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On the realization space of the cube

Abstract. We prove that the realization space of the d-dimensional cube is contractible. For this we first show that any two realizations are connected by a finite sequence of projective transformations and normal transformations. As an application we use this fact to define an analog of the connected sum construction for cubical d-polytopes, and apply this construction to certain cubical d-polytopes to conclude that the rays spanned by f-vectors of cubical d-polytopes are dense in Adin's cone. The connectivity result on cubes extends to any product of simplices, and further it shows that the respective realization spaces are contractible.

Keywords. Cubical polytopes, realization space, face numbers, connected sum

1. Introduction

Perhaps the most natural transformations of polytopes that preserve the combinatorial type, namely the facial structure, are projective transformations and normal transformations. Loosely speaking, the former are given by perspective transformation from one hyperplane where the polytope lies to another hyperplane, while the latter are given by scaling the outer normal vectors to facets so that facets do not degenerate. While the former are connected to the projective linear group acting on vector spaces, the latter is connected to the Chow cohomology of toric varieties, and in particular inherits an algebra structure via the Minkowski sum [5]. (By *polytope* we always mean a convex polytope.)

The simplex, of any fixed dimension, is *projectively unique*, namely, any simplex can be continuously transformed to any other simplex of the same dimension by a homotopy of projective transformations. Thus, any two *simplicial* polytopes, after applying an appropriate projective transformation to one of them, can be glued along a common facet

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whose hyperplane separates them, to produce again a convex polytope. This realizes the connected sum operation geometrically.

However, the d-cube is not projectively unique for $d \geq 3$; this can be seen even by dimension count: the realization space of the (combinatorial) d-cube has dimension larger than the dimension of the space of projective transformations. Indeed, the group of projective transformations on $\mathbb{R}^d$ is of dimension $d(d+2)$, while the realization space of the d-cube has dimension $2d^2$. In particular, we cannot realize the connected sum operation geometrically for cubical d-polytopes, $d \geq 4$.

We enlarge the set of transformations by adding normal transformations to the generating set. While the first author mentioned this theorem in passing, assuming it had to be known, it was to our surprise that the following results appear to be new, even the qualitative assertion in (a).

Theorem 1 (Cubes are normal-projectively unique). *Fix a dimension d.*

(a) *For any two realizations of the d-cube, one can be obtained from the other by a composition of finitely many transformations, each either projective or normal. In fact, $8d$ transformations suffice.*

(b) *The constructed algorithm transforms cubes continuously to the standard cube. In particular, we obtain a deformation retraction to a point. Thus, the realization space of cubes is contractible.*

Let us stress that we stay entirely inside the space of cubes. Every transformation takes us from one cube to another; none of the projective transformations results in an unbounded polytope.

As a corollary of the quantitative assertion in (a), we obtain a cubical analog of the connected sum construction, at a small price.

Theorem 2. (a) (Bounded towers) *For any two realizations C_1 and C_2 of the $(d-1)$-cube, there exists a cubical d-polytope C made up of m ($m \leq 4d$) d-cubes stacked one on top of the other, such that C_1 and C_2 are projectively equivalent to its bottom and top facets, resp. Call C a d-tower of m cubes.*

(b) (Cubical connected sum) *For any two cubical d-polytopes P_1 and P_2, and facets F_i of P_i, $i = 1, 2$, there exists a projective transformation ϕ and a d-tower T of at most $4d$ cubes such that $P := P_1 \cup T \cup \phi(P_2)$ is convex, and $P_1 \cap T = F_1$ and $\phi(P_2) \cap T = \phi(F_2)$ are the top and bottom facets of T respectively.*

We apply this cubical connected sum operation to the cubical polytopes constructed recently in [2]; the f-vectors of the latter approach the extremal rays of Adin's cone, which is conjectured to contain all f-vectors of d-polytopes [1]. The following density result for f-vectors of cubical polytopes then follows: Let $\square^d$ denote the d-cube and $f(P)$ denote the f-vector of a polytope P. Let $\mathcal{A}_d$ be the Adin cone (its apex is $f(\square^d)$ and its dimension is $\lfloor d/2 \rfloor$ by the cubical Dehn–Sommerville relations [1]).

Theorem 3 (Ray density in Adin's cone). *For any $\varepsilon > 0$ and any $x \in \mathcal{A}_d$ there exists a cubical d-polytope P such that the angle $\measuredangle x f(\Box^d) f(P)$ is smaller than ε.*

Lastly, we note that our cubical connected sum construction endows the set of f-vectors of cubical d-polytopes with the structure of an affine semigroup (see [9]).

Outline. In Section 2 we give preliminaries; we prove Theorem 1 in Section 3, Theorem 2 in Section 4 and Theorem 3 in Section 5. We conclude with generalizations and related open questions in Section 6.

2. Preliminaries

For further background on polytopes see e.g. [8].

2.1. Two notions of equivalence of d-polytopes

Let $P = \{x \in \mathbb{R}^d \mid Ax \leq b\}$ be a d-polytope, with the origin in its *interior* P°. Denote by $r_1, \ldots, r_m$ the rows of A. By scaling we may assume these are the *facet outer normals*. The *polar polytope*

$$P^\triangle = \{y \in \mathbb{R}^d \mid \langle y, x \rangle \leq 1 \text{ for all } x \in P\} = \operatorname{conv}(p_1, \ldots, p_m)$$

is the d-polytope with vertices $p_1 = \frac{1}{b_1} r_1, \ldots, p_m = \frac{1}{b_m} r_m$.

A *projective transformation* is a map

$$\varphi : \mathbb{R}^d \to \mathbb{R}^d \quad \text{defined by} \quad x \mapsto \frac{Ax + b}{c^T x + \alpha},$$

where $(\cdot)^T$ denotes the transpose, for some $A \in M_{d \times d}(\mathbb{R})$, $b, c \in \mathbb{R}^d$, and $\alpha \in \mathbb{R}$ that satisfy

$$\det \begin{pmatrix} A & b \\ c^T & \alpha \end{pmatrix} \neq 0.$$

These transformations form a group under composition.

Definition 4. Two d-polytopes P and Q are *projectively equivalent* if there is a projective transformation φ such that $Q = \varphi(P)$.

We will need another notion of equivalence:

Definition 5. Two combinatorially equivalent d-polytopes P and Q are *normally equivalent* if additionally their unit facet normals of corresponding facets coincide.[1]

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[1] In the literature sometimes a weaker notion of normal equivalence occurs, when the combinatorial isomorphism condition is dropped, and one requires only that the set of unit outer normals of P coincides with that of Q. We however need the stronger notion in the definition, which is commonly used.

In this case we say $Q = \psi(P)$ for a normal transformation ψ. Thus, given a polytope P, any two polytopes normally equivalent to it differ by a normal transformation. On the dual polytopes $P^{\triangle}$, $Q^{\triangle}$ we say $Q^{\triangle} = \psi^{\triangle}(P^{\triangle})$ for a *ray transformation* $\psi^{\triangle}$ (it scales the vertices along the rays from the origin while preserving the combinatorial type).

2.2. Connected sums of d-polytopes

Suppose P and Q are d-polytopes whose intersection is a common facet $F = P \cap Q$ of both. If $R = P \cup Q$ is convex then its proper faces are precisely the proper faces of either P or Q, excluding F:

$$\mathrm{faces}(R) = (\mathrm{faces}(P) \cup \mathrm{faces}(Q)) \setminus \{F\}.$$

The following lemma, a proof of which can be found in [7, Lemma 3.2.4], tells us when and how the connected sum of two polytopes can be formed.

Lemma 6. *Let P and Q be d-polytopes that have projectively equivalent facets F_1 and F_2 respectively. Then there exists a projective transformation φ such that $P \cap \varphi(Q) = F_1 = \varphi(F_2)$ and $R = P \cup \varphi(Q)$ is convex.*

The combinatorial type of R in Lemma 6 is called the *connected sum* of P and Q along F_1 and F_2, denoted $P \#_{F_1 \sim F_2} Q$, or simply $P \#_F Q$ when the faces F_1, F_2 combinatorially isomorphic to F are understood.

2.3. Cubical polytopes

We give just a brief reminder of the definitions of a cubical d-polytope and its h^c-vector and g^c-vector. For more details, in particular, for the construction used in Section 5, see [2].

A d-polytope Q is *cubical* if each of its proper faces is combinatorially a cube. Its *f-polynomial* is defined by (note the shift of index!)

$$f(Q, t) = \sum_{i=0}^{d-1} f_i t^i$$

where $f_i = f_i(Q)$ is the number of i-dimensional faces of Q.

We then define the *short cubical h-polynomial*

$$h^{\mathrm{sc}}(Q, t) = (1-t)^{d-1} f\left(Q, \frac{2t}{1-t}\right),$$

and the *cubical h-polynomial*

$$h^c(Q, t) = \sum_{i=0}^{d} h_i^c t^i = \frac{t(1-t)^{d-1}}{1+t} f\left(Q, \frac{2t}{1-t}\right) + 2^{d-1} \frac{1-(-t)^{d+1}}{1+t}.$$

Adin [1] has shown that $h^c(Q, t)$ is symmetric, that is,

$$h_i^c = h_{d-i}^c \quad (0 \le i \le d).$$

These $\lceil d/2 \rceil$ equations are the *cubical Dehn–Sommerville relations*. We thus define the *cubical g-vector* $g^c(Q) = (g_0^c, \ldots, g_{\lfloor d/2 \rfloor}^c)$ by

$$g_0^c = h_0^c = 2^{d-1}, \quad g_i^c = h_i^c - h_{i-1}^c \quad \text{for } 1 \le i \le \lfloor d/2 \rfloor.$$

Adin conjectured

Conjecture 7 ([1, Question 2]). *For a cubical d-polytope we have*

$$g_i^c \ge 0 \quad (1 \le i \le \lfloor d/2 \rfloor). \tag{2.1}$$

The cone (2.1) is the nonnegative orthant in $\mathbb{R}^{\lfloor d/2 \rfloor}$, and its image under the map transforming g^c-vectors back into f-vectors yields the Adin cone $\mathcal{A}_d$ in $\mathbb{R}^d$.

In [2], for each $1 \le i \le \lfloor d/2 \rfloor$, the authors exhibit a sequence of cubical d-polytopes whose corresponding sequence of g^c-vectors approaches the ray spanned by e_i. This translates into sequences of f-vectors approaching the extremal rays of $\mathcal{A}_d$.

3. Any two combinatorial d-cubes are related by normal and projective transformations

We will use the following lemma, which describes the effect of a projective transformation on the polar polytope.

Lemma 8. *Let P be a d-polytope with $0 \in P^\circ$, and $P^\triangle = \text{conv}(p_1, \ldots, p_m)$. Then for any $v \in P^\circ$ there exists a d-polytope Q which is projectively equivalent to P, and $Q^\triangle = \text{conv}(p_1 + v, \ldots, p_m + v)$.*

Proof. Consider the effect of a projective transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ that takes P to Q (with $0 \in Q^\circ$) on the polar polytopes $P^\triangle$ and $Q^\triangle$. It is easy check that the map

$$\varphi^\triangle : \mathbb{R}^d \to \mathbb{R}^d \quad \text{defined by} \quad x \mapsto \frac{A^T x - c}{-b^T x + \alpha}$$

is a projective transformation that satisfies

$$\varphi^\triangle(Q^\triangle) = P^\triangle.$$

Denote $\varphi^{-\triangle} = (\varphi^\triangle)^{-1}$, so that

$$Q^\triangle = \varphi^{-\triangle}(P^\triangle).$$

Taking $A = I_{d \times d}$, $b = 0$, $c = v$, and $\alpha = 1$ produces a projective transformation φ for which

$$\varphi^{-\triangle}(x) = x + v$$

and the claim follows. ∎

Let $Q = \{x \in \mathbb{R}^d \mid Ax \le b\}$ be a combinatorial d-cube, with the origin in its interior, and $r_1, \ldots, r_{2d}$ the rows of A, that is, the facet outer normals. We may assume that they are ordered by pairs of combinatorially opposite facets, that is, r_i is normal to a facet opposite to the facet normal to r_{i+1}, for $i = 1, 3, \ldots, 2d - 1$. The polar polytope $Q^\triangle$ is the combinatorial d-crosspolytope with vertices $p_1 = \frac{1}{b_1} r_1, \ldots, p_{2d} = \frac{1}{b_{2d}} r_{2d}$, and we denote by $l_i = [p_{2i-1}, p_{2i}]$, for $1 \le i \le d$, the line segments connecting the pairs of opposite vertices. The following proposition proves Theorem 1 (a).

Proposition 9. *Let Q and Q' be two combinatorial d-cubes. Then there is a sequence $\phi_1, \ldots, \phi_s$ ($s \le 8d - 1$) of projective and normal transformations such that*

$$Q' = (\phi_s \circ \cdots \circ \phi_1)(Q).$$

Proof. We present the sequence in terms of the polar d-crosspolytopes. For each pair of antipodal vertices of $P := Q^\triangle$ we perform a sequence of four transformations, alternating between projective and ray transformations, arriving at a d-crosspolytope which is ray equivalent to the standard d-crosspolytope, namely the convex hull of the standard basis elements and their minuses. We refer to the sequence of four transformations for the i-th pair of antipodal vertices as the *i-th iteration*. We denote the crosspolytope obtained after the i-th iteration by $P^{(i)}$, its vertices by $p_1^{(i)}, \ldots, p_{2d}^{(i)}$ and the line segments connecting its pairs of opposite vertices $p_{2j-1}^{(i)}, p_{2j}^{(i)}$ by $l_j^{(i)}$ for $j = 1, \ldots, d$.

(1) Use Lemma 8 to translate the crosspolytope $P^{(i-1)}$ so that the origin lies in the interior of the line segment $l_i^{(i-1)} = [p_{2i-1}^{(i-1)}, p_{2i}^{(i-1)}]$, say in its midpoint to make a canonical choice; this projective transformation produces a polytope P'.

(2) For $P' = \{x \mid Ax \le b\}$ with vertex notation as in $P^{(i-1)}$, choose $c_{2i-1} \ge 1/b_{2i-1}$ so that there exists an affine hyperplane H_i orthogonal to l_i, which strictly separates $q_{2i-1} := c_{2i-1} r_{2i-1}$ from $\mathrm{Vert}(P') \setminus \{p_{2i-1}\}$. To make a canonical choice, let c be the infimum of the possible values for such c_{2i-1}'s, fix $c_{2i-1} = c + 1$ and fix the H_i as above that intersects the ray spanned by q_{2i-1} at $(c + 0.5) r_{2i-1}$.

Set $P'' = \mathrm{conv}(q_{2i-1} \cup P')$. Then P'' is ray equivalent to P'.

(3) Again denote the vertices of P'' by p_i, in correspondence with the vertices of $P^{(i-1)}$, so $p_{2i-1} = q_{2i-1}$. Use again Lemma 8 to move the origin close enough to p_{2i-1} along the segment l_i, that is, so that the origin and p_{2i-1} are on the same side of the hyperplane H_i of step (2). To make a canonical choice, move the origin to $(c + 0.7) r_{2i-1}$. Then

$$H_i \cap P'' \cong \mathrm{conv}(H_i \cap \mathrm{Span}(p_j) \mid j \in [2d] \setminus \{2i - 1, 2i\}).$$

(Here $\cong$ means combinatorially equivalent). The resulting polytope P''' is projectively equivalent to P''.

(4) Set $q_j := H_i \cap \mathrm{Span}(p_j)$ for $j \in [2d] \setminus \{2i - 1, 2i\}$. Then

$$P^{(i)} = \mathrm{conv}(q_1, \ldots, q_{2i-2}, p_{2i-1}, p_{2i}, q_{2i+1}, \ldots, q_{2d})$$

is ray equivalent to P'''.

The resulting crosspolytope $P^{(i)}$ has the property that all line segments except $l_i^{(i)}$ lie on the hyperplane H_i which is orthogonal to $l_i^{(i)}$. Furthermore, for all previous line segments $l_1^{(i)}, \ldots, l_{i-1}^{(i)}$, the same property, achieved at the previous iterations, i.e., all other line segments lie on the respective hyperplanes $H_1, \ldots, H_{i-1}$, still holds, because (i) these hyperplanes are spanned by rays, and the new points we choose at the i-th iteration are on the rays, and further (ii) the first up to $(i-1)$-th segments are just translated in the i-th iteration.

After performing this process for every pair of antipodal vertices we obtain a combinatorial d-crosspolytope with segments $l_1, \ldots, l_d$ such that, for each $1 \leq i \leq d$, there exists an affine hyperplane H_i which is orthogonal to l_i and contains all other segments l_j, $j \neq i$. It follows that the segments $l_1, \ldots, l_d$ all intersect in a point, and are pairwise orthogonal.

To see that the line segments are pairwise orthogonal, note that if the line segment l_i was orthogonal to the line segment l_j before performing the i-th iteration, then the new line segment $l_i^{(i)}$ is orthogonal to the new line segment $l_j^{(i)}$.

To see that all line segments intersect in a point, consider the affine space spanned by the line segments, constructed sequentially. We start with some line segment (it spans an affine space of dimension 1), then add a second line segment, which can raise the dimension by 0, 1 or 2, and so on. Note that at each step the dimension cannot grow by 0 (because each line segment is orthogonal to the space spanned by all other segments), and since we have d line segments in $\mathbb{R}^d$ the total dimension is at most d, so at each step the dimension cannot grow by 2 either, thus, at each step, the dimension grows by 1. Note that this argument is valid for any ordering of the line segments, so any two of the segments intersect in a point. In view of their pairwise orthogonality, this is the same point for all pairs of segments.

We perform the same procedure for $Q'^{\triangle}$ to get a combinatorial d-crosspolytope which is normally equivalent to the standard d-crosspolytope. To finish, we do a final normal transformation to concatenate the two sequences of transformations performed on Q and on Q'. In fact, the resulting three normal transformations in a row can be replaced by a single one. This algorithm gives $s = 8d - 1$. ∎

In the following figure we give an illustration of a single iteration for an octahedron. The line segment $l = vv'$ is colored black. The red lines represent the rays from the origin (the red point) on which the vertices lie.

Perform ϕ_1: the origin now lies on l.

Perform ϕ_2: the vertex v is moved along its ray so that a hyperplane as in the next step exists.

Perform ϕ_3: the hyperplane H_l is orthogonal to l, and separates v and the origin from the other vertices.

Perform ϕ_4: the resulting octahedron has the property that all diagonals, besides l, lie on a hyperplane which is orthogonal to l.
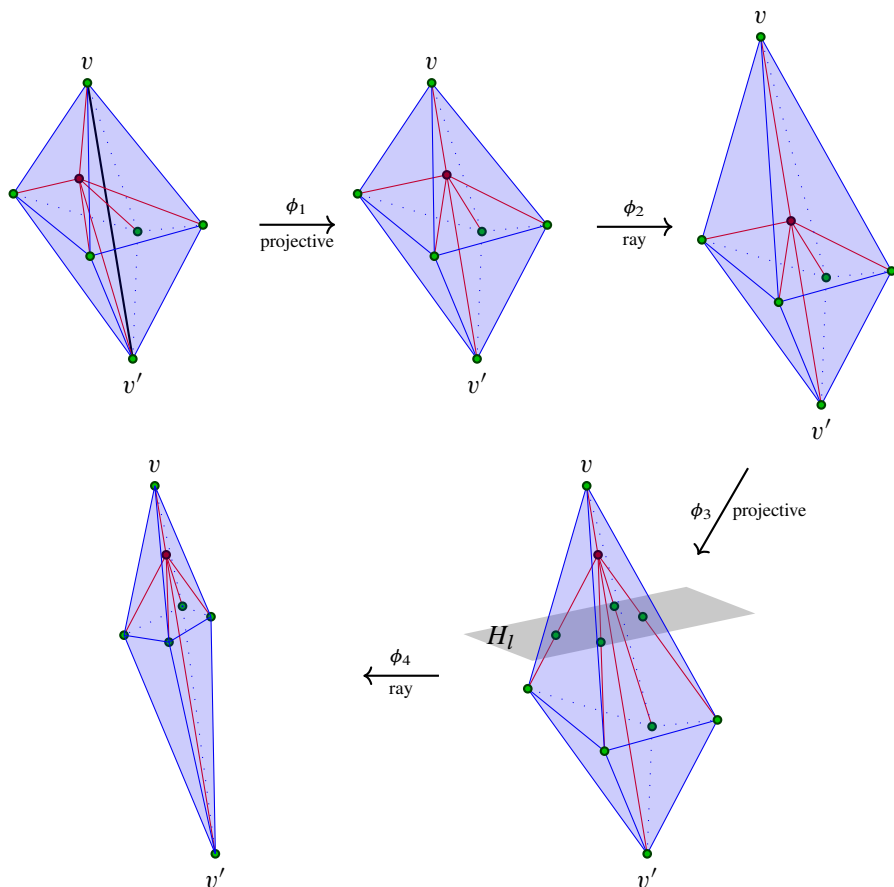
We now deduce Theorem 1 (b), that is, we show that the realization space R is contractible. Consider the point p in R corresponding to the standard cube. Consider the sequence of $4d + 2$ transformations that take the point $x \in R$ to the point p: after performing the duals of the first $4d$ transformations in the above proof, of the last two transformations, the first is normal that changes a box to a unit cube, and the second is projective, in fact an isometry to the standard cube (axis parallel, unit, with center of mass at the origin). This sequence can be seen as a continuous path in R from x to p, where we take the dual of each translation and each ray scaling done linearly in unit time; likewise for the last two transformations. The resulting map

$$f : R \times [0, 4d + 2] \to R$$

is a homotopy from the identity map on R to the constant map with value p: it is indeed continuous by the canonical choices in steps (1)–(4) for each iteration of the algorithm in the proof of Proposition 9. (If y is a cube nearby x, then for any vertex in x its unique nearby vertex in y gets the same combinatorial labeling. Given the algorithm for x, this determines the algorithm for y, and hence the path from y to p in R.) ∎

4. A cubical connector d-polytope and the C-connected sum

Definition 10. A d-*tower* of s cubes is a cubical stacked d-polytope T obtained by stacking on the facet opposite to the facet stacked on in the previous step.

Explicitly, for $s = 1$ it is just a d-cube. Mark some two opposite facets as bottom and top. For $s > 1$, a d-tower of s cubes is obtained from a d-tower of $s - 1$ cubes with bottom facet F and top facet F' by stacking a d-cube onto F'. Then the polytope T has a unique *bottom* facet and a unique *top* facet.

Given two combinatorial $(d - 1)$-cubes Q_1 and Q_2, we use Proposition 9 to construct a d-tower having bottom facet Q'_1 and top facet Q'_2, with Q'_i projectively equivalent to Q_i, $i = 1, 2$. The following lemma shows how to translate each normal transformation from Proposition 9 into a d-tower of one cube.

Lemma 11. *Let Q_1 and Q_2 be two combinatorial $(d - 1)$-cubes which are normally equivalent. Then there exists a d-cube Q in which Q_1 and Q_2 (both realized in $\mathbb{R}^d$) are opposite facets.*

Proof. Assume that both Q_1 and Q_2 are realized in $\mathbb{R}^d$ on the last coordinate $= 0$ hyperplane. Lift the vertices of Q_2 (say to height 1), and take the convex hull, denoted by Q.

Here is an explicit description of Q: Let $A_1, A_2 \in \mathbb{R}^{(2d-2)\times(d-1)}$ and $b_1, b_2 \in \mathbb{R}^{2d-2}$ be such that

$$Q_1 = \{x \in \mathbb{R}^{d-1} \mid A_1 x \le b_1\}, \quad Q_2 = \{x \in \mathbb{R}^{d-1} \mid A_2 x \le b_2\}.$$

The fact that Q_1 and Q_2 are normally equivalent means that $A_1 = A_2$. We define

$$A = \begin{pmatrix} & A_1 & & b_1 - b_2 \\ \hline 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \hline 1 \\ 0 \end{pmatrix}, \tag{4.1}$$

and

$$Q = \{x \in \mathbb{R}^d \mid Ax \le b\}. \qquad \blacksquare$$

Applying Lemma 11 to each of the normal transformations in Proposition 9, and Lemma 6 to glue each such new d-cube to the previously constructed polytope so that the result is again a convex polytope, we deduce Theorem 2 (here s is as in Proposition 9):

Corollary 12. *Let Q and Q' be two combinatorial $(d - 1)$-cubes. Then there is a d-tower of $4d$ cubes with bottom facet projectively equivalent to Q and top facet projectively equivalent to Q'. We call this tower a* cubical connector *and denote it $C(Q, Q')$.*

Definition 13. Let Q_1 and Q_2 be cubical d-polytopes. Let F_1 be a facet of Q_1, F_2 a facet of Q_2, and $C = C(F_1, F_2)$ an appropriate cubical connector (a tower of $4d$ cubes). The **C**-*connected sum* $Q = Q_1 \# Q_2$ is the cubical d-polytope obtained by taking the connected sum $Q_1 \#_{F_1} C \#_{F_2} Q_2$.

5. Filling the g^c-cone

We apply the connected sum construction to appropriate AKN-polytopes (see [2]), thus obtaining sequences of cubical d-polytopes with corresponding g^c-vector sequences approaching any ray in the nonnegative orthant of $\mathbb{R}^{\lfloor d/2 \rfloor}$.

Lemma 14. *Let $Q_1 \,\#\, Q_2$ be a C-connected sum. Then*

$$
\begin{aligned}
g_1^c(Q_1 \,\#\, Q_2) &= g_1^c(Q_1) + g_1^c(Q_2) + (4d+1)2^{d-1}, \\
g_i^c(Q_1 \,\#\, Q_2) &= g_i^c(Q_1) + g_i^c(Q_2) \quad (2 \le i \le \lfloor d/2 \rfloor).
\end{aligned}
\tag{5.1}
$$

Proof. Let us first observe that for the (usual) connected sum $Q \,\#_F\, Q'$, when Q and Q' are cubical d-polytopes we have

$$
f(Q \,\#_F\, Q', t) = f(Q, t) + f(Q', t) - f(\square^{d-1}, t) - t^{d-1}
$$

and

$$
h^{sc}(Q \,\#_F\, Q', t) = (1-t)^d f\left(Q \,\#_F\, Q', \frac{2t}{1-t} \right).
$$

It follows that

$$
h^{sc}(Q \,\#_F\, Q', t) = h^{sc}(Q, t) + h^{sc}(Q', t) + (t-1)(h^{sc}(\square^{d-1}, t) + 2^{d-1}t^{d-1}),
$$

and so, for $1 \le i \le d - 2$, we have

$$
\begin{aligned}
h_i^{sc}(Q \,\#_F\, Q') &= h_i^{sc}(Q) + h_i^{sc}(Q') + h_{i-1}^{sc}(\square^{d-1}) - h_i^{sc}(\square^{d-1}) \\
&= h_i^{sc}(Q) + h_i^{sc}(Q').
\end{aligned}
$$

It immediately follows that

$$
\begin{aligned}
h_i^c(Q \,\#_F\, Q') &= h_i^c(Q) + h_i^c(Q') \quad (1 \le i \le d-1), \\
g_i^c(Q \,\#_F\, Q') &= g_i^c(Q) + g_i^c(Q') \quad (2 \le i \le \lfloor d/2 \rfloor).
\end{aligned}
$$

We can now analyze the C-connected sum $Q_1 \,\#\, Q_2$. Since

$$
Q_1 \,\#\, Q_2 = Q_1 \,\#_{F_1}\, C \,\#_{F_2}\, Q_2,
$$

we have

$$
g_i^c(Q_1 \,\#\, Q_2) = g_i^c(Q_1) + g_i^c(C) + g_i^c(Q_2) \quad (2 \le i \le \lfloor d/2 \rfloor),
$$

and since C is cubical stacked we have $g_i^c(C) = 0$ for $2 \le i \le \lfloor d/2 \rfloor$, so we obtain (5.1) as required for $2 \le i \le \lfloor d/2 \rfloor$. For $i = 1$ one computes directly, using $g_1^c = f_0 - 2^d$. ∎

The following proves Theorem 3:

Proposition 15. *Let r be any ray in the nonnegative orthant in $\mathbb{R}^{\lfloor d/2 \rfloor}$. Then there exists a sequence $\{Q_n\}_{n=1}^\infty$ of cubical d-polytopes with the sequence $\{g^c(Q_n)\}_{n=1}^\infty$ approaching r.*

Proof. To construct the sequence Q_n approaching r, the ray spanned by $(s_1, \ldots, s_{\lfloor d/2 \rfloor})$, we start by constructing a sequence having the correct ratio between the $\lfloor d/2 \rfloor$-th coordinate and the $(\lfloor d/2 \rfloor - 1)$-th coordinate: Take

$$Q_m = Q(\lfloor d/2 \rfloor, d, m), \quad m \to \infty, \quad \text{and} \quad Q'_l = Q(\lfloor d/2 \rfloor - 1, d, l), \quad l \to \infty,$$

and recall from [2] that

$$g^c_{\lfloor d/2 \rfloor}(Q_m) = 2^m m^{\lfloor d/2 \rfloor - 1} + o(2^m m^{\lfloor d/2 \rfloor - 1}), \quad g^c_{\lfloor d/2 \rfloor - 1}(Q_m) = o(2^m m^{\lfloor d/2 \rfloor - 1}),$$
$$g^c_{\lfloor d/2 \rfloor}(Q'_l) = 0, \quad g^c_{\lfloor d/2 \rfloor - 1}(Q'_l) = 2^l l^{\lfloor d/2 \rfloor - 2} + o(2^l l^{\lfloor d/2 \rfloor - 2}).$$

Let $c = s_{\lfloor d/2 \rfloor - 1}/s_{\lfloor d/2 \rfloor}$. For each $m \geq d$, let $l = \lceil \log c + m + \log m \rceil$, and take the corresponding subsequence of Q'_l's (we abuse notation and denote it again by Q'_l). We have

$$\begin{aligned}
g^c_{\lfloor d/2 \rfloor - 1}(Q'_l) &= 2^l l^{\lfloor d/2 \rfloor - 2} + o(2^l l^{\lfloor d/2 \rfloor - 2}) \\
&= 2^{\lceil \log c + m + \log m \rceil} (\lceil m + \log m + \log c \rceil)^{\lfloor d/2 \rfloor - 2} + o(2^l l^{\lfloor d/2 \rfloor - 2}) \\
&= c 2^m m^{\lfloor d/2 \rfloor - 1} + o(2^m m^{\lfloor d/2 \rfloor - 1})
\end{aligned}$$

and so taking

$$Q_n = Q_m \# Q'_l, \quad n \to \infty,$$

using Lemma 14 we obtain

$$\lim_{n \to \infty} \frac{g^c_{\lfloor d/2 \rfloor - 1}(Q_n)}{g^c_{\lfloor d/2 \rfloor}(Q_n)} = \frac{s_{\lfloor d/2 \rfloor - 1}}{s_{\lfloor d/2 \rfloor}}.$$

Do the same with the new sequence and an AKN-sequence approaching the $(\lfloor d/2 \rfloor - 2)$-th ray, etc. Note that proceeding in this way (from the last coordinate backwards) does not influence the ratios already taken care of, because the g^c-entries are 0 after the dominating coordinate in the AKN construction.

Since the formula for g^c_1 in Lemma 14 is different, we use $c = \frac{s_1}{s_2} + 2^{d-m}(1 - \frac{1}{m})$ in the last step. ∎

6. Concluding remarks: Generalizations and open questions

Let us start off by remarking that the bound on the number of iterated projections and normal transformations may not be optimal, and the reason for this may lie in the fact that we are not allowing the full action by projective transformations and normal transformations, as we want to stay in the world of polytopes. Indeed, purely from a naive dimension count for the realization space of the d-cube $(2d^2)$ compared to the projective linear group $(d(d+2))$ it might be possible that only a constant number of these operations suffice (namely three, projective followed by normal followed by projective). We leave this as an open problem.

Problem 16. *How many normal and projective transformations are needed to transform any combinatorial cube into the standard one?*

Second, there is the natural question of more classes of combinatorial types of polytopes that are connected by normal and projective transformations. Let us call those polytopes PN-*unique*. Dually, let us call polytopes *weakly*-PR-*unique* if they are related by projective transformation, and a movement of vertices along the rays they generate within the same combinatorial type. But in the dual, this permits moving some facet hyperplanes to infinity. If we want the dual to PN-uniqueness, then we add the condition that the origin has to be in the interior of the polytope at all times; we call such polytopes PR-*unique*. Then the PR-unique polytopes are precisely dual to the PN-unique polytopes.

We note the following simple fact about simplicial stacking (connected sum with a simplex S) on PR-unique polytopes:

Proposition 17. *If P is PR-unique and F a simplex facet of P, then $P \#_F S$ is PR-unique.*

Proof. Do PR-transformations so that the P part has the correct shape, then get the new vertex v to its desired position u with transformations that do not affect the P part: this can always be done with a sequence of three PR-transformations. For example, scale v by ε so that εv is close enough to F, namely so that the line through u and εv intersects the interior of F, say at w. Then move the origin to w, then scale εv to u. ∎

We immediately conclude:

Corollary 18. *Every polygon, and more generally every stacked polytope, is PR-unique. In particular, in every dimension $d \geq 2$, there are infinitely many combinatorial types of PR-unique polytopes.*

This is in contrast to projectively unique polytopes, which are only finitely many in dimension 2 and 3. (However, in sufficiently large fixed dimension d there exist projectively unique d-polytopes with arbitrarily many vertices – this was proved for $d \geq 69$ by Adiprasito and Ziegler [3], answering a question of Perles and Shephard [6].)

We use the notation of [4] about free joins and subdirect sums, and note the following:

Proposition 19. *The free join of two polytopes P and Q is weakly-PR-unique if and only if both components are.*

This follows easily, as we may act on each component separately. The same is not true for PR-uniqueness, and therefore PN-uniqueness. A counterexample is the cone over the crosspolytope. Indeed, this follows from the following observation, which is straightforward from the definitions:

Lemma 20. *If P is PR-unique then every facet of P is projectively unique.*

The next result holds by following the proof of Proposition 9 for the cube case.

Theorem 21. *The subdirect sum of a* PR-*unique polytope with a simplex is* PR-*unique, and vice versa. Dually, the subdirect product with a simplex is* PN-*unique if and only if the original polytope is.*

This is especially interesting if one considers only those polytopes that are obtained as products of simplices. These are PN-unique by the above theorem (and include the cube). Moreover, the algorithm described in Proposition 9 goes through verbatim, and is continuously dependent on the starting geometry. Hence, we find once again that the realization space of such polytopes is contractible (a fact not known for general PN-unique or PR-unique polytopes). We end with a question:

Problem 22. *Is the dodecahedron* PN-*unique?*

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