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Nonuniqueness in law of stochastic 3D Navier–Stokes equations

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Abstract. We consider the stochastic Navier–Stokes equations in three dimensions and prove that the law of analytically weak solutions is not unique. In particular, we focus on three examples of a stochastic perturbation: an additive, a linear multiplicative and a nonlinear noise of cylindrical type, all driven by a Wiener process. In these settings, we develop a stochastic counterpart of the convex integration method introduced recently by Buckmaster and Vicol. This permits us to construct probabilistically strong and analytically weak solutions defined up to a suitable stopping time. In addition, these solutions fail to satisfy the corresponding energy inequality at a prescribed time with a prescribed probability. Then we introduce a general probabilistic construction used to extend the convex integration solutions beyond the stopping time and in particular to the whole time interval $[0, \infty)$. Finally, we show that their law is distinct from the law of solutions obtained by Galerkin approximation. In particular, nonuniqueness in law holds on an arbitrary time interval $[0, T]$, $T > 0$.

Keywords. Stochastic Navier–Stokes equations, nonuniqueness in law, convex integration

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1. Introduction

The fundamental problems in fluid dynamics remain largely open. On the theoretical side, existence and smoothness of solutions to the three-dimensional incompressible Navier–Stokes system is one of the Millennium Prize Problems. An intimately related question is that of uniqueness of solutions. Intuitively, smooth solutions are unique whereas uniqueness for less regular solutions, such as weak solutions, is very challenging and not true for a number of models.

A revolutionary step was made through the method of convex integration by De Lellis and Székelyhidi Jr. [15–17]. They were able to construct infinitely many weak solutions to the incompressible Euler system which dissipate energy and even satisfy various additional criteria such as a global or local energy inequality. After this breakthrough, an avalanche of excitement and intriguing results followed, proving existence of solutions with often rather pathological behavior. In particular, it is nowadays well understood that the compressible counterpart of the Euler system is desperately ill-posed: even certain smooth initial data give rise to infinitely many weak solutions satisfying an energy inequality; see Chiodaroli et al. [11]. Very recently, the nonuniqueness of weak solutions to the incompressible Navier–Stokes equations was obtained by Buckmaster and Vicol [8]; see also Buckmaster, Colombo and Vicol [5].

In view of these substantial theoretical difficulties, it is natural to believe that a certain probabilistic description is indispensable and may eventually help with the nonuniqueness issue. In particular, it is essential to develop a suitable probabilistic understanding of the deterministic systems, in order to capture their chaotic and intrinsically random nature after the blow-up and loss of uniqueness. Moreover, there is evidence that a suitable stochastic perturbation may provide a regularizing effect on deterministically ill-posed problems, in particular those involving transport, as shown, e.g., by Flandoli, Gubinelli and Priola [20] and Flandoli and Luo [21]. Also a linear multiplicative noise as treated in the present paper has a certain stabilizing effect on the three-dimensional Navier–Stokes system; see Röckner, Zhu and Zhu [40].

On the other hand, an external stochastic forcing is often included in the system of governing equations, taking additional model uncertainties into account. Mathematically, this introduces new phenomena and raises basic questions of solvability of the system, i.e. existence and uniqueness of solutions, as well as their long time behavior. In particular, the question of uniqueness of the probability measures induced by solutions, the so-called uniqueness in law, has been a longstanding open problem.

In the present paper, we prove that nonuniqueness in law holds for the stochastic three-dimensional Navier–Stokes system posed on a periodic domain in a class of analytically weak solutions. This system governs the time evolution of the velocity u of a viscous incompressible fluid under stochastic perturbations. It reads

$$\begin{aligned} du - \nu\Delta u dt + \operatorname{div}(u\otimes u)dt + \nabla P dt &= G(u)dB, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.1}$$

where $G(u)dB$ represents a stochastic force acting on the fluid and $\nu > 0$ is the kinematic viscosity.

We particularly focus on three examples of stochastic forcing, namely, an additive noise driven by a cylindrical Wiener process B of trace class, i.e.,

$$G(u)dB = GdB = \sum_{i=1}^{\infty} G^i dB_i, \quad G^i = G^i(x), \quad \operatorname{Tr}(GG^*) < \infty, \tag{1.2}$$

and a linear multiplicative noise driven by a real-valued Wiener process B_1, i.e.,

$$G(u)dB = u dB_1, \tag{1.3}$$

and finally a nonlinear noise of cylindrical type

$$\begin{aligned} G(u)dB &= \Big(\sum_{j=1}^{m} g_{ij}(\langle u,\varphi_1\rangle,\ldots,\langle u,\varphi_{k_{ij}}\rangle)dB_j\Big)_i, \\ g_{ij} &\in C_b^3(\mathbb{T}^3\times\mathbb{R}^{k_{ij}};\mathbb{R}), \quad \varphi_i \in C^\infty(\mathbb{T}^3), \end{aligned} \tag{1.4}$$

where $B = (B_j)$ is an m-dimensional Wiener process and $g_{\cdot j}$ is divergence-free with respect to the spatial variable in $\mathbb{T}^3$.

In these three settings, we develop a stochastic counterpart of the convex integration method introduced by Buckmaster and Vicol [7] and construct analytically weak solutions with unexpected behavior defined up to suitable stopping times. The striking feature of these solutions is that they are probabilistically strong, i.e., adapted to the given Wiener process. This severely contradicts the general belief present within the SPDE community, namely, that probabilistically strong solutions and uniqueness in law could help with the uniqueness problem for the Navier–Stokes system.

We say that uniqueness in law holds for a system of SPDEs if the probability law induced by the solutions is uniquely determined. On the other hand, we say that pathwise uniqueness holds true if any two solutions coincide almost surely. There are explicit examples of stochastic differential equations (SDEs) where pathwise uniqueness does not hold but uniqueness in law is valid. Pathwise uniqueness for the stochastic Navier–Stokes system essentially poses the same difficulties as uniqueness in the deterministic setting. As a consequence, there has been a clear hope that showing uniqueness in law for the Navier–Stokes system might be easier than proving pathwise uniqueness. Furthermore, Yamada–Watanabe–Engelbert's theorem states that, for a certain class of SDEs, pathwise uniqueness is equivalent to uniqueness in law and existence of a probabilistically strong solution; see Kurtz [35] and Cherny [9]. This suggests another possible way towards pathwise uniqueness, provided one could prove uniqueness in law.

Our main result proves the above hopes wrong, at least for a certain class of analytically weak solutions. However, the question of uniqueness of the so-called Leray solutions remains an outstanding open problem. In particular, we show that nonuniqueness in law for analytically weak solutions holds true on an arbitrary time interval $[0, T]$, $T > 0$. This trivially implies pathwise nonuniqueness. More precisely, we construct a deterministic divergence-free initial condition $u(0) \in L^2$ which gives rise to two solutions to the Navier–Stokes system (1.1) with distinct laws. One of the solutions is constructed by means of the convex integration method whereas the other one is a solution obtained by a classical compactness argument via Galerkin approximation; see e.g. [19].

We note that the solutions obtained by Galerkin approximation are clearly more physical as they correspond to Leray solutions in the deterministic setting and satisfy the energy inequality. However, these solutions are not probabilistically strong as the adaptedness with respect to the given noise is lost within the stochastic compactness method. On the other hand, convex integration permits one to construct adapted solutions up to a stopping time but they behave in an unphysical way with respect to the energy inequality. Moreover, spatial regularity is worse as we can only prove that they belong to H^γ for a certain $\gamma > 0$ small.

1.1. Main results

Even though the main result, i.e., nonuniqueness in law, is the same in the three settings (1.2), (1.3) and (1.4), the proofs are different. The additive noise case is easier and we present a direct construction of two solutions with different laws. This is not possible in the case of a linear multiplicative noise where the proof becomes more involved. The nonlinear case is even more challenging and requires tools from the theory of rough paths. For notational simplicity, we suppose from now on that $\nu = 1$.

1.1.1. Additive noise. Consider the stochastic Navier–Stokes system driven by an additive noise on $\mathbb{T}^3$, which reads

$$\begin{aligned} du - \Delta u dt + \operatorname{div}(u \otimes u) dt + \nabla P dt &= dB, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.5}$$

where B is a GG^*-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and G is a Hilbert–Schmidt operator from L^2 to L^2. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the normal filtration generated by B, that is, the canonical right continuous filtration augmented by all the $\mathbf{P}$-negligible events.

Our first result in this setting is the existence of a probabilistically strong solution which is defined up to a stopping time and which violates the corresponding energy inequality.

Theorem 1.1. *Suppose that* $\operatorname{Tr}(GG^*) < \infty$. *Let* $T > 0$, $K > 1$ *and* $\kappa \in (0,1)$ *be given. Then there exist* $\gamma \in (0,1)$ *and a* $\mathbf{P}$*-a.s. strictly positive stopping time* $\mathfrak{t}$ *satisfying* $\mathbf{P}(\mathfrak{t} \geq T) > \kappa$ *such that the following holds true: There exists an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* u *which belongs to* $C([0,\mathfrak{t}]; H^\gamma)$ $\mathbf{P}$*-a.s. and is an analytically weak solution to* (1.5) *with* $u(0)$ *deterministic. In addition,*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[0,\mathfrak{t}]} \|u(t)\|_{H^\gamma} < \infty, \tag{1.6}$$

and

$$\|u(T)\|_{L^2} > K\|u(0)\|_{L^2} + K(T \operatorname{Tr}(GG^*))^{1/2} \quad \text{on the set } \{\mathfrak{t} \geq T\}. \tag{1.7}$$

The proof of this result relies on the convex integration method, and the stopping time is employed in the construction in order to control the noise in various bounds. While this result readily implies nonuniqueness in law for solutions defined on the random time interval $[0,\mathfrak{t}]$, our main result is more general: we prove nonuniqueness in law on an arbitrary time interval or more generally up to an arbitrary stopping time.

Theorem 1.2. *Suppose that* $\operatorname{Tr}(GG^*) < \infty$. *Then nonuniqueness in law holds for the Navier–Stokes system* (1.5) *on* $[0,\infty)$. *Furthermore, for every given* $T > 0$, *nonuniqueness in law holds for the Navier–Stokes system* (1.5) *on* $[0,T]$.

In order to derive the result of Theorem 1.2 from Theorem 1.1, it is necessary to extend the convex integration solutions to the whole time interval $[0,\infty)$. To this end, we present a general probabilistic construction which connects the law of solutions defined up to a stopping time to a law of a solution obtained by the classical compactness argument. The principal difficulty is to allow for the concatenation of solutions at a random time. Since the stopping time $\mathfrak{t}$ is defined in terms of the solution u, we work with the notion of martingale solution which is defined as the law of a solution u. Consequently, we are able to obtain nonuniqueness in law, i.e., nonuniqueness of martingale solutions directly, as opposed to the case of a linear multiplicative noise.

1.1.2. Linear multiplicative noise. Consider the stochastic Navier–Stokes equation driven by a linear multiplicative noise on $\mathbb{T}^3$, which reads

$$\begin{aligned} du - \Delta u dt + \operatorname{div}(u \otimes u) dt + \nabla P dt &= u dB, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.8}$$

where B is a real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. As above, we denote by $(\mathcal{F}_t)_{t\geq 0}$ the normal filtration generated by B. The main results in this case are as follows.

Theorem 1.3. *Let $T > 0$, $K > 1$ and $\kappa \in (0,1)$ be given. Then there exist $\gamma \in (0,1)$ and a $\mathbf{P}$-a.s. strictly positive stopping time $\mathfrak{t}$ satisfying $\mathbf{P}(\mathfrak{t} \geq T) > \kappa$ such that the following holds true: There exists an $(\mathcal{F}_t)_{t\geq 0}$-adapted process u which belongs to $C([0,\mathfrak{t}]; H^\gamma)$ $\mathbf{P}$-a.s. and is an analytically weak solution to (1.8) with $u(0)$ deterministic. In addition,*

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{t\in[0,\mathfrak{t}]} \|u(t)\|_{H^\gamma} < \infty,$$

and

$$\|u(T)\|_{L^2} > K e^{T/2} \|u(0)\|_{L^2} \quad \text{on the set } \{\mathfrak{t} \geq T\}.$$

Theorem 1.4. *Nonuniqueness in law holds for the Navier–Stokes system (1.8) on $[0,\infty)$. Furthermore, for every given $T > 0$, nonuniqueness in law holds for the Navier–Stokes system (1.8) on $[0,T]$.*

In contrast to the additive noise setting, the stopping time $\mathfrak{t}$ in the case of the linear multiplicative noise is a function of B and not a function of the solution u. As a consequence, we are forced to work with the notion of a probabilistically weak solution which governs the joint law of (u, B). We extend our method of concatenation of two solutions to connect the probabilistically weak solution obtained through Theorem 1.3 to a probabilistically weak solution obtained by compactness. Accordingly, we first only deduce joint nonuniqueness in law, i.e., nonuniqueness of probabilistically weak solutions. Finally, we prove that joint nonuniqueness in law implies nonuniqueness in law, concluding the proof of Theorem 1.4. This relies on a generalization of the result of Cherny [9] to the infinite-dimensional setting, which is interesting in its own right; see Appendix C.

1.1.3. Nonlinear noise. We consider the Navier–Stokes equations

$$\begin{aligned} du - \Delta u dt + \operatorname{div}(u \otimes u) dt + \nabla P dt &= G(u) dB, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.9}$$

with $G(u)$ defined via (1.4) and B an m-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and we denote by $(\mathcal{F}_t)_{t\geq 0}$ its normal filtration. In this setting, we apply convex integration in order to establish the following results.

Theorem 1.5. *Let $T > 0$, $K > 1$ and $\kappa \in (0,1)$ be given. Then there exist $\gamma \in (0,1)$ and a $\mathbf{P}$-a.s. strictly positive stopping time $\mathfrak{t}$ satisfying $\mathbf{P}(\mathfrak{t} \geq T) > \kappa$ such that the following* *holds true: There exists an* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process* u *which belongs to* $C([0,\mathfrak{t}];L^2)\cap L^2([0,\mathfrak{t}];H^\gamma)$ $\mathbf{P}$*-a.s. and is an analytically weak solution to* (1.9) *with* $u(0)$ *deterministic. In addition, for* $q\in\mathbb{N}$*,*

$$\mathbf{E}\left[\sup_{r\in[0,t\wedge\mathfrak{t}]}\|u(r)\|_{L^2}^{2q}+\int_0^{t\wedge\mathfrak{t}}\|u(r)\|_{H^\gamma}^2\,dr\right]\leq C_{t,q} \tag{1.10}$$

for some constant $C_{t,q}$*, and*

$$\mathbf{E}[\mathbf{1}_{\mathfrak{t}\geq T}\|u(T)\|_{L^2}^2]>K\|u(0)\|_{L^2}^2+KTC_G \tag{1.11}$$

with

$$C_G=(2\pi)^3\sum_{i=1}^{3}\sum_{j=1}^{m}\|g_{ij}\|_{C^0}^2.$$

Theorem 1.6. *Nonuniqueness in law holds for the Navier–Stokes system* (1.9) *on* $[0,\infty)$*. Furthermore, for every given* $T>0$*, nonuniqueness in law holds for the Navier–Stokes system* (1.9) *on* $[0,T]$*.*

This nonlinear case presents further challenges which do not appear in the previous settings of additive and linear multiplicative noise. First of all, there is no obvious transformation of the SPDEs into a PDE with random coefficients. Consequently, it is necessary to employ rough path theory in order to obtain pathwise control of the stochastic integral in the convex integration scheme. This is the reason why we restricted ourselves to the cylindrical noise of the form (1.4). Nevertheless, a more general noise could be considered provided the corresponding rough path estimate is valid.

Using rough path theory to control the stochastic integral requires the so-called iterated integral of B against B to be included in the path space. Accordingly, the stopping time $\mathfrak{t}$ is a function of $(B,\int B\otimes dB)$. Since we have to define the corresponding stopping time on the canonical path space, the difficulty lies in how to define the iterated stochastic integral on the path space without the use of any probability measure. Indeed, due to the low time regularity of the Wiener process, the stochastic integral cannot be defined by purely analytical means and probability theory is required in a nontrivial way. We overcome this by introducing a notion of generalized probabilistically weak solution which takes this issue into account.

We note that in order to apply rough path theory it is essential that the intermittent jets possess sufficient time regularity, namely, we require complementary Young regularity to the Brownian motion, i.e. $\alpha_0=2/3+\kappa$ for $\kappa>0$ small. To this end, it is necessary to lower spatial regularity and we derive new bounds for the intermittent jets in Lemma B.2. They lead to the convergence of v_q in $C^{\alpha_0}([0,\mathfrak{t}];B^{-5-\delta}_{1,1})$; see (8.5). This is the reason for restricting to the case of a cylindrical noise, i.e. one which smoothens in the spatial variable. Other cases of spatially smoothing noise can be treated similarly.

Remark 1.7. Let us emphasize that if we directly tried to apply convex integration without using stopping time, we would have to take expectation to control the stochastic integral. As convex integration is an iteration procedure, we would have to include L^p-moment estimates for arbitrary p, which is typically achieved by the Burkholder–Davis–Gundy inequality. However, as the implicit constant here depends on p, the estimates would blow up during the iteration scheme.

Remark 1.8. Our convex integration schemes in the case of additive and nonlinear noise could be understood as follows. In addition to the principal part of the perturbation $w_{q+1}^{(p)}$, the incompressibility corrector $w_{q+1}^{(c)}$ and the temporal corrector $w_{q+1}^{(t)}$ as appearing in the deterministic literature, we introduce a *stochastic corrector* $w_{q+1}^{(s)}$. Its role is to add noise scale by scale as one proceeds through the iteration. More precisely, for the original equation for u, we construct iterations u_q given by

$$\begin{aligned} u_{q+1} = v_{q+1} + z_{q+1} &= v_\ell + w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + z_{q+1} \\ &= (v_\ell + z_\ell) + w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + (z_{q+1} - z_\ell) \\ &= u_\ell + w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(s)}, \end{aligned}$$

where $w_{q+1}^{(s)} = z_{q+1} - z_\ell$ is the stochastic corrector. In the case of additive noise, we set $z_{q+1} = \mathbb{P}_{\leq f(q+1)} z$ (i.e. a suitable truncation in Fourier space) with

$$dz - \Delta z dt = G dB,$$

whereas in the case of nonlinear noise we define

$$dz_{q+1} - \Delta z_{q+1} dt = G(v_q + z_{q+1}) dB.$$

Due to the dependence on v_{q-1}, z_q diverges in C^1 but converges in L^2. When we need to control the C^1-norm of z_q in the estimates of the Reynolds stress, we can always use a small constant from v_q to absorb the blow-up of this norm.

Finally, we note that due to its particular structure, the linear multiplicative noise case is different in this respect. Here, the perturbations are additionally randomized multiplicatively by e^B in the following way:

$$u_{q+1} = e^B v_\ell + e^B w_{q+1}^{(p)} + e^B w_{q+1}^{(c)} + e^B w_{q+1}^{(t)}.$$

1.2. Further relevant literature

Stochastic Navier–Stokes equations driven by a trace-class noise have been the subject of a large number of works. The reader is referred e.g. to [18, 19, 29] and the references therein. In the two-dimensional case, existence and uniqueness of strong solutions was obtained if the noisy forcing term is white in time and colored in space. In the three-dimensional case, existence of martingale solutions was proved in [13,22,26]. Furthermore, ergodicity was proved if the system is driven by a nondegenerate trace-class noise [13,22,42]. Navier–Stokes equations driven by space-time white noise are also considered in [12,50], and the system is studied in the context of rough path theory in [31,32].

The linear multiplicative noise (1.3) can be seen as a damping term: it is shown in [40] that it prevents the system from exploding with large probability. In a more recent work, Flandoli and Luo [21] proved that one kind of transport noise improves the vorticity blow-up in 3D Navier–Stokes equations with large probability. In [2], a global solution starting from small initial data was constructed for 3D Navier–Stokes equations in vorticity formulation driven by a linear multiplicative noise. However, the solutions are not adapted to the filtration generated by the noise and the stochastic integral should be understood in a rough path sense (see [37, 41] for more general noise). By the methods in [2, 37], adapted solutions up to a stopping time can also be obtained. However, existence of globally defined probabilistically strong solutions to the stochastic Navier–Stokes system without any stopping time remains a challenging open problem. Finally, we note that convex integration has already been applied in a stochastic setting, namely, to the isentropic Euler system in [4] and to the full Euler system in [10].

1.3. Relevant literature update

In the first version of the present paper uploaded to arXiv we established nonuniqueness in law only for a spatially regular additive noise (namely, $\mathrm{Tr}((-\Delta)^{3/2+2\sigma}GG^*) < \infty$) and a linear multiplicative noise. Our method was then applied to several other fluid models driven by these noises [33, 34, 39, 44–48]. In particular, in [33] we studied the question of well-posedness for stochastic Euler equations from various perspectives. In [34] we proved existence and nonuniqueness of global-in-time probabilistically strong and Markov solutions to the stochastic Navier–Stokes system. In the present version of the manuscript we are for the first time able to prove nonuniqueness in law for the Navier–Stokes system with a nonlinear stochastic perturbation.

1.4. Organization of the paper

In Section 2, we collect the notations used throughout. Sections 3 and 4 are devoted to the proof of our first main result, Theorem 1.2, nonuniqueness in law for the case of an additive noise. First, in Section 3 we introduce the notion of martingale solution and present a general method of extending martingale solutions defined up to a stopping time to the whole time interval $[0, \infty)$. This is then applied to solutions obtained through the convex integration technique, and nonuniqueness in law is shown in Section 3.3. Convex integration solutions are constructed in Section 4, which proves Theorem 1.1. A similar structure can be found in Sections 5 and 6 devoted to the setting of a linear multiplicative noise. This relies on the notion of probabilistically weak solution and a general concatenation procedure presented in Section 5.2. Application to convex integration solutions together with the proof of Theorem 1.4 can be found in Section 5.3. Convex integration in this setting is applied in Section 6, where Theorem 1.3 is established. In Sections 7 and 8, we prove the results for the nonlinear noise. In Appendix A, we collect several auxiliary results concerning stability of martingale, probabilistically weak as well as generalized probabilistically weak solutions. In Appendix B, the construction of intermittent jets needed for convex integration is recalled. In Appendix C, we show that nonuniqueness in law implies joint nonuniqueness in law in a general infinite-dimensional SPDE setting. Finally, Appendix D is devoted to the rough path analysis required in the nonlinear setting.

2. Notations

2.1. Function spaces

Throughout, we write $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. Given a Banach space E with a norm $\|\cdot\|_E$ and $T > 0$, we write $C_T E = C([0,T]; E)$ for the space of continuous functions from $[0,T]$ to E, equipped with the supremum norm $\|f\|_{C_T E} = \sup_{t\in[0,T]} \|f(t)\|_E$. We also use CE or $C([0,\infty); E)$ to denote the space of continuous functions from $[0,\infty)$ to E. For $\alpha \in (0,1)$ we define $C^\alpha_T E$ as the space of α-Hölder continuous functions from $[0,T]$ to E, endowed with the seminorm $\|f\|_{C^\alpha_T E} = \sup_{s,t\in[0,T], s\neq t} \frac{\|f(s)-f(t)\|_E}{|t-s|^\alpha}$. When $E = \mathbb{R}$ we write C^α_T. We also use $C^\alpha_{\mathrm{loc}} E$ for the space of functions from $[0,\infty)$ to E satisfying $f|_{[0,T]} \in C^\alpha_T E$ for all $T > 0$. For $p \in [1,\infty]$ we write $L^p_T E = L^p([0,T]; E)$ for the space of L^p-integrable functions from $[0,T]$ to E, equipped with the usual L^p-norm. We also use $L^p_{\mathrm{loc}}([0,\infty); E)$ to denote the space of functions f from $[0,\infty)$ to E satisfying $f|_{[0,T]} \in L^p_T E$ for all $T > 0$. We use L^p to denote the set of standard L^p-integrable functions from $\mathbb{T}^3$ to $\mathbb{R}^3$. For $s > 0$, $p > 1$ we set $W^{s,p} := \{f \in L^p : \|(I-\Delta)^{s/2} f\|_{L^p} < \infty\}$ with the norm $\|f\|_{W^{s,p}} = \|(I-\Delta)^{s/2} f\|_{L^p}$. Set $L^2_\sigma = \{u \in L^2 : \operatorname{div} u = 0\}$. For $s > 0$, $H^s := W^{s,2} \cap L^2_\sigma$. For $s < 0$ define H^s to be the dual space of H^{-s}. We also use the Besov space $B^\beta_{p,q}$, $\beta \in \mathbb{R}$, defined as the closure of smooth functions with respect to the $B^\beta_{p,q}$-norm

$$\|f\|_{B^\beta_{p,q}} := \Big(\sum_{j\geq -1} 2^{\beta j q} \|\Delta_j f\|^q_{L^p}\Big)^{1/q},$$

with Δ_j, $j \in \mathbb{N}_0 \cup \{-1\}$, being the usual Littlewood–Paley blocks.

We set $\|f\|_{C^N_{t,x}} = \sum_{0\leq n+|\alpha|\leq N} \|\partial^n_t D^\alpha f\|_{L^\infty_t L^\infty}$. For a Polish space H we also use $\mathcal{B}(H)$ to denote the σ-algebra of Borel sets in H.

2.2. Probabilistic elements

Let $\Omega_0 := C([0,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma)$ and let $\mathscr{P}(\Omega_0)$ denote the set of all probability measures on $(\Omega_0, \mathcal{B})$ with $\mathcal{B}$ being the Borel σ-algebra coming from the topology of locally uniform convergence on Ω_0. Let $x : \Omega_0 \to H^{-3}$ denote the canonical process on Ω_0 given by

$$x_t(\omega) = \omega(t).$$

Similarly, for $t \geq 0$ we define $\Omega_t := C([t,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([t,\infty); L^2_\sigma)$ equipped with its Borel σ-algebra $\mathcal{B}^t$ which coincides with $\sigma\{x(s) : s \geq t\}$. Finally, we define the canonical filtration $\mathcal{B}^0_t := \sigma\{x(s) : s \leq t\}$, $t \geq 0$, as well as its right continuous version $\mathcal{B}_t :=$

$\bigcap_{s>t} \mathcal{B}_s^0$, $t \geq 0$. For a given probability measure P we use E^P to denote the expectation under P.

For a Hilbert space U, let $L_2(U; L^2_\sigma)$ be the space all Hilbert–Schmidt operators from U to L^2_σ with the norm $\|\cdot\|_{L_2(U;L^2_\sigma)}$. Let $G : L^2_\sigma \to L_2(U; L^2_\sigma)$ be $\mathcal{B}(L^2_\sigma)/\mathcal{B}(L_2(U; L^2_\sigma))$-measurable. In the following, we assume

$$\|G(x)\|_{L_2(U;L^2_\sigma)} \leq C(1 + \|x\|_{L^2})$$

for every $x \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and if in addition $y_n \to y$ in L^2 then

$$\lim_{n\to\infty} \|G(y_n)^* x - G(y)^* x\|_U = 0,$$

where the asterisk denotes the adjoint operator.

Suppose there is another Hilbert space U_1 such that the embedding $U \subset U_1$ is Hilbert–Schmidt. Let $\bar{\Omega} := C([0,\infty); H^{-3} \times U_1) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma \times U_1)$ and let $\mathscr{P}(\bar{\Omega})$ denote the set of all probability measures on $(\bar{\Omega}, \bar{\mathcal{B}})$ with $\bar{\mathcal{B}}$ being the Borel σ-algebra coming from the topology of locally uniform convergence on $\bar{\Omega}$. Let $(x, y) : \bar{\Omega} \to H^{-3} \times U_1$ denote the canonical process on $\bar{\Omega}$ given by

$$(x_t(\omega), y_t(\omega)) = \omega(t).$$

For $t \geq 0$ we define the σ-algebra $\bar{\mathcal{B}}^t = \sigma\{(x(s), y(s)) : s \geq t\}$. Finally, we define the canonical filtration $\bar{\mathcal{B}}_t^0 := \sigma\{(x(s), y(s)) : s \leq t\}$, $t \geq 0$, as well as its right continuous version $\bar{\mathcal{B}}_t := \bigcap_{s>t} \bar{\mathcal{B}}_s^0$, $t \geq 0$.

3. Nonuniqueness in law I: the case of an additive noise

3.1. Martingale solutions

Let us begin with a definition of martingale solution on $[0,\infty)$. In what follows, we fix $\gamma \in (0,1)$.

Definition 3.1. Let $s \geq 0$ and $x_0 \in L^2_\sigma$. A probability measure $P \in \mathscr{P}(\Omega_0)$ is a *martingale solution to the Navier–Stokes system* (1.1) *with initial value* x_0 *at time* s provided

(M1) $P(x(t) = x_0, 0 \leq t \leq s) = 1$, and for any $n \in \mathbb{N}$,

$$P\left\{x \in \Omega_0 : \int_0^n \|G(x(r))\|^2_{L_2(U;L^2_\sigma)}\, dr < \infty\right\} = 1.$$

(M2) For every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and all $t \geq s$ the process

$$M^i_{t,s} := \langle x(t) - x(s), e_i\rangle + \int_s^t \langle \mathrm{div}(x(r) \otimes x(r)) - \Delta x(r), e_i\rangle\, dr$$

is a continuous square integrable $(\mathcal{B}_t)_{t\geq s}$-martingale under P with quadratic variation process given by $\int_s^t \|G(x(r))^* e_i\|^2_U\, dr$.

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$,

$$E^P\left(\sup_{r\in[0,t]} \|x(r)\|_{L^2}^{2q} + \int_s^t \|x(r)\|_{H^\gamma}^2\, dr\right) \leq C_{t,q}(\|x_0\|_{L^2}^{2q} + 1),$$

where E^P denotes the expectation under P.

In particular, we observe that in the context of Definition 3.1 for the additive noise case, i.e. for G independent of x, if $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of L^2_σ consisting of eigenvectors of GG^* then $M_{t,s} := \sum_{i\in\mathbb{N}} M^i_{t,s} e_i$ is a GG^*-Wiener process starting from s with respect to the filtration $(\mathscr{B}_t)_{t\geq s}$ under P.

Similarly, we may define martingale solutions up to a stopping time $\tau : \Omega_0 \to [0, \infty]$. To this end, we define the space of trajectories stopped at τ by

$$\Omega_{0,\tau} := \{\omega(\cdot \wedge \tau(\omega)) : \omega \in \Omega_0\}.$$

We note that due to the Borel measurability of τ, the set $\Omega_{0,\tau} = \{\omega \in \Omega_0 : x(t,\omega) = x(t \wedge \tau(\omega), \omega), \forall t \geq 0\}$ is a Borel subset of Ω_0, hence $\mathscr{P}(\Omega_{0,\tau}) \subset \mathscr{P}(\Omega_0)$.

Definition 3.2. Let $s \geq 0$ and $x_0 \in L^2_\sigma$. Let $\tau \geq s$ be a $(\mathscr{B}_t)_{t\geq s}$-stopping time. A probability measure $P \in \mathscr{P}(\Omega_{0,\tau})$ is a *martingale solution to the Navier–Stokes system* (1.1) *on* $[s, \tau]$ *with initial value* x_0 *at time* s provided

(M1) $P(x(t) = x_0, 0 \leq t \leq s) = 1$ and for any $n \in \mathbb{N}$,

$$P\left\{x \in \Omega_0 : \int_0^{n\wedge\tau} \|G(x(r))\|^2_{L_2(U;L^2_2)}\, dr < \infty\right\} = 1.$$

(M2) For every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and all $t \geq s$ the process

$$M^i_{t\wedge\tau,s} := \langle x(t\wedge\tau) - x_0, e_i\rangle + \int_s^{t\wedge\tau} \langle \mathrm{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle\, dr$$

is a continuous square integrable $(\mathscr{B}_t)_{t\geq s}$-martingale under P with quadratic variation process given by $\int_s^{t\wedge\tau} \|G(x(r))^* e_i\|^2_U\, dr$.

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$,

$$E^P\left(\sup_{r\in[0,t\wedge\tau]} \|x(r)\|_{L^2}^{2q} + \int_s^{t\wedge\tau} \|x(r)\|_{H^\gamma}^2\, dr\right) \leq C_{t,q}(\|x_0\|_{L^2}^{2q} + 1),$$

where E^P denotes the expectation under P.

The following result provides the existence of martingale solutions as well as stability of the set of all martingale solutions. A similar result can be found in [22, 26] but in the present paper we require in addition stability with respect to the initial time. For completeness, we include the proof in Appendix A.

Theorem 3.1. *For every $(s, x_0) \in [0,\infty) \times L^2_\sigma$, there exists $P \in \mathscr{P}(\Omega_0)$ which is a martingale solution to the Navier–Stokes system* (1.1) *starting at time s from the initial condition x_0 in the sense of Definition* 3.1*. The set of all such martingale solutions with the same $C_{t,q}$ in* (M3) *of Definition* 3.1 *is denoted by $\mathscr{C}(s, x_0, C_{t,q})$.*

Let $(s_n, x_n) \to (s, x_0)$ in $[0,\infty) \times L^2_\sigma$ as $n \to \infty$ and let $P_n \in \mathscr{C}(s_n, x_n, C_{t,q})$. Then there exists a subsequence n_k such that the sequence $\{P_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to some $P \in \mathscr{C}(s, x_0, C_{t,q})$.

For completeness, let us recall the definition of uniqueness in law.

Definition 3.3. We say that *uniqueness in law* holds for (1.1) if martingale solutions starting from the same initial distribution are unique.

Now, we have all in hand to proceed with the proof of our first main result, Theorem 1.2. On the one hand, by classical arguments as in Theorem 3.1 we obtain existence of a martingale solution to (1.1) which satisfies the corresponding energy inequality. On the other hand, for the case of an additive noise, Theorem 1.1 provides a stopping time $\mathfrak{t}$ such that there exists an $(\mathcal{F}_t)_{t\geq 0}$-adapted analytically weak solution $u \in C([0,\mathfrak{t}]; H^\gamma)$ to (1.5) which violates the energy inequality. The main idea is to construct a martingale solution which is defined on the full interval $[0,\infty)$ and preserves the properties of the adapted solution on $[0,\mathfrak{t}]$, that is, the energy inequality is not satisfied in this random time interval. To this end, the essential point is to make use of adaptedness of solutions obtained through Theorem 1.1 and connect them to ordinary martingale solutions obtained by Theorem 3.1. The difficulty is that the connection has to happen at a random time, which only turns out to be a stopping time with respect the right continuous filtration $(\mathcal{B}_t)_{t\geq 0}$. Consequently, the classical martingale theory of Stroock and Varadhan [43] does not apply and we are facing a number of measurability issues which have to be carefully treated.

3.2. General construction for martingale solutions

First, we present an auxiliary result which is then used to extend martingale solutions defined up to a stopping time τ to the whole interval $[0,\infty)$. To this end, we denote by $\mathcal{B}_\tau$ the σ-field associated to the stopping time τ. The results of this section apply to a general form of noise in (1.1); the restriction to an additive noise is only required in Section 3.3 below in order to apply the result of Theorem 1.1.

Proposition 3.2. *Let τ be a bounded $(\mathcal{B}_t)_{t\geq 0}$-stopping time. Then for every $\omega \in \Omega_0$ there exists $Q_\omega \in \mathscr{P}(\Omega_0)$ such that for $\omega \in \{x(\tau) \in L^2_\sigma\}$,*

$$Q_\omega\big(\omega' \in \Omega_0 : x(t,\omega') = \omega(t) \text{ for } 0 \leq t \leq \tau(\omega)\big) = 1, \tag{3.1}$$

$$Q_\omega(A) = R_{\tau(\omega),x(\tau(\omega),\omega)}(A) \quad \text{for all } A \in \mathcal{B}^{\tau(\omega)}, \tag{3.2}$$

where $R_{\tau(\omega),x(\tau(\omega),\omega)} \in \mathscr{P}(\Omega_0)$ is a martingale solution to the Navier–Stokes system (1.1) *starting at time $\tau(\omega)$ from the initial condition $x(\tau(\omega),\omega)$. Furthermore, for every $B \in \mathcal{B}$ the mapping $\omega \mapsto Q_\omega(B)$ is $\mathcal{B}_\tau$-measurable.*

Proof. We have to be able to select from the set of all martingale solutions in a measurable way. To this end, we observe that as a consequence of stability with respect to the initial time and the initial condition in Theorem 3.1, for every $(s, x_0) \in [0, \infty) \times L^2_\sigma$ the set $\mathscr{C}(s, x_0, C_{t,q})$ of all martingale solutions to (1.1) with the same $C_{t,q}$ is compact with respect to weak convergence of probability measures. Let $\mathrm{Comp}(\mathscr{P}(\Omega_0))$ denote the space of all compact subsets of $\mathscr{P}(\Omega_0)$ equipped with the Hausdorff metric. Using the stability from Theorem 3.1 together with [43, Lemma 12.1.8] we find that the map

$$[0, \infty) \times L^2_\sigma \to \mathrm{Comp}(\mathscr{P}(\Omega_0)), \quad (s, x_0) \mapsto \mathscr{C}(s, x_0, C_{t,q}),$$

is Borel measurable. Accordingly, [43, Theorem 12.1.10] gives the existence of a measurable selection. More precisely, there exists a Borel measurable map

$$[0, \infty) \times L^2_\sigma \to \mathscr{P}(\Omega_0), \quad (s, x_0) \mapsto R_{s,x_0},$$

such that $R_{s,x_0} \in \mathscr{C}(s, x_0, C_{t,q})$ for all $(s, x_0) \in [0, \infty) \times L^2_\sigma$.

As the next step, we recall that the canonical process x on Ω_0 is continuous in H^{-3}, hence $x : [0, \infty) \times \Omega_0 \to H^{-3}$ is progressively measurable with respect to the canonical filtration $(\mathcal{B}^0_t)_{t\geq 0}$, and consequently also with respect to the right continuous filtration $(\mathcal{B}_t)_{t\geq 0}$. In addition, τ is a stopping time with respect to $(\mathcal{B}_t)_{t\geq 0}$. Therefore, it follows from [43, Lemma 1.2.4] that both τ and $x(\tau(\cdot), \cdot)$ are $\mathcal{B}_\tau$-measurable. Furthermore, $L^2_\sigma \subset H^{-3}$ continuously and densely, and by Kuratowski's measurability theorem we know $L^2_\sigma \in \mathcal{B}(H^{-3})$ and $\mathcal{B}(L^2_\sigma) = \mathcal{B}(H^{-3}) \cap L^2_\sigma$, which implies that $1_{\{x(\tau)\in L^2_\sigma\}} \in \mathcal{B}_\tau$. Therefore, $x(\tau(\cdot), \cdot)1_{\{x(\tau)\in L^2_\sigma\}} : \Omega_0 \to L^2_\sigma$ is $\mathcal{B}_\tau$-measurable, where $\mathcal{B}_\tau$ is the σ-algebra associated to τ. Combining this with the measurability of the selection $(s, x_0) \mapsto R_{s,x_0}$ constructed above, we deduce that

$$\Omega_0 \to \mathscr{P}(\Omega_0), \quad \omega \mapsto R_{\tau(\omega),x(\tau(\omega),\omega)1_{\{x(\tau(\omega),\omega)\in L^2_\sigma\}}}, \tag{3.3}$$

is $\mathcal{B}_\tau$-measurable as a composition of $\mathcal{B}_\tau$-measurable mappings. Recall that for every $\omega \in \Omega_0 \cap \{x(\tau) \in L^2_\sigma\}$ this mapping gives a martingale solution starting at the deterministic time $\tau(\omega)$ from the deterministic initial condition $x(\tau(\omega), \omega)$. Hence, for $\omega\in\{x(\tau)\in L^2_\sigma\}$,

$$R_{\tau(\omega),x(\tau(\omega),\omega)}\big(\omega' \in \Omega_0 : x(\tau(\omega), \omega') = x(\tau(\omega), \omega)\big) = 1.$$

Now, we apply [43, Lemma 6.1.1] to deduce that for every $\omega \in \Omega_0 \cap \{x(\tau) \in L^2_\sigma\}$ there is a unique probability measure

$$\delta_\omega \otimes_{\tau(\omega)} R_{\tau(\omega),x(\tau(\omega),\omega)} \in \mathscr{P}(\Omega_0) \tag{3.4}$$

such that for every $\omega \in \Omega_0 \cap \{x(\tau) \in L^2_\sigma\}$, (3.1) and (3.2) hold. This permits us to concatenate, at the deterministic time $\tau(\omega)$, the Dirac mass δ_ω with the martingale solution $R_{\tau(\omega),x(\tau(\omega),\omega)}$. Define

$$Q_\omega = \begin{cases} \delta_\omega \otimes_{\tau(\omega)} R_{\tau(\omega),x(\tau(\omega),\omega)} & \text{if } \omega \in \{x(\tau) \in L^2_\sigma\}, \\ \delta_{x(\cdot\wedge\tau(\omega))} & \text{otherwise.} \end{cases}$$

In order to show that the mapping $\omega \mapsto Q_\omega(B)$ is $\mathcal{B}_\tau$-measurable for every $B \in \mathcal{B}$, it is enough to consider sets of the form $A = \{x(t_1) \in \Gamma_1, \dots, x(t_n) \in \Gamma_n\}$ where $n \in \mathbb{N}$, $0 \le t_1 < \dots < t_n$, and $\Gamma_1, \dots, \Gamma_n \in \mathcal{B}(H^{-3})$. Then by the definition of Q_ω, we have

$$\begin{aligned}
\delta_\omega \otimes_{\tau(\omega)} R_{\tau(\omega),x(\tau(\omega),\omega)}(A) = {} & \mathbf{1}_{[0,t_1)}(\tau(\omega)) R_{\tau(\omega),x(\tau(\omega),\omega)}(A) \\
& + \sum_{k=1}^{n-1} \mathbf{1}_{[t_k,t_{k+1})}(\tau(\omega))\, \mathbf{1}_{\Gamma_1}(x(t_1,\omega)) \cdots \mathbf{1}_{\Gamma_k}(x(t_k,\omega)) \\
& \quad \times R_{\tau(\omega),x(\tau(\omega),\omega)}\big(x(t_{k+1}) \in \Gamma_{k+1}, \dots, x(t_n) \in \Gamma_n\big) \\
& + \mathbf{1}_{[t_n,\infty)}(\tau(\omega))\, \mathbf{1}_{\Gamma_1}(x(t_1,\omega)) \cdots \mathbf{1}_{\Gamma_n}(x(t_n,\omega)).
\end{aligned}$$

Here the right hand side multiplied by $\mathbf{1}_{\{x(\tau) \in L^2_\sigma\}}$ is $\mathcal{B}_\tau$-measurable as a consequence of the $\mathcal{B}_\tau$-measurability of (3.3) and τ. Moreover, $\delta_{x(\cdot \wedge \tau(\omega))}$ is $\mathcal{B}_\tau$-measurable as a consequence of the $\mathcal{B}_\tau$-measurability of $x(\tau \wedge \cdot)$. Thus the final result follows from $\{x(\tau) \in L^2_\sigma\}$ being $\mathcal{B}_\tau$-measurable. ∎

Remark 3.3. If P is a martingale solution up to a stopping time τ, our ultimate goal is to make use of Proposition 3.2 in order to define a probability measure

$$P \otimes_\tau R(\cdot) := \int_{\Omega_0} Q_\omega(\cdot)\, P(d\omega)$$

and show that it is a martingale solution on $[0,\infty)$ in the sense of Definition 3.1 which coincides with P up to time τ. However, due to the fact that τ is only a stopping time with respect to the right continuous filtration $(\mathcal{B}_t)_{t \ge 0}$, (3.1) does not suffice to show that $(Q_\omega)_{\omega \in \Omega_0}$ is a conditional probability distribution of $P \otimes_\tau R$ given $\mathcal{B}_\tau$. More precisely, we cannot prove that for every $A \in \mathcal{B}_\tau$ and $B \in \mathcal{B}$,

$$P \otimes_\tau R(A \cap B) = \int_A Q_\omega(B)\, P(d\omega).$$

This is the reason why the corresponding results of [43], namely Theorem 6.1.2 and in particular Theorem 1.2.10 leading to the desired martingale property (M2), cannot be applied. It will be seen below in Proposition 3.4 that an additional condition on Q_ω, i.e., (3.5), is necessary in order to guarantee (M1)–(M3). To conclude this remark, we note that measurability of the mapping $\omega \mapsto Q_\omega(B)$ in a certain sense is only needed to define the integral in (3.6). Since we do not show that $(Q_\omega)_{\omega \in \Omega_0}$ is a conditional probability distribution, the $\mathcal{B}_\tau$-measurability from Proposition 3.2 is actually not used in the following.

Proposition 3.4. *Let $x_0 \in L^2_\sigma$. Let P be a martingale solution to the Navier–Stokes system (1.1) on $[0,\tau]$ starting at time 0 from the initial condition x_0. In addition to the assumptions of Proposition 3.2, suppose that there exists a Borel set $\mathcal{N} \subset \Omega_{0,\tau}$ such that $P(\mathcal{N}) = 0$ and for every $\omega \in \mathcal{N}^c$,*

$$Q_\omega\big(\omega' \in \Omega_0 : \tau(\omega') = \tau(\omega)\big) = 1. \tag{3.5}$$

Then the probability measure $P \otimes_\tau R \in \mathscr{P}(\Omega_0)$ *defined by*

$$P \otimes_\tau R(\cdot) := \int_{\Omega_0} Q_\omega(\cdot)\, P(d\omega) \tag{3.6}$$

satisfies $P \otimes_\tau R = P$ *on the* σ*-algebra* $\sigma(x(t \wedge \tau) : t \geq 0)$ *and it is a martingale solution to the Navier–Stokes system* (1.1) *on* $[0, \infty)$ *with initial condition* x_0.

Proof. First, we observe that due to (3.5) and (3.1), we have $P \otimes_\tau R(A) = P(A)$ for every Borel set $A \in \sigma(x(t \wedge \tau) : t \geq 0)$. It remains to verify that the measure $P \otimes_\tau R$ satisfies (M1)–(M3) in Definition 3.1 with $s = 0$. The first condition in (M1) follows easily since by construction $P \otimes_\tau R(x(0) = x_0) = P(x(0) = x_0) = 1$; the second one follows from (M3) and the assumption on G. In order to show (M3), we write

$$\begin{aligned} E^{P\otimes_\tau R}&\Big(\sup_{r\in[0,t]} \|x(r)\|_{L^2}^{2q} + \int_0^t \|x(r)\|_{H^\gamma}^2\, dr\Big) \\ &\leq E^{P\otimes_\tau R}\Big(\sup_{r\in[0,t\wedge\tau]} \|x(r)\|_{L^2}^{2q} + \int_0^{t\wedge\tau} \|x(r)\|_{H^\gamma}^2\, dr\Big) \\ &\quad + E^{P\otimes_\tau R}\Big(\sup_{r\in[t\wedge\tau,t]} \|x(r)\|_{L^2}^{2q} + \int_{t\wedge\tau}^t \|x(r)\|_{H^\gamma}^2\, dr\Big). \end{aligned}$$

Here, the first term on the right hand side can be estimated due to the bound (M3) for P, whereas the second term can be bounded based on (M3) for R. Then by (3.5),

$$\begin{aligned} E^{P\otimes_\tau R}&\Big(\sup_{r\in[0,t]} \|x(r)\|_{L^2}^{2q} + \int_0^t \|x(r)\|_{H^\gamma}^2\, dr\Big) \\ &\leq C(\|x_0\|_{L^2}^{2q} + 1) + C(E^P \|x(\tau)\|_{L^2}^{2q} + 1) \leq C(\|x_0\|_{L^2}^{2q} + 1). \end{aligned}$$

In the last step, we have used the fact that τ is bounded together with (M3) for P.

Finally, we shall verify (M2). To this end, we recall that since P is a martingale solution on $[0, \tau]$, the process $M^i_{t\wedge\tau,0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\geq 0}$-martingale under P with quadratic variation process given by $\int_0^{t\wedge\tau} \|G(x(r))^* e_i\|_U^2\, dr$. On the other hand, since for every $\omega \in \Omega_0$, the probability measure $R_{\tau(\omega),x(\tau(\omega),\omega)}$ is a martingale solution starting at time $\tau(\omega)$ from the initial condition $x(\tau(\omega), \omega)$, the process $M^i_{t,t\wedge\tau(\omega)}$ is a continuous square integrable $(\mathcal{B}_t)_{t\geq\tau(\omega)}$-martingale under $R_{\tau(\omega),x(\tau(\omega),\omega)}$ with quadratic variation process given by $\int_{t\wedge\tau(\omega)}^t \|G(x(r))^* e_i\|_U^2\, dr$, $t \geq \tau(\omega)$. In other words, the process $M^i_{t,0} - M^i_{t\wedge\tau(\omega),0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\geq 0}$-martingale under $R_{\tau(\omega),x(\tau(\omega),\omega)}$ with quadratic variation process given by $\int_{t\wedge\tau(\omega)}^t \|G(x(r))^* e_i\|_U^2\, dr$.

Next, we will show that $M^i_{t,0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\geq 0}$-martingale under $P \otimes_\tau R$ with quadratic variation process given by $\int_0^t \|G(x(r))^* e_i\|_U^2\, dr$. To this end, let $s \leq t$ and $A \in \mathcal{B}_s$. We first prove that

$$E^{Q_\omega}[M^i_{t,0}\mathbf{1}_A] = E^{Q_\omega}[M^i_{(t\wedge\tau(\omega))\vee s,0}\mathbf{1}_A]. \tag{3.7}$$

In fact, it is enough to consider sets of the form $A = \{x(t_1) \in \Gamma_1, \dots, x(t_n) \in \Gamma_n\}$ where $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n \leq s$, and $\Gamma_1, \dots, \Gamma_n \in \mathcal{B}(H^{-3})$. For more general $A \in \mathcal{B}_s$ we could use approximation and the continuity of $M^i_{\cdot,0}$. Then by the definition of Q_ω and using the martingale property with respect to $R_{\tau(\omega),x(\tau(\omega),\omega)}$, which is valid for $t \geq \tau(\omega)$, we have

$$
\begin{aligned}
&E^{Q_\omega}[(M^i_{t,0} - M^i_{(t\wedge\tau(\omega))\vee s,0})\,\mathbf{1}_A] \\
&= \mathbf{1}_{[0,t_1)}(\tau(\omega))E^{R_{\tau(\omega),x(\tau(\omega),\omega)}}[(M^i_{t,0} - M^i_{s,0})\,\mathbf{1}_A] \\
&\quad + \sum_{k=1}^{n-1} \mathbf{1}_{[t_k,t_{k+1})}(\tau(\omega))\,\mathbf{1}_{\Gamma_1}(x(t_1,\omega))\cdots\mathbf{1}_{\Gamma_k}(x(t_k,\omega)) \\
&\qquad\qquad\qquad \times E^{R_{\tau(\omega),x(\tau(\omega),\omega)}}((M^i_{t,0} - M^i_{s,0})\,\mathbf{1}_{x(t_{k+1})\in\Gamma_{k+1},\dots,x(t_n)\in\Gamma_n}) \\
&+ \mathbf{1}_{[t_n,\infty)}(\tau(\omega))\,\mathbf{1}_{\Gamma_1}(x(t_1,\omega))\cdots\mathbf{1}_{\Gamma_n}(x(t_n,\omega)) \times E^{R_{\tau(\omega),x(\tau(\omega),\omega)}}(M^i_{t,0} - M^i_{(t\wedge\tau(\omega))\vee s,0}) \\
&= 0.
\end{aligned}
$$

Now (3.7) follows.

Then it follows from (3.6) and (3.4) that

$$
\begin{aligned}
E^{P\otimes_\tau R}[M^i_{t,0}\mathbf{1}_A] &= \int_{\Omega_0} E^{Q_\omega}[M^i_{t,0}\mathbf{1}_A]\,P(d\omega) \\
&= \int_{\Omega_0} E^{\delta_\omega\otimes_{\tau(\omega)}R_{\tau(\omega),x(\tau(\omega),\omega)}}[M^i_{t,0}\mathbf{1}_A]\,P(d\omega).
\end{aligned}
$$

According to (3.7) and then using the key assumption (3.5) we further deduce that

$$
\begin{aligned}
E^{P\otimes_\tau R}[M^i_{t,0}\mathbf{1}_A] &= \int_{\Omega_0} E^{\delta_\omega\otimes_{\tau(\omega)}R_{\tau(\omega),x(\tau(\omega),\omega)}}[M^i_{(t\wedge\tau(\omega))\vee s,0}\mathbf{1}_A]P(d\omega) \\
&= E^{P\otimes_\tau R}[M^i_{(t\wedge\tau)\vee s,0}\mathbf{1}_A] \\
&= E^{P\otimes_\tau R}[M^i_{t\wedge\tau,0}\mathbf{1}_{A\cap\{\tau>s\}}] + E^{P\otimes_\tau R}[M^i_{s,0}\mathbf{1}_{A\cap\{\tau\leq s\}}].
\end{aligned}
$$

Finally, using the martingale property up to τ with respect to P, we get

$$
\begin{aligned}
E^{P\otimes_\tau R}[M^i_{t,0}\mathbf{1}_A] &= E^{P\otimes_\tau R}[M^i_{s,0}\mathbf{1}_{A\cap\{\tau>s\}}] + E^{P\otimes_\tau R}[M^i_{s,0}\mathbf{1}_{A\cap\{\tau\leq s\}}] \\
&= E^{P\otimes_\tau R}[M^i_{s,0}\mathbf{1}_A].
\end{aligned}
$$

Hence M^i is a $(\mathcal{B}_t)_{t\geq 0}$-martingale with respect to $P\otimes_\tau R$. In order to identify its quadratic variation, we proceed similarly and write

$$
\begin{aligned}
&E^{P\otimes_\tau R}\left[\left((M^i_{t,0})^2 - \int_0^t \|G(x(r))^* e_i\|_U^2\,dr\right)\mathbf{1}_A\right] \\
&\quad= \int_{\Omega_0} E^{Q_\omega}\left[\left((M^i_{t,0} - M^i_{t\wedge\tau(\omega),0})^2 - \int_{t\wedge\tau(\omega)}^t \|G(x(r))^* e_i\|_U^2\,dr\right)\mathbf{1}_A\right]P(d\omega) \\
&\qquad + \int_{\Omega_0} E^{Q_\omega}\left[\left((M^i_{t\wedge\tau(\omega),0})^2 - \int_0^{t\wedge\tau(\omega)} \|G(x(r))^* e_i\|_U^2\right)\mathbf{1}_A\right]P(d\omega) \\
&\qquad + 2\int_{\Omega_0} E^{Q_\omega}[(M^i_{t\wedge\tau(\omega),0}(M^i_{t,0} - M^i_{t\wedge\tau(\omega),0}))\,\mathbf{1}_A]\,P(d\omega) \\
&\quad=: J_1 + J_2 + J_3.
\end{aligned}
$$

Here, due to the martingale property with respect to R and P similar to (3.7), we obtain

$$J_1 = \int_{\Omega_0} E^{Q_\omega}\Big[\Big((M^i_{t\wedge\tau(\omega)\vee s,0} - M^i_{t\wedge\tau(\omega),0})^2 - \int_{t\wedge\tau(\omega)}^{t\wedge\tau(\omega)\vee s} \|G(x(r))^* e_i\|_U^2\,dr\Big)\mathbf{1}_A\Big]\,P(d\omega),$$

$$J_2 = \int_{\Omega_0} E^{Q_\omega}\Big[\Big((M^i_{s\wedge\tau(\omega),0})^2 - \int_0^{s\wedge\tau(\omega)} \|G(x(r))^* e_i\|_U^2\,dr\Big)\mathbf{1}_A\Big]\,P(d\omega),$$

$$J_3 = 2\int_{\Omega_0} E^{Q_\omega}[M^i_{t\wedge\tau(\omega),0}(M^i_{t\wedge\tau(\omega)\vee s,0} - M^i_{t\wedge\tau(\omega),0})\,\mathbf{1}_A]\,P(d\omega).$$

Combining these calculations and using (3.5) as above we finally deduce that

$$\begin{aligned}
&E^{P\otimes_\tau R}\Big[\Big((M^i_{t,0})^2 - \int_0^t \|G(x(r))^* e_i\|_U^2\,dr\Big)\mathbf{1}_A\Big] \\
&\quad = E^{P\otimes_\tau R}\Big[\Big((M^i_{s\wedge\tau,0})^2 - \int_0^{s\wedge\tau} \|G(x(r))^* e_i\|_U^2\,dr\Big)\mathbf{1}_A\Big] \\
&\qquad + E^{P\otimes_\tau R}\Big[\Big((M^i_{s,0} - M^i_{\tau,0})^2 - \int_\tau^s \|G(x(r))^* e_i\|_U^2\,dr\Big)\mathbf{1}_{A\cap\{\tau\le s\}}\Big] \\
&\qquad + 2E^{P\otimes_\tau R}[M^i_{\tau,0}(M^i_{s,0} - M^i_{\tau,0})\,\mathbf{1}_{A\cap\{\tau\le s\}}] \\
&\quad = E^{P\otimes_\tau R}\Big[\Big((M^i_{s,0})^2 - \int_0^s \|G(x(r))^* e_i\|_U^2\,dr\Big)\mathbf{1}_A\Big],
\end{aligned}$$

which completes the proof of (M2). ■

As the next step, we present an auxiliary result which allows us to show that for weakly continuous stochastic processes, hitting times of open sets are stopping times with respect to the corresponding right continuous filtration. Here we want to emphasize that the filtration $(\mathcal{B}_t)_{t\ge 0}$ used below is not the augmented one since we have to consider different probabilities. As a consequence, we have to be careful about making any conclusions about stopping times.

Lemma 3.5. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a stochastic basis. Let H_1, H_2 be separable Hilbert spaces such that the embedding $H_1 \subset H_2$ is continuous. Suppose that there exists $\{h_k\}_{k\in\mathbb{N}} \subset H_2^* \subset H_1^*$ such that for all $f \in H_1$,*

$$\|f\|_{H_1} = \sup_{k\in\mathbb{N}} h_k(f).$$

Suppose X is an $(\mathcal{F}_t)_{t\ge 0}$-adapted stochastic process with trajectories in $C([0,\infty); H_2)$. Let $L > 0$ and $\alpha \in (0,1)$. Then

$$\tau_1 := \inf\{t \ge 0 : \|X(t)\|_{H_1} > L\} \quad \textit{and} \quad \tau_2 := \inf\{t \ge 0 : \|X\|_{C_t^\alpha H_1} > L\}$$

are $(\mathcal{F}_{t+})_{t\ge 0}$-stopping times where $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$.

We note that in the above result, the process X a priori need not take values in H_1. In other words, without additional regularity of the trajectories of X, we simply have $\tau_1 = \tau_2 = 0$. However, in the application of Lemma 3.5 in the proof of Theorem 1.2 below, additional regularity will be known a.s. under a suitable probability measure.

Proof of Lemma 3.5. In the proof we use $X^\omega(s)$ to denote $X(s,\omega)$. First, we observe that the trajectories of X are lower semicontinuous in H_1 in the following sense:

$$\begin{aligned}\|X(t)\|_{H_1} &= \sup_{k\in\mathbb{N}} h_k(X(t)) = \sup_{k\in\mathbb{N}} \lim_{s\to t} h_k(X(s)) \le \liminf_{s\to t} \sup_{k\in\mathbb{N}} h_k(X(s)) \\ &\le \liminf_{s\to t} \|X(s)\|_{H_1},\end{aligned} \tag{3.8}$$

where $t \ge 0$. Note that since by assumption we only know that X takes values in $H_2 \supset H_1$, the H_1-norms appearing in (3.8) may be infinite. Next, for $t > 0$ we have

$$\{\tau_1 \ge t\} = \bigcap_{s\in[0,t]} \{\|X(s)\|_{H_1} \le L\} = \bigcap_{s\in[0,t]\cap\mathbb{Q}} \{\|X(s)\|_{H_1} \le L\} \in \mathcal{F}_t.$$

Indeed, to show the first equality, we observe that the right hand side is a subset of the left one. For the converse inclusion, we know that $\{\tau_1 > t\}$ is a subset of the right hand side. Now, we consider $\omega \in \{\tau_1 = t\}$. In this case, $\|X^\omega(s)\|_{H_1} \le L$ for every $s \in [0,t)$. Thus, there exists a sequence $t_k \uparrow t$ such that $\|X^\omega(t_k)\|_{H_1} \le L$ and by the lower semicontinuity of X it follows that $\|X^\omega(t)\|_{H_1} \le L$. The second equality is also a consequence of lower semicontinuity. Indeed, if ω belongs to the right hand side, then for $s \in [0,t]$, $s \notin \mathbb{Q}$, there is a sequence $(s_k)_{k\in\mathbb{N}} \subset [0,t] \cap \mathbb{Q}$, $s_k \to s$, such that $\|X^\omega(s_k)\|_{H_1} \le L$. Hence $\|X^\omega(s)\|_{H_1} \le L$ and ω belongs to the left hand side as well. Therefore, we deduce that

$$\{\tau_1 \le t\} = \bigcap_{\varepsilon>0} \{\tau_1 < t+\varepsilon\} \in \mathcal{F}_{t+},$$

which proves that τ_1 is an $(\mathcal{F}_{t+})_{t\ge0}$-stopping time.

We proceed similarly for τ_2. By the same argument as in (3.8) we find that also the time increments of X are lower semicontinuous in H_1. More precisely, for $t_1, t_2 \ge 0$,

$$\|X(t_1) - X(t_2)\|_{H_1} \le \liminf_{s_1\to t_1,\, s_2\to t_2} \|X(s_1) - X(s_2)\|_{H_1},$$

and as a consequence if $t_1 \ne t_2$ then

$$\frac{\|X(t_1) - X(t_2)\|_{H_1}}{|t_1 - t_2|^\alpha} \le \liminf_{\substack{s_1\to t_1,\, s_2\to t_2 \\ s_1\ne s_2}} \frac{\|X(s_1) - X(s_2)\|_{H_1}}{|s_1 - s_2|^\alpha}.$$

This implies for $t > 0$ that

$$\{\tau_2 \ge t\} = \{\|X\|_{C_t^\alpha H_1} \le L\} = \bigcap_{s_1\ne s_2\in[0,t]\cap\mathbb{Q}} \left\{ \frac{\|X(s_1) - X(s_2)\|_{H_1}}{|s_1 - s_2|^\alpha} \le L \right\} \in \mathcal{F}_t, \tag{3.9}$$

Indeed, for the first equality, the inclusion $\supset$ is immediate, because the process $t \mapsto \|X\|_{C_t^\alpha H_1}$ is nondecreasing. For the converse inclusion, we know that $\{\tau_2 > t\}$ is a subset of the right hand side. Let $\omega \in \{\tau_2 = t\}$. Then there is a sequence $t_k \uparrow t$ such that $\|X^\omega\|_{C_{t_k}^\alpha H} \leq L$ and we have

$$\begin{aligned}\sup_{s_1 \neq s_2 \in [0,t]} \frac{\|X^\omega(s_1) - X^\omega(s_2)\|_{H_1}}{|s_1 - s_2|^\alpha} &\leq \sup_{s_1 \neq s_2 \in [0,t]} \liminf_{k \to \infty} \frac{\|X^\omega(s_1 \wedge t_k) - X^\omega(s_2 \wedge t_k)\|_{H_1}}{|s_1 \wedge t_k - s_2 \wedge t_k|^\alpha} \\ &\leq \sup_{k \in \mathbb{N}} \sup_{s_1 \neq s_2 \in [0,t_k]} \frac{\|X^\omega(s_1) - X^\omega(s_2)\|_{H_1}}{|s_1 - s_2|^\alpha} \leq L.\end{aligned}$$

We deduce that $\|X^\omega\|_{C_t^\alpha H_1} \leq L$, hence ω also belongs to the set on the right hand side of the first equality in (3.9). The second equality in (3.9) follows by a similar argument. Therefore, we conclude that τ_2 is an $(\mathcal{F}_{t+})_{t \geq 0}$-stopping time. ■

3.3. Application to solutions obtained through Theorem 1.1

As the first step, we decompose the Navier–Stokes system (1.5) into two parts, one of which is linear and contains the stochastic integral, whereas the other one is a nonlinear but random PDE. More precisely, we consider

$$\begin{aligned} dz - \Delta z + \nabla P_1 dt &= dB, \\ \operatorname{div} z &= 0, \\ z(0) &= 0, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \partial_t v - \Delta v + \operatorname{div}((v + z) \otimes (v + z)) + \nabla P_2 &= 0, \\ \operatorname{div} v &= 0, \end{aligned} \tag{3.11}$$

where P_1 and P_2 denote the associated pressure terms. Note that the initial value for v was not given in advance but it was part of the construction in Theorem 1.1. This decomposition allows us to separate the difficulties coming from the stochastic perturbation from those originating in the nonlinearity.

Now, we fix a GG^*-Wiener process B defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and we denote by $(\mathcal{F}_t)_{t \geq 0}$ its normal filtration, i.e. the canonical filtration of B augmented by all the $\mathbf{P}$-negligible sets. This filtration is right continuous. We recall that using the factorization method it is standard to derive regularity of the stochastic convolution z which solves the linear equation (3.10) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$. In particular, the following result follows from [14, Theorems 5.14, 5.16] together with the Kolmogorov continuity criterion.

Proposition 3.6. *Suppose that* $\mathrm{Tr}(GG^*) < \infty$. *Then for all* $\delta \in (0, 1/2)$ *and* $T > 0$,

$$E^{\mathbf{P}}[\|z\|_{C_T H^{1-\delta}} + \|z\|_{C_T^{1/2-\delta} L^2}] < \infty.$$

As the next step, for every $\omega \in \Omega_0$ we define a process $M^{\omega}_{t,0}$ similarly to Definition 3.1, that is,

$$M^{\omega}_{t,0} := \omega(t) - \omega(0) + \int_0^t [\mathbb{P}\,\mathrm{div}(\omega(r) \otimes \omega(r)) - \Delta\omega(r)]\,dr \tag{3.12}$$

and for every $\omega \in \Omega_0$ we let

$$Z^{\omega}(t) := M^{\omega}_{t,0} + \int_0^t \mathbb{P}\Delta e^{(t-r)\Delta} M^{\omega}_{r,0}\,dr. \tag{3.13}$$

The idea behind these definitions is as follows. The process M is defined in terms of the canonical process x and hence its definition makes sense for every $\omega \in \Omega_0$, i.e. without the reference to any probability measure. Consequently, the same applies to Z. In addition, if P is a martingale solution to the Navier–Stokes system (1.5), the process M is a GG^*-Wiener process under P. Hence we may apply integration by parts to show that Z solves (3.10) with B replaced by M. In other words, under P, Z is almost surely equal to a stochastic convolution, i.e., we have

$$Z(t) = \int_0^t \mathbb{P}e^{(t-r)\Delta}\,dM_{r,0} \quad P\text{-a.s.}$$

In addition, by definition of Z and M together with the regularity of trajectories in Ω_0, it follows that for every $\omega \in \Omega_0$, $Z^{\omega} \in C([0,\infty); H^{-3})$. For $n \in \mathbb{N}$, $L > 0$ and for $\delta \in (0, 1/12)$ to be determined below we define

$$\begin{aligned}\tau^n_L(\omega) = \inf&\left\{t \geq 0 : \|Z^{\omega}(t)\|_{H^{1-\delta}} > \frac{(L-1/n)^{1/4}}{C_S}\right\}\\ &\wedge \inf\left\{t > 0 : \|Z^{\omega}\|_{C_t^{1/2-2\delta}L^2} > \frac{(L-1/n)^{1/2}}{C_S}\right\} \wedge L,\end{aligned}$$

where C_S is the Sobolev constant for $\|f\|_{L^\infty} \leq C_S\|f\|_{H^{(3+\sigma)/2}}$ with $\sigma > 0$. We observe that the sequence $(\tau^n_L)_{n\in\mathbb{N}}$ is nondecreasing and define

$$\tau_L := \lim_{n\to\infty} \tau^n_L. \tag{3.14}$$

Note that without additional regularity of the trajectory ω, we have $\tau^n_L(\omega) = 0$. However, under P we may use the regularity assumption on G to deduce that $Z \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}}L^2$ P-a.s. By Lemma 3.5 we find that τ^n_L is a $(\mathcal{B}_t)_{t\geq 0}$-stopping time and consequently also τ_L is a $(\mathcal{B}_t)_{t\geq 0}$-stopping time as an increasing limit of stopping times. We emphasize that we need to introduce the stopping time on the path space without using any probability. The introduction of τ^n_L is to approximate τ_L which coincides with the stopping time T_L introduced in (4.2) below under the law of the convex integration solution. Moreover, τ^n_L is defined on the path space and is a stopping time by Lemma 3.5. We cannot directly prove that T_L is a stopping time on the path space without using the continuity property of the Brownian motion.

As the next step, we apply Theorem 1.1 on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. We note that the stopping time t from the statement of Theorem 1.1 is given by T_L for a sufficiently large $L > 1$, defined in (4.2) below. We recall that u is adapted to $(\mathcal{F}_t)_{t\geq 0}$ which is an essential property employed in what follows. We denote by P the law of u and prove the following result.

Proposition 3.7. *The probability measure P is a martingale solution to the Navier–Stokes system* (1.5) *on $[0, \tau_L]$ in the sense of Definition* 3.2*, where τ_L was defined in* (3.14)*.*

Proof. Recall that the stopping time T_L was defined in (4.2) in terms of the process z, the solution to the linear equation (3.10). Theorem 1.1 yields the existence of a solution u to the Navier–Stokes system (1.5) on $[0, T_L]$ such that $u(\cdot \wedge T_L) \in \Omega_0$ $\mathbf{P}$-a.s. We will now prove that

$$\tau_L(u) = T_L \quad \mathbf{P}\text{-a.s.} \tag{3.15}$$

To this end, we observe that due to the definition of M in (3.12) and Z in (3.13) together with the fact that u solves the Navier–Stokes system (1.5) on $[0, T_L]$, we have

$$Z^u(t) = z(t) \quad \text{ for } t \in [0, T_L]\ \mathbf{P}\text{-a.s.} \tag{3.16}$$

Since $z \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2$ $\mathbf{P}$-a.s. according to Proposition 3.6, the trajectories of the processes

$$t \mapsto \|z(t)\|_{H^{1-\delta}} \quad \text{and} \quad t \mapsto \|z\|_{C_t^{1/2-2\delta} L^2}$$

are $\mathbf{P}$-a.s. continuous. It follows from the definition of T_L that one of the following three statements holds $\mathbf{P}$-a.s.:

$$\text{either} \quad T_L = L \quad \text{or} \quad \|z(T_L)\|_{H^{1-\delta}} \geq L^{1/4}/C_S \quad \text{or} \quad \|z\|_{C^{1/2-2\delta}_{T_L} L^2} \geq L^{1/2}/C_S.$$

Therefore, as a consequence of (3.16), we deduce that $\tau_L(u) \leq T_L$ $\mathbf{P}$-a.s. Suppose now that $\tau_L(u) < T_L$ on a set of positive probability $\mathbf{P}$. Then on this set, either

$$\begin{aligned} &\|z(\tau_L(u))\|_{H^{1-\delta}} = \|Z^u(\tau_L(u))\|_{H^{1-\delta}} \geq L^{1/4}/C_S, \quad \text{or} \\ &\|Z^u\|_{C^{1/2-2\delta}_{\tau_L(u)} L^2} = \|z\|_{C^{1/2-2\delta}_{\tau_L(u)} L^2} \geq L^{1/2}/C_S, \end{aligned}$$

which however contradicts the definition of T_L. Hence we have proved (3.15).

Recall that τ_L is a $(\mathcal{B}_t)_{t\geq 0}$-stopping time. We intend to show that P is a martingale solution to the Navier–Stokes system (1.5) on $[0, \tau_L]$ in the sense of Definition 3.2. First, we observe that it can be seen from the construction in Theorem 1.1 that the initial value $u(0) = v(0) + z(0) = v(0)$ is indeed deterministic. Hence condition (M1) follows. However, we note that the initial value $v(0)$ cannot be prescribed in advance. In other words, Theorem 1.1 does not yield a solution to the Cauchy problem, it only provides the existence of an initial condition for which a solution violating the energy inequality exists.

For an appropriate choice of the constant $C_{t,q}$ in Definition 3.2, which has to depend on the constant C_L in (1.6) in Theorem 1.1, condition (M3) also follows.

Let us now verify (M2). To this end, let $s \le t$ and let g be a bounded, real-valued, $\mathcal{B}_s$-measurable and continuous function on Ω_0. Since $u(\cdot \wedge T_L)$ is an $(\mathcal{F}_t)_{t\ge0}$-adapted process and (3.15) holds, we deduce that $u(\cdot \wedge \tau_L(u))$ is also $(\mathcal{F}_t)_{t\ge0}$-adapted. Consequently, the composition $g(u(\cdot \wedge \tau_L(u)))$ is $\mathcal{F}_s$-measurable. On the other hand, we know that under $\mathbf{P}$, $M^{u,i}_{t\wedge\tau_L(u),0} = \langle B_{t\wedge\tau_L(u)}, e_i\rangle$ is an $(\mathcal{F}_t)_{t\ge0}$-martingale. Its quadratic variation process is given by $\|Ge_i\|^2_{L^2}(t \wedge \tau_L(u))$. Therefore, we have

$$E^P[M^i_{t\wedge\tau_L,0}g] = E^{\mathbf{P}}[M^{u,i}_{t\wedge\tau_L(u),0}g(u)] = E^{\mathbf{P}}[M^{u,i}_{s\wedge\tau_L(u),0}g(u)] = E^P[M^i_{s\wedge\tau_L,0}g]$$

and by similar arguments we also find that

$$E^P\big[\big((M^i_{t\wedge\tau_L,0})^2 - (t\wedge\tau_L)\|Ge_i\|^2_{L^2}\big)g\big] = E^P\big[\big((M^i_{s\wedge\tau_L,0})^2 - (s\wedge\tau_L)\|Ge_i\|^2_{L^2}\big)g\big].$$

Accordingly, the process $M^i_{t\wedge\tau_L,0}$ is a continuous square integrable $(\mathcal{B}_t)_{t\ge0}$-martingale under P with quadratic variation process given by $\|Ge_i\|^2_{L^2}(t \wedge \tau_L)$, and (M2) in Definition 3.2 follows. ∎

At this point, we are already able to deduce that martingale solutions on $[0, \tau_L]$ in the sense of Definition 3.2 are not unique. However, we aim at a stronger result, namely that globally defined martingale solutions on $[0, \infty)$ in the sense of Definition 3.1 are not unique. Moreover, we will prove that for an arbitrary time interval $[0, T]$, the martingale solutions on $[0, T]$ are not unique. To this end, we will extend P to a martingale solution on $[0, \infty)$ through the procedure developed in Section 3.2. More precisely, as an immediate corollary of Proposition 3.7 and the fact that τ_L is a $(\mathcal{B}_t)_{t\ge0}$-stopping time, we may apply Proposition 3.2. In particular, we construct Q_ω for all $\omega \in \Omega_0$. In view of Proposition 3.4, (M1)–(M3) follow once we verify condition (3.5) for Q_ω. This will be achieved in the following result.

Proposition 3.8. *The probability measure $P \otimes_{\tau_L} R$ is a martingale solution to the Navier–Stokes system* (1.5) *on* $[0, \infty)$ *in the sense of Definition* 3.1.

Proof. In light of Propositions 3.2 and 3.4, it only remains to establish (3.5). Due to (3.15) and (3.16), we know that

$$\begin{aligned} P\big(\omega : Z^\omega(\cdot \wedge \tau_L(\omega)) \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}}L^2\big) \\ &= \mathbf{P}\big(Z^u(\cdot \wedge \tau_L(u)) \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}}L^2\big) \\ &= \mathbf{P}\big(z(\cdot \wedge T_L) \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}}L^2\big) = 1. \end{aligned}$$

This means that there exists a P-measurable set $\mathcal{N} \subset \Omega_{0,\tau_L}$ such that $P(\mathcal{N}) = 0$ and for $\omega \in \mathcal{N}^c$,

$$Z^\omega_{\cdot\wedge\tau_L(\omega)} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}}L^2. \tag{3.17}$$

On the other hand, it follows from (3.13) that for every $\omega' \in \Omega_0$,

$$
\begin{aligned}
&Z^{\omega'}(t) - Z^{\omega'}(t \wedge \tau_L(\omega)) \\
&\qquad = M^{\omega'}_{t,0} - e^{(t-t\wedge\tau_L(\omega))\Delta} M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_{t\wedge\tau_L(\omega)}^{t} \mathbb{P}\Delta e^{(t-s)\Delta} M^{\omega'}_{s,0}\, ds \\
&\qquad\quad + (e^{(t-t\wedge\tau_L(\omega))\Delta} - I)\Big[M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_0^{t\wedge\tau_L(\omega)} \mathbb{P}\Delta e^{(t\wedge\tau_L(\omega)-s)\Delta} M^{\omega'}_{s,0}\, ds \Big] \\
&\qquad = \mathbb{Z}^{\omega'}_{\tau_L(\omega)}(t) + (e^{(t-t\wedge\tau_L(\omega))\Delta} - I) Z^{\omega'}(t\wedge\tau_L(\omega))
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbb{Z}^{\omega'}_{\tau_L(\omega)}(t) &= M^{\omega'}_{t,0} - e^{(t-t\wedge\tau_L(\omega))\Delta} M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_{t\wedge\tau_L(\omega)}^{t} \mathbb{P}\Delta e^{(t-s)\Delta} M^{\omega'}_{s,0}\, ds \\
&= M^{\omega'}_{t,0} - M^{\omega'}_{t\wedge\tau_L(\omega),0} + \int_{t\wedge\tau_L(\omega)}^{t} \mathbb{P}\Delta e^{(t-s)\Delta} (M^{\omega'}_{s,0} - M^{\omega'}_{s\wedge\tau_L(\omega),0})\, ds.
\end{aligned}
$$

Since $M_{\cdot,0} - M_{\cdot\wedge\tau_L(\omega),0}$ is $\mathcal{B}^{\tau_L(\omega)}$-measurable, we know that $\mathbb{Z}^{\omega'}_{\tau_L(\omega)}$ is $\mathcal{B}^{\tau_L(\omega)}$-measurable.

Using (3.1) and (3.2) we find that for all $\omega \in \Omega_0$,

$$
\begin{aligned}
&Q_\omega(\omega' \in \Omega_0 : Z^{\omega'}_{\cdot} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2) \\
&\quad = Q_\omega(\omega' \in \Omega_0 : Z^{\omega'}_{\cdot\wedge\tau_L(\omega)} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2,\ \mathbb{Z}^{\omega'}_{\tau_L(\omega)} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2) \\
&\quad = \delta_\omega(\omega' \in \Omega_0 : Z^{\omega'}_{\cdot\wedge\tau_L(\omega)} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2) \\
&\qquad\qquad \times R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}(\omega' \in \Omega_0 : \mathbb{Z}^{\omega'}_{\tau_L(\omega)} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2).
\end{aligned}
$$

Here the first factor on the right hand side equals 1 for all $\omega \in \mathcal{N}^c$ due to (3.17). Since for $\omega \in \{x(\tau) \in L^2_\sigma\}$, $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$ is a martingale solution to the Navier–Stokes system (1.5) starting at the deterministic time $\tau_L(\omega)$ from the deterministic initial condition $x(\tau_L(\omega),\omega)$, the process $\omega' \mapsto M^{\omega'}_{\cdot,0} - M^{\omega'}_{\cdot\wedge\tau_L(\omega),0}$ is a GG^*-Wiener process starting from $\tau_L(\omega)$ with respect to $(\mathcal{B}_t)_{t\geq 0}$ under the measure $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$. Due to the regularity of its covariance we deduce that also the second factor equals 1. Indeed, for $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$-a.e. ω' we have

$$
\mathbb{Z}^{\omega'}_{\tau_L(\omega)}(t) = \int_0^t \mathbb{P} e^{(t-s)\Delta}\, d(M^{\omega'}_{s,0} - M^{\omega'}_{s\wedge\tau_L(\omega),0}),
$$

and the regularity of this stochastic convolution follows again from Proposition 3.6. In particular, for $R_{\tau_L(\omega),x(\tau_L(\omega),\omega)}$-a.e. ω',

$$
\mathbb{Z}^{\omega'}_{\tau_L(\omega)} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2.
$$

To summarize, we have proved that for all $\omega \in \mathcal{N}^c \cap \{x(\tau) \in L^2_\sigma\}$,

$$
Q_\omega(\omega' \in \Omega_0 : Z^{\omega'}_{\cdot} \in CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2) = 1.
$$

As a consequence, for all $\omega \in \mathcal{N}^c \cap \{x(\tau) \in L^2_\sigma\}$ there exists a measurable set N_ω such that $Q_\omega(N_\omega) = 0$ and for all $\omega' \in N^c_\omega$ the trajectory $t \mapsto Z^{\omega'}(t)$ belongs to $CH^{1-\delta} \cap C^{1/2-\delta}_{\mathrm{loc}} L^2$. Therefore, by (3.14) we deduce that $\tau_L(\omega') = \bar{\tau}_L(\omega')$ for all $\omega' \in N^c_\omega$ where

$$\begin{aligned}\bar{\tau}_L(\omega') := {} & \inf\{t \geq 0 : \|Z^{\omega'}(t)\|_{H^{1-\delta}} \geq L^{1/4}/C_S\} \\ & \wedge \inf\{t \geq 0 : \|Z^{\omega'}\|_{C_t^{1/2-2\delta}L^2} \geq L^{1/2}/C_S\} \wedge L.\end{aligned}$$

This implies that for $t < L$,

$$\begin{aligned}&\{\omega' \in N^c_\omega : \tau_L(\omega') \leq t\} \\ &\quad = \Big\{\omega' \in N^c_\omega : \sup_{s\in\mathbb{Q}, s\leq t} \|Z^{\omega'}(s)\|_{H^{1-\delta}} \geq L^{1/4}/C_S\Big\} \\ &\qquad \cup \Big\{\omega' \in N^c_\omega : \sup_{s_1\neq s_2\in\mathbb{Q}\cap[0,t]} \frac{\|Z^{\omega'}(s_1) - Z^{\omega'}(s_2)\|_{L^2}}{|s_1 - s_2|^{1/2-2\delta}} \geq L^{1/2}/C_S\Big\} \\ &\quad =: N^c_\omega \cap A_t.\end{aligned} \tag{3.18}$$

Finally, we deduce that for all $\omega \in \mathcal{N}^c \cap \{x(\tau) \in L^2_\sigma\}$ with $P(x(\tau) \in L^2_\sigma) = 1$,

$$\begin{aligned}&Q_\omega(\omega' \in \Omega_0 : \tau_L(\omega') = \tau_L(\omega)) = Q_\omega(\omega' \in N^c_\omega : \tau_L(\omega') = \tau_L(\omega)) \\ &\quad = Q_\omega\big(\omega' \in N^c_\omega : \omega'(s) = \omega(s),\, 0 \leq s \leq \tau_L(\omega),\, \tau_L(\omega') = \tau_L(\omega)\big) = 1,\end{aligned} \tag{3.19}$$

where we have used (3.1) and the fact that (3.18) implies

$$\{\omega' \in N^c_\omega : \bar{\tau}_L(\omega') = \tau_L(\omega)\} = N^c_\omega \cap \Big(A_{\tau_L(\omega)} \setminus \bigcup_{n=1}^{\infty} A_{\tau_L(\omega)-1/n}\Big) \in N^c_\omega \cap \mathcal{B}^0_{\tau_L(\omega)},$$

and $Q_\omega(A_{\tau_L(\omega)} \setminus \bigcup_{n=1}^\infty A_{\tau_L(\omega)-1/n}) = 1$. This verifies condition (3.5) in Proposition 3.4 and as a consequence $P \otimes_{\tau_L} R$ is a martingale solution to the Navier–Stokes system (1.5) on $[0,\infty)$ in the sense of Definition 3.1. ∎

Remark 3.9. The property (3.19) is essential for showing that the concatenated probability measure satisfies (M1)–(M3). This is the reason why we had to introduce $\bar{\tau}_L$ and make use of the continuity of Z under the law of a martingale solution, which is different from the original regularity of Z following merely from its definition (3.13) together with the regularity of trajectories in Ω_0. Without the improved regularity, we could only prove that τ_L is a stopping time with respect to the right continuous filtration $(\mathcal{B}_t)_{t\geq 0}$, and the dependence on the right limit does not allow one to establish (3.19).

Finally, we have all in hand to conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $T > 0$ be arbitrary, let $\kappa = 1/2$ and $K = 2$. Based on Theorem 1.1 and Proposition 3.8 there exists $L > 1$ and a measure $P \otimes_{\tau_L} R$ which is a martingale solution to the Navier–Stokes system (1.5) on $[0,\infty)$ and it coincides on the random interval $[0, \tau_L]$ with the law of the solution constructed through Theorem 1.1.

The martingale solution $P \otimes_{\tau_L} R$ starts from a certain deterministic initial value $x_0 = v(0) \in L^2_\sigma$ dictated by the construction in Theorem 1.1. The key result is the failure of the energy inequality at time T formulated in (1.7) on the set $\{T_L \geq T\} \subset \Omega$. In view of (3.6), (3.19) and (3.15) we obtain, by (1.7) and the choice of $K = 2$,

$$
\begin{aligned}
E^{P\otimes_{\tau_L} R}[\|x(T)\|^2_{L^2}] &= E^{P\otimes_{\tau_L} R}[\mathbf{1}_{\{\tau_L \geq T\}}\|x(T)\|^2_{L^2}] + E^{P\otimes_{\tau_L} R}[\mathbf{1}_{\{\tau_L < T\}}\|x(T)\|^2_{L^2}] \\
&\geq \int_{\Omega_0} E^{Q_\omega}[\mathbf{1}_{\{\tau_L \geq T\}}\|x(T)\|^2_{L^2}]\, P(d\omega) > 2(\|x_0\|^2_{L^2} + T\,\mathrm{Tr}(GG^*)).
\end{aligned}
$$

On the other hand, by a classical compactness argument based on Galerkin approximation we may construct another martingale solution $\tilde{P}$ which starts from the same deterministic initial condition x_0 and which satisfies the energy inequality

$$
E^{\tilde{P}}[\|x(T)\|^2_{L^2}] \leq \|x_0\|^2_{L^2} + T\,\mathrm{Tr}(GG^*).
$$

Therefore, the two martingale solutions $P \otimes_{\tau_L} R$ and $\tilde{P}$ are distinct and nonuniqueness in law holds for the Navier–Stokes system (1.5). ■

4. Proof of Theorem 1.1

In this section we fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let B be a GG^*-Wiener process on $(\Omega, \mathcal{F}, \mathbf{P})$. We let $(\mathcal{F}_t)_{t\geq 0}$ be the normal filtration generated by B, that is, the canonical right continuous filtration augmented by all the $\mathbf{P}$-negligible sets. In order to verify that the solution constructed in this section is a martingale solution before a suitable stopping time, it is essential that the solution is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, which corresponds to a probabilistically strong solution. In the following, we construct a probabilistically strong solution before a stopping time. Furthermore, the solution does not satisfy the energy inequality.

We intend to develop an iteration procedure leading to the proof of Theorem 1.1. More precisely, we apply the convex integration method to the nonlinear equation (3.11). The iteration is indexed by a parameter $q \in \mathbb{N}_0$. We consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}} \subset \mathbb{N}$ which diverges to ∞, and a sequence $\{\delta_q\}_{q\in\mathbb{N}} \subset (0, 1)$ which is decreasing to 0. We choose $a, b \in \mathbb{N}$ and $\beta \in (0, 1)$ and let

$$
\lambda_q = a^{b^q}, \quad \delta_q = \lambda_q^{-2\beta},
$$

where β will be chosen sufficiently small and a as well as b will be chosen sufficiently large. At each step q, a pair $(v_q, \mathring{R}_q)$ is constructed solving the system

$$
\begin{aligned}
\partial_t v_q - \Delta v_q + \mathrm{div}((v_q + z_q) \otimes (v_q + z_q)) + \nabla p_q &= \mathrm{div}\, \mathring{R}_q, \\
\mathrm{div}\, v_q &= 0,
\end{aligned}
\tag{4.1}
$$

with $z_q = \mathbb{P}_{\leq f(q)} z$ for $f(q) = \lambda_{q+1}^{\alpha/8}$ and $\mathbb{P}_{\leq f(q)}$ being the Fourier multiplier operator, which projects a function onto its Fourier frequencies $\leq f(q)$ in absolute value. Hence, in addition to the conditions below, we need $b\alpha/8 \in \mathbb{N}$ such that $f(q) \in \mathbb{N}$. By the Sobolev embedding we know $\|f\|_{L^\infty} \leq C_S \|f\|_{H^{(3+\sigma)/2}}$ for $\sigma > 0$, where we choose $C_S \geq 1$. For $L > 1$ and $0 < \delta < 1/12$ define

$$T_L := \inf\{t \geq 0 : \|z(t)\|_{H^{1-\delta}} \geq L^{1/4}/C_S\} \wedge \inf\{t \geq 0 : \|z\|_{C_t^{1/2-2\delta}L^2} \geq L^{1/2}/C_S\} \wedge L. \tag{4.2}$$

According to Proposition 3.6, the stopping time T_L is $\mathbf{P}$-a.s. strictly positive and $T_L \uparrow \infty$ as $L \to \infty$ $\mathbf{P}$-a.s. Moreover, for $t \in [0, T_L]$,

$$\begin{aligned} &\|z(t)\|_{H^{1-\delta}} \leq L^{1/4}/C_S, \quad \|z\|_{C_t^{1/2-2\delta}L^2} \leq L^{1/2}/C_S, \quad \|z_q\|_{C_t^{1/2-2\delta}L^2} \leq L^{1/2}/C_S, \\ &\|z_q(t)\|_{L^\infty} \leq L^{1/4}\lambda_{q+1}^{\alpha/8}, \quad \|\nabla z_q(t)\|_{L^\infty} \leq L^{1/4}\lambda_{q+1}^{\alpha/4}, \quad \|z_q\|_{C_t^{1/2-2\delta}L^\infty} \leq \lambda_{q+1}^{\alpha/4} L^{1/2}. \end{aligned} \tag{4.3}$$

Let $M_0(t) = L^4 e^{4Lt}$. By induction on q we assume the following bounds for the iterations $(v_q, \mathring{R}_q)$: if $t \in [0, T_L]$ then

$$\begin{aligned} \|v_q\|_{C_t L^2} &\leq M_0(t)^{1/2}\Big(1 + \sum_{1\leq r\leq q} \delta_r^{1/2}\Big) \leq 2M_0(t)^{1/2}, \\ \|v_q\|_{C^1_{t,x}} &\leq M_0(t)^{1/2}\lambda_q^4, \\ \|\mathring{R}_q\|_{C_t L^1} &\leq M_0(t) c_R \delta_{q+1}. \end{aligned} \tag{4.4}$$

Here we define $\sum_{1\leq r\leq 0} := 0$, and $c_R > 0$ is a sufficiently small universal constant given in (4.28) and (4.37) below. In addition, we use $\sum_{r\geq 1} \delta_r^{1/2} \leq \sum_{r\geq 1} a^{-rb\beta} = \frac{a^{-\beta b}}{1-a^{-\beta b}} < 1/2$, which boils down to the requirement

$$a^{\beta b} > 3, \tag{4.5}$$

which we assume from now on. The iteration will be initiated through the following result which also establishes compatibility conditions between the parameters L, a, β, b essential for what follows.

Lemma 4.1. *For $L > 1$ define*

$$v_0(t, x) = \frac{L^2 e^{2Lt}}{(2\pi)^{3/2}} (\sin(x_3), 0, 0).$$

Then the associated Reynolds stress is given by[1]

$$\begin{aligned} \mathring{R}_0(t, x) = {} & \frac{(2L+1)L^2 e^{2Lt}}{(2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & -\cos(x_3) \\ 0 & 0 & 0 \\ -\cos(x_3) & 0 & 0 \end{pmatrix} \\ & + v_0 \mathring{\otimes} z_0 + z_0 \mathring{\otimes} v_0 + z_0 \mathring{\otimes} z_0. \end{aligned} \tag{4.6}$$

[1] We denote by $\mathring{\otimes}$ the trace-free part of the tensor product.

Moreover, all the estimates in (4.4) *on level* $q = 0$ *for* $(v_0, \mathring{R}_0)$*, as well as* (4.5)*, are valid provided*

$$45 \cdot (2\pi)^{3/2} < 5 \cdot (2\pi)^{3/2} a^{2\beta b} \leq c_R L \leq c_R \left(\frac{(2\pi)^{3/2} a^4}{2} - 1 \right). \tag{4.7}$$

In particular, we require

$$c_R L > 45 \cdot (2\pi)^{3/2}. \tag{4.8}$$

Furthermore, the initial values $v_0(0, x)$ *and* $\mathring{R}_0(0, x)$ *are deterministic.*

Proof. The first bound in (4.4) follows immediately since

$$\|v_0(t)\|_{L^2} = \frac{L^2 e^{2Lt}}{\sqrt{2}} \leq M_0(t)^{1/2}.$$

For the second bound, we have

$$\|v_0\|_{C^1_{t,x}} \leq M_0(t)^{1/2} \frac{2(1+L)}{(2\pi)^{3/2}} \leq M_0(t)^{1/2} \lambda_0^4 = M_0(t)^{1/2} a^4$$

provided

$$\frac{2(1+L)}{(2\pi)^{3/2}} \leq a^4. \tag{4.9}$$

A direct computation implies that the corresponding Reynolds stress is given by (4.6) and we obtain

$$\|\mathring{R}_0(t)\|_{L^1} \leq (2\pi)^{3/2} M_0(t)^{1/2} 2(2L+1) + 2M_0(t)^{1/2} L^{1/4} + L^{1/2}.$$

Therefore, the desired third bound in (4.4) holds provided

$$\|\mathring{R}_0(t)\|_{L^1} \leq 5 \cdot (2\pi)^{3/2} M_0(t)/L \leq M_0(t) c_R \delta_1 = M_0(t) c_R a^{-2\beta b},$$

which requires $5 \cdot (2\pi)^{3/2} L^{-1} \leq c_R a^{-2\beta b}$. Here we have used (4.8) in the first inequality. Combining this condition with (4.9), we obtain the requirement

$$5 \cdot (2\pi)^{3/2} a^{2\beta b} \leq c_R L \leq c_R \left(\frac{(2\pi)^{3/2} a^4}{2} - 1 \right). \tag{4.10}$$

In particular, we require that

$$c_R L > 5 \cdot (2\pi)^{3/2}, \tag{4.11}$$

otherwise the left inequality in (4.10) cannot be fulfilled. Under these conditions, all the estimates in (4.4) are valid on level $q = 0$. Taking into account (4.5), conditions (4.10) and (4.11) are strengthened to (4.7) and (4.8) from the statement of the lemma, and the proof is complete. ∎

The key result of this section which is used to prove Theorem 1.1 is the following.

Proposition 4.2 (Main iteration). *Let $L > 1$ satisfying (4.8) be given and let $(v_q, \mathring{R}_q)$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution to (4.1) satisfying (4.4). Then there exists a choice of parameters a, b, β such that (4.7) is fulfilled and there exist $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(v_{q+1}, \mathring{R}_{q+1})$ which solve (4.1), obey (4.4) at level $q+1$ and for $t \in [0, T_L]$ we have*

$$\|v_{q+1}(t) - v_q(t)\|_{L^2} \leq M_0(t)^{1/2}\delta_{q+1}^{1/2}. \tag{4.12}$$

Furthermore, if $v_q(0)$ and $\mathring{R}_q(0)$ are deterministic, so are $v_{q+1}(0)$ and $\mathring{R}_{q+1}(0)$.

The proof of Proposition 4.2 is presented in Section 4.1. At this point, we take Proposition 4.2 for granted and apply it in order to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof relies on the above described iteration procedure. More precisely, our goal is to prove that for $L > 1$ satisfying (4.8), Lemma 4.1 and Proposition 4.2 give rise to an $(\mathcal{F}_t)_{t\geq 0}$-adapted analytically weak solution v to the transformed problem (3.11). By possibly increasing the value of L, the corresponding solution v fails to satisfy a suitable energy inequality at the given time T. Finally, again by possibly making L bigger, we verify that $u := v + z$ and $\mathfrak{t} := T_L$ fulfill all the requirements in the statement of the theorem.

Starting from $(v_0, \mathring{R}_0)$ given in Lemma 4.1, the iteration of Proposition 4.2 yields a sequence $(v_q, \mathring{R}_q)$ satisfying (4.4) and (4.12). By interpolation we deduce that the following series is summable for $\gamma \in (0, \frac{\beta}{4+\beta})$ and $t \in [0, T_L]$:

$$\begin{aligned}\sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{H^\gamma} &\lesssim \sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{L^2}^{1-\gamma} \|v_{q+1}(t) - v_q(t)\|_{H^1}^{\gamma} \\ &\lesssim M_0(t) \sum_{q\geq 0} \delta_{q+1}^{\frac{1-\gamma}{2}} \lambda_{q+1}^{4\gamma} \lesssim M_0(t).\end{aligned}$$

Thus we obtain a limiting solution $v = \lim_{q\to\infty} v_q$, which lies in $C([0, T_L]; H^\gamma)$. Since v_q is $(\mathcal{F}_t)_{t\geq 0}$-adapted for every $q \geq 0$, the limit v is $(\mathcal{F}_t)_{t\geq 0}$-adapted as well. Furthermore, v is an analytically weak solution to (3.11) since $\lim_{q\to\infty} \mathring{R}_q = 0$ in $C([0, T_L]; L^1)$ and $\lim_{q\to\infty} z_q = z$ in $C([0, T_L]; L^2)$. In addition, there exists a deterministic constant C_L such that

$$\|v(t)\|_{H^\gamma} \leq C_L \quad \text{for all } t \in [0, T_L]. \tag{4.13}$$

Let us now show that the constructed solution v fails to satisfy the corresponding energy inequality at time T. Namely, we will show

$$\|v(T)\|_{L^2} > (\|v(0)\|_{L^2} + L)e^{LT}. \tag{4.14}$$

According to (4.12), in view of $b^{q+1} \geq b(q+1)$ which holds if $b \geq 2$ and then applying (4.5), for all $t \in [0, T_L]$ we obtain

$$\begin{aligned} \|v(t) - v_0(t)\|_{L^2} &\leq \sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{L^2} \leq M_0(t)^{1/2} \sum_{q\geq 0} \delta_{q+1}^{1/2} \\ &\leq M_0(t)^{1/2} \sum_{q\geq 0} (a^{-\beta b})^{q+1} = M_0(t)^{1/2} \frac{a^{-\beta b}}{1 - a^{-\beta b}} < \frac{1}{2} M_0(t)^{1/2}. \end{aligned}$$

Consequently,

$$\begin{aligned} (\|v(0)\|_{L^2} + L)e^{LT} &\leq (\|v_0(0)\|_{L^2} + \|v(0) - v_0(0)\|_{L^2} + L)e^{LT} \\ &\leq \left(\frac{3}{2} M_0(0)^{1/2} + L\right) e^{LT}, \end{aligned}$$

which we want to estimate (strictly) by

$$\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) M_0(T)^{1/2} \leq \|v_0(T)\|_{L^2} - \|v(T) - v_0(T)\|_{L^2} \leq \|v(T)\|_{L^2}$$

on the set $\{T_L \geq T\} \subset \Omega$. In view of the definition of $M_0(t)$, this is indeed possible provided

$$\frac{3}{2} + \frac{1}{L} < \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) e^{LT}. \tag{4.15}$$

In other words, given $T > 0$ and the universal constant $c_R > 0$, we can choose $L = L(T, c_R) > 1$ large enough so that (4.8) as well as (4.15) hold and consequently (4.14) is satisfied. Moreover, in view of Proposition 3.6 and the definition of the stopping times (4.2), we observe that for a given $T > 0$ we may possibly increase L so that the set $\{T_L \geq T\}$ satisfies $\mathbf{P}(T_L \geq T) > \kappa$.

Let us now define $u := v + z$. Then u is $(\mathcal{F}_t)_{t\geq 0}$-adapted, solves the Navier–Stokes system (1.5) and we deduce from (4.13) together with (4.3) that (1.6) holds true. To verify (1.7), we use (4.3) and apply (4.14) on $\{T_L \geq T\}$ to obtain

$$\|u(T)\|_{L^2} \geq \|v(T)\|_{L^2} - \|z(T)\|_{L^2} > (\|v(0)\|_{L^2} + L)e^{LT} - L^{1/2}/C_S.$$

Thus, since $u(0) = v(0)$ we may possibly increase the value of L depending on K and $\mathrm{Tr}(GG^*)$ to deduce the desired lower bound (1.7). The initial value $v(0)$ is deterministic by our construction. Finally, we set $\mathfrak{t} := T_L$, which finishes the proof. ∎

To summarize the above discussion, first we fix the parameter L large enough in dependence on T, c_R, κ, K and $\mathrm{Tr}(GG^*)$. Then we apply Proposition 4.2 and deduce the result of Theorem 1.1. It remains to prove Proposition 4.2 and to verify that the parameters a, b, β can be appropriately chosen.

4.1. The main iteration – proof of Proposition 4.2

The proof of Proposition 4.2 proceeds along the lines of [7, Section 7]. We have to track the proof carefully to make the construction in each step $(\mathcal{F}_t)_{t\geq 0}$-adapted and the initial value $v(0)$ deterministic. In the course of the proof we will need to adjust the value of the parameters a, b, β as further conditions on these parameters will appear. The parameter L is given and will be kept fixed. In addition, we have to make sure that the condition (4.7), which is essential for the failure of the energy inequality in Theorem 1.1, is not violated. However, we observe that the right inequality in (4.7) remains valid if we increase a. In other words, given L we find the minimal value of a for which this inequality holds and from now on we may increase a as we wish. On the other hand, increasing a or b can in principle cause problems in the left inequality in (4.7), but here we may make the parameter β smaller so that the inequality remains true. To summarize, we may freely increase a or b at the cost of making β smaller.

4.1.1. Choice of parameters. In the following, additional parameters will be indispensable and their value has to be carefully chosen in order to respect all the compatibility conditions appearing in the estimations below. First, for a sufficiently small $\alpha \in (0,1)$ to be chosen below, we let $\ell \in (0,1)$ be a small parameter satisfying

$$\ell\lambda_q^4 \leq \lambda_{q+1}^{-\alpha}, \quad \ell^{-1} \leq \lambda_{q+1}^{2\alpha}, \quad 4L \leq \ell^{-1}. \tag{4.16}$$

In particular, we define

$$\ell := \lambda_{q+1}^{-3\alpha/2}\lambda_q^{-2}. \tag{4.17}$$

The last condition in (4.16) together with (4.7) leads to

$$45 \cdot (2\pi)^{3/2} < 5 \cdot (2\pi)^{3/2} a^{2\beta b} \leq c_R L \leq c_R \frac{a^4 \cdot (2\pi)^{3/2} - 1}{2}.$$

We remark that the reasoning from the beginning of Section 4.1 remains valid for this new condition: we may freely increase a provided we make β smaller at the same time. In addition, we will require $\alpha b > 16$ and $\alpha > 18\beta b$.

In order to verify the inductive estimates (4.4) in Sections 4.1.4 and 4.1.6, it will also be necessary to absorb various expressions including $M_0(t)^{1/2}$ for all $t \in [0, T_L]$. Since the stopping time T_L is bounded by L, this reduces to absorbing $M_0(L)^{1/2}$, and it will be seen that the strongest such requirement is

$$M_0(L)^{1/2}\lambda_{q+1}^{13\alpha-1/7} \leq c_R\delta_{q+2}/10, \tag{4.18}$$

needed in Section 4.1.6. In other words,

$$L^2 e^{2L^2} a^{b(13\alpha-1/7+2b\beta)} \ll 1$$

and choosing $b = 8 \cdot 14^2 L^2$, $L \in \mathbb{N}$, (this choice comes from the fact that with our choice of α below we want to guarantee that $\alpha b > 16$, as well as the fact that b is a multiple of 7 needed for the choice of parameters needed for the intermittent jets below, cf. Appendix B) and $e^2 \leq a$ leads to

$$ba^{b/14}a^{b(13\alpha-1/7+2b\beta)} \ll 1.$$

In view of $\alpha > 18\beta b$, this can be achieved by choosing a large enough and $\alpha = 14^{-2}$. This choice also satisfies $\alpha b > 16$ required above, and the condition $\alpha > 18\beta b$ can be achieved by choosing β small. It is also compatible with all the other requirements needed below.

From now on, the parameters α and b remain fixed and the free parameters are a and β for which we already have a lower, respectively upper, bound. Below, we will possibly increase a and decrease β at the same time in order to preserve all the above conditions and to fulfil further conditions appearing below.

4.1.2. Mollification. We intend to replace v_q by a mollified velocity field v_ℓ. To this end, we extend $z_q(t) = z_q(0)$, $v_q(t) = v_q(0)$ for $t < 0$ and let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers on $\mathbb{R}^3$, and let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers with support on $\mathbb{R}^+$. We define a mollification of v_q, $\mathring{R}_q$ and z_q in space and time by convolution as follows:

$$v_\ell = (v_q *_x \phi_\ell) *_t \varphi_\ell, \quad \mathring{R}_\ell = (\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell, \quad z_\ell = (z_q *_x \phi_\ell) *_t \varphi_\ell,$$

where $\phi_\ell = \frac{1}{\ell^3}\phi(\frac{\cdot}{\ell})$ and $\varphi_\ell = \frac{1}{\ell}\varphi(\frac{\cdot}{\ell})$. Since the mollifier φ_ℓ is supported on $\mathbb{R}^+$, it is easy to see that z_ℓ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and so are v_ℓ and $\mathring{R}_\ell$. Since φ_ℓ is supported on $\mathbb{R}^+$, if the initial values $v_q(0)$ and $\mathring{R}_q(0)$ are deterministic, so are $v_\ell(0)$ and $\mathring{R}_\ell(0)$, $\partial_t \mathring{R}_\ell(0)$. Moreover, $z_q(0) = 0$ implies that $z_\ell(0)$ and $R_{\mathrm{com}}(0)$ given below are deterministic as well. Then using (4.1) we find that $(v_\ell, \mathring{R}_\ell)$ satisfies

$$\begin{aligned} \partial_t v_\ell - \Delta v_\ell + \operatorname{div}((v_\ell + z_\ell) \otimes (v_\ell + z_\ell)) + \nabla p_\ell &= \operatorname{div}(\mathring{R}_\ell + R_{\mathrm{com}}), \\ \operatorname{div} v_\ell &= 0, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} R_{\mathrm{com}} &= (v_\ell + z_\ell) \mathring{\otimes} (v_\ell + z_\ell) - ((v_q + z_q) \mathring{\otimes} (v_q + z_q)) *_x \phi_\ell *_t \varphi_\ell, \\ p_\ell &= (p_q *_x \phi_\ell) *_t \varphi_\ell - \tfrac{1}{3}(|v_\ell + z_\ell|^2 - (|v_q + z_q|^2 *_x \phi_\ell) *_t \varphi_\ell). \end{aligned}$$

By using (4.4) and (4.16) we find, for $t \in [0, T_L]$,

$$\begin{aligned} \|v_q - v_\ell\|_{C_t L^2} \lesssim \|v_q - v_\ell\|_{C^0_{t,x}} \lesssim \ell \|v_q\|_{C^1_{t,x}} \leq \ell \lambda_q^4 M_0(t)^{1/2} &\leq M_0(t)^{1/2} \lambda_{q+1}^{-\alpha} \\ &\leq \tfrac{1}{4} M_0(t)^{1/2} \delta_{q+1}^{1/2}, \end{aligned} \tag{4.20}$$

where we use the fact that $\alpha > \beta$ and we choose a large enough in order to absorb the implicit constant. In addition, for $t \in [0, T_L]$,

$$\|v_\ell\|_{C_t L^2} \leq \|v_q\|_{C_t L^2} \leq M_0(t)^{1/2}\Big(1 + \sum_{1\leq r\leq q} \delta_r^{1/2}\Big), \tag{4.21}$$

and for $N \geq 1$,

$$\|v_\ell\|_{C^N_{t,x}} \lesssim \ell^{-N+1} \|v_q\|_{C^1_{t,x}} \leq \ell^{-N+1} \lambda_q^4 M_0(t)^{1/2} \leq M_0(t)^{1/2} \ell^{-N} \lambda_{q+1}^{-\alpha}, \tag{4.22}$$

where we have chosen a large enough to absorb the implicit constant.

4.1.3. Construction of v_{q+1}. Let us now proceed with the construction of the perturbation w_{q+1} which then defines the next iteration by $v_{q+1} := v_\ell + w_{q+1}$. To this end, we make use of the construction of the intermittent jets [7, Section 7.4], which we recall in

Appendix B. In particular, the building blocks $W_{(\xi)} = W_{\xi,r_\perp,r_\|,\lambda,\mu}$ for $\xi \in \Lambda$ are defined in (B.3) and the set Λ is introduced in Lemma B.1. The necessary estimates are collected in (B.7). For the intermittent jets we choose the following parameters:

$$\lambda = \lambda_{q+1}, \quad r_\| = \lambda_{q+1}^{-4/7}, \quad r_\perp = r_\|^{-1/4}\lambda_{q+1}^{-1} = \lambda_{q+1}^{-6/7}, \quad \mu = \lambda_{q+1} r_\| r_\perp^{-1} = \lambda_{q+1}^{9/7}. \tag{4.23}$$

It is required that b is a multiple of 7 to ensure that $\lambda_{q+1} r_\perp = a^{b^{q+1}/7} \in \mathbb{N}$.

In order to define the amplitude functions, let χ be a smooth function such that

$$\chi(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1, \\ z & \text{if } z \ge 2, \end{cases}$$

and $z \le 2\chi(z) \le 4z$ for $z \in (1,2)$. We then define, for $t \in [0, T_L]$ and $\omega \in \Omega$,

$$\rho(\omega,t,x) = 4c_R\delta_{q+1}M_0(t)\chi\big((c_R\delta_{q+1}M_0(t))^{-1}|\mathring{R}_\ell(\omega,t,x)|\big),$$

which is $(\mathcal{F}_t)_{t\ge0}$-adapted and we have

$$\left|\frac{\mathring{R}_\ell(\omega,t,x)}{\rho(\omega,t,x)}\right| = \frac{1}{4}\,\frac{(c_R\delta_{q+1}M_0(t))^{-1}|\mathring{R}_\ell(\omega,t,x)|}{\chi((c_R\delta_{q+1}M_0(t))^{-1}|\mathring{R}_\ell(\omega,t,x)|)} \le \frac{1}{2}.$$

Note that if $\mathring{R}_\ell(0,x)$ and $\partial_t\mathring{R}_\ell(0,x)$ are deterministic, so are $\rho(0,x)$ and $\partial_t\rho(0,x)$. Moreover, for any $p \in [1,\infty]$ and $t \in [0, T_L]$ we have

$$\|\rho\|_{C_tL^p} \le 16\big((8\pi^3)^{1/p}c_R\delta_{q+1}M_0(t) + \|\mathring{R}_\ell\|_{C_tL^p}\big). \tag{4.24}$$

Furthermore, by mollification estimates, the embedding $W^{4,1} \subset L^\infty$ and (4.4) we obtain, for $N \ge 0$ and $t \in [0, T_L]$,

$$\|\mathring{R}_\ell\|_{C^N_{t,x}} \lesssim \ell^{-4-N}c_R\delta_{q+1}M_0(t),$$

and by repeated application of the chain rule (see [6, Proposition C.1]) we obtain

$$\begin{aligned}\|\rho\|_{C^N_{t,x}} &\lesssim \ell^{-4-N}c_R\delta_{q+1}M_0(t) + (c_R\delta_{q+1}M_0(t))^{-N+1}\ell^{-5N}(c_R\delta_{q+1}M_0(t))^N \\ &\lesssim \ell^{-4-5N}c_R\delta_{q+1}M_0(t),\end{aligned} \tag{4.25}$$

where we have used the fact that $\frac{d}{dt}M_0(t) = 4LM_0(t)$ with $4L \le \ell^{-1}$ and the implicit constants are independent of ω.

As the next step, we define the amplitude functions

$$a_{(\xi)}(\omega,t,x) := a_{\xi,q+1}(\omega,t,x) := \rho(\omega,t,x)^{1/2}\gamma_\xi\left(\mathrm{Id} - \frac{\mathring{R}_\ell(\omega,t,x)}{\rho(\omega,t,x)}\right)(2\pi)^{-3/4}, \tag{4.26}$$

where γ_ξ is introduced in Lemma B.1. Since ρ and $\mathring{R}_\ell$ are $(\mathcal{F}_t)_{t\ge0}$-adapted, we know that also $a_{(\xi)}$ is $(\mathcal{F}_t)_{t\ge0}$-adapted. If $\mathring{R}_\ell(0,x)$ and $\partial_t\mathring{R}_\ell(0,x)$ are deterministic, so are $a_{(\xi)}(0,x)$ and $\partial_t a_{(\xi)}(0,x)$. By (B.5) we have

$$(2\pi)^{3/2}\sum_{\xi\in\Lambda} a_{(\xi)}^2 \fint_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(\xi)}\,dx = \rho\,\mathrm{Id} - \mathring{R}_\ell, \tag{4.27}$$

and using (4.24) for $t \in [0, T_L]$,

$$\|a_{(\xi)}\|_{C_tL^2} \leq \|\rho\|_{C_tL^1}^{1/2} \|\gamma_\xi\|_{C^0(B_{1/2}(\mathrm{Id}))} \leq \frac{4c_R^{1/2}(8\pi^3+1)^{1/2}M}{8|\Lambda|(8\pi^3+1)^{1/2}} M_0(t)^{1/2}\delta_{q+1}^{1/2}$$
$$\leq \frac{c_R^{1/4} M_0(t)^{1/2}\delta_{q+1}^{1/2}}{2|\Lambda|}, \tag{4.28}$$

where we choose c_R to be a small universal constant to absorb M and we use M to denote the universal constant as in Lemma B.1. Furthermore, by using the fact that ρ is bounded from below by $4c_R\delta_{q+1}M_0(t)$, we deduce by similar arguments to those in (4.25) that for $t \in [0, T_L]$ and $N \geq 0$,

$$\|a_{(\xi)}\|_{C^N_{t,x}} \leq \ell^{-2-5N} c_R^{1/4}\delta_{q+1}^{1/2} M_0(t)^{1/2}. \tag{4.29}$$

With these preparations in hand, we define the principal part $w_{q+1}^{(p)}$ of the perturbation w_{q+1} as

$$w_{q+1}^{(p)} := \sum_{\xi\in\Lambda} a_{(\xi)} W_{(\xi)}. \tag{4.30}$$

If $\mathring{R}_\ell(0,x)$ and $\partial_t\mathring{R}_\ell(0,x)$ are deterministic, so are $w_{q+1}^{(p)}(0,x)$ and $\partial_t w_{q+1}^{(p)}(0,x)$. Since the coefficients $a_{(\xi)}$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted and $W_{(\xi)}$ is a deterministic function we deduce that $w_{q+1}^{(p)}$ is also $(\mathcal{F}_t)_{t\geq0}$-adapted. Moreover, according to (4.27) and (B.4) it follows that

$$w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell = \sum_{\xi\in\Lambda} a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}) + \rho\,\mathrm{Id}, \tag{4.31}$$

where $\mathbb{P}_{\neq0} f := f - \mathcal{F}f(0) = f - (2\pi)^{3/2} \fint_{\mathbb{T}^3} f$.

We also define an incompressibility corrector by

$$w_{q+1}^{(c)} := \sum_{\xi\in\Lambda} \mathrm{curl}(\nabla a_{(\xi)} \times V_{(\xi)}) + \nabla a_{(\xi)} \times \mathrm{curl}\, V_{(\xi)} + a_{(\xi)} W_{(\xi)}^{(c)}, \tag{4.32}$$

with $W_{(\xi)}^{(c)}$ and $V_{(\xi)}$ being given in (B.6). Since $a_{(\xi)}$ is $(\mathcal{F}_t)_{t\geq0}$-adapted and $W_{(\xi)}, W_{(\xi)}^{(c)}$ and $V_{(\xi)}$ are deterministic functions we know that $w_{q+1}^{(c)}$ is also $(\mathcal{F}_t)_{t\geq0}$-adapted. If $\mathring{R}_\ell(0,x)$ and $\partial_t\mathring{R}_\ell(0,x)$ are deterministic, so are $w_{q+1}^{(c)}(0,x)$ and $\partial_t w_{q+1}^{(c)}(0,x)$. By a direct computation we deduce that

$$w_{q+1}^{(p)} + w_{q+1}^{(c)} = \sum_{\xi\in\Lambda} \mathrm{curl}\,\mathrm{curl}(a_{(\xi)} V_{(\xi)}),$$

hence

$$\mathrm{div}(w_{q+1}^{(p)} + w_{q+1}^{(c)}) = 0.$$

We also introduce a temporal corrector

$$w_{q+1}^{(t)} := -\frac{1}{\mu} \sum_{\xi\in\Lambda} \mathbb{P}\mathbb{P}_{\neq0}(a_{(\xi)}^2 \phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi), \tag{4.33}$$

where $\mathbb{P}$ is the Helmholtz projection. If $\mathring{R}_\ell(0,x)$ and $\partial_t \mathring{R}_\ell(0,x)$ are deterministic, so is $w^{(t)}_{q+1}(0,x)$. As above, $w^{(t)}_{q+1}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and by a direct computation we obtain

$$
\begin{aligned}
&\partial_t w^{(t)}_{q+1} + \sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}(a^2_{(\xi)} \operatorname{div}(W_{(\xi)} \otimes W_{(\xi)}))\\
&\quad = -\frac{1}{\mu}\sum_{\xi\in\Lambda} \mathbb{P}\mathbb{P}_{\neq 0}\partial_t(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi) + \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}(a^2_{(\xi)}\partial_t(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi))\\
&\quad = (\mathrm{Id}-\mathbb{P})\frac{1}{\mu}\sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}\partial_t(a^2_{(\xi)}\phi^2_{(\xi)}\psi^2_{(\xi)}\xi) - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathbb{P}_{\neq 0}(\partial_t a^2_{(\xi)}(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi)).
\end{aligned}
\tag{4.34}
$$

Note that the first term on the right hand side can be viewed as a pressure term ∇p_1.

Finally, the total perturbation w_{q+1} is defined by

$$
w_{q+1} := w^{(p)}_{q+1} + w^{(c)}_{q+1} + w^{(t)}_{q+1}, \tag{4.35}
$$

which is mean zero, divergence-free and $(\mathcal{F}_t)_{t\geq 0}$-adapted. If $\mathring{R}_\ell(0,x)$ and $\partial_t \mathring{R}_\ell(0,x)$ are deterministic, so is $w_{q+1}(0,x)$. The new velocity v_{q+1} is defined as

$$
v_{q+1} := v_\ell + w_{q+1}. \tag{4.36}
$$

Thus, it is also $(\mathcal{F}_t)_{t\geq 0}$-adapted. If $\mathring{R}_q(0,x)$ and $v_q(0,x)$ are deterministic, so is $v_{q+1}(0,x)$.

4.1.4. Verification of the inductive estimates for v_{q+1}. Next, we verify the inductive estimates (4.4) on level $q+1$ for v and we prove (4.12). First, we recall the following result from [7, Lemma 7.4].

Lemma 4.3. *Fix integers $N,\kappa \geq 1$ and let $\zeta > 1$ be such that*

$$
\frac{2\pi\sqrt{3}\zeta}{\kappa} \leq \frac{1}{3} \quad \textit{and} \quad \zeta^4 \frac{(2\pi\sqrt{3}\zeta)^N}{\kappa^N} \leq 1.
$$

Let $p\in\{1,2\}$ and let f be a $\mathbb{T}^3$-periodic function such that there exists a constant $C_f > 0$ such that

$$
\|D^j f\|_{L^p} \leq C_f \zeta^j
$$

for all $0 \leq j \leq N+4$. In addition, let g be a $(\mathbb{T}/\kappa)^3$-periodic function. Then

$$
\|fg\|_{L^p} \lesssim C_f \|g\|_{L^p},
$$

where the implicit constant is universal.

This result will be used to bound $w^{(p)}_{q+1}$ in L^2, whereas for the other L^p-norms we apply a different approach. By (4.28) and (4.29) we obtain, for $t\in[0,T_L]$,

$$
\|D^j a_{(\xi)}\|_{C_t L^2} \lesssim \frac{c_R^{1/4} M_0(t)^{1/2}}{2|\Lambda|}\delta^{1/2}_{q+1}\ell^{-8j},
$$

which combined with Lemma 4.3 for $\zeta = \ell^{-8}$ yields, for $t \in [0, T_L]$,

$$\|w_{q+1}^{(p)}\|_{C_tL^2} \le \sum_{\xi\in\Lambda} \frac{1}{2|\Lambda|} c_R^{1/4} M_0(t)^{1/2}\delta_{q+1}^{1/2}\|W_{(\xi)}\|_{C_tL^2} \le \frac{1}{2}M_0(t)^{1/2}\delta_{q+1}^{1/2}, \tag{4.37}$$

where we use $c_R^{1/4}$ to absorb the universal constant and the fact that due to (B.3) together with the normalizations (B.1), (B.2) we have $\|W_{(\xi)}\|_{L^2} \simeq 1$ uniformly in all the parameters involved.

For a general L^p norm we apply (B.7) and (4.29) to deduce that for $t \in [0, T_L]$ and $p \in (1, \infty)$,

$$\begin{aligned}
\|w_{q+1}^{(p)}\|_{C_tL^p} &\lesssim \sum_{\xi\in\Lambda} \|a_{(\xi)}\|_{C^0_{t,x}}\|W_{(\xi)}\|_{C_tL^p} \\
&\lesssim M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-2}r_\perp^{2/p-1}r_\|^{1/p-1/2}, \qquad (4.38)\\
\|w_{q+1}^{(c)}\|_{C_tL^p} &\lesssim \sum_{\xi\in\Lambda}(\|a_{(\xi)}\|_{C^0_{t,x}}\|W_{(\xi)}^{(c)}\|_{C_tL^p} + \|a_{(\xi)}\|_{C^2_{t,x}}\|V_{(\xi)}\|_{C_tW^{1,p}}) \\
&\lesssim M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-12}r_\perp^{2/p-1}r_\|^{1/p-1/2}(r_\perp r_\|^{-1} + \lambda_{q+1}^{-1}) \\
&\lesssim M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-12}r_\perp^{2/p}r_\|^{1/p-3/2}, \qquad (4.39)
\end{aligned}$$

and

$$\begin{aligned}
\|w_{q+1}^{(t)}\|_{C_tL^p} &\lesssim \mu^{-1}\sum_{\xi\in\Lambda}\|a_{(\xi)}\|^2_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{L^{2p}}\|\psi_{(\xi)}\|^2_{C_tL^{2p}} \\
&\lesssim \delta_{q+1}M_0(t)\ell^{-4}r_\perp^{2/p-1}r_\|^{1/p-2}(\mu^{-1}r_\perp^{-1}r_\|) \\
&= M_0(t)\delta_{q+1}\ell^{-4}r_\perp^{2/p-1}r_\|^{1/p-2}\lambda_{q+1}^{-1}. \qquad (4.40)
\end{aligned}$$

We note that for $p = 2$, (4.38) provides a worse bound than (4.37) which was based on Lemma 4.3. Since by (4.18), $M_0(L)^{1/2}\lambda_{q+1}^{4\alpha-1/7} < 1$, we see that for $t \in [0, T_L]$,

$$\begin{aligned}
&\|w_{q+1}^{(c)}\|_{C_tL^p} + \|w_{q+1}^{(t)}\|_{C_tL^p} \\
&\quad\lesssim M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-2}r_\perp^{2/p-1}r_\|^{1/p-1/2}(\ell^{-10}r_\perp r_\|^{-1} + M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-2}r_\|^{-3/2}\lambda_{q+1}^{-1}) \\
&\quad\lesssim M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-2}r_\perp^{2/p-1}r_\|^{1/p-1/2}, \qquad (4.41)
\end{aligned}$$

where we use (4.16) and the fact that $\lambda_{q+1}^{20\alpha-2/7} < 1$ by our choice of α. The bound (4.41) will be used below in the estimation of the Reynolds stress.

Combining (4.37), (4.39) and (4.40) we obtain, for $t \in [0, T_L]$,

$$\begin{aligned}
\|w_{q+1}\|_{C_tL^2} &\le M_0(t)^{1/2}\delta_{q+1}^{1/2}(1/2 + C\ell^{-12}r_\perp r_\|^{-1} + CM_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-4}r_\|^{-3/2}\lambda_{q+1}^{-1}) \\
&\le M_0(t)^{1/2}\delta_{q+1}^{1/2}(1/2 + C\lambda_{q+1}^{24\alpha-2/7} + CM_0(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{8\alpha-1/7}) \\
&\le \tfrac{3}{4}M_0(t)^{1/2}\delta_{q+1}^{1/2}, \qquad (4.42)
\end{aligned}$$

where by (4.18) we choose β small enough and a large enough such that

$$C\lambda_{q+1}^{24\alpha-2/7} \le 1/8, \quad CM_0(L)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{8\alpha-1/7} \le 1/8.$$

The bound (4.42) can be directly combined with (4.21) and the definition (4.36) of the velocity v_{q+1} to deduce the first bound in (4.4) on level $q+1$. Indeed, for $t \in [0, T_L]$,

$$\|v_{q+1}\|_{C_tL^2} \le \|v_\ell\|_{C_tL^2} + \|w_{q+1}\|_{C_tL^2} \le M_0(t)^{1/2}\Big(1 + \sum_{1\le r\le q+1} \delta_r^{1/2}\Big).$$

In addition, (4.42) together with (4.20) yields, for $t \in [0, T_L]$,

$$\|v_{q+1} - v_q\|_{C_tL^2} \le \|w_{q+1}\|_{C_tL^2} + \|v_\ell - v_q\|_{C_tL^2} \le M_0(t)^{1/2}\delta_{q+1}^{1/2},$$

hence (4.12) holds.

As the next step, we shall verify the second bound in (4.4). Using (4.29) and (B.7) we get, for $t \in [0, T_L]$,

$$\begin{aligned}
\|w_{q+1}^{(p)}\|_{C^1_{t,x}} &\le \sum_{\xi\in\Lambda} \|a_{(\xi)}\|_{C^1_{t,x}} \|W_{(\xi)}\|_{C^1_{t,x}} \\
&\lesssim M_0(t)^{1/2}\ell^{-7}r_\perp^{-1}r_\|^{-1/2}\lambda_{q+1}\left(1 + \frac{r_\perp\mu}{r_\|}\right) \\
&\lesssim M_0(t)^{1/2}\ell^{-7}r_\perp^{-1}r_\|^{-1/2}\lambda_{q+1}^2, \qquad (4.43)\\
\|w_{q+1}^{(c)}\|_{C^1_{t,x}} &\lesssim \sum_{\xi\in\Lambda}\big(\|a_{(\xi)}\|_{C^1_{t,x}}\|W_{(\xi)}^{(c)}\|_{C^1_{t,x}} + \|a_{(\xi)}\|_{C^3_{t,x}}(\|V_{(\xi)}\|_{C^1_tC^1_x} + \|V_{(\xi)}\|_{C_tC^2_x})\big) \\
&\lesssim M_0(t)^{1/2}\ell^{-17}r_\|^{-3/2}\left(\mu + \frac{r_\perp\mu\lambda_{q+1}}{r_\|}\right) \\
&\lesssim M_0(t)^{1/2}\ell^{-17}r_\|^{-3/2}\lambda_{q+1}^2, \qquad (4.44)
\end{aligned}$$

and

$$\begin{aligned}
\|w_{q+1}^{(t)}\|_{C^1_{t,x}} &\le \frac{1}{\mu}\sum_{\xi\in\Lambda}[\|a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\|_{C_tW^{1+\alpha,p}} + \|a_{(\xi)}^2\phi_{(\xi)}^2\psi_{(\xi)}^2\|_{C^1_tW^{\alpha,p}}] \\
&\le \frac{1}{\mu}\sum_{\xi\in\Lambda}\big(\|a_{(\xi)}\|_{C^0_{t,x}}\|a_{(\xi)}\|_{C^{1+\alpha}_{t,x}}\|\phi_{(\xi)}\|^2_{L^\infty}\|\psi_{(\xi)}\|^2_{C_tL^\infty} \\
&\quad + \|a_{(\xi)}\|_{C^1_{t,x}}\|a_{(\xi)}\|_{C^0_{t,x}}\|\phi_{(\xi)}\|_{L^\infty}(\|\phi_{(\xi)}\|_{W^{1+\alpha,\infty}}\|\psi_{(\xi)}\|^2_{C_tL^\infty} \\
&\qquad + \|\phi_{(\xi)}\|_{W^{\alpha,\infty}}\|\psi_{(\xi)}\|_{C_tL^\infty}\|\psi_{(\xi)}\|_{C^1_tL^\infty}) \\
&\quad + \|a_{(\xi)}\|_{C^1_{t,x}}\|a_{(\xi)}\|_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{L^\infty}(\|\psi_{(\xi)}\|_{C_tL^\infty}\|\psi_{(\xi)}\|_{C_tW^{1+\alpha,p}} \\
&\qquad + \|\psi_{(\xi)}\|_{C^1_tL^\infty}\|\psi_{(\xi)}\|_{C_tW^{\alpha,p}} + \|\psi_{(\xi)}\|_{C_tL^\infty}\|\psi_{(\xi)}\|_{C^1_tW^{\alpha,p}})\big) \\
&\lesssim \frac{1}{\mu}M_0(t)\ell^{-9}r_\perp^{-2}r_\|^{-1}\lambda_{q+1}^{1+\alpha}\left(1 + \frac{r_\perp\mu}{r_\|}\right) \\
&\lesssim M_0(t)\ell^{-9}r_\perp^{-1}r_\|^{-2}\lambda_{q+1}^{1+\alpha}, \qquad (4.45)
\end{aligned}$$

where we have chosen p large enough and applied the Sobolev embedding in the first inequality in (4.45) needed because $\mathbb{P}\mathbb{P}_{\neq 0}$ is not a bounded operator on C^0; in the last inequality we have used interpolation and an extra λ_{q+1}^{α} appeared. Combining (4.22) and (4.43)–(4.45) with (4.16) we obtain, for $t \in [0, T_L]$,

$$\begin{aligned}
\|v_{q+1}\|_{C^1_{t,x}} &\leq \|v_\ell\|_{C^1_{t,x}} + \|w_{q+1}\|_{C^1_{t,x}} \\
&\leq M_0(t)^{1/2}\big(\lambda_{q+1}^{\alpha} + C\lambda_{q+1}^{14\alpha+22/7} + C\lambda_{q+1}^{34\alpha+20/7} + CM_0(t)^{1/2}\lambda_{q+1}^{19\alpha+3}\big) \\
&\leq M_0(t)^{1/2}\lambda_{q+1}^{4},
\end{aligned}$$

where we use (4.18) to have

$$CM_0(L)^{1/2} \leq \tfrac{1}{2}\lambda_{q+1}^{1-19\alpha}.$$

Thus, the second estimate in (4.4) holds true on level $q+1$.

We conclude this part with further estimates of the perturbations $w_{q+1}^{(p)}$, $w_{q+1}^{(c)}$ and $w_{q+1}^{(t)}$, which will be used below in order to bound the Reynolds stress $\mathring{R}_{q+1}$ and to establish the final estimate in (4.4) on level $q+1$. By a similar approach to (4.38), (4.39), (4.40), we derive the following estimates: for $t \in [0, T_L]$ by using (4.16), (4.29) and (B.7),

$$\begin{aligned}
&\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_t W^{1,p}} \\
&\quad\leq \sum_{\xi\in\Lambda} \|\mathrm{curl}\,\mathrm{curl}(a_{(\xi)}V_{(\xi)})\|_{C_t W^{1,p}} \\
&\quad\lesssim \sum_{\xi\in\Lambda} \big(\|a_{(\xi)}\|_{C^3_{t,x}}\|V_{(\xi)}\|_{C_t L^p} + \|a_{(\xi)}\|_{C^2_{t,x}}\|V_{(\xi)}\|_{C_t W^{1,p}} \\
&\qquad + \|a_{(\xi)}\|_{C^1_{t,x}}\|V_{(\xi)}\|_{C_t W^{2,p}} + \|a_{(\xi)}\|_{C^0_{t,x}}\|V_{(\xi)}\|_{C_t W^{3,p}}\big) \\
&\quad\lesssim M_0(t)^{1/2} r_\perp^{2/p-1} r_\|^{1/p-1/2} (\ell^{-17}\lambda_{q+1}^{-2} + \ell^{-12}\lambda_{q+1}^{-1} + \ell^{-7} + \ell^{-2}\lambda_{q+1}) \\
&\quad\lesssim M_0(t)^{1/2} r_\perp^{2/p-1} r_\|^{1/p-1/2} \ell^{-2}\lambda_{q+1},
\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}
\|w_{q+1}^{(t)}\|_{C_t W^{1,p}} &\leq \frac{1}{\mu}\sum_{\xi\in\Lambda}\big(\|a_{(\xi)}\|_{C^0_{t,x}}\|a_{(\xi)}\|_{C^1_{t,x}}\|\phi_{(\xi)}\|_{L^{2p}}^2\|\psi_{(\xi)}\|_{C_t L^{2p}}^2 \\
&\quad + \|a_{(\xi)}\|_{C^0_{t,x}}^2\|\phi_{(\xi)}\|_{L^{2p}}\|\nabla\phi_{(\xi)}\|_{L^{2p}}\|\psi_{(\xi)}\|_{C_t L^{2p}}^2 \\
&\quad + \|a_{(\xi)}\|_{C^0_{t,x}}^2\|\phi_{(\xi)}\|_{L^{2p}}^2\|\nabla\psi_{(\xi)}\|_{C_t L^{2p}}\|\psi_{(\xi)}\|_{C_t L^{2p}}\big) \\
&\lesssim \frac{M_0(t)}{\mu} r_\perp^{2/p-2} r_\|^{1/p-1}(\ell^{-9} + \ell^{-4}\lambda_{q+1}) \\
&\lesssim M_0(t) r_\perp^{2/p-2} r_\|^{1/p-1}\ell^{-4}\lambda_{q+1}^{-2/7}.
\end{aligned} \tag{4.47}$$

4.1.5. Definition of the Reynolds stress $\mathring{R}_{q+1}$. Subtracting system (4.19) from (4.1) at level $q+1$, we obtain

$$
\begin{aligned}
&\operatorname{div} \mathring{R}_{q+1} - \nabla p_{q+1} \\
&\quad = \underbrace{-\Delta w_{q+1} + \partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \operatorname{div}((v_\ell + z_\ell) \otimes w_{q+1} + w_{q+1} \otimes (v_\ell + z_\ell))}_{\operatorname{div}(R_{\mathrm{lin}}) + \nabla p_{\mathrm{lin}}} \\
&\qquad + \underbrace{\operatorname{div}\big((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\big)}_{\operatorname{div}(R_{\mathrm{cor}}) + \nabla p_{\mathrm{cor}}} \\
&\qquad + \underbrace{\operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell) + \partial_t w_{q+1}^{(t)}}_{\operatorname{div}(R_{\mathrm{osc}}) + \nabla p_{\mathrm{osc}}} \\
&\qquad + \underbrace{\operatorname{div}(v_{q+1} \otimes z_{q+1} - v_{q+1} \otimes z_\ell + z_{q+1} \otimes v_{q+1} - z_\ell \otimes v_{q+1} + z_{q+1} \otimes z_{q+1} - z_\ell \otimes z_\ell)}_{\operatorname{div}(R_{\mathrm{com1}}) + \nabla p_{\mathrm{com1}}} \\
&\qquad + \operatorname{div}(R_{\mathrm{com}}) - \nabla p_\ell.
\end{aligned} \tag{4.48}
$$

We recall the inverse divergence operator $\mathcal{R}$ of [7, Section 5.6], which acts on vector fields v with $\int_{\mathbb{T}^3} v \, dx = 0$ as

$$
(\mathcal{R} v)^{kl} = (\partial_k \Delta^{-1} v^l + \partial_l \Delta^{-1} v^k) - \tfrac{1}{2} (\delta_{kl} + \partial_k \partial_l \Delta^{-1}) \operatorname{div} \Delta^{-1} v
$$

for $k, l \in \{1, 2, 3\}$. Then $\mathcal{R} v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$, and $\mathcal{R}$ is a right inverse of the div operator, i.e. $\operatorname{div}(\mathcal{R} v) = v$. By using $\mathcal{R}$ we define

$$
\begin{aligned}
R_{\mathrm{lin}} &:= -\mathcal{R} \Delta w_{q+1} + \mathcal{R} \partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)}) + (v_\ell + z_\ell) \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} (v_\ell + z_\ell), \\
R_{\mathrm{cor}} &:= (w_{q+1}^{(c)} + w_{q+1}^{(t)}) \mathring{\otimes} w_{q+1} + w_{q+1}^{(p)} \mathring{\otimes} (w_{q+1}^{(c)} + w_{q+1}^{(t)}), \\
R_{\mathrm{com1}} &:= v_{q+1} \mathring{\otimes} z_{q+1} - v_{q+1} \mathring{\otimes} z_\ell + z_{q+1} \mathring{\otimes} v_{q+1} - z_\ell \mathring{\otimes} v_{q+1} + z_{q+1} \mathring{\otimes} z_{q+1} \\
&\qquad - z_\ell \mathring{\otimes} z_\ell.
\end{aligned}
$$

We observe that if $\mathring{R}_q(0, x)$ and $v_q(0, x)$ are deterministic, the same is valid for the above defined error terms $R_{\mathrm{lin}}(0, x)$, $R_{\mathrm{cor}}(0, x)$ and $R_{\mathrm{com1}}(0, x)$.

In order to define the remaining oscillation error from the third line in (4.48), we apply (4.31) and (4.34) to obtain

$$
\begin{aligned}
&\operatorname{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell) + \partial_t w_{q+1}^{(t)} \\
&= \sum_{\xi \in \Lambda} \operatorname{div}\big(a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\big) + \nabla \rho + \partial_t w_{q+1}^{(t)} \\
&= \sum_{\xi \in \Lambda} \mathbb{P}_{\neq 0}\big(\nabla a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\big) + \nabla \rho + \sum_{\xi \in \Lambda} \mathbb{P}_{\neq 0}\big(a_{(\xi)}^2 \operatorname{div}(W_{(\xi)} \otimes W_{(\xi)})\big) + \partial_t w_{q+1}^{(t)} \\
&= \sum_{\xi \in \Lambda} \mathbb{P}_{\neq 0}\big(\nabla a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\big) + \nabla \rho + \nabla p_1 - \frac{1}{\mu} \sum_{\xi \in \Lambda} \mathbb{P}_{\neq 0}\big(\partial_t a_{(\xi)}^2 (\phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi)\big).
\end{aligned}
$$

Therefore,

$$R_{\rm osc} := \sum_{\xi\in\Lambda} \mathcal{R}\big(\nabla a_{(\xi)}^2 \mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\big) - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathcal{R}\big(\partial_t a_{(\xi)}^2 (\phi_{(\xi)}^2 \psi_{(\xi)}^2 \xi)\big) =: R_{\rm osc}^{(x)} + R_{\rm osc}^{(t)},$$

which is also deterministic at time 0. Finally, we define the Reynolds stress on level $q+1$ by

$$\mathring{R}_{q+1} := R_{\rm lin} + R_{\rm cor} + R_{\rm osc} + R_{\rm com} + R_{\rm com1}.$$

We note that by construction, $\mathring{R}_{q+1}(0,x)$ is deterministic.

4.1.6. Verification of the inductive estimate (4.4) *for* $\mathring{R}_{q+1}$. To conclude the proof of Proposition 4.2, we shall verify the third estimate in (4.4). To this end, we estimate each term in the definition of $\mathring{R}_{q+1}$ separately.

In the following we choose $p = \frac{32}{32-7\alpha} > 1$ so that in particular $r_\perp^{2/p-2} r_\parallel^{1/p-1} \leq \lambda_{q+1}^{\alpha}$. For the linear error we apply (4.4) to obtain, for $t\in[0,T_L]$,

$$\begin{aligned}
\|R_{\rm lin}\|_{C_tL^p} &\lesssim \|\mathcal{R}\Delta w_{q+1}\|_{C_tL^p} + \|\mathcal{R}\partial_t (w_{q+1}^{(p)} + w_{q+1}^{(c)})\|_{C_tL^p} \\
&\quad + \|(v_\ell + z_\ell) \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} (v_\ell + z_\ell)\|_{C_tL^p} \\
&\lesssim \|w_{q+1}\|_{C_tW^{1,p}} + \sum_{\xi\in\Lambda} \|\partial_t \operatorname{curl}(a_{(\xi)} V_{(\xi)})\|_{C_tL^p} \\
&\quad + M_0(t)^{1/2} (\lambda_q^4 + \lambda_{q+1}^{\alpha/8}) \|w_{q+1}\|_{C_tL^p},
\end{aligned}$$

where by (B.7) and (4.29),

$$\begin{aligned}
&\sum_{\xi\in\Lambda} \|\partial_t \operatorname{curl}(a_{(\xi)} V_{(\xi)})\|_{C_tL^p} \\
&\qquad \leq \sum_{\xi\in\Lambda} \big(\|a_{(\xi)}\|_{C_tC_x^1} \|\partial_t V_{(\xi)}\|_{C_tW^{1,p}} + \|\partial_t a_{(\xi)}\|_{C_tC_x^1} \|V_{(\xi)}\|_{C_tW^{1,p}} \big) \\
&\qquad \lesssim M_0(t)^{1/2} \ell^{-7} r_\perp^{2/p} r_\parallel^{1/p-3/2} \mu + M_0(t)^{1/2} \ell^{-12} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda_{q+1}^{-1}.
\end{aligned}$$

In view of (4.46), (4.47) as well as (4.38), (4.41), we deduce that for $t\in[0,T_L]$,

$$\begin{aligned}
\|R_{\rm lin}\|_{C_tL^p} &\lesssim M_0(t)^{1/2} \ell^{-2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda_{q+1} + M_0(t) \ell^{-4} r_\perp^{2/p-2} r_\parallel^{1/p-1} \lambda_{q+1}^{-2/7} \\
&\quad + M_0(t)^{1/2} \ell^{-7} r_\perp^{2/p} r_\parallel^{1/p-3/2} \mu + M_0(t)^{1/2} \ell^{-12} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} \lambda_{q+1}^{-1} \\
&\quad + M_0(t) \ell^{-2} r_\perp^{2/p-1} r_\parallel^{1/p-1/2} (\lambda_q^4 + \lambda_{q+1}^{\alpha/8}) \\
&\lesssim M_0(t)^{1/2} \lambda_{q+1}^{5\alpha-1/7} + M_0(t) \lambda_{q+1}^{9\alpha-2/7} \\
&\quad + M_0(t)^{1/2} \lambda_{q+1}^{15\alpha-1/7} + M_0(t)^{1/2} \lambda_{q+1}^{25\alpha-15/7} \\
&\leq M_0(t) c_R \delta_{q+2}/5.
\end{aligned}$$

Here, we have taken a sufficiently large and β sufficiently small.

The corrector error is estimated using (4.38)–(4.41), for $t \in [0, T_L]$, as

$$\begin{aligned}
&\|R_{\mathrm{cor}}\|_{C_t L^p} \\
&\quad \le \|w^{(c)}_{q+1} + w^{(t)}_{q+1}\|_{C_t L^{2p}} \|w_{q+1}\|_{C_t L^{2p}} + \|w^{(c)}_{q+1} + w^{(t)}_{q+1}\|_{C_t L^{2p}} \|w^{(p)}_{q+1}\|_{C_t L^{2p}} \\
&\quad \lesssim M_0(t)\big(\ell^{-12} r_\perp^{1/p} r_\parallel^{1/(2p)-3/2} + \ell^{-4} M_0(t)^{1/2} r_\perp^{1/p-1} r_\parallel^{1/(2p)-2} \lambda_{q+1}^{-1}\big) \\
&\qquad \times \ell^{-2} r_\perp^{1/p-1} r_\parallel^{1/(2p)-1/2} \\
&\quad \lesssim M_0(t)\big(\ell^{-14} r_\perp^{2/p-1} r_\parallel^{1/p-2} + \ell^{-6} M_0(t)^{1/2} r_\perp^{2/p-2} r_\parallel^{1/p-5/2} \lambda_{q+1}^{-1}\big) \\
&\quad \lesssim M_0(t)\big(\lambda_{q+1}^{29\alpha-2/7} + M_0(t)^{1/2} \lambda_{q+1}^{13\alpha-1/7}\big) \\
&\quad \le M_0(t) c_R \delta_{q+2}/5.
\end{aligned}$$

Here we use (4.18) to have $M_0(L)^{1/2}\lambda_{q+1}^{13\alpha-1/7} \le c_R\delta_{q+2}/10$.

Finally, we proceed with the oscillation error R_{osc} and we focus on $R^{(x)}_{\mathrm{osc}}$ first. Since $W_{(\xi)}$ is $(\mathbb{T}/(r_\perp\lambda_{q+1}))^3$-periodic, we deduce that

$$\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)}) = \mathbb{P}_{\ge r_\perp \lambda_{q+1}/2}(W_{(\xi)} \otimes W_{(\xi)}),$$

where $\mathbb{P}_{\ge r} = \mathrm{Id} - \mathbb{P}_{<r}$ and $\mathbb{P}_{<r}$ denotes the Fourier multiplier operator which projects a function onto its Fourier frequencies $< r$ in absolute value. We also recall the following results from [7, Lemma 7.5].

Lemma 4.4. *Fix parameters $1 \le \zeta < \kappa$, $p \in (1,2]$, and assume there exists $N \in \mathbb{N}$ such that $\zeta^N \le \kappa^{N-2}$. Let $a \in C^N(\mathbb{T}^3)$ be such that there exists $C_a > 0$ with*

$$\|D^j a\|_{C^0} \le C_a \zeta^j$$

for all $0 \le j \le N$. Assume that $f \in L^p(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3} a(x)\mathbb{P}_{\ge\kappa} f(x)\,dx = 0$. Then

$$\big\||\nabla|^{-1}(a\mathbb{P}_{\ge\kappa} f)\big\|_{L^p} \lesssim C_a \frac{\|f\|_{L^p}}{\kappa},$$

where the implicit constant depends only on p and N.

Using Lemma 4.4 with $a = \nabla a^2_{(\xi)}$ for $C_a = M_0(t)\ell^{-9}$, $\zeta = \ell^{-5}$, $\kappa = r_\perp\lambda_{q+1}$ and any $N \ge 3$, we have

$$\begin{aligned}
\|R^{(x)}_{\mathrm{osc}}\|_{C_t L^p} &\le \sum_{\xi\in\Lambda} \big\|\mathcal{R}\big(\nabla a^2_{(\xi)} \mathbb{P}_{\ge r_\perp\lambda_{q+1}/2}(W_{(\xi)} \otimes W_{(\xi)})\big)\big\|_{C_t L^p} \\
&\lesssim M_0(t)\ell^{-9} \frac{\|W_{(\xi)} \otimes W_{(\xi)}\|_{C_t L^p}}{r_\perp\lambda_{q+1}} \lesssim M_0(t)\ell^{-9} \frac{\|W_{(\xi)}\|^2_{C_t L^{2p}}}{r_\perp\lambda_{q+1}} \\
&\lesssim M_0(t)\ell^{-9} r_\perp^{2/p-2} r_\parallel^{1/p-1} (r_\perp^{-1}\lambda_{q+1}^{-1}) \\
&\lesssim M_0(t)\ell^{-9} \lambda_{q+1}^{\alpha} (r_\perp^{-1}\lambda_{q+1}^{-1}) \\
&\lesssim M_0(t)\lambda_{q+1}^{19\alpha-1/7} \le M_0(t) c_R \delta_{q+2}/10.
\end{aligned}$$

For the second term $R_{\mathrm{osc}}^{(t)}$ we use Fubini's theorem to integrate along the orthogonal directions of $\phi_{(\xi)}$ and $\psi_{(\xi)}$ and use (B.7) to deduce

$$\begin{aligned}\|R_{\mathrm{osc}}^{(t)}\|_{C_tL^p} &\leq \mu^{-1}\sum_{\xi\in\Lambda}\|\partial_t a_{(\xi)}^2\|_{C^0_{t,x}}\|\phi_{(\xi)}\|^2_{C_tL^{2p}}\|\psi_{(\xi)}\|^2_{C_tL^{2p}}\\ &\lesssim M_0(t)\mu^{-1}\ell^{-9}r_\perp^{2/p-2}r_\parallel^{1/p-1}\lesssim M_0(t)\lambda_{q+1}^{19\alpha-9/7}\leq M_0(t)c_R\delta_{q+2}/10.\end{aligned}$$

In view of the standard mollification estimates we use (4.3) to find that for $t\in[0,T_L]$,

$$\begin{aligned}\|R_{\mathrm{com}}\|_{C_tL^1} &\lesssim \ell(\|v_q\|_{C^1_{t,x}}+\|z_q\|_{C_tC^1})(\|v_q\|_{C_tL^2}+\|z_q\|_{C_tL^2})\\ &\quad+\ell^{1/2-2\delta}(\|z_q\|_{C_t^{1/2-2\delta}L^\infty}+\|v\|_{C^1_{t,x}})(\|v_q\|_{C_tL^2}+\|z_q\|_{C_tL^2})\\ &\lesssim 2\ell(\lambda_q^4+\lambda_{q+1}^{\alpha/4})M_0(t)+\ell^{1/2-2\delta}(\lambda_{q+1}^{\alpha/4}+\lambda_q^4)M_0(t)\leq M_0(t)c_R\delta_{q+2}/5,\end{aligned}$$

where $\delta<1/12$ and we require that $\ell^{1/2-2\delta}(\lambda_{q+1}^{\alpha/4}+\lambda_q^4)<c_R\delta_{q+2}/10$, i.e.

$$\lambda_{q+1}^{2\beta b-\alpha/2}\lambda_q^{-2/3}(\lambda_{q+1}^{\alpha/4}+\lambda_q^4)\ll 1.$$

With the choice of ℓ in (4.17) and since we postulated that $\alpha>18\beta b$ and $\alpha b>16$, this can indeed be achieved by possibly increasing a and consequently decreasing β. Finally, we use (4.3) to obtain, for $t\in[0,T_L]$,

$$\begin{aligned}\|R_{\mathrm{com}1}\|_{C_tL^1} &\lesssim (\|v_{q+1}\|_{C_tL^2}+\|z_{q+1}\|_{C_tL^2}+\|z_\ell\|_{C_tL^2})\|z_\ell-z_{q+1}\|_{C_tL^2}\\ &\lesssim M_0(t)^{1/2}\|z_\ell-z_{q+1}\|_{C_tL^2}\lesssim M_0(t)^{1/2}(\|z_\ell-z_q\|_{C_tL^2}+\|z_{q+1}-z_q\|_{C_tL^2})\\ &\lesssim M_0(t)(\ell^{1/2-2\delta}+\lambda_{q+1}^{-\frac{\alpha}{8}(1-\delta)})\leq M_0(t)c_R\delta_{q+2}/5,\end{aligned}$$

where we use

$$\lambda_{q+1}^{2b\beta-\frac{\alpha}{8}(1-\delta)}\ll 1,$$

which holds because $\alpha>18\beta b$. Summarizing the above estimates we obtain

$$\|\mathring{R}_{q+1}\|_{C_tL^1}\leq M_0(t)c_R\delta_{q+2},$$

which is the desired last bound in (4.4). The proof of Proposition 4.2 is complete.

5. Nonuniqueness in law II: the case of a linear multiplicative noise

5.1. *Probabilistically weak solutions*

In the case of an additive noise, the stopping times employed in convex integration can be regarded as functions of the solution u. This does not follow a priori from their definition (4.2), but can be seen from (3.13) and (3.16). Accordingly, it was possible to prove nonuniqueness of martingale solutions in the sense of Definition 3.1 directly. However, the situation is rather different in the case of a linear multiplicative noise. Indeed, the stopping times are functions of the driving noise B, which is not a function of u, and therefore it is necessary to work with the extended canonical space $\bar{\Omega}$ including trajectories of both the solution u and the noise B.

To this end, we define the notion of probabilistically weak solution. In the first step, we then establish joint nonuniqueness in law: we show that the joint law of (u, B) is not unique. In the second step, we extend the finite-dimensional result of Cherny [9] to a general SPDE setting (see Appendix C), proving that uniqueness in law implies joint uniqueness in law. This permits us to deduce the desired nonuniqueness of martingale solutions stated in Theorem 1.4.

To avoid confusion, we point out that the two notions of solution, i.e. martingale solution and probabilistically weak solution, are equivalent. The only reason why the proof of nonuniqueness in law from Section 3 does not apply to the case of linear multiplicative noise is the different definition of stopping times. Conversely, the proof of the present section applies to the additive noise case as well. However, it is more complicated than the direct proof in Section 3 which does not rely on the generalization of Cherny's result in Theorem C.1.

Definition 5.1. Let $s \geq 0$ and $x_0 \in L^2_\sigma$, $y_0 \in U_1$. A probability measure $P \in \mathscr{P}(\bar{\Omega})$ is a *probabilistically weak solution* to the Navier–Stokes system (1.1) with initial value (x_0, y_0) at time s provided

(M1) $P(x(t) = x_0, y(t) = y_0, 0 \leq t \leq s) = 1$ and for any $n \in \mathbb{N}$,

$$P\left\{(x, y) \in \bar{\Omega} : \int_0^n \|G(x(r))\|^2_{L_2(U;L^\sigma_2)}\, dr < \infty\right\} = 1.$$

(M2) Under P, y is a cylindrical $(\bar{\mathscr{B}}_t)_{t\geq s}$-Wiener process on U starting from y_0 at time s and for every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and all $t \geq s$,

$$\langle x(t) - x(s), e_i\rangle + \int_s^t \langle \operatorname{div}(x(r) \otimes x(r)) - \Delta x(r), e_i\rangle\, dr = \int_s^t \langle e_i, G(x(r)) dy_r\rangle.$$

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$,

$$E^P\left(\sup_{r\in[0,t]} \|x(r)\|^{2q}_{L^2} + \int_s^t \|x(r)\|^2_{H^\gamma}\, dr\right) \leq C_{t,q}(\|x_0\|^{2q}_{L^2} + 1).$$

For the application to the Navier–Stokes system, we will again require a definition of probabilistically weak solutions defined up to a stopping time τ. To this end, we set

$$\bar{\Omega}_\tau := \{\omega(\cdot \wedge \tau(\omega)) : \omega \in \bar{\Omega}\}.$$

Definition 5.2. Let $s \geq 0$ and $x_0 \in L^2_\sigma$, $y_0 \in U_1$. Let $\tau \geq s$ be a $(\bar{\mathscr{B}}_t)_{t\geq s}$-stopping time. A probability measure $P \in \mathscr{P}(\bar{\Omega}_\tau)$ is a *probabilistically weak solution* to the Navier–Stokes system (1.1) *on* $[s, \tau]$ with initial value (x_0, y_0) at time s provided

(M1) $P(x(t) = x_0, y(t) = y_0, 0 \leq t \leq s) = 1$ and for any $n \in \mathbb{N}$,

$$P\left\{(x, y) \in \bar{\Omega} : \int_0^{n\wedge\tau} \|G(x(r))\|^2_{L_2(U;L^\sigma_2)}\, dr < \infty\right\} = 1.$$

(M2) Under P, $\langle y(\cdot\wedge\tau), l\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq s}$-martingale starting from y_0 at time s with quadratic variation process given by $(t\wedge\tau-s)\|l\|_U^2$ for $l\in U$. For every $e_i\in C^\infty(\mathbb{T}^3)\cap L^2_\sigma$ and $t\geq s$,

$$\langle x(t\wedge\tau)-x(s), e_i\rangle+\int_s^{t\wedge\tau}\langle \mathrm{div}(x(r)\otimes x(r))-\Delta x(r), e_i\rangle\,dr \\ =\int_s^{t\wedge\tau}\langle e_i, G(x(r))dy_r\rangle.$$

(M3) For any $q\in\mathbb{N}$ there exists a positive real function $t\mapsto C_{t,q}$ such that for all $t\geq s$,

$$E^P\left(\sup_{r\in[0,t\wedge\tau]}\|x(r)\|_{L^2}^{2q}+\int_s^{t\wedge\tau}\|x(r)\|_{H^\gamma}^2\,dr\right)\leq C_{t,q}(\|x_0\|_{L^2}^{2q}+1).$$

Similarly to Theorem 3.1 we obtain the following existence and stability result. The proof is presented in Appendix A.

Theorem 5.1. *For every $(s,x_0,y_0)\in[0,\infty)\times L^2_\sigma\times U_1$, there exists $P\in\mathscr{P}(\bar\Omega)$ which is a probabilistically weak solution to the Navier–Stokes system (1.1) starting at time s from the initial condition (x_0,y_0) in the sense of Definition 5.1. The set of all such solutions with the same implicit constant $C_{t,q}$ is denoted by $\mathscr{W}(s,x_0,y_0,C_{t,q})$.*

Let $(s_n,x_n,y_n)\to(s,x_0,y_0)$ in $[0,\infty)\times L^2_\sigma\times U_1$ as $n\to\infty$ and let $P_n\in\mathscr{W}(s_n,x_n,y_n,C_{t,q})$. Then there exists a subsequence n_k such that the sequence $\{P_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to some $P\in\mathscr{W}(s,x_0,y_0,C_{t,q})$.

As in the case of additive noise, the nonuniqueness in law stated in Theorem 1.4 means nonuniqueness of martingale solutions in the sense of Definition 3.1. Nonuniqueness of probabilistically weak solutions corresponds to joint nonuniqueness in law.

Definition 5.3. We say that *joint uniqueness in law* holds for (1.1) if probabilistically weak solutions starting from the same initial distribution are unique.

5.2. General construction for probabilistically weak solutions

The overall strategy is similar to Section 3.2: in the first step, we shall extend probabilistically weak solutions defined up to a $(\bar{\mathcal{B}}_t)_{t\geq 0}$-stopping time τ to the whole interval $[0,\infty)$. We denote by $\bar{\mathcal{B}}_\tau$ the σ-field associated to τ.

Proposition 5.2. *Let τ be a bounded $(\bar{\mathcal{B}}_t)_{t\geq 0}$-stopping time. Then for every $\omega\in\bar\Omega$ there exists $Q_\omega\in\mathscr{P}(\bar\Omega)$ such that for $\omega\in\{x(\tau)\in L^2_\sigma\}$,*

$$Q_\omega\big(\omega'\in\bar\Omega:(x,y)(t,\omega')=(x,y)(t,\omega)\ \textit{for}\ 0\leq t\leq\tau(\omega)\big)=1, \tag{5.1}$$

$$Q_\omega(A)=R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}(A)\quad \textit{for all}\ A\in\bar{\mathcal{B}}^{\tau(\omega)}, \tag{5.2}$$

where $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}\in\mathscr{P}(\bar\Omega)$ is a probabilistically weak solution to the Navier–Stokes system (1.1) starting at time $\tau(\omega)$ from $(x(\tau(\omega),\omega),y(\tau(\omega),\omega))$. Furthermore, for every $B\in\bar{\mathcal{B}}$ the mapping $\omega\mapsto Q_\omega(B)$ is $\bar{\mathcal{B}}_\tau$-measurable.

Proof. The proof is identical to the proof of Proposition 3.2 applied to the extended path space $\bar{\Omega}$ instead of Ω_0 and making use of Theorem 5.1 instead of Theorem 3.1. ∎

We proceed with a result which is analogous to Proposition 3.4.

Proposition 5.3. *Let $x_0 \in L^2_\sigma$. Let P be a probabilistically weak solution to the Navier–Stokes system (1.1) on $[0,\tau]$ starting at time 0 from the initial condition $(x_0, 0)$. In addition to the assumptions of Proposition 5.2, suppose that there exists a Borel set $\mathcal{N} \subset \bar{\Omega}_\tau$ such that $P(\mathcal{N}) = 0$ and for every $\omega \in \mathcal{N}^c$,*

$$Q_\omega\big(\omega' \in \bar{\Omega} : \tau(\omega') = \tau(\omega)\big) = 1. \tag{5.3}$$

Then the probability measure $P \otimes_\tau R \in \mathscr{P}(\bar{\Omega})$ defined by

$$P \otimes_\tau R(\cdot) := \int_{\bar{\Omega}} Q_\omega(\cdot)\, P(d\omega)$$

satisfies $P \otimes_\tau R = P$ on $\sigma\{x(t\wedge\tau), y(t\wedge\tau) : t \geq 0\}$ and is a probabilistically weak solution to the Navier–Stokes system (1.1) on $[0,\infty)$ with initial condition $(x_0, 0)$.

Proof. The fact that $P \otimes_\tau R(A) = P(A)$ for every Borel set $A \in \sigma\{(x(t\wedge\tau), y(t\wedge\tau)) : t \geq 0\}$, and property (M1), follow directly from the construction together with (5.3). In order to show (M3), we write

$$\begin{aligned}
&E^{P\otimes_\tau R}\bigg(\sup_{r\in[0,t]} \|x(r)\|_{L^2}^{2q} + \int_0^t \|x(r)\|_{H^\gamma}^2\, dr\bigg) \\
&\qquad\qquad \leq E^{P\otimes_\tau R}\bigg(\sup_{r\in[0,t\wedge\tau]} \|x(r)\|_{L^2}^{2q} + \int_0^{t\wedge\tau} \|x(r)\|_{H^\gamma}^2\, dr\bigg) \\
&\qquad\qquad\quad + E^{P\otimes_\tau R}\bigg(\sup_{r\in[t\wedge\tau,t]} \|x(r)\|_{L^2}^{2q} + \int_{t\wedge\tau}^t \|x(r)\|_{H^\gamma}^2\, dr\bigg) \\
&\qquad\qquad \leq C(\|x_0\|_{L^2}^{2q} + 1) + C(E^P\|x(\tau)\|_{L^2}^{2q} + 1) \\
&\qquad\qquad \leq C(\|x_0\|_{L^2}^{2q} + 1),
\end{aligned}$$

where we use (M3) for P and for R, (5.3) and the boundedness of the stopping time τ.

For (M2), we first recall that since P is a probabilistically weak solution on $[0,\tau]$, the process $\langle y_{t\wedge\tau}, l\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq 0}$-martingale under P with quadratic variation process given by $(t\wedge\tau)\|l\|_U^2$. On the other hand, since for every $\omega \in \bar{\Omega}$, the probability measure $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}$ is a probabilistically weak solution starting at time $\tau(\omega)$ from the initial condition $(x(\tau(\omega),\omega), y(\tau(\omega),\omega))$, the process $\langle y_t - y_{t\wedge\tau(\omega)}, l\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq\tau(\omega)}$-martingale under $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}$ with quadratic variation process given by $(t-\tau(\omega))\|l\|_U^2$, $t \geq \tau(\omega)$. Then by the same arguments as in the proof of Proposition 3.4 we deduce that under $P \otimes_\tau R$, the process $\langle y, l\rangle_U$ is a continuous square integrable $(\bar{\mathcal{B}}_t)_{t\geq 0}$-martingale with quadratic variation process given by $t\|l\|_U^2$, $t \geq 0$, which implies that y is a cylindrical $(\bar{\mathcal{B}}_t)_{t\geq 0}$-Wiener process on U.

Furthermore, under P, for every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and all $t \geq 0$,

$$M^{x,y,i}_{t\wedge\tau,0} := \langle x(t\wedge\tau) - x(0), e_i\rangle + \int_0^{t\wedge\tau} \langle \mathrm{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle\, dr$$
$$= \int_0^{t\wedge\tau} \langle e_i, G(x(r))dy(r)\rangle.$$

On the other hand, for $\omega \in \bar{\Omega}$, under $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)}$ we have, for $t \geq \tau(\omega)$,

$$M^{x,y,i}_{t,t\wedge\tau} := \langle x(t) - x(\tau(\omega)), e_i\rangle + \int_{\tau(\omega)}^{t} \langle \mathrm{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle\, dr$$
$$= \int_{\tau(\omega)}^{t} \langle e_i, G(x(r))dy(r)\rangle.$$

Therefore, we obtain

$$P\otimes_\tau R\left\{M^{x,y,i}_{t,0} = \int_0^t \langle e_i, G(x(r))dy(r)\rangle,\ e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0\right\}$$
$$= \int_{\bar{\Omega}} dP(\omega) Q_\omega \left\{ M^{x,y,i}_{t,t\wedge\tau(\omega)} = \int_{t\wedge\tau(\omega)}^{t} \langle e_i, G(x(r))dy(r)\rangle,\right.$$
$$\left. M^{x,y,i}_{t\wedge\tau(\omega),0} = \int_0^{t\wedge\tau(\omega)} \langle e_i, G(x(r))dy(r)\rangle,\ e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0\right\}.$$

Now, using (5.3) and (5.2) we obtain

$$\int_{\bar{\Omega}} dP(\omega) Q_\omega \left\{ M^{x,y,i}_{t,t\wedge\tau(\omega)} = \int_{t\wedge\tau(\omega)}^{t} \langle e_i, G(x(r))dy(r)\rangle,\ e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0\right\}$$
$$= \int_{\bar{\Omega}} dP(\omega) R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega)} \left\{ M^{x,y,i}_{t,t\wedge\tau(\omega)} = \int_{t\wedge\tau(\omega)}^{t} \langle e_i, G(x(r))dy(r)\rangle,\right.$$
$$\left. e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0\right\}$$
$$= 1,$$

and using (5.3) and (5.1) we deduce

$$\int_{\bar{\Omega}} dP(\omega) Q_\omega \left\{ M^{x,y,i}_{t\wedge\tau(\omega),0} = \int_0^{t\wedge\tau(\omega)} \langle e_i, G(x(r))dy(r)\rangle,\ e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0\right\}$$
$$= P\left\{ M^{x,y,i}_{t\wedge\tau,0} = \int_0^{t\wedge\tau} \langle e_i, G(x(r))dy(r)\rangle,\ e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0\right\} = 1.$$

In view of the elementary inequality for probability measures, $Q_\omega(A\cap B) \geq 1 - Q_\omega(A^c) - Q_\omega(B^c)$, we finally deduce that $P\otimes_\tau R$-a.s.,

$$M^{x,y,i}_{t,0} = \int_0^t \langle e_i, G(x(r))dy(r)\rangle \quad \text{for all } e_i \in C^\infty(\mathbb{T}^3)\cap L^2_\sigma,\ t\geq 0,$$

and hence condition (M2) follows. ∎

5.3. Application to solutions obtained through Theorem 1.3

The general construction presented in Section 5.2 applies to a general infinite-dimensional stochastic perturbation of the Navier–Stokes system. From now on, we restrict ourselves to the setting of a linear multiplicative noise. In particular, the driving Wiener process is real-valued and consequently $U = U_1 = \mathbb{R}$.

For $n \in \mathbb{N}$, $L > 1$ and $\delta \in (0, 1/12)$ we define

$$\begin{aligned}\tau_L^n(\omega) &= \inf\{t \geq 0 : |y(t,\omega)| > (L - 1/n)^{1/4}\}\\ &\qquad \wedge \inf\{t > 0 : \|y(t,\omega)\|_{C_t^{1/2-2\delta}} > (L-1/n)^{1/2}\} \wedge L.\end{aligned}$$

Then the sequence $\{\tau_L^n\}_{n\in\mathbb{N}}$ is nondecreasing and we define

$$\tau_L := \lim_{n\to\infty} \tau_L^n. \tag{5.4}$$

Without additional regularity of the process y, we have $\tau_L^n(\omega) = 0$. By Lemma 3.5 we find that τ_L^n is a $(\bar{\mathcal{B}}_t)_{t\geq 0}$-stopping time and hence so is τ_L as an increasing limit of stopping times.

Now, we fix a real-valued Wiener process B defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and we denote by $(\mathcal{F}_t)_{t\geq 0}$ its normal filtration. On this stochastic basis, we apply Theorem 1.3 and denote by u the corresponding solution to the Navier–Stokes system (1.8) on $[0, T_L]$, where the stopping time T_L is defined in (6.3). We recall that u is adapted to $(\mathcal{F}_t)_{t\geq 0}$, which is an essential property employed to prove the martingale property in the proof of Proposition 5.4. We denote by P the law of (u, B) and obtain the following result by similar arguments to the proof of Proposition 3.7.

Proposition 5.4. *The probability measure P is a probabilistically weak solution to the Navier–Stokes system* (1.8) *on $[0, \tau_L]$ in the sense of Definition 5.2, where τ_L was defined in* (5.4).

Proof. The proof is similar to the proof of Proposition 3.7 once we note that

$$y(t, (u, B)) = B(t) \quad \text{for } t \in [0, T_L]\ \mathbf{P}\text{-a.s.}$$

In particular, property (M2) in Definition 5.2 follows since (u, B) satisfies (1.8). ∎

Proposition 5.5. *The probability measure $P \otimes_{\tau_L} R$ is a probabilistically weak solution to the Navier–Stokes system* (1.8) *on $[0,\infty)$ in the sense of Definition 5.1.*

Proof. In light of Propositions 5.2 and 5.3, it only remains to establish (5.3), which follows by similar arguments to those in the proof of Proposition 3.8. ∎

Finally, we have all in hand to conclude the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $T > 0$ be arbitrary. Let $\kappa = 1/2$ and $K = 2$ and apply Theorem 1.3 and Proposition 5.5. As in the proof of Theorem 1.2 it follows that the constructed probability measure $P \otimes_{\tau_L} R$ satisfies

$$P \otimes_{\tau_L} R(\tau_L \geq T) = \mathbf{P}(T_L \geq T) > 1/2,$$

and consequently

$$E^{P\otimes_{\tau_L}R}[\|x(T)\|_{L^2}^2] > 2e^T\|x_0\|_{L^2}^2.$$

The initial value $x_0 = v(0) \in L^2_\sigma$ is given through the construction in Theorem 1.3. However, based on Galerkin approximation one can construct a probabilistically weak solution $\tilde{P}$ to (1.8) starting from the same initial value as $P \otimes_{\tau_L} R$. In addition, this solution satisfies the usual energy inequality, that is,

$$E^{\tilde{P}}[\|x(T)\|_{L^2}^2] \le e^T\|x_0\|_{L^2}^2.$$

Therefore, the two probabilistically weak solutions are distinct and as a consequence joint nonuniqueness in law, i.e. nonuniqueness of probabilistically weak solutions, holds for the Navier–Stokes system (1.8). In view of Theorem C.1 we finally deduce the desired nonuniqueness in law, i.e., nonuniqueness of martingale solutions. ■

6. Proof of Theorem 1.3

As the first step, we transform (1.8) to a random PDE. To this end, we consider the stochastic process

$$\theta(t) = e^{B_t}, \quad t \ge 0,$$

and define $v := \theta^{-1}u$. Then by Itô's formula we obtain

$$\begin{aligned} \partial_t v + \tfrac{1}{2}v - \Delta v + \theta\,\mathrm{div}(v\otimes v) + \theta^{-1}\nabla P &= 0,\\ \mathrm{div}\, v &= 0. \end{aligned} \tag{6.1}$$

Our aim is to develop an induction argument as in Section 4 and apply it to (6.1). At each step $q \in \mathbb{N}_0$, a pair $(v_q, \mathring{R}_q)$ is constructed solving the system

$$\begin{aligned} \partial_t v_q + \tfrac{1}{2}v_q - \Delta v_q + \theta\,\mathrm{div}(v_q\otimes v_q) + \nabla p_q &= \mathrm{div}\,\mathring{R}_q,\\ \mathrm{div}\, v_q &= 0. \end{aligned} \tag{6.2}$$

We choose suitable parameters $a \in \mathbb{N}$ and $b \in \mathbb{N}$ sufficiently large and a parameter $\beta \in (0,1)$ sufficiently small and define

$$\lambda_q = a^{b^q}, \quad \delta_q = \lambda_q^{-2\beta}.$$

The necessary stopping times T_L are now defined in terms of the Wiener process B as

$$T_L := \inf\{t > 0 : |B(t)| \ge L^{1/4}\} \wedge \inf\{t > 0 : \|B\|_{C_t^{1/2-2\delta}} \ge L^{1/2}\} \wedge L \tag{6.3}$$

for $L > 1$ and $\delta \in (0, 1/12)$. As a consequence, for $t \in [0, T_L]$,

$$|B(t)| \le L^{1/4}, \quad \|B\|_{C_t^{1/2-2\delta}} \le L^{1/2}, \tag{6.4}$$

which implies

$$\|\theta\|_{C_t^{1/2-2\delta}} + |\theta(t)| + |\theta^{-1}(t)| \le 3L^{1/2}e^{L^{1/4}} =: m_L^2. \tag{6.5}$$

We also define

$$M_0(t) := e^{4Lt+2L}. \tag{6.6}$$

For the induction, we will assume the following bounds for $(v_q, \mathring{R}_q)$ which are valid for $t \in [0, T_L]$:

$$\begin{aligned} \|v_q\|_{C_t L^2} &\leq m_L M_0(t)^{1/2}\Big(1 + \sum_{1\leq r\leq q} \delta_r^{1/2}\Big) \leq 2m_L M_0(t)^{1/2}, \\ \|v_q\|_{C^1_{t,x}} &\leq m_L M_0(t)^{1/2}\lambda_q^4, \\ \|\mathring{R}_q\|_{C_t L^1} &\leq c_R M_0(t)\delta_{q+1}. \end{aligned} \tag{6.7}$$

Here $\sum_{1\leq r\leq 0} \delta_r^{1/2} := 0$, $c_R > 0$ is a sufficiently small universal constant given in (6.22), (6.24) and we have used the fact that $\sum_{r\geq 1} \delta_r^{1/2} \leq \sum_{r\geq 1} a^{-rb\beta} = \frac{a^{-\beta b}}{1-a^{-\beta b}} < 1/2$ and

$$a^{\beta b} > 3 \tag{6.8}$$

in the first inequality. The following result sets the starting point of our iteration procedure and gives the key compatibility conditions between the parameters L, a, β, b.

Lemma 6.1. *Let $L > 1$ and define*

$$v_0(t,x) := \frac{m_L e^{2Lt+L}}{(2\pi)^{3/2}}(\sin(x_3), 0, 0).$$

Then the associated Reynolds stress is given by

$$\mathring{R}_0(t,x) = \frac{m_L(2L+3/2)e^{2Lt+L}}{(2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & -\cos(x_3) \\ 0 & 0 & 0 \\ -\cos(x_3) & 0 & 0 \end{pmatrix}.$$

The initial values $v_0(0,x)$ and $\mathring{R}_0(0,x)$ are deterministic. Moreover, all the estimates in (6.7) on level $q = 0$ for $(v_0, \mathring{R}_0)$ as well as (6.8) are valid provided

$$18 \cdot (2\pi)^{3/2}\sqrt{3} < 2 \cdot (2\pi)^{3/2}\sqrt{3}\, a^{2\beta b} \leq \frac{c_R e^L}{L^{1/4}(2L+3/2)e^{\frac{1}{2}L^{1/4}}}, \quad 4L \leq a^4. \tag{6.9}$$

In particular, the minimal lower bound for L is given through

$$18 \cdot (2\pi)^{3/2}\sqrt{3} < \frac{c_R e^L}{L^{1/4}(2L+3/2)e^{\frac{1}{2}L^{1/4}}}. \tag{6.10}$$

Proof. We observe that for $t \in [0, T_L]$,

$$\|v_0(t)\|_{L^2} = \frac{m_L e^{2Lt+L}}{\sqrt{2}} \leq m_L M_0(t)^{1/2}, \quad \|v_0\|_{C^1_{t,x}} \leq 4L m_L e^{2Lt+L} \leq m_L M_0(t)^{1/2}\lambda_0^4,$$

provided

$$4L \leq a^4. \tag{6.11}$$

The associated Reynolds stress can be directly computed and admits the bound

$$\|\mathring{R}_0(t)\|_{L^1} \leq 2 \cdot (2\pi)^{3/2} m_L (2L + 3/2) e^{2Lt+L} \leq M_0(t) c_R \delta_1$$

provided

$$2 \cdot (2\pi)^{3/2} \sqrt{3}\, L^{1/4} (2L + 3/2) e^{1/2L^{1/4}} \leq e^L c_R a^{-2\beta b}. \tag{6.12}$$

Under conditions (6.11) and (6.12) all the estimates in (6.7) are valid on level $q = 0$. Combining (6.11), (6.12) with (6.8) we arrive at (6.9), (6.10) from the statement of the lemma. ∎

We note that the compatibility conditions (6.9), (6.10) are similar in spirit to the corresponding conditions in the additive noise case, i.e. (4.7), (4.8). In other words, (6.10) gives the minimal admissible lower bound for L. Then based on the second condition in (6.9) we obtain a minimal admissible lower bound for a. Whenever we need to increase a or b in the course of the main iteration process below, we have to decrease the value of β simultaneously so that the first condition in (6.9) is not violated.

Proposition 6.2 (Main iteration). *Let $L > 1$ satisfying (6.10) be given and let $(v_q, \mathring{R}_q)$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution to (6.2) satisfying (6.7). Then there exists a choice of parameters a, b, β such that (6.9) is fulfilled and there exist $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(v_{q+1}, \mathring{R}_{q+1})$ which solve (6.2), obey (6.7) at level $q + 1$ and for $t \in [0, T_L]$ we have*

$$\|v_{q+1}(t) - v_q(t)\|_{L^2} \leq m_L M_0(t)^{1/2} \delta_{q+1}^{1/2}. \tag{6.13}$$

Furthermore, if $v_q(0)$ and $\mathring{R}_q(0)$ are deterministic, so are $v_{q+1}(0)$ and $\mathring{R}_{q+1}(0)$.

The proof of Proposition 6.13 is presented in Section 6.1 below. Based on this result, we are able to conclude the proof of Theorem 1.3.

Proof of Theorem 1.3. Starting from $(v_0, \mathring{R}_0)$ given in Lemma 6.1 and using Proposition 6.2 we obtain a sequence $(v_q, \mathring{R}_q)$ satisfying (6.7) and (6.13). By interpolation, it follows that for $\gamma \in (0, \frac{\beta}{4+\beta})$ and $t \in [0, T_L]$,

$$\begin{aligned}\sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{H^\gamma} &\lesssim \sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{L^2}^{1-\gamma} \|v_{q+1}(t) - v_q(t)\|_{H^1}^{\gamma} \\ &\lesssim m_L M_0(t)^{1/2}.\end{aligned}$$

Therefore, the sequence v_q converges to a limit $v \in C([0, T_L]; H^\gamma)$ which is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Furthermore, we know that v is an analytically weak solution to (6.1) with a deterministic initial value, since due to (6.7) we have $\lim_{q\to\infty} \mathring{R}_q = 0$ in $C([0, T_L]; L^1)$. According to (6.13) and (6.8), it follows that for $t \in [0, T_L]$,

$$\begin{aligned}\|v(t) - v_0(t)\|_{L^2} &\leq \sum_{q\geq 0} \|v_{q+1}(t) - v_q(t)\|_{L^2} \leq m_L M_0(t)^{1/2} \sum_{q\geq 0} \delta_{q+1}^{1/2} \\ &\leq \tfrac{1}{2} m_L M_0(t)^{1/2}.\end{aligned}$$

Now, we show that for a given $T > 0$ we can choose $L = L(T) > 1$ large enough so that v fails to satisfy the corresponding energy inequality at time T, namely,

$$\|v(T)\|_{L^2} > e^{2L^{1/2}} \|v(0)\|_{L^2} \tag{6.14}$$

on the set $\{T_L \geq T\}$. To this end, we observe that

$$e^{2L^{1/2}} \|v(0)\|_{L^2} \leq e^{2L^{1/2}} (\|v_0(0)\|_{L^2} + \|v(0) - v_0(0)\|_{L^2}) \leq e^{2L^{1/2}} \frac{3}{2} m_L M_0(0)^{1/2}.$$

On the other hand, we obtain, on $\{T_L \geq T\}$,

$$\begin{aligned}\|v(T)\|_{L^2} &\geq (\|v_0(T)\|_{L^2} - \|v(T) - v_0(T)\|_{L^2}) \geq \Big(\frac{1}{\sqrt{2}} - \frac{1}{2}\Big) m_L M_0(T)^{1/2} \\ &> e^{2L^{1/2}} \frac{3}{2} m_L M_0(0)^{1/2}\end{aligned}$$

provided

$$\Big(\frac{1}{\sqrt{2}} - \frac{1}{2}\Big) e^{2LT} > \frac{3}{2} e^{2L^{1/2}}. \tag{6.15}$$

Hence (6.14) follows for a suitable choice of L satisfying additionally (6.15). Furthermore, for a given $T > 0$ we can increase L if necessary so that $\mathbf{P}(T_L \geq T) > \kappa$.

To conclude the proof, we define $u := \theta v$ and observe that $u(0) = v(0)$. In addition, u is $(\mathcal{F}_t)_{t\geq 0}$-adapted and solves the original Navier–Stokes system (1.5). Then in view of (6.14) and the fact that $|\theta_T| \geq e^{-L^{1/4}}$ on the set $\{T_L \geq T\}$ due to (6.4), we obtain

$$\|u(T)\|_{L^2} = |\theta(T)|\, \|v(T)\|_{L^2} > e^{L^{1/2}} \|u(0)\|_{L^2} \quad \text{on } \{T_L \geq T\}.$$

Choosing L sufficiently large in dependence on K and T from the statement of the theorem, the desired lower bound follows. Finally, setting $\mathrm{t} := T_L$ completes the proof. ∎

6.1. *The main iteration – proof of Proposition 6.2*

The overall strategy of the proof is similar to Section 4.1 but modifications are required since the approximate system on level q has a different form. As in Section 4.1, we have to make sure that the construction is $(\mathcal{F}_t)_{t\geq 0}$-adapted at each step.

6.1.1. Choice of parameters. We choose a small parameter $\ell \in (0,1)$ as in Section 4.1.1: for a sufficiently small $\alpha \in (0,1)$ to be chosen below, we let $\ell \in (0,1)$ be a small parameter defined in (4.17) and satisfying (4.16). We note that the compatibility conditions (6.9) and (6.10) and the last condition in (4.16) can all be fulfilled provided we make a large enough and β small enough at the same time. In addition, we will require $\alpha b > 16$ and $\alpha > 8\beta b$.

In order to verify the inductive estimates (6.7) we need to absorb various expressions including $m_L^4 M_0(t)^{1/2}$ for all $t \in [0, T_L]$. To this end, we need to change the condition (4.18) in Section 4.1.1 to

$$C m_L^4 \ell^{1/3} \lambda_q^4 \leq c_R \delta_{q+2}/5, \quad m_L^4 M_0(L)^{1/2} \lambda_{q+1}^{13\alpha - 1/7} \leq c_R \delta_{q+2}/10, \quad m_L \leq \ell^{-1}. \tag{6.16}$$

In other words, we need

$$
\begin{aligned}
&9Le^{2L^{1/4}}a^{b(-\frac{\alpha}{2}+\frac{10}{3b}+2b\beta)} \ll 1,\\
&9Le^{2L^{1/4}}e^{2L^2+L}a^{b(13\alpha-\frac{1}{7}+2b\beta)} \ll 1,\\
&\sqrt{3}\,L^{1/4}e^{1/2L^{1/4}} \leq a^{2+\frac{3\alpha}{2}\cdot 7L^2}.
\end{aligned}
$$

Choosing $b=(7L^2)\vee(17\cdot 14^2)$, in view of $\alpha>8\beta b$, (6.16) can be achieved by choosing a large enough and $\alpha=14^{-2}$. This choice also satisfies $\alpha b>16$ required above, and the condition $\alpha>8\beta b$ can be achieved by choosing β small. It is also compatible with all the other requirements needed below.

6.1.2. Mollification. As the next step, we define space-time mollifications of v_q and $\mathring{R}_q$ and a time mollification of θ as follows:

$$
v_\ell=(v_q *_x \phi_\ell) *_t \varphi_\ell,\quad \mathring{R}_\ell=(\mathring{R}_q *_x \phi_\ell) *_t \varphi_\ell,\quad \theta_\ell=e^B *_t \varphi_\ell.
$$

By choosing time mollifiers that are compactly supported in $\mathbb{R}^+$, the mollification preserves $(\mathcal{F}_t)_{t\geq 0}$-adaptedness. If the initial data $v_q(0)$, $\mathring{R}_q(0)$ are deterministic, so are $v_\ell(0)$ and $\mathring{R}_\ell(0)$, $\partial_t\mathring{R}_\ell(0)$. Then using (6.2) we find that $(v_\ell,\mathring{R}_\ell)$ satisfies

$$
\begin{aligned}
\partial_t v_\ell+\tfrac{1}{2}v_\ell-\Delta v_\ell+\theta_\ell\operatorname{div}(v_\ell\otimes v_\ell)+\nabla p_\ell&=\operatorname{div}(\mathring{R}_\ell+R_{\mathrm{com}}),\\
\operatorname{div} v_\ell&=0,
\end{aligned}
$$

where

$$
\begin{aligned}
R_{\mathrm{com}}&=\theta_\ell(v_\ell\mathring{\otimes}v_\ell)-(\theta v_q\mathring{\otimes}v_q) *_x \phi_\ell *_t \varphi_\ell,\\
p_\ell&=(p_q *_x \phi_\ell) *_t \varphi_\ell-\tfrac{1}{3}\big(\theta_\ell|v_\ell|^2-(\theta|v_q|^2 *_x \phi_\ell) *_t \varphi_\ell\big).
\end{aligned}
$$

With this setting, the counterparts of the estimates (4.20), (4.21) and (4.22) are obtained the same way only replacing $M_0(t)^{1/2}$ by $m_L M_0(t)^{1/2}$. In particular,

$$
\|v_q-v_\ell\|_{C_tL^2}\leq \tfrac{1}{4}m_L M_0(t)^{1/2}\delta_{q+1}^{1/2}, \tag{6.17}
$$

$$
\|v_\ell\|_{C_tL^2}\leq m_L M_0(t)^{1/2}\Big(1+\sum_{1\leq r\leq q}\delta_r^{1/2}\Big)\leq 2m_L M_0(t)^{1/2}, \tag{6.18}
$$

$$
\|v_\ell\|_{C^N_{t,x}}\leq m_L M_0(t)^{1/2}\ell^{-N}\lambda_{q+1}^{-\alpha}. \tag{6.19}
$$

6.1.3. Construction of v_{q+1}. We recall that the intermittent jets $W_{(\xi)}$ and the corresponding estimates are summarized in Appendix B. The parameters $\lambda, r_\|, r_\perp, \mu$ are chosen as in (4.23) and we define χ and ρ to be the same functions as in Section 4.1.3 with $M_0(t)$ given by (6.6). Now, we define the modified amplitude functions

$$
\begin{aligned}
\bar{a}_{(\xi)}(\omega,t,x):=\bar{a}_{\xi,q+1}(\omega,t,x)&:=\theta_\ell^{-1/2}\rho(\omega,t,x)^{1/2}\gamma_\xi\bigg(\mathrm{Id}-\frac{\mathring{R}_\ell(\omega,t,x)}{\rho(\omega,t,x)}\bigg)(2\pi)^{-3/4}\\
&=\theta_\ell^{-1/2}a_{\xi,q+1}(\omega,t,x),
\end{aligned}
\tag{6.20}
$$

where γ_ξ is introduced in Lemma B.1 and $a_{\xi,q+1}$ is as in Section 4.1.3 with $M_0(t)$ given in (6.6). Since ρ, θ_ℓ and $\mathring{R}_\ell$ are $(\mathcal{F}_t)_{t\geq 0}$-adapted, we know $\bar{a}_{(\xi)}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Note that since $\theta_\ell(0)$ and $\partial_t\theta_\ell(0)$ are deterministic, if $\mathring{R}_\ell(0)$ and $\partial_t\mathring{R}_\ell(0)$ are deterministic, so are $\bar{a}_\xi(0)$ and $\partial_t\bar{a}_\xi(0)$. By (B.5) we have

$$(2\pi)^{3/2}\sum_{\xi\in\Lambda}\bar{a}^2_{(\xi)}\fint_{\mathbb{T}^3}W_{(\xi)}\otimes W_{(\xi)}\,dx=\theta_\ell^{-1}(\rho\,\mathrm{Id}-\mathring{R}_\ell), \tag{6.21}$$

and for $t\in[0,T_L]$,

$$\begin{aligned}\|\bar{a}_{(\xi)}\|_{C_tL^2}&\leq\|\theta_\ell^{-1/2}\|_{C_t}\|\rho\|^{1/2}_{C_tL^1}\|\gamma_\xi\|_{C^0(B_{1/2}(\mathrm{Id}))}\\&\leq\frac{4c_R^{1/2}(8\pi^3+1)^{1/2}M}{8|\Lambda|(8\pi^3+1)^{1/2}}m_LM_0(t)^{1/2}\delta^{1/2}_{q+1}\leq\frac{c_R^{1/4}m_LM_0(t)^{1/2}\delta^{1/2}_{q+1}}{2|\Lambda|},\end{aligned}\tag{6.22}$$

where we choose c_R as a small universal constant to absorb M, the universal constant from Lemma B.1, and we apply the bound $|\theta_\ell^{-1}|\leq m_L^2$. Furthermore, since ρ is bounded from below by $4c_R\delta_{q+1}M_0(t)$, we obtain, for $t\in[0,T_L]$,

$$\|\bar{a}_{(\xi)}\|_{C^N_{t,x}}\lesssim\ell^{-2-5N}c_R^{1/4}m_LM_0(t)^{1/2}\delta^{1/2}_{q+1} \tag{6.23}$$

for $N\geq 0$, where we have used (6.5) and $4L\leq\ell^{-1}$ and the derivative of $\theta_\ell^{-1/2}$ gives an extra $\ell^{-1}m_L^4$ and $m_L\leq\ell^{-1}$.

As the next step, we define w_{q+1} much as in Section 4.1.3. In particular, first we define the principal part $w^{(p)}_{q+1}$ of w_{q+1} as (4.30) with $a_{(\xi)}$ replaced by $\bar{a}_{(\xi)}$ given in (6.20). Then it follows from (6.21) that

$$\theta_\ell w^{(p)}_{q+1}\otimes w^{(p)}_{q+1}+\mathring{R}_\ell=\theta_\ell\sum_{\xi\in\Lambda}\bar{a}^2_{(\xi)}\mathbb{P}_{\neq 0}(W_{(\xi)}\otimes W_{(\xi)})+\rho\,\mathrm{Id}.$$

The incompressible corrector $w^{(c)}_{q+1}$ is therefore defined as in (4.32) again with $a_{(\xi)}$ replaced by $\bar{a}_{(\xi)}$. The temporal corrector $w^{(t)}_{q+1}$ is now defined as in (4.33) with $a_{(\xi)}$ given in (4.26) for $M_0(t)$ from (6.6). Note that for the temporal corrector we use the original amplitude functions $a_{(\xi)}$ from Section 4.1.3 (only using a different function $M_0(t)$), since we need the extra θ_ℓ to obtain a suitable cancelation. The total velocity increment w_{q+1} and the new velocity v_{q+1} are then given by

$$w_{q+1}:=w^{(p)}_{q+1}+w^{(c)}_{q+1}+w^{(t)}_{q+1},\quad v_{q+1}:=v_\ell+w_{q+1}.$$

Both are $(\mathcal{F}_t)_{t\geq 0}$-adapted, divergence-free and w_{q+1} is mean zero. If $v_q(0)$, $\mathring{R}_\ell(0)$ are deterministic, so is $v_{q+1}(0)$.

6.1.4. Verification of the inductive estimates for v_{q+1}. For the counterparts of the estimates (4.37)–(4.47), the main difference now is the extra m_L appearing in the bounds (6.22) and (6.23) for $\bar{a}_{(\xi)}$. Therefore, many of the estimates remain valid with $M_0(t)^{1/2}$ replaced by $m_LM_0(t)^{1/2}$, only the bounds for the temporal corrector $w^{(t)}_{q+1}$ do not change.

More precisely, for $t \in [0, T_L]$ we obtain

$$\|w_{q+1}^{(p)}\|_{C_tL^2} \le \tfrac{1}{2}m_L M_0(t)^{1/2}\delta_{q+1}^{1/2}, \tag{6.24}$$

$$\|w_{q+1}^{(p)}\|_{C_tL^p} \lesssim m_L M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}, \tag{6.25}$$

$$\|w_{q+1}^{(c)}\|_{C_tL^p} \lesssim m_L M_0(t)^{1/2}\delta_{q+1}^{1/2}\ell^{-12}r_\perp^{2/p}r_\parallel^{1/p-3/2}, \tag{6.26}$$

$$\|w_{q+1}^{(t)}\|_{C_tL^p} \lesssim M_0(t)\delta_{q+1}\ell^{-4}r_\perp^{2/p-1}r_\parallel^{1/p-2}\lambda_{q+1}^{-1}. \tag{6.27}$$

Combining (6.24), (6.26) and (6.27) then leads to

$$\begin{aligned}\|w_{q+1}\|_{C_tL^2} &\le m_L M_0(t)^{1/2}\delta_{q+1}^{1/2}(1/2 + C\lambda_{q+1}^{24\alpha-2/7} + CM_0(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{8\alpha-1/7})\\ &\le \tfrac{3}{4}m_L M_0(t)^{1/2}\delta_{q+1}^{1/2},\end{aligned} \tag{6.28}$$

where we use (6.16) to bound $CM_0(L)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{8\alpha-1/7} \le 1/8$.

As a consequence of (6.28) and (6.18), the first bound in (6.7) on level $q+1$ readily follows. In addition, (6.28) together with (6.17) implies (6.13) from the statement of the proposition. In order to verify the second bound in (6.7), we observe that similarly to (4.43)–(4.45), for $t \in [0, T_L]$,

$$\|w_{q+1}^{(p)}\|_{C^1_{t,x}} \lesssim m_L M_0(t)^{1/2}\ell^{-7}r_\perp^{-1}r_\parallel^{-1/2}\lambda_{q+1}^2, \tag{6.29}$$

$$\|w_{q+1}^{(c)}\|_{C^1_{t,x}} \lesssim m_L M_0(t)^{1/2}\ell^{-17}r_\parallel^{-3/2}\lambda_{q+1}^2, \tag{6.30}$$

$$\|w_{q+1}^{(t)}\|_{C^1_{t,x}} \lesssim M_0(t)\ell^{-9}r_\perp^{-1}r_\parallel^{-2}\lambda_{q+1}^{1+\alpha}. \tag{6.31}$$

Combining (6.29)–(6.31) with (6.19) and taking (6.16) into account, the second bound in (6.7) follows.

In order to control the Reynolds stress below, we observe that similarly to (4.46), (4.47), the following bounds hold true for $t \in [0, T_L]$ and $p \in (1, \infty)$:

$$\|w_{q+1}^{(p)} + w_{q+1}^{(c)}\|_{C_tW^{1,p}} \le m_L M_0(t)^{1/2}r_\perp^{2/p-1}r_\parallel^{1/p-1/2}\ell^{-2}\lambda_{q+1}, \tag{6.32}$$

$$\|w_{q+1}^{(t)}\|_{C_tW^{1,p}} \le M_0(t)r_\perp^{2/p-2}r_\parallel^{1/p-1}\ell^{-4}\lambda_{q+1}^{-2/7}. \tag{6.33}$$

6.1.5. Definition of the Reynolds stress $\mathring{R}_{q+1}$. As before, we know

$$\begin{aligned}&\operatorname{div}\mathring{R}_{q+1} - \nabla p_{q+1}\\ &\quad= \underbrace{\tfrac{1}{2}w_{q+1} - \Delta w_{q+1} + \partial_t(w_{q+1}^{(p)} + w_{q+1}^{(c)}) + \theta_\ell \operatorname{div}(v_\ell \otimes w_{q+1} + w_{q+1} \otimes v_\ell)}_{\operatorname{div}(R_{\mathrm{lin}})+\nabla p_{\mathrm{lin}}}\\ &\quad+ \underbrace{\theta_\ell \operatorname{div}\big((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)})\big)}_{\operatorname{div}(R_{\mathrm{cor}}+\nabla p_{\mathrm{cor}})}\\ &\quad+ \underbrace{\operatorname{div}(\theta_\ell w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} + \mathring{R}_\ell) + \partial_t w_{q+1}^{(t)}}_{\operatorname{div}(R_{\mathrm{osc}})+\nabla p_{\mathrm{osc}}}\\ &\quad+ \underbrace{(\theta - \theta_\ell)\operatorname{div}(v_{q+1} \mathring{\otimes} v_{q+1})}_{\operatorname{div}(R_{\mathrm{com1}})+\nabla p_{\mathrm{com1}}} + \operatorname{div}(R_{\mathrm{com}}) - \nabla p_\ell.\end{aligned}$$

Therefore, applying the inverse divergence operator $\mathcal{R}$ we define

$$\begin{aligned}R_{\mathrm{lin}} &:= \tfrac{1}{2}\mathcal{R}w_{q+1} - \mathcal{R}\Delta w_{q+1} + \mathcal{R}\partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1})\\ &\quad + \theta_\ell v_\ell \mathring{\otimes} w_{q+1} + \theta_\ell w_{q+1} \mathring{\otimes} v_\ell,\\ R_{\mathrm{cor}} &:= \theta_\ell(w^{(c)}_{q+1} + w^{(t)}_{q+1}) \mathring{\otimes} w_{q+1} + \theta_\ell w^{(p)}_{q+1} \mathring{\otimes} (w^{(c)}_{q+1} + w^{(t)}_{q+1}),\\ R_{\mathrm{com1}} &:= (\theta_\ell - \theta)(v_{q+1} \mathring{\otimes} v_{q+1}).\end{aligned}$$

And similarly to Section 4.1.5 we have

$$R_{\mathrm{osc}} := \sum_{\xi\in\Lambda} \mathcal{R}\big(\nabla a^2_{(\xi)}\mathbb{P}_{\neq 0}(W_{(\xi)} \otimes W_{(\xi)})\big) - \frac{1}{\mu}\sum_{\xi\in\Lambda} \mathcal{R}\big(\partial_t a^2_{(\xi)}(\phi^2_{(\xi)}\psi^2_{(\xi)}\xi)\big),$$

with $a_{(\xi)}$ given in (4.26) for $M_0(t)$ from (6.6). Hence the bounds for R_{osc} are the same as in Section 4.1.6. The Reynolds stress on level $q+1$ is then defined as

$$\mathring{R}_{q+1} := R_{\mathrm{lin}} + R_{\mathrm{osc}} + R_{\mathrm{cor}} + R_{\mathrm{com}} + R_{\mathrm{com1}}.$$

6.1.6. Verification of the inductive estimate for $\mathring{R}_{q+1}$. In the following we estimate the remaining terms in $\mathring{R}_{q+1}$ separately. We choose $p = \frac{32}{32-7\alpha} > 1$.

For the linear error, for $t \in [0, T_L]$ have

$$\begin{aligned}&\|R_{\mathrm{linear}}\|_{C_tL^p}\\ &\quad\lesssim \|w_{q+1}\|_{C_tW^{1,p}} + \|\mathcal{R}\partial_t(w^{(p)}_{q+1} + w^{(c)}_{q+1})\|_{C_tL^p}\\ &\qquad + m_L^2\|v_\ell \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} v_\ell\|_{C_tL^p}\\ &\quad\lesssim \|w_{q+1}\|_{C_tW^{1,p}} + \sum_{\xi\in\Lambda}\|\partial_t \operatorname{curl}(\bar{a}_{(\xi)}V_{(\xi)})\|_{C_tL^p} + \lambda_q^4 m_L^3 M_0(t)^{1/2}\|w_{q+1}\|_{C_tL^p}.\end{aligned}$$

Hence using (6.32), (6.33), (6.23), (B.7) and (6.25)–(6.27) we have, for $t \in [0, T_L]$,

$$\begin{aligned}&\|R_{\mathrm{linear}}\|_{C_tL^p}\\ &\quad\lesssim m_L M_0(t)^{1/2}\ell^{-2}r_\perp^{2/p-1}r_\|^{1/p-1/2}\lambda_{q+1} + M_0(t)\ell^{-4}r_\perp^{2/p-2}r_\|^{1/p-1}\lambda_{q+1}^{-2/7}\\ &\qquad + m_L M_0(t)^{1/2}\ell^{-7}r_\perp^{2/p}r_\|^{1/p-3/2}\mu + m_L M_0(t)^{1/2}\ell^{-12}r_\perp^{2/p-1}r_\|^{1/p-1/2}\lambda_{q+1}^{-1}\\ &\qquad + m_L^4 M_0(t)\ell^{-2}r_\perp^{2/p-1}r_\|^{1/p-1/2}\lambda_q^4\\ &\quad\lesssim m_L M_0(t)^{1/2}\lambda_{q+1}^{5\alpha-1/7} + M_0(t)\lambda_{q+1}^{9\alpha-2/7} + m_L M_0(t)^{1/2}\lambda_{q+1}^{15\alpha-1/7}\\ &\qquad + m_L M_0(t)^{1/2}\lambda_{q+1}^{25\alpha-15/7} + m_L^4 M_0(t)\lambda_{q+1}^{7\alpha-8/7}\\ &\quad\leq M_0(t)c_R\delta_{q+2}/5,\end{aligned}$$

where we use the fact that a is sufficiently large and β is suffciently small, in particular, (6.16) holds.

The corrector error is estimated by (6.25)–(6.27) for $t \in [0, T_L]$ as follows:

$$
\begin{aligned}
&\|R_{\mathrm{cor}}\|_{C_tL^p} \\
&\leq m_L^2 \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{C_tL^{2p}} \|w_{q+1}\|_{C_tL^{2p}} + m_L^2 \|w_{q+1}^{(c)} + w_{q+1}^{(t)}\|_{C_tL^{2p}} \|w_{q+1}^{(p)}\|_{C_tL^{2p}} \\
&\lesssim m_L^4 M_0(t)(\ell^{-12} r_\perp^{1/p} r_\parallel^{1/(2p)-3/2} + \ell^{-4} M_0(t)^{1/2} r_\perp^{1/p-1} r_\parallel^{1/(2p)-2} \lambda_{q+1}^{-1}) \\
&\quad \times \ell^{-2} r_\perp^{1/p-1} r_\parallel^{1/(2p)-1/2} \\
&\lesssim m_L^4 M_0(t)(\ell^{-14} r_\perp^{2/p-1} r_\parallel^{1/p-2} + \ell^{-6} M_0(t)^{1/2} r_\perp^{2/p-2} r_\parallel^{1/p-5/2} \lambda_{q+1}^{-1}) \\
&\leq m_L^4 M_0(t)(\lambda_{q+1}^{29\alpha-2/7} + M_0(t)^{1/2} \lambda_{q+1}^{13\alpha-1/7}) \leq M_0(t) c_R \delta_{q+2}/5,
\end{aligned}
$$

where we again use (6.16) to have $m_L^4 \lambda_{q+1}^{29\alpha-2/7} + m_L^4 M_0(L)^{1/2} \lambda_{q+1}^{13\alpha-1/7} \leq c_R \delta_{q+2}/5$.

In view of a standard mollification estimate we deduce that, for $t \in [0, T_L]$,

$$
\begin{aligned}
|\theta_\ell(t) - \theta(t)| &\leq \ell^{1/2-2\delta} L^{1/2} e^{L^{1/2}} \leq \ell^{1/2-2\delta} m_L^2, \\
\|R_{\mathrm{com}}\|_{C_tL^1} &\lesssim m_L^2 \ell \|v_q\|_{C_{t,x}^1} \|v_q\|_{C_tL^2} + \ell^{1/2-2\delta} m_L^4 M_0(t) \lambda_q^4 \\
&\lesssim \ell^{1/2-2\delta} m_L^4 M_0(t) \lambda_q^4 \leq M_0(t) c_R \delta_{q+2}/5,
\end{aligned}
$$

where $\delta \in (0, 1/12)$ and we choose a large enough to have

$$
C\ell^{1/2-2\delta} m_L^4 \lambda_q^4 < \frac{c_R}{5} \lambda_{q+2}^{-2\beta}. \tag{6.34}
$$

With the choice of ℓ and since we postulated that $\alpha > 8\beta b$ and $\alpha b > 16$, this can indeed be achieved by possibly increasing a and consequently decreasing β.

The second commutator error can be estimated for $t \in [0, T_L]$ as follows:

$$
\|R_{\mathrm{com1}}\|_{C_tL^1} \leq \ell^{1/2-2\delta} m_L^4 M_0(t) \leq M_0(t) c_R \delta_{q+2}/5,
$$

where we use (6.34) to have $\ell^{1/2-2\delta} m_L^4 < \frac{c_R}{5} \delta_{q+2}$.

Thus, collecting the above estimates we obtain the desired third bound in (6.7), and the proof of Proposition 6.2 is complete.

7. Nonuniqueness in law III: the case of a nonlinear noise

The treatment of a nonlinear noise requires more input coming from the driving Brownian motion, namely, the corresponding iterated integral of B against B as known in the theory of rough paths. This is reflected through an additional variable $\mathbb{Y}$ included in the path space. Furthermore, we include a variable Z which is used to control the first step of the iteration scheme defined via (8.1), (8.2) below, namely to control z_0. This is just an auxiliary point, which by Corollary 7.1 does not restrict the final nonuniqueness in law result.

In what follows, we therefore use the following notations. Let

$$
\tilde{\Omega} := C([0,\infty); H^{-3} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times L^2_\sigma)
$$

and let $\mathscr{P}(\widetilde{\Omega})$ denote the set of all probability measures on $(\widetilde{\Omega}, \widetilde{\mathscr{B}})$ with $\widetilde{\mathscr{B}}$ being the Borel σ-algebra coming from the natural topology on $\widetilde{\Omega}$. Let $(x, y, \mathbb{Y}, Z) : \widetilde{\Omega} \to H^{-3} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times H^{-3}$ denote the canonical process on $\widetilde{\Omega}$ given by

$$(x_t(\omega), y_t(\omega), \mathbb{Y}_t(\omega), Z_t(\omega)) = \omega(t).$$

For $t \geq 0$ we define the σ-algebra $\widetilde{\mathscr{B}}^t = \sigma\{(x(s), y(s), \mathbb{Y}(s), Z(s)) : s \geq t\}$. Finally, we define the canonical filtration $\widetilde{\mathscr{B}}_t^0 := \sigma\{(x(s), y(s), \mathbb{Y}(s), Z(s)) : s \leq t\}$, $t \geq 0$, as well as its right continuous version $\widetilde{\mathscr{B}}_t := \bigcap_{s>t} \widetilde{\mathscr{B}}_s^0$, $t \geq 0$. In this section we choose $U = \mathbb{R}^m$.

7.1. *Generalized probabilistically weak solutions*

Accordingly, we need to generalize our notion of solution, taking the additional variables $\mathbb{Y}$ and Z into account. We fix a deterministic function $v_0 \in C^1_{t,x}$, which will be chosen in Lemma 8.1 below. In order to define the stopping time in the same path space, the process Z solves the SPDE

$$dZ_t - \Delta Z_t = G(v_0 + Z_t) dB_t, \quad \operatorname{div} Z = 0. \tag{7.1}$$

By [36, Theorem 4.2.5], the solution to (7.1) belongs to $C([0, \infty); L^2_\sigma)$ a.s.

Definition 7.1. Let $s \geq 0$ and $x_0 \in L^2_\sigma$, $y_0 \in \mathbb{R}^m$, $\mathbb{Y}_0 \in \mathbb{R}^{m\times m}$, $Z_0 \in L^2_\sigma$. A probability measure $P \in \mathscr{P}(\widetilde{\Omega})$ is a *generalized probabilistically weak solution* to the Navier–Stokes system (1.1) with initial value $(x_0, y_0, \mathbb{Y}_0, Z_0)$ at time s provided

(M1) $P(x(t) = x_0, y(t) = y_0, \mathbb{Y}(t) = \mathbb{Y}_0, Z(t) = Z_0, 0 \leq t \leq s) = 1$ and for any $n \in \mathbb{N}$,

$$P\left\{(x, y, \mathbb{Y}, Z) \in \widetilde{\Omega} : \int_0^n \|G(x(r))\|^2_{L_2(\mathbb{R}^m; L^2_\sigma)}\, dr < \infty,\ Z \in CL^2_\sigma\right\} = 1.$$

(M2) Under P, y is a $(\widetilde{\mathscr{B}}_t)_{t\geq s}$-Brownian motion in $\mathbb{R}^m$ starting from y_0 at time s and for every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and all $t \geq s$,

$$\langle x(t) - x(s), e_i\rangle + \int_s^t \langle \operatorname{div}(x(r)\otimes x(r)) - \Delta x(r), e_i\rangle\, dr = \int_s^t \langle e_i, G(x(r)) dy(r)\rangle,$$

$$\mathbb{Y}(t) - \mathbb{Y}(s) = \int_s^t y(r) \otimes dy(r),$$

$$\langle Z(t) - Z(s), e_i\rangle - \int_s^t \langle \Delta Z(r), e_i\rangle\, dr = \int_s^t \langle e_i, G(v_0 + Z(r)) dy(r)\rangle.$$

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$,

$$E^P\left(\sup_{r\in[0,t]} \|x(r)\|^{2q}_{L^2} + \int_s^t \|x(r)\|^2_{H^\gamma}\, dr\right) \leq C_{t,q}(\|x_0\|^{2q}_{L^2} + 1),$$

$$E^P\left(\sup_{r\in[0,t]} \|Z(r)\|^{2q}_{L^2} + \int_s^t \|\nabla Z(r)\|^2_{L^2}\, dr\right) \leq C_{t,q}(\|Z_0\|^{2q}_{L^2} + 1).$$

By the assumption (1.4) on G, pathwise uniqueness holds for (7.1). Hence, since v_0 is deterministic, the law of Z is uniquely determined by the Brownian motion B according to the Yamada–Watanabe Theorem. Under (1.4), we know that the constant $C_{t,q}$ is independent of v_0.

Define $y_{r,t} := y(t) - y(r)$ and $\mathbb{X}_{r,t} := \mathbb{Y}(t) - \mathbb{Y}(r) - y(r) \otimes (y(t) - y(r))$. Note that under a generalized probabilistically weak solution P, the pair $(y, \mathbb{X})$ can be viewed as a rough path, concretely, it is the Itô lift of an m-dimensional Brownian motion. In particular, Chen's relation holds true:

$$\delta\mathbb{X}_{r,\theta,t} = \mathbb{X}_{r,t} - \mathbb{X}_{r,\theta} - \mathbb{X}_{\theta,t} = y_{r,\theta} \otimes y_{\theta,t}, \quad r \leq \theta \leq t. \tag{7.2}$$

For the application to the Navier–Stokes system, we will again require a definition of generalized probabilistically weak solutions defined up to a stopping time τ. To this end, we set

$$\widetilde{\Omega}_\tau := \{\omega(\cdot \wedge \tau(\omega)) : \omega \in \widetilde{\Omega}\}.$$

Definition 7.2. Let $s \geq 0$ and $x_0 \in L^2_\sigma$, $y_0 \in \mathbb{R}^m$, $\mathbb{Y}_0 \in \mathbb{R}^{m\times m}$, $Z_0 \in L^2_\sigma$. Let $\tau \geq s$ be a $(\widetilde{\mathscr{B}}_t)_{t\geq s}$-stopping time. A probability measure $P \in \mathscr{P}(\widetilde{\Omega}_\tau)$ is a *generalized probabilistically weak solution* to the Navier–Stokes system (1.1) *on* $[s, \tau]$ with initial value $(x_0, y_0, \mathbb{Y}_0, Z_0)$ at time s provided

(M1) $P(x(t) = x_0, y(t) = y_0, \mathbb{Y}(t) = \mathbb{Y}_0, Z(t) = Z_0, 0 \leq t \leq s) = 1$ and for any $n \in \mathbb{N}$,

$$P\left\{(x, y, \mathbb{Y}, Z) \in \widetilde{\Omega} : \int_0^{n\wedge\tau} \|G(x(r))\|^2_{L_2(\mathbb{R}^m;L^\sigma_2)}\, dr < \infty,\ Z \in CL^2_\sigma\right\} = 1.$$

(M2) Under P, $y = (y_i)_{i=1}^m$ and each component $y_i(\cdot \wedge \tau), i = 1, \ldots, m$, is a continuous square integrable $(\widetilde{\mathscr{B}}_t)_{t\geq s}$-martingale starting from y_0^i at time s with cross variation process between y_i and y_j given by $(t \wedge \tau - s)\delta_{ij}$. For every $e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma$ and all $t \geq s$,

$$\langle x(t\wedge\tau) - x(s), e_i\rangle + \int_s^{t\wedge\tau} \langle \mathrm{div}(x(r) \otimes x(r)) - \Delta x(r), e_i\rangle\, dr = \int_s^{t\wedge\tau} \langle e_i, G(x(r))dy(r)\rangle,$$

$$\mathbb{Y}(t\wedge\tau) - \mathbb{Y}(s) = \int_s^{t\wedge\tau} y(r) \otimes dy(r),$$

$$\langle Z(t\wedge\tau) - Z(s), e_i\rangle - \int_s^{t\wedge\tau} \langle \Delta Z(r), e_i\rangle\, dr = \int_s^{t\wedge\tau} \langle e_i, G(v_0 + Z(r))dy(r)\rangle.$$

(M3) For any $q \in \mathbb{N}$ there exists a positive real function $t \mapsto C_{t,q}$ such that for all $t \geq s$,

$$E^P\left(\sup_{r\in[0,t\wedge\tau]} \|x(r)\|^{2q}_{L^2} + \int_s^{t\wedge\tau} \|x(r)\|^2_{H^\gamma}\, dr\right) \leq C_{t,q}(\|x_0\|^{2q}_{L^2} + 1),$$

$$E^P\left(\sup_{r\in[0,t\wedge\tau]} \|Z(r)\|^{2q}_{L^2} + \int_s^{t\wedge\tau} \|\nabla Z(r)\|^2_{L^2}\, dr\right) \leq C_{t,q}(\|Z_0\|^{2q}_{L^2} + 1).$$

It is easy to see the following relation between generalized probabilistically weak solutions and probabilistically weak solutions.

Corollary 7.1. *Let P be a generalized probabilistically weak solution starting from $(x_0, y_0, 0, 0)$ at time s. Then the canonical process $\mathbb{Y}$ and Z under P is a measurable function of y. In other words, P is fully determined by the joint probability law of x, y and can be identified with a probability measure on the reduced path space $\bar{\Omega}$. Hence P is a probabilistically weak solution with initial value (x_0, y_0) at time s.*

Conversely, let $(x_0, y_0) \in L^2_\sigma \times \mathbb{R}^m$ be given, let $P \in \mathscr{P}(\bar{\Omega})$ be a probabilistically weak solution and define P-a.s. for $t \geq s$,

$$\mathbb{Y}(t) := \int_s^t y(r) \otimes dy(r),$$

and set Z to be the unique probabilistically strong solution to (7.1) with B replaced by y and $Z(s) = 0$. Let Q be the joint law of $(x, y, \mathbb{Y}, Z)$ under P. Then $Q \in \mathscr{P}(\widetilde{\Omega})$ gives rise to a generalized probabilistically weak solution starting from the initial value $(x_0, y_0, 0, 0)$ at time s.

Similarly to Theorem 3.1, the following existence and stability result holds. We prove it in Appendix A.

Theorem 7.2. *For every $(s, x_0, y_0, \mathbb{Y}_0, Z_0) \in [0, \infty) \times L^2_\sigma \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times L^2_\sigma$, there exists $P \in \mathscr{P}(\widetilde{\Omega})$ which is a generalized probabilistically weak solution to the Navier–Stokes system (1.1) starting at time s from the initial condition $(x_0, y_0, \mathbb{Y}_0, Z_0)$ in the sense of Definition 7.1. The set of all such solutions with the same implicit constant $C_{t,q}$ is denoted by $\mathscr{GW}(s, x_0, y_0, \mathbb{Y}_0, Z_0, C_{t,q})$.*

Let

$$(s_n, x_n, y_n, \mathbb{Y}_n, Z_n) \to (s, x_0, y_0, \mathbb{Y}_0, Z_0) \quad \text{in } [0, \infty) \times L^2_\sigma \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times L^2_\sigma$$

as $n \to \infty$ and let

$$P_n \in \mathscr{GW}(s_n, x_n, y_n, \mathbb{Y}_n, Z_n, C_{t,q}).$$

Then there exists a subsequence n_k such that the sequence $\{P_{n_k}\}_{k\in\mathbb{N}}$ converges weakly to some $P \in \mathscr{GW}(s, x_0, y_0, \mathbb{Y}_0, Z_0, C_{t,q})$.

By Corollary 7.1 nonuniqueness of generalized probabilistically weak solutions from $(x_0, y_0, 0, 0)$ implies joint nonuniqueness in law from (x_0, y_0) in the sense of Definition 5.3.

7.2. *General construction for generalized probabilistically weak solutions*

The overall strategy is similar to Section 3.2. In the first step, we shall extend generalized probabilistically weak solutions defined up to a $(\widetilde{\mathscr{B}}_t)_{t\geq 0}$-stopping time τ to the whole interval $[0, \infty)$. We denote by $\widetilde{\mathscr{B}}_\tau$ the σ-field associated to τ.

Proposition 7.3. *Let τ be a bounded $(\widetilde{\mathcal{B}}_t)_{t\geq 0}$-stopping time. Then for every $\omega \in \widetilde{\Omega}$ there exists $Q_\omega \in \mathscr{P}(\widetilde{\Omega})$ such that for $\omega \in \{x(\tau) \in L^2_\sigma, Z(\tau) \in L^2_\sigma\}$,*

$$Q_\omega\big(\omega' \in \widetilde{\Omega} : (x, y, \mathbb{Y}, Z)(t, \omega') = (x, y, \mathbb{Y}, Z)(t, \omega) \text{ for } 0 \leq t \leq \tau(\omega)\big) = 1, \quad (7.3)$$

$$Q_\omega(A) = R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega),\mathbb{Y}(\tau(\omega),\omega),Z(\tau(\omega),\omega)}(A) \quad \text{for all } A \in \mathcal{B}^{\tau(\omega)}, \quad (7.4)$$

where $R_{\tau(\omega),x(\tau(\omega),\omega),y(\tau(\omega),\omega),\mathbb{Y}(\tau(\omega),\omega),Z(\tau(\omega),\omega)} \in \mathscr{P}(\widetilde{\Omega})$ is a generalized probabilistically weak solution to the Navier–Stokes system (1.1) starting at time $\tau(\omega)$ from the initial condition

$$(x(\tau(\omega), \omega), y(\tau(\omega), \omega), \mathbb{Y}(\tau(\omega), \omega), Z(\tau(\omega), \omega)).$$

Furthermore, for every $B \in \widetilde{\mathcal{B}}$ the mapping $\omega \mapsto Q_\omega(B)$ is $\widetilde{\mathcal{B}}_\tau$-measurable.

Proof. The proof is identical to the proof of Proposition 3.2 applied to the extended path space $\widetilde{\Omega}$ instead of Ω_0 and making use of Theorem 7.2 instead of Theorem 3.1. ∎

We proceed with an analogue to Proposition 5.3.

Proposition 7.4. *Let $x_0 \in L^2_\sigma$. Let P be a generalized probabilistically weak solution to the Navier–Stokes system (1.1) on $[0, \tau]$ starting at time 0 from the initial condition $(x_0, 0, 0, 0)$. In addition to the assumptions of Proposition 7.3, suppose that there exists a Borel set $\mathcal{N} \subset \widetilde{\Omega}_\tau$ such that $P(\mathcal{N}) = 0$ and for every $\omega \in \mathcal{N}^c$,*

$$Q_\omega(\omega' \in \widetilde{\Omega} : \tau(\omega') = \tau(\omega)) = 1. \quad (7.5)$$

Then the probability measure $P \otimes_\tau R \in \mathscr{P}(\widetilde{\Omega})$ defined by

$$P \otimes_\tau R(\cdot) := \int_{\widetilde{\Omega}} Q_\omega(\cdot)\, P(d\omega)$$

satisfies $P \otimes_\tau R = P$ on the σ-algebra $\sigma(x(t \wedge \tau), y(t \wedge \tau), \mathbb{Y}(t \wedge \tau), Z(t \wedge \tau) : t \geq 0)$ and is a generalized probabilistically weak solution to the Navier–Stokes system (1.1) on $[0, \infty)$ with initial condition $(x_0, 0, 0, 0)$.

Proof. Most of the proof follows exactly the same argument as in Proposition 5.3. For (M2), we consider the z part and similar to the proof of Proposition 5.3 we obtain

$$\begin{aligned}
&P \otimes_\tau R\left\{\mathbb{Y}(t) - \mathbb{Y}(0) = \int_0^t y(r) \otimes dy(r),\ t \geq 0\right\}\\
&\quad= \int_{\widetilde{\Omega}} dP(\omega) Q_\omega\left\{\mathbb{Y}(t) - \mathbb{Y}(t \wedge \tau(\omega)) = \int_{t\wedge\tau(\omega)}^t y(r) \otimes dy(r),\right.\\
&\qquad\qquad \left.\mathbb{Y}(t \wedge \tau(\omega)) - \mathbb{Y}(0) = \int_0^{t\wedge\tau(\omega)} y(r) \otimes dy(r),\ t \geq 0\right\} = 1,
\end{aligned}$$

and

$$\begin{aligned}
&P \otimes_\tau R\left\{\langle Z(t) - Z(0), e_i\rangle - \int_0^t \langle Z(r), \Delta e_i\rangle\, dr\right.\\
&\qquad \left.= \int_0^t \langle e_i, G(v_0 + Z(r)) dy(r)\rangle,\ e_i \in C^\infty(\mathbb{T}^3) \cap L^2_\sigma,\ t \geq 0\right\} = 1.
\end{aligned}$$

Hence condition (M2) follows. ∎

7.3. Application to solutions obtained through Theorem 1.5

For $\alpha \in (0,1)$ we denote

$$\|\mathbb{X}\|_{\alpha,[0,T]} := \sup_{0\le r<t\le T} \frac{|\mathbb{X}_{r,t}|}{|r-t|^{\alpha}},$$

and for $n \in \mathbb{N}$, $L > 1$ and $\delta \in (0, 1/12)$ we define

$$\begin{aligned}\tau_L^n(\omega) = {} & \inf\{t > 0 : \|y(\omega)\|_{C_t^{1/2-2\delta}} > (\ln\ln L - 1/n)\} \wedge \ln\ln L \\ & \wedge \inf\{T \ge 0 : \|\mathbb{X}\|_{1-4\delta,[0,T]} > (\ln\ln L - 1/n)\} \\ & \wedge \inf\{t \ge 0 : \|Z(t)\|_{L^2} > L - 1/n\}.\end{aligned}$$

Then the sequence $\{\tau_L^n\}_{n\in\mathbb{N}}$ is nondecreasing and we define

$$\tau_L := \lim_{n\to\infty} \tau_L^n. \tag{7.6}$$

Without additional regularity of the process y, we have $\tau_L^n(\omega) = 0$. By Lemma 3.5 we deduce that τ_L^n is a $(\widetilde{\mathcal{B}}_t)_{t\ge 0}$-stopping time and consequently so is τ_L as an increasing limit of stopping times.

On a stochastic basis $(\Omega, \mathcal{F}, \mathbf{P})$, we apply Theorem 1.5 and denote by u and z_0 the corresponding solutions to the Navier–Stokes system (1.9) and to the linear equation (8.1) with $q = 0$ on $[0, T_L]$, where the stopping time T_L is defined in (8.3) below. We recall that u is adapted to $(\mathcal{F}_t)_{t\ge 0}$. The process z_0 is the unique probabilistically strong solution, hence is also adapted to $(\mathcal{F}_t)_{t\ge 0}$. We denote by P the law of $(u, B, \int_0^{\cdot} B\,dB, z_0)$ and obtain the following result by similar arguments to those in the proof of Proposition 3.7.

Proposition 7.5. *The probability measure P is a generalized probabilistically weak solution to the Navier–Stokes system (1.9) with initial condition $(u(0), 0, 0, 0)$ on $[0, \tau_L]$ in the sense of Definition 7.2, where τ_L was defined in (7.6).*

Proof. The proof is similar to the proof of Proposition 3.7 once we note that from the definition of the canonical process we have

$$y\Big(t, \Big(u, B, \int B\,dB, z_0\Big)\Big) = B(t), \quad \mathbb{Y}\Big(t, \Big(u, B, \int B\,dB, z_0\Big)\Big) = \int_0^t B_r\,dB_r,$$

$$Z\Big(t, \Big(u, B, \int B\,dB, z_0\Big)\Big) = z_0(t),$$

$$\mathbb{X}_{s,t}\Big(u, B, \int B\,dB, z_0\Big) = \int_s^t B_{s,r} \otimes dB_r =: \mathbb{B}_{s,t} \quad \text{for } s,t \in [0, T_L]\ \mathbf{P}\text{-a.s.},$$

and by Chen's relation (7.2) and the definition of Z, the functions

$$t \mapsto \|B\|_{C_t^{1/2-2\delta}}, \quad t \mapsto \|\mathbb{B}\|_{1-4\delta,[0,t]}, \quad t \mapsto \|z_0(t)\|_{L^2}$$

are continuous $\mathbf{P}$-a.s. In particular, property (M2) in Definition 7.2 follows since $(u, B, \int B\,dB)$ satisfies (1.9) and (z_0, B) satisfies (8.1) with $q = 0$. ■

Proposition 7.6. *The probability measure $P \otimes_{\tau_L} R$ is a generalized probabilistically weak solution to the Navier–Stokes system* (1.9) *on* $[0,\infty)$ *in the sense of Definition* 7.1.

Proof. In light of Propositions 7.3 and 7.4, it only remains to establish (7.5). By Theorem 1.5 and Proposition 7.5 we know that

$$P\big(\omega : y(\cdot \wedge \tau_L) \in C^{1/2-\delta}\mathbb{R}^m, Z(\cdot \wedge \tau_L) \in CL^2_\sigma\big) = 1,$$

and by $\mathbb{X}_{s,t\wedge\tau_L} = \int_s^{t\wedge\tau_L} y_{s,r} \otimes dy_r$ for $t > s$, we have

$$P(\omega : \|\mathbb{X}\|_{1-2\delta,[0,\tau_L]} < \infty) = 1.$$

In other words, there exists a P-measurable set $\mathcal{N} \subset \widetilde{\Omega}$ such that $P(\mathcal{N}) = 0$ and for $\omega \in \mathcal{N}^c$,

$$y(\cdot \wedge \tau_L(\omega)) \in C^{1/2-\delta}\mathbb{R}^m, \quad \|\mathbb{X}(\omega)\|_{1-2\delta,[0,\tau_L(\omega)]} < \infty, \quad Z(\cdot \wedge \tau_L(\omega)) \in CL^2_\sigma.$$

Moreover, by Chen's relation, for all $\omega \in \mathcal{N}^c \cap \{x(\tau_L) \in L^2_\sigma, Z(\tau_L) \in L^2_\sigma\}$ we have

$$\begin{aligned}
&Q_\omega\big(\omega' : \|\mathbb{X}\|_{1-2\delta,[0,T]} < \infty,\ y \in C_T^{1/2-\delta}\mathbb{R}^m,\ Z \in C_T L^2_\sigma,\ T > 0\big)\\
&= Q_\omega\big(\omega' : \|\mathbb{X}\|_{1-2\delta,[0,\tau_L(\omega)]} < \infty,\ \|\mathbb{X}\|_{1-2\delta,[\tau_L(\omega)\wedge T,T]} < \infty,\\
&\qquad\qquad y \in C_T^{1/2-\delta}\mathbb{R}^m,\ Z \in C_T L^2_\sigma,\ T > 0\big)\\
&= \delta_\omega\big(\omega' : y(\cdot \wedge \tau_L(\omega)) \in C^{1/2-\delta}\mathbb{R}^m,\ Z(\cdot \wedge \tau_L(\omega)) \in CL^2_\sigma,\ \|\mathbb{X}\|_{1-2\delta,[0,\tau_L(\omega)]} < \infty\big)\\
&\quad \times R_{\tau_L(\omega),(x,y,\mathbb{Y},Z)(\tau_L(\omega),\omega)}\big(\omega' : y - y(\cdot \wedge \tau_L(\omega)) \in C_T^{1/2-\delta}\mathbb{R}^m,\\
&\qquad\qquad Z - Z(\cdot \wedge \tau_L(\omega)) \in C_T L^2_\sigma,\ \|\mathbb{X}\|_{1-2\delta,[\tau_L(\omega)\wedge T,T]} < \infty, T > 0\big).
\end{aligned}$$

Here the first factor on the right hand side equals 1 for all $\omega \in \mathcal{N}^c$. Since $R_{\tau_L(\omega),(x,y,\mathbb{Y},Z)(\tau_L(\omega),\omega)}$ is a generalized probabilistically weak solution starting at the deterministic time $\tau_L(\omega)$ from the deterministic initial condition $(x, y, \mathbb{Y}, Z)(\tau_L(\omega), \omega)$, the process $\omega' \mapsto y - y(\cdot \wedge \tau_L(\omega))$ is a $(\widetilde{\mathcal{B}}_t)_{t\geq 0}$-Wiener process starting from $\tau_L(\omega)$ and $\mathbb{X}_{s,t} = \int_s^t y_{s,r} \otimes dy_r$ for $t > s \geq \tau_L(\omega)$ under the measure $R_{\tau_L(\omega),(x,y,\mathbb{Y},Z)(\tau_L(\omega),\omega)}$. Thus we deduce that also the second factor equals 1. To summarize, we have proved that for all $\omega \in \mathcal{N}^c \cap \{x(\tau_L) \in L^2_\sigma,\ Z(\tau_L) \in L^2_\sigma\}$,

$$Q_\omega(\omega' : y \in C_T^{1/2-\delta}\mathbb{R}^m,\ Z \in C_T L^2_\sigma,\ \|\mathbb{X}\|_{1-2\delta,[0,T]} < \infty,\ T > 0) = 1.$$

Therefore for all $\omega \in \mathcal{N}^c \cap \{x(\tau_L) \in L^2_\sigma$ and $Z(\tau_L) \in L^2_\sigma\}$, $\|y\|_{C_T^{1/2-2\delta}}$, $\|\mathbb{X}\|_{1-4\delta,[0,T]}$ are continuous with respect to T and $Z \in CL^2_\sigma$. The proof is completed by the same argument as for Proposition 3.8. ∎

Now, we are ready to conclude the proof of Theorem 1.6.

Proof of Theorem 1.6. By (1.11) with $K = 2$ and an argument as in the proof of Theorem 1.2, we deduce that the generalized probabilistically weak solution $P \otimes_{\tau_L} R$ does not satisfy the energy inequality. The solutions obtained in Theorem 7.2 satisfy the usual energy inequality by Galerkin approximation. Hence the two generalized probabilistically weak solutions starting from the initial condition $(u(0), 0, 0, 0)$ are distinct, which by Corollary 7.1 implies joint nonuniqueness in law, i.e. nonuniqueness of probabilistically weak solutions. In view of Theorem C.1 we finally deduce the desired nonuniqueness in law, i.e. nonuniqueness of martingale solutions. ■

8. Proof of Theorem 1.5

In this section, we fix an m-dimensional Brownian motion B on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with its normal filtration $(\mathcal{F}_t)_{t\geq 0}$ and assume the coefficient G satisfies (1.4). The principal difference between this setting and the setting of additive or linear multiplicative noise is that no transformation of the SPDE (1.9) into a PDE with random coefficients is available. Therefore we introduce a convex integration scheme which at each step additionally solves a parametrized stochastic Stokes equation with a nonlinear Itô noise. To be more precise, let $v_{-1} \equiv v_0$ be given in Lemma 8.1 below. At each step $q \in \mathbb{N}_0$, we construct a triple $(z_q, v_q, \mathring{R}_q)$ solving

$$\begin{aligned} dz_q - \Delta z_q dt &= G(v_{q-1} + z_q) dB, \\ \operatorname{div} z_q &= 0, \\ z_q(0) &= 0, \end{aligned} \tag{8.1}$$

$$\begin{aligned} \partial_t v_q - \Delta v_q + \operatorname{div}((v_q + z_q) \otimes (v_q + z_q)) + \nabla p_q &= \operatorname{div} \mathring{R}_q, \\ \operatorname{div} v_q &= 0. \end{aligned} \tag{8.2}$$

In order to obtain the desired iterative estimates (4.4) from the random PDE (8.2), it is necessary to have pathwise control of each z_q. This is not possible using stochastic Itô integration theory. Instead, we make use of rough path theory which we present in Appendix D. As explained in Section D.1, if v_{q-1} is adapted, then the unique rough path solution z_q of (8.1) coincides $\mathbf{P}$-a.s. with the unique stochastic solution coming from stochastic Itô integral theory. In particular, z_q is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process. This in turn permits one to conclude that the next iteration v_q is $(\mathcal{F}_t)_{t\geq 0}$-adapted as well.

As before, we consider an increasing sequence $\{\lambda_q\}_{q\in\mathbb{N}} \subset \mathbb{N}$ which diverges to ∞, and a sequence $\{\delta_q\}_{q\in\mathbb{N}} \subset (0, 1)$ which is decreasing to 0. For $a, b \in \mathbb{N}$ and $\beta \in (0, 1)$ to be chosen below we define

$$\lambda_q = a^{b^q}, \quad \delta_q = \lambda_q^{-2\beta}.$$

It will be seen that β will be chosen sufficiently small and a as well as b will be chosen sufficiently large. Set $\mathbb{B}_{s,t} := \int_s^t (B_r - B_s) \otimes dB_r$ and define, for $L > 1$ and $0 < \delta < 1/12$,

$$\begin{aligned} T_L &:= T_L^1 \wedge T_L^2, \\ T_L^1 &:= \inf\{t \geq 0 : \|B\|_{C_t^{1/2-2\delta}} \geq \ln\ln L\} \\ &\qquad \wedge \inf\{t \geq 0 : \|\mathbb{B}\|_{1-4\delta,[0,t]} \geq \ln\ln L\} \wedge \ln\ln L, \\ T_L^2 &:= \inf\{t \geq 0 : \|z_0(t)\|_{L^2} \geq L\}. \end{aligned} \tag{8.3}$$

By the definition of Brownian motion and the properties of the solution to the heat equation (8.1) for $q = 0$, the stopping time T_L is **P**-a.s. strictly positive, and $T_L \uparrow \infty$ as $L \to \infty$ **P**-a.s. Moreover, by Theorem D.9, for $t \in [0, T_L]$ with L large enough and $q \in \mathbb{N}$,

$$\begin{aligned}
&\|z_q\|_{C_t C^1} \leq L^{1/4}(1 + \|v_{q-1}\|_{C^1_{t,x}}), \quad \|z_q\|_{C_t^{1/2-2\delta} L^\infty} \leq L^{1/2}(1 + \|v_{q-1}\|_{C^1_{t,x}}), \\
&\|z_0\|_{C_t L^2} \leq L, \quad \|z_0\|_{C_t C^1} \leq L^{1/4}(1 + \|v_0\|_{C^1_{t,x}}), \\
&\|z_0\|_{C_t^{1/2-2\delta} L^\infty} \leq L^{1/2}(1 + \|v_0\|_{C^1_{t,x}}).
\end{aligned} \tag{8.4}$$

Let $M_0(t) = L^4 e^{4Lt}$. We postulate that the iterative bounds (4.4) hold for $(v_q, \mathring{R}_q)$.

Lemma 8.1. *For $L > 1$ define*

$$v_0(t, x) = \frac{L^2 e^{2Lt}}{(2\pi)^{3/2}} (\sin(x_3), 0, 0),$$

and let z_0 solve (8.1) with $v_{-1} \equiv v_0$. Then the associated Reynolds stress is given by

$$\mathring{R}_0(t, x) = \frac{(2L + 1)L^2 e^{2Lt}}{(2\pi)^{3/2}} \begin{pmatrix} 0 & 0 & -\cos(x_3) \\ 0 & 0 & 0 \\ -\cos(x_3) & 0 & 0 \end{pmatrix} + v_0 \mathring{\otimes} z_0 + z_0 \mathring{\otimes} v_0 + z_0 \mathring{\otimes} z_0.$$

Moreover, all the estimates in (4.4) on level $q = 0$ for $(v_0, \mathring{R}_0)$ as well as (4.5) are valid provided (4.7) and (8.4) hold. In particular, we require that (4.8) holds. Furthermore, the initial values $v_0(0, x)$ and $\mathring{R}_0(0, x)$ are deterministic.

Proof. The proof follows from the same argument as in Lemma 4.1. ■

As part of the following result we also control the difference $z_{q+1} - z_q$ on the time interval $[0, t]$, $t \in [0, T_L]$, in a pathwise manner in the space of controlled rough paths. We refer the reader to Appendix D and in particular to Definition D.2 where the corresponding norm $\|\cdot\|_{\bar{B},2\alpha,\gamma}$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $\gamma \in \mathbb{R}$, is defined. We denote by $\|\cdot\|_{\bar{B},2\alpha,\gamma,t}$ the norm in $D^{2\alpha}_{\bar{B},\gamma}([0, t])$. In general, $\|\cdot\|_{\bar{B},2\alpha,\gamma}$ is the norm for the pair process (z, z'), where z' is the Gubinelli derivative of the controlled rough path z. In the following we use it for z_q with $z'_q = G(z_q + v_{q-1})$ and we will write $\|z_q\|_{\bar{B},2\alpha,\gamma,t}$ instead of $\|z_q, z'_q\|_{\bar{B},2\alpha,\gamma,t}$ in this section.

Proposition 8.2 (Main iteration). *Let $L > 1$ satisfying (4.8) be given and let $(z_q, v_q, \mathring{R}_q)$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted solution to (8.1), (8.2) satisfying (4.4). Then there exists a choice of parameters a, b, β such that (4.7) is fulfilled and there exist $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $(z_{q+1}, v_{q+1}, \mathring{R}_{q+1})$ which solve (8.1), (8.2), obey (4.4) at level $q + 1$ and for $t \in [0, T_L]$ we have for $\delta > 0$, $q \in \mathbb{N}_0$, $\alpha_0 = 2/3 + \kappa$, and $\kappa > 0$ small enough,*

$$\|v_{q+1}(t) - v_q(t)\|_{L^2} \leq M_0(t)^{1/2} \delta_{q+1}^{1/2}, \quad \|v_{q+1} - v_q\|_{C_t^{\alpha_0} B^{-5-\delta}_{1,1}} \leq M_0(t)^{1/2} \delta_{q+1}^{1/2}, \tag{8.5}$$

$$\|z_{q+1} - z_q\|_{\bar{B},\alpha_0,2\alpha_0,t} \leq M_0(t)^{1/2} \delta_{q+1}^{1/2}. \tag{8.6}$$

Furthermore, if $v_q(0)$ and $\mathring{R}_q(0)$ are deterministic, so are $v_{q+1}(0)$ and $\mathring{R}_{q+1}(0)$.

Having Proposition 8.2 at hand, we may prove Theorem 1.5.

Proof of Theorem 1.5. The proof mostly uses exactly the same argument as in the proof of Theorem 1.1. Starting from $(z_0, v_0, \mathring{R}_0)$ given in Lemma 8.1, Proposition 8.2 gives a sequence $(z_q, v_q, \mathring{R}_q)$ satisfying (4.4) and (8.5). Hence, as in the proof of Theorem 1.1, we obtain a limiting solution $v = \lim_{q\to\infty} v_q$ which lies in $C([0,T_L]; H^\gamma)$ and (4.13) holds. Since v_q is $(\mathcal{F}_t)_{t\geq 0}$-adapted for every $q \geq 0$, so is the limit v. By (8.6) we obtain the convergence of z_q in $\bar{D}^{\alpha_0}_{B,2\alpha_0}([0,T_L])$ introduced in Definition D.2. We denote by z the limit and note that it is also $(\mathcal{F}_t)_{t\geq 0}$-adapted as a limit of adapted processes.

Hence we can take the limit in (8.1), (8.2) and conclude that $u = v + z$ satisfies the Navier–Stokes system (1.9) in the analytically weak sense before time T_L. In order to pass to the limit in the stochastic integral, we recall that by Section D.1 the rough integral in (8.1) on level q coincides with the Itô stochastic integral. By the $\mathbf{P}$-a.s. convergence of v_q and z_q, we may therefore pass to the limit $\lim_{q\to\infty}\int G(v_q + z_{q+1})\,dB = \int G(v+z)\,dB$ in $L^2(\Omega)$ in the Itô formulation. Moreover, the limit stochastic integral again coincides with the corresponding rough path integral.

By the same argument as in the proof of Theorem 1.1 we obtain

$$\|v(T)\|_{L^2} > (\|v(0)\|_{L^2} + L)e^{LT} \quad \text{on } \{T_L \geq T\}. \tag{8.7}$$

In other words, given $T > 0$ and the universal constant $c_R > 0$, we can choose $L = L(T, c_R) > 1$ large enough so that (4.8) and (4.15) hold and consequently (8.7) is satisfied. Moreover, in view of the definition of the stopping times (8.3), for a given $T > 0$ we may increase L if necessary so that $\mathbf{P}(T_L \geq T) > \kappa$.

To verify (1.10) and (1.11), we use Itô's formula for z_q and let $q \to \infty$ to have, for all $p \in \mathbb{N}$,

$$\mathbf{E}\Big[\sup_{t\in[0,T_L]} \|z(t)\|_{L^2}^{2p} + \int_0^{T_L} \|z(t)\|_{H^1}^2\,dt\Big] \leq C_{t,p},$$

which combined with (4.13) implies (1.10). We also have

$$\mathbf{E}[\|z(T)\|_{L^2}^2] \leq C_G T.$$

We then apply (8.7) on $\{T_L \geq T\}$ together with $\frac{1}{2}v^2 \leq z^2 + u^2$ to obtain

$$\begin{aligned}\mathbf{E}[\mathbf{1}_{T_L\geq T}\|u(T)\|_{L^2}^2] &\geq \tfrac{1}{2}\mathbf{E}[1_{T_L\geq T}\|v(T)\|_{L^2}^2] - \mathbf{E}[\|z(T)\|_{L^2}^2]\\ &> \tfrac{1}{2}\kappa(\|v(0)\|_{L^2} + L)^2 e^{2LT} - C_G T.\end{aligned}$$

Thus, since $u(0) = v(0)$ we may increase L if necessary, depending on K and C_G, in order to get the desired lower bound (1.11). The initial value $v(0)$ is deterministic by construction. Finally, we set $\mathfrak{t} := T_L$, which finishes the proof. ∎

8.1. The main iteration – proof of Proposition 8.2

8.1.1. Choice of parameters. Let us summarize how the parameters need to be chosen in order to fulfill all the compatibility conditions below. First, for a sufficiently small

$\alpha \in (0,1)$ to be chosen later, we define $\ell \in (0,1)$ as in (4.17). The last condition in (4.16) together with (4.7) leads to

$$45 \cdot (2\pi)^{3/2} < 5 \cdot (2\pi)^{3/2} a^{2\beta b} \le c_R L \le c_R \frac{a^4 \cdot (2\pi)^{3/2} - 1}{2}.$$

We remark that the reasoning from the beginning of Section 4.1 remains valid for this new condition: we may freely increase a provided we make β smaller at the same time. In addition, we will require $\alpha b > 32$ and $\alpha > 16\beta b^2$.

In order to verify the inductive estimates (4.4), especially dealing with the terms with rough paths, it will also be necessary to absorb various expressions including $M_0(t)^{M_0(t)^7}$ with $t = \ln\ln L$. It will be seen that the strongest such requirement is, for $q \in \mathbb{N}$,

$$\begin{aligned} &L^2 M_0(\ln\ln L)^{M_0(\ln\ln L)^7} \lambda_q^{20\alpha - 1/42} \le c_R \delta_{q+2}/10, \\ &L^2 M_0(\ln\ln L)^{M_0(\ln\ln L)^7} (\lambda_q^{-3\alpha/2} \lambda_{q-1}^{-2})^{1/6} \lambda_{q-1}^4 \le c_R \delta_{q+2}/10, \end{aligned} \tag{8.8}$$

needed in the estimates of R_{q+1}. In other words, for $\bar{M}_L := \lceil L^2 M_0(\ln\ln L)^{M_0(\ln\ln L)^7} \rceil$,

$$\bar{M}_L a^{20\alpha - \frac{1}{42} + 2b^2\beta} \ll 1, \quad \bar{M}_L a^{-\frac{b\alpha}{4} + \frac{11}{3} + 2b^3\beta} \ll 1,$$

and choosing $b = 28\bar{M}_L \vee (33 \cdot 28 \cdot 80)$ (this choice comes from the fact that with our choice of α below we want to guarantee that $\alpha b > 32$ as well as the fact that b is a multiple of 28 needed for the parameters in the intermittent jets, cf. Appendix B) and choosing a large such that $\bar{M}_L < b \le a^{\alpha/2}$ leads to

$$a^{20\alpha + \alpha/2 - \frac{1}{42} + 2b^2\beta} \ll 1, \quad a^{\alpha/2 - \frac{b\alpha}{4} + \frac{11}{3} + 2b^3\beta} \ll 1.$$

In view of $\alpha > 16\beta b^2$, this can be achieved by choosing a large enough and $\alpha = 80^{-1} \cdot 28^{-1}$. This choice also satisfies $\alpha b > 32$ required above, and the condition $\alpha > 16\beta b^2$ can be achieved by choosing β small. It is also compatible with all the other requirements needed below.

From now on, the parameters α and b remain fixed and the free parameters are a and β for which we already have a lower, respectively upper, bound. In what follows, we will possibly increase a and decrease β at the same time in order to preserve all the above conditions and to fulfil further conditions appearing below.

8.1.2. Verification of the inductive estimates for v_{q+1}. Note that $v_\ell, z_\ell, \mathring{R}_\ell$ are defined as in Section 4.1.2. We could check that v_ℓ satisfies the equation

$$\begin{aligned} \partial_t v_\ell - \Delta v_\ell + \operatorname{div}((v_\ell + z_\ell) \otimes (v_\ell + z_\ell)) + \nabla p_\ell &= \operatorname{div}(\mathring{R}_\ell + R_{\mathrm{com}}), \\ \operatorname{div} v_\ell &= 0 \end{aligned}$$

with

$$R_{\mathrm{com}} = (v_\ell + z_\ell) \mathring{\otimes} (v_\ell + z_\ell) - ((v_q + z_q) \mathring{\otimes} (v_q + z_q)) *_x \phi_\ell *_t \varphi_\ell.$$

Hence, the estimates (4.20)–(4.22) hold in this case.

For the intermittent jets we choose the following parameters:

$$\lambda = \lambda_{q+1}, \quad r_{\|} = \lambda_{q+1}^{-4/7}, \quad r_{\perp} = \lambda_{q+1}^{-27/28}, \quad \mu = \lambda_{q+1}^{9/7}, \tag{8.9}$$

where we note in particular the new value of $r_\perp$ as compared to (4.23). This is needed in order to obtain the sufficient (Young) time regularity of $v_{q+1} - v_q$ in (8.5), which in turn is employed for the rough path control of $z_{q+1} - z_q$ in Section 8.1.3 below.

Next, we define $\rho, a_{(\xi)}, w_{q+1}^{(p)}, w_{q+1}^{(c)}$ and $w_{q+1}^{(t)}$ as in Section 4.1.3 with the new parameters given in (8.9). By exactly the same argument as in Section 4.1.3, all the estimates and equalities in (4.24)–(4.36) hold. When applying Lemma 4.3 we choose $\zeta = \ell^{-8}$ with $(\ell^{-8})^5 < (\lambda_{q+1} r_\perp) = \lambda_{q+1}^{1/28}$, where we use $\alpha = \frac{1}{28\cdot 80}$. Hence, (4.37)–(4.39) also hold. The estimation in (4.40) follows the same way except for the last equality, i.e. we have

$$\|w_{q+1}^{(t)}\|_{C_t L^p} \lesssim \delta_{q+1} M_0(t) \ell^{-4} r_\perp^{2/p-1} r_\|^{1/p-2} (\mu^{-1} r_\perp^{-1} r_\|).$$

By the choice of the parameters, we have $M_0(\ln\ln L)^{1/2}\lambda_{q+1}^{4\alpha-1/28} < 1$, which implies that (4.41) holds in this case. As a result, for $t \in [0, T_L]$ we obtain

$$\begin{aligned}\|w_{q+1}\|_{C_t L^2} &\leq M_0(t)^{1/2}\delta_{q+1}^{1/2}(1/2 + C\lambda_{q+1}^{24\alpha-11/28} + C M_0(t)^{1/2}\delta_{q+1}^{1/2}\lambda_{q+1}^{8\alpha-1/28}) \\ &\leq \tfrac{3}{4} M_0(t)^{1/2}\delta_{q+1}^{1/2}.\end{aligned}$$

Hence the first inequality in (8.5) holds.

8.1.3. Estimate of $\|z_q - z_{q+1}\|_{\bar{B},\alpha_0,2\alpha_0,t}$. In the following, we intend to estimate $\|v_q - v_{q+1}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}}$ for some $\alpha_0 > 2/3$ and $\gamma > 0$. This is required for the rough path estimate of $\|z_q - z_{q+1}\|_{\bar{B},\alpha_0,2\alpha_0,t}$ on the time interval $[0,t]$; cf. Theorem D.10.

To this end, we first estimate $\|w_{q+1}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}}$. The idea is to gain some negative power of λ_{q+1} from $\|W_{(\xi)}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}}$. By paraproduct estimates similar to [28, Lemma A.7] and applying Lemma B.2 we deduce that for any $\delta > 0$ and $\gamma = 5 + \delta$,

$$\begin{aligned}\|W_{(\xi)}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}} &\lesssim \|\psi_{(\xi)}\|_{C_t^{\alpha_0} H^{\gamma+\delta}} \|\phi_{(\xi)}\|_{H^{-\gamma}}, \\ &\lesssim \mu^{\alpha_0}(\lambda_{q+1} r_\perp / r_\|)^{\alpha_0+\gamma+\delta}(\lambda_{q+1})^{-\gamma} r_\perp^{-\delta} \lesssim \lambda_{q+1}^{(53\alpha_0+44\delta-11\gamma)/28}.\end{aligned}$$

Using the paraproduct estimates again we have, for $\gamma = 5 + \delta$,

$$\begin{aligned}\|w_{q+1}^{(p)}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}} &\lesssim \|a_{(\xi)}\|_{C_{x,t}^{\alpha_0+\gamma+\delta}} \|W_{(\xi)}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}} \\ &\lesssim \delta_{q+1}^{1/2} M_0(t)^{1/2} \ell^{-2-5\lceil \alpha_0+\gamma+\delta\rceil} \lambda_{q+1}^{(53\alpha_0+44\delta-11\gamma)/28},\end{aligned}$$

where $\lceil x \rceil$ denote the smallest integer greater than x. In view of (B.7), we find (applying interpolation similarly to Lemma B.2)

$$\begin{aligned}\|w_{q+1}^{(c)}\|_{C_t^{\alpha_0} L^1} &\lesssim \|a_{(\xi)}\|_{C_{x,t}^{\alpha_0}} \|W_{(\xi)}^{(c)}\|_{C_t^{\alpha_0} L^1} + \|a_{(\xi)}\|_{C_{x,t}^{2+\alpha_0}} \|V_{(\xi)}\|_{C_t^{\alpha_0} W^{1,1}} \\ &\lesssim \delta_{q+1}^{1/2} M_0(t)^{1/2} \ell^{-17} r_\perp r_\|^{1/2} (r_\perp r_\|^{-1} + \lambda_{q+1}^{-1}) \Big(\frac{r_\perp \lambda_{q+1}\mu}{r_\|}\Big)^{\alpha_0} \\ &\lesssim \delta_{q+1}^{1/2} M_0(t)^{1/2} \ell^{-17} \lambda_{q+1}^{53\alpha_0/28 - 23/14}.\end{aligned}$$

We choose $p = \frac{35}{35-14\alpha} > 1$ so that in particular $r_\perp^{2/p-2} r_\parallel^{1/p-1} \le \lambda_{q+1}^\alpha$. Then for $\alpha_0 = 2/3 + \kappa$ with $\kappa > 0$ small we obtain

$$\begin{aligned}\|w_{q+1}^{(t)}\|_{C_t^{\alpha_0} L^p} &\lesssim \mu^{-1} \|a_{(\xi)}\|_{C_{x,t}^{\alpha_0}} \|a_{(\xi)}\|_{C_{x,t}} \|\phi_{(\xi)}\|_{L^{2p}}^2 \|\psi_{(\xi)}\|_{C_t^{\alpha_0} L^{2p}} \|\psi_{(\xi)}\|_{C_t L^{2p}} \\ &\lesssim \delta_{q+1} M_0(t) \ell^{-9} r_\perp^{2/p-2} r_\parallel^{1/p-1} \mu^{-1} \left(\frac{r_\perp \lambda_{q+1} \mu}{r_\parallel} \right)^{\alpha_0} \\ &\lesssim \delta_{q+1} M_0(t) \ell^{-9} \lambda_{q+1}^{53\alpha_0/28 + \alpha - 9/7}\end{aligned}$$

and then using $\alpha_0 = 2/3 + \kappa$ we get $53\alpha_0/28 - 9/7 = -1/42 + 53\kappa/28$, so finally

$$\|w_{q+1}^{(t)}\|_{C_t^{\alpha_0} L^p} \lesssim \delta_{q+1} M_0(t) \ell^{-9} \lambda_{q+1}^{-1/42+53\kappa/28+\alpha}.$$

We also have

$$\|v_q - v_\ell\|_{C_t^{\alpha_0} L^2} \lesssim \|v_q - v_\ell\|_{C_t L^2}^{1-\alpha_0} \|v_q - v_\ell\|_{C_t^1 L^2}^{\alpha_0} \lesssim \ell^{1-\alpha_0} \|v_q\|_{C_{t,x}^1} \le \ell^{1-\alpha_0} \lambda_q^4 M_0(t)^{1/2}.$$

Combining the above estimates we obtain, for $\alpha_0 = 2/3 + \kappa$, $\gamma = 5 + \delta$,

$$\begin{aligned}&\|v_q - v_{q+1}\|_{C_t^{\alpha_0} B_{1,1}^{-\gamma}} \\ &\lesssim M_0(t)^{1/2} (\ell^{1-\alpha_0} \lambda_q^4 + \ell^{-17} \lambda_{q+1}^{53\alpha_0/28 - 23/14} + M_0(t)^{1/2} \ell^{-9} \lambda_{q+1}^{-1/42+53\kappa/28+\alpha} \\ &\qquad + \ell^{-2-5\lceil \alpha_0+\gamma+\delta \rceil} \lambda_{q+1}^{(53\alpha_0+44\delta-11\gamma)/28}) \\ &\lesssim M_0(t)^{1/2} (\ell^{1-\alpha_0} \lambda_q^4 + M_0(t)^{1/2} \ell^{-9} \lambda_{q+1}^{-1/42+53\kappa/28+\alpha}) \le M_0(t)^{1/2} \delta_{q+1}^{1/2}.\end{aligned}$$

In the last inequality we have used $\ell^{1-\alpha_0} \lambda_q^4 \le (\lambda_{q+1}^{-3\alpha/2} \lambda_q^{-2})^{1/6} \lambda_q^4 \le \lambda_{q+1}^{-\beta b}$, where the first inequality follows from the definition of ℓ together with the fact that $1 - \alpha_0 > 1/6$, and the second inequality is implied by the second inequality in (8.8) with q replaced by $q + 1$.

Then the second inequality in (8.5) holds.

By the choice of v_0 in Lemma 8.1 we obtain

$$\|v_q\|_{C_t^{\alpha_0} B_{1,1}^{-5-\delta}} \lesssim \|v_0\|_{C_t^{\alpha_0} B_{1,1}^{-5-\delta}} + \sum_{k \ge 0} \|v_k - v_{k+1}\|_{C_t^{\alpha_0} B_{1,1}^{-5-\delta}} \le (2L+2) M_0(t)^{1/2}.$$

Moreover, by Theorem D.10 we have

$$\|z_q\|_{C_t L^2} \le \|z_q\|_{\bar{B},\alpha_0,2\alpha_0,t} \le (2L+2) L M_0(t)^{1/2}, \tag{8.10}$$

We intend to combine the above two estimates with the last inequality in Theorem D.10. To this end, we observe that $\bar{N} = M_0(\ln \ln L)$ in Theorem D.10 and the right hand side of the estimate can be controlled by $\bar{N}^{(6.5)T\bar{N}^{6.5}}$. Then we could choose L large enough to have it controlled by $\bar{N}^{\bar{N}^7}$. As a consequence, we choose κ small satisfying $53\kappa/28 \le \alpha/2$ and (8.8) to see for $q \in \mathbb{N}$ that

$$\begin{aligned}&\|z_q - z_{q+1}\|_{\bar{B},\alpha_0,2\alpha_0,t} \\ &\lesssim M_0(t)^{1/2} (\ell_0^{1-\alpha_0} \lambda_{q-1}^4 + \ell_0^{-9} \lambda_q^{-1/42+53\kappa/28+\alpha} M_0(t)^{1/2}) M_0(\ln \ln L)^{M_0(\ln \ln L)^7} \\ &\le M_0(t)^{1/2} \delta_{q+1}^{1/2} \end{aligned} \tag{8.11}$$

with $\ell_0 = \lambda_q^{-3\alpha/2}\lambda_{q-1}^{-2}$. For $q = 0$ nothing needs to be proven since $z_1 = z_0$. Hence, (8.6) holds.

8.1.4. Conclusion

Proof of Proposition 8.2. Recall that we changed the parameter $r_\perp$ to a smaller value. Hence, (4.43)–(4.45) hold, which implies that for $t \in [0, T_L]$,

$$\begin{aligned}\|v_{q+1}\|_{C^1_{t,x}} &\le \|v_\ell\|_{C^1_{t,x}} + \|w_{q+1}\|_{C^1_{t,x}}\\ &\le M_0(t)^{1/2}\big(\lambda_{q+1}^{\alpha} + C\lambda_{q+1}^{14\alpha+3+1/4} + C\lambda_{q+1}^{34\alpha+20/7} + CM_0(t)^{1/2}\lambda_{q+1}^{19\alpha+3+3/28}\big)\\ &\le M_0(t)^{1/2}\lambda_{q+1}^4,\end{aligned}$$

where we use $CM_0(\ln\ln L)^{1/2} \le \frac{1}{2}\lambda_{q+1}^{25/28-19\alpha}$. Thus, the second estimate in (4.4) holds true on level $q+1$. Moreover, the estimates (4.46)–(4.47) hold.

In the following we control $\mathring{R}_{q+1}$. We choose $p = \frac{35}{35-14\alpha} > 1$ so that $r_\perp^{2/p-2}r_\parallel^{1/p-1} \le \lambda_{q+1}^{\alpha}$. First, we recall that $r_\perp$ becomes smaller and p is close to 1. Hence $r_\perp^{2/p-1}$ becomes smaller and we use $r_\perp^{2/p-2}r_\parallel^{1/p-1} \le \lambda_{q+1}^{\alpha}$. As a result, the bounds for $R_{\rm osc}^{(t)}$ and the terms not involving z in $R_{\rm lin}$ do not change. For $R_{\rm cor}$ we have

$$\begin{aligned}&\|R_{\rm cor}\|_{C_tL^p}\\ &\quad\lesssim M_0(t)\big(\ell^{-12}r_\perp^{1/p}r_\parallel^{1/(2p)-3/2} + \ell^{-4}M_0(t)^{1/2}r_\perp^{1/p-1}r_\parallel^{1/(2p)-1/2}\mu^{-1}r_\perp^{-1}r_\parallel^{-1/2}\big)\\ &\qquad\times\ell^{-2}r_\perp^{1/p-1}r_\parallel^{1/(2p)-1/2}\\ &\quad\lesssim M_0(t)\big(\ell^{-14}r_\perp^{2/p-1}r_\parallel^{1/p-2} + \ell^{-6}M_0(t)^{1/2}r_\perp^{2/p-3}r_\parallel^{1/p-3/2}\mu^{-1}\big)\\ &\quad\lesssim M_0(t)\big(\lambda_{q+1}^{29\alpha-11/28} + M_0(t)^{1/2}\lambda_{q+1}^{13\alpha-1/28}\big) \le M_0(t)c_R\delta_{q+2}/5.\end{aligned}$$

As before, we also have

$$\begin{aligned}\|R_{\rm osc}^{(x)}\|_{C_tL^p} &\lesssim M_0(t)\ell^{-9}r_\perp^{2/p-2}r_\parallel^{1/p-1}(r_\perp^{-1}\lambda_{q+1}^{-1}) \lesssim M_0(t)\ell^{-9}\lambda_{q+1}^{\alpha}(r_\perp^{-1}\lambda_{q+1}^{-1})\\ &\lesssim M_0(t)\lambda_{q+1}^{19\alpha-1/28} \le M_0(t)c_R\delta_{q+2}/10.\end{aligned}$$

In the following it suffices to consider the terms containing z in $R_{\rm lin}$, $R_{\rm com}$ and $R_{\rm com1}$. Recall that we have

$$\begin{aligned}z_\ell &= (z_q *_x \phi_\ell) *_t \varphi_\ell,\\ R_{\rm com} &= (v_\ell + z_\ell)\mathring{\otimes}(v_\ell + z_\ell) - ((v_q + z_q)\mathring{\otimes}(v_q + z_q)) *_x \phi_\ell *_t \varphi_\ell,\\ R_{\rm com1} &= v_{q+1}\mathring{\otimes}z_{q+1} - v_{q+1}\mathring{\otimes}z_\ell + z_{q+1}\mathring{\otimes}v_{q+1} - z_\ell\mathring{\otimes}v_{q+1} + z_{q+1}\mathring{\otimes}z_{q+1}\\ &\quad - z_\ell\mathring{\otimes}z_\ell.\end{aligned}$$

In order to estimate the remaining term in $R_{\rm lin}$,

$$R_{\rm lin}^1 := (v_\ell + z_\ell)\mathring{\otimes}w_{q+1} + w_{q+1}\mathring{\otimes}(v_\ell + z_\ell),$$

we use (8.4) as well as (4.16) to obtain, for $q \in \mathbb{N}$,

$$\begin{aligned}\|R^1_{\mathrm{lin}}\|_{C_tL^p} &\lesssim \|(v_\ell + z_\ell) \mathring{\otimes} w_{q+1} + w_{q+1} \mathring{\otimes} (v_\ell + z_\ell)\|_{C_tL^p}\\ &\lesssim M_0(t)^{1/2}(\lambda_q^4 + \lambda_{q-1}^4 L^{1/4})\|w_{q+1}\|_{C_tL^p}\\ &\lesssim M_0(t)\ell^{-2} r_\perp^{2/p-1} r_\|^{1/p-1/2}(\lambda_q^4 + \lambda_{q-1}^4 L^{1/4}) \lesssim M_0(t)\lambda_{q+1}^{6\alpha-5/4}\\ &\leq M_0(t)c_R\delta_{q+2}/10,\end{aligned}$$

where we have used $\lambda_{q-1}^4 L^{1/4} \leq \lambda_q^4$. For $q = 0$ we have

$$\begin{aligned}\|R^1_{\mathrm{lin}}\|_{C_tL^p} &\lesssim M_0(t)\ell^{-2} r_\perp^{2/p-1} r_\|^{1/p-1/2}(\lambda_0^4 + \lambda_0^4 L^{1/4}) \lesssim M_0(t)\lambda_0^4 L^{1/4}\lambda_1^{5\alpha-5/4}\\ &\leq M_0(t)c_R\delta_{q+2}/10,\end{aligned}$$

where the last inequality follows from $\alpha b > 32$ and $\alpha > 16\beta b^2$.

In view of the standard mollification estimates we deduce, for $t \in [0, T_L]$ and $q = 0$,

$$\|R_{\mathrm{com}}\|_{C_tL^1} \lesssim \ell^{1/2-2\delta}\lambda_0^4 L^{1/4} L^2 M_0(t) \leq M_0(t)c_R\delta_2/5,$$

where we use $\alpha b > 32$ and $\alpha > 16\beta b^2$ in the last inequality. For $q \in \mathbb{N}$ we have

$$\begin{aligned}\|R_{\mathrm{com}}\|_{C_tL^1} &\lesssim \ell(\|v_q\|_{C^1_{t,x}} + \|z_q\|_{C_tC^1})(\|v_q\|_{C_tL^2} + \|z_q\|_{C_tL^2})\\ &\quad + \ell^{1/2-2\delta}(\|z_q\|_{C_t^{1/2-2\delta}L^\infty} + \|v_q\|_{C^1_{t,x}})(\|v_q\|_{C_tL^2} + \|z_q\|_{C_tL^2})\\ &\lesssim \ell\lambda_q^4 L^2 M_0(t) + \ell^{1/2-2\delta}\lambda_q^4 L^2 M_0(t) \leq M_0(t)c_R\delta_{q+2}/5,\end{aligned}$$

where we have used (8.10) and $\lambda_{q-1}^4 L^{1/2} \leq \lambda_q^4$ and $\delta < 1/12$ and we require $\ell^{1/2-2\delta}\lambda_q^4 L^2 < c_R\delta_{q+2}/10$, i.e.

$$L^2\lambda_{q+1}^{-\alpha/2}\lambda_q^{-2/3}\lambda_q^4 < \lambda_{q+1}^{-2\beta b},$$

with the choice of ℓ in (4.17), and the exponents were obtained with the choice $\delta = 1/12$. Since we postulated that $\alpha b > 32$, this can indeed be achieved by possibly increasing a and consequently decreasing β.

Finally, we use (8.4) and (8.8) to obtain, for $t \in [0, T_L]$ and $q \in \mathbb{N}$,

$$\begin{aligned}&\|R_{\mathrm{com}1}\|_{C_tL^1}\\ &\quad\lesssim \|v_{q+1}\|_{C_tL^2}\|z_{q+1} - z_\ell\|_{C_tL^2} + (\|z_{q+1}\|_{C_tL^2} + \|z_\ell\|_{C_tL^2})\|z_{q+1} - z_\ell\|_{C_tL^2}\\ &\quad\lesssim M_0(t)(\ell_0^{1-\alpha_0}\lambda_{q-1}^4 + M_0(t)^{1/2}\ell_0^{-9}\lambda_q^{-1/42+53\kappa/28+\alpha} + \ell^{1/2-2\delta}\lambda_{q-1}^4)\\ &\qquad\times M_0(\ln\ln L)^{M_0(\ln\ln L)^7}L^2\\ &\quad\leq M_0(t)c_R\delta_{q+2}/5,\end{aligned}$$

with $\ell_0 = \lambda_q^{-3\alpha/2}\lambda_{q-1}^{-2}$ and where we have used (8.11), yielding for $q \in \mathbb{N}$,

$$\begin{aligned}\|z_{q+1} - z_\ell\|_{C_tL^2} &\leq \|z_{q+1} - z_q\|_{C_tL^2} + \|z_q - z_\ell\|_{C_tL^2}\\ &\lesssim M_0(t)^{1/2}(\ell_0^{1-\alpha_0}\lambda_{q-1}^4 + \ell^{-9}\lambda_q^{-1/42+53\kappa/28+\alpha}M_0(t)^{1/2}\\ &\qquad + \ell^{1/2-2\delta}\lambda_{q-1}^4)M_0(\ln\ln L)^{M_0(\ln\ln L)^7}.\end{aligned}$$

For $q = 0$ we use (8.4) to get

$$\begin{aligned}\|z_1 - z_\ell\|_{C_t L^2} &\leq \|z_1 - z_0\|_{C_t L^2} + \|z_0 - z_\ell\|_{C_t L^2} \leq \ell^{1/2-2\delta} L^{1/2}(1 + \|v_0\|_{C^1_{t,x}}) \\ &\leq \ell^{1/2-2\delta} L^{1/2} \lambda_0^4 M_0(t)^{1/2},\end{aligned}$$

which combined with (8.10) implies, for $t \in [0, T_L]$ and $q = 0$,

$$\|R_{\mathrm{com}1}\|_{C_t L^1} \lesssim M_0(t)\ell^{1/2-2\delta}\lambda_0^4 L^{5/2} \lesssim M_0(t) c_R \delta_2/5.$$

The proof is complete. ∎

Appendix A. Proof of Theorems 3.1, 5.1 and 7.2

Let us begin with the following tightness result.

Lemma A.1. *Let $\{(s_n, x_n)\}_{n\in\mathbb{N}} \subset [0,\infty) \times L^2_\sigma$ be such that $(s_n, x_n) \to (s, x_0)$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a family of probability measures on Ω_0 satisfying, for all $n \in \mathbb{N}$,*

$$P_n(x(t) = x_n, 0 \leq t \leq s_n) = 1, \tag{A.1}$$

and for some $\gamma, \kappa > 0$ and any $T > 0$,

$$\sup_{n\in\mathbb{N}} E^{P_n}\left(\sup_{t\in[0,T]} \|x(t)\|_{L^2} + \sup_{r\neq t\in[0,T]} \frac{\|x(t) - x(r)\|_{H^{-3}}}{|t-r|^\kappa} + \int_{s_n}^T \|x(r)\|^2_{H^\gamma}\, dr\right) < \infty. \tag{A.2}$$

Then $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{S} := C([0,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma)$.

Proof. In view of the uniform bound (A.2), the canonical process under the measure P_n is bounded in $L^\infty_{\mathrm{loc}}([0,\infty); L^2) \cap C^\kappa_{\mathrm{loc}}([0,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([s_n,\infty); H^\gamma)$ and the bounds are uniform in n. We recall that a set $K \subset \mathbb{S}$ is compact provided

$$K_T := \{f|_{[0,T]} : f \in K\} \subset C([0,T]; H^{-3}) \cap L^2(0,T; L^2_\sigma)$$

is compact for every $T > 0$. In addition, for every $T > 0$, the embedding

$$L^\infty(0,T; L^2) \cap C^\kappa([0,T]; H^{-3}) \cap L^2([0,T]; H^\gamma) \subset C([0,T]; H^{-3}) \cap L^2(0,T; L^2_\sigma)$$

is compact (see e.g. [3, Section 1.8.2]). This implies that also the embedding of the local-in-time spaces

$$L^\infty_{\mathrm{loc}}([0,\infty); L^2) \cap C^\kappa_{\mathrm{loc}}([0,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); H^\gamma) \subset \mathbb{S}$$

is compact. This result, however, cannot be applied directly in order to prove the claim of the lemma due to the fact that the uniform H^γ regularity in (A.2) only holds on the time intervals $[s_n, T]$. The idea is instead to use (A.1) which says that under each measure P_n the canonical process is constant on $[0, s_n]$ and its value equals to x_n. Together with the fact that $(s_n, x_n) \to (s, x_0)$ in $[0,\infty) \times L^2_\sigma$, the desired compactness then follows.

To be more precise, we fix $\epsilon > 0$ and any $k \in \mathbb{N}$ with $k \geq k_0 := \sup_{n\in\mathbb{N}} s_n$, we may choose $R_k > 0$ sufficiently large such that

$$P_n\left(x \in \Omega_0 : \sup_{t\in[0,k]} \|x(t)\|_{L^2} + \sup_{r\neq t\in[0,k]} \frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^\kappa} + \int_{s_n}^k \|x(r)\|_{H^\gamma}^2 \, dr > R_k\right) \leq \epsilon/2^k.$$

Now, we set $\Omega_n := \{x \in \Omega_0 : x(t) = x_n,\ 0 \leq t \leq s_n\}$ and define

$$K := \bigcup_{n\in\mathbb{N}} \bigcap_{\substack{k\in\mathbb{N}\\ k\geq k_0}} \left\{x \in \Omega_n : \sup_{t\in[0,k]} \|x(t)\|_{L^2} + \sup_{r\neq t\in[0,k]} \frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^\kappa} + \int_{s_n}^k \|x(r)\|_{H^\gamma}^2 \, dr \leq R_k\right\}. \quad \text{(A.3)}$$

By Chebyshev's inequality together with (A.2), it follows that

$$\sup_{n\in\mathbb{N}} P_n(\bar{K}^c) \leq \sup_{n\in\mathbb{N}} P_n(K^c) \leq \epsilon,$$

so it only remains to show that $\bar{K}$ is compact in $\mathbb{S}$. As mentioned above, it is sufficient to prove that for every $k \geq k_0$, the set of restrictions of functions in K to $[0,k]$ is relatively compact in $\mathbb{S}_k := C([0,k]; H^{-3}) \cap L^2(0,k; L^2_\sigma)$.

To this end, let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence in K. If there exists $N \in \mathbb{N}$ such that $x_m \in \Omega_N$ for infinitely many m, the result can be obtained by a standard argument based on the compact embedding discussed above. If this is not true, we may assume without loss of generality that $x_m \in \Omega_m$. The compactness in $C([0,k]; H^{-3})$ is a direct consequence of the bound

$$\sup_{t\in[0,k]} \|x_m(t)\|_{L^2} + \sup_{r\neq t\in[0,k]} \frac{\|x_m(t)-x_m(r)\|_{H^{-3}}}{|t-r|^\kappa} \leq R_k$$

and the compact embedding

$$L^\infty(0,k; L^2) \cap C^\kappa([0,k]; H^{-3}) \subset C([0,k]; H^{-3}).$$

Consequently, we can find a subsequence x_{m_l} such that

$$\lim_{l,n\to\infty} \sup_{t\in[0,k]} \|x_{m_l}(t) - x_{m_n}(t)\|_{H^{-3}} = 0. \quad \text{(A.4)}$$

With this in hand, we deduce

$$\begin{aligned}\int_0^k &\|x_{m_l}(t) - x_{m_n}(t)\|_{L^2}^2 \, dt \\ &\leq \int_0^{s_{m_l}\wedge s_{m_n}} \|x_{m_l}(t) - x_{m_n}(t)\|_{L^2}^2 \, dt \\ &\quad + \int_{s_{m_l}\wedge s_{m_n}}^{s_{m_l}\vee s_{m_n}} \|x_{m_l}(t) - x_{m_n}(t)\|_{L^2}^2 \, dt + \int_{s_{m_l}\vee s_{m_n}}^k \|x_{m_l}(t) - x_{m_n}(t)\|_{L^2}^2 \, dt\end{aligned}$$

$$
\begin{aligned}
&\leq k\|x_{m_l}(0)-x_{m_n}(0)\|_{L^2}^2+4R_k^2(s_{m_l}\vee s_{m_n}-s_{m_l}\wedge s_{n_m})\\
&\quad+\varepsilon\int_{s_{m_l}\vee s_{m_n}}^k\|x_{m_l}(t)-x_{m_n}(t)\|_{H^\gamma}^2\,dt+C_\varepsilon k\sup_{t\in[0,k]}\|x_{m_l}(t)-x_{m_n}(t)\|_{H^{-3}}^2\\
&\leq k\|x_{m_l}(0)-x_{m_n}(0)\|_{L^2}^2+4R_k^2(s_{m_l}\vee s_{m_n}-s_{m_l}\wedge s_{m_n})\\
&\quad+4\varepsilon R_k+C_\varepsilon k\sup_{t\in[0,k]}\|x_{m_l}(t)-x_{m_n}(t)\|_{H^{-3}}^2\to 0
\end{aligned}
$$

as $m_l, m_n \to \infty$, where we have used interpolation and Young's inequality in the second step and (A.4) in the last step. Now the proof is complete. ■

Proof of Theorem 3.1. The existence of a martingale solution can be easily deduced by Galerkin approximation and the same arguments as in [22,26]. The stability of martingale solutions with respect to the initial time and initial condition will be proved based on Lemma A.1.

First, we prove that $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{S} := C([0,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma)$. To this end, we denote $F(x) := -\mathbb{P}\operatorname{div}(x\otimes x)+\Delta x$. Since for every $n\in\mathbb{N}$, the measure P_n is a martingale solution to (1.1) starting from the initial condition x_n at time s_n in the sense of Definition 3.1, we know that for $t\in[s_n,\infty)$,

$$
x(t)=x_n+\int_{s_n}^t F(x(r))\,dr+M^x_{t,s_n}\qquad P_n\text{-a.s.},
$$

where $t\mapsto M^{x,i}_{t,s_n}=\langle M^x_{t,s_n},e_i\rangle$, $x\in\Omega_0$, is a continuous square integrable martingale with respect to $(\mathcal{B}_t)_{t\geq s_n}$ with quadratic variation process $t\mapsto\int_{s_n}^t\|G(x(r))^*e_i\|_U^2\,dr$. Moreover, according to (M3), for every $p>1$,

$$
\begin{aligned}
E^{P_n}\Big[\sup_{r\neq t\in[s_n,T]}\frac{\|\int_r^t F(x(l))\,dl\|_{H^{-3}}^p}{|t-r|^{p-1}}\Big]&\leq E^{P_n}\Big[\int_{s_n}^t\|F(x(r))\|_{H^{-3}}^p\,dr\Big]\\
&\lesssim\|x_n\|_{L^2}^{2p}+1,
\end{aligned}
$$

where the implicit constant is universal and therefore independent of n since all P_n share the same $C_{t,q}$. By the condition on G we have, for every $p>1$,

$$
\begin{aligned}
E^{P_n}\|M_{t,s_n}-M_{r,s_n}\|_{L^2}^{2p}&\leq C_pE^{P_n}\Big(\int_r^t\|G(x(l))\|^2_{L_2(U;L^2_\sigma)}\,dl\Big)^p\\
&\leq C_p|t-r|^{p-1}E^{P_n}\int_r^t\|G(x(l))\|^{2p}_{L_2(U;L^2_\sigma)}\,dl\\
&\leq C_p|t-r|^{p-1}E^{P_n}\int_r^t(\|x(l)\|_{L^2}^{2p}+1)\,dl\leq C_p|t-r|^{p-1}(\|x_n\|_{L^2}^{2p}+1).
\end{aligned}
$$

By Kolmogorov's criterion, for any $\alpha\in(0,\frac{p-1}{2p})$ we get

$$
E^{P_n}\Big[\sup_{r\neq t\in[0,T]}\frac{\|M_{t,s_n}-M_{r,s_n}\|_{L^2}}{|t-r|^{p\alpha}}\Big]\leq C_p(\|x_n\|_{L^2}^{2p}+1).
$$

Combining the above estimates, we conclude that for all $\kappa \in (0, 1/2)$,

$$\sup_{n\in\mathbb{N}} E^{P_n}\left[\sup_{r\neq t\in[0,T]} \frac{\|x(t)-x(r)\|_{H^{-3}}}{|t-r|^{\kappa}}\right] < \infty. \tag{A.5}$$

Combining (A.5), (M3) and Lemma A.1 it follows that $\{P_n\}_{n\in\mathbb{N}}$ is tight in $\mathbb{S}$.

Without loss of generality, we may assume that P_n converges weakly to some probability measure $P \in \mathscr{P}(\Omega_0)$. It remains to prove that $P \in \mathscr{C}(s, x_0, C_{t,q})$. By Skorokhod's representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ and $\mathbb{S}$-valued random variables $\tilde{x}_n$ and $\tilde{x}$ such that

(i) $\tilde{x}_n$ has law P_n for each $n \in \mathbb{N}$,

(ii) $\tilde{x}_n \to \tilde{x}$ in $\mathbb{S}$ $\tilde{P}$-a.s., and $\tilde{x}$ has law P.

Since the initial conditions x_n as well as the initial times s_n are deterministic, we find by (i), (ii), and (M1) applied to P_n that

$$\begin{aligned} P(x(t) = x_0,\, 0 \leq t \leq s) &= \tilde{P}(\tilde{x}(t) = x_0,\, 0 \leq t \leq s) \\ &= \lim_{n\to\infty} \tilde{P}(\tilde{x}_n(t) = x_n,\, 0 \leq t \leq s_n) \\ &= \lim_{n\to\infty} P_n(x(t) = x_n,\, 0 \leq t \leq s_n) = 1. \end{aligned}$$

As the next step, we verify (M2) for P. We know that under $\tilde{P}$, according to the convergence in (ii), for every $e_i \in C^{\infty}(\mathbb{T}^3)$ we have

$$\langle \tilde{x}_n(t), e_i\rangle \to \langle \tilde{x}(t), e_i\rangle, \quad \int_{s_n}^t \langle F(\tilde{x}_n(r)), e_i\rangle\, dr \to \int_s^t \langle F(\tilde{x}(r)), e_i\rangle\, dr \quad \tilde{P}\text{-a.s.}$$

This implies that for every $t \in [s, \infty)$ and every $p > 1$,

$$\begin{aligned} &\sup_{n\in\mathbb{N}} E^{\tilde{P}}[|M^{\tilde{x}_n,i}_{t,s_n}|^{2p}] \leq C \sup_{n\in\mathbb{N}} E^{P_n}\left[\left(\int_{s_n}^t \|G(x(r))\|^2_{L_2(U;L^2_\sigma)}\, dr\right)^p\right] < \infty, \\ &\lim_{n\to\infty} E^{\tilde{P}}[|M^{\tilde{x}_n,i}_{t,s_n} - M^{\tilde{x},i}_{t,s}|] = 0. \end{aligned} \tag{A.6}$$

Let $t > r \geq s$ and g be any bounded and real-valued $\mathscr{B}_r$-measurable continuous function on $\mathbb{S}$. Using (A.6) we know that

$$\begin{aligned} E^P[(M^{x,i}_{t,s} - M^{x,i}_{r,s})g(x)] &= E^{\tilde{P}}[(M^{\tilde{x},i}_{t,s} - M^{\tilde{x},i}_{r,s})g(\tilde{x})] \\ &= \lim_{n\to\infty} E^{\tilde{P}}[(M^{\tilde{x}_n,i}_{t,s_n} - M^{\tilde{x}_n,i}_{r,s_n})g(\tilde{x}_n)] \\ &= \lim_{n\to\infty} E^{P_n}[(M^{x,i}_{t,s_n} - M^{x,i}_{r,s_n})g(x)] = 0. \end{aligned}$$

Consequently,

$$E^P[M^{x,i}_{t,s}|\mathscr{B}_r] = M^{x,i}_{r,s},$$

hence $t \mapsto M^i_{t,s}$ is a $(\mathscr{B}_t)_{t\geq s}$-martingale under P. Similarly,

$$\lim_{n\to\infty} E^{\tilde{P}}[|M^{\tilde{x}_n,i}_{t,s_n} - M^{\tilde{x},i}_{t,s}|^2] = 0,$$

which gives

$$E^P\left[(M^{x,i}_{t,s})^2 - \int_s^t \|G(x(l))^* e_i\|_U^2\, dl \,\middle|\, \mathcal{B}_r\right] = (M^{x,i}_{r,s})^2 - \int_r^t \|G(x(l))^* e_i\|_U^2\, dl,$$

and (M2) follows.

Finally, we verify (M3). Define

$$S(t,s,x) := \sup_{r\in[0,t]} \|x(r)\|_{L^2}^{2q} + \int_s^t \|x(r)\|_{H^\gamma}^2\, dr,$$

It is easy to see that $x \mapsto S(t,s,x)$ is lower semicontinuous on $\mathbb{S}$. Hence, by Fatou's lemma,

$$\begin{aligned} E^P[S(t,s,x)] = E^{\tilde{P}}[S(t,s,\tilde{x})] &\le \liminf_{n\to\infty} E^{\tilde{P}}[S(t,s_n,\tilde{x}_n)] \\ &\le C_{t,q} \liminf_{n\to\infty} (\|x_n\|_{L^2}^{2q} + 1) \\ &< \infty. \end{aligned}$$

The proof is complete. ■

Proof of Theorem 5.1. The existence of a probabilistically weak solution can be easily deduced from Theorem 3.1 and the martingale representation theorem (see [14]). The stability of weak solutions with respect to the initial time and initial condition will be proved as in Theorem 3.1. First, we prove that the set $\{P_n\}_{n\in\mathbb{N}}$ is tight in

$$\bar{\mathbb{S}} := C([0,\infty); H^{-3} \times U_1) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma \times U_1).$$

To this end, we denote $F(x) := -\mathbb{P}\,\mathrm{div}(x \otimes x) + \Delta x$ and recall that for every $n \in \mathbb{N}$, the measure P_n is a probabilistically weak solution to (1.1) starting from the initial condition x_n at time s_n in the sense of Definition 5.1. Thus, for $t \in [s_n,\infty)$,

$$x(t) = x_n + \int_{s_n}^t F(x(r))\, dr + \int_{s_n}^t G(x(r))\, dy(r) \quad P_n\text{-a.s.}$$

where under P_n the process y is a cylindrical Wiener process on U starting from y_n at time s_n. In other words, under P_n the process $t \mapsto y(t+s_n) - y_n$ is a cylindrical Wiener process on U starting at time 0 from the initial value 0. Since the law of the Wiener process is unique and tight, for a given $\epsilon > 0$ there exists a compact set $K_1 \subset C([0,\infty); U_1) \cap L^2_{\mathrm{loc}}([0,\infty); U_1)$ such that

$$\sup_{n\in\mathbb{N}} P_n\big(y(\cdot + s_n) - y_n \in K_1^c\big) \le \epsilon.$$

Let us now define

$$\begin{aligned} K_2 := \bigcup_{n\in\mathbb{N}} \big\{ & y \in C([0,\infty); U_1) : \\ & y(t+s_n) - y_n \in K_1 \text{ for } t \in [0,\infty),\ y(t) = y_n \text{ for } t \in [0,s_n] \big\}. \end{aligned}$$

Then

$$\sup_{n\in\mathbb{N}} P_n(\bar{K}_2^c) \le \sup_{n\in\mathbb{N}} P_n(y(\cdot + s_n) - y_n \in K_1^c) \le \epsilon \tag{A.7}$$

and we claim that K_2 is relatively compact in $C([0,\infty); U_1) \subset L^2_{\mathrm{loc}}([0,\infty); U_1)$. Indeed, let $\{y^m\}_{m\in\mathbb{N}}$ be a sequence in K_2. Then for every $m \in \mathbb{N}$ there exist $n_m \in \mathbb{N}$ and $y^{m,n_m} \in K_1$ such that

$$y^m(t + s_{n_m}) - y_{n_m} = y^{m,n_m}(t) \quad \text{for } t \in [0,\infty), \quad y^m(t) = y_{n_m} \quad \text{for } t \in [0, s_{n_m}].$$

If there exists $N \in \mathbb{N}$ such that $n_m = N$ for infinitely many $m \in \mathbb{N}$ then the relative compactness of $\{y^m\}_{m\in\mathbb{N}}$ follows directly from the fact that the corresponding sequence $\{y^{m,n_m}\}_{m\in\mathbb{N}}$ is relatively compact due to compactness of K_1. If such an N does not exist, then by passing to a subsequence and relabeling we can assume without loss of generality that $n_m = m$. In addition, for $t \in [s_m, \infty)$,

$$y^m(t) = y^{m,m}(t - s_m) + y_m.$$

Hence using the relative compactness of

$$\{y^{m,m}\}_{m\in\mathbb{N}} \subset K_1 \quad \text{and} \quad \{(s_m, y_m)\}_{m\in\mathbb{N}} \subset [0,\infty) \times U_1,$$

we finally deduce that the given sequence $\{y^m\}_{m\in\mathbb{N}}$ is relatively compact.

Now, we recall that the set K defined in the course of the proof of Theorem 3.1 in (A.3) is relatively compact in $C([0,\infty); H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma)$. Chebyshev's inequality again shows that

$$\sup_{n\in\mathbb{N}} P_n(\bar{K}^c) \le \sup_{n\in\mathbb{N}} P_n(K^c) \le \epsilon. \tag{A.8}$$

Hence the set $K \times K_2$ is relatively compact in $\bar{\mathbb{S}}$ and the desired tightness follows from (A.7) and (A.8).

Without loss of generality, we may assume that P_n converges weakly to some probability measure P. It remains to prove that $P \in \mathscr{W}(s, x_0, y_0, C_{t,q})$. By Skorokhod's representation theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $\bar{\mathbb{S}}$-valued random variables $(\tilde{x}_n, \tilde{y}_n)$ and $(\tilde{x}, \tilde{y})$ such that

(i) $(\tilde{x}_n, \tilde{y}_n)$ has law P_n for each $n \in \mathbb{N}$,

(ii) $(\tilde{x}_n, \tilde{y}_n) \to (\tilde{x}, \tilde{y})$ in $\bar{\mathbb{S}}$ $\tilde{P}$-a.s., and $(\tilde{x}, \tilde{y})$ has law P.

Let $(\tilde{\mathcal{F}}_t)_{t\ge0}$ be the $\tilde{P}$-augmented canonical filtration of the process $(\tilde{x}, \tilde{y})$. Then it is easy to see that $\tilde{y}$ is a cylindrical Wiener process on U with respect to $(\tilde{\mathcal{F}}_t)_{t\ge0}$. In fact, let $t > s$ and g be any bounded and real-valued $\bar{\mathcal{B}}_s$-measurable continuous function on $\bar{\Omega}$. We have

$$\begin{aligned} E^P[(y(t) - y(s))g(x,y)] &= E^{\tilde{P}}[(\tilde{y}(t) - \tilde{y}(s)r)g(\tilde{x}, \tilde{y})] \\ &= \lim_{n\to\infty} E^{\tilde{P}}[(\tilde{y}_n(t) - \tilde{y}_n(s))g(\tilde{x}_n, \tilde{y}_n)] = 0, \end{aligned}$$

and similarly for $y_i = \langle y, l_i\rangle$ with $\{l_i\}$ an orthonormal basis in U,

$$E^P[(y_i(t)y_j(t) - y_i(s)y_j(s) - \delta_{ij}(t-s))g(x,y)] = 0,$$

We then find that y is a cylindrical Wiener process on U with respect to $(\bar{\mathcal{B}}_t)_{t\geq 0}$ under P.

Conditions (M1) and (M3) follow similarly to the proof of Theorem 3.1. Finally, we shall verify (M2) for P. We know that under $\tilde{P}$, by the convergence in (ii), for every $e_i \in C^\infty(\mathbb{T}^3)$ we have

$$\langle \tilde{x}_n(t), e_i\rangle \to \langle \tilde{x}(t), e_i\rangle, \qquad \int_{s_n}^t \langle F(\tilde{x}_n(r)), e_i\rangle\, dr \to \int_s^t \langle F(\tilde{x}(r)), e_i\rangle\, dr \quad \tilde{P}\text{-a.s.}$$

Define

$$M_{t,s}^{x,i} = \left\langle x(t) - x(s) - \int_s^t F(x(r))\, dr, e_i\right\rangle.$$

Then for every $t \in [s,\infty)$ and every $p \in (1,\infty)$ we have

$$\begin{aligned} &\sup_{n\in\mathbb{N}} E^{\tilde{P}}[|M_{t,s_n}^{\tilde{x}_n,i}|^{2p}] \leq C \sup_{n\in\mathbb{N}} E^{P_n}\left[\left(\int_{s_n}^t \|G(x(r))\|_{L_2(U;L^2_\sigma)}^2\, dr\right)^p\right] < \infty, \\ &\lim_{n\to\infty} E^{\tilde{P}}[|M_{t,s_n}^{\tilde{x}_n,i} - M_{t,s}^{\tilde{x},i}|^2] = 0. \end{aligned} \tag{A.9}$$

Let $t > r \geq s$ and g be any bounded continuous function on $\bar{\mathbb{S}}$. Using (A.9) we know

$$\begin{aligned} E^P[(M_{t,s}^{x,i} - M_{r,s}^{x,i})g(x|_{[0,r]}, y|_{[0,r]})] &= E^{\tilde{P}}[(M_{t,s}^{\tilde{x},i} - M_{r,s}^{\tilde{x},i})g(\tilde{x}|_{[0,r]}, \tilde{y}|_{[0,r]})] \\ &= \lim_{n\to\infty} E^{\tilde{P}}[(M_{t,s_n}^{\tilde{x}_n,i} - M_{r,s_n}^{\tilde{x}_n,i})g(\tilde{x}_n|_{[0,r]}, \tilde{y}_n|_{[0,r]})] \\ &= \lim_{n\to\infty} E^{P_n}[(M_{t,s_n}^{x,i} - M_{r,s_n}^{x,i})g(x|_{[0,r]}, y|_{[0,r]})] = 0. \end{aligned}$$

Consequently, $t \mapsto M_{t,s}^i$ is a $(\bar{\mathcal{B}}_t)_{t\geq s}$-martingale under P. Similarly,

$$E^P\left[(M_{t,s}^{x,i})^2 - \int_s^t \|G(x)^* e_i\|_U^2\, dl \,\middle|\, \bar{\mathcal{B}}_r\right] = (M_{r,s}^{x,i})^2 - \int_s^r \|G(x)^* e_i\|_U^2\, dl,$$

which identifies the quadratic variation of $t \mapsto M_{t,s}^i$. It remains to identify the cross variation of this process with the cylindrical Wiener process y under P. To this end, we let $\{l_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of U and define $y_j = \langle y, l_j\rangle_U$. Then we deduce that

$$\begin{aligned} &E^P\left[M_{t,s}^{x,i}(y_j(t) - y_j(s)) - \int_s^t \langle G^*(x)e_i, l_j\rangle_U\, dl \,\middle|\, \bar{\mathcal{B}}_r\right] \\ &\qquad = M_{r,s}^{x,i}(y_j(r) - y_j(s)) - \int_s^r \langle G^*(x)e_i, l_j\rangle_U\, dl. \end{aligned}$$

Thus, the quadratic variation process of $M_{t,s}^{x,i} - \int_s^t \langle e_i, G(x)dy\rangle$ is 0, which implies (M2). The proof is complete. ∎

Proof of Theorem 7.2. The existence of a generalized probabilistically weak solution follows from a similar argument to the proof of Theorem 5.1 and defining $\mathbb{Y}(t) = \mathbb{Y}_0 + \int_s^t y(r) \otimes dy(r)$. By [36, Theorem 4.2.5] we have $P(Z \in CL^2_\sigma) = 1$. The stability of solutions with respect to the initial time and initial condition follows in a similar way to Theorem 5.1. First, it follows as in Theorem 5.1 that $\{P_n\}_{n\in\mathbb{N}}$ is tight in

$$\widetilde{\mathbb{S}} := C_{\mathrm{loc}}([0,\infty); H^{-3} \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times H^{-3}) \cap L^2_{\mathrm{loc}}([0,\infty); L^2_\sigma \times \mathbb{R}^m \times \mathbb{R}^{m\times m} \times L^2_\sigma).$$

Without loss of generality, we may assume that P_n converges weakly to some probability measure P. It remains to prove that $P \in \mathscr{GW}(s, x_0, y_0, \mathbb{Y}_0, Z_0, C_{t,q})$. By Skorokhod's representation theorem, there exists a probability space $(\Omega', \mathcal{F}', P')$ and $\widetilde{\mathbb{S}}$-valued random variables $(\tilde{x}_n, \tilde{y}_n, \tilde{\mathbb{Y}}_n, \tilde{Z}_n)$ and $(\tilde{x}, \tilde{y}, \tilde{\mathbb{Y}}, \tilde{Z})$ such that

(i) $(\tilde{x}_n, \tilde{y}_n, \tilde{\mathbb{Y}}_n, \tilde{Z}_n)$ has law P_n for each $n \in \mathbb{N}$,

(ii) $(\tilde{x}_n, \tilde{y}_n, \tilde{\mathbb{Y}}_n, \tilde{Z}_n) \to (\tilde{x}, \tilde{y}, \tilde{\mathbb{Y}}, \tilde{Z})$ in $\widetilde{\mathbb{S}}$ P'-a.s., and $(\tilde{x}, \tilde{y}, \tilde{\mathbb{Y}}, \tilde{Z})$ has law P.

In the following we verify (M1)–(M3) for P.

For (M1), using the convergence in (i) above, we have

$$\begin{aligned} &P(x(t) = x_0,\ y(t) = y_0,\ \mathbb{Y}(t) = \mathbb{Y}_0,\ Z(t) = Z_0,\ 0 \le t \le s) \\ &\qquad = P'(\tilde{x}(t) = x_0,\ \tilde{y}(t) = y_0,\ \tilde{\mathbb{Y}}(t) = \mathbb{Y}_0,\ \tilde{Z}(t) = Z_0,\ 0 \le t \le s) = 1. \end{aligned}$$

By condition (1.4) on G, $P(\int_0^T \|G(x(r))\|^2_{L_2(\mathbb{R}^m; L^2_\sigma)}\, dr < \infty) = 1$ for every $T > 0$. Condition (M3) follows by similar arguments to the proof of Theorem 3.1. Then by [36, Theorem 4.2.5] we have $P(Z \in CL^2_\sigma) = 1$.

For (M2), we write $\mathbb{Y} = (\mathbb{Y}_{ij})$, $y = (y_i)$ and $\tilde{\mathbb{Y}} = (\tilde{\mathbb{Y}}_{ij})$, $\tilde{y} = (\tilde{y}_i)$ and $\tilde{\mathbb{Y}}_n = (\tilde{\mathbb{Y}}_{n,ij})$, $\tilde{y}_n = (\tilde{y}_{n,i})$. Similarly to the proof of Theorem 5.1 we find that y is a $(\widetilde{\mathcal{B}}_t)_{t\ge s}$- $\mathbb{R}^m$-valued Brownian motion.

Next, we shall prove

$$P\left(\mathbb{Y}(t) - \mathbb{Y}(s) = \int_s^t y(r) \otimes dy(r)\right) = 1. \tag{A.10}$$

To this end, we need to verify that

- the quadratic variation process of $\mathbb{Y}_{ij}$ is given by $\int_s^t y_i(r)^2 dr$,
- the cross variation of $\mathbb{Y}_{ij}$ with y_k is given by $\int_s^t y_i(r)\, dr\, \delta_{jk}$.

Let $t > r \ge s$ and g be any bounded and real-valued $\widetilde{\mathcal{B}}_r$-measurable continuous function on $\widetilde{\Omega}$. Then

$$\begin{aligned} &E^P\left[\left(\mathbb{Y}_{ij}(t)^2 - \mathbb{Y}_{ij}(r)^2 - \int_r^t y_i(l)^2\, dl\right) g(x, y, \mathbb{Y})\right] \\ &\qquad = E^{P'}\left[\left(\tilde{\mathbb{Y}}_{ij}(t)^2 - \tilde{\mathbb{Y}}_{ij}(r)^2 - \int_r^t \tilde{y}_i(l)^2\, dl\right) g(\tilde{x}, \tilde{y}, \tilde{\mathbb{Y}})\right] \\ &\qquad = \lim_{n\to\infty} E^{P'}\left[\left(\tilde{\mathbb{Y}}_{n,ij}(t)^2 - \tilde{\mathbb{Y}}_{n,ij}(r)^2 - \int_r^t \tilde{y}_{n,i}(r)^2\, dr\right) g(\tilde{x}_n, \tilde{y}_n, \tilde{\mathbb{Y}}_n)\right] = 0, \end{aligned}$$

and

$$
\begin{aligned}
&E^P\Big[\Big(\mathbb{Y}_{ij}(t)y_k(t) - \mathbb{Y}_{ij}(r)y_k(r) - \delta_{kj}\int_r^t y_i(l)\,dl\Big)g(x,y,\mathbb{Y})\Big] \\
&\quad = E^{P'}\Big[\Big(\tilde{\mathbb{Y}}_{ij}(t)\tilde{y}_k(t) - \tilde{\mathbb{Y}}_{ij}(r)\tilde{y}_k(r) - \delta_{kj}\int_r^t \tilde{y}_i(l)\,dl\Big)g(\tilde{x},\tilde{y},\tilde{\mathbb{Y}})\Big] \\
&\quad = \lim_{n\to\infty} E^{P'}\Big[\Big(\tilde{\mathbb{Y}}_{n,ij}(t)\tilde{y}_{n,k}(t) - \tilde{\mathbb{Y}}_{n,ij}(r)\tilde{y}_{n,k}(r) - \delta_{kj}\int_r^t \tilde{y}_{n,i}(l)\,dl\Big)g(\tilde{x}_n,\tilde{y}_n,\tilde{\mathbb{Y}}_n)\Big] \\
&\quad = 0.
\end{aligned}
$$

Hence the quadratic variation process of $\mathbb{Y}(t) - \mathbb{Y}(s) - \int_s^t y(r)\otimes dy(r)$ is zero and (A.10) follows. Similarly,

$$
P\Big(Z(t) - Z(s) - \int_s^t \Delta Z(r)\,dr = \int_s^t G(v_0 + Z(r))\,dy(r)\Big) = 1.
$$

The rest follows by the same argument as in the proof of Theorem 5.1. ∎

Appendix B. Intermittent jets

In this part we recall the construction of intermittent jets from [7, Section 7.4] and derive a new estimate in Lemma B.2. We point out that the construction is entirely deterministic, that is, none of the functions below depends on ω. Let us begin with the following geometric lemma which can be found in [7, Lemma 6.6].

Lemma B.1. *Denote by $\overline{B_{1/2}}(\mathrm{Id})$ the closed ball of radius $1/2$ around the identity matrix Id in the space of 3×3 symmetric matrices. There exists $\Lambda \subset \mathbb{S}^2\cap\mathbb{Q}^3$ such that for each $\xi\in\Lambda$ there exists a C^∞-function $\gamma_\xi : \overline{B_{1/2}}(\mathrm{Id}) \to \mathbb{R}$ such that*

$$
R = \sum_{\xi\in\Lambda} \gamma_\xi(R)^2(\xi\otimes\xi)
$$

for every symmetric matrix satisfying $|R-\mathrm{Id}|\le 1/2$. For $C_\Lambda = 8|\Lambda|(1+8\pi^3)^{1/2}$, where $|\Lambda|$ is the cardinality of the set Λ, define the constant

$$
M = C_\Lambda \sup_{\xi\in\Lambda}\Big(\|\gamma_\xi\|_{C^0} + \sum_{|j|\le N}\|D^j\gamma_\xi\|_{C^0}\Big).
$$

For each $\xi\in\Lambda$ let $A_\xi\in\mathbb{S}^2\cap\mathbb{Q}^3$ be an orthogonal vector to ξ. Then for each $\xi\in\Lambda$, $\{\xi, A_\xi, \xi\times A_\xi\}\subset\mathbb{S}^2\cap\mathbb{Q}^3$ is an orthonormal basis for $\mathbb{R}^3$. We denote by n_* the smallest natural number such that

$$
\{n_*\xi, n_*A_\xi, n_*\xi\times A_\xi\}\subset\mathbb{Z}^3 \quad \textit{for every } \xi\in\Lambda.
$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with support in a ball of radius 1. We define $\Phi := -\Delta\chi$ and $\phi := -\Delta\Phi = (-\Delta)^2\chi$ and we normalize so that

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \phi(x_1, x_2)^2\, dx_1\, dx_2 = 1. \tag{B.1}$$

This particular form of ϕ given through the function χ did not appear in [7]. It is needed below in Lemma B.2 which we apply in Section 8.1 with $\gamma = 5 + \delta$, i.e. $l = 2$. By definition we know $\int_{\mathbb{R}^2} \phi\, dx = 0$. Define $\psi : \mathbb{R} \to \mathbb{R}$ to be a smooth, mean zero function with support in the ball of radius 1 satisfying

$$\frac{1}{2\pi}\int_{\mathbb{R}} \psi(x_3)^2\, dx_3 = 1. \tag{B.2}$$

For parameters $r_\perp, r_\| > 0$ such that

$$r_\perp \ll r_\| \ll 1,$$

we define the rescaled cut-off functions

$$\phi_{r_\perp}(x_1, x_2) = \frac{1}{r_\perp}\phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right), \quad \psi_{r_\|}(x_3) = \frac{1}{r_\|^{1/2}}\psi\left(\frac{x_3}{r_\|}\right),$$
$$\Phi_{r_\perp}(x_1, x_2) = \frac{1}{r_\perp}\Phi\left(\frac{x_1}{r_\perp}, \frac{x_2}{r_\perp}\right).$$

We periodize $\phi_{r_\perp}$, $\Phi_{r_\perp}$ and $\psi_{r_\|}$ so that they are viewed as periodic functions on $\mathbb{T}^2$, $\mathbb{T}^2$ and $\mathbb{T}$ respectively.

Consider a large real number λ such that $\lambda r_\perp \in \mathbb{N}$, and a large time oscillation parameter $\mu > 0$. For every $\xi \in \Lambda$ we introduce

$$\begin{aligned}
\psi_{(\xi)}(t, x) &:= \psi_{\xi, r_\perp, r_\|, \lambda, \mu}(t, x) := \psi_{r_\|}(n_* r_\perp \lambda(x \cdot \xi + \mu t)),\\
\Phi_{(\xi)}(x) &:= \Phi_{\xi, r_\perp, \lambda}(x) := \Phi_{r_\perp}\big(n_* r_\perp \lambda(x - \alpha_\xi) \cdot A_\xi, n_* r_\perp \lambda(x - \alpha_\xi) \cdot (\xi \times A_\xi)\big),\\
\phi_{(\xi)}(x) &:= \phi_{\xi, r_\perp, \lambda}(x) := \phi_{r_\perp}\big(n_* r_\perp \lambda(x - \alpha_\xi) \cdot A_\xi, n_* r_\perp \lambda(x - \alpha_\xi) \cdot (\xi \times A_\xi)\big),
\end{aligned}$$

where $\alpha_\xi \in \mathbb{R}^3$ are shifts to ensure that $\{\Phi_{(\xi)}\}_{\xi \in \Lambda}$ have mutually disjoint supports.

The intermittent jets $W_{(\xi)} : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}^3$ are defined as in [7, Section 7.4]:

$$W_{(\xi)}(t, x) := W_{\xi, r_\perp, r_\|, \lambda, \mu}(t, x) := \xi \psi_{(\xi)}(t, x)\phi_{(\xi)}(x). \tag{B.3}$$

By the choice of α_ξ we have

$$W_{(\xi)} \otimes W_{(\xi')} \equiv 0 \quad \text{for } \xi \neq \xi' \in \Lambda, \tag{B.4}$$

and by the normalizations (B.1) and (B.2) we obtain

$$\fint_{\mathbb{T}^3} W_{(\xi)}(t, x) \otimes W_{(\xi)}(t, x)\, dx = \xi \otimes \xi.$$

These facts combined with Lemma B.1 imply that

$$\sum_{\xi\in\Lambda}\gamma_\xi(R)^2 \fint_{\mathbb{T}^3} W_{(\xi)}(t,x)\otimes W_{(\xi)}(t,x)\,dx = R \tag{B.5}$$

for every symmetric matrix R satisfying $|R-\mathrm{Id}|\leq 1/2$. Since $W_{(\xi)}$ are not divergence-free, we introduce the corrector term

$$W^{(c)}_{(\xi)} := \frac{1}{n_*^2\lambda^2}\nabla\psi_{(\xi)}\times \operatorname{curl}(\Phi_{(\xi)}\xi) = \operatorname{curl}\operatorname{curl} V_{(\xi)} - W_{(\xi)} \tag{B.6}$$

with

$$V_{(\xi)}(t,x) := \frac{1}{n_*^2\lambda^2}\xi\psi_{(\xi)}(t,x)\Phi_{(\xi)}(x).$$

Thus we have

$$\operatorname{div}(W_{(\xi)} + W^{(c)}_{(\xi)}) \equiv 0.$$

Next, we recall the key bounds from [7, Section 7.4]. For $N, M\geq 0$ and $p\in[1,\infty]$ the following holds provided $r_\|^{-1}\ll r_\perp^{-1}\ll\lambda$:

$$\begin{aligned}
&\|\nabla^N\partial_t^M\psi_{(\xi)}\|_{C_tL^p}\lesssim r_\|^{1/p-1/2}\left(\frac{r_\perp\lambda}{r_\|}\right)^N\left(\frac{r_\perp\lambda\mu}{r_\|}\right)^M,\\
&\|\nabla^N\phi_{(\xi)}\|_{L^p}+\|\nabla^N\Phi_{(\xi)}\|_{L^p}\lesssim r_\perp^{2/p-1}\lambda^N,\\
&\|\nabla^N\partial_t^M W_{(\xi)}\|_{C_tL^p}+\frac{r_\|}{r_\perp}\|\nabla^N\partial_t^M W^{(c)}_{(\xi)}\|_{C_tL^p}+\lambda^2\|\nabla^N\partial_t^M V_{(\xi)}\|_{C_tL^p}\\
&\qquad\qquad\lesssim r_\perp^{2/p-1}r_\|^{1/p-1/2}\lambda^N\left(\frac{r_\perp\lambda\mu}{r_\|}\right)^M,
\end{aligned}\tag{B.7}$$

where the implicit constants may depend on p, N and M, but are independent of $\lambda, r_\perp, r_\|, \mu$.

Finally, we establish two additional estimates employed in Section 8.1.

Lemma B.2. *Let $\alpha_0\in[0,1]$ and $\gamma,\delta>0$. Suppose that $l=\frac{\gamma-1-\delta}{2}\in\mathbb{N}$ and $\phi=(-\Delta)^l\chi$ for a smooth function χ with support in a ball of radius 1. Then*

$$\|\psi_{(\xi)}\|_{C_T^{\alpha_0}H^{\gamma+\delta}}\lesssim\mu^{\alpha_0}\left(\frac{r_\perp\lambda}{r_\|}\right)^{\alpha_0+\gamma+\delta},\quad \|\phi_{(\xi)}\|_{H^{-\gamma}}\lesssim\lambda^{-\gamma}r_\perp^{-\delta}.$$

Proof. The first bound is a consequence of (B.7) and interpolation

$$\|\psi_{(\xi)}\|_{H^{\gamma+\delta}}\lesssim\|\psi_{(\xi)}\|_{L^2}^{1-(\gamma+\delta)/\lceil\gamma+\delta\rceil}\|\psi_{(\xi)}\|_{H^{\lceil\gamma+\delta\rceil}}^{(\gamma+\delta)/\lceil\gamma+\delta\rceil}\lesssim\left(\frac{r_\perp\lambda}{r_\|}\right)^{\gamma+\delta},$$

hence using interpolation again leads to

$$\|\psi_{(\xi)}\|_{C_T^{\alpha_0}H^{\gamma+\delta}}\lesssim\|\psi_{(\xi)}\|_{C_TH^{\gamma+\delta}}^{1-\alpha_0}\|\psi_{(\xi)}\|_{C_T^1H^{\gamma+\delta}}^{\alpha_0}\lesssim\mu^{\alpha_0}\left(\frac{r_\perp\lambda}{r_\|}\right)^{\alpha_0+\gamma+\delta}.$$

Let us now show the estimate for the $H^{-\gamma}$-norm of $\phi_{(\xi)}$. We view $\phi_{(\xi)}$ as a periodic function on $\mathbb{R}^3$ and we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{(\xi)}(x)e^{-ik\cdot x}\,dx &= e^{-ik\cdot\alpha_\xi}\int_{\mathbb{R}^3}\phi_{r_\perp}(n_*\lambda r_\perp u_1, n_*\lambda r_\perp u_2)e^{-iAk\cdot u}\,du\\
&= \delta_0((Ak)_3)e^{-ik\cdot\alpha_\xi}\int_{\mathbb{R}^2}\phi_{r_\perp}(n_*\lambda r_\perp u_1, n_*\lambda r_\perp u_2)e^{-i[(Ak)_1u_1+(Ak)_2u_2]}\,du_1\,du_2\\
&= \delta_0((Ak)_3)e^{-ik\cdot\alpha_\xi}(n_*\lambda r_\perp)^{-2}\int_{\mathbb{R}^2}\phi_{r_\perp}(u_1,u_2)e^{-i[\frac{(Ak)_1}{n_*\lambda r_\perp}u_1+\frac{(Ak)_2}{n_*\lambda r_\perp}u_2]}\,du_1\,du_2\\
&= \delta_0((Ak)_3)e^{-ik\cdot\alpha_\xi}\sum_{m\in 2\pi n_*\lambda r_\perp\mathbb{Z}^2}\hat{\phi}_{r_\perp}\left(\frac{(Ak)_1}{n_*\lambda r_\perp},\frac{(Ak)_2}{n_*\lambda r_\perp}\right)\delta_0(((Ak)_1,(Ak)_2)+m),\end{aligned}$$

where in the first equality we use $u=(u_1,u_2,u_3)=(x\cdot A_\xi, x\cdot(\xi\times A_\xi), x\cdot\xi):=Ax$ with A being an orthonormal matrix and $\hat{\phi}_{r_\perp}(k)=\int_{\mathbb{T}^2}\phi_{r_\perp}(x)e^{-ik\cdot x}\,dx$. Then

$$\begin{aligned}&(1-\Delta)^{-\gamma/2}\phi_{(\xi)}(x)\\
&\quad=\sum_{m\in 2\pi n_*\lambda r_\perp\mathbb{Z}^2}e^{-iA^*(m,0)\cdot\alpha_\xi}(1+|m|^2)^{-\gamma/2}\hat{\phi}_{r_\perp}\left(\frac{m}{n_*\lambda r_\perp}\right)e^{iA^*(m,0)\cdot x}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\phi_{(\xi)}\|_{H^{-\gamma}}^2 &\lesssim \sum_{m\in 2\pi n_*\lambda r_\perp\mathbb{Z}^2}(1+|m|^2)^{-\gamma}\left|\hat{\phi}_{r_\perp}\left(\frac{m}{n_*\lambda r_\perp}\right)\right|^2\\
&= \sum_{m\in 2\pi n_*\lambda r_\perp\mathbb{Z}^2\backslash\{0\}}(1+|m|^2)^{-\gamma}\left|\hat{\phi}_{r_\perp}\left(\frac{m}{n_*\lambda r_\perp}\right)\right|^2\\
&\lesssim (\lambda r_\perp)^{-2\gamma}\sum_{k\in 2\pi\mathbb{Z}^2\backslash\{0\}}|k|^{-2\gamma}|\hat{\phi}_{r_\perp}(k)|^2.\end{aligned}$$

Here, in the equality we have used the fact that $\phi_{r_\perp}$ has zero mean. Moreover, for $l=\frac{\gamma-1-\delta}{2}\in\mathbb{N}$ and $\phi=(-\Delta)^l\chi$ with a smooth compactly supported function χ we have

$$\hat{\phi}_{r_\perp}(k)=r_\perp\int_{\mathbb{R}^2}\phi(x)e^{-ikr_\perp\cdot x}\,dx=r_\perp\int_{\mathbb{R}^2}\chi(-\Delta)^l e^{-ikr_\perp\cdot x}\,dx,$$

which implies that for $l=\frac{\gamma-1-\delta}{2}$ and $\delta>0$,

$$\begin{aligned}\|\phi_{(\xi)}\|_{H^{-\gamma}}^2 &\lesssim (\lambda r_\perp)^{-2\gamma}r_\perp^2\sum_{k\in 2\pi\mathbb{Z}^2\backslash\{0\}}|k|^{-2\gamma}(\|\chi\|_{L^1}(|k|r_\perp)^{2l})^2\\
&\lesssim \lambda^{-2\gamma}(r_\perp)^{-2\delta}\sum_{k\in 2\pi\mathbb{Z}^2\backslash\{0\}}|k|^{-2-2\delta}\|\chi\|_{L^1}^2\\
&\lesssim \lambda^{-2\gamma}(r_\perp)^{-2\delta}.\end{aligned}$$

∎

Appendix C. Uniqueness in law implies joint uniqueness in law

In this part we will extend the result of Cherny [9] to a general infinite-dimensional setting. A generalization to a semigroup framework in Banach spaces was proved by Ondreját [38]. Let U, U_1, H and H_1 be separable Hilbert spaces and suppose that the embedding $U \subset U_1$ is Hilbert–Schmidt and the embedding $H \subset H_1$ is continuous. Consider the SPDE of the form

$$dX = F(X)dt + G(X)dB, \quad X(0) = x \in H, \tag{C.1}$$

where $F : H \to H_1$ is $\mathcal{B}(H)/\mathcal{B}(H_1)$-measurable and B is a cylindrical Wiener process on U which is defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. In other words, B can be viewed as a continuous process taking values in U_1 and we assume that for $x \in H$, $G(x)$ is a Hilbert–Schmidt operator from U to H. Solutions to (C.1) are then understood in the following sense.

Definition C.1. A pair (X, B) is a *solution* to (C.1) provided there exists a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ such that

(H1) B is a cylindrical $(\mathcal{F}_t)_{t\geq 0}$-Wiener process on U;
(H2) X is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process in $C([0,\infty); H_1)$ P-a.s.;
(H3) $F(X) \in L^1_{\mathrm{loc}}([0,\infty); H_1)$ and $G(X) \in L^2_{\mathrm{loc}}([0,\infty); L_2(U, H))$ P-a.s.;
(H4) P-a.s. we have, for all $t \in [0,\infty)$,

$$X_t = x + \int_0^t F(X_s)\, ds + \int_0^t G(X_s)\, dB_s.$$

Let us now recall the definition of uniqueness in law and of joint uniqueness in law.

Definition C.2. We say that *uniqueness in law* holds for (C.1) if for any two solutions (X, B) and $(\tilde{X}, \tilde{B})$ starting from the same initial distribution, one has $\mathrm{Law}(X) = \mathrm{Law}(\tilde{X})$. We say that *joint uniqueness in law* holds for (C.1) if for any two solutions (X, B) and $(\tilde{X}, \tilde{B})$ starting from the same initial distribution, one has $\mathrm{Law}(X, B) = \mathrm{Law}(\tilde{X}, \tilde{B})$.

Clearly, joint uniqueness in law implies uniqueness in law. The following result shows that the two notions are in fact equivalent for SPDEs of the form (C.1).

Theorem C.1. *Suppose that uniqueness in law holds for* (C.1). *Then joint uniqueness in law holds for* (C.1).

Set $E = L_2(U; H)$. Since E is separable, $C([0,t]; E)$ is dense in $L^2([0,t]; E)$. By the same argument as in [9, Lemma 3.2], we can prove the following result.

Lemma C.2. *Let $t > 0$ and $f \in L^2([0,t]; E)$. For $k \in \mathbb{N}$, set*

$$f^{(k)}(s) = \begin{cases} 0 & \text{if } s \in [0, t/k], \\ \frac{k}{t} \int_{(i-1)t/k}^{it/k} f(r)\, dr & \text{if } s \in (it/k, (i+1)t/k] \ (i = 1, \dots, k-1). \end{cases}$$

Then $f^{(k)} \to f$ in $L^2([0,t]; E)$.

By Lemma C.2 and the same argument as in [9, Lemma 3.3], we obtain the following.

Lemma C.3. *Let (X, B) be a solution to (C.1) defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Let $(Q_\omega)_{\omega\in\Omega}$ be a conditional probability distribution of (X, B) given $\mathcal{F}_0$.*[2] *Let Y be the coordinate process with values in H_1 and let Z be the coordinate process with values in U_1. Let $(\mathcal{H}_t)_{t\geq 0}$ be the canonical filtration on $C([0,\infty); H_1 \times U_1)$ and denote $\mathcal{H} = \bigvee_{t\geq 0} \mathcal{H}_t$. Then for P-a.e. $\omega \in \Omega$ the pair (Y, Z) is a solution to (C.1) on the stochastic basis $(C([0,\infty); H_1 \times U_1), \mathcal{H}, (\mathcal{H}_t)_{t\geq 0}, Q_\omega)$.*

Proof of Theorem C.1. Let (X, B) be a solution to (C.1) on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Let $\{\beta^k\}_{k\in\mathbb{N}}$ and $\{\bar\beta^k\}_{k\in\mathbb{N}}$ be two families of independent real-valued Wiener processes defined on another stochastic basis $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P')$ and set

$$(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde P) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', (\mathcal{F}_t \otimes \mathcal{F}'_t)_{t\geq 0}, P \otimes P').$$

All the processes $X, B, \beta^k, \bar\beta^k$, $k \in \mathbb{N}$, can be defined on $\tilde\Omega$ in the obvious way. Assume that the cylindrical Wiener process B admits a decomposition $B = \sum_{k=1}^\infty \alpha^k l_k$, where $\{\alpha^k\}_{k\in\mathbb{N}}$ is a family of independent real-valued Wiener processes and $\{l_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in U. Let $\varphi(x)$ be the orthogonal projection from U to $(\ker G(x))^\perp$ and $\psi(x)$ be the orthogonal projection from U to $\ker G(x)$. Then set

$$\begin{aligned}
\varphi_s &:= \varphi(X_s), \quad \psi_s := \psi(X_s),\\
V_t &:= \sum_{k=1}^\infty \Big[\int_0^t \varphi_s \, d\alpha_s^k \, l_k + \int_0^t \psi_s \, d\beta_s^k \, l_k\Big],\\
\bar V_t &:= \sum_{k=1}^\infty \Big[\int_0^t \varphi_s \, d\bar\beta_s^k \, l_k + \int_0^t \psi_s \, d\alpha_s^k \, l_k\Big].
\end{aligned}$$

In the following, $\langle\!\langle \cdot, \cdot \rangle\!\rangle_t$ denotes the cross variation process at time t. We obtain

$$\begin{aligned}
&\langle\!\langle \langle V, l_i\rangle_U, \langle V, l_j\rangle_U \rangle\!\rangle_t\\
&\quad = \sum_{k=1}^\infty \Big[\int_0^t \langle \varphi_s l_k, l_i\rangle_U \langle \varphi_s l_k, l_j\rangle_U \, ds + \int_0^t \langle \psi_s l_k, l_i\rangle_U \langle \psi_s l_k, l_j\rangle_U \, ds\Big]\\
&\quad = \int_0^t [\langle \varphi_s l_i, \varphi_s l_j\rangle_U + \langle \psi_s l_i, \psi_s l_j\rangle_U] \, ds = \int_0^t \langle (\varphi_s + \psi_s) l_i, (\varphi_s + \psi_s) l_j\rangle_U \, ds\\
&\quad = \int_0^t \langle l_i, l_j\rangle_U \, ds = \delta_{ij} t.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\langle\!\langle \langle V, l_i\rangle_U, \langle \bar V, l_j\rangle_U \rangle\!\rangle_t &= 0,\\
\langle\!\langle \langle \bar V, l_i\rangle_U, \langle \bar V, l_j\rangle_U \rangle\!\rangle_t &= \delta_{ij} t.
\end{aligned}$$

[2] Here, we consider (X, B) as a $C([0,\infty); H_1 \times U_1)$-valued process.

As a consequence, under $\tilde{P}$ the process $(V, \bar{V})$ is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-cylindrical Wiener process on $U \times U$. Moreover, for any $t \geq 0$, we have

$$\int_0^t G(X_s)\,dB_s = \int_0^t G(X_s)\varphi_s\,dB_s = \int_0^t G(X_s)\,dV_s.$$

Hence (X, V) is a solution to (C.1) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$.

Consider now the filtration

$$\mathcal{G}_s = \tilde{\mathcal{F}}_s \vee \sigma(\bar{V}_t : t \geq 0) = \tilde{\mathcal{F}}_s \vee \sigma(\bar{V}_t - \bar{V}_s : t \geq s), \quad s \geq 0.$$

Since $\tilde{\mathcal{F}}_s$ and $\sigma(V_t - V_s : t \geq s) \vee \sigma(\bar{V}_t - \bar{V}_s : t \geq s)$ are independent, the process V is a cylindrical $(\mathcal{G}_t)_{t\geq 0}$-Wiener process on U under $\tilde{P}$. Thus (X, V) is a solution to (C.1) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{G}}_t)_{t\geq 0}, \tilde{P})$.

Let $(Q_{\tilde{\omega}})_{\tilde{\omega}\in\tilde{\Omega}}$ be a conditional probability distribution of (X, V) given $\mathcal{G}_0$. By Lemma C.3, for $\tilde{P}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$, the pair (Y, Z) is a solution to (C.1) on $(C([0,\infty); H \times U), \mathcal{H}, (\mathcal{H}_t)_{t\geq 0}, Q_{\tilde{\omega}})$. As uniqueness in law holds for (C.1), the probability law induced by Y on each of these stochastic bases, i.e. $Q_{\tilde{\omega}} \circ Y^{-1}$, is the same for $\tilde{P}$-a.e. $\tilde{\omega} \in \tilde{\Omega}$. Since this is the conditional probability distribution of X given $\mathcal{G}_0$, it follows that the process X is independent of $\mathcal{G}_0$. In particular, X and $\bar{V}$ are independent. Let $\chi(x)$ be the pseudo-inverse of $G(x)$ (see e.g. [36, Appendix C] for more details); then $\chi(x)G(x) = \varphi(x)$. Set $\chi_s := \chi(X_s)$. Thus,

$$\int_0^t \varphi_s\,dB_s = \int_0^t \chi_s G(X_s)\,dB_s = \int_0^t \chi_s\,dM_s,$$

where

$$M_t = \int_0^t G(X_s)\,dB_s = X_t - x - \int_0^t F(X_s)\,ds.$$

Accordingly, we obtain

$$B_t = \int_0^t \varphi_s\,dB_s + \int_0^t \psi_s\,dB_s = \int_0^t \chi_s\,dM_s + \int_0^t \psi_s\,d\bar{V}_s.$$

The process M is a measurable functional of X, while $\bar{V}$ is independent of X. Thus the distribution $\mathrm{Law}(X, B)$ is unique. ■

Appendix D. Analysis of rough partial differential equations

In this section, we employ the theory of rough paths to derive estimates for the following rough partial differential equation. Assume that $v \in C^1_{t,x}$ and z solves the system

$$\begin{aligned} dz &= \Delta z\,dt + G(v+z)\,dB, \\ \operatorname{div} z &= 0, \\ z(0) &= z_0, \end{aligned} \tag{D.1}$$

with $\operatorname{div} z_0 = 0$. Then we have

$$z(t) = P_t z_0 + \int_0^t P_{t-s} G(v+z)\, dB_s,$$

where $P_t = e^{t\Delta}$ is the heat semigroup. The nonlinearity G in (D.1) is defined through

$$G(u) = \big(g_{ij}(\cdot, \langle u, \varphi_1^{ij}\rangle, \dots, \langle u, \varphi_{k_{ij}}^{ij}\rangle)\big) \tag{D.2}$$

with $g_{ij} \in C_b^3(\mathbb{T}^3 \times \mathbb{R}^{k_{ij}})$, $\varphi_\ell^{ij} \in C^\infty(\mathbb{T}^3)$, $i = 1, \dots, 3$, $j = 1, \dots, m$, $\ell = 1, \dots, k_{ij}$, i.e. the functions g_{ij} and their derivatives up to order 3 are bounded and $g_{\cdot j}$ is divergence-free with respect to the spatial variable in $\mathbb{T}^3$.

The driving process B is an m-dimensional Brownian motion and we view it as a rough path. To this end, fix $\alpha \in (\frac{1}{3}, \frac{1}{2})$. We use $\rho_\alpha(B)$ to denote its α-Hölder rough path seminorm which is given by

$$\rho_\alpha(B) = \sup_{0 \le s < t \le T} \frac{|B_t - B_s|}{|t-s|^\alpha} + \sup_{0 \le s < t \le T} \frac{|\int_s^t (B_r - B_s) \otimes dB_r|}{|t-s|^{2\alpha}}.$$

The first component of the rough path is denoted by $B_{s,t} := B_t - B_s$ and we understand the iterated integral $\mathbb{B}_{s,t} := \int_s^t (B_r - B_s) \otimes dB_r$ in the Itô sense. However, the results of this section apply mutatis mutandis to other rough path lifts of the Brownian motion as well as general rough paths.

Let C^β denote the closure of smooth functions with respect to the usual Hölder norm. We also use the Hölder–Besov space $\mathscr{C}^\beta$, $\beta \in \mathbb{R}$, defined by the closure of smooth functions with respect to the $B^\beta_{\infty,\infty}$-norm

$$\|f\|_{\mathscr{C}^\beta} := \|f\|_{B^\beta_{\infty,\infty}} = \sup_{j \in \mathbb{N}_0 \cup \{-1\}} 2^{\beta j} \|\Delta_j f\|_{L^\infty}$$

with Δ_j, $j \in \mathbb{N}_0 \cup \{-1\}$, being the usual Littlewood–Paley blocks. For any $0 < \beta \notin \mathbb{N}$ it is well known (see [1, p. 99]) that $\|f\|_{C^\beta} \asymp \|f\|_{\mathscr{C}^\beta}$. For a path h defined on $[0, T]$, we denote its increment $h_t - h_s$ by $h_{s,t}$.

We also recall the following smoothing effect from the heat semigroup (see e.g. [49, Lemma 2.8]), which is used in the following proof.

Lemma D.1. *Let $T > 0$.*

(i) *For any $\theta > 0$, $\alpha \in \mathbb{R}$, and $t \in [0, T]$,*

$$\|P_t f\|_{\mathscr{C}^{\theta+\alpha}} \lesssim t^{-\theta/2} \|f\|_{\mathscr{C}^\alpha}, \quad \|P_t f\|_{H^{\theta+\alpha}} \lesssim t^{-\theta/2} \|f\|_{H^\alpha}, \tag{D.3}$$

with the implicit constant independent of f.

(ii) *For any $0 < \theta < 2$ and $t \in [0, T]$,*

$$\|P_t f - f\|_{L^\infty} \lesssim t^{\theta/2} \|f\|_{\mathscr{C}^\theta}, \quad \|P_t f - f\|_{L^2} \lesssim t^{\theta/2} \|f\|_{H^\theta}, \tag{D.4}$$

with the implicit constant independent of f.

Now, we introduce the definition of a controlled rough path adapted to our purposes (see also [27]).

Definition D.1. Let $\gamma \in \mathbb{R}$. We call a pair (y, y') a *controlled rough path in the* $\mathcal{C}^\gamma$*-scale* provided $(y, y') \in C_T\mathcal{C}^\gamma \times (C_T\mathcal{C}^{\gamma-2\alpha} \cap C_T^\alpha L^\infty)$ and the remainder

$$(s,t) \mapsto R^y_{s,t} := y_{s,t} - y'_s B_{s,t} \tag{D.5}$$

belongs to $C^{2\alpha}_{2,T} L^\infty$, the space of 2-index maps on $[0,T]^2$ with values in L^∞ such that

$$\|R^y\|_{2\alpha,L^\infty} = \sup_{0\le s<t\le T} \frac{\|R^y_{s,t}\|_{L^\infty}}{|t-s|^{2\alpha}} < \infty.$$

The space of controlled rough paths in the $\mathcal{C}^\gamma$-scale is denoted by $D^{2\alpha}_{B,\gamma}$ and endowed with the norm

$$\|y, y'\|_{B,2\alpha,\gamma} = \|y\|_{C_T\mathcal{C}^\gamma} + \|y'\|_{C_T\mathcal{C}^{\gamma-2\alpha}} + \|y'\|_{C^\alpha_T L^\infty} + \|R^y\|_{2\alpha,L^\infty}.$$

We also present the corresponding definition with $\mathcal{C}^\gamma$ replaced by H^γ.

Definition D.2. Let $\gamma \in \mathbb{R}$. We call a pair (y, y') a *controlled rough path in the* H^γ*-scale* provided $(y, y') \in C_T H^\gamma \times (C_T H^{\gamma-2\alpha} \cap C^\alpha_T L^2)$ and the remainder

$$(s,t) \mapsto R^y_{s,t} := y_{s,t} - y'_s B_{s,t} \tag{D.6}$$

belongs to $C^{2\alpha}_{2,T} L^2$, the space of 2-index maps on $[0,T]^2$ with values in L^2 such that

$$\|R^y\|_{2\alpha,L^2} = \sup_{0\le s<t\le T} \frac{\|R^y_{s,t}\|_{L^2}}{|t-s|^{2\alpha}} < \infty.$$

The space of controlled rough paths in the H^γ-scale is denoted by $\bar{D}^{2\alpha}_{B,\gamma}$ and endowed with the norm

$$\|y, y'\|_{\bar{B},2\alpha,\gamma} = \|y\|_{C_T H^\gamma} + \|y'\|_{C_T H^{\gamma-2\alpha}} + \|y'\|_{C^\alpha_T L^2} + \|R^y\|_{2\alpha,L^2}.$$

The following integration lemma is a version of [25, Theorem 4.5] adapted to our setting.

Lemma D.2. *Let* $\sigma \in [0,\alpha)$ *and* $(y, y') \in D^{2\alpha}_{B,4\alpha-2\sigma}$. *Then the integral*

$$\int_0^t P(t-r) y_r \, dB_r := \lim_{|\pi|\to 0} \sum_{[s,r]\in\pi} P(t-s)(y_s B_{s,r} + y'_s \mathbb{B}_{s,r})$$

exists as an element of $\mathcal{C}^{-2\kappa}$ *for* $\kappa > 1 - 3\alpha + \sigma$ *where the limit is taken over partitions* π *of* $[0,t]$ *with vanishing mesh size. Moreover, for every* $0 \le \theta < 1$,

$$\begin{aligned}\Big\| \int_s^t P(t-r) y_r \, dB_r - P(t-s) y_s B_{s,t} - P(t-s) y'_s \mathbb{B}_{s,t} \Big\|_{\mathcal{C}^{4\alpha-2\theta}} \\ \lesssim \|y, y'\|_{B,2\alpha,4\alpha-2\sigma} |t-s|^{(\alpha-\sigma+\theta)\wedge(3\alpha)} \rho_\alpha(B).\end{aligned}$$

Here the implicit constant is independent of y, $\rho_\alpha(B)$.

Proof. The proof follows the ideas of the usual sewing lemma which has already appeared in many variants; see e.g. [23, Lemma 4.2], [24, Theorem 2.4]. The key computation is the following.

Let

$$\xi_{s,t} := y_s B_{s,t} + y'_s \mathbb{B}_{s,t} =: \xi^1_{s,t} + \xi^2_{s,t},$$

which gives

$$\delta\xi_{s,u,t} := \xi_{s,t} - \xi_{s,u} - \xi_{u,t} = -R^y_{s,u} B_{u,t} - y'_{s,u}\mathbb{B}_{u,t} =: h^1_{s,u,t} + h^2_{s,u,t},$$

where the first equality is the definition of increment of a 2-index map ξ. Consider dyadic partitions $\pi_k = \{s = t_0 < t_1 < \cdots < t_{2^k} = t\}$ with $t_i = s + 2^{-k} i(t-s)$, let

$$I_k := \sum_{[u,v]\in\pi_k} P_{t-u}\xi_{u,v},$$

and denote $m = (u+v)/2$. Then we have

$$\begin{aligned}
I_k - I_{k+1} &= \sum_{[u,v]\in\pi_k} P_{t-u}\delta\xi_{u,m,v} + P_{t-m}(P_{m-u} - I)\xi_{m,v} \\
&= \sum_{[u,v]\in\pi_k} P_{t-u} h^1_{u,m,v} + \sum_{[u,v]\in\pi_k} P_{t-u} h^2_{u,m,v} \\
&\quad + \sum_{[u,v]\in\pi_k} P_{t-m}(P_{m-u} - I)\xi^1_{m,v} + \sum_{[u,v]\in\pi_k} P_{t-m}(P_{m-u} - I)\xi^2_{m,v} \\
&= \sum_{i=1}^4 J_i.
\end{aligned}$$

By (D.3) and (D.4) we have, for $2\alpha - 1 < \beta < 3\alpha - \sigma - 1$ and $\beta \le 2\alpha - \theta$,

$$\begin{aligned}
\|J_4\|_{\mathscr{C}^{4\alpha-2\theta}} &\lesssim \sum_{[u,v]\in\pi_k} (t-m)^{\beta-2\alpha+\theta} \|(P_{m-u} - I)\xi^2_{m,v}\|_{C^{2\beta}} \\
&\lesssim \sum_{[u,v]\in\pi_k} (t-m)^{\beta-2\alpha+\theta}(m-u)^{\alpha-\sigma-\beta} \|y'\|_{C_T\mathscr{C}^{2\alpha-2\sigma}} (v-m)^{2\alpha}\rho_\alpha(B) \\
&\lesssim \|y'\|_{C_T\mathscr{C}^{2\alpha-2\sigma}}\rho_\alpha(B) 2^{-k(3\alpha-\beta-1-\sigma)}|t-s|^{3\alpha-\beta-1-\sigma} \sum_{[u,v]\in\pi_k} (t-m)^{\beta-2\alpha+\theta}(m-u) \\
&\lesssim \|y'\|_{C_T\mathscr{C}^{2\alpha-2\sigma}}\rho_\alpha(B) 2^{-k(3\alpha-\beta-1-\sigma)}|t-s|^{\alpha-\sigma+\theta},
\end{aligned}$$

where in the last inequality above, in view of the condition $\beta - 2\alpha + \theta > -1$, we estimated the Riemann sum by the corresponding integral (using convexity of the integrand) and integrated. Similarly, for $2\alpha - 1 < \beta < 3\alpha - \sigma - 1$ and $\beta \le 2\alpha - \theta$ we have

$$\begin{aligned}
\|J_3\|_{\mathscr{C}^{4\alpha-2\theta}} &\lesssim \sum_{[u,v]\in\pi_k} (t-m)^{\beta-2\alpha+\theta}(m-u)^{2\alpha-\sigma-\beta} \|y\|_{C_T\mathscr{C}^{4\alpha-2\sigma}}\rho_\alpha(B)(v-m)^{\alpha} \\
&\lesssim \|y\|_{C_T\mathscr{C}^{4\alpha-2\sigma}}\rho_\alpha(B) 2^{-k(3\alpha-\beta-1-\sigma)}|t-s|^{3\alpha-\beta-1-\sigma} \sum_{[u,v]\in\pi_k} (t-m)^{\beta-2\alpha+\theta}(m-u) \\
&\lesssim \|y\|_{C_T\mathscr{C}^{4\alpha-2\sigma}}\rho_\alpha(B) 2^{-k(3\alpha-\beta-1-\sigma)}|t-s|^{\alpha-\sigma+\theta}.
\end{aligned}$$

Moreover, by (D.3) and (D.4),

$$
\begin{aligned}
\|J_1\|_{\mathcal{C}^{4\alpha-2\theta}} &\lesssim \sum_{[u,v]\in\pi_k} (t-u)^{(-2\alpha+\theta)\wedge 0}\|h^1_{u,m,v}\|_{L^\infty} \\
&\lesssim \sum_{[u,v]\in\pi_k} (t-u)^{(-2\alpha+\theta)\wedge 0}(m-u)^{2\alpha}\|R^y\|_{2\alpha,L^\infty}(v-m)^{\alpha}\rho_\alpha(B) \\
&\lesssim \|R^y\|_{2\alpha,L^\infty}2^{-k(3\alpha-1)}|t-s|^{3\alpha-1}\sum_{[u,v]\in\pi_k}(t-m)^{(-2\alpha+\theta)\wedge 0}(m-u)\rho_\alpha(B) \\
&\lesssim \|R^y\|_{2\alpha,L^\infty}2^{-k(3\alpha-1)}|t-s|^{(\alpha+\theta)\wedge(3\alpha)}\rho_\alpha(B).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|J_2\|_{\mathcal{C}^{4\alpha-2\theta}} &\lesssim \sum_{[u,v]\in\pi_k} (t-u)^{(-2\alpha+\theta)\wedge 0}\|h^2_{u,m,v}\|_{L^\infty} \\
&\lesssim \sum_{[u,v]\in\pi_k} (t-u)^{(-2\alpha+\theta)\wedge 0}(m-u)^{\alpha}\|y'\|_{C^\alpha_T L^\infty}\rho_\alpha(B)(v-m)^{2\alpha} \\
&\lesssim \|y'\|_{C^\alpha_T L^\infty}\rho_\alpha(B)2^{-k(3\alpha-1)}|t-s|^{(\alpha+\theta)\wedge(3\alpha)}.
\end{aligned}
$$

Thus the result follows by summing over k and taking the limit. In particular, the lower bound for κ from the statement of the lemma comes from the requirement that $\alpha-\sigma+\theta>1$. ∎

As the corresponding semigroup estimates remain the same in the H^γ-scale, we also obtain the following result.

Lemma D.3. *Let $\sigma\in[0,\alpha)$ and $(y,y')\in\bar{D}^{2\alpha}_{B,4\alpha-2\sigma}$. Then the integral*

$$
\int_0^t P(t-r)y_r\,dB_r := \lim_{|\pi|\to 0}\sum_{[s,r]\in\pi} P(t-s)(y_sB_{s,r}+y'_s\mathbb{B}_{s,r})
$$

exists as an element of $H^{-2\kappa}$ for $\kappa>1-3\alpha-\sigma$, where the limit is taken over partitions π of $[0,t]$ with vanishing mesh size. Moreover, for every $0\le\theta<1$,

$$
\begin{aligned}
&\left\|\int_s^t P(t-r)y_r\,dB_r - P(t-s)y_sB_{s,t} - P(t-s)y'_s\mathbb{B}_{s,t}\right\|_{H^{4\alpha-2\theta}} \\
&\qquad\qquad\lesssim \|y,y'\|_{\bar{B},2\alpha,4\alpha-2\sigma}|t-s|^{(\alpha-\sigma+\theta)\wedge(3\alpha)}\rho_\alpha(B).
\end{aligned}
$$

Here the implicit constant is independent of y, $\rho_\alpha(B)$.

By a similar argument to that for [30, Lemma 3.5] we obtain the following result:

Lemma D.4. *Let $T\in(0,1]$, $\sigma\in[0,\alpha)$ and $(y,y')\in D^{2\alpha}_{B,4\alpha-2\sigma}$. Then*

$$
(z,z') = \left(\int_0^\cdot P(\cdot-s)y_s\,dB_s, y\right)\in D^{2\alpha}_{B,4\alpha}
$$

and

$$\|z, z'\|_{B,2\alpha,4\alpha} \lesssim (1+\rho_\alpha(B))(\|y_0\|_{\mathscr{C}^{4\alpha-2\sigma}} + \|y_0'\|_{\mathscr{C}^{2\alpha-2\sigma}} + T^{\alpha(\alpha-\sigma)/(2\alpha-\sigma)}\|y, y'\|_{B,2\alpha,4\alpha-2\sigma}).$$

Here the implicit constant is independent of $y, \rho_\alpha(B)$.

Proof. By (D.5) we first have

$$\begin{aligned}\|z'\|_{C_T^\alpha L^\infty} = \|y\|_{C_T^\alpha L^\infty} &\lesssim (\|y'\|_{C_T L^\infty} + \|y_0\|_{L^\infty})\rho_\alpha(B) + \|R^y\|_{\alpha,L^\infty}\\ &\lesssim (1+\rho_\alpha(B))(\|y_0'\|_{L^\infty} + \|y_0\|_{L^\infty} + T^\alpha\|y, y'\|_{B,2\alpha,4\alpha-2\sigma}).\end{aligned}$$

The desired bound for the Gubinelli derivative $z' = y$ in $C_T\mathscr{C}^{2\alpha}$ follows from

$$\|z'\|_{C_T^{\alpha(\alpha-\sigma)/(2\alpha-\sigma)}\mathscr{C}^{2\alpha}} \lesssim \|y\|^{\alpha/(2\alpha-\sigma)}_{C_T\mathscr{C}^{4\alpha-2\sigma}}\|y\|^{(\alpha-\sigma)/(2\alpha-\sigma)}_{C_T^\alpha L^\infty}.$$

In order to bound z in $C_T\mathscr{C}^{4\alpha}$, we write

$$z_t = \left(\int_0^t P(t-s)y_s\,dB_s - P(t)y_0B_{0,t} - P(t)y_0'\mathbb{B}_{0,t}\right) + P(t)y_0B_{0,t} + P(t)y_0'\mathbb{B}_{0,t}.$$

We apply Lemma D.2 with $\theta = 0$ to control the first term for $0 \le t \le T$:

$$\left\|\int_0^t P(t-s)y_s\,dB_s - P(t)y_0B_{0,t} - P(t)y_0'\mathbb{B}_{0,t}\right\|_{\mathscr{C}^{4\alpha}} \lesssim T^{\alpha-\sigma}\|y, y'\|_{B,2\alpha,4\alpha-2\sigma}\rho_\alpha(B),$$

and for the remaining two we estimate by (D.3) as follows:

$$\begin{aligned}\|P_ty_0'\mathbb{B}_{0,t}\|_{\mathscr{C}^{4\alpha}} &\lesssim t^{-\alpha-\sigma+2\alpha}\|y_0'\|_{\mathscr{C}^{2\alpha-2\sigma}}\rho_\alpha(B) \lesssim \|y_0'\|_{\mathscr{C}^{2\alpha-2\sigma}}\rho_\alpha(B),\\ \|P_ty_0B_{0,t}\|_{\mathscr{C}^{4\alpha}} &\lesssim t^{-\sigma+\alpha}\|y_0\|_{\mathscr{C}^{4\alpha-2\sigma}}\rho_\alpha(B) \lesssim \|y_0\|_{\mathscr{C}^{4\alpha-2\sigma}}\rho_\alpha(B).\end{aligned}$$

It remains to control the 2α-Hölder norm of R^z in L^∞. We have

$$\begin{aligned}R^z_{s,t} &= \left(\int_s^t P(t-r)y_r\,dB_r - P(t-s)y_sB_{s,t} - P(t-s)y_s'\mathbb{B}_{s,t}\right)\\ &\quad + (P(t-s)-\mathrm{Id})y_sB_{s,t} + (P(t-s)-\mathrm{Id})\int_0^s P(s-r)y_r\,dB_r + P(t-s)y_s'\mathbb{B}_{s,t}\\ &= I_1 + \cdots + I_4.\end{aligned}$$

Applying Lemma D.2 with $\theta = 2\alpha-\kappa$, $\kappa > 0$ small enough we obtain, for $0 \le t \le T$,

$$\begin{aligned}\|I_1\|_{L^\infty} \lesssim \|I_1\|_{\mathscr{C}^{2\kappa}} &\lesssim \|y, y'\|_{B,2\alpha,4\alpha-2\sigma}|t-s|^{3\alpha-\sigma-\kappa}\rho_\alpha(B)\\ &\lesssim T^{\alpha-\sigma-\kappa}\|y, y'\|_{B,2\alpha,4\alpha-2\sigma}|t-s|^{2\alpha}\rho_\alpha(B),\end{aligned}$$

whereas the remaining terms are estimated by (D.3) and (D.4) as follows:

$$\|I_2\|_{L^\infty} \lesssim |t-s|^{3\alpha-\sigma}\|y_s\|_{\mathscr{C}^{4\alpha-2\sigma}}\rho_\alpha(B) \lesssim T^{\alpha-\sigma}\|y, y'\|_{B,2\alpha,4\alpha-2\sigma}|t-s|^{2\alpha}\rho_\alpha(B),$$

$$\|I_3\|_{L^\infty} \lesssim |t-s|^{2\alpha}\left\|\int_0^s P(s-r)y_r\,dB_r\right\|_{\mathscr{C}^{4\alpha}} = \|z_s\|_{\mathscr{C}^{4\alpha}}|t-s|^{2\alpha},$$

which combined with the above estimate for $z \in C_T C^{4\alpha}$ yields the desired bound for I_3, and finally

$$\|I_4\|_{L^\infty} \lesssim |t-s|^{2\alpha} \|y'_s\|_{L^\infty} \rho_\alpha(B) \lesssim (T^\alpha \|y, y'\|_{B,2\alpha,4\alpha-2\sigma} + \|y'_0\|_{L^\infty})|t-s|^{2\alpha}.$$

The claim follows. ■

By the same arguments, we deduce the H^γ-counterpart of Lemma D.4.

Lemma D.5. *Let $T \in (0,1]$, $\sigma \in [0,\alpha)$ and $(y, y') \in \bar{D}^{2\alpha}_{B,4\alpha-2\sigma}$. Then*

$$(z, z') = \left(\int_0^\cdot P(\cdot - s) y_s \, dB_s, y \right) \in \bar{D}^{2\alpha}_{B,4\alpha}$$

and

$$\|z, z'\|_{\bar{B},2\alpha,4\alpha} \lesssim (1 + \rho_\alpha(B))(\|y_0\|_{H^{4\alpha-2\sigma}} + \|y'_0\|_{H^{2\alpha-2\sigma}} + T^{\alpha(\alpha-\sigma)/(2\alpha-\sigma)} \|y, y'\|_{\bar{B},2\alpha,4\alpha-2\sigma}).$$

Here the implicit constant is independent of y, $\rho_\alpha(B)$.

By a similar argument to that for [30, Lemma 3.6] we obtain the following result.

Lemma D.6. *Let G satisfy assumption (D.2) and $(y, G(v+y)) \in D^{2\alpha}_{B,4\alpha}$. Then for $\sigma \in [0,\alpha)$ we have $(G(v+y), DG(v+y)G(v+y)) \in D^{2\alpha}_{B,4\alpha-2\sigma}$ and*

$$\|G(v+y), DG(v+y)G(v+y)\|_{B,2\alpha,4\alpha-2\sigma} \lesssim (1 + \|y, G(v+y)\|_{B,2\alpha,4\alpha} + \|v\|_{C^1_{T,x}})(1+\rho_\alpha(B))^2.$$

Moreover, if $(\tilde{y}, G(v+\tilde{y})) \in D^{2\alpha}_{B,4\alpha}$ then

$$\begin{aligned}
&\|G(v+y) - G(v+\tilde{y}), DG(v+y)G(v+y) - DG(v+\tilde{y})G(v+\tilde{y})\|_{B,2\alpha,4\alpha-2\sigma} \\
&\quad \lesssim (1 + \|y, G(v+y)\|_{B,2\alpha,4\alpha} + \|\tilde{y}, G(v+\tilde{y})\|_{B,2\alpha,4\alpha} + \|v\|_{C^1})(1+\rho_\alpha(B))^2 \\
&\qquad \times (\|y - \tilde{y}, G(v+y) - G(v+\tilde{y})\|_{B,2\alpha,4\alpha}).
\end{aligned}$$

Remark D.7. It will be seen in the proof below that due to the definition of the coefficient G in (D.2), the spatial regularity of the controlled rough path $(G(v+y), DG(v+y)G(v+y))$ actually only depends on the spatial regularity of the functions g_{ij} and not on the spatial regularity of v, y. Consequently, the claimed space regularity of order $4\alpha - 2\sigma$ was only taken for convenience in order to follow more easily the arguments of [30].

Proof of Lemma D.6. For simplicity we only concentrate on the case $G(y) = g(\cdot, \langle y, \varphi \rangle)$ with φ smooth and $g \in C^3_b$. First, we observe that as a consequence of (D.5) with $y' = G(v+y)$ we have

$$\|y\|_{C^\alpha_T L^\infty} \lesssim (\|y'\|_{C_T \mathscr{C}^{2\alpha}} + \|y_0\|_{L^\infty} + \|R^y\|_{\alpha,L^\infty})(1+\rho_\alpha(B)).$$

Then, since the spatial dependence of $G(v+y)$ only depends on the spatial dependence of g, we get

$$\|G(v+y)\|_{C_T\mathscr{C}^{4\alpha-2\sigma}} \lesssim 1,$$

and since $DG(v+y)G(v+y) = \partial g(\cdot, \langle v+y,\varphi\rangle)\langle g(\cdot,\langle v+y,\varphi\rangle),\varphi\rangle$ where ∂ denotes the derivative with respect to the variable in place of the inner product, we have

$$\begin{aligned}\|DG(v+y)G(v+y)\|_{C_T^\alpha L^\infty} &\lesssim 1 + \|y\|_{C_T^\alpha L^\infty} + \|v\|_{C^1_{T,x}} \\ &\lesssim (1 + \|y, G(v+y)\|_{B,2\alpha,4\alpha} + \|v\|_{C^1_{T,x}})(1+\rho_\alpha(B)), \\ \|DG(v+y)G(v+y)\|_{C_T\mathscr{C}^{2\alpha-2\sigma}} &\lesssim 1.\end{aligned}$$

Moreover,

$$\begin{aligned} R^G_{s,t} &= g(\langle v_t + y_t,\varphi\rangle) - g(\langle v_s+y_s,\varphi\rangle) - \partial g(\langle v_s + y_s,\varphi\rangle)\langle g(\langle v_s+y_s,\varphi\rangle)B_{s,t},\varphi\rangle \\ &= \int_0^1 \big[\partial g(\langle v_s + y_s + r(v_{s,t}+y_{s,t}),\varphi\rangle)\langle v_{s,t}+y_{s,t},\varphi\rangle - \partial g(\langle v_s+y_s,\varphi\rangle)\big]\,dr \\ &\qquad\qquad \times \langle g(\langle v_s+y_s,\varphi\rangle)B_{s,t},\varphi\rangle \\ &= \int_0^1 \partial g(\langle v_s+y_s+r(v_{s,t}+y_{s,t}),\varphi\rangle)\langle v_{s,t},\varphi\rangle\,dr \\ &\quad + \int_0^1 [\partial g(\langle v_s+y_s+r(v_{s,t}+y_{s,t}),\varphi\rangle) - \partial g(\langle v_s+y_s,\varphi\rangle)]\,dr \\ &\qquad\qquad \times \langle g(\langle v_s+y_s,\varphi\rangle)B_{s,t},\varphi\rangle \\ &\quad + \int_0^1 \partial g(\langle v_s+y_s+r(v_{s,t}+y_{s,t}),\varphi\rangle)\langle R^y_{s,t},\varphi\rangle\,dr. \end{aligned} \tag{D.7}$$

Consequently, we deduce

$$\begin{aligned}\|R^G_{s,t}\|_{2\alpha,L^\infty} &\lesssim (\|v\|_{C^1_{T,x}} + \|y\|_{C^\alpha_T L^\infty})(1+\rho_\alpha(B)) + \|R^y\|_{2\alpha,L^\infty} \\ &\lesssim (1 + \|v\|_{C^1_{T,x}} + \|y, G(v+y)\|_{B,2\alpha,4\alpha})(1+\rho_\alpha(B))^2.\end{aligned}$$

Thus the proof of the first result is complete. The second one is a simpler version of the argument in the proof of Lemma D.8 below, so we leave it to the reader. ■

Lemma D.8. *Let G satisfy assumption (D.2) and $(y, G(v+y)) \in \bar{D}^{2\alpha}_{B,4\alpha}$. Then for $\sigma \in [0,\alpha)$, we have $(G(v+y), DG(v+y)G(v+y)) \in \bar{D}^{2\alpha}_{B,4\alpha-2\sigma}$, and for $\gamma > 0$,*

$$\begin{aligned}&\|G(v+y), DG(v+y)G(v+y)\|_{\bar{B},2\alpha,4\alpha-2\sigma} \\ &\qquad\lesssim (1 + \|y, G(v+y)\|_{\bar{B},2\alpha,4\alpha} + \|v\|_{C^{2\alpha}_T H^{-\gamma}})(1+\rho_\alpha(B))^2.\end{aligned}$$

Moreover, if $(\tilde{y}, G(\tilde{v}+\tilde{y})) \in \bar{D}^{2\alpha}_{B,4\alpha}$, then for $\gamma > 0$,

$$\begin{aligned}&\|G(v+y) - G(\tilde{v}+\tilde{y}), DG(v+y)G(v+y) - DG(\tilde{v}+\tilde{y})G(\tilde{v}+\tilde{y})\|_{\bar{B},2\alpha,4\alpha-2\sigma} \\ &\lesssim (1 + \|y, G(v+y)\|_{\bar{B},2\alpha,4\alpha} + \|\tilde{y}, G(\tilde{v}+\tilde{y})\|_{\bar{B},2\alpha,4\alpha} + \|v\|_{C^{2\alpha}_T H^{-\gamma}} + \|\tilde{v}\|_{C^{2\alpha}_T H^{-\gamma}}) \\ &\quad \times (1+\rho_\alpha(B))^2(\|y-\tilde{y}, G(v+y) - G(\tilde{v}+\tilde{y})\|_{\bar{B},2\alpha,4\alpha} + \|v-\tilde{v}\|_{C^{2\alpha}_T H^{-\gamma}}).\end{aligned}$$

Proof. For notational simplicity we again focus only on the case $G(y) = g(\cdot, \langle y, \varphi\rangle)$ with φ smooth and $g \in C_b^3$; the general case follows the same argument. The first estimate is similar to Lemma D.6. Now we prove the second one. First, we observe that as a consequence of (D.6) with $y' = G(v + y)$ we have

$$\|y - \tilde{y}\|_{C_T^\alpha L^2} \lesssim \|y - \tilde{y}, G(v + y) - G(\tilde{v} + \tilde{y})\|_{\bar{B},2\alpha,4\alpha}(1 + \rho_\alpha(B)).$$

Then (as in Remark D.7, also here the spatial regularity of v, y does not influence the estimate)

$$\|G(y + v) - G(\tilde{y} + \tilde{v})\|_{C_T H^{4\alpha-2\sigma}} \lesssim \|y - \tilde{y}\|_{C_T H^{4\alpha-2\sigma}} + \|v - \tilde{v}\|_{C_T H^{-\gamma}},$$

and

$$\begin{aligned}&\|DG(v + y)G(v + y) - DG(\tilde{v} + \tilde{y})G(\tilde{v} + \tilde{y})\|_{C_T H^{2\alpha-2\sigma}}\\ &\qquad\qquad\lesssim \|y - \tilde{y}\|_{C_T H^{2\alpha-2\sigma}} + \|v - \tilde{v}\|_{C_T H^{-\gamma}}.\end{aligned}$$

Moreover,

$$\begin{aligned}&(DG(v + y)G(v + y) - DG(\tilde{v} + \tilde{y})G(\tilde{v} + \tilde{y}))_{s,t}\\
&= \int_0^1 \big(\partial^2 g(\langle y_s + v_s + r(y + v)_{s,t}, \varphi\rangle) - \partial^2 g(\langle \tilde{y}_s + \tilde{v}_s + r(\tilde{y} + \tilde{v})_{s,t}, \varphi\rangle)\big)\, dr\\
&\qquad\qquad \times \langle (y + v)_{s,t}, \varphi\rangle \langle G_t(v + y), \varphi\rangle\\
&\quad + \int_0^1 \partial^2 g(\langle \tilde{y}_s + \tilde{v}_s + r(\tilde{y} + \tilde{v})_{s,t}, \varphi\rangle)\, dr\, \langle (y + v)_{s,t} - (\tilde{y} + \tilde{v})_{s,t}, \varphi\rangle \langle G_t(v + y), \varphi\rangle\\
&\quad + \int_0^1 \partial^2 g(\langle \tilde{y}_s + \tilde{v}_s + r(\tilde{y} + \tilde{v})_{s,t}, \varphi\rangle)\, dr\, \langle (\tilde{y} + \tilde{v})_{s,t}, \varphi\rangle \langle G_t(v + y) - G_t(\tilde{v} + \tilde{y}), \varphi\rangle\\
&\quad + (\partial g(\langle y_s + v_s, \varphi\rangle) - \partial g(\langle \tilde{y}_s + \tilde{v}_s, \varphi\rangle))\langle G(v + y)_{s,t}, \varphi\rangle\\
&\quad + \partial g(\langle \tilde{y}_s + \tilde{v}_s, \varphi\rangle)\langle G(v + y)_{s,t} - G(\tilde{v} + \tilde{y})_{s,t}, \varphi\rangle,\end{aligned}$$

which implies

$$\begin{aligned}&\|DG(v + y)G(v + y) - DG(\tilde{v} + \tilde{y})G(\tilde{v} + \tilde{y})\|_{C_T^\alpha L^2}\\
&\lesssim (1 + \|y, G(v + y)\|_{\bar{B},2\alpha,4\alpha} + \|\tilde{y}, G(\tilde{v} + \tilde{y})\|_{\bar{B},2\alpha,4\alpha} + \|v\|_{C_T^{2\alpha} H^{-\gamma}} + \|\tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}})\\
&\quad \times (1 + \rho_\alpha(B))^2(\|y - \tilde{y}, G(v + y) - G(\tilde{v} + \tilde{y})\|_{\bar{B},2\alpha,4\alpha} + \|v - \tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}}).\end{aligned}$$

Furthermore, for $R^G_{s,t}$ in (D.7) we have $R^G_{s,t} - \tilde{R}^G_{s,t} = I_1 + I_2 + I_3$, with I_i corresponding to the difference of the last three lines:

$$\begin{aligned}I_1 + I_3 &= \int_0^1 \big(\partial g(\langle v_s + y_s + r(v_{s,t} + y_{s,t}), \varphi\rangle) - \partial g(\langle \tilde{v}_s + \tilde{y}_s + r(\tilde{v}_{s,t} + \tilde{y}_{s,t}), \varphi\rangle)\big)\\
&\qquad\qquad \times \langle v_{s,t} + R^y_{s,t}, \varphi\rangle\, dr\\
&\quad + \int_0^1 \partial g(\langle \tilde{v}_s + \tilde{y}_s + r(\tilde{v}_{s,t} + \tilde{y}_{s,t}), \varphi\rangle)\langle v_{s,t} - \tilde{v}_{s,t} + R^y_{s,t} - R^{\tilde{y}}_{s,t}, \varphi\rangle\, dr,\end{aligned}$$

$$\begin{aligned}I_2 = &\int_0^1 \int_0^1 \partial^2 g(\langle v_s + y_s + r\theta(v_{s,t} + y_{s,t}), \varphi\rangle)\, d\theta\, dr\, \langle v_{s,t} + y_{s,t}, \varphi\rangle \\ &\qquad \times \langle g(\langle v_s + y_s, \varphi\rangle) B_{s,t}, \varphi\rangle \\ &- \int_0^1 \int_0^1 \partial^2 g(\langle \tilde{v}_s + \tilde{y}_s + r\theta(\tilde{v}_{s,t} + \tilde{y}_{s,t}), \varphi\rangle)\, d\theta\, dr\, \langle \tilde{v}_{s,t} + \tilde{y}_{s,t}, \varphi\rangle \\ &\qquad \times \langle g(\langle \tilde{v}_s + \tilde{y}_s, \varphi\rangle) B_{s,t}, \varphi\rangle.\end{aligned}$$

Then

$$\begin{aligned}&\|R^G - \tilde{R}^G\|_{2\alpha,L^2} \\ &\lesssim \big(1 + \|y, G(v+y)\|_{\bar{B},2\alpha,4\alpha} + \|\tilde{y}, G(\tilde{v}+\tilde{y})\|_{\bar{B},2\alpha,4\alpha} + \|v\|_{C_T^{2\alpha} H^{-\gamma}} + \|\tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}}\big) \\ &\quad \times (1 + \rho_\alpha(B))^2 (\|y - \tilde{y}, G(v+y) - G(\tilde{v}+\tilde{y})\|_{\bar{B},2\alpha,4\alpha} + \|v - \tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}}).\end{aligned}$$

Thus, the proof is complete. ∎

Thus, combining Lemma D.4, Lemma D.6 and a similar argument to that in [30, Lemma 3.8, Theorem 3.9] we obtain the following result.

Theorem D.9. *Let $T > 0$ and G satisfy assumption (D.2). Then there exists a unique global solution $(z, G(z+v)) \in D^{2\alpha}_{B,4\alpha}([0,T])$ to (D.1). Moreover, the solution satisfies, for $\sigma \in [0,\alpha)$,*

$$\|z\|_{C_T \mathcal{C}^{4\alpha}} \le (\|z_0\|_{\mathcal{C}^{4\alpha}} + 1 + \|v\|_{C^1_{T,x}}) N^{TN^{\frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}}},$$

$$\|z\|_{C_T^\alpha L^\infty} \le (\|z_0\|_{C^{4\alpha}} + 1 + \|v\|_{C^1_{T,x}}) TN^{N^{\frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}} T + \frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}},$$

where $N = C(1 + \rho_\alpha(B))^3$ for some constant C independent of B, z, v.

Proof. By Lemmas D.4 and D.6, for $\theta = \frac{\alpha(\alpha-\sigma)}{2\alpha-\sigma}$ and $z' = G(z+v)$ we have

$$\|z, z'\|_{B,2\alpha,4\alpha} \le C(1 + \rho_\alpha(B))^3 (\|z_0\|_{\mathcal{C}^{4\alpha}} + 1 + T^\theta \|z, z'\|_{B,2\alpha,4\alpha} + T^\theta \|v\|_{C^1_{T,x}}).$$

Then we can choose $\bar{T}$ such that $1/4 < C(1 + \rho_\alpha(B))^3 \bar{T}^\theta \le 1/2$. Thus

$$\|z, z'\|_{B,2\alpha,4\alpha,[0,\bar{T}]} \le 2C(1 + \rho_\alpha(B))^3 (\|z_0\|_{\mathcal{C}^{4\alpha}} + 1) + \|v\|_{C^1_{T,x}}.$$

Here $\|z, z'\|_{B,2\alpha,4\alpha,[s,t]}$ is the norm for the space $D^{2\alpha}_{B,4\alpha}$ on the time interval $[s,t]$. Starting from $\bar{T}$ we obtain

$$\|z, z'\|_{B,2\alpha,4\alpha,[\bar{T},2\bar{T}]} \le N(N(\|z_0\|_{\mathcal{C}^{4\alpha}} + 1) + \|v\|_{C^1_{T,x}} + 1) + \|v\|_{C^1_{T,x}}.$$

Then we know we have at most l-fold iteration with $l < (4N)^{1/\theta} T$ and we obtain

$$\|z\|_{C_T \mathcal{C}^{4\alpha}} \lesssim (\|z_0\|_{\mathcal{C}^{4\alpha}} + 1 + \|v\|_{C^1_{T,x}}) N^{TN^{1/\theta}},$$

$$\|z, z'\|_{B,2\alpha,4\alpha,[0,T]} \le (\|z_0\|_{\mathcal{C}^{4\alpha}} + 1 + \|v\|_{C^1_{T,x}}) TN^{N^{1/\theta} T + 1/\theta}.$$

Here we may increase the constant C in the expression of N. ∎

Analogously, we obtain the result in the H^γ-scale.

Theorem D.10. *Let $T > 0$ and G satisfy assumption (D.2). Then there exists a unique global solution $(z, G(z+v)) \in \bar{D}^{2\alpha}_{B,4\alpha}([0,T])$ to (D.1). Moreover, the solution satisfies, for $\sigma \in [0,\alpha)$,*

$$\|z\|_{C_T H^{4\alpha}} \le (\|z_0\|_{H^{4\alpha}} + 1 + \|v\|_{C_T^{2\alpha} H^{-\gamma}}) N^{TN^{\frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}}},$$

$$\|z, z'\|_{\bar{B},2\alpha,4\alpha} \le (\|z_0\|_{H^{4\alpha}} + 1 + \|v\|_{C_T^{2\alpha} H^{-\gamma}}) TN^{N^{\frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}} T + \frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}}.$$

where $z' = G(z+v)$, $N = C(1+\rho_\alpha(B))^3$ for some constant C independent of B, z, v. Furthermore, let $z, \tilde{z}$ be two solutions corresponding to $v, \tilde{v}$ respectively. Then

$$\|z - \tilde{z}\|_{C_T H^{4\alpha}} \lesssim \|v - \tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}} \bar{N}^{\bar{N}^{\frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}} T},$$

$$\|z - \tilde{z}, z' - \tilde{z}'\|_{\bar{B},2\alpha,4\alpha} \lesssim \|v - \tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}} T\bar{N}^{T\bar{N}^{\frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}} + \frac{2\alpha-\sigma}{\alpha(\alpha-\sigma)}}.$$

Here

$$\begin{aligned}\bar{N} &= C(1 + \|z, z'\|_{\bar{B},2\alpha,4\alpha} + \|\tilde{z}, \tilde{z}'\|_{\bar{B},2\alpha,4\alpha} + \|v\|_{C_T^{2\alpha} H^{-\gamma}} + \|\tilde{v}\|_{C_T^{2\alpha} H^{-\gamma}}) \\ &\quad \times (1 + \rho_\alpha(B))^3.\end{aligned}$$

D.1. Back to Itô stochastic integration

If v was $(\mathcal{F}_t)_{t\ge0}$-adapted, then equation (D.1) can be solved in the Itô sense as well. As expected, the solutions z obtained from these two approaches coincide **P**-a.s., which can be seen as follows. Equation (D.1) has a unique stochastic solution z^{sto} adapted to the filtration $(\mathcal{F}_t)_{t\ge0}$. In view of [23, Proposition 5.1], z^{sto} solves (D.1) also in the rough path sense **P**-a.s. On the other hand, for ω from the set of full probability where the rough path lift $(B, \mathbb{B})$ is constructed, we obtain a rough path solution $z(\omega)$. By uniqueness for the rough path formulation of (D.1), $z(\omega) = z^{\mathrm{sto}}(\omega)$ on a set of full probability. As a consequence, the rough path solution may be regarded as an adapted stochastic process.

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