcal{S} by

$$\mathbf{P}_{\mathcal{S}}(\mathbf{d}) = H(\mathbf{d}) / \sum_{\mathbf{d}' \in \Omega} H(\mathbf{d}') = \frac{H(\mathbf{d})}{\mathbf{E}_{\mathcal{B}_m} \tilde{H}}. \quad (4.6)$$

Let G be the graph with vertex set \mathfrak{X} and with two vertices (sequences) adjacent if they are of the form $\mathbf{d} - \mathbf{e}_a, \mathbf{d} - \mathbf{e}_b$ for some $a, b \in [n]$ and odd \mathbf{d} .

We need to estimate the probability of \mathfrak{X} in the two probability spaces (see Step 4 in the template). In $\mathcal{G}(n, m)$ each vertex degree is distributed hypergeometrically with expected value $d = 2m/n$. Also note that, letting $\Delta^* = \Delta(\mathbf{d}^*)$, we have by definition $\Delta_1 \geq \Delta^* + n^{\varepsilon/12} \sqrt{\Delta^*} \geq d + n^{\varepsilon/12} \sqrt{\Delta^*}$. Thus, for $\mathbf{d} \in \Omega$,

$$\mathbf{P}_{\mathcal{S}'}(d_i > \Delta_1) \leq \mathbf{P}_{\mathcal{S}'}(d_i > d + n^{\varepsilon/12} \sqrt{\Delta^*}) = o(n^{-\omega})$$

by [21, Theorems 2.10 and 2.1] for example (and noting $\Delta^* \rightarrow \infty$). The union bound, applied to each i , now gives $\mathbf{P}_{\mathcal{S}'}(\mathfrak{X}) = 1 - o(n^{-\omega})$.

For similar reasons $\mathbf{P}_{\mathcal{B}_m}(\mathfrak{X}) = 1 - o(n^{-\omega})$. To deal with the exponential factor $\tilde{H}(\mathbf{d})$, we claim that if \mathbf{d} is chosen according to $\mathcal{B}_m(n)$ then $\gamma_2(\mathbf{d}) = \mu(1-\mu)(1 + O(\xi))$ with probability $1 - o(n^{-\omega})$, where $\xi = O(n^{\varepsilon-1/2})$. Indeed, this follows from the forthcoming Lemma 6.2 (ii) in which we may take $\alpha = (1 + 1/\sqrt{d})(\log n)^2/\sqrt{n}$ and note that $(\log n)^3 = o(dn)$ is implied by $m/\sqrt{n} \rightarrow \infty$. Thus, for such $\mathbf{d} \in \mathcal{B}_m(n)$, the exponential factor $\tilde{H}(\mathbf{d})$ is $1 + O(\xi)$ with probability $1 - o(n^{-\omega})$, and it is always at most $e^{1/4}$. We deduce that $\mathbf{E}_{\mathcal{B}_m} \tilde{H} = 1 + O(\xi)$, and additionally $\mathbf{P}_{\mathcal{S}}(\mathfrak{X}) = 1 - o(n^{-\omega})$. Thus, we may

set $\varepsilon_0 = O(1/n)$ in Lemma 2.1 (with apologies to the function $n^{-\omega}$, ending its life in this proof dominated by $1/n$).

To apply Lemma 2.1 (see Step 5 in the template above), the final condition we need to show is that the ratios of probabilities satisfy

$$\frac{\mathbf{P}_{S'}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{S'}(\mathbf{d} - \mathbf{e}_b)} = e^{O(\delta)} \frac{\mathbf{P}_S(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_S(\mathbf{d} - \mathbf{e}_b)} \quad (4.7)$$

whenever $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$ are elements of \mathfrak{D}^+ , for a particular $\delta = \delta(\Delta_1)$ independent of \mathbf{d} and specified below, where the constant implicit in $O()$ is independent of \mathbf{d} and \mathbf{d}^* .

To evaluate the right hand side of (4.7) we observe that

$$\frac{\mathbf{P}_S(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_S(\mathbf{d} - \mathbf{e}_b)} = \frac{H(\mathbf{d} - \mathbf{e}_a)}{H(\mathbf{d} - \mathbf{e}_b)} = \frac{d_a(n - d_b)}{d_b(n - d_a)} \exp\left(\frac{(d_a - d_b)\gamma_2}{d^2(1 - \mu')^2} + O\left(\frac{\Delta^2}{(dn)^2}\right)\right) \quad (4.8)$$

for all $\mathbf{d} \in Q_1^1$ by (2.1), where d , γ_2 and Δ are defined with respect to \mathbf{d} , and $\mu' = \mu(\mathbf{d} - \mathbf{e}_a)$. Note that $\mu' = d(1 + O(1/nd))/(n - 1)$ and $d \leq \Delta + 1$, since $\mathbf{d} \in Q_1^1$. Thus, we also have $\gamma_2 = (M_2 + M_1 - dM_1)/(n - 1)^2 = O(d\Delta/n)$. Hence, the argument of the exponential factor in (4.8) is

$$\frac{(d_a - d_b)\gamma_2}{d^2} + O(\Delta^2/n^2) = \frac{(d_a - d_b)(M_2 + M_1 - dM_1)}{M_1^2} + O(\Delta^2/n^2).$$

Combining this with (4.8) and Claim 4.2 it follows that for all $\mathbf{d} \in Q_1^1$,

$$R_{ab}(\mathbf{d}) = \frac{H(\mathbf{d} - \mathbf{e}_a)}{H(\mathbf{d} - \mathbf{e}_b)} \left(1 + O\left(\frac{\Delta_1^6}{d^3 n^2}\right)\right), \quad (4.9)$$

where we use $(n - d_b)/(n - d_a) = \exp((d_a - d_b)/n + O(\Delta^2/n^2))$, $M_2 = O(\Delta M_1)$ and $\Delta^4/M_1^2 \leq \Delta^6/d^3 n^2$, $\Delta \leq \Delta_1$, and the most significant error term derives from Claim 4.2.

Equation (4.9) now implies that

$$\frac{\mathbf{P}_{S'}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{S'}(\mathbf{d} - \mathbf{e}_b)} = R_{ab}(\mathbf{d}) = e^{O(\delta)} \frac{\mathbf{P}_S(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_S(\mathbf{d} - \mathbf{e}_b)}$$

whenever $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$ are elements of $\mathfrak{X} = \mathfrak{D}^+$, where we may take $\delta = \Delta_1^6/d^3 n^2$.

It is clear that the diameter of G is at most $r := m = nd/2$. Lemma 2.1 then implies that $P_{S'}(\mathbf{d}) = e^{O(r\delta + \varepsilon_0)} P_S(\mathbf{d})$ for $\mathbf{d} \in \mathfrak{D}^+$. To proceed from here, since we found that $\mathbf{E}_{\mathcal{B}_m} \bar{H} = 1 + O(\xi)$, equation (4.6) implies $P_S(\mathbf{d}) = H(\mathbf{d})(1 + O(\xi))$ for $\mathbf{d} \in \mathfrak{D}^+$. Hence,

$$P_{S'}(\mathbf{d}^*) = e^{O(r\delta + \varepsilon_0 + \xi)} H(\mathbf{d}^*). \quad (4.10)$$

Note that $\xi = O(n^{\varepsilon-1/2})$, and $r\delta + \varepsilon_0 = O(\Delta_1^6/d^2 n) = O((\Delta(\mathbf{d}^*)^6 + n^\varepsilon)/d^2 n)$. The theorem follows since $S' = \mathcal{D}(\mathcal{G}(n, m))$. \blacksquare

The result in Conjecture 1.2 or (1.2) follows from this in the sparse case, with different error terms, as long as d is appreciably above $1/\sqrt{n}$. For smaller d , the analysis could be adjusted to obtain results, but the random graph is quite uninteresting here, typically

having most vertices of degree 0, and the rest of degree 1 except for perhaps a few vertices of degree 2.

We note that the above result applies in the case of d -regular graphs only for $d = o(n^{1/4})$, far short of $o(\sqrt{n})$ as reached in [31]. It is also quite straightforward to reach past \sqrt{n} using our method, by carrying the calculations a little further, iterating several more the recursive equations that are only used twice in the proof above. In fact, this is how we first obtained the formulae for P and R in later sections. Having derived those “limiting” formulae, our proofs can completely avoid considering the iterated versions of the formulae, as shown in the next section.

5. Function operators and fixed points

In the previous section we used two iterations of the recursive equations from Proposition 3.1, each time applying them to all degree sequences at a certain distance from a root sequence \mathbf{d} . This allowed us to determine the ratio $\mathcal{N}(\mathbf{d} - \mathbf{e}_a)/\mathcal{N}(\mathbf{d} - \mathbf{e}_b)$ up to negligible error terms. For denser graphs we would need an unbounded number of iterations to obtain the desired precision of about $O(1/n\sqrt{d})$ since the improvement is $O(\Delta/n)$ each time. Instead of doing this, we define operators based on the recursive identities from Proposition 3.1 and study their behaviour on input functions that are close to the desired functions.

Let $\mathbb{Z}_{\geq 0}^n$ denote the set of non-negative integer sequences of length n . For a given integer n and a set $\mathcal{A} \subseteq \binom{[n]}{2}$ we define $\vec{\mathcal{A}}$ to be the set of ordered pairs (u, v) with $\{u, v\} \in \mathcal{A}$. Ordered pairs are needed here because, although the functions of interest are symmetric in the sense that the probability of an edge uv is the same as vu , our approximations to the probability do not obey this symmetry. Similarly, let $\vec{\mathcal{A}}_2$ denote the set of ordered triples (u, v, w) with u, v and w all distinct and $\{u, v\}, \{v, w\} \in \mathcal{A}$.

Suppose that we are given $\mathbf{p} : \vec{\mathcal{A}} \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$, $\mathbf{y} : \vec{\mathcal{A}}_2 \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\mathbf{r} : [n]^2 \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$. We write $\mathbf{p}_{av}(\mathbf{d})$ for $\mathbf{p}(a, v, \mathbf{d})$ (where $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$), and remind the reader that in this paper, a subscript av always denotes an ordered pair rather than an edge. Similarly, we write $\mathbf{y}_{avb}(\mathbf{d})$ for $\mathbf{y}(a, v, b, \mathbf{d})$ and $\mathbf{r}_{ab}(\mathbf{d})$ for $\mathbf{r}(a, b, \mathbf{d})$. We also define an associated function $\mathbf{bad}(\mathbf{p}, \mathbf{y})$ as follows. For $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ and $a, b \in [n]$ with $a \neq b$, set $\mathbf{bad}(\mathbf{p}, \mathbf{y})(a, a, \mathbf{d}) = 0$ and

$$\mathbf{bad}(\mathbf{p}, \mathbf{y})(a, b, \mathbf{d}) = \frac{1}{d_a} \left(\sum_{v \in \mathcal{A}(a) \setminus \mathcal{A}(b)} \mathbf{p}_{av}(\mathbf{d}) + \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} \mathbf{y}_{avb}(\mathbf{d}) \right). \quad (5.1)$$

We define operators $\mathcal{P}(\mathbf{p}, \mathbf{r})$, $\mathcal{Y}(\mathbf{p}, \mathbf{y})$ and $\mathcal{R}(\mathbf{p}, \mathbf{y})$, acting on \mathbf{p} , \mathbf{y} and \mathbf{r} as above, as follows. For $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ and $a, v, b \in [n]$, we set

$$\mathcal{P}(\mathbf{p}, \mathbf{r})(a, v, \mathbf{d}) = d_v \left(\sum_{b \in \mathcal{A}(v)} \mathbf{r}_{ba}(\mathbf{d} - \mathbf{e}_v) \frac{1 - \mathbf{p}_{bv}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{1 - \mathbf{p}_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \right)^{-1} \quad \text{for } (a, v) \in \vec{\mathcal{A}}, \quad (5.2)$$

$$\mathcal{Y}(\mathbf{p}, \mathbf{y})(a, v, b, \mathbf{d}) = \frac{\mathbf{p}_{av}(\mathbf{d}) \cdot (\mathbf{p}_{bv}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - \mathbf{y}_{avb}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v))}{1 - \mathbf{p}_{av}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \quad \text{for } (a, v, b) \in \vec{\mathcal{A}}_2, \quad (5.3)$$

$$\mathcal{R}(\mathbf{p}, \mathbf{y})(a, b, \mathbf{d}) = \frac{d_a}{d_b} \cdot \frac{1 - \mathbf{bad}(\mathbf{p}, \mathbf{y})(a, b, \mathbf{d} - \mathbf{e}_b)}{1 - \mathbf{bad}(\mathbf{p}, \mathbf{y})(b, a, \mathbf{d} - \mathbf{e}_a)}. \quad (5.4)$$

Proposition 3.1 says that in a certain sense, the probability and ratio functions P_{av} , Y_{avb} and R_{ab} are fixed points of the operators \mathcal{P} , \mathcal{Y} and \mathcal{R} . It is very useful for us that these operators are “contractive”, in a certain sense, in a neighbourhood of this fixed point. Unfortunately, the concept of contraction which we have here uses a slightly different metric before and after applying the operators, stemming from the fact that the value of $\mathcal{R}(\mathbf{p}, \mathbf{y})$ at a point (a, b, \mathbf{d}) depends on the values of \mathbf{p} and \mathbf{y} at the points (c, w, \mathbf{d}') and (c, w, h, \mathbf{d}') for several \mathbf{d}' in a neighbourhood of \mathbf{d} . This makes it difficult to define a true and useful contraction mapping. Nevertheless, we can exploit the useful features of the situation using the following lemma.

Definition 5.1. Let $\mathcal{D}_0 \subseteq \mathbb{Z}_{\geq 0}^n$ and let $\mu \in \mathbb{R}$. We use $\Pi_\mu(\mathcal{D}_0)$ to denote the set of pairs of functions (\mathbf{p}, \mathbf{y}) with $\mathbf{p} : \mathcal{A} \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\mathbf{y} : \vec{\mathcal{A}}_2 \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}$ such that for all even $\mathbf{d} \in \mathcal{D}_0$, we have

- (Pa) $0 \leq \mathbf{p}_{av}(\mathbf{d}) \leq \mu$ for all $(a, v) \in \vec{\mathcal{A}}$,
- (Pb) $\sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} \mathbf{y}_{avb}(\mathbf{d}) \leq \mu d_a$ for all $a \neq b \in [n]$,
- (Pc) $0 \leq \mathbf{y}_{avb}(\mathbf{d}) \leq \mu \mathbf{p}_{bv}(\mathbf{d})$ for all $(a, v, b) \in \vec{\mathcal{A}}_2$.

We denote by $Q_s^0(\mathbf{d})$, $Q_s^1(\mathbf{d}) \subseteq \mathbb{Z}^n$ the set of even and odd, respectively, vectors of arbitrary integers that have L_1 distance at most s from \mathbf{d} . Recall that we use $1 \pm \xi$ to denote a quantity in the interval $[1 - \xi, 1 + \xi]$.

Lemma 5.2. *There is a constant $C > 0$ such that the following holds. Let n be an integer and $\mathcal{A} \subseteq \binom{[n]}{2}$. Let $\mathbf{d} = \mathbf{d}(n) \in \mathbb{Z}_{\geq 0}^n$ satisfy $|\mathcal{A}(a) \setminus \mathcal{A}(b)| < d_a$ whenever $\mathcal{A}(a) \cap \mathcal{A}(b) \neq \emptyset$. Let $0 < \xi \leq 1$ and $0 < \mu_0 = \mu_0(n) < C$. Let $(\mathbf{p}, \mathbf{y}), (\mathbf{p}', \mathbf{y}') \in \Pi_{\mu_0}(Q_2^0(\mathbf{d}))$, and let $\mathbf{r}, \mathbf{r}' : [n]^2 \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}$. Let $a, v, b \in [n]$.*

- (a) *If \mathbf{d} is odd, and $\mathcal{A}(a) \cap \mathcal{A}(b) \neq \emptyset$, $\mathbf{p}_{cw}(\mathbf{d}') = \mathbf{p}'_{cw}(\mathbf{d}')(1 \pm \xi)$ for all $(c, w) \in \vec{\mathcal{A}}$ and all $\mathbf{d}' \in Q_1^0(\mathbf{d})$, and $\mathbf{y}_{cwh}(\mathbf{d}') = \mathbf{y}'_{cwh}(\mathbf{d}')(1 \pm \xi)$ for all $(c, w, h) \in \vec{\mathcal{A}}_2$ and all $\mathbf{d}' \in Q_1^0(\mathbf{d})$, then*

$$\mathcal{R}(\mathbf{p}, \mathbf{y})_{ab}(\mathbf{d}) = \mathcal{R}(\mathbf{p}', \mathbf{y}')_{ab}(\mathbf{d})(1 + O(\mu_0 \xi)).$$

- (b) *If \mathbf{d} is even, and $(a, v) \in \vec{\mathcal{A}}$, $\mathbf{p}_{cv}(\mathbf{d}') = \mathbf{p}'_{cv}(\mathbf{d}')(1 \pm \xi)$ for all $c \in \mathcal{A}(v)$ and all $\mathbf{d}' \in Q_2^0(\mathbf{d})$, and $\mathbf{r}_{ca}(\mathbf{d}') = \mathbf{r}'_{ca}(\mathbf{d}')(1 \pm \mu_0 \xi)$ for all $c \in \mathcal{A}(v)$ and all $\mathbf{d}' \in Q_1^1(\mathbf{d})$, then*

$$\mathcal{P}(\mathbf{p}, \mathbf{r})_{av}(\mathbf{d}) = \mathcal{P}(\mathbf{p}', \mathbf{r}')_{av}(\mathbf{d})(1 + O(\mu_0 \xi)).$$

- (c) *If \mathbf{d} is even, and $(a, v, b) \in \vec{\mathcal{A}}_2$, $\mathbf{p}_{cv}(\mathbf{d}') = \mathbf{p}'_{cv}(\mathbf{d}')(1 \pm \mu_0 \xi)$ for all $c \in \mathcal{A}(v)$ and all $\mathbf{d}' \in Q_2^0(\mathbf{d})$, and $\mathbf{y}_{cwh}(\mathbf{d}') = \mathbf{y}'_{cwh}(\mathbf{d}')(1 \pm \xi)$ for all $(c, w, h) \in \vec{\mathcal{A}}_2$ and all*

$\mathbf{d}' \in Q_2^0(\mathbf{d})$, then

$$\mathcal{Y}(\mathbf{p}, \mathbf{y})_{avb}(\mathbf{d}) = \mathcal{Y}(\mathbf{p}', \mathbf{y}')_{avb}(\mathbf{d})(1 + O(\mu_0\xi)).$$

The constants implicit in $O(\cdot)$ are absolute.

Lemma 5.2 indicates that applying the operators once (one after the other) to two functions that are close in terms of relative error yields functions that are closer by a factor of μ_0 , where in our applications of the lemma, μ_0 will be a (very) loose upper bound on the density of the graphs of interest. We define suitable distance functions below to make this idea precise. We also remark that in (a) we restrict to $\mathcal{A}(a) \cap \mathcal{A}(b) \neq \emptyset$ since otherwise no vertex v satisfies $avb \in \vec{\mathcal{A}}_2$. This will be useful for an application to bipartite graphs in a later paper. Note that when $\mathcal{A} = \binom{[n]}{2}$ we have $\mathcal{A}(a) \cap \mathcal{A}(b) \neq \emptyset$ for all $a, b \in [n]$.

Proof of Lemma 5.2. For (a), since $\mathbf{p}_{cw}(\mathbf{d}') = \mathbf{p}'_{cw}(\mathbf{d}')(1 \pm \xi)$ and $\mathbf{y}_{cwh}(\mathbf{d}') = \mathbf{y}'_{cwh}(\mathbf{d}')(1 \pm \xi)$, from (5.1) and the non-negativity of the functions in Π_{μ_0} we obtain

$$\mathbf{bad}(\mathbf{p}, \mathbf{y})(a, b, \mathbf{d}') = \mathbf{bad}(\mathbf{p}', \mathbf{y}')(a, b, \mathbf{d}')(1 + O(\xi))$$

for all $\mathbf{d}' \in Q_1^0(\mathbf{d})$. Additionally, for such \mathbf{d}' , the assumption that $(\mathbf{p}', \mathbf{y}') \in \Pi_{\mu_0}(Q_2^0(\mathbf{d}))$, together with the assumption that $|\mathcal{A}(a) \setminus \mathcal{A}(b)| < d_a$ for $\mathcal{A}(a) \cap \mathcal{A}(b) \neq \emptyset$, implies that $\mathbf{bad}(\mathbf{p}', \mathbf{y}')(a, b, \mathbf{d}') = O(\mu_0)$. Thus $\mathbf{bad}(\mathbf{p}, \mathbf{y})(a, b, \mathbf{d}') = \mathbf{bad}(\mathbf{p}', \mathbf{y}')(a, b, \mathbf{d}') + O(\xi\mu_0)$ for all $\mathbf{d}' \in Q_1^0(\mathbf{d})$. Hence, with μ_0 sufficiently small to ensure $\mathbf{bad}(\mathbf{p}, \mathbf{y})(b, a, \mathbf{d} - \mathbf{e}_a) < 1/2$ say, part (a) follows from (5.4).

The equation for $\mathcal{P}(\mathbf{p}, \mathbf{r})$ in (b) follows similarly from (5.2) since both \mathbf{p}_{bv} and \mathbf{p}_{av} are bounded by μ_0 , so in particular the denominator in (5.2) is bounded away from 0. Finally, (c) follows easily via (5.3) and noting the bound of $\mu_0\mathbf{p}_{bv}(\mathbf{d}')$ on $\mathbf{y}_{avb}(\mathbf{d}')$. ■

Fix $\Omega^{(0)} \subseteq \mathbb{Z}_{\geq 0}^n$ for the following definitions. Let $\Omega^{(s)}$ denote the set of all $\mathbf{d} \in \Omega^{(0)}$ for which $Q_s^0(\mathbf{d}), Q_s^1(\mathbf{d}) \subseteq \Omega^{(0)}$. We will make use of these sets as restricted domains for functions of \mathbf{d} that refer to slightly altered sequences \mathbf{d}' . Define a set of distance functions, indexed by s , on the set of all (\mathbf{p}, \mathbf{y}) for which $\mathbf{p} : \vec{\mathcal{A}} \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$ and $\mathbf{y} : \vec{\mathcal{A}}_2 \times \mathbb{Z}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}$ by

$$\chi^{(s)}((\mathbf{p}, \mathbf{y}), (\mathbf{p}', \mathbf{y}')) = \max \{ \chi_1^{(s)}(\mathbf{p}, \mathbf{p}'), \chi_2^{(s)}(\mathbf{y}, \mathbf{y}') \}, \quad \text{where} \quad (5.5)$$

$$\chi_1^{(s)}(\mathbf{p}, \mathbf{p}') = \sup \{ |\log(\mathbf{p}_{cw}(\mathbf{d})/\mathbf{p}'_{cw}(\mathbf{d}))| : (c, w) \in \vec{\mathcal{A}}, \mathbf{d} \in \Omega^{(s)} \}, \quad (5.6)$$

$$\chi_2^{(s)}(\mathbf{y}, \mathbf{y}') = \sup \{ |\log(\mathbf{y}_{cwh}(\mathbf{d})/\mathbf{y}'_{cwh}(\mathbf{d}))| : (c, w, h) \in \vec{\mathcal{A}}_2, \mathbf{d} \in \Omega^{(s)} \}. \quad (5.7)$$

If any denominator is 0 we define $\chi_i^{(s)}$, $i = 1, 2$, to be ∞ at that point, so these are extended metrics. Clearly $\chi^{(s)}$ is non-increasing in s unless $\Omega^{(s)}$ is empty, in which case we set $\chi^{(s)} = 0$. Also, define the compositional operator

$$\mathcal{C}(\mathbf{p}, \mathbf{y}) = (\hat{\mathbf{p}}, \mathcal{Y}(\hat{\mathbf{p}}, \mathbf{y}))$$

where $\hat{\mathbf{p}} = \mathcal{P}(\mathbf{p}, \mathcal{R}(\mathbf{p}, \mathbf{y}))$.

Lemma 5.3. *There is a constant $C > 0$ such that the following holds. Let n and \mathcal{A} be as in Lemma 5.2, and let $\Omega^{(0)} \subseteq \mathbb{Z}_{\geq 0}^n$. Let $\xi \leq 1/2$ and $0 < \mu_0 = \mu_0(n) < C$. Assume that $s \geq 0$ is such that $(\mathbf{p}, \mathbf{y}), (\mathbf{p}', \mathbf{y}') \in \Pi_{\mu_0}(\Omega^{(s)})$, and $\chi^{(s)}((\mathbf{p}, \mathbf{y}), (\mathbf{p}', \mathbf{y}')) \leq \xi$. Then $\chi^{(s+4)}(\mathcal{C}(\mathbf{p}, \mathbf{y}), \mathcal{C}(\mathbf{p}', \mathbf{y}')) = O(\mu_0 \xi)$.*

Proof. Let $\mathbf{d} \in \Omega^{(s+1)}$ be odd. Then $Q_2^0(\mathbf{d}) = Q_1^0(\mathbf{d}) \subseteq \Omega^{(s)}$, and so $(\mathbf{p}, \mathbf{y}), (\mathbf{p}', \mathbf{y}') \in \Pi_{\mu_0}(Q_2^0(\mathbf{d}))$, as required for the lemma. Since $\chi^{(s)}((\mathbf{p}, \mathbf{y}), (\mathbf{p}', \mathbf{y}')) \leq \xi \leq 1/2$, we have $\mathbf{p}_{cw}(\mathbf{d}') = \mathbf{p}'_{cw}(\mathbf{d}')(1 \pm 2\xi)$ for all $(c, w) \in \vec{\mathcal{A}}$ and $\mathbf{d}' \in Q_1^0(\mathbf{d})$, and a similar statement holds for \mathbf{y}, \mathbf{y}' . We can thus apply Lemma 5.2 (a), and defining $\mathbf{r} = \mathcal{R}(\mathbf{p}, \mathbf{y})$ and $\mathbf{r}' = \mathcal{R}(\mathbf{p}', \mathbf{y}')$ deduce that $\mathbf{r}_{ab}(\mathbf{d}) = \mathbf{r}'_{ab}(\mathbf{d})(1 + O(\mu_0 \xi))$ for all such \mathbf{d} and for a, b as in that lemma. Preparing for the next step, note that the error term $O(\mu_0 \xi)$ is $\pm \xi$ for C sufficiently small.

Now let $\mathbf{d} \in \Omega^{(s+2)}$ be even and define $\hat{\mathbf{p}} = \mathcal{P}(\mathbf{p}, \mathbf{r})$ and $\hat{\mathbf{p}}' = \mathcal{P}(\mathbf{p}', \mathbf{r}')$. Applying Lemma 5.2 (b) in a similar way gives

$$\hat{\mathbf{p}}_{cw}(\mathbf{d}) = \hat{\mathbf{p}}'_{cw}(\mathbf{d})(1 + O(\mu_0 \xi))$$

for $(c, w) \in \vec{\mathcal{A}}$, and again we may assume the error term is $\pm \xi$.

Finally, we may repeat the process with $\mathbf{d} \in \Omega^{(s+4)}$, and use Lemma 5.2 (c) to deduce that

$$\mathcal{Y}(\hat{\mathbf{p}}, \mathbf{y})_{avb}(\mathbf{d}) = \mathcal{Y}(\hat{\mathbf{p}}', \mathbf{r}')_{avb}(\mathbf{d})(1 + O(\mu_0 \xi))$$

for all $(a, v, b) \in \vec{\mathcal{A}}_2$ for such \mathbf{d} . Since the error terms $\mu_0 \xi < C/2$ in the last two conclusions can be made arbitrarily small by taking C small, we have $\log(1 + O(\mu_0 \xi)) = O(\mu_0 \xi)$ as required to deduce the lemma. \blacksquare

6. Proof of the binomial model in the graph case

In this section we prove Conjecture 1.3 for p in the ‘‘gap range’’ which we can describe as $o(n^{-1/2}) < p < c/\log n$. Before doing so we need concentration results for some functions $f(\mathbf{d})$ when \mathbf{d} either has independent binomial entries, or is the degree sequence of $\mathcal{G}(n, m)$. In [32, Theorem 3.4] it was essentially shown that, when \mathbf{d} has independent binomial entries, $\sigma^2 = \sigma^2(\mathbf{d}) = \sum_{i=1}^n (d_i - d)^2/n$ is concentrated. We give a more efficient proof of the crucial part of this, using the following result, which we will also apply to the degree sequence of $\mathcal{G}(n, m)$. This is a direct corollary of Theorem (7.4) and Example (7.3) of McDiarmid [27]. However, since the constants there are not explicit and the framework makes the proof not so easily accessible, we give a proof here.

Lemma 6.1 (McDiarmid). *Let $c > 0$ and let f be a function defined on the set of subsets of some set U such that $|f(S) - f(T)| \leq c$ whenever $|S| = |T| = m$ and $|S \cap T| = m - 1$. Let S be a randomly chosen m -subset of U . Then for all $\alpha > 0$ we have*

$$\mathbf{P}(|f(S) - \mathbf{E}f(S)| \geq \alpha c \sqrt{m}) \leq 2 \exp(-2\alpha^2).$$

Proof. Consider a process in which the random subset S is generated by inserting m distinct elements one after another, each randomly chosen from the remaining available ones. Let S_k denote the k th subset formed in this process, $0 \leq k \leq m$. Consider the Doob martingale process determined by $Y_k = \mathbf{E}(f(S_m) \mid S_0, \dots, S_k) = \mathbf{E}(f(S_m) \mid S_k)$. Given S_{k-1} , let $\mathbf{X} = X_k, \dots, X_m$ denote the remaining elements added in the process. Let \mathbf{X}_0 be the random sequence \mathbf{X} conditioned on $X_k = x_k \in U$, and \mathbf{X}_1 the random sequence \mathbf{X} conditioned on $X_k = x'_k \in U$. Then \mathbf{X}_0 can be coupled with \mathbf{X}_1 by interchanging x_k and x'_k wherever they occur in \mathbf{X}_0 . The values of $f(S_m)$ in the two elements of a couple pair differ by at most c by assumption. Since each possible realisation of X_k, \dots, X_m has the same probability, it follows that

$$|\mathbf{E}(f(S_m) \mid S_{k-1} \wedge X_k = x_k) - \mathbf{E}(f(S_m) \mid S_{k-1} \wedge X_k = x'_k)| \leq c,$$

and hence $|Y_{k-1} - Y_k| \leq c$. Azuma's inequality (see, e.g., [21]), or alternatively [27, Corollary (6.10)], now completes the proof. ■

Recall that by ω we denote a function that tends to ∞ arbitrarily slowly with n , and that $\mathcal{B}_m(n)$ is a sequence of n i.i.d. random variables each distributed as $\text{Bin}(n-1, p)$ conditioned on $\sum d_i = 2m$.

Lemma 6.2. *Define $\mathbf{d} = (d_1, \dots, d_n)$ as either (a) the degree sequence of a random graph in $\mathcal{G}(n, m)$, or (b) a sequence in $\mathcal{B}_m(n)$. Let $d = 2m/n$. Then*

(i) *for $1 \leq i \leq n$ and all $\alpha > 0$ we have*

$$\mathbf{P}(|d_i - d| \geq \alpha) \leq 2 \exp\left(-\frac{\alpha^2}{2(d + \alpha/3)}\right);$$

(ii) *if $(\log n)^3 = o(dn)$ and α satisfies $(\log n)/\sqrt{n} + (\log n)^{3/2}/\sqrt{dn} = o(\alpha)$ then*

$$\mathbf{P}(|\sigma^2 - \mathbf{Var} d_1| \geq \alpha d + 1/n) = o(n^{-\omega}).$$

Moreover, $\mathbf{Var} d_1 = d(n-d)/n + O(d/n)$.

Proof. We deal with the graph case (a) first. Each vertex degree d_i is distributed hypergeometrically with parameters $\binom{n}{2}, m, n-1$ and expected value d , and hence (i) holds by [21, Theorems 2.10 and 2.1]. For a graph G with degrees d_1, \dots, d_n define

$$f = f(G) = \sum_{i=1}^n \min\{(d_i - d)^2, x\},$$

where $x > 1$ is specified below. Then increasing or decreasing the value of a single d_j by 1 whilst holding d fixed can only change f by at most $(\sqrt{x})^2 - (\sqrt{x} - 1)^2 < 2\sqrt{x}$. Since G is determined by a random m -subset of all possible edges, Lemma 6.1 applies with $c = 8\sqrt{x}$ (as each edge in the symmetric difference of S and T affects two vertex degrees). Replacing α appropriately gives

$$\mathbf{P}(|f(G) - \mathbf{E}f(G)| \geq \alpha dn) \leq 2 \exp(-\alpha^2 dn/32x) = o(n^{-\omega})$$

provided that $\alpha^2 dn/x \log n \rightarrow \infty$. On the other hand, let A denote the event that $\max_i |d_i - d| \geq \sqrt{x}$. By (i) and the union bound applied over all n values of i , we have $\mathbf{P}(A) = o(n^{-\omega})$ as long as we choose $x = \omega(d \log n + (\log n)^2)$. By the bound on α , there exists x satisfying both conditions, and at this point we set x as such. Provided that A does not hold, we have $f(G) = \sum (d_i - d)^2 = n\sigma^2$. We thus conclude

$$\mathbf{P}(|\sigma^2 - \mathbf{E}f(G)/n| \geq \alpha d) = o(n^{-\omega}) + O(1)\mathbf{P}(f(G) \neq n\sigma^2) = o(n^{-\omega}).$$

Now evidently

$$|\mathbf{E}f(G) - n\mathbf{E}\sigma^2| = O(n^3)\mathbf{P}(f(G) \neq n\sigma^2) = o(n^{-\omega})$$

and thus

$$\mathbf{P}(|\sigma^2 - \mathbf{E}\sigma^2| \geq \alpha d + 1/2n) = o(n^{-\omega}).$$

Noting that $\mathbf{E}(d_i - d)^2 = \mathbf{Var} d_1$, we obtain part (ii) for (a). The estimate for $\mathbf{Var} d_1$ follows from the standard formula for variance of this hypergeometric random variable.

For the binomial random variable case (b), essentially the same argument applies for both (i) and (ii), by regarding d_1, \dots, d_n each as a sum of $n - 1$ independent indicator variables. Conditioning on the sum being $2m$ is equivalent to a uniformly random selection of a $2m$ -subset of the $n(n - 1)$ indicator variables. ■

We shall see that the following establishes Conjecture 1.3 in the gap range with explicit error terms. Recall that given a sequence \mathbf{d} we write $M_1 = M_1(\mathbf{d}) = \sum_i d_i$, $d = d(\mathbf{d}) = M_1/n$, $\mu = \mu(\mathbf{d}) = d/(n - 1)$, and $\sigma^2(\mathbf{d}) = \frac{1}{n} \sum_{i=1}^n (d_i - d)^2$ where n is the length of the sequence \mathbf{d} .

Theorem 6.3. *Let n and m be integers, and assume that $\mu_1 = 2m/n(n - 1)$ satisfies $(\log n)^\omega/n \leq \mu_1 = o(1/(\log n)^{3/4})$. Let \mathfrak{D} be the set of sequences \mathbf{d} of length n satisfying the following for some constant $C \geq 2$:*

- (i) $M_1(\mathbf{d}) = 2m$ (and thus $\mu = \mu(\mathbf{d}) = \mu_1$ and $d = d(\mathbf{d}) = 2m/n$),
- (ii) $|d_i - d| \leq C\sqrt{d \log n}$ for all $i \in [n]$,
- (iii) $\sigma^2(\mathbf{d}) \leq 2d$.

Then

- (a) in each of the models $\mathcal{B}_m(n)$ and $\mathcal{D}(\mathcal{G}(n, m))$ we have $\mathbf{P}(\mathfrak{D}) \geq 1 - n^{-h(C)}$, where $h(x) \rightarrow \infty$ as $x \rightarrow \infty$,
- (b) for $\mathbf{d} = \mathbf{d}(n) \in \mathfrak{D}$ we have

$$\begin{aligned} \mathbf{P}_{\mathcal{D}(\mathcal{G}(n, m))}(\mathbf{d}) &= \mathbf{P}_{\mathcal{B}_m}(\mathbf{d}) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1 - \mu)^2}\right) \\ &\quad \times \left(1 + O\left(\frac{1}{\sqrt{d}} + \frac{d\sqrt{\log n}}{n} + \frac{d^2(\log n)^{3/2}}{n^2}\right)\right), \end{aligned}$$

where $\gamma_2 = \gamma_2(\mathbf{d}) = \frac{1}{(n-1)^2} \sum_i (d_i - d)^2$.

We note that the constant implicit in $O(\cdot)$ in (b) of course can, and in fact does, depend on C .

Proof. Let Ω be the underlying set of $\mathcal{B}_m(n)$ and let $\mathbf{d} \in \Omega$. We will consider \mathbf{d} chosen either according to $\mathcal{D}(\mathcal{G}(n, m))$ or $\mathcal{B}_m(n)$. By definition, $M_1(\mathbf{d}) = 2m$ for all $\mathbf{d} \in \Omega$. Apply Lemma 6.2 (i) with $\alpha = C\sqrt{d \log n}$ and the union bound to see that in both $\mathcal{D}(\mathcal{G}(n, m))$ and $\mathcal{B}_m(n)$, with probability at least $1 - n^{-f(C)}$, for all $i \in [n]$ we have $|d_i - d| \leq C\sqrt{d \log n}$ where we may take $f(C) = C^2/3 - 1$. Now, apply Lemma 6.2 (ii) with $\alpha = 1/2$ and note that $d = \mu_1(n-1) > (\log n)^2$ by the theorem's hypothesis to find that $\sigma^2(\mathbf{d}) \leq 2d$ with probability at least $1 - n^{-\omega}$. Therefore, \mathbf{d} satisfies (ii) and (iii) with probability at least $1 - n^{-h(C)}$, for some function $h(C) \rightarrow \infty$ as $C \rightarrow \infty$, and \mathbf{d} satisfies (i) always. Hence (a) follows.

For (b) we first consider the ratio for adjacent degree sequences (see Step 1 in the template given at the start of Section 4). Let Q_1^1 be the set of sequences $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ such that $\mathbf{d} - \mathbf{e}_a \in \mathfrak{D}$ for some $a \in [n]$. Recall that $P_{av}(\mathbf{d})$ denotes the probability that the edge av is present in a graph $G \in \mathcal{G}(\mathbf{d})$ and that

$$R_{ab}(\mathbf{d}) = \frac{\mathbf{P}_{S'}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{S'}(\mathbf{d} - \mathbf{e}_b)},$$

where $S' = \mathcal{D}(\mathcal{G}(n, m))$. We now present functions P^{gr} , R^{gr} and Y^{gr} that approximate the probability and ratio functions P , R and Y sufficiently well. For $a, v, b \in [n]$ set

$$P_{av}^{\text{gr}}(\mathbf{d}) = \frac{d_a d_v}{d(n-1)} \left(1 - \frac{(d_a - d)(d_v - d)}{d(n-1-d)} \right), \quad (6.1)$$

$$R_{ab}^{\text{gr}}(\mathbf{d}) = \frac{d_a(n-d_b)}{d_b(n-d_a)} \left(1 + \frac{d_a - d_b}{d^2 n} \sigma^2(\mathbf{d}) \right), \quad (6.2)$$

$$Y_{avb}^{\text{gr}}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d}) P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) (1 + 1/n). \quad (6.3)$$

Claim 6.4. *Uniformly for $\mathbf{d} = \mathbf{d}(n) \in \mathfrak{D}$ and $a \neq v \in [n]$,*

$$P_{av}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d})(1 + O(\eta_1 + \eta_2)), \quad (6.4)$$

and uniformly for all $\mathbf{d} \in Q_1^1$ and for all $a, b \in [n]$,

$$R_{ab}(\mathbf{d}) = R_{ab}^{\text{gr}}(\mathbf{d})(1 + O(\eta_1 + \mu_1 \eta_2)), \quad (6.5)$$

where $\eta_1 = 1/m + \varepsilon \mu_1/n$ and $\eta_2 = \varepsilon/n + \varepsilon^3 \mu_1^2$, with $\varepsilon = C\sqrt{(\log n)/d}$, where $d = 2m/n$.

The proof will show that an analogous statement also holds for Y_{avb} . Note that ε is simply the upper bound on the relative degree spread of a sequence \mathbf{d} in \mathfrak{D} implied by (ii).

Proof of Claim 6.4. To show that P and P^{gr} (and R and R^{gr}) are $(\eta_1 + \eta_2)$ -close in the sense of (6.4) and (6.5), we consider the operator \mathcal{C} as defined for Lemma 5.3. We first observe that \mathcal{C} fixes (P, Y) , where in this context we regard P to be the function \mathbf{p} with $\mathbf{p}_{av} = P_{av}$ for all appropriate a and v , and similarly Y to be \mathbf{y} with $\mathbf{y}_{avb} = Y_{avb}$, by

Proposition 3.1. We will next use the contraction property of \mathcal{C} as expressed in Lemma 5.3 to show that for any integer $k > 0$, $\mathcal{C}^k(P^{\text{gr}}, Y^{\text{gr}})$ and $\mathcal{C}^k(P, Y)$ are $O(\mu)^k$ -close. We will also show that P^{gr} and $\mathcal{C}^k(P^{\text{gr}})$ are $(\eta_1 + \eta_2)$ -close. These observations will then be shown to imply Claim 6.4.

Fix $k_0 = 4 \log n$ and $r = 4k_0 + 4 = O(\log n)$. Let $\Omega^{(0)}$ be the set of sequences $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ that are at L_1 distance at most r from a sequence in \mathfrak{D} . Let $\mu_0 = 5\mu_1$, and define $\Omega^{(s)}$ as in Lemma 5.3 to be the set of sequences $\mathbf{d} \in \Omega^{(0)}$ at L_1 distance at least $s + 1$ from all sequences outside $\Omega^{(0)}$.

Towards applying Lemma 5.3 we first establish that (P, Y) and $(P^{\text{gr}}, Y^{\text{gr}})$ are elements of $\Pi_{\mu_0}(\Omega^{(2)})$ (see Definition 5.1). Note that for $\mathbf{d} \in \Omega^{(0)}$, the values of $d(\mathbf{d})$ and $\mu(\mathbf{d})$ are asymptotically equal to $2m/n$ and μ_1 , respectively, since $M_1(\mathbf{d}) = M_1(\mathbf{d}_0) + O(\log n)$ for some sequence $\mathbf{d}_0 \in \mathfrak{D}$. Furthermore, we note that condition (ii) of the theorem, together with the lower bound on μ_1 in the theorem statement, implies that for all even $\mathbf{d} \in \Omega^{(0)}$, $d_i \sim \mu n$ uniformly for all i . To bound $P_{av}(\mathbf{d})$, we first observe, using $\mu \geq (\log n)^3/n$ and assumption (ii) of the theorem, that Lemma 2.4 (a) implies $\mathcal{N}(\mathbf{d}) > 0$ for all even $\mathbf{d} \in \Omega^{(0)}$. After this, for n sufficiently large, Lemma 2.3, together with the fact that $d_i \sim \mu n$ uniformly for all i , implies that for all distinct $a, v \in [n]$,

$$P_{av}(\mathbf{d}) \leq \frac{\mu}{1 - \mu}(1 + o(1)) < \frac{5\mu}{4} \quad \text{for all even } \mathbf{d} \in \Omega^{(0)}, \quad (6.6)$$

where for the last inequality we use $\mu \sim \mu_1 < 1/6$, say. Since $\Omega^{(2)} \subseteq \Omega^{(0)}$ and $5\mu/4 < \mu_0$ this establishes requirement (IIa) for P in the definition of $\Pi_{\mu_0}(\Omega^{(2)})$. Now restrict slightly to $\mathbf{d} \in \Omega^{(2)}$. By definition $Y_{avb}(\mathbf{d})$ is the probability that both edges av and bv are present. Hence Proposition 3.1 (c) implies (with the above bounds on $P_{av}(\mathbf{d})$ applying for all $\mathbf{d} \in \Omega^{(0)}$) that $0 \leq Y_{avb}(\mathbf{d}) = Y_{bva}(\mathbf{d}) \leq 3\mu_1 P_{bv}(\mathbf{d})/2$ (easily) assuming, as we may, that μ_1 is sufficiently small. Thus (P, Y) satisfies condition (IIc) for membership of $\Pi_{\mu_0}(\Omega^{(2)})$, and also

$$\sum_{v \in [n] \setminus \{a, b\}} Y_{avb}(\mathbf{d}) \leq \sum_{v \in [n] \setminus \{a, b\}} \frac{3P_{bv}(d)\mu_1}{2} \leq 3\mu_1 d_a(1 + o(1))$$

using $P_{bv}(\mathbf{d}) \leq 2\mu_1 \sim 2d_a/n$, which follows from (6.6) and $\mu \sim \mu_1$. As $4\mu_1 < \mu_0$, this shows Y satisfies condition (IIb) for membership of $\Pi_{\mu_0}(\Omega^{(2)})$ when n is sufficiently large. To see that $(P^{\text{gr}}, Y^{\text{gr}})$ is also in $\Pi_{\mu_0}(\Omega^{(2)})$ we recall that $d_i \sim \mu n$ uniformly for all i for all $\mathbf{d} \in \Omega^{(0)}$. Thus, by definition (6.1) we have $P_{av}^{\text{gr}}(\mathbf{d}) \sim \mu$ for all distinct $a, v \in [n]$ and all $\mathbf{d} \in \Omega^{(0)}$. Properties (IIa)–(IIc) follow directly from this fact and the definition (6.3) since $\mu \sim \mu_1 = \mu_0/5$.

Now for large n and distinct $a, v \in [n]$ we have $P_{av}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d})(1 \pm 1)$ for all even $\mathbf{d} \in \Omega^{(0)}$ since $P_{av}^{\text{gr}}(\mathbf{d}) \sim \mu$ and by (6.6). Also, $0 \leq Y_{avb}(\mathbf{d}) \leq 3\mu_1 P_{bv}(\mathbf{d})/2$ implies $Y_{avb}(\mathbf{d}) = Y_{avb}^{\text{gr}}(\mathbf{d})(1 \pm 1)$ for all even $\mathbf{d} \in \Omega^{(2)}$. We may now apply Lemma 5.2 (a) with $\xi = 1$ for any odd $\mathbf{d} \in \Omega^{(3)}$ to deduce that

$$\mathcal{R}(P, Y)_{ab}(\mathbf{d}) = \mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d})(1 + O(\mu_0))$$

for all $a, b \in [n]$. Writing \mathbf{r} for $\mathcal{R}(P, Y)$ and \mathbf{r}' for $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})$ we see from this and Lemma 5.2 (b) that

$$\mathcal{P}(P, \mathbf{r})_{av}(\mathbf{d}) = \mathcal{P}(P^{\text{gr}}, \mathbf{r}')_{av}(\mathbf{d})(1 + O(\mu_0))$$

for all even $\mathbf{d} \in \Omega^{(4)}$, all distinct $a, v \in [n]$. Next applying Lemma 5.2 (c) in the same way to even $\mathbf{d} \in \Omega^{(6)}$ gives

$$\mathcal{Y}(\hat{\mathbf{p}}, Y)_{avb}(\mathbf{d}) = \mathcal{Y}(\hat{\mathbf{p}}', Y^{\text{gr}})_{avb}(\mathbf{d})(1 + O(\mu_0))$$

for all even $\mathbf{d} \in \Omega^{(6)}$ and all distinct $a, v, b \in [n]$, where $\hat{\mathbf{p}} = \mathcal{P}(P, \mathbf{r})$ and $\hat{\mathbf{p}}' = \mathcal{P}(P^{\text{gr}}, \mathbf{r}')$. Recalling the definition of the distance functions $\chi^{(s)}$ and the definition of the operator \mathcal{C} we see that this is equivalent to the statement $\chi^{(6)}(\mathcal{C}(P, Y), \mathcal{C}(P^{\text{gr}}, Y^{\text{gr}})) = O(\mu_0)$. Making $k - 1$ iterated applications of Lemma 5.3 with ever-decreasing ξ produces

$$\chi^{(4k+2)}(\mathcal{C}^{k_0}(P, Y), \mathcal{C}^{k_0}(P^{\text{gr}}, Y^{\text{gr}})) = O(\mu_0)^k.$$

Finally, $\mathcal{C}(P^{\text{gr}}, Y^{\text{gr}})(\mathbf{d})$ can be estimated by straightforward expansions using the following: uniformly for all $\mathbf{d} \in \Omega^{(0)}$,

- (a) $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d}) = R_{ab}^{\text{gr}}(\mathbf{d})(1 + O(\eta_1))$ for all $a, b \in [n]$;
- (b) $\mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d})(1 + O(\eta_2))$ for all distinct $a, v \in [n]$;
- (c) $\mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d}) = Y_{avb}^{\text{gr}}(\mathbf{d})(1 + O(\eta_2))$ for all distinct $a, v, b \in [n]$.

This is justified by Lemma A.1 in Appendix A, after making two observations. One is that the error term $\varepsilon^4 d^2 / n^2$ can be dropped because $\varepsilon^3 d = o(1)$. The second is that the error terms η_1 and η_2 are now defined with reference to sequences which are at distance $O(\log n)$ from the sequences in \mathfrak{D} and are thus asymptotically the same as the stated values. (In fact, a better approximation is proved in Lemma 7.1 using computer assistance.) Applying (a), (b) and (c) in turn, recalling Lemma 5.2 to handle the small error terms, shows that $\chi^{(6)}(\mathcal{C}(P^{\text{gr}}, Y^{\text{gr}}), (P^{\text{gr}}, Y^{\text{gr}})) = O(\eta_1 + \eta_2)$. Using Lemma 5.3 repeatedly, and bounding the total distance moved during the iterations as for a contraction mapping (as the sum of a geometric series), this gives

$$\chi^{(r-2)}((P^{\text{gr}}, Y^{\text{gr}}), \mathcal{C}^{k_0}(P^{\text{gr}}, Y^{\text{gr}})) = O(\eta_1 + \eta_2).$$

Combining this with the above bound on $\chi^{(4k+2)}$ when $k = k_0$, and with $\mathcal{C}(P, Y) = (P, Y)$ and the triangle inequality, gives $\chi^{(r-2)}((P, Y), (P^{\text{gr}}, Y^{\text{gr}})) = O(\eta_1 + \eta_2) + O(\mu_0)^{k_0}$. This implies (6.4) for all even $\mathbf{d} \in \Omega^{(r-2)}$ since $k_0 = 4 \log n$, and we may assume $O(\mu_0) < 1/e$ say since $\mu_0 = 5\mu_1 \rightarrow 0$. Note that $\mathfrak{D} \subseteq \Omega^{(r)} \subseteq \Omega^{(r-2)}$ by definition. For (6.5), we now use the fact that (6.4) holds for all even $\mathbf{d} \in \Omega^{(r-2)}$ to deduce from Lemma 5.2 (a) that

$$\mathcal{R}(P, Y)_{ab}(\mathbf{d}) = \mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d})(1 + O(\mu_0\eta_1 + \mu_0\eta_2))$$

for all odd $\mathbf{d} \in \Omega^{(r-1)}$. This, together with (a) above and the facts that $\mathcal{R}(P, Y) = R$ and $\mu_0 = 5\mu_1$, implies (6.5) for all $\mathbf{d} \in \mathcal{Q}_1^1 \subseteq \Omega^{(r-1)}$. \blacksquare

Since $\varepsilon = C\sqrt{(\log n)/d}$ and $d < n/\sqrt{\log n}$, the claim gives

$$\frac{\mathbf{P}_{\mathcal{S}'}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{\mathcal{S}'}(\mathbf{d} - \mathbf{e}_b)} = R_{ab}(\mathbf{d}) = R_{ab}^{\text{gr}}(\mathbf{d}) \left(1 + O\left(\frac{1}{dn} + \frac{(\sqrt{d \log n})^3}{n^3} + \frac{\sqrt{d \log n}}{n^2} \right) \right) \quad (6.7)$$

uniformly for all $\mathbf{d} \in Q_1^1$.

We now move to Steps 2 and 3 in the template in Section 4. Let $H(\mathbf{d}) = \mathbf{P}_{\mathcal{B}_m}(\mathbf{d}) \tilde{H}(\mathbf{d})$ be the conjectured formula in the right hand side of (b) (without error terms), where

$$\tilde{H}(\mathbf{d}) = \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\mu^2(1-\mu)^2} \right).$$

Define the probability spaces \mathcal{S} and \mathcal{S}' exactly as in the proof of Theorem 4.1 with the same underlying set Ω . That is,

$$\mathbf{P}_{\mathcal{S}}(\mathbf{d}) = H(\mathbf{d}) / \sum_{\mathbf{d}' \in \Omega} H(\mathbf{d}') = \frac{H(\mathbf{d})}{\mathbf{E}_{\mathcal{B}_m} \tilde{H}}$$

and $\mathcal{S}' = \mathcal{D}(\mathcal{G}(n, m))$. Also define the graph G as before, with vertex set $\mathfrak{X} := \mathfrak{D}$, and with an edge joining each pair of sequences in \mathfrak{D} of the form $\mathbf{d} - \mathbf{e}_a$ and $\mathbf{d} - \mathbf{e}_b$ for some $a \neq b$. The L_1 distance from a sequence $\mathbf{d} \in G$ to the constant sequence (d, \dots, d) is $\sum_i |d_i - d|$, which is at most $n\sqrt{2d}$ by (iii) and Cauchy's inequality. Some vertex of G has L_1 distance at most n from this constant sequence. It follows that the diameter of G is $r = O(n\sqrt{d})$.

We claim (see Step 4 of the template) that \mathfrak{X} has probability at least $1 - \varepsilon_0$ for some suitably chosen ε_0 in both \mathcal{S} and \mathcal{S}' . Note that $\mathbf{P}_{\mathcal{S}'}(\mathfrak{X}) \geq 1 - n^{-h(C)}$ and $\mathbf{P}_{\mathcal{B}_m}(\mathfrak{X}) \geq 1 - n^{-h(C)}$ by (a) proved above. Furthermore, if $\mathbf{d} \in \mathcal{B}_m(n)$ then $\gamma_2(\mathbf{d}) = \frac{n}{(n-1)^2} \sigma^2(\mathbf{d}) = \mu(1-\mu)(1 + O(\xi))$ with probability $1 - o(n^{-\omega})$, where $\xi = (\log n)^2/\sqrt{n}$ (this is the more precise implication of Lemma 6.2(ii) applied with $\alpha = (\log n)^2/\sqrt{n}$). Thus, for such \mathbf{d} in $\mathcal{B}_m(n)$, the exponential factor $\tilde{H}(\mathbf{d})$ is $1 + O(\xi)$ with probability $1 - o(n^{-\omega})$. Therefore,

$$\mathbf{E}_{\mathcal{B}_m} \tilde{H} = 1 + O(\xi) \quad (6.8)$$

and thus $\mathbf{P}_{\mathcal{S}}(\mathfrak{X}) \geq 1 - n^{-h(C)}$. It follows that, as in the proof of Theorem 4.1, we may use $\varepsilon_0 = n^{-1}$ in Lemma 2.1.

We now move to Step 5 in the template. For $\mathbf{d} \in Q_1^1$,

$$\frac{\mathbf{P}_{\mathcal{S}}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{\mathcal{S}}(\mathbf{d} - \mathbf{e}_b)} = \frac{H(\mathbf{d} - \mathbf{e}_a)}{H(\mathbf{d} - \mathbf{e}_b)} = \frac{d_a(n - d_b)}{d_b(n - d_a)} \exp\left(\frac{(d_a - d_b)\gamma_2}{d^2(1 - \mu')^2} + O\left(\frac{\Delta^2}{(dn)^2} \right) \right) \quad (6.9)$$

by (2.1), where $\mu' = \mu(\mathbf{d} - \mathbf{e}_a)$ and γ_2, d and Δ are defined with respect to \mathbf{d} as in (2.1). Note that the first term in the exponential in (6.9) is $O(\sqrt{(\log n)/dn^2})$ for $\mathbf{d} \in Q_1^1$ by (ii) and (iii) and since $\mu' = \mu(1 + 1/2m)$. Thus,

$$\frac{(d_a - d_b)\gamma_2}{d^2(1 - \mu')^2} = \frac{d_a - d_b}{(dn)^2} \sum_i (d_i - d)^2 + O\left(\frac{\sqrt{d \log n}}{n^2} \right),$$

and the second term, $\Delta^2/(dn)^2$, is $O(1/n^2)$ by assumption (ii). We can now infer from the definition of R^{gr} that (6.9) is equivalent to

$$\frac{\mathbf{P}_{\mathcal{S}}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{\mathcal{S}}(\mathbf{d} - \mathbf{e}_b)} = R_{ab}^{\text{gr}}(\mathbf{d}) \left(1 + O\left(\frac{\sqrt{d \log n}}{n^2}\right) \right). \quad (6.10)$$

This together with (6.7) gives

$$\frac{\mathbf{P}_{\mathcal{S}'}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{\mathcal{S}'}(\mathbf{d} - \mathbf{e}_b)} = e^{O(\delta)} \frac{\mathbf{P}_{\mathcal{S}}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{\mathcal{S}}(\mathbf{d} - \mathbf{e}_b)}$$

with

$$\delta = \frac{1}{dn} + \frac{(\sqrt{d \log n})^3}{n^3} + \frac{\sqrt{d \log n}}{n^2}$$

for $\mathbf{d} \in Q_1^1$. Therefore, by Lemma 2.1 and (6.8),

$$\mathbf{P}_{\mathcal{S}'}(\mathbf{d}) = \mathbf{P}_{\mathcal{S}}(\mathbf{d}) e^{O(r\delta + \varepsilon_0)} = H(\mathbf{d})(1 + O(\xi + r\delta + \varepsilon_0))$$

for all $\mathbf{d} \in \mathfrak{D}$, which proves (b) since $\xi = (\log n)^2/\sqrt{n}$,

$$r\delta = O(d^{-1/2} + d\sqrt{\log n}/n + d^2(\log n)^{3/2}/n^2)$$

and $\varepsilon_0 = 1/n$. ■

It is a simple exercise in analysis to see that the theorem implies Conjecture 1.3 in the gap range: the truth of the theorem itself implies a slightly altered version of the theorem's statement (b), in which C is a function of n that tends ("slowly") to ∞ . (The same can be done with the constant in the $O(\cdot)$ if desired.) The fact that $h(C) \rightarrow \infty$ then shows that the asymptotic approximation (1.2) holds for the sequences \mathbf{d} in a suitable set $R_p(n)$. All that remains is to note that the distribution of m in $\mathcal{D}(\mathcal{G}(n, p))$ is identical to that in \mathcal{E}'_p , and that the latter restricted by $\sum d_i = 2m$ is identical to $\mathcal{B}_m(n)$.

Corollary 6.5. *Conjecture 1.3 holds.*

We remark that one can avoid the sharp concentration results that we used, instead employing only variance via Chebyshev's inequality, at the expense of relaxing the $o(n^{-\omega})$ error in the conjecture to $o(1)$. The result would still be interesting; we leave the details to the reader.

7. A wider range of degrees: proof of Theorem 1.4

In this section we prove Theorem 1.4. Compared with Theorem 6.3, some crucial differences that affect the argument include μ being permitted to have constant size, the allowable degree spread being d^α for $\alpha > 1/2$, and the transfer of σ^2 from explicit bounds to a term in the ratio formula.

The proof has the same structure as for Theorem 6.3. The crucial change required is to redefine the approximations, P^{gr} , R^{gr} and Y^{gr} , of the probability and the ratio functions

so that the error functions corresponding to η_1 and η_2 in Claim 6.4 satisfy $\eta_1 + \mu\eta_2 = o(1/nd^\alpha)$. This error bound is necessary to obtain a final formula with $1 + o(1)$ error, since with the range of degrees under consideration, the diameter of the graph G of degree sequences (see the proof of Theorem 6.3) is up to $r = O(nd^\alpha)$.

To define the approximations, we write P^{gr} , R^{gr} and Y^{gr} parametrised to facilitate identifying negligible terms. We define the expressions

$$\pi = \mu(1 + \varepsilon_a)(1 + \varepsilon_v) \left(1 + \frac{-\mu\varepsilon_a\varepsilon_v + (\varepsilon_a + \varepsilon_v)\sigma^2/dn}{1 - \mu} + \frac{\varepsilon_a + \varepsilon_v}{n-1} \right),$$

$$\rho = \frac{1 + \varepsilon_a}{1 + \varepsilon_b} \cdot \frac{1 - \mu(1 + \varepsilon_b) + 1/n}{1 - \mu(1 + \varepsilon_a) + 1/n} \left(1 + \frac{(\varepsilon_a - \varepsilon_b)\sigma^2}{(1 - \mu)^2 dn} \right).$$

When referring to π , we list only an initial segment of parameters $\varepsilon_a, \varepsilon_v, \mu, \sigma^2, d$ that are different from the ones in the definitions above. So for instance $\pi(x, y)$ stands for π with $\varepsilon_a, \varepsilon_v$ replaced by x, y . Similarly for ρ and the parameters $\varepsilon_a, \varepsilon_b, \mu, \sigma^2, d$. Recall that we consider sequences \mathbf{d} of length n and $\mu = d/(n-1)$.

For this section, we define P^{gr} and R^{gr} by

$$P_{av}^{\text{gr}} = \pi, \quad R_{ab}^{\text{gr}} = \rho$$

for all $a, v, b \in [n]$, where $\varepsilon_i = (d_i - d)/d$ and $\sigma^2 = \sigma^2(\mathbf{d}) = \sum (d_i - d)^2/n$, so that P_{av}^{gr} etc. are functions of degree sequences. Furthermore, for all $a, v, b \in [n]$ we define

$$Y_{avb}^{\text{gr}}(\mathbf{d}) = \pi(\varepsilon_a, \varepsilon_v)\pi(\varepsilon_b, \varepsilon_v - \delta) \cdot \left(1 + \frac{1 + \varepsilon_a - \mu(1 + \varepsilon_a + \varepsilon_b)}{(n-1)(1-\mu)} \right).$$

Note also that d and μ were specified in the theorem statement (determined by m), but with a slight notational abuse, for the following lemma, given any sequence \mathbf{d} of length n we define $\mu = \mu(\mathbf{d}) = \frac{1}{2}M_1(\mathbf{d})/\binom{n}{2}$ so that the average is $d = \mu(n-1)$. In the following lemma, the parity of \mathbf{d} is immaterial, though it will only be applied for odd \mathbf{d} in (a) and even \mathbf{d} in (b) and (c).

Lemma 7.1. *Let n be an integer and let $1/2 \leq \alpha < 3/5$. Let $\mathcal{A} = \binom{[n]}{2}$ and let $\mathbf{d} = \mathbf{d}(n)$ be a sequence of length n with average \bar{d} such that $\mu_1 = \mu(\mathbf{d}) = \bar{d}/(n-1) < 1/4$, and assume that for all $1 \leq i \leq n$ we have $|d_i - \bar{d}| \leq \varepsilon\bar{d}$, where $\varepsilon = \bar{d}^{\alpha-1} > 0$. Then*

- (a) $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d}) = R_{ab}^{\text{gr}}(\mathbf{d})(1 + O(\mu_1\varepsilon^4))$ for all $a, b \in [n]$,
- (b) $\mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d})(1 + O(\mu_1\varepsilon^4))$ for all distinct $a, v \in [n]$,
- (c) $\mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d}) = Y_{avb}^{\text{gr}}(\mathbf{d})(1 + O(\mu_1\varepsilon^4))$ for all distinct $a, v, b \in [n]$.

We have used a computer algebra system to assist with the calculations in the proof of this lemma; the corresponding Maple worksheet can be accessed by following this article's DOI.

Proof of Lemma 7.1. In the calculations below, the following approximations of P^{gr} and π , respectively, will often be convenient. Let \mathbf{d}' be a sequence of length n that is at L_1 distance $O(1)$ from \mathbf{d} , and with $d'_a = d_a - j_a$ and $d'_v = d_v - j_v$. Here and in

the following, the bare symbols μ, d, ε_a and so on are defined with respect to the original sequence \mathbf{d} , whilst μ', d', ε'_a , etc., are defined with respect to the average degree of \mathbf{d}' . For such a sequence \mathbf{d}' we find that $\mu(\mathbf{d}')$ is $\mu' = \mu_1 + O(1/n^2)$ since the average d' of \mathbf{d}' is $\bar{d} + O(1/n)$. Therefore, the variable ε'_a defined as $(d'_a - d')/d'$ is equal to $\varepsilon_a - j_a\delta + O(\mu\delta^2)$, where $\delta = 1/\bar{d}$, and the analogous formula holds for ε_v . Similarly, $\sigma' = \sigma^2(\mathbf{d}') = \sigma^2(\mathbf{d}) + O(\varepsilon\mu_1)$. Thus

$$P_{av}^{\text{gr}}(\mathbf{d}') = \pi(\varepsilon'_a, \varepsilon'_v, \mu', \sigma', d') = \pi(\varepsilon_a - j_a\delta, \varepsilon_v - j_v\delta)(1 + O(\xi)), \quad (7.1)$$

where here and below $\xi = \mu_1\varepsilon^4$ (note that $\varepsilon \geq \bar{d}^{-1/2}$ by assumption so that $1/\bar{d}n = O(\mu_1\varepsilon^4)$). In other words, the changes from μ_1, \bar{d} , and σ^2 to μ', d' , and σ' are negligible in the formula for P^{gr} .

For (a), we note first that $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{aa}(\mathbf{d}) = 1 = R_{aa}^{\text{gr}}(\mathbf{d})$ by definition of ρ and \mathcal{R} in (5.4). Assume now that $a \neq b$. Using (5.4) to evaluate $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d})$, we estimate the expression $\mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b) = \mathbf{bad}(P^{\text{gr}}, Y^{\text{gr}})(a, b, \mathbf{d} - \mathbf{e}_b)$, for which, in turn, we need to estimate $\sum Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b)$, where the sum is over all $v \in [n]$ such that both av and bv are allowable (see (5.1)). By definition and (7.1),

$$Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) = \pi(\varepsilon_a, \varepsilon_v)\pi(\varepsilon_b - \delta, \varepsilon_v - \delta) \cdot \left(1 + \frac{1 + \varepsilon_a - \mu_1(1 + \varepsilon_a + \varepsilon_b)}{(n-1)(1 - \mu_1)} + O(\xi)\right),$$

where we use $\varepsilon_a, \varepsilon_b$ and μ_1 in the third factor (rather than the altered versions ε'_a etc.) using the same reasoning as in the lead-up to (7.1). Consider expanding this expression for $Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b)$ ignoring terms of order ε^4 , and hence also ignoring δ^2 and $\varepsilon^2\delta$, since $\varepsilon^2 \geq 1/\bar{d}$. A convenient way to do this is to make substitutions $\varepsilon_v = y_1\varepsilon_v, \delta = y_1^2\delta, \mu_1 = y_2\mu_1, 1/n = y_1^2y_2/n$, and so on (for instance, σ^2/dn is $O(\varepsilon^2\mu_1)$), where y_1 represents a parameter of size $O(\varepsilon)$ and y_2 of size $O(\mu_1)$, and then expand about $y_1 = 0$. We note by inspection that $Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b)$ has μ^2 as a factor since $\pi(\varepsilon_a, \varepsilon_v)$ and $\pi(\varepsilon_b - \delta, \varepsilon_v - \delta)$ each have μ as a factor. So the terms in its expansion have the corresponding upper bound $O(\mu^2\varepsilon^i)$ on their absolute sizes. We also note that in expanding a rational function about a non-singular point, the error in the Taylor expansion is bounded by a multiple of the least significant terms omitted. This avoids any need to bound higher derivatives explicitly. In this way, expanding after these $\{y_1, y_2\}$ substitutions, and noting that $1/n = O(\mu_1\varepsilon^2)$, we obtain

$$Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) = J + O(\mu_1^2\varepsilon^4),$$

where J is a polynomial of degree 3 in y_1 . (Unfortunately J is too large to write here.) Next, removing the ‘‘sizing’’ variables y_i from J by setting them equal to 1, and then expanding the result about $\varepsilon_v = 0$ and retaining all terms of total degree at most 3 in ε_v , we get

$$Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) = c_0 + c_1\varepsilon_v + c_2\varepsilon_v^2 + O(\mu_1^2\varepsilon^4),$$

where the functions c_0, c_1 , and c_2 are independent of ε_v . (By calculation, the third order term turns out to be absorbed by the error term.) Then considering the definition of

$\mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b)$ in (5.1) we find that the second summation in that definition can be written as

$$\begin{aligned} \Sigma_{\mathbf{bad}} &:= \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) \\ &= \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} (c_0 + c_1 \varepsilon_v + c_2 \varepsilon_v^2 + O(\mu_1^2 \varepsilon^4)) \\ &= nc_0 + n\sigma^2 \delta^2 c_2 - 2c_0 - c_1(\varepsilon_a + \varepsilon_b) - c_2(\varepsilon_a^2 + \varepsilon_b^2) + O(n\mu_1^2 \varepsilon^4), \end{aligned}$$

where in the last inequality we use $\sum_{v \in [n]} \varepsilon_v = 0$ and $\sum_{v \in [n]} \varepsilon_v^2 = n\sigma^2 \delta^2$ (and we recall that the ε_v 's are defined with respect to \mathbf{d}). Noting that $\mathcal{A}(a) \setminus \mathcal{A}(b) = \{b\}$, we can write $\mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b)$ in (5.4), by using (5.1) and (7.1), as

$$\mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b) = \frac{1}{d_a} (\Sigma_{\mathbf{bad}} + \pi(\varepsilon_a, \varepsilon_b - \delta) + O(\mu_1 \xi)),$$

where $d_a = (1 + \varepsilon_a) \bar{d}$. Note that the error term from $\Sigma_{\mathbf{bad}}$ produces an absolute error term of size $O(\mu_1 \varepsilon^4) = O(\xi)$ in $\mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b)$ since $n/d_a \sim 1/\mu_1$. Substituting the above expression, stripped of its error terms, into

$$\frac{\mathcal{R}(P^{\text{gr}})_{ab}(\mathbf{d})}{\rho} - 1 = \frac{1}{\rho} \cdot \frac{(1 + \varepsilon_a)(1 - \mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b))}{(1 + \varepsilon_b)(1 - \mathbf{bad}(b, a, \mathbf{d} - \mathbf{e}_a))} - 1$$

and simplifying gives a rational function \hat{F} . That is, $\mathcal{R}(P^{\text{gr}})_{ab}(\mathbf{d})/\rho - 1 = \hat{F} + O(\xi)$. After inserting the size variables y_1 and y_2 into \hat{F} as specified above, and simplifying, we find it has y_2 as a factor (of multiplicity 1), and its denominator is non-zero at $y_1 = 0$. Then expanding the expression in powers of y_1 shows that $\hat{F} = O(y_1^4)$. Along with the extra factor y_2 , this implies $\hat{F} = O(\xi)$. Thus, part (a) follows.

To prove part (b) note that, analogous to (7.1), if $\mathbf{d}' = \mathbf{d} - \mathbf{e}_v$ we also have, for $b \neq v$,

$$R_{ab}^{\text{gr}}(\mathbf{d}') = \rho(\varepsilon'_a, \varepsilon'_b, \mu', (\sigma^2)', d') = \rho \cdot (1 + O(\mu_1 \varepsilon^4)),$$

where we also use the fact that $a \neq v$. Therefore, by definition (5.2) and (7.1),

$$\begin{aligned} \mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) &= d_v \left(\sum_{b \in \mathcal{A}(v)} R_{ba}^{\text{gr}}(\mathbf{d} - \mathbf{e}_v) \frac{1 - P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \right)^{-1} \\ &= d_v \left(\sum_{b \in \mathcal{A}(v)} \rho(\varepsilon_b, \varepsilon_a) \cdot \frac{1 - \pi(\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)} (1 + O(\mu_1 \varepsilon^4)) \right)^{-1}. \end{aligned} \tag{7.2}$$

By expanding in ε_b we obtain

$$\rho(\varepsilon_b, \varepsilon_a) \cdot \frac{1 - \pi(\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)} = K + O(\varepsilon^4),$$

where K is a polynomial in ε_b of degree at most 3. Calculations using the size variables y_1 and y_2 as above show that $K = k_0 + k_1 \varepsilon_b + k_2 \varepsilon_b^2 + O(\varepsilon^4)$ for some k_i independent

of ε_b . Also, recall that we have $\mathcal{A}(v) = [n] \setminus \{v\}$. So the main summation over b in (7.2) can be evaluated (noting again that $\sum_b \varepsilon_b^2 = n\sigma^2\delta^2$) as

$$nk_0 + n\sigma^2\delta^2k_2 - k_0 - k_1\varepsilon_v - k_2\varepsilon_v^2$$

with relative error $O(\varepsilon^4)$, noting that K has constant order, where we use $\sum_{b \in [n]} \varepsilon_b = 0$. Using the size variables y_1 and y_2 as described above, we find that $\mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) = \pi(1 + O(\mu_1\varepsilon^4))$ for $a \neq v$, with the extra factor μ_1 arising in the error term in the same way as for \mathcal{R} in part (a). Part (b) follows.

For part (c) consider $Z = \mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d})/Y_{avb}^{\text{gr}}(\mathbf{d}) - 1$. By definition of \mathcal{Y} , P^{gr} , and Y^{gr} we can write Z as

$$Z = \frac{P_{av}^{\text{gr}}(\mathbf{d})(P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_b))}{Y_{avb}^{\text{gr}}(\mathbf{d})(1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v))} - 1. \quad (7.3)$$

Then, using (7.1), replace $P_{av}^{\text{gr}}(\mathbf{d})$ by $\pi(\varepsilon_a, \varepsilon_v)$, $P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)$ by $\pi(\varepsilon_b, \varepsilon_v - \delta)(1 + O(\xi))$, $P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)$ by $\pi(\varepsilon_a - \delta, \varepsilon_v - \delta)(1 + O(\xi))$, and, using the same argument as leading to (7.1), replace $Y_{avb}^{\text{gr}}(\mathbf{d} - j\mathbf{e}_a - j\mathbf{e}_b)$ by

$$\begin{aligned} & \pi(\varepsilon_a - j\delta, \varepsilon_v - j\delta)\pi(\varepsilon_b, \varepsilon_v - (j+1)\delta) \\ & \times \left(1 + \frac{1 + \varepsilon_a - j\delta - \mu_1(1 + \varepsilon_a - j\delta + \varepsilon_b)}{(n-1)(1 - \mu_1)} + O(\xi) \right) \end{aligned}$$

for $j \in \{0, 1\}$. Now, as for parts (a) and (b), use the sizing variables y_1 and y_2 and expand about $y_1 = 0$ (note that both the denominator and numerator of the fraction in Z are asymptotically equal to μ_1^2). We then find that Z is of size $O(\mu_1\varepsilon^4)$, which proves (c). ■

To prove Theorem 1.4, we set $\mathfrak{Y} = \mathfrak{D}$ and then follow the proof of Theorem 6.3, referring to the set \mathfrak{D} within it as \mathfrak{D}' . Since $\mathfrak{D}' \subseteq \mathfrak{D}$, Theorem 6.3 (a) implies $\mathbf{P}(\mathfrak{Y}) = 1 - o(1)$ in both models $\mathcal{B}_m(n)$ and $\mathcal{D}(\mathcal{G}(n, m))$. Actually, in the current setting we permit higher values of μ , which is now only bounded above by a small constant, but the earlier proof applies equally well for this extended range.

Next define Q_1^1 as in the proof Theorem 6.3 to be the set of sequences $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ such that $\mathbf{d} - \mathbf{e}_a \in \mathfrak{D}$ for some $a \in [n]$. Then the proof of Claim 6.4 (which assumed only the upper bound $1/4$ on μ) applies, with adjustment to the error terms η_i using Lemma 7.1 in place of Lemma A.1, to show that

$$P_{av}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d})(1 + O(\mu_1\varepsilon^4)), \quad (7.4)$$

$$R_{ab}(\mathbf{d}) = R_{ab}^{\text{gr}}(\mathbf{d})(1 + O(\mu_1\varepsilon^4)), \quad (7.5)$$

uniformly for all $\mathbf{d} \in \mathfrak{D}$ (or $\mathbf{d} \in Q_1^1$, respectively), and all appropriate a, v and b . Using the latter together with the definition of ρ , we find in place of (6.7) (with the same definitions of \mathcal{S} and \mathcal{S}') that

$$\frac{\mathbf{P}_{\mathcal{S}'}(\mathbf{d} - \mathbf{e}_a)}{\mathbf{P}_{\mathcal{S}'}(\mathbf{d} - \mathbf{e}_b)} = R_{ab}^{\text{gr}}(\mathbf{d})(1 + O(\mu_1\varepsilon^4 + 1/n^2)). \quad (7.6)$$

Then for the independent binomial probability space \mathcal{S} , (6.9) is only altered by the γ_2 term cancelling with the σ^2 term in the definition of π . Thus we obtain (6.10) with error term reduced to $O(1/n^2)$.

In this application of Lemma 2.1, the diameter of the auxiliary graph G is $r = O(nd^\alpha)$ and $\delta = \mu_1 \varepsilon^4 + 1/n^2$ from (7.6). We may again use $\varepsilon_0 = 1/n$. The result is

$$\mathbf{P}_{\mathcal{S}'}(\mathbf{d}) = \mathbf{P}_{\mathcal{S}}(\mathbf{d})e^{O(r\delta + \varepsilon_0)} = H(\mathbf{d})(1 + O(\xi + r\delta + \varepsilon_0)),$$

with $\xi = (\log n)^2/\sqrt{n}$ entering as before. The theorem follows, since $d^{5\alpha-3} \geq d^\alpha/n$. ■

8. Concluding remarks

In the main result, Theorem 1.4, the upper bound μ_0 on the density μ can probably be set equal to any constant less than $1/2$, at the expense of only slight changes to the proof. Since other results cover this range of density, we do not follow this line any further. On the other hand, various aspects of the proofs can be improved with some straightforward work, to obtain a wider range of degrees and smaller error terms, and we plan to pursue this elsewhere.

The approach of Sections 3 and 5 can be applied to other problems. We plan to apply the method in order to prove related binomial-based models for the degree sequences of random bipartite graphs, loopless directed graphs, and hypergraphs. The approach can also be used to make a major advance in asymptotic enumeration of Latin rectangles.

Appendix A. Approximating the operator fixed points

Here we prove the estimates (a)–(c) inside the proof of Claim 6.4. Recall the definitions of P^{gr} , Y^{gr} and R^{gr} in (6.1)–(6.3), and of $\sigma^2(\mathbf{d})$. Recall also that \mathfrak{D} is a set of sequences of length n satisfying $|d_i - d| \leq C\sqrt{d} \log n$ for all $i \in [n]$ (for some constant C) and that $\sigma^2(\mathbf{d}) \leq 2d$ by the assumptions of Theorem 6.3. Furthermore, recall that we set $k_0 = 4 \log n$ and that $\Omega^{(0)}$ is the set of sequences $\mathbf{d} \in \mathbb{Z}_{\geq 0}^n$ that are at L_1 distance at most $r = 4k_0 + 4$ from a sequence in \mathfrak{D} . Therefore, (a)–(c) follow from the following lemma with $\varepsilon = (C + o(1))\sqrt{(\log n)/d}$.

Lemma A.1. *Let n be an integer and let $\mathcal{A} = \binom{[n]}{2}$. Let \mathbf{d} be a sequence of length n with average d such that $d/(n-1) < 1/4$ and $\sigma^2(\mathbf{d}) = O(d)$, and assume that for all $1 \leq i \leq n$ we have $|d_i - d| \leq \varepsilon d$, where $\varepsilon = \varepsilon(n) > 0$ is bounded above by a sufficiently small constant. Then*

- (a) $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d}) = R_{ab}^{\text{gr}}(\mathbf{d})(1 + O(\eta_1))$ for all $a, b \in [n]$,
 - (b) $\mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) = P_{av}^{\text{gr}}(\mathbf{d})(1 + O(\eta_2))$ for all distinct $a, v \in [n]$,
 - (c) $\mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d}) = Y_{avb}^{\text{gr}}(\mathbf{d})(1 + O(\eta_2))$ for all distinct $a, v, b \in [n]$,
- where $\eta_1 = 1/dn + \varepsilon d/n^2 + \varepsilon^4 d^2/n^2$ and $\eta_2 = 1/dn + \varepsilon/n + \varepsilon^3 d^2/n^2$.

Proof. For completeness, this proof duplicates some steps in the proof of Lemma 7.1. We first reparametrise P^{gr} , R^{gr} and Y^{gr} to facilitate identifying negligible terms. Write $\mu = \mu(\mathbf{d})$ for the “density” $d/(n-1)$ of a graph with degree sequence \mathbf{d} , and note that, by assumption, $\mu \leq 1/4$. Define the sequence $(\varepsilon_1, \dots, \varepsilon_n)$ of relative deviations from the average degree, that is,

$$\varepsilon_a = (d_a - d)/d \quad \text{for } 1 \leq a \leq n.$$

Note that $|\varepsilon_a| \leq \varepsilon$ for each a . Set $\sigma^2 = \sigma^2(\mathbf{d})$ and

$$\begin{aligned} \pi(\varepsilon_a, \varepsilon_v, \mu) &= \mu(1 + \varepsilon_a)(1 + \varepsilon_v) \cdot \left(1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu}\right), \\ \rho(\varepsilon_a, \varepsilon_b, \mu, \sigma^2, d, n) &= \frac{1 + \varepsilon_a}{1 + \varepsilon_b} \cdot \frac{1 - \mu(1 + \varepsilon_b)}{1 - \mu(1 + \varepsilon_a)} \cdot \left(1 + (\varepsilon_a - \varepsilon_b) \frac{\sigma^2}{dn}\right), \end{aligned}$$

Note that with this definition of π we have $P_{av}^{\text{gr}}(\mathbf{d}) = \pi$ for all distinct $a, v \in [n]$. As in Section 7, when referring to π and ρ , we list only an initial segment of parameters containing all those that are different from the ones in the definitions above.

In the calculations below the following approximations of π will often be convenient. Let \mathbf{d}' be a sequence that is at L_1 distance $O(1)$ from \mathbf{d} , and with $d'_a = d_a - j_a$ and $d'_v = d_v - j_v$. For such a sequence, μ becomes

$$\mu' = \mu + O(1/n^2)$$

since d changes by $O(1/n)$. Therefore, the variable ε_a changes to $\varepsilon_a - j_a \delta + O(\mu \delta^2)$, where $\delta = 1/d$, and the analogous equation holds for ε_v . (Here and in the following, the bare symbols μ, d, ε_a and so on are defined with respect to the original sequence \mathbf{d} , whilst μ', d', ε'_a etc. are defined with respect to the average degree of \mathbf{d}' .) Thus

$$\begin{aligned} P_{av}^{\text{gr}}(\mathbf{d}') &= \pi(\varepsilon'_a, \varepsilon'_v, \mu') \\ &= \pi(\varepsilon_a - j_a \delta, \varepsilon_v - j_v \delta)(1 + O(\mu \delta^2)), \end{aligned} \tag{A.1}$$

from which we see that the small changes in ε_b for $b \neq a, v$ have negligible effect.

For (a), if $a = b$ then

$$\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{aa}(\mathbf{d}) = 1 = R_{ab}^{\text{gr}}(\mathbf{d})$$

by definition; see (5.4) and (6.2). So assume $a \neq b$. Using (5.4) to evaluate $\mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d})$, we estimate the expression

$$\mathbf{bad}(P^{\text{gr}}, Y^{\text{gr}})(a, b, \mathbf{d} - \mathbf{e}_b) = \mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b),$$

for which, in turn, we need to estimate $\sum Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b)$, where the sum is over all $v \in [n]$ such that both av and bv are allowable (see (5.1)). By definition (6.3),

$$\begin{aligned} Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) &= P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_a - \mathbf{e}_v)(1 + 1/n) \\ &= \pi \cdot \pi(\varepsilon_b - \delta, \varepsilon_v - \delta)(1 + \mu \delta + O(\mu \delta^2)), \end{aligned} \tag{A.2}$$

by (A.1) and as $\mu\delta = 1/(n-1)$. As $\mathcal{A}(a) \cap \mathcal{A}(b) = [n] \setminus \{a, b\}$, defining $\xi = \mu\delta^2 + \mu\varepsilon\delta$ this gives

$$\begin{aligned}
\frac{1}{d_a} \sum_{v \in \mathcal{A}(a) \cap \mathcal{A}(b)} Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b) &= \frac{1}{d_a} \sum_{\substack{1 \leq v \leq n \\ v \neq a, b}} \pi \cdot \pi(\varepsilon_b - \delta, \varepsilon_v - \delta)(1 + \mu\delta + O(\mu\delta^2)) \\
&= \frac{\mu(1 + \varepsilon_b - \delta)}{n-1} (1 + \mu\delta + O(\xi)) \\
&\quad \times \sum_{\substack{1 \leq v \leq n \\ v \neq a, b}} (1 + \varepsilon_v)(1 + \varepsilon_v - \delta) \left(1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu}\right) \left(1 - \frac{\varepsilon_b \varepsilon_v \mu}{1 - \mu}\right) \\
&= \frac{\mu(1 + \varepsilon_b - \delta)}{n-1} (1 + \mu\delta + O(\xi)) \\
&\quad \times \sum_{\substack{1 \leq v \leq n \\ v \neq a, b}} (1 - \delta + c_1 \varepsilon_v + (1 + O(\varepsilon\mu))\varepsilon_v^2 + O(\varepsilon^4\mu)), \tag{A.3}
\end{aligned}$$

where $c_1 = c_1(\delta, \varepsilon_a, \mu)$ is some suitable function independent of ε_v that satisfies $c_1 = O(1)$. Note that $\sum_{v=1}^n \varepsilon_v = \sum_{v=1}^n (d - d_v)/d = 0$ and $\sum_{v=1}^n \varepsilon_v^2 = n\sigma^2/d^2$, by definition of σ^2 . Hence

$$\begin{aligned}
(1 + \mu\delta + O(\xi)) \sum_{\substack{1 \leq v \leq n \\ v \neq a, b}} (1 - \delta + c_1 \varepsilon_v + (1 + O(\varepsilon\mu))\varepsilon_v^2 + O(\varepsilon^4\mu)) \\
= (1 + \mu\delta)((n-2)(1-\delta) - c_1(\varepsilon_a + \varepsilon_b) + (1 + O(\varepsilon\mu))(n\sigma^2/d^2 - \varepsilon_a^2 - \varepsilon_b^2)) \\
+ O(n\varepsilon^4\mu + n\xi).
\end{aligned}$$

On the other hand, noting that for (5.1) in this case $\mathcal{A}(a) \setminus \mathcal{A}(b) = \{b\}$ (so the first summation only has one term) and $\mathbf{p}_{ab} = P_{ab}^{\text{gr}}$, we compute, using (A.1), that

$$\frac{\mathbf{p}_{ab}(\mathbf{d} - \mathbf{e}_b)}{d_a} = \frac{\pi(\varepsilon_a, \varepsilon_b - \delta)}{\mu(n-1)(1 + \varepsilon_a)} (1 + O(\mu\delta^2)) = \frac{1 + \varepsilon_b}{n-1} (1 + O(\varepsilon^2\mu + \mu\delta^2)).$$

Thus, from (5.1),

bad($a, b, \mathbf{d} - \mathbf{e}_b$)

$$\begin{aligned}
&= \frac{1 + \varepsilon_b}{n-1} + \frac{\mu(1 + \varepsilon_b - \delta)}{n-1} (1 + \mu\delta)((n-1)(1-\delta) - 1 + n\sigma^2/d^2) \\
&\quad + O\left(\mu\delta^2 + \frac{\varepsilon\mu}{n} + \varepsilon^4\mu^2\right) \\
&= \mu(1 + \varepsilon_b) + \frac{\sigma^2}{dn}(1 + \varepsilon_b) - \frac{1}{n} + O\left(\mu\delta^2 + \frac{\varepsilon\mu}{n} + \varepsilon^4\mu^2\right), \tag{A.4}
\end{aligned}$$

where we use $\sigma^2 = O(d)$ and $\delta = 1/d = 1/\mu(n-1)$ and note some non-trivial cancellations. The analogous formula is obtained for **bad**($b, a, \mathbf{d} - \mathbf{e}_a$) by swapping indices.

Hence

$$\begin{aligned} \mathcal{R}(P^{\text{gr}}, Y^{\text{gr}})_{ab}(\mathbf{d}) &= \frac{d_a}{d_b} \cdot \frac{1 - \mathbf{bad}(a, b, \mathbf{d} - \mathbf{e}_b)}{1 - \mathbf{bad}(b, a, \mathbf{d} - \mathbf{e}_a)} \\ &= \frac{1 + \varepsilon_a}{1 + \varepsilon_b} \cdot \frac{1 - \mu(1 + \varepsilon_b)}{1 - \mu(1 + \varepsilon_a)} \cdot \left(1 + (\varepsilon_a - \varepsilon_b) \frac{\sigma^2}{dn} + O\left(\mu\delta^2 + \frac{\varepsilon\mu}{n} + \varepsilon^4\mu^2\right) \right) \\ &= R_{ab}^{\text{gr}}(\mathbf{d}) + O\left(\mu\delta^2 + \frac{\varepsilon\mu}{n} + \varepsilon^4\mu^2\right). \end{aligned}$$

This proves part (a) of the lemma.

To prove part (b) note that, analogous to (A.1) we also have, for $a, b \neq v$,

$$R_{ab}^{\text{gr}}(\mathbf{d} - \mathbf{e}_v) = \rho(\varepsilon'_a, \varepsilon'_b, \mu', (\sigma^2)', d') = \rho \cdot (1 + O(\mu\delta^2)).$$

(In particular, $(\sigma^2)' - \sigma^2 = O(\max |d_i - d|/n) = O(\varepsilon d/n)$.) Therefore, by definition (5.2),

$$\begin{aligned} \mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) &= d_v \left(\sum_{b \in [n] \setminus \{v\}} R_{ba}^{\text{gr}}(\mathbf{d} - \mathbf{e}_v) \frac{1 - P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_b - \mathbf{e}_v)}{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \right)^{-1} \\ &= d_v \left(\sum_{b \in [n] \setminus \{v\}} \rho \cdot \frac{1 - \pi(\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)} (1 + O(\mu\delta^2)) \right)^{-1} \\ &= d_v (1 + \varepsilon_a) \frac{1 - \pi(\varepsilon_a - \delta, \varepsilon_v - \delta)}{1 - \mu(1 + \varepsilon_a)} \left(1 + O\left(\frac{\varepsilon}{n} + \mu\delta^2\right) \right) \\ &\quad \times \left(\sum_{b \in [n] \setminus \{v\}} (1 + \varepsilon_b) \frac{1 - \pi(\varepsilon_b - \delta, \varepsilon_v - \delta)}{1 - \mu(1 + \varepsilon_b)} \right)^{-1}, \quad (\text{A.5}) \end{aligned}$$

where we have used $\sigma^2 = O(d)$ and $\pi(\cdot, \cdot) = O(\mu)$. Straightforward calculations show that for $c \in \{a, b\}$,

$$\begin{aligned} \frac{1 - \pi(\varepsilon_c - \delta, \varepsilon_v - \delta)}{1 - \mu(1 + \varepsilon_c)} &= 1 + \frac{\mu}{1 - \mu} (2\delta - \varepsilon_v - \varepsilon_v \varepsilon_c) + O\left(\frac{\varepsilon}{n} + \varepsilon^3\mu^2 + \mu\delta^2\right) \\ &= \left(1 - \frac{\mu\varepsilon_v\varepsilon_c}{1 - \mu} \right) \left(1 - \frac{\mu(\varepsilon_v - 2\delta)}{1 - \mu} + O\left(\frac{\varepsilon}{n} + \varepsilon^3\mu^2 + \mu\delta^2\right) \right). \quad (\text{A.6}) \end{aligned}$$

Therefore, (A.5) is equivalent to

$$\begin{aligned} \mathcal{P}(P^{\text{gr}}, R^{\text{gr}})_{av}(\mathbf{d}) &= d_v (1 + \varepsilon_a) \left(1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu} \right) \left(1 + O\left(\frac{\varepsilon}{n} + \mu\delta^2 + \varepsilon^3\mu^2\right) \right) \sum_{\substack{1 \leq b \leq n \\ b \neq v}} (1 + \varepsilon_b) \left(1 - \frac{\varepsilon_b \varepsilon_v \mu}{1 - \mu} \right) \\ &= \mu(1 + \varepsilon_v)(1 + \varepsilon_a) \left(1 - \frac{\varepsilon_a \varepsilon_v \mu}{1 - \mu} \right) \left(1 + O\left(\frac{\varepsilon}{n} + \mu\delta^2 + \varepsilon^3\mu^2\right) \right) \\ &= \pi \cdot \left(1 + O\left(\frac{\varepsilon}{n} + \mu\delta^2 + \varepsilon^3\mu^2\right) \right), \end{aligned}$$

where in the second equality we use $\sum_b \varepsilon_b = 0$ and $\sum_b \varepsilon_b^2 = O(1/\mu)$.

For part (c) we have, by definition of \mathcal{Y} in (5.3),

$$\mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d}) = \frac{P_{av}^{\text{gr}}(\mathbf{d})(P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) - Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v))}{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \quad (\text{A.7})$$

for distinct $a, v, b \in [n]$. By definition of Y^{gr} in (6.3), and (A.1), we obtain as in (A.2)

$$\begin{aligned} Y_{avb}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) &= P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)P_{bv}^{\text{gr}}(\mathbf{d} - 2\mathbf{e}_a - 2\mathbf{e}_v)(1 + 1/n) \\ &= \pi(\varepsilon_a - \delta, \varepsilon_v - \delta) \cdot \pi(\varepsilon_b, \varepsilon_v - 2\delta)(1 + \mu\delta + O(\xi)) \\ &= \pi(\varepsilon_b, \varepsilon_v - \delta) \cdot \pi(\varepsilon_a - \delta, \varepsilon_v - 2\delta)(1 + \mu\delta + O(\xi)) \\ &= P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - 2\mathbf{e}_v)(1 + \mu\delta + O(\xi)). \end{aligned}$$

Plugging this into (A.7) we obtain

$$\begin{aligned} \mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d}) &= P_{av}^{\text{gr}}(\mathbf{d})P_{bv}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v) \frac{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - 2\mathbf{e}_v)(1 + \mu\delta + O(\xi))}{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)}. \end{aligned}$$

Now straightforward calculations show that

$$\begin{aligned} &\frac{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - 2\mathbf{e}_v)(1 + \mu\delta + O(\xi))}{1 - P_{av}^{\text{gr}}(\mathbf{d} - \mathbf{e}_a - \mathbf{e}_v)} \\ &= \frac{1 - \mu(1 + \varepsilon_a - \delta)(1 + \varepsilon_v - 2\delta)\left(1 + \frac{\varepsilon_a\varepsilon_v\mu}{1-\mu} + \mu\delta + O(\xi)\right)}{1 - \mu(1 + \varepsilon_a - \delta)(1 + \varepsilon_v - \delta)\left(1 + \frac{\varepsilon_a\varepsilon_v\mu}{1-\mu} + O(\xi)\right)} \\ &= \frac{1 - B + \delta\mu - \mu/n + O(\xi)}{1 - B} \\ &= 1 + \mu\delta + O(\xi), \end{aligned}$$

where

$$B = \mu(1 + \varepsilon_a - \delta)(1 + \varepsilon_v - \delta)\left(1 + \frac{\varepsilon_a\varepsilon_v\mu}{1-\mu}\right) = \mu + O(\mu\varepsilon).$$

Thus it follows that $\mathcal{Y}(P^{\text{gr}}, Y^{\text{gr}})_{avb}(\mathbf{d}) = Y_{avb}^{\text{gr}}(\mathbf{d})(1 + O(\xi))$ and we are done. \blacksquare

Acknowledgements. We would like to thank Nina Kamčev for suggesting the use of the variable Y in the recursive equations along with P and R . Our original proof of the main theorem in an earlier arXiv version [26] of this paper was able to restrict to two variables, but at the expense of some extra complexity. This research was done when the first author was affiliated with the School of Mathematics, Monash University, Melbourne, VIC 3800, Australia.

Funding. The first author was partially supported by a DECRA Fellowship from the Australian Research Council. The second author was supported by an ARC Australian Laureate Fellowship.

References

- [1] Barbour, A. D.: [Poisson convergence and random graphs](#). Math. Proc. Cambridge Philos. Soc. **92**, 349–359 (1982) Zbl 0498.60016 MR 671189

- [2] Barbour, A. D., Holst, L., Janson, S.: *Poisson approximation*. Oxford Studies in Probability 2, The Clarendon Press, Oxford University Press, New York (1992) Zbl 0746.60002 MR 1163825
- [3] Barbour, A. D., Karoński, M., Ruciński, A.: [A central limit theorem for decomposable random variables with applications to random graphs](#). J. Combin. Theory Ser. B **47**, 125–145 (1989) Zbl 0689.05042 MR 1047781
- [4] Barvinok, A., Hartigan, J. A.: [The number of graphs and a random graph with a given degree sequence](#). Random Structures Algorithms **42**, 301–348 (2013) Zbl 1264.05125 MR 3039682
- [5] Bauerschmidt, R., Knowles, A., Yau, H.-T.: [Local semicircle law for random regular graphs](#). Comm. Pure Appl. Math. **70**, 1898–1960 (2017) Zbl 1372.05194 MR 3688032
- [6] Bender, E. A., Canfield, E. R.: [The asymptotic number of labeled graphs with given degree sequences](#). J. Combin. Theory Ser. A **24**, 296–307 (1978) Zbl 0402.05042 MR 505796
- [7] Blitstein, J., Diaconis, P.: [A sequential importance sampling algorithm for generating random graphs with prescribed degrees](#). Internet Math. **6**, 489–522 (2010) Zbl 1238.60084 MR 2809836
- [8] Bollobás, B.: [A probabilistic proof of an asymptotic formula for the number of labelled regular graphs](#). Eur. J. Combin. **1**, 311–316 (1980) Zbl 0457.05038 MR 595929
- [9] Bollobás, B.: [Vertices of given degree in a random graph](#). J. Graph Theory **6**, 147–155 (1982) Zbl 0499.05056 MR 655200
- [10] Bollobás, B.: [Random graphs](#). 2nd ed., Cambridge Studies in Advanced Mathematics 73, Cambridge University Press, Cambridge (2001) Zbl 0979.05003 MR 1864966
- [11] Burstein, D., Rubin, J.: [Degree switching and partitioning for enumerating graphs to arbitrary orders of accuracy](#). arXiv:1511.03738 (2015)
- [12] Cain, J., Wormald, N.: [Encores on cores](#). Electron. J. Combin. **13**, art. 81, 13 pp. (2006) Zbl 1112.05094 MR 2255423
- [13] Chen, Y., Diaconis, P., Holmes, S. P., Liu, J. S.: [Sequential Monte Carlo methods for statistical analysis of tables](#). J. Amer. Statist. Assoc. **100**, 109–120 (2005) Zbl 1117.62310 MR 2156822
- [14] Dudek, A., Frieze, A., Ruciński, A., Šileikis, M.: [Embedding the Erdős–Rényi hypergraph into the random regular hypergraph and Hamiltonicity](#). J. Combin. Theory Ser. B **122**, 719–740 (2017) Zbl 1350.05110 MR 3575225
- [15] Erdős, P., Rényi, A.: [On random graphs. I](#). Publ. Math. Debrecen **6**, 290–297 (1959) Zbl 0092.15705 MR 120167
- [16] Erdős, P., Rényi, A.: [On the strength of connectedness of a random graph](#). Acta Math. Acad. Sci. Hungar. **12**, 261–267 (1961) Zbl 0103.16302 MR 130187
- [17] Gao, P., Wormald, N.: [Enumeration of graphs with a heavy-tailed degree sequence](#). Adv. Math. **287**, 412–450 (2016) Zbl 1327.05155 MR 3422681
- [18] Isaev, M., McKay, B. D.: [Complex martingales and asymptotic enumeration](#). Random Structures Algorithms **52**, 617–661 (2018) Zbl 1396.05010 MR 3809691
- [19] Janson, S.: [The probability that a random multigraph is simple](#). Combin. Probab. Comput. **18**, 205–225 (2009) Zbl 1216.05145 MR 2497380
- [20] Janson, S.: [The probability that a random multigraph is simple. II](#). J. Appl. Probab. **51A**, 123–137 (2014) Zbl 1309.05162 MR 3317354
- [21] Janson, S., Łuczak, T., Ruciński, A.: [Random graphs](#). Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York (2000) Zbl 0968.05003 MR 1782847
- [22] Karoński, M., Ruciński, A.: [Poisson convergence and semi-induced properties of random graphs](#). Math. Proc. Cambridge Philos. Soc. **101**, 291–300 (1987) Zbl 0627.60016 MR 870602

- [23] Kim, J. H., Vu, V. H.: [Sandwiching random graphs: universality between random graph models](#). *Adv. Math.* **188**, 444–469 (2004) Zbl [1050.05111](#) MR [2087234](#)
- [24] Koren, M.: [Extreme degree sequences of simple graphs](#). *J. Combin. Theory Ser. B* **15**, 213–224 (1973) Zbl [0268.05123](#) MR [329967](#)
- [25] Krivelevich, M., Sudakov, B., Vu, V. H., Wormald, N. C.: [Random regular graphs of high degree](#). *Random Structures Algorithms* **18**, 346–363 (2001) Zbl [0996.05106](#) MR [1839497](#)
- [26] Liebenau, A., Wormald, N.: [Asymptotic enumeration of graphs by degree sequence, and the degree sequence of a random graph](#). arXiv:[1702.08373v2](#) (2018)
- [27] McDiarmid, C.: [On the method of bounded differences](#). In: *Surveys in combinatorics, 1989* (Norwich, 1989), London Mathematical Society Lecture Note Series 141, Cambridge University Press, Cambridge, 148–188 (1989) Zbl [0712.05012](#) MR [1036755](#)
- [28] McKay, B. D.: [Asymptotics for symmetric 0-1 matrices with prescribed row sums](#). *Ars Combin.* **19**, 15–25 (1985) Zbl [0568.05032](#) MR [790916](#)
- [29] McKay, B. D.: [Subgraphs of random graphs with specified degrees](#). In: *Proceedings of the International Congress of Mathematicians* (Hyderabad, 2010), Volume IV, Hindustan Book Agency, New Delhi, 2489–2501 (2010) Zbl [1231.05238](#) MR [2827981](#)
- [30] McKay, B. D., Wormald, N. C.: [Asymptotic enumeration by degree sequence of graphs of high degree](#). *Eur. J. Combin.* **11**, 565–580 (1990) Zbl [0739.05043](#) MR [1078713](#)
- [31] McKay, B. D., Wormald, N. C.: [Asymptotic enumeration by degree sequence of graphs with degrees \$o\(n^{1/2}\)\$](#) . *Combinatorica* **11**, 369–382 (1991) Zbl [0742.05047](#) MR [1137769](#)
- [32] McKay, B. D., Wormald, N. C.: [The degree sequence of a random graph. I. The models](#). *Random Structures Algorithms* **11**, 97–117 (1997) Zbl [0884.05081](#) MR [1610253](#)
- [33] Palka, Z.: [On the number of vertices of given degree in a random graph](#). *J. Graph Theory* **8**, 167–170 (1984) Zbl [0532.05052](#) MR [732030](#)
- [34] Palka, Z.: [Extreme degrees in random graphs](#). *J. Graph Theory* **11**, 121–134 (1987) Zbl [0672.05069](#) MR [889344](#)
- [35] Pittel, B., Wormald, N. C.: [Counting connected graphs inside-out](#). *J. Combin. Theory Ser. B* **93**, 127–172 (2005) Zbl [1057.05044](#) MR [2117934](#)
- [36] Read, R.: [Some enumeration problems in graph theory](#). Ph.D. thesis, University of London (1958)
- [37] Stein, C.: [A way of using auxiliary randomization](#). In: *Probability theory* (Singapore, 1989), de Gruyter, Berlin, 159–180 (1992) Zbl [0759.62008](#) MR [1188718](#)
- [38] Stein, C. M.: [Asymptotic evaluation of the number of Latin rectangles](#). *J. Combin. Theory Ser. A* **25**, 38–49 (1978) Zbl [0429.05022](#) MR [499035](#)
- [39] Wormald, N.: [Some problems in the enumeration of labelled graphs](#). Ph.D. thesis, University of Newcastle (1978)