



Iordan Ganev · David Jordan · Pavel Safronov

# The quantum Frobenius for character varieties and multiplicative quiver varieties

Received October 21, 2021; revised February 8, 2022

**Abstract.** We prove that quantized multiplicative quiver varieties, quantum character varieties, and Kauffman bracket skein algebras each define sheaves of Azumaya algebras over the smooth loci of their corresponding classical moduli spaces. In the case of skein algebras this establishes a strong form of the Unicity Conjecture of Bonahon and Wong.

Our proofs exploit a strong compatibility between quantum Hamiltonian reduction and the quantum Frobenius homomorphism as well as a natural nondegeneracy condition satisfied by each of the classical Hamiltonian spaces. We therefore introduce the concepts of Frobenius quantum moment maps, Frobenius Poisson orders and their Hamiltonian reductions, and apply them to the study of Azumaya loci.

**Keywords:** character variety, quantum groups, skein algebra, Azumaya algebra, quantum Frobenius, quiver variety.

---

## 1. Introduction

In this paper we study *quantized character varieties*, and *quantized multiplicative quiver varieties*, when the quantization parameter is a root of unity. Our main results describe these quantized moduli spaces as Azumaya algebras – meaning that, étale-locally, they are sheaves of matrix algebras – over an explicitly given open locus on the spectrum of their centers. In the quiver examples, this spectrum is the classical multiplicative quiver variety, while for character variety examples it is the classical character variety.

As a key application, we also treat the so-called Kauffman bracket skein algebra of a surface. The ‘Unicity Conjecture’ of Bonahon–Wong asserts that the Kauffman bracket skein algebra is Azumaya over some non-empty open subset [14–16]; in this form it was

---

Iordan Ganev: Institute for Computing and Information Sciences, Radboud University, 6525 EC Nijmegen, The Netherlands; [iordan.ganev@gmail.com](mailto:iordan.ganev@gmail.com)

David Jordan: School of Mathematics, University of Edinburgh, Edinburgh EH9 3J2, UK; [d.jordan@ed.ac.uk](mailto:d.jordan@ed.ac.uk)

Pavel Safronov: School of Mathematics, University of Edinburgh, Edinburgh EH9 3J2, UK; [p.safronov@ed.ac.uk](mailto:p.safronov@ed.ac.uk)

*Mathematics Subject Classification 2020:* 16G20 (primary); 17B37, 18M15 (secondary).

first proved by Frohman, Kania-Bartoszyńska, and Lê [29], who showed that the set of Azumaya points was non-empty and open but did not determine the locus precisely. We improve upon their result by showing that the Azumaya locus in fact contains the smooth locus.

Both classes of quantizations are constructed via the process of quantum Hamiltonian reduction, and in both cases the theory of Poisson orders equips the quantization with a connection over an open symplectic leaf. Our primary techniques, therefore, exploit several remarkable compatibilities between Lusztig’s quantum Frobenius homomorphism on the quantum group, the theory of Poisson orders, and the procedure of quantum Hamiltonian reduction. To this end, we develop the frameworks of Frobenius Poisson orders and their Frobenius quantum Hamiltonian reductions. We exploit the Frobenius Poisson order to determine the Azumaya points before reduction, and then we show that Frobenius Poisson orders descend to Poisson orders on the Hamiltonian reduction, and hence conclude that the Azumaya algebras we constructed before reduction descend to Azumaya algebras on the Hamiltonian reduction. A further remarkable feature of both classes of examples is that they are *non-degenerate* Hamiltonian  $G$ -Poisson spaces, which allows us to describe the symplectic leaves explicitly in terms of the classical multiplicative moment map.

We first recall our two motivating classes of examples, then state our main results concerning them, and then discuss the general results and techniques in more detail.

### 1.1. Character varieties and their quantization

Given a reductive group  $G$ , the  $G$ -character stack  $Ch_G(S)$  of a connected topological surface  $S$  (possibly with boundary) is the moduli space of  $G$ -local systems on  $S$ , equivalently of representations  $\pi_1(S) \rightarrow G$ , modulo the conjugation action. To construct  $Ch_G(S)$ , one considers first the *framed* character variety,  $Ch_G^{\text{fr}}(S)$ , consisting of  $G$ -local systems with a fixed trivialization at one point; this is an affine variety, equipped with a  $G$ -action by changes of framing. The framed character variety carries the Fock–Rosly Poisson structure, and admits a multiplicative moment map  $Ch_G^{\text{fr}}(S) \rightarrow G$ . The character variety  $Ch_G(S)$  is then the quotient of  $Ch_G^{\text{fr}}(S)$  by the  $G$ -action.

Given now a *closed* surface  $S$  of genus  $g$ , let us denote by  $S^\circ$  the surface obtained by removing some small disk. An important observation is that the character variety of  $S$  is obtained from the character variety of  $S^\circ$  via a procedure of group-valued Hamiltonian reduction [3, 5, 63]. This means that  $Ch_G^{\text{fr}}(S^\circ)$  carries a multiplicative moment map valued in the group  $G$ , which records the holonomy around the boundary. The multiplicative Hamiltonian reduction then computes the joint effect of attaching the disk and quotienting by the  $G$ -action. The character variety obtained in this way is complicated – it is singular in general, and may have several irreducible components. There is an important open subset of the character variety called the ‘good locus’; it consists of the closed orbits whose stabilizer is the center. It is empty in genus 1, but non-empty for genus greater than 1 (and in fact dense for  $G = \text{GL}_N, \text{SL}_N$ ).

Functorial quantizations of character varieties were introduced in [10]; it was also proved there that the framed quantizations of punctured surfaces could be described alge-

braically via (mild generalizations of) certain ‘moduli’ algebras  $A_S$  defined combinatorially by Alekseev, Grosse, and Schomerus [1]. In the case of closed surfaces, the resulting quantized character varieties were shown in [11] to admit a description via quantum Hamiltonian reduction of the algebras  $A_S$ , echoing the classical construction. In particular, in [10] a ‘distinguished object’ was introduced – a non-commutative stand-in for the structure sheaf – whose endomorphism algebra gives a quantization of the affine character variety, and which is computed via quantum Hamiltonian reduction.

In the case  $G = \mathrm{SL}_2$ , the Kauffman bracket skein algebra provides another celebrated quantization of the character variety. The Kauffman bracket skein algebra  $K_A(S)$  with parameter  $A \in \mathbb{C}^\times$  is the vector space spanned by isotopy classes of links  $\langle L \rangle$  drawn in the cylinder  $S \times I$  over the surface, modulo the relations

$$\begin{aligned} \langle \bigcirc \rangle &= 1, & \langle L \cup \bigcirc \rangle &= (-A^2 - A^{-2})\langle L \rangle, \\ \langle \text{X} \rangle &= A \langle \text{X} \rangle + A^{-1} \langle \text{X} \rangle, \end{aligned}$$

where the diagrams represent links which are as depicted in some oriented 3-ball, and identical outside of it. The algebra structure on  $K_A(S)$  is obtained by vertically stacking links in  $S$ . It was shown by Turaev [69] that the Kauffman bracket skein algebra provides a deformation quantization of the  $\mathrm{SL}_2(\mathbb{C})$ -character variety of  $S$ .

### 1.2. Multiplicative quiver varieties and their quantization

Let  $Q$  be a quiver with dimension vector  $\mathbf{d}$ . The multiplicative quiver variety is a moduli space of representations of a doubled quiver satisfying certain moment map relations, first introduced in [23]. It is constructed by first recalling that the collection of representations of the doubled quiver of  $Q$  with dimension vector  $\mathbf{d}$  forms a product of matrix spaces. The framed representation variety  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d})$  is an open locus of this product of matrix spaces, defined by the non-vanishing of certain determinants, and admits a multiplicative moment map to the gauge group  $\mathbb{G}^{\mathbf{d}} = \prod \mathrm{GL}_{d_v}$ , where the product runs over the set of vertices of  $Q$ . The multiplicative quiver variety  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$  is the multiplicative GIT Hamiltonian reduction of the framed representation variety  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d})$  by the gauge group  $\mathbb{G}^{\mathbf{d}}$  at a moment parameter  $\xi$ , and with stability parameter  $\theta$  (see [71, 76]):

$$\mathcal{M}(Q, \mathbf{d}, \xi, \theta) = \mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d}) //_{\xi, \theta} \mathbb{G}^{\mathbf{d}}.$$

According to [23], certain special cases of multiplicative quiver varieties yield moduli spaces of  $\mathrm{GL}_N$ -connections with irregular singularities or, equivalently, moduli spaces of representations of  $\pi_1(S)$ , with prescribed monodromies around the punctures.

Quantizations of multiplicative quiver varieties were introduced in [37]. The construction involved first quantizing  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d})$  via an algebra  $\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$ , then quantizing the moment map, and defining  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$  as a quantum Hamiltonian reduction. For a more thorough recollection of multiplicative quiver varieties and their quantization, see Section 5. The algebras  $\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$  give deformations of  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d})$ , which are flat

over the ring  $\mathbb{C}[q, q^{-1}]$ . In [37] it was shown that the quantum Hamiltonian reductions of  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  are formally flat (i.e., they are flat when tensored over the ring  $\mathbb{C}[[\hbar]]$ , where  $\hbar = \log q$ ) whenever the classical multiplicative moment map is flat; by [23] this can be read off from the dimension vector and the moment map parameters. However, flatness of the quantum Hamiltonian reduction over  $\mathbb{C}[q, q^{-1}]$  remains unsettled.

### 1.3. Main results: Examples and applications

Our main results in the context of the preceding examples are as follows:

**Theorem 1.1.** *Let  $G$  be a connected reductive group and  $q$  a primitive  $\ell$ -th root of unity, which together satisfy Assumption 3.1. Let  $S$  be a closed topological surface of genus  $g$ , and denote by  $S^\circ$  the surface obtained by removing some open disk from  $S$ .*

- (1) *The moduli algebra  $A_{S^\circ}$  is finitely generated over its center, which is isomorphic to the coordinate ring of the classical framed  $G$ -character variety  $\text{Ch}_G^{\text{fr}}(S^\circ) \cong G^{2g}$ .*
- (2) *The Azumaya locus of the moduli algebra  $A_{S^\circ}$  coincides with the preimage of the open cell  $G^\circ \subset G$  under the classical moment map  $\text{Ch}_G^{\text{fr}}(S^\circ) \rightarrow G$ .*
- (3) *The quantized character variety of the closed surface  $S$  is finitely generated over its center, which is isomorphic to the coordinate ring of the classical character variety. It may be constructed as a Frobenius quantum Hamiltonian reduction of  $A_{S^\circ}$ .*
- (4) *The quantized character variety of the closed surface  $S$  is Azumaya over the entire ‘good locus’ of  $\text{Ch}_G(S)$ .*

We remark that the proofs of statements (3) and (4) apply identically to the *twisted character variety* of  $S$ , where we take  $G = \text{GL}_N$ , and we require that the holonomy around the boundary of  $S^\circ$  is a primitive  $N$ -th root of unity [34]. In this case all points are ‘good’ in the above sense and the quotient is smooth; we obtain in this way an Azumaya algebra defined over the entire twisted character variety.

Our techniques apply, in particular, to skein algebras such as the Kauffman bracket skein algebra. In a series of papers [14–16] it was proved that the Kauffman bracket skein algebra at roots of unity has a central subalgebra isomorphic to the functions on the classical  $\text{SL}_2(\mathbb{C})$ -character variety, and in [29] it was proved that this is in fact the whole center and moreover that the Azumaya locus is open and dense. We prove the following theorem in Section 4.4:

**Theorem 1.2.** *Suppose that  $\ell > 2$  is an odd integer, and that  $\zeta$  is a primitive  $\ell$ -th root of unity. Then the skein algebra  $K_\zeta(S)$  is Azumaya over the whole smooth locus of  $\text{Ch}_{\text{SL}_2}(S)$ .*

**Remark 1.3.** While Theorem 1.2 fits naturally in the broader framework we develop in the paper, those readers who are only interested in the proof of Theorem 1.2 (i.e., in the precise determination of the non-singular locus), and who are already familiar with [29], may wish to skip directly to Section 4.4, referring back to Sections 2 and 3 only as needed.

It is remarkable that the proof of the corresponding ‘unicity theorem’ for quantum character varieties in Theorem 1.1 is uniform in all groups, and follows from functoriality and generalities about quantum Hamiltonian reduction, in contrast to the hands-on algebraic methods in the skein literature. Precise determination of the Azumaya locus, as well as an extension of the unicity theorem to more general groups in this way, was a major motivation for this work.

Turning now to the quiver examples, we have the following result.

**Theorem 1.4.** *Let  $\ell > 1$  be an odd integer, and  $q$  a primitive  $\ell$ -th root of unity.*

- (1) *The algebra  $\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$  is finitely generated over a central subalgebra, which is isomorphic to the coordinate ring  $\mathcal{O}(\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d}))$  of the classical framed multiplicative quiver variety.*
- (2)  *$\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$  is Azumaya over the preimage in  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d})$  of the big cell  $G^\circ \subset G$  under the multiplicative moment map  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d}) \rightarrow G$ .*
- (3) *Frobenius quantum Hamiltonian reduction defines a coherent sheaf of algebras over the classical multiplicative quiver variety  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$ , which is Azumaya over the locus  $\mathcal{M}^s(Q, \mathbf{d}, \xi, \theta)$  of  $\theta$ -stable representations.*

In particular, these theorems identify the center of the algebra  $\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$  (resp.  $A_{S^\circ}$ ) with functions on  $\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d})$  (resp.  $\overline{\mathrm{Ch}_G(S)}$ ), and likewise identify the center of the quantized multiplicative quiver variety (resp. quantum character variety) with the affinization of  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$  (resp. of the classical character variety). It is already difficult to compute such centers directly, and our results imply, by the flatness in  $q$ , that the center is trivial away from roots of unity, a fact which is again not easy to see directly.

The Azumaya property asserts moreover that each sheaf is étale-locally the endomorphism algebra of a vector bundle on the classical variety, or equivalently, that the fiber of the algebra at each point is a matrix algebra over  $\mathbb{C}$ . In each statement, the second assertion is derived from the first via the process of Frobenius quantum Hamiltonian reduction, which is described in the next subsection.

**Remark 1.5.** Modules over an Azumaya algebra form an invertible sheaf of categories which is locally trivial for the étale topology, i.e., a gerbe. It would be interesting to understand this gerbe more conceptually. In a related direction, we expect that modules over the quantum group at a root of unity form a factorizable category relative to its Müger center. Using the results of Gwilliam–Scheimbauer [33], one may prove that this implies that this category defines an invertible object in the Morita category of braided monoidal categories relative to the classical representation category of the group, i.e., it might be interpreted as a ‘higher Azumaya algebra’. In particular, this will formally imply that its factorization homology over a topological surface  $S$  forms an invertible sheaf of categories over the classical character variety  $\mathrm{Ch}_G(S)$ .

#### 1.4. Main results: Methods and general results

The proofs of our main results are rooted in a collection of beautiful ideas emerging from the literature on quantum groups and geometric representation theory, most notably the seminal paper [12] where the Hamiltonian reduction of Azumaya algebras in characteristic  $p$  was first carried out for differential operators with applications to Cherednik algebras, and [74], where a  $q$ -analog was developed to study  $q$ -difference operators and double affine Hecke algebras at roots of unity. Similar techniques were used in the study of hypertoric varieties in positive characteristic [68], and of their  $q$ -analogues, quantum multiplicative hypertoric varieties [30].

A significant technical portion of the paper develops a framework generalizing these examples to our setting. Let us therefore review some of the ingredients here, many of which are well-known, and some are new.

*Integral forms of the quantum group.* Let  $G$  be a connected reductive group, with  $\mathfrak{g}$  its Lie algebra, and  $U = U(\mathfrak{g})$  its universal enveloping algebra. For the remainder of the paper we will reserve the letter  $q$  to denote a complex root of unity, and  $\ell$  to denote its order. We will assume  $G$  and  $\ell$  satisfy a number of mild assumptions (see Assumption 3.1).

We shall require several related forms of the quantum group associated to  $\mathfrak{g}$ . Our basic reference is [21]. For us, the quantum group at generic parameters refers to the Drinfeld–Jimbo rational form of the quantum group, defined over the base ring  $\mathcal{K} = \mathbb{C}(t)$  of rational functions in a variable  $t$  with coefficients in  $\mathbb{C}$ ; we will denote this rational form of the quantum group by  $U_{\mathcal{K}}$ . It is generated by the quantum Cartan subalgebra, isomorphic to the group algebra of the coweight lattice of  $G$ , and by the Serre generators  $E_{\alpha}$  and  $F_{\alpha}$  for each positive simple root  $\alpha$ . We do not recall the relations in detail for general  $\mathfrak{g}$  (see instead [21, Chapter 9]), because we will only use some essential functorial properties, which we detail later in this section.

In addition to the rational form of the quantum group, we consider the so-called divided powers integral form<sup>1</sup> of the quantum group, introduced by Lusztig [51], and defined as follows. Let  $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$  denote the subring of  $\mathcal{K}$  consisting of Laurent polynomials in  $t$ , and consider the  $\mathcal{R}$ -subalgebra  $U_{\mathcal{R}}^L$  of  $U_{\mathcal{K}}$  generated by the quantum Cartan generators and an integer family of *divided powers*  $E_{\alpha}^{(r)} = \frac{E_{\alpha}^r}{[r]_{\alpha}!}$  for each Serre generator. Here  $[r]_{\alpha}$  denotes the quantum integer  $\frac{t^{rd_{\alpha}} - t^{-rd_{\alpha}}}{t^{d_{\alpha}} - t^{-d_{\alpha}}}$ ,  $d_{\alpha} = (\alpha, \alpha)/2$  and  $[r]_{\alpha}!$  denotes the quantum factorial  $[r]_{\alpha}[r-1]_{\alpha} \cdots [2]_{\alpha}[1]_{\alpha}$ . We reserve the notation  $U_q^L$  for the base-changed algebra,

$$U_q^L = U_{\mathcal{R}}^L \otimes_{\mathcal{R}} \mathbb{C},$$

via the algebra homomorphism from  $\mathcal{R}$  to  $\mathbb{C}$  given by  $t \mapsto q$ , our chosen root of unity. Finally, we denote by  $u_q$  the small quantum group, which we regard as a subalgebra of  $U_q^L$  generated by the ‘undivided powers’  $E_{\alpha}^{(1)}$ ,  $F_{\alpha}^{(1)}$ , together with the quantum Cartan generators.

---

<sup>1</sup>For simplicity we study integral forms over the ring  $\mathbb{C}[t^{\pm 1}]$  rather than  $\mathbb{Z}[t^{\pm 1}]$ .

**Definition 1.6.** Let  $\text{Rep}_{\mathcal{R}}(G)$ ,  $\text{Rep}_{\mathcal{K}}(G)$ ,  $\text{Rep}_q(G)$ ,  $\text{Rep}(u_q(\mathfrak{g}))$ , and  $\text{Rep}(G)$  denote the categories of locally finite-dimensional modules, respectively, of Lusztig’s integral form  $U_{\mathcal{R}}^L$  (which are inductive limits of submodules which are finitely generated projective over  $\mathcal{R}$ ), the rational form  $U_{\mathcal{K}}$ , its specialization  $U_q^L$ , the small quantum group  $u_q$ , and the classical enveloping algebra  $U$ , such that the weights of the Cartan subalgebra lie in the weight lattice of  $G$ .

*The quantum Frobenius.* An important feature of Lusztig’s integral form is that it admits a homomorphism of Hopf algebras,

$$\text{Fr}: U_q^L \rightarrow U,$$

uniquely defined so that for all simple roots  $\alpha$ , we have

$$\text{Fr}(E_{\alpha}^{(n)}) = \begin{cases} E_{\alpha}^{(n/\ell)}, & \ell \mid n, \\ 0, & \ell \nmid n, \end{cases} \quad \text{Fr}(F_{\alpha}^{(n)}) = \begin{cases} F_{\alpha}^{(n/\ell)}, & \ell \mid n, \\ 0, & \ell \nmid n. \end{cases}$$

The ‘quantum Frobenius’ map  $\text{Fr}$  is a surjective homomorphism of quasi-triangular Hopf algebras, whose kernel is the two-sided ideal generated by the augmentation ideal  $\ker(\varepsilon)$  of the small quantum group  $u_q(\mathfrak{g})$ . Thus, we have Lusztig’s resulting ‘short exact sequence’ of Hopf algebras,

$$u_q \rightarrow U_q^L \xrightarrow{\text{Fr}} U. \quad (1.1)$$

Basic references for the quantum Frobenius include [7, 21, 44–46, 52, 53, 57].

This setup gives rise to a remarkable adjoint pair of braided tensor functors,

$$\text{Fr}^*: \text{Rep}(G) \rightarrow \text{Rep}_q(G), \quad \text{Fr}_*: \text{Rep}_q(G) \rightarrow \text{Rep}(G),$$

the pullback via  $\text{Fr}$  and the passage to  $u_q$ -invariants, respectively. We note that  $\text{Fr}_* \circ \text{Fr}^*$  is the identity functor on  $\text{Rep}(G)$ ; in particular, the functors  $(\text{Fr}^*, \text{Fr}_*)$  form an adjoint pair. The functor  $\text{Fr}^*$  is braided monoidal, and maps into the Müger center of  $\text{Rep}_q(G)$ . That is, for any object  $V$  in  $\text{Rep}(G)$  and any object  $W$  in  $\text{Rep}_q(G)$ , the two braidings,

$$\sigma_{\text{Fr}^*(V), W} \text{ and } \sigma_{W, \text{Fr}^*(V)}^{-1}: \text{Fr}^*(V) \otimes W \rightarrow W \otimes \text{Fr}^*(V),$$

coincide. In fact, each is the switch-of-factors map on the underlying vector space.

*Frobenius Poisson orders.* The notion of a Poisson order, introduced in [18], consists of a non-commutative algebra  $A$ , a central subalgebra  $Z$  of  $A$ , a Poisson bracket on  $Z$ , and a linear map from  $Z$  to  $\text{Der}(A)$ . The general and elegant formalism developed in *loc. cit.* produces isomorphisms between fibers of any two points in the same symplectic core of  $\text{Spec}(Z)$ , in particular on any two points of the open symplectic leaf. For instance, if  $A$  is generically Azumaya over  $Z$ , it is in fact Azumaya over the whole open symplectic leaf. Poisson orders have been applied more recently to the theory of discriminants of PI algebras [19, 58] and Sklyanin algebras [75].

The general method of obtaining Poisson orders is as follows. Suppose  $A_{\mathcal{R}}$  is a family of associative algebras parametrized by  $t \in \mathcal{R} = \mathbb{C}[t, t^{-1}]$  together with a central subal-



gebra  $Z \subset A_q$  at the special value  $t = q$ . Then under a mild assumption (which is in particular satisfied when  $Z$  is the entire center of  $A_q$ , see Proposition 3.23) we get the structure of a Poisson order on the pair  $(A_q, Z)$ .

In our examples,  $A_{\mathcal{R}}$  is a family of algebras in  $\text{Rep}_{\mathcal{R}}(G)$ , so we combine the quantum Frobenius map and the notion of a Poisson order into the notion of a Frobenius Poisson order. Namely, a Frobenius Poisson order consists of an algebra  $A_q \in \text{Rep}_q(G)$ , a Poisson algebra  $Z \in \text{Rep}(G)$  and a central embedding  $\text{Fr}^*(Z) \subset A_q$  such that  $(A_q, Z)$  forms a Poisson order. If  $A_{\mathcal{R}}$  is a flat family of associative algebras in  $\text{Rep}_{\mathcal{R}}(G)$ , and  $Z \in \text{Rep}(G)$  is a central subalgebra at the special value  $t = q$ , then under the same mild assumption  $(A_q, Z)$  forms a Frobenius Poisson order (see Proposition 3.24).

*Frobenius quantum Hamiltonian reduction.* If  $G$  is a Poisson–Lie group and  $\mathcal{Z} = \text{Spec } Z$  is a Poisson  $G$ -variety equipped with a classical moment map  $\mu: \mathcal{Z} \rightarrow G$  (we recall the relevant formalism of group-valued moment maps in Section 2), the affine quotient  $\mathcal{Z} // G = \mu^{-1}(e)/G = \text{Spec } \mathcal{O}(\mu^{-1}(e))^G$  carries a natural structure of a Poisson variety; see Proposition 2.9.

Similarly, if  $A_q$  is an algebra in  $\text{Rep}_q(G)$  equipped with a quantum moment map  $\mu_q: \mathcal{O}_q(G) \rightarrow A_q$  from the reflection equation algebra  $\mathcal{O}_q(G)$  (see Sections 3.3 and 3.4 for the relevant definitions), then we can form the quantum Hamiltonian reduction  $A_q // U_q^L(\mathfrak{g})$ , which is still an associative algebra (see Proposition 3.13).

We are interested in obtaining a Poisson order structure on the Hamiltonian reduction of  $(A_q, Z)$ , so we need to assume the two moment maps are compatible with each other: this leads to the notion of a Frobenius quantum moment map for a Frobenius Poisson order  $(A_q, Z)$ . Namely, we assume that the composite  $\mathcal{O}(G) \subset \mathcal{O}_q(G) \xrightarrow{\mu_q} A_q$  factors through the central subalgebra  $Z \subset A_q$ , which gives rise to a classical moment map  $\mathcal{Z} \rightarrow G$ .

Our first result in this direction is that given a Frobenius Poisson order  $(A_q, Z)$  equipped with such a Frobenius quantum moment map satisfying some extra compatibilities (see Definition 3.20 for the notion of a  $G$ -Hamiltonian Frobenius Poisson order) the Hamiltonian reduction  $(A_q // U_q^L(\mathfrak{g}), \mathcal{Z} // G)$  becomes a Poisson order (see Proposition 3.21).

Our second result in this direction is the following mechanism for obtaining Hamiltonian Frobenius Poisson orders (see Proposition 3.28). Suppose  $A_{\mathcal{R}} \in \text{Rep}_{\mathcal{R}}(G)$  is a flat family of algebras such that the map on invariants  $(A_{\mathcal{R}})^{U_{\mathcal{R}}^L(\mathfrak{g})} \rightarrow (A_q)^{U_q^L(\mathfrak{g})}$  is surjective,  $Z \subset A_q$  is a central subalgebra and  $\mu_{\mathcal{R}}: \mathcal{O}_{\mathcal{R}}(G) \rightarrow A_{\mathcal{R}}$  is a quantum moment map. Then  $(A_q, Z)$  becomes a Hamiltonian Frobenius Poisson order. In examples we check the surjectivity assumption using a good filtration on  $A_{\mathcal{R}}$  (see Section 3.2).

*Distinguished fibers.* In each of our examples there is a distinguished point – corresponding to the trivial local system, and the trivial quiver representation, respectively, where the Azumaya property can be checked directly: for character varieties this proceeds by reducing to the small quantum group, while for quiver varieties it requires a long computation.

*Non-degenerate  $G$ -Hamiltonian varieties.* In order to fully exploit the method of Poisson orders, we require a description of the open symplectic leaf of the framed moduli space.



It turns out that in our examples these can be described purely in terms of the moment map, because both of our examples are *non-degenerate* Poisson  $G$ -varieties as defined in [3]. For framed character varieties, this is proved in *loc. cit.*, while for framed quiver varieties, we use a convenient characterization of non-degeneracy due to Li-Bland–Ševera [48] to prove that they are non-degenerate by a direct computation in coordinates. We show (Theorem 2.15) that given a non-degenerate Poisson  $G$ -variety  $X$  equipped with a moment map  $\mu: X \rightarrow G$ , the open symplectic leaf of  $G$  is given by the preimage  $\mu^{-1}(G^\circ)$  of the big cell  $G^\circ = B_+ B_- \subset G$ .

*Langlands duality at even roots of unity.* Throughout the paper we make simplifying assumptions on  $G$  and the order  $\ell$  of the root of unity  $q$ , e.g. that  $G$  in the semisimple case is of adjoint type and  $\ell$  is odd. This ensures that the Müger center of  $\text{Rep}_q(G)$  is identified with the symmetric monoidal category  $\text{Rep}(G)$  equipped with the obvious braiding. This no longer holds if we relax the assumptions on  $G$  and  $\ell$ .

Let us now assume that  $\ell$  is divisible by 4 and  $G = \text{Spin}(2n + 1)$  for  $n \geq 1$ . Then the Müger center of  $\text{Rep}_q(G)$  coincides with  $\text{Rep}_q({}^L G)$ , where  ${}^L G = \text{PSp}(2n)$ , the adjoint group of type  $C_n$ . Therefore, the quantum character variety for  $G$  then forms a sheaf of algebras over the classical character variety for  ${}^L G$ .

In these cases one still has a factorizable braided monoidal category  $\text{Vect} \otimes_{\text{Rep}({}^L G)} \text{Rep}_q(G)$ , but it is not given by modules over a Hopf algebra. We expect that our approach nevertheless admits a minor modification which would prove that the quantum character variety is Azumaya over the classical character variety for a more general class of  $G$  and  $\ell$ .

In the case of skein algebras, the results of [29] also apply in the case of even order roots of unity, and so we can apply the method of Poisson orders there as well. We hope to return to this in future work as well.

### 1.5. Outline

We briefly describe the contents of this paper. In Section 2, we recall factorizable Poisson–Lie groups (Section 2.1), multiplicative moment maps (Section 2.2), and non-degenerate Poisson  $G$ -varieties (Section 2.3). The main result therein (Theorem 2.15, Section 2.4) is a description of symplectic leaves of a non-degenerate Poisson  $G$ -variety  $X$  as the preimages under the multiplicative moment map  $\mu: X \rightarrow G$  of the  $G^*$ -orbits on  $G$  (i.e., orbits under the dressing action), where  $G^*$  is the Poisson–Lie dual group of  $G$ .

In Section 3, we first consider reflection equation algebras, for generic parameters, for Lusztig’s integral form, and at a root of unity (Section 3.3). We then set up the notions of Frobenius Poisson orders and Frobenius quantum moment maps (Section 3.5), and discuss degenerations of quantum groups and quantum algebras in light of these notions (Section 3.6). We also formulate the procedure of Frobenius quantum Hamiltonian reduction and prove that Azumaya algebras descend to Azumaya algebras under this procedure (Theorem 3.32, Section 3.8).

In Section 4, we apply the techniques developed in Section 3 to the setting of character varieties. After recalling the construction of quantum character varieties and stacks,

we show that the framed character variety, together with its ‘quantum character sheaf’, form a Frobenius Poisson order (Section 4.1). We exhibit a Frobenius quantum moment map in this setting (Section 4.2), and run the procedure of Frobenius quantum Hamiltonian reduction (Section 4.3) to obtain Azumaya algebras over classical affine character varieties (Theorem 4.5).

In Section 5, we turn our attention to multiplicative quiver varieties and their quantizations at a root of unity. We then recall in Section 5.1 the construction of the multiplicative quiver variety, as a Hamiltonian reduction of the framed multiplicative quiver variety. We recall in Section 5.2 the quantization of the framed multiplicative quiver variety, and in Section 5.2 we identify the central subalgebra, establish the existence of good filtrations, and finally construct the Frobenius Poisson order, in Theorem 5.21. In Section 5.4 we show that the resulting Poisson  $G$ -variety is non-degenerate (Theorem 5.23), and in Section 5.5 we prove that the zero representation is an Azumaya point. In Section 5.6, we construct the Frobenius quantum moment map (Theorem 5.31). In Section 5.7, we define the quantized multiplicative quiver variety associated to an arbitrary GIT parameter  $\theta$  via Frobenius quantum Hamiltonian reduction. Finally, in Section 5.8, we bring all the elements together to establish the Azumaya property on the smooth locus of the quantum multiplicative quiver variety (Theorem 1.4).

## 2. Multiplicative Hamiltonian reduction

The goal of this section is to collect some useful results on moment maps for factorizable Poisson–Lie groups. Throughout this section, we assume that  $G$  is an arbitrary connected reductive group. For  $x \in \mathfrak{g}$  we denote by  $x^L, x^R \in T_G$  the left- and right-invariant vector fields with value  $x$  at the unit.

We assume the reader is familiar with the theory of quasi-Poisson groups and quasi-Poisson spaces; see e.g. [2] and [64, Section 4.1] for references. Let us recall that a *Poisson  $G$ -variety* is a  $G$ -variety  $X$  equipped with a Poisson structure such that the action map  $G \times X \rightarrow X$  is Poisson. All varieties we consider in this section are affine.

**Remark 2.1.** Much of the formalism of  $G$ -valued Poisson moment maps developed in this preliminary section constitutes a minor modification of two related and well-known notions – the  $G^*$ -valued moment maps developed in [49], and the quasi-Poisson moment maps developed in [2–4]. We nevertheless include precise statements and selected proofs in our setting for the sake of completeness.

### 2.1. Factorizable Poisson–Lie groups

Fix a non-degenerate element  $c \in \text{Sym}^2(\mathfrak{g})^G$ . Let  $\phi = [c_{12}, c_{13}] \in \bigwedge^3(\mathfrak{g})^G$ . With respect to a basis  $\{e_a\}$  of  $\mathfrak{g}$  we may write

$$[e_a, e_b] = \sum_c f_{ab}^c e_c, \quad c = \sum_{a,b} c^{ab} e_a \otimes e_b, \quad \phi = \frac{1}{6} \sum_{a,b,c,d,e} c^{ae} c^{bd} f_{ed}^c e_a \wedge e_b \wedge e_c.$$

We have the following quasi-Poisson structures:

- Denote by  $G_\phi$  the quasi-Poisson group  $G$  equipped with the zero bivector and trivector  $\phi$ .
- Consider  $G$  as a  $G$ -variety under conjugation and equip it with the bivector

$$\pi_{\text{STS},\phi} = \sum_{a,b} c^{ab} e_a^R \wedge e_b^L.$$

By [3, Proposition 3.1] we get a quasi-Poisson  $G_\phi$ -variety that we denote by  $G_{\text{STS},\phi}$ .

**Definition 2.2.** A *classical twist* is an element  $t \in \bigwedge^2(\mathfrak{g})$  satisfying the equation

$$\frac{1}{2}[t, t] = -\phi.$$

Equivalently, we may say that the classical  $r$ -matrix  $r = t + c$  satisfies the classical Yang–Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

**Example 2.3.** In the case  $\mathfrak{g} = \mathfrak{gl}_N$  we use the classical  $r$ -matrix

$$r = \frac{1}{2} \sum_i \mathcal{E}_i^i \otimes \mathcal{E}_i^i + \sum_{i>j} \mathcal{E}_i^j \otimes \mathcal{E}_j^i,$$

where  $\mathcal{E}_i^j$  is the elementary matrix with 1 in the  $i$ -th row and  $j$ -th column.

Using  $t$  we may perform the following twists:

- Twist the quasi-Poisson structure  $G_\phi$  into a Poisson–Lie structure

$$\pi_{\text{Sk}} = t^L - t^R$$

on  $G$  that we denote by  $G_{\text{Sk}}$ . We call Poisson–Lie structures obtained in this manner *factorizable*.

- Twist the quasi-Poisson  $G_\phi$ -variety  $G_{\text{STS},\phi}$  to a Poisson  $G_{\text{Sk}}$ -variety  $G_{\text{STS}}$  with the Poisson bivector

$$\pi_{\text{STS}} = \pi_{\text{STS},\phi} - t^{\text{ad}},$$

where  $(-)^{\text{ad}}$  is the map  $\bigwedge^\bullet(\mathfrak{g}) \rightarrow \Gamma(G, \bigwedge^\bullet T_G)$  given by differentiating the conjugation action of  $G$  on itself. The Poisson structure  $\pi_{\text{STS}}$  was introduced in [66].

The classical twist gives  $\mathfrak{g}$  the structure of a Lie bialgebra and provides an embedding of Lie algebras  $\mathfrak{g}^* \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ . In particular, the action of  $\mathfrak{g} \oplus \mathfrak{g}$  on  $G$  by left and right translations induces a  $\mathfrak{g}^*$ -action on  $G$  which is known as the  *Dressing action*. Since the diagonal subalgebra  $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$  and  $\mathfrak{g}^* \subset \mathfrak{g} \oplus \mathfrak{g}$  are transverse Lagrangians, the dressing  $\mathfrak{g}^*$ -action on  $G$  is free at the unit  $e \in G$ . In particular, there is an open orbit  $G^\circ \subset G$  of  $e \in G$ . The following is shown in [4, Section 4].

**Proposition 2.4.** *The symplectic leaves of  $G_{\text{STS}}$  are the connected components of the intersections of the conjugacy classes of  $G$  with the dressing  $\mathfrak{g}^*$ -orbits.*

We will assume that the Lie algebra  $\mathfrak{g}^*$  integrates to a group  $G^*$  and that there is a  $\mathfrak{g}$ -action on  $G^*$  such that the map  $G^* \rightarrow G$  is  $\mathfrak{g}$ -equivariant.

**Example 2.5.** Suppose  $G$  is a connected reductive group equipped with a choice of a maximal torus  $T \subset G$  and a Borel subgroup  $B_+ \subset G$  containing it. Let  $c \in \text{Sym}^2(\mathfrak{g})^G$  be a non-degenerate element. Let  $\Delta^+$  be the set of positive roots and pick an orthonormal basis  $\{k_i\}$  of  $\text{Lie}(T)$ . Then  $\{e_\alpha, f_\alpha, k_i\}$  is a basis of  $\mathfrak{g}$ , where  $e_\alpha$  for  $\alpha \in \Delta^+$  is a standard basis of  $\text{Lie}(B_+)$ . Then we have the *standard*  $r$ -matrix

$$r_{\text{std}} = \frac{1}{2} \sum_i k_i \otimes k_i + \sum_{\alpha \in \Delta^+} d_\alpha f_\alpha \otimes e_\alpha.$$

Let  $B_-$  be the opposite Borel subgroup. Denote by  $p_\pm: B_\pm \rightarrow T$  the abelianization maps. Then the dual Poisson–Lie group is given by

$$G^* = \{(b_+, b_-) \in B_+ \times B_- \mid p_+(b_+)p_-(b_-) = 1\}.$$

The open subscheme  $G^\circ \subset G$ , the image of  $G^* \subset G$ , is given by the subscheme  $B_+ B_- \subset G$  which is isomorphic to the big Bruhat cell. Note that in the case  $G = \text{GL}_n$  we have  $Z(\text{GL}_n) \subset G^\circ$ .

## 2.2. Multiplicative moment maps

Fix a factorizable Poisson–Lie structure on  $G$ . Pick a basis  $\{e_i\}$  of  $\mathfrak{g}$  and let  $\{e^i\}$  be the dual basis of  $\mathfrak{g}^*$ . Denote by  $\tilde{a}: \mathfrak{g}^* \rightarrow \Gamma(G, T_G)$  the dressing  $\mathfrak{g}^*$ -action on  $G$ .

**Definition 2.6.** Let  $X$  be a Poisson  $G$ -variety. A  $G$ -equivariant map  $\mu: X \rightarrow G$  is a *moment map* if for every  $h \in \mathcal{O}(G)$  and  $a \in \mathcal{O}(X)$  we have

$$\{\mu(h), a\} = \sum_i \mu(\tilde{a}(e^i).h) e_i.a. \quad (2.1)$$

In this case we say that  $X$  is a *Hamiltonian  $G$ -Poisson variety*.

**Proposition 2.7.** Suppose  $X$  is a Poisson  $G$ -variety and  $\mu: X \rightarrow G$  is a moment map. Then  $\mu: X \rightarrow G_{\text{STS}}$  is a Poisson morphism.

*Proof.* By [64, Proposition 4.18], our definition of moment maps agrees with the definition of Alekseev and Kosmann-Schwarzbach [2]. Now, let  $X'$  be the quasi-Poisson  $G_\phi$ -variety obtained by twisting  $X$  using  $-t \in \wedge^2(\mathfrak{g})$ . Then by [3, Proposition 3.3] the map  $\mu: X' \rightarrow G_{\text{STS}, \phi}$  is quasi-Poisson and hence after twisting back we find that  $X \rightarrow G_{\text{STS}}$  is Poisson.  $\blacksquare$

Using a moment map we may construct a Hamiltonian reduction of  $X$ .

**Definition 2.8.** Let  $\xi \in G$  be an element in the center and  $X$  a Poisson  $G$ -variety equipped with a moment map  $\mu: X \rightarrow G$ . The *Hamiltonian reduction* is

$$X //_{\xi} G = \text{Spec } \mathcal{O}(\mu^{-1}(\xi))^G.$$

Hamiltonian reduction carries a natural Poisson structure constructed in the following way. Let  $I \subset \mathcal{O}(X)$  be the ideal defining  $\mu^{-1}(\xi)$ . Since  $G$  is reductive, the natural morphism

$$\mathcal{O}(X)^G/I^G \rightarrow (\mathcal{O}(X)/I)^G$$

is an isomorphism. So, it is enough to construct a Poisson structure on  $\mathcal{O}(X)^G/I^G$ .

**Proposition 2.9.**  $\mathcal{O}(X)^G \subset \mathcal{O}(X)$  is a Poisson subalgebra and  $I^G \subset \mathcal{O}(X)^G$  is a Poisson ideal.

Therefore, we obtain a natural Poisson structure on  $\mathcal{O}(X)^G/I^G$ , i.e.,  $X//G$  is a Poisson variety.

### 2.3. Non-degenerate Poisson $G$ -varieties

Let  $a: \mathfrak{g} \rightarrow \Gamma(X, T_X)$  be the  $\mathfrak{g}$ -action on a  $G$ -variety  $X$ . The following was introduced in [3, Definition 9.1].

**Definition 2.10.** Let  $G$  be a quasi-Poisson group and  $(X, \pi_X)$  a smooth quasi-Poisson  $G$ -variety.  $X$  is *non-degenerate* if the map

$$(a, \pi_X^\#): \mathfrak{g} \oplus T_X^* \rightarrow T_X$$

is surjective.

**Remark 2.11.** Clearly, if  $X$  is a Poisson  $G$ -variety, the locus where the Poisson structure is symplectic is contained in the non-degeneracy locus.

**Lemma 2.12.** Let  $G$  be a quasi-Poisson group,  $(X, \pi_X)$  a quasi-Poisson  $G$ -variety, and  $t \in \bigwedge^2(\mathfrak{g})$ . Then  $(X, \pi_X)$  is non-degenerate if and only if its twist  $(X, \pi_X - a(t))$  is non-degenerate.

*Proof.* Indeed, the images of  $\pi_X^\#$  and  $\pi_X^\# - a(t)^\#$  differ by the image of  $a$ . ■

We have the following convenient criterion to determine whether a Poisson  $G$ -variety is non-degenerate, proven in [48, Theorem 3].

**Proposition 2.13.** Let  $r$  be a classical  $r$ -matrix defining a factorizable Poisson–Lie structure on  $G$  and suppose  $(X, \pi_X)$  is a Poisson  $G$ -space equipped with a moment map  $\mu: X \rightarrow G$ . Denote  $\tilde{\pi}_X = \pi_X + a(r) \in \Gamma(X, T_X \otimes T_X)$  and  $\tilde{\pi}_X^\#: T_X^* \rightarrow T_X$  the induced map. Then  $(X, \pi_X)$  is non-degenerate if and only if  $\tilde{\pi}_X^\#: T_X^* \rightarrow T_X$  is an isomorphism.

Let us now show that non-degeneracy is preserved by fusion. Consider the element

$$\psi = \sum_{a,b} r^{ab} e_a^1 \wedge e_b^2 \in \bigwedge^2(\mathfrak{g} \oplus \mathfrak{g}),$$

where  $r$  is the classical  $r$ -matrix.

**Proposition 2.14.** *Suppose  $(X, \pi_X)$  is a Poisson  $G \times G$ -variety and consider its fusion  $X_{\text{fus}} = (X, \pi_X - a(\psi))$ .*

- (1)  *$X_{\text{fus}}$  is a Poisson  $G$ -variety with respect to the diagonal  $G$ -action.*
- (2) *Suppose  $\mu_1 \times \mu_2: X \rightarrow G \times G$  is a moment map. Then  $\mu_{\text{fus}} = \mu_1 \mu_2: X_{\text{fus}} \rightarrow G$  is a moment map for the diagonal  $G$ -action.*
- (3) *Suppose  $X$  is equipped with a moment map and is non-degenerate. Then  $X_{\text{fus}}$  is non-degenerate.*

We omit the proof, which is standard.

#### 2.4. Symplectic leaves of non-degenerate Poisson $G$ -varieties

Non-degenerate Poisson  $G$ -varieties equipped with moment maps have an easy description of symplectic leaves. Let  $a_G: \mathfrak{g} \rightarrow \Gamma(G, T_G)$  be the adjoint action.

**Theorem 2.15.** *Let  $X$  be a non-degenerate Poisson  $G$ -variety equipped with a moment map  $\mu: X \rightarrow G$ . Then the symplectic leaves of  $X$  are given by the connected components of the preimages of the dressing  $\mathfrak{g}^*$ -orbits  $G_\alpha \subset G$ .*

We believe this result to be well-known to experts, but we have not found a satisfactory reference, so we include a complete proof.

*Proof of Theorem 2.15.* The moment map equation is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} T_{X,x}^* & \xrightarrow{-\pi_X^\#} & T_{X,x} \\ a^* \downarrow & & \downarrow d\mu_x \\ \mathfrak{g}^* & \xrightarrow{\tilde{a}} & T_{G,\mu(x)} \end{array}$$

In particular, symplectic leaves of  $X$  are contained in the preimages  $\mu^{-1}(G_\alpha)$ . Since  $X$  is non-degenerate, it is enough to prove that if  $y \in \mathfrak{g}$  is such that  $a_G(y)$  is in the image of  $\tilde{a}$ , then  $a(y)$  is in the image of  $\pi_X^\#$ .

By Proposition 2.7 the moment map  $\mu: X \rightarrow G_{\text{STS}}$  is Poisson, so we have a commutative diagram

$$\begin{array}{ccc} T_{X,x}^* & \xrightarrow{\pi_X^\#} & T_{X,x} \\ \mu^* \uparrow & & \downarrow d\mu_x \\ T_{G,\mu(x)}^* & \xrightarrow{\pi_{\text{STS}}^\#} & T_{G,\mu(x)} \end{array}$$

Since  $a_G(y)$  is in the image of  $\tilde{a}$ , by Proposition 2.4 the element  $a_G(y)$  lies in the image of  $\pi_{\text{STS}}^\#: T_{G,\mu(x)}^* \rightarrow T_{G,\mu(x)}$  and hence  $a(y)$  is in the image of  $\pi_X^\#$  by the above commutative diagram. ■

In particular, under the above assumptions  $\mu^{-1}(G^\circ) \subset X$  is an open symplectic leaf. We will now use this observation to show that the Hamiltonian reduction of a non-degenerate Poisson  $G$ -variety is symplectic. Denote by  $\pi: \mu^{-1}(\xi) \rightarrow X // G$  the projection.

**Proposition 2.16.** *Let  $X$  be a non-degenerate Poisson  $G$ -variety equipped with a moment map  $\mu: X \rightarrow G$  and  $\xi \in G^\circ \subset G$  a point lying in the center. Suppose  $U \subset \mu^{-1}(\xi)$  and  $V \subset X // G$  are open subsets such that  $\pi: U \rightarrow V$  is a  $G$ -torsor. Then  $V \subset X // G$  is a smooth symplectic variety.*

We omit the proof, which is standard.

In most examples we will consider the  $G$ -action on  $X$  has a generic stabilizer, so let us explain how to deal with that.

**Proposition 2.17.** *Let  $G$  be a factorizable Poisson–Lie group,  $X$  a Poisson  $G$ -variety, and  $G_{\text{stab}} \subset G$  a normal subgroup which acts trivially on  $X$ . Assume the pairing on  $\mathfrak{g}$  restricts to a non-degenerate pairing on  $\mathfrak{g}_{\text{stab}}$ . Denote  $\bar{G} = G/G_{\text{stab}}$ .*

- (1) *If  $\mu: X \rightarrow G$  is a moment map for the  $G$ -action on  $X$ , then  $\bar{\mu}: X \rightarrow G \rightarrow \bar{G}$  is a moment map for the  $\bar{G}$ -action on  $X$ .*
- (2) *If  $X$  is a non-degenerate Poisson  $G$ -variety, then it is a non-degenerate Poisson  $\bar{G}$ -variety.*

*Proof.* (1) Let  $r \in \mathfrak{g} \otimes \mathfrak{g}$  be the classical  $r$ -matrix defining the factorizable Poisson–Lie structure on  $G$  and denote by  $\bar{r} \in \bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}$  its image. Then it is still a classical  $r$ -matrix and by assumption its symmetric part is non-degenerate. Therefore,  $\bar{G}$  inherits the structure of a factorizable Poisson–Lie group.

Moreover, the natural diagram

$$\begin{array}{ccc} \bar{\mathfrak{g}}^* & \longrightarrow & \bar{\mathfrak{g}} \oplus \bar{\mathfrak{g}} \\ \downarrow & & \uparrow \\ \mathfrak{g}^* & \longrightarrow & \mathfrak{g} \oplus \mathfrak{g} \end{array}$$

is commutative. Therefore,  $\mathcal{O}(\bar{G}) \rightarrow \mathcal{O}(G)$  is  $\bar{\mathfrak{g}}$ -equivariant, which implies the first claim.

(2) The image of  $\bar{\mathfrak{g}} \rightarrow \Gamma(X, T_X)$  coincides with the image of  $\mathfrak{g} \rightarrow \Gamma(X, T_X)$  which implies the second claim. ■

Recall that the usual moment map on a symplectic  $G$ -variety is unique up to addition of a character. A similar statement is true for multiplicative moment maps on non-degenerate Poisson  $G$ -varieties.

**Proposition 2.18.** *Let  $X$  be a connected non-degenerate Poisson  $G$ -variety and let  $\mu_1, \mu_2: X \rightarrow G$  be moment maps. Then there is an element  $g \in G^*$  landing in the center  $Z(G) \subset G$  under the map  $G^* \rightarrow G$  such that  $\mu_2(x) = g\mu_1(x)$ .*



*Proof.* Consider an open dense subset  $U = \mu_1^{-1}(G^\circ) \cap \mu_2^{-1}(G^\circ) \subset X$ . It will be enough to establish the claim on this open subset.

We have  $\mu_1, \mu_2: U \rightarrow G^\circ$ . We may choose a connected étale cover  $\tilde{U} \rightarrow U$  and lift the moment maps as

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{\mu}_i} & G^* \\ \downarrow & & \downarrow \\ U & \xrightarrow{\mu_i} & G^\circ \end{array}$$

Let  $\theta \in \Omega^1(G^*; \mathfrak{g}^*)$  be the left-invariant Maurer–Cartan form on  $G^*$ . It is shown in [2, Example 5.1.3] (see also [64, Lemma 4.22]) that the moment map equation (2.1) reduces to

$$a(v) = \pi_{\tilde{U}}^\#(\tilde{\mu}_i^*(\theta, v)),$$

where  $v \in \mathfrak{g}$  and  $a: \mathfrak{g} \rightarrow \Gamma(\tilde{U}, T_{\tilde{U}})$  is the action map. By Theorem 2.15,  $\tilde{U}$  is symplectic, so the above equation implies that

$$\tilde{\mu}_1^*(\theta, v) = \tilde{\mu}_2^*(\theta, v).$$

If we write  $\tilde{\mu}_2(x) = g(x)\tilde{\mu}_1(x)$  for some function  $g: \tilde{U} \rightarrow G^*$ , then the above equation implies that  $g$  is locally constant. Since  $\tilde{U}$  is connected,  $g$  is constant. Therefore,  $\mu_2(x) = g\mu_1(x)$  for some  $g \in G^*$ . Since both  $\mu_1$  and  $\mu_2$  are  $G$ -equivariant and the map  $G^* \rightarrow G$  is  $\mathfrak{g}$ -equivariant, the image of  $g \in G^*$  under  $G^* \rightarrow G$  is invariant under conjugation, i.e., it lies in the center  $Z(G) \subset G$ . ■

### 3. Frobenius compatibilities

In this section we spell out a number of compatibility conditions between the quantum Frobenius homomorphism and each of the following: quantum moment maps, Poisson orders, and Hamiltonian reductions. We use these compatibilities to sheafify the quantum Hamiltonian reduction over the classical multiplicative Hamiltonian reduction, and show that the reduction procedure preserves the Azumaya property on fibers. Throughout this section we fix a connected reductive group  $G$  and a primitive  $\ell$ -th root of unity  $q$ .

#### 3.1. Factorizability of the small quantum group

**Assumption 3.1.** *The group  $G$  and the number  $\ell$  satisfy the following conditions:*

- $\ell$  is odd.
- $G$  is a product of groups of the following kinds:  $\mathrm{GL}_n$  and a simple group whose determinant of the Cartan matrix is coprime to  $\ell$ .
- If  $G$  contains a factor of type  $G_2$ , then  $\ell$  is not divisible by 3.

The reason we consider these restrictions is due to the following statement.

**Theorem 3.2.** *Suppose  $(G, \ell)$  satisfy Assumption 3.1. Then there is an equivalence between  $\text{Rep}(G)$  and the Müger center of  $\text{Rep}_q(G)$  such that the functor  $\text{Rep}(G) \rightarrow \text{Rep}_q(G)$  is ribbon. Moreover, there is a factorizable Hopf algebra  $u_q(\mathfrak{g})$ , the small quantum group, together with an equivalence of ribbon categories  $\text{Rep}(u_q(\mathfrak{g})) \cong \text{Vect} \otimes_{\text{Rep}(G)} \text{Rep}_q(G)$ .*

*Proof.* Let  $X$  be the weight lattice of  $G$  and let  $\alpha_i \in X$  be the simple roots. Fix a  $\mathbb{Q}$ -valued symmetric bilinear pairing on  $X$  such that  $2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = a_{ij}$  is the Cartan matrix. We normalize  $(-, -)$  so that the norm squared of a short root is 2. Let

$$d_i = (\alpha_i, \alpha_i)/2.$$

In general the pairing  $(-, -)$  is valued in  $\frac{1}{D}\mathbb{Z}$ , where  $D$  divides the determinant of the Cartan matrix. By assumptions  $\ell$  is coprime to  $D$ . Thus, by the Chinese remainder theorem we may choose  $v \in \mathbb{C}$  such that  $v^D = q$  and  $v^\ell = 1$ . For two weights  $\mu, \nu \in X$  we set by definition  $q^{(\mu, \nu)} = v^{D(\mu, \nu)}$ . The braiding  $\sigma_{W_1, W_2}$  for  $W_1, W_2 \in \text{Rep}_q(G)$  is given by

$$\sigma_{W_1, W_2}(w_1 \otimes w_2) = q^{-(\mu, \nu)} \Theta(w_1 \otimes w_2),$$

where  $\Theta$  is the so-called quasi  $R$ -matrix (see [53, Chapter 4]) and  $w_i \in W_i$  are vectors of weights  $\mu, \nu \in X$  respectively.

The simple objects in  $\text{Rep}_q(G)$  are parametrized by dominant highest weights  $\lambda \in X$ . It is shown in [57, Theorem 4.3] that the Müger center is the tensor category generated by simples with highest weight in the sublattice  $X^M \subset X$  defined in the following way:

$$X^M = \{\lambda \in X \mid 2D(\lambda, \mu) \in \ell\mathbb{Z} \ \forall \mu \in X\}.$$

Suppose  $X^M = \ell X$ . Then Lusztig's quantum Frobenius map [53, Chapter 35] defines a monoidal functor  $\text{Rep}(G) \rightarrow \text{Rep}_q(G)$  where objects in  $\text{Rep}(G)$  have weights in the lattice  $X^M$  which by the above result of [57] lands in the Müger center.

We note that  $\text{Fr}^*$  is fully faithful by its construction as a pullback along an algebra homomorphism, so we need only identify its image with the Müger center. From the explicit formula for the quasi  $R$ -matrix  $\Theta$  it is clear that it annihilates all representations of the form  $\text{Fr}^*(V)$ . Therefore,  $\text{Fr}^*: \text{Rep}(G) \rightarrow \text{Rep}_q(G)$  is a braided monoidal functor iff  $q^{(\mu, \nu)} = 1$  for all  $\mu, \nu \in X^M$ . But  $\ell$  is coprime to  $2D$ , so  $2D(\lambda, \mu) \in \ell\mathbb{Z}$  implies that  $(\lambda, \mu) \in \ell\mathbb{Z}$  for all  $\lambda, \mu \in X^M$ . The fact that under the condition  $X^M = \ell X$  the functor  $\text{Fr}^*: \text{Rep}(G) \rightarrow \text{Rep}_q(G)$  is ribbon is proved as in [57, Lemma 6.1].

We will now check the conditions  $X^M = \ell X$  in our cases.

- Suppose  $G = \text{GL}_n$ . The weight lattice is  $X = \mathbb{Z}^n$  with generators  $\{e_i\}$  such that  $(e_i, e_j) = \delta_{ij}$ . The simple roots are  $\alpha_i = e_i - e_{i+1}$  for  $1 \leq i < n$ . Let  $\lambda = \sum n_i e_i$ . Then  $2(\lambda, e_j) = 2n_j$ . So,  $X^M = \ell X$ .
- Suppose  $G$  is a simple simply-connected group. Then the weight lattice  $X$  is generated by the fundamental weights  $\omega_i$ . Let  $b_{ij}$  be the inverse of the Cartan matrix  $a_{ij}$ . Consider

$$\Omega_{ij} = (\omega_i, \omega_j) = \sum_{k, l} b_{ik} b_{jl} d_k a_{kl}.$$

Let  $\Omega_{ij}^{-1}$  be its inverse. Note that if  $G$  is simply-laced (i.e.,  $d_k = 1$ ), we get  $\Omega_{ij} = b_{ij}$  and  $\Omega_{ij}^{-1} = a_{ij}$ .

Consider  $\lambda = \sum_i n_i \omega_i$ . Then  $\lambda \in X^M$  iff

$$2(\lambda, \omega_j) = 2 \sum_i n_i \Omega_{ij} = \ell m_j$$

for some integers  $m_j$ , i.e.,

$$n_i = \frac{\ell}{2} \sum_j \Omega_{ij}^{-1} m_j.$$

By an explicit calculation, one sees that  $\Omega_{ij}^{-1} \in \frac{1}{4}\mathbb{Z}$  if  $G$  is not of type  $G_2$  and  $\Omega_{ij}^{-1} \in \frac{1}{3}\mathbb{Z}$  if  $G$  is of type  $G_2$ . In these cases  $\ell$  is coprime to the denominators of  $\Omega_{ij}^{-1}$ , and hence  $n_i$  is divisible by  $\ell$ , i.e.,  $X^M \subset \ell X$ .

Since  $\ell$  is coprime to  $D$ ,  $2D\ell(\lambda, \mu)$  is divisible by  $\ell$  for all  $\lambda, \mu \in X$ . Therefore,  $\ell X \subset X^M$ .

• Suppose  $G$  is an arbitrary simple group. The inclusion  $\ell X \subset X^M$  is proved identically to the simply-connected case.

Now consider the symmetric pairing  $A: X/\ell X \otimes X/\ell X \rightarrow \mathbb{Z}/\ell\mathbb{Z}$  given by

$$A(\lambda, \mu) = 2D(\lambda, \mu),$$

which is well-defined due to the inclusion  $\ell X \subset X^M$ . Then  $X^M = \ell X$  iff  $A$  is non-degenerate.

Let  $X_{\text{sc}} \supset X$  be the lattice spanned by the fundamental weights. The index of  $X \subset X_{\text{sc}}$  divides  $D$  and hence is coprime to  $\ell$  by assumption. Therefore, the inclusion  $X/\ell X \rightarrow X_{\text{sc}}/\ell X_{\text{sc}}$  is an isomorphism. This means that the pairing  $A$  is non-degenerate for  $X$  iff  $A$  is non-degenerate for  $X_{\text{sc}}$ , which we have already checked above.

The equivalence  $\text{Vect} \otimes_{\text{Rep}(G)} \text{Rep}_q(G) \cong \text{Rep}(u_q(\mathfrak{g}))$  of ribbon categories is the content of [6, 7] and [57, Section 5]. ■

Note that some results related to the previous theorem were previously obtained in [62], [70, Chapter XI.6.3], [46]. The case of even  $\ell$  is considered in [57].

### 3.2. Good filtrations

We will require at a few places the notion of a good filtration, also known as a dual-Weyl or co-Weyl filtration on a  $U_q^{\mathbb{L}}(\mathfrak{g})$ -module. The reader may safely skip this material at a first read, and may consult [36, 59] for further details.

Let  $U_{\mathcal{R}}^{\mathbb{L}}(\mathfrak{b})$  denote the quantum Borel subalgebra generated by the  $E_{\alpha}^{(k)}$ 's and the quantum Cartan subalgebra.

**Definition 3.3.** Let  $\lambda$  denote a dominant integral weight, regarded as a character of  $U_{\mathcal{R}}^{\mathbb{L}+}(\mathfrak{g})$ .

- (1) The *Verma module*  $M(\lambda)$  is the  $U_{\mathcal{R}}^L$ -module induced from  $U_{\mathcal{R}}^L(\mathfrak{b})$  to  $U_{\mathcal{R}}^L(\mathfrak{g})$ , i.e.

$$M(\lambda) = U_{\mathcal{R}}^L(\mathfrak{g}) \otimes_{U_{\mathcal{R}}^L(\mathfrak{b})} \mathbb{C}\lambda.$$

- (2) The *Weyl module*  $\Delta(\lambda)$  is the universal finitely generated  $\mathcal{R}$ -module quotient of  $M(\lambda)$ , i.e., for any finitely other generated  $\mathcal{R}$ -module quotient of  $M(\lambda) \rightarrow V$ , we have a unique factorization  $M(\lambda) \rightarrow \Delta(\lambda) \rightarrow V$ .

- (3) The *dual Weyl module*  $\nabla(\lambda)$  is the  $U_{\mathcal{R}}^L(\mathfrak{g})$ -module coinduced from  $U_{\mathcal{R}}^L(\mathfrak{b})$  to  $U_{\mathcal{R}}^L(\mathfrak{g})$ , i.e.,

$$\nabla(\lambda) = \text{Hom}_{U_{\mathcal{R}}^L(\mathfrak{b})}(M(0), \lambda),$$

with left  $U_{\mathcal{R}}^L(\mathfrak{g})$ -action obtained by precomposing the evident right  $U_{\mathcal{R}}^L(\mathfrak{g})$ -action with the antipode  $S$ .

- (4) A *Weyl filtration* on a  $U_{\mathcal{R}}^L(\mathfrak{g})$ -module  $M$  is a filtration  $F_{\bullet}M$  with successive quotients  $F_i/F_{i-1}$  isomorphic to some  $\Delta(\lambda)$ .
- (5) A *good* (or *dual-Weyl*) filtration on a  $U_{\mathcal{R}}^L(\mathfrak{g})$ -module  $M$  is a filtration  $F_{\bullet}M$  with successive quotients  $F_i/F_{i-1}$  isomorphic to some  $\nabla(\lambda)$ .

**Proposition 3.4** ([59], see also [74, Section A.9]). (1) *We have an isomorphism  $\nabla(\lambda)^* \cong \Delta(-\omega_0\lambda)$ .*

- (2) *The tensor product of finite-dimensional modules with good (resp. Weyl) filtration admits a good (resp. Weyl) filtration.*
- (3) *If  $V_{\mathcal{R}}$  is a module with good filtration, then the specialization map  $(V_{\mathcal{R}})^{U_{\mathcal{R}}^L} \rightarrow (V_q)^{U_q^L}$  is a surjection.*

### 3.3. Reflection equation algebras

In the geometric representation theory of quantum groups, an important role is played by a canonical ad-equivariant quantization of the coordinate algebra of  $G$ , the so-called ‘reflection equation’ algebra. In particular, the reflection equation algebra is the domain for quantum moment maps, so its behavior at a root of unity will be important for the notion of Frobenius quantum moment maps and their Hamiltonian reduction.

**3.3.1. Generic parameters.** Before discussing the somewhat more complicated behavior of the reflection equation algebra at a root of unity, we recall the situation for generic parameters. Specifically, we discuss the relations between the three equivalent constructions of the rational form  $\mathcal{O}_t(G)$  of the reflection equation algebra: (1) the functorial/co-end construction of Lyubashenko–Majid [54], (2) the matrix coefficient/braided dual formulation of Donin–Kulish–Mudrov [25], and finally (3) the identification with a subalgebra of the quantized enveloping algebra of ad-locally finite elements, via the so-called Rosso isomorphism, due to Joseph–Letzter and Rosso [39, 61]. We follow the exposition of [10].

The functorial/co-end construction works for any presentable rigid braided tensor category  $\mathcal{C}$ . Let  $T : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}$  be the tensor multiplication functor, which we will always

assume to be cocontinuous. The braiding on  $\mathcal{C}$  equips  $T$  moreover with the structure of a tensor functor, while the cocontinuity of  $T$  endows it with a right adjoint  $T^R$ , which is lax monoidal. It follows that the associated co-end,  $TT^R(\mathbf{1}_{\mathcal{C}})$ , is a bialgebra object in  $\mathcal{C}$ . Explicitly, it may be realized as a quotient of the sum of  $V^* \otimes V$ , where  $V$  ranges over all objects of  $\mathcal{C}$ . In addition, for any object  $W \in \mathcal{C}$  there is a ‘field-goal’ isomorphism [11, Corollary 4.6]

$$\tau_W: TT^R(\mathbf{1}_{\mathcal{C}}) \otimes W \xrightarrow{\sim} W \otimes TT^R(\mathbf{1}_{\mathcal{C}})$$

given diagrammatically by

$$\tau_W = \begin{array}{c} W \quad V^* \quad V \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \\ V^* \quad V \quad W \end{array}. \quad (3.1)$$

When  $\mathcal{C}$  is the category of locally finite modules for a quasi-triangular Hopf algebra  $H$ , then classical results of Majid–Lyubashenko [54] identify the co-end algebra with the so called *braided dual*  $\tilde{H}$ . This is the subalgebra of  $H^*$  spanned by matrix coefficients of finite-dimensional  $H$ -modules, with a natural multiplication structure which invokes the braiding on  $\mathcal{C}$ , hence the universal  $R$ -matrix for  $H$ .

When  $H = U_{\mathcal{K}}(\mathfrak{g})$  is the rational form of the quantum group (see Section 1.4 above),  $\mathcal{C}$  is the category  $\text{Rep}_{\mathcal{K}}(G)$  of locally finite modules for the quantum group  $U_{\mathcal{K}}$ , and the co-end/braided dual algebra is denoted  $\mathcal{O}_{\mathcal{K}}(G)$ . Regarded as an algebra in vector spaces equipped with a compatible  $U_{\mathcal{K}}$ -action, the algebra  $\mathcal{O}_{\mathcal{K}}(G)$  is spanned by the matrix coefficients of irreducible representations, and admits a Peter–Weyl decomposition,

$$\mathcal{O}_{\mathcal{K}}(G) \cong \bigoplus_{\lambda} V_{\lambda}^* \otimes V_{\lambda},$$

where the sum is over the dominant integral weights.

If we take  $H = U_{\mathcal{R}}^L(\mathfrak{g})$ , Lusztig’s integral form, we have a natural coaction  $\Delta: \mathcal{O}_{\mathcal{R}}(G) \rightarrow U_{\mathcal{R}}^L(\mathfrak{g}) \otimes \mathcal{O}_{\mathcal{R}}(G)$  realizing  $\mathcal{O}_{\mathcal{R}}(G)$  as a  $U_{\mathcal{R}}^L(\mathfrak{g})$ -comodule algebra (see e.g. [43]). Applying the natural evaluation homomorphism  $\mathcal{O}_{\mathcal{R}}(G) \rightarrow \mathcal{R}$  to the second factor, we obtain the Rosso homomorphism  $\mathcal{O}_{\mathcal{R}}(G) \rightarrow U_{\mathcal{R}}^L(\mathfrak{g})$  [39]. For  $h \in \mathcal{O}_{\mathcal{R}}(G)$  we denote the coaction by  $\Delta(h) = h_{(1)} \otimes h_{(2)}$ . Then the ‘field-goal’ isomorphism for  $v \in V$  is given by

$$\tau_V(h \otimes v) = (h_{(1)} \triangleright v) \otimes h_{(2)}.$$

Finally, let us recall that at generic parameters, the category  $\text{Rep}_{\mathcal{K}}(G)$  is *factorizable*. This implies, among other things, that the map  $\mathcal{O}_{\mathcal{K}}(G) \rightarrow U_{\mathcal{K}}(\mathfrak{g})$  is injective, embedding  $\mathcal{O}_{\mathcal{K}}(G)$  as a subalgebra of ad-locally finite elements of the quantum group. We note that this is a strong self-duality property, which holds only for  $t$  generic; the failure of this property at roots of unity, and the precise way in which it fails, will be an important point for what follows.

**3.3.2. Root of unity parameters.** Let us now consider the divided powers quantum group  $U_q^L(\mathfrak{g})$ , specialized to an odd root of unity,  $q^\ell = 1$  with  $\ell > 1$ . We denote by  $\mathcal{O}_q(G)$  the

co-end algebra of  $\text{Rep}_q(G)$ , defined functorially as in the generic case. Two very special phenomena occur at this specialization.

The first is that  $\mathcal{O}_q(G)$  develops a large  $\ell$ -center, in addition to its Harish-Chandra center. This is best understood as a consequence of the quantum Frobenius homomorphism, as follows. Firstly, the natural tensor structure on  $\text{Fr}^*$  induces an isomorphism  $\text{Fr}^* \circ T \cong T \circ (\text{Fr}^* \boxtimes \text{Fr}^*)$  (where we abuse notation and write  $T$  for tensor multiplication in both tensor categories). Recalling the identifications  $\mathcal{O}(G) = TT^R(\mathbf{1}_{\text{Rep}(G)})$ ,  $\mathcal{O}_q(G) = TT^R(\mathbf{1}_{\text{Rep}_q(G)})$  and finally  $\text{Fr}^*(\mathbf{1}_{\text{Rep}(G)}) = \mathbf{1}(\text{Rep}_q(G))$ , we immediately obtain a bialgebra monomorphism in  $\text{Rep}_q(G)$ ,

$$i : \text{Fr}^*(\mathcal{O}(G)) \hookrightarrow \mathcal{O}_q(G).$$

Moreover, because the image of  $\text{Fr}^*$  lies in the Müger center of  $\text{Rep}_q(G)$ , it follows that  $\text{Fr}^*(\mathcal{O}(G))$  is a central subalgebra of  $\mathcal{O}_q(G)$ . It is therefore natural to view  $\mathcal{O}_q(G)$  as a sheaf of algebras over  $G$ .

**Remark 3.5.** Let us note in passing that, while the above formulation gives a simple monadic construction of the algebra monomorphism  $i$ , which indeed sends the Peter–Weyl component  $V^* \otimes V$  to  $\text{Fr}^*(V) \otimes \text{Fr}^*(V)$ , it is rather difficult to express it directly in elementary generators-and-relations terms (for example, see [22, Remarks 3.13, 3.14]). This monadic formulation of the homomorphism  $i$  is exploited repeatedly in Section 4.

The second special phenomenon at a root of unity is that factorizability breaks down. For this reason, the Rosso map  $\mathcal{O}_q(G) \rightarrow U_q^L(\mathfrak{g})$  is not injective. Rather, it factors as a surjection onto the small quantum group  $u_q(\mathfrak{g}) \subset U_q^L(\mathfrak{g})$ , and its kernel is the augmentation ideal of  $\mathcal{O}(G)$ .

In a similar way, the  $U_q^L(\mathfrak{g})$ -coaction on  $\mathcal{O}_q(G)$  is induced from a  $u_q(\mathfrak{g})$ -coaction on  $\mathcal{O}_q(G)$ . Moreover, the  $u_q(\mathfrak{g})$ -coaction on  $\mathcal{O}(G) \subset \mathcal{O}_q(G)$  is trivial. So, for a point  $\xi \in G$  we have a  $u_q(\mathfrak{g})$ -comodule algebra

$$o_q(G; \xi) = \mathcal{O}_q(G) \otimes_{\mathcal{O}(G)} \mathbb{C},$$

where the  $\mathcal{O}(G)$ -action on  $\mathbb{C}$  is via the character  $\xi^*: \mathcal{O}(G) \rightarrow \mathbb{C}$ .

**Remark 3.6.** We note that for any representation  $V$  of  $U_q^L(\mathfrak{g})$ , the representation  $V^* \otimes V$  has weights entirely in the root lattice. In particular, since  $\mathcal{O}_q(G)$  is a co-end of such representations, its weights are also contained in the root lattice.

We record the following important modification of the Peter–Weyl theorem.

**Theorem 3.7** ([36, 59]). *The algebra  $\mathcal{O}_q(G)$  admits a good filtration with successive quotients*

$$\nabla(-\omega_0\lambda) \otimes \nabla(\lambda).$$

*Proof.* In the parallel setting of algebraic groups defined in characteristic  $p$ , this is proved in [36, Proposition 4.20]. Following the discussion in [59], the proof is identical for the quantum group at a root of unity. ■

### 3.4. Quantum moment maps

Let  $H$  be a Hopf algebra over a commutative ring  $k$  and  $H'$  a  $k$ -algebra equipped with an  $H$ -comodule and an  $H$ -module structure which preserve the algebra structure. We will be interested in the following three cases:

- (1)  $k = \mathcal{R}$ ,  $H = U_{\mathcal{R}}^L(\mathfrak{g})$  and  $H' = \mathcal{O}_{\mathcal{R}}(G)$ .
- (2)  $k = \mathbb{C}$ ,  $H = U_q^L(\mathfrak{g})$  and  $H' = \mathcal{O}_q(G)$ .
- (3)  $k = \mathbb{C}$ ,  $H = u_q(\mathfrak{g})$  and  $H' = o_q(G; \xi)$ , where  $\xi \in G$  is a point in the center.

**Definition 3.8.** Let  $A$  be an  $H$ -module algebra. A map  $\mu: H' \rightarrow A$  of  $H$ -module algebras is a *quantum moment map* if for every  $h \in H'$  and  $a \in A$  we have

$$\mu(h)a = (h_{(1)} \triangleright a)\mu(h_{(2)}), \quad (3.2)$$

where  $h_{(1)} \otimes h_{(2)} \in H \otimes H'$  is the  $H$ -coaction on  $h \in H'$ .

**Remark 3.9.** In the case  $H = U_{\mathcal{R}}^L(\mathfrak{g})$ ,  $H' = \mathcal{O}_{\mathcal{R}}(G)$  and  $A \in \text{Rep}_{\mathcal{R}}(G)$  we may rephrase the moment map condition as the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{R}}(G) \otimes A_{\mathcal{R}} & \xrightarrow{\mu \otimes \text{Id}} & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \\ \downarrow \tau_A & & \searrow m \\ & & A_{\mathcal{R}} \\ & \nearrow m & \\ A_{\mathcal{R}} \otimes \mathcal{O}_{\mathcal{R}}(G) & \xrightarrow{\text{Id} \otimes \mu} & A_{\mathcal{R}} \otimes A_{\mathcal{R}} \end{array}$$

as explained in [64, Theorem 3.10].

**Remark 3.10.** The first notion of quantum moment maps was due to Lu [50], and is the case  $H' = H$  of Definition 3.8. The notion was generalized in [74, Section 1.5], to allow  $H$  to be replaced by its coideal subalgebra  $H'$  – the main example of interest there was the coideal subalgebra of ad-locally finite elements.

We take the viewpoint that the natural codomain of a quantum moment map is the braided dual algebra  $H' = \mathcal{O}_{\mathcal{R}}(G)$ , or its specialization  $\mathcal{O}_q(G)$ . At generic parameters,  $\mathcal{O}_{\mathcal{K}}(G)$  is indeed identified via the Rosso isomorphism with the subalgebra of locally finite elements, and so Varagnolo–Vasserot’s notion suffices. At roots of unity, the Rosso homomorphism still exists but is no longer an embedding, hence we do not wish to require  $H'$  to be a coideal.

We will also need a construction of fusion of quantum moment maps. Suppose  $A, B$  are two algebras in  $\text{Rep}_{\mathcal{R}}(G)$ . Then  $A \otimes B$  is naturally an algebra in  $\text{Rep}_q(G)$  with multiplication

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\text{Id} \otimes \sigma_{A,B}^{-1} \otimes \text{Id}} (A \otimes A) \otimes (B \otimes B) \xrightarrow{m_A \otimes m_B} A \otimes B, \quad (3.3)$$



As for classical moment maps, quantum moment maps are used to construct a Hamiltonian reduction of  $A$ . Fix a map  $\xi: H' \rightarrow k$  of  $H$ -module algebras and denote  $I' = \ker(\xi)$ , the corresponding  $H$ -invariant two-sided ideal.

**Definition 3.12.** Let  $A$  be an  $H$ -module algebra equipped with a quantum moment map  $\mu: H' \rightarrow A$ . The *quantum Hamiltonian reduction* is

$$A//H = (A/A\mu(I'))^H.$$

Note that  $I_q = A\mu(I') \subset A$  is only a left ideal, so  $A/A\mu(I')$  is merely an  $A$ -module. Nevertheless,  $A//H$  carries a natural algebra structure that we explain presently. For  $a \in A$ , we write  $[a]$  for the image of  $a$  in  $A/I_q$ .

**Proposition 3.13.** *Let  $a, b \in A$ , and suppose  $[a]$  and  $[b]$  lie in  $A//H$ . Then  $[ab]$  also lies in  $A//H$ . There is a well-defined algebra structure on  $A//H$  given by*

$$A//H \otimes A//H \rightarrow A//H, \quad [a] \otimes [b] \mapsto [ab].$$

*Proof.* To show that the element  $[ab] \in A/I_q$  lies in  $A//H$ , first note that, for any  $h \in H$ , we have  $h \triangleright a \in \varepsilon(h)a + I_q$ , and similarly for  $b$ . Consequently,

$$\begin{aligned} h \triangleright ab &= (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \\ &\in (\varepsilon(h_{(1)})a + I_q)(\varepsilon(h_{(2)})b + I_q) = \varepsilon(h)ab + aI_q + I_qb + I_q. \end{aligned}$$

Since  $I_q$  is a left ideal, we have  $aI_q \subseteq I_q$ . We show below that  $I_qb \subseteq I_q$ . From this it follows that  $h \triangleright ab \in \varepsilon(h)ab + I_q$ , and so  $[ab]$  lies in  $A//H$ .

Next, we prove that  $I_qb \subseteq I_q$ . It suffices to show that  $\mu(h)b \in I_q$  for all  $h \in I'$ . Fix  $h \in I'$  and recall the moment map equation:  $\mu(h)b = (h_{(1)} \triangleright b)\mu(h_{(2)})$ . The fact that  $[b]$  lies in  $A//H$  implies that  $h_{(1)} \triangleright b = \varepsilon(h_{(1)})b + x$  for some  $x \in I_q$ . We write  $x = \sum_i x_i \mu(h_i)$  for some  $x_i \in A$  and  $h_i \in I'$ . We have

$$\begin{aligned} \mu(h)b &= (h_{(1)} \triangleright b)\mu(h_{(2)}) = \varepsilon(h_{(1)})b\mu(h_{(2)}) + \sum_i x_i \mu(h_i)\mu(h_{(2)}) \\ &= b\mu(h) + \sum_i x_i \mu(h_i h_{(2)}). \end{aligned}$$

In the last expression, the first term lies in  $I_q$  since  $h \in I'$ , and the second term lies in  $I_q$  since each  $h_i$  is in the two-sided ideal  $I'$ . We conclude that  $I_qb \subseteq I_q$ .

It remains to show that the product map is well-defined, that is,  $[ab]$  is independent of the lifts of  $[a], [b] \in A/I_q$  to  $A$ . Indeed, if  $a', b' \in A$  are such that  $[a] = [a']$  and  $[b] = [b']$ , then  $a'b'$  is contained in  $(a + I_q)(b + I_q) = ab + aI_q + I_qb + I_q = ab + I_q$ , and so  $[a'b'] = [ab]$ . ■

### 3.5. Frobenius Poisson orders and their quantum Hamiltonian reduction

In this section we combine Poisson and associative algebras into the notion of a Poisson order and, correspondingly, classical and quantum moment maps into the notion of a Hamiltonian Frobenius Poisson order. Finally, we explain that the quantum Hamiltonian reduction of a Hamiltonian Frobenius Poisson order carries naturally the structure of a Poisson order.

**Definition 3.14.** A *Poisson order* is given by the following data:

- An algebra  $A$ .
- A commutative algebra  $Z$  equipped with a Poisson structure. We denote  $\mathcal{Z} = \text{Spec } Z$ .
- A central embedding  $Z \subset A$  such that  $A$  is finitely generated as a  $Z$ -module.
- A  $\mathbb{C}$ -linear map  $D: Z \rightarrow \text{Der}(A)$  which satisfies

$$D(z_1)(z_2) = \{z_1, z_2\}$$

for every  $z_1, z_2 \in Z$ .

**Remark 3.15.** We will identify Poisson order structures where  $D$  differs by inner derivations. See Proposition 3.24 for our source of Poisson orders where the corresponding structure  $D$  is only defined modulo inner derivations.

We have the following important theorem about Poisson orders [18, Theorem 4.2].

**Theorem 3.16.** Suppose  $(A, Z = \mathcal{O}(\mathcal{Z}))$  is a Poisson order with  $\mathcal{Z}$  smooth. Moreover, assume  $\mathcal{Z}_0 \subset \mathcal{Z}$  is an open symplectic leaf. Then  $A|_{\mathcal{Z}_0}$  is locally free and its fibers at any two closed points are isomorphic as algebras.

**Definition 3.17.** A *Frobenius Poisson order* is given by the following data:

- An algebra  $A_q \in \text{Rep}_q(G)$ .
- A Poisson algebra  $Z$  equipped with a Poisson  $G$ -action. As before, we have a Poisson  $G$ -variety  $\mathcal{Z} = \text{Spec } Z$ .
- A central embedding  $\text{Fr}^*(Z) \subset A_q$  in  $\text{Rep}_q(G)$ .
- A linear map  $D: Z \rightarrow \text{Der}(A_q)$  giving  $(A_q, Z)$  the structure of a Poisson order.

**Remark 3.18.** Since the  $R$ -matrix on  $U_q^L(\mathfrak{g})$  is expressed via the generators of the small quantum group, the subalgebra  $\text{Fr}^*(Z) \subset A_q$  is central in  $\text{Rep}_q(G)$  if and only if it is central in the category of vector spaces.

We may now define the notion of a moment map for a Frobenius Poisson order.

**Definition 3.19.** Let  $(A_q, Z)$  be a Frobenius Poisson order. We say it is *weakly  $G$ -Hamiltonian* if there is a quantum moment map  $\mu_q: \mathcal{O}_q(G) \rightarrow A_q$  (Definition 3.8) and a classical moment map  $\mu: \mathcal{Z} \rightarrow G$  (Definition 2.6) such that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & A_q \\ \mu^* \uparrow & & \uparrow \mu_q \\ \mathcal{O}(G) & \longrightarrow & \mathcal{O}_q(G) \end{array}$$

commutes.

**Definition 3.20.** A  *$G$ -Hamiltonian Frobenius Poisson order* is a weakly  $G$ -Hamiltonian Frobenius Poisson order which satisfies the following properties:

- (1)  $D(z)(\mu_q(h)) = 0$  for every  $z \in Z^G$  and  $h \in \mathcal{O}_q(G)$ .
- (2)  $D(z)(-): A_q \rightarrow A_q$  is  $U_q^L(\mathfrak{g})$ -equivariant for every  $z \in Z^G$ .

An important observation for us is that, given a weakly  $G$ -Hamiltonian Frobenius Poisson order, its quantum Hamiltonian reduction contains a natural central subalgebra, which identifies with the classical Hamiltonian reduction of  $Z$  by  $G$ .

In more detail, choose a character  $\xi_q$  of  $\mathcal{O}_q(G)$ , and denote its restriction to  $\mathcal{O}(G)$  by  $\xi$ . Then  $\xi_q$  induces a character of  $\mathcal{O}_q(G; \xi) = \mathcal{O}_q(G) \otimes_{\mathcal{O}(G)} \mathbb{C}$ . Denote by  $I_q \subset A_q$  the quantum moment map ideal consisting of elements of the form  $a\mu_q(h)$  where  $a \in A_q$  and  $h \in \mathcal{O}_q(G)$  is such that  $\xi_q(h) = 0$ . We may identify

$$A_q // U_q^L(\mathfrak{g}) = (A_q / I_q)^{U_q^L(\mathfrak{g})} \cong (((A_q \otimes_{\mathcal{O}(G)} \mathbb{C}) \otimes_{\mathcal{O}_q(G; \xi)} \mathbb{C})^{u_q(\mathfrak{g})})^G. \quad (3.4)$$

In other words, we may perform the quantum Hamiltonian reduction of  $A_q$  in stages, first with respect to  $u_q(\mathfrak{g})$  and then with respect to  $G$ .

$A_q \otimes_{\mathcal{O}(G)} \mathbb{C}$  defines a sheaf of algebras over  $\mu^{-1}(\xi)$ . Denote by  $\mathcal{E}^\xi$  the sheaf of algebras over  $\mu^{-1}(\xi)$  obtained as the quantum Hamiltonian reduction

$$\mathcal{E}^\xi = (A_q \otimes_{\mathcal{O}(G)} \mathbb{C}) // u_q(\mathfrak{g}).$$

Denote by  $\mathcal{A}^\xi$  the sheaf of algebras on  $Z // G$  defined by  $A_q // U_q^L(\mathfrak{g})$ . By (3.4) we have an isomorphism of algebras

$$\mathcal{A}^\xi \cong \pi_*(\mathcal{E}^\xi)^G$$

over  $Z // G$ .

Furthermore, we obtain a Poisson structure on  $Z // G$  by Proposition 2.9, an algebra structure on  $A_q // U_q^L(\mathfrak{g})$  by Proposition 3.13 and a central embedding  $Z // G \subset A_q // U_q^L(\mathfrak{g})$ . We will now show that this can be enhanced to the structure of a Poisson order.

**Proposition 3.21.** *Suppose  $(A_q, Z)$  is a  $G$ -Hamiltonian Frobenius Poisson order where  $Z$  is Noetherian. Then  $(A_q // U_q^L(\mathfrak{g}), Z // G)$  is a Poisson order.*

*Proof.* Let  $I \subset \mathcal{O}(Z)$  be the ideal of functions vanishing on  $\mu^{-1}(\xi)$  and  $I_q \subset A_q$  the left ideal of elements of the form  $a\mu_q(h)$  with  $a \in A_q$  and  $h \in \mathcal{O}_q(G)$  with  $\xi_q(h) = 0$ .

Recall from Proposition 2.9 that the Poisson structure on  $\mathcal{O}(Z // G)$  is uniquely characterized by the property that  $\mathcal{O}(Z // G) \cong Z^G / I^G \leftarrow Z^G \subset Z$  is a diagram of Poisson algebras.

Consider  $z \in Z^G$  and  $a\mu_q(h) \in I_q$ . Then

$$D(z)(a\mu_q(h)) = D(z)(a)\mu_q(h) + aD(z)(\mu_q(h)) = D(z)(a)\mu_q(h).$$

Therefore,  $D(z)(I_q) \subset I_q$  and hence  $D$  descends to a map

$$D: Z^G \otimes A_q / I_q \rightarrow A_q / I_q.$$

Consider an element  $a \in A_q$  and  $[a] \in (A_q/I_q)^{U_q^L(\mathfrak{g})}$  whose images in  $A_q/I_q$  coincide. Then for  $h \in U_q^L(\mathfrak{g})$  we have  $h \triangleright a = \varepsilon(h)a + I_q$ . For  $z \in Z^G$  we therefore get

$$\begin{aligned} h \triangleright D(z)(a) &= D(z)(h \triangleright a) \\ &= \varepsilon(h)D(z)(a) + D(z)(I_q) \\ &= \varepsilon(h)D(z)(a) + I_q. \end{aligned}$$

In other words,  $D(z)(a)$  descends to  $(A_q/I_q)^{U_q^L(\mathfrak{g})}$  and so  $D$  gives a linear map

$$D: Z^G \otimes (A_q/I_q)^{U_q^L(\mathfrak{g})} \rightarrow (A_q/I_q)^{U_q^L(\mathfrak{g})}.$$

By construction,  $D$  defines a linear map  $D: Z^G \rightarrow \text{Der}(A_q // U_q^L(\mathfrak{g}))$  and restricts to the Poisson bracket  $Z^G \otimes Z^G/I^G \rightarrow Z^G/I^G$  on  $Z^G/I^G \cong (Z/I)^G \subset A_q // U_q^L(\mathfrak{g})$ . Since  $Z^G \rightarrow Z^G/I^G$  is surjective, this defines the required structure of a Poisson order.

By assumption  $A_q$  is finitely generated over  $Z$ . Therefore,  $A_q/I_q$  is finitely generated over  $Z/I$ . Since  $Z/I$  is Noetherian,  $(A_q/I_q)^{u_q(\mathfrak{g})} \subset A_q/I_q$  is also finitely generated over  $Z/I$ . Since  $G$  is reductive, we conclude that  $((A_q/I_q)^{u_q(\mathfrak{g})})^G = A // U_q^L(\mathfrak{g})$  is finitely generated over  $(Z/I)^G = Z // G$ . ■

### 3.6. Classical degenerations of quantum algebras

The notion of a  $G$ -Hamiltonian Frobenius Poisson order requires strong compatibility between several structures. In this section we show that these compatibilities arise very naturally when specializing algebras from  $\text{Rep}_{\mathcal{R}}(G)$  to  $\text{Rep}_q(G)$ . We begin by showing that degenerating  $U_{\mathcal{R}}^L(\mathfrak{g})$  at a root of unity induces extra structures on  $G$ , such as the Poisson–Lie structure (see [24, Section 8]). Recall from [53, Chapter 32] that to construct the braiding on  $\text{Rep}_{\mathcal{R}}(G)$  we need to fix a bilinear pairing on  $(-, -)$  on the character lattice. We normalize it so that the short roots have square length 2.

Let  $V_{\mathcal{R}}, W_{\mathcal{R}} \in \text{Rep}_{\mathcal{R}}(G)$  be two representations flat over  $\mathcal{R}$  and let

$$R_{V_{\mathcal{R}}, W_{\mathcal{R}}}: V_{\mathcal{R}} \otimes W_{\mathcal{R}} \rightarrow V_{\mathcal{R}} \otimes W_{\mathcal{R}}$$

be the isomorphism given by the braiding in  $\text{Rep}_{\mathcal{R}}(G)$  precomposed with the tensor flip. Let  $\pi: V_{\mathcal{R}} \rightarrow V_q$  be the specialization map taking  $t$  to  $q$ , and similarly for  $W_{\mathcal{R}}$ . Moreover, assume we have two representations  $V, W \in \text{Rep}(G)$  and embeddings  $\text{Fr}^*(V) \subset V_q$  and  $\text{Fr}^*(W) \subset W_q$ .

Choose an element  $v \otimes w \in V_{\mathcal{R}} \otimes W_{\mathcal{R}}$  such that

$$\pi(v) \otimes \pi(w) \in V \otimes W \subset V_q \otimes W_q.$$

Then

$$R_{V_{\mathcal{R}}, W_{\mathcal{R}}}(v \otimes w)|_{t=q} = \pi(v) \otimes \pi(w)$$

since  $\text{Fr}^*: \text{Rep}(G) \rightarrow \text{Rep}_q(G)$  is braided monoidal and the braiding on  $\text{Rep}(G)$  is the tensor flip.

**Proposition 3.22.** *With the above assumptions we have*

$$r(\pi(v) \otimes \pi(w)) = \pi \left( \frac{R_{V_{\mathcal{R}}, W_{\mathcal{R}}}(v \otimes w) - v \otimes w}{t - q} \right),$$

where  $r = -2\ell^2 q^{-1} r_{\text{std}} \in \mathfrak{g} \otimes \mathfrak{g}$  and  $r_{\text{std}}$  is the standard  $r$ -matrix from Example 2.5 defined by the bilinear pairing  $\ell^2(-, -)$  on  $\mathfrak{g}$ .

*Proof.* Let us recall the precise construction of the braiding on  $\text{Rep}_{\mathcal{R}}(G)$  from [53, Chapter 32]. Fix a reduced expression for the longest element of the Weyl group which gives rise to an ordering of the set  $\Delta_+$  of positive roots. For a positive root  $\alpha \in \Delta_+$  let  $d_\alpha = (\alpha, \alpha)/2$ . Consider the quasi  $R$ -matrix (see [53, Chapter 4] and [41, 47])

$$\Theta = \prod_{\alpha \in \Delta_+} \sum_n (-1)^n t^{-n(n-1)d_\alpha/2} \{n\}_\alpha F_\alpha^{(n)} \otimes E_\alpha^{(n)},$$

where

$$\{n\}_\alpha = \prod_{a=1}^n (t^{ad_\alpha} - t^{-ad_\alpha}).$$

If we assume that  $v \in V_{\mathcal{R}}, w \in W_{\mathcal{R}}$  have weights  $\mu$  and  $\nu$  respectively, then

$$R_{V_{\mathcal{R}}, W_{\mathcal{R}}}(v \otimes w) = t^{-(\mu, \nu)} \Theta(v \otimes w).$$

By our assumption on the group  $G$ , under the quantum Frobenius map the character lattice of  $G$  used in the definition of  $\text{Rep}(G)$  is the  $\ell$ -scaled character lattice used in the definition of  $\text{Rep}_q(G)$ . Therefore,  $q^{-(\mu, \nu)} = 1$ .

Since  $V, W \in \text{Rep}(G)$ , the only terms in  $\Theta$  not vanishing to order  $t - q$  occur for  $n = \ell$ . Therefore, the linear term in  $t - q$  coming from  $\Theta$  is

$$-\frac{2\ell^2}{q} \sum_{\alpha \in \Delta_+} (d_\alpha f_\alpha \otimes e_\alpha)(\pi(v) \otimes \pi(w)).$$

The linear term in  $t - q$  coming from  $t^{-(\mu, \nu)}$  is  $-q^{-1}(\mu, \nu)$ . Combining the two terms we exactly recover the formula for the standard  $r$ -matrix  $r_{\text{std}}$ .  $\blacksquare$

Next, we show that  $\mathfrak{g}$  carries a natural structure of a factorizable Lie bialgebra. Let  $p: U_{\mathcal{R}}^{\text{L}}(\mathfrak{g}) \rightarrow U_q^{\text{L}}(\mathfrak{g})$  be the specialization map taking  $t$  to  $q$ . Since  $U_{\mathcal{R}}^{\text{L}}(\mathfrak{g})$  is free as an  $\mathcal{R}$ -module, we may choose an isomorphism  $U_{\mathcal{R}}^{\text{L}}(\mathfrak{g}) \cong U_q^{\text{L}}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{R}$  which is the identity at  $t = q$ . For an element  $x \in U_{\mathcal{R}}^{\text{L}}(\mathfrak{g})$  such that  $\text{Fr}(p(x)) \in \mathfrak{g} \subset U(\mathfrak{g})$ , we find that

$$\delta(\pi(x)) = \text{Fr} \left( p \left( \frac{\Delta(x) - \Delta^{\text{op}}(x)}{t - q} \right) \right)$$

defines a Lie cobracket on  $\mathfrak{g}$  (see e.g. [27, Proposition 9.1]). Since the braiding on  $\text{Rep}_{\mathcal{R}}(G)$  is an isomorphism of  $U_{\mathcal{R}}^{\text{L}}(\mathfrak{g})$ -modules, by Proposition 3.22 we get

$$\delta(\pi(x)) = [r, \pi(x) \otimes 1 + 1 \otimes \pi(x)],$$

i.e.,  $\delta$  defines a factorizable Lie bialgebra structure. In particular, we obtain a factorizable Poisson–Lie structure on  $G$ .

Suppose  $A_{\mathcal{R}}$  is an  $\mathcal{R}$ -algebra. Denote by  $\pi: A_{\mathcal{R}} \rightarrow A_q$  evaluation at  $q$ , and fix an arbitrary  $\mathbb{C}$ -linear section  $\sigma: A_q \rightarrow A_{\mathcal{R}}$  to  $\pi$ .

**Proposition 3.23** ([18, Section 2.2], [35, Proposition 2.6 (2)]). *Suppose that  $A_{\mathcal{R}}$  is flat as an  $\mathcal{R}$ -module and fix a central subalgebra  $Z \subset A_q$ .*

(1) *There exists a linear map  $D: Z \rightarrow \text{Der}(A_q)$  satisfying*

$$D(z)(x) = \pi \left( \frac{\sigma(z)\sigma(x) - \sigma(x)\sigma(z)}{t - q} \right) \quad (3.5)$$

*for all  $x \in A_q, z \in Z$ . The linear map  $D$  is independent of the section  $\sigma$  up to an inner derivation.*

(2) *If in addition  $D(Z)(Z) \subset Z$ , then  $(A_q, \text{Spec}(Z))$  defines a Poisson order.*

(3) *The condition in (2) is automatic in the case  $Z = Z(A_q)$ .*

**Proposition 3.24.** *Suppose  $A_{\mathcal{R}}$  and  $Z \subset A_q$  satisfy the assumptions of Proposition 3.23. In addition, we make the following assumptions:*

- $A_{\mathcal{R}}$  is an algebra in  $\text{Rep}_{\mathcal{R}}(G)$ .
- $Z \subset A_q$  is  $U_q^{\mathbb{L}}(\mathfrak{g})$ -equivariant and  $u_q(\mathfrak{g})$ -invariant.

*Then  $(A_q, Z)$  is a Frobenius Poisson order.*

*Proof.* Since  $Z$  is  $u_q(\mathfrak{g})$ -invariant, it becomes an algebra in  $\text{Rep}(G)$ . So, we have an infinitesimal action map  $a: \mathfrak{g} \rightarrow \text{Der}(Z)$ . To show that  $\text{Spec}(Z)$  is a Poisson  $G$ -variety, it is enough to work infinitesimally and show that  $Z$  is a Poisson  $\mathfrak{g}$ -algebra, i.e., for every  $x \in \mathfrak{g}$  and  $a, b \in Z$  we have

$$x.\{a, b\} - \{x.a, b\} - \{a, x.b\} = (x_{[1]}.a)(x_{[2]}.b),$$

where  $\delta(x) = x_{[1]} \otimes x_{[2]}$  is the Lie cobracket on  $\mathfrak{g}$ .

The fact that  $A_{\mathcal{R}}$  is an algebra in  $U_{\mathcal{R}}^{\mathbb{L}}(\mathfrak{g})$  gives the relation

$$h \triangleright (ab) = h_{(1)} \triangleright a \cdot h_{(2)} \triangleright b$$

for  $h \in U_{\mathcal{R}}^{\mathbb{L}}(\mathfrak{g})$  and  $a, b \in A_{\mathcal{R}}$ . Now suppose  $\pi(a), \pi(b) \in Z$  and  $\text{Fr}(p(h)) = x \in \mathfrak{g} \subset U(\mathfrak{g})$ . Taking the commutator we get

$$h \triangleright [a, b] = h_{(1)} \triangleright a \cdot h_{(2)} \triangleright b - h_{(1)} \triangleright b \cdot h_{(2)} \triangleright a.$$

This relation is trivial modulo  $(t - q)$  while the  $(t - q)$  term gives

$$x.\{\pi(a), \pi(b)\} = \{x.\pi(a), \pi(b)\} + \{\pi(a), x.\pi(b)\} + (x_{[1]}. \pi(a))(x_{[2]}. \pi(b)). \quad \blacksquare$$

We will now show that the degeneration at a root of unity is compatible with fusion. Suppose  $A_{\mathcal{R}}, B_{\mathcal{R}}$  are two algebras in  $\text{Rep}_{\mathcal{R}}(G)$ . Then  $A_{\mathcal{R}} \otimes B_{\mathcal{R}}$  is naturally an algebra in  $\text{Rep}_{\mathcal{R}}(G)$ . Let  $p: A_{\mathcal{R}} \rightarrow A_q$  and  $p: B_{\mathcal{R}} \rightarrow B_q$  be the maps given by evaluation at  $t = q$ .



Suppose  $A_{\mathcal{R}}, B_{\mathcal{R}}$  are flat over  $\mathcal{R}$ . Suppose  $Z_{A_q} \subset A_q$  and  $Z_{B_q} \subset B_q$  are two central  $U_q^L(\mathfrak{g})$ -equivariant and  $u_q(\mathfrak{g})$ -invariant subalgebras. Then  $Z_{A_q} \otimes Z_{B_q} \subset A_q \otimes B_q$  is central. So, by Proposition 3.24 we obtain a Frobenius Poisson order structure on  $(A_q \otimes B_q, \text{Spec}(Z_{A_q}) \times \text{Spec}(Z_{B_q}))$ .

**Proposition 3.25.** *The Poisson  $G$ -variety structure on  $\text{Spec}(Z_{A_q}) \times \text{Spec}(Z_{B_q})$  obtained by degenerating the algebra  $A_{\mathcal{R}} \otimes B_{\mathcal{R}}$  coincides with the fusion of the Poisson  $G$ -varieties  $\text{Spec}(Z_{A_q})$  and  $\text{Spec}(Z_{B_q})$ .*

*Proof.* By definition, the Poisson structure on  $Z_{A_q} \otimes Z_{B_q}$  is given as follows. Suppose  $\pi(z_1), \pi(z_2) \in Z_{A_q}$  and  $\pi(w_1), \pi(w_2) \in Z_{B_q}$ . Then

$$\{\pi(z_1) \otimes \pi(w_1), \pi(z_2) \otimes \pi(w_2)\} = \pi \left( \frac{(z_1 \otimes w_1)(z_2 \otimes w_2) - (z_2 \otimes w_2)(z_1 \otimes w_1)}{t - q} \right).$$

The product on the right-hand side is the braided tensor product of algebras given by equation (3.3) defined using the braiding on  $\text{Rep}_{\mathcal{R}}(G)$ . Its first-order term in  $(t - q)$  can therefore be read off from Proposition 3.22 and we get

$$\begin{aligned} \{\pi(z_1) \otimes \pi(w_1), \pi(z_2) \otimes \pi(w_2)\} &= \{\pi(z_1), \pi(z_2)\} \otimes \pi(w_1)\pi(w_2) \\ &\quad + \pi(z_1)\pi(z_2) \otimes \{\pi(w_1), \pi(w_2)\} \\ &\quad - \pi(z_1)\pi(w_2) \cdot a(r)(\pi(z_2) \otimes \pi(w_1)) \\ &\quad + \pi(z_2)\pi(w_1) \cdot a(r)(\pi(z_1) \otimes \pi(w_2)), \end{aligned}$$

which coincides with the fusion as defined in Proposition 2.14. ■

Finally, a degeneration of quantum moment maps gives rise to classical moment maps.

**Proposition 3.26.** *Suppose  $A_{\mathcal{R}}$  and  $Z \subset A_q$  satisfy the assumptions of Proposition 3.24. In addition, suppose there is a quantum moment map  $\mu_{\mathcal{R}}: \mathcal{O}_{\mathcal{R}}(G) \rightarrow A_{\mathcal{R}}$  and a  $G$ -equivariant map  $\mu: Z \rightarrow G$  such that the diagram*

$$\begin{array}{ccc} \mathcal{O}_q(G) & \xrightarrow{\mu_q} & A_q \\ \uparrow & & \uparrow \\ \mathcal{O}(G) & \xrightarrow{\mu^*} & Z \end{array}$$

*commutes. Then  $\mu$  is a moment map for the Poisson  $G$ -action on  $Z$  and  $(A_q, Z)$  is a weakly  $G$ -Hamiltonian Frobenius Poisson order.*

*Proof.* Let us first check that  $\mu$  satisfies the classical moment map condition (2.1). Consider  $z \in A_{\mathcal{R}}$  and  $h \in \mathcal{O}_{\mathcal{R}}(G)$ . The quantum moment map equation gives

$$\mu_{\mathcal{R}}(h)z = (h_{(1)} \triangleright z)\mu_{\mathcal{R}}(h_{(2)}),$$

which is equivalent to

$$[\mu_{\mathcal{R}}(h), z] = (h_{(1)} \triangleright z - \varepsilon(h_{(1)})z)\mu_{\mathcal{R}}(h_{(2)}).$$

If we assume  $\pi(z) \in Z \subset A_q$  and  $\pi(h) \in \mathcal{O}(G) \subset \mathcal{O}_q(G)$ , the above equation is trivial modulo  $(t - q)$  while the linear  $(t - q)$  term gives

$$\{\mu(\pi(h)), \pi(z)\} = \sum_i \mu(\tilde{a}(e^i) \cdot \pi(h)) e_i \cdot \pi(z),$$

where we use the fact that  $\mu(\pi(h)) = \mu_q(\pi(h))$  by assumption and that the first-order term in  $h_{(1)} \triangleright z - \varepsilon(h_{(1)})z$  gives the  $\mathfrak{g}^*$ -action on  $\mathcal{O}(G)$  (see [64, Theorem 4.27] for a similar degeneration at  $q = 1$ ). ■

Observe that the classical moment map  $\mu: Z \rightarrow G$  in the above statement is unique if it exists and its existence boils down to the following property of the quantum moment map:  $\mu_q(\mathcal{O}(G)) \subset Z$ . We will now show that if  $Z$  is the whole center, this condition is automatic.

**Lemma 3.27.** *Suppose  $A_q$  is an algebra in  $\text{Rep}_q(G)$  equipped with a quantum moment map  $\mu_q: \mathcal{O}_q(G) \rightarrow A_q$ . Then  $\mu_q(\mathcal{O}(G)) \subset Z(A_q)$ .*

*Proof.* Recall that for  $W \in \text{Rep}_q(G)$  we have the field-goal isomorphism  $\tau_W: \mathcal{O}_q(G) \otimes W \rightarrow W \otimes \mathcal{O}_q(G)$  given by (3.1). The subalgebra  $\mathcal{O}(G) \subset \mathcal{O}_q(G)$  is generated by  $V^* \otimes V$  where  $V$  lies in the Müger center of  $\text{Rep}_q(G)$ . Therefore,  $\tau_W$  on  $\mathcal{O}(G) \subset \mathcal{O}_q(G)$  restricts to the braiding  $\mathcal{O}(G) \otimes W \rightarrow W \otimes \mathcal{O}(G)$  in  $\text{Rep}_q(G)$ . Since  $\mathcal{O}(G)$  lies in  $u_q(\mathfrak{g})$ -invariants of  $\mathcal{O}_q(G)$ , the above braiding in  $\text{Rep}_q(G)$  coincides with the tensor flip. Then the claim follows immediately from the interpretation of the quantum moment map equation using the field-goal isomorphism given by Remark 3.9. ■

We end with an important result which explains when a weakly  $G$ -Hamiltonian Frobenius Poisson order is in fact  $G$ -Hamiltonian.

**Proposition 3.28.** *Suppose  $A_{\mathcal{R}}$  and  $Z$  satisfy the assumptions of Proposition 3.26. In addition, suppose that the morphism  $(A_{\mathcal{R}})^{U_{\mathcal{R}}^L(\mathfrak{g})} \rightarrow (A_q)^{U_q^L(\mathfrak{g})}$  is surjective. Then  $(A_{\mathcal{R}}, Z)$  is a  $G$ -Hamiltonian Frobenius Poisson order.*

*Proof.* We have  $Z^G \subset (A_q)^{U_q^L(\mathfrak{g})}$  and so we may consider lifts of elements of  $Z^G$  to  $U_{\mathcal{R}}^L(\mathfrak{g})$ -invariant elements of  $A_{\mathcal{R}}$  in the definition of  $D$ . Now consider  $z \in (A_{\mathcal{R}})^{U_{\mathcal{R}}^L(\mathfrak{g})}$  such that  $\pi(z) \in Z^G$ ,  $a \in A$  and  $h \in U_{\mathcal{R}}^L(\mathfrak{g})$ . Then

$$h \triangleright [z, a] = [z, h \triangleright a].$$

This shows that  $D(\pi(z))(-)$  is  $U_q^L(\mathfrak{g})$ -equivariant.

For  $h \in \mathcal{O}_{\mathcal{R}}(G)$  we have

$$\mu_{\mathcal{R}}(h)z = z\mu_{\mathcal{R}}(h)$$

by the quantum moment map equation. Therefore,  $D(\pi(z))(\mu_q(\pi(h))) = 0$ . ■

### 3.7. Hamiltonian reduction of matrix algebras

Before we state our main result on Azumaya algebras, we will prove two preliminary claims on matrix algebras and their reductions.

Let  $k$  be a commutative ring,  $H$  a Hopf  $k$ -algebra and  $H'$  a  $k$ -algebra together with a structure of an  $H$ -module and an  $H$ -comodule as in Section 3.4. The following proposition is a version of [12, Lemma 3.4.4] and [74, Proposition 1.5.2].

**Proposition 3.29.** *Let  $R$  be a commutative  $k$ -algebra and  $V$  a finitely generated projective  $R$ -module. Suppose  $\xi: H' \rightarrow k$  is a map of  $H$ -module algebras such that the composite*

$$H' \xrightarrow{\Delta} H \otimes H' \xrightarrow{\text{Id} \otimes \xi} H$$

*is an isomorphism. Suppose  $A = \text{End}_R(V)$  is an  $H$ -module algebra equipped with a quantum moment map  $\mu: H' \rightarrow \text{End}_R(V)$ . Denote by  $(V^*)_{H'}$  the module of  $\xi$ -twisted  $H'$ -coinvariants. Then we have an isomorphism of  $R$ -algebras*

$$\text{End}_R(V) // H \cong \text{End}_R((V^*)_{H'})^{\text{op}}.$$

We believe the proof is well-known to experts, but as we did not find a satisfactory reference, we include a proof here.

*Proof of Proposition 3.29.* We begin by computing the coinvariants  $\text{End}_R(V) \otimes_{H'} k$ . The moment map  $\mu: H' \rightarrow \text{End}_R(V)$  gives  $V$  the structure of an  $H'$ -module. Since we are considering the  $H'$ -action on  $\text{End}_R(V)$  given by right multiplication, it is given by precomposition with the moment map, i.e.,  $H'$  acts on the source. Since  $V$  is finitely generated projective, we may identify  $\text{End}_R(V) \cong \text{End}_R(V^*)^{\text{op}}$ , where now the  $H'$ -action is on the target. Since  $V^*$  is also finitely generated projective,  $\text{Hom}_R(V^*, -)$  preserves colimits, and hence

$$\text{End}_R(V^*) \otimes_{H'} k \cong \text{Hom}_R(V^*, V^* \otimes_{H'} k) = \text{Hom}_R(V^*, (V^*)_{H'}).$$

For  $a \in A \otimes_{H'} k$  the quantum moment map equation gives

$$\mu(h)a = (h_{(1)} \triangleright a)\xi(h_{(2)}).$$

Since the map  $H' \rightarrow H$  given by  $h \mapsto h_{(1)}\xi(h_{(2)})$  is an isomorphism by assumption, we may identify  $H$ -invariants with  $\xi$ -twisted  $H'$ -invariants where  $H'$  acts on  $A \otimes_{H'} k$  on the left. The functor  $\text{Hom}_R(-, (V^*)_{H'})$  sends coinvariants to invariants, so we conclude

$$\text{End}_R(V^*)^{\text{op}} // H = (\text{End}_R(V^*)^{\text{op}} \otimes_{H'} k)^H \cong \text{Hom}_R((V^*)_{H'}, (V^*)_{H'})^{\text{op}}. \quad \blacksquare$$

Now suppose  $\mathcal{C}$  is a braided tensor category and  $V$  a dualizable object. Then the internal endomorphism algebra  $E(V) = V \otimes V^*$  carries a natural algebra structure, where  $V^*$  is the left dual of  $V$ . We then have the following result [73, Proposition 2.3], which states that the braided tensor product of matrix algebras is again a matrix algebra.

**Proposition 3.30.** *Let  $\mathcal{C}$  be a braided tensor category and  $V, W \in \mathcal{C}$  dualizable objects. Then the morphism*

$$\phi = 1 \otimes \sigma_{W \otimes W^*, V^*}: V \otimes W \otimes W^* \otimes V^* \rightarrow V \otimes V^* \otimes W \otimes W^*$$

*defines an isomorphism of algebras  $E(V \otimes W) \cong E(V) \otimes E(W)$ .*

### 3.8. Summary of results

Let us now assemble all the tools above to state our main general result, a framework for constructing Azumaya algebras via quantum Hamiltonian reduction. We fix the following data:

- $A_{\mathcal{R}}$  is an algebra in  $\text{Rep}_{\mathcal{R}}(G)$  flat over  $\mathcal{R}$ . Denote by  $A_q$  its specialization at  $t = q$ .
- $\mu_{\mathcal{R}}: \mathcal{O}_{\mathcal{R}}(G) \rightarrow A_{\mathcal{R}}$  is a quantum moment map.
- $Z \subset A_q$  is a central,  $U_q^{\mathbb{L}}(\mathfrak{g})$ -equivariant and  $u_q(\mathfrak{g})$ -invariant subalgebra. Set  $\mathcal{Z} = \text{Spec}(Z)$ . We assume that  $\mathcal{Z}$  is smooth and connected and  $A_q$  is a finitely generated projective  $Z$ -module.
- $G_{\text{stab}} \subset G$  is a normal subgroup which acts trivially on  $\mathcal{Z}$  and such that the pairing on  $\mathfrak{g}$  restricts to a non-degenerate pairing on  $\mathfrak{g}_{\text{stab}} \subset \mathfrak{g}$ . Denote  $\bar{G} = G/G_{\text{stab}}$ .

**Lemma 3.31.** *Suppose  $p \in \mathcal{Z}$  is a closed point such that  $A_q|_p$  is a matrix algebra. Then  $Z$  coincides with the center of  $A_q$ .*

*Proof.* By assumption  $A_q$  is a finitely generated projective  $Z$ -module. The Azumaya condition is open, so there is an open dense subset  $\mathcal{Z}^{\circ} \subset \mathcal{Z}$  such that  $A_q|_{\mathcal{Z}^{\circ}}$  is Azumaya over  $\mathcal{Z}^{\circ}$ . In particular, the center of  $A_q|_{\mathcal{Z}^{\circ}}$  coincides with  $\mathcal{O}(\mathcal{Z}^{\circ})$ .

Now suppose  $z \in Z(A_q)$ . Then its image in  $A_q|_{\mathcal{Z}^{\circ}}$  is also in the center, i.e., it lies in  $\mathcal{O}(\mathcal{Z}^{\circ})$ . The image of  $z$  under  $A_q \rightarrow A_q/Z$  is therefore zero generically on  $\mathcal{Z}$  and since  $A_q/Z$  is flat over  $Z$ , it has to be zero. ■

Since  $Z = Z(A_q)$ , Proposition 3.23 provides the structure of a Poisson order on  $(A_q, Z)$ . Moreover, combining Lemma 3.27 and Proposition 3.26 we see that  $(A_q, Z)$  is a weakly  $G$ -Hamiltonian Frobenius Poisson order with a classical moment map  $\mu: \mathcal{Z} \rightarrow G$ . We fix the following additional data:

- $G_0 \subset G$  is a subvariety such that the composite  $G_0 \rightarrow G \rightarrow \bar{G}$  is étale and such that the moment map  $\mu: \mathcal{Z} \rightarrow G$  factors through  $G_0$ .
- $\xi_q: \mathcal{O}_q(G) \rightarrow \mathbb{C}$  is a map of  $U_q^{\mathbb{L}}(\mathfrak{g})$ -module algebras. Denote by  $\xi \in G$  the point corresponding to the composite  $\mathcal{O}(G) \rightarrow \mathcal{O}_q(G) \xrightarrow{\xi} \mathbb{C}$ . We assume  $\xi$  lies in the intersection of the big cell  $G^{\circ} \subset G$  and  $G_0 \subset G$ .
- $U \subset \mu^{-1}(\xi)$  is a non-empty open subset consisting of stable points (i.e., each point lies in a closed  $G$ -orbit) with stabilizer  $G_{\text{stab}}$ . Let  $\mathcal{M}^{\xi} \subset \mathcal{Z} // G$  denote the image of  $U$  under the projection  $\mu^{-1}(\xi) \rightarrow \mathcal{Z} // G$ . Note that by Luna's étale slice theorem the projection  $U \rightarrow \mathcal{M}^{\xi}$  is a  $\bar{G}$ -torsor (see [26, Proposition 5.7]).

**Theorem 3.32.** *Consider  $A_{\mathcal{R}}, \mu_q, \xi_q, Z, U$  as above and assume that the map  $(A_{\mathcal{R}})^{U_{\mathcal{R}}^{\mathbb{L}}(\mathfrak{g})} \rightarrow (A_q)^{U_q^{\mathbb{L}}(\mathfrak{g})}$  is surjective. Then  $(A_q, \mathcal{Z})$  defines a  $G$ -Hamiltonian Frobenius Poisson order. Assume further that:*

- (1) *The Poisson  $G$ -variety structure on  $\mathcal{Z}$  is non-degenerate.*
- (2)  *$G_{\text{stab}}$  acts trivially on  $(A_q/I_q)^{u_q(\mathfrak{g})}$ .*
- (3) *The point  $p$  from Lemma 3.31 lies in  $\mu^{-1}(G^{\circ})$ .*

Then  $A_q$  is a sheaf of Azumaya algebras over  $\mu^{-1}(G^\circ)$  and its Hamiltonian reduction  $\mathcal{A}^\xi|_{\mathcal{M}^\xi}$  is a sheaf of Azumaya algebras over  $\mathcal{M}^\xi \subset \mathcal{Z} // G$ .

*Proof.* By Proposition 3.28 the pair  $(A_q, \mathcal{Z})$  is a  $G$ -Hamiltonian Frobenius Poisson order. By Theorem 2.15,  $\mu^{-1}(G^\circ) \subset \mathcal{Z}$  is an open symplectic leaf, which is clearly smooth. Therefore, by Theorem 3.16 the sheaf  $A_q$  is Azumaya over  $\mu^{-1}(G^\circ)$ .

Denote  $\bar{\mu}: X \rightarrow G \rightarrow \bar{G}$  which is a moment map for the  $\bar{G}$ -action on  $X$  by Proposition 2.17. Let  $\bar{\xi} \in \bar{G}$  be the image of  $\xi$  under  $G \rightarrow \bar{G}$ . By assumption  $\mu^{-1}(\xi) \rightarrow \bar{\mu}^{-1}(\bar{\xi})$  is étale. Therefore, applying Proposition 2.16 to the  $\bar{G}$ -Hamiltonian reduction we deduce that the  $G$ -Hamiltonian reduction  $\mathcal{M}^\xi$  is a smooth symplectic variety.

By Proposition 3.24,  $\mathcal{A}^\xi$  defines a Poisson order over  $\mathcal{Z} // G$ . Since  $\mathcal{M}^\xi$  is symplectic (in particular, smooth), Theorem 3.16 implies that to establish an Azumaya property it is enough to establish it generically. Since  $\pi: U \rightarrow \mathcal{M}^\xi$  is a  $\bar{G}$ -torsor, we have  $\pi^*(\mathcal{A}^\xi|_{\mathcal{M}^\xi}) \cong \mathcal{E}|_U$ . Therefore, the Azumaya property of  $\mathcal{A}^\xi$  over  $\mathcal{M}^\xi$  is equivalent to that of  $\mathcal{E}$  over  $U$ , which is therefore also enough to establish generically.

Let  $\xi_q^{-1}$  be the character of  $\mathcal{O}_q(G)$  obtained by precomposing  $\xi_q$  with the antipode on  $\mathcal{O}_q(G)$ . Then the unit algebra  $\mathbb{C} \in \text{Rep}_q(G)$  carries a moment map  $\xi_q^{-1}: \mathcal{O}_q(G) \rightarrow \mathbb{C}$ . Let us denote it by  $\mathbb{C}_{\xi_q^{-1}}$ . Then we may identify the quantum Hamiltonian reduction of  $A$  along  $\xi_q$  with the quantum Hamiltonian reduction of  $A \otimes \mathbb{C}_{\xi_q^{-1}}$  along the trivial character  $\varepsilon: \mathcal{O}_q(G) \rightarrow \mathbb{C}$ . In particular, this allows us to assume  $\xi_q = \varepsilon$  and  $\xi = e \in G$  is the identity element.

Choose an étale cover  $p: Y \rightarrow U$  over which  $A_q/I$  splits as  $\text{End}(\mathcal{V})$  for a vector bundle  $\mathcal{V}$  over  $Y$ . The pullback  $p^*\mathcal{E}$  is given by the Hamiltonian reduction  $\text{End}(\mathcal{V}) // u_q(\mathfrak{g})$ . By Theorem 3.2,  $u_q(\mathfrak{g})$  is a factorizable Hopf algebra, so the Rosso homomorphism  $o_q(G; e) \rightarrow u_q(\mathfrak{g})$  is an isomorphism. Therefore, by Proposition 3.29,  $\text{End}(\mathcal{V}) // u_q(\mathfrak{g}) \cong \text{End}(\mathcal{W})$ , where  $\mathcal{W} = (\mathcal{V})_{u_q(\mathfrak{g})}^*$ . The sheaf  $\mathcal{W}$  is coherent over a smooth scheme, so by generic flatness (see [32, Théorème 6.9.1]) it is generically a vector bundle. Therefore,  $p^*\mathcal{E}$  is generically a matrix algebra over  $Y$ , so  $\mathcal{E}$  is generically an Azumaya algebra over  $U$ . ■

**Lemma 3.33.** *Suppose  $(A_q, \mathcal{Z})$  is a  $G$ -Hamiltonian Frobenius Poisson order. Then the Azumaya locus of  $A_q$  is contained within  $\mu^{-1}(G^\circ)$ .*

*Proof.* Recall from [38, 39, 42] that we have an element denoted  $K^{-\rho}$  in  $\mathcal{O}_q(G)$ , which acts diagonalizably on weight vectors  $v_\lambda$  via  $K^{-\rho}v_\lambda = q^{-\langle \rho, \lambda \rangle}v_\lambda$ . When  $q$  is generic, this element corresponds to a monomial in  $U_q(\mathfrak{t})$  under the Rosso isomorphism, but it is well-defined as an element of  $\mathcal{O}_q(G)$  for all  $q$ . The element  $K^{\ell\rho}$  lies in the central subalgebra  $\mathcal{O}(G) \subset \mathcal{O}_q(G)$ , where it generates the vanishing ideal of the complement to  $G^\circ \subset G$ . Now let  $p \in \text{Spec}(\mathcal{Z})$  be any point in the preimage of the complement  $G \setminus G^\circ$  of the Bruhat cell. Then  $\mu_q(K^{-\rho})$  defines a non-zero element of the fiber, which  $q$ -commutes with elements of  $A$ , and therefore defines a non-zero two-sided ideal in the fiber. This ideal is proper, because  $(K^{-\rho})^\ell$  is zero by assumption. Since finite-dimensional matrix algebras cannot contain non-trivial two-sided ideals, we conclude that  $p$  cannot lie in the Azumaya locus. ■

#### 4. Quantum character varieties

In this section we implement the program of Section 3 to show that quantum character varieties of closed surfaces form Azumaya algebras over their classical degenerations. We begin by recalling the construction of quantum character stacks and varieties, referring to [10] for more details.

Let  $\mathbf{Mfld}_{\text{or}}^2$  denote the topological category with objects being oriented surfaces (possibly with boundary), and with morphisms being oriented embeddings. Let  $\text{Disk}_{\text{or}}^2$  denote full subcategory consisting of disjoint unions of disks; hence  $\text{Disk}_{\text{or}}^2$  is a model for the framed  $E_2$ -operad.

A ribbon tensor category  $\mathcal{A}$  determines a functor from  $\text{Disk}_{\text{or}}^2$  to the  $(2, 1)$ -category  $\mathbf{Pr}$  of locally presentable linear categories. Examples include  $\text{Rep}_{\mathcal{K}}(G)$  for generic quantum parameter, the category  $\text{Rep}_{\mathcal{R}}(G)$  of representations of Lusztig's integral form, its specialization  $\text{Rep}_q(G)$  at a root of unity parameter  $q$ , and finally  $\text{Rep}(G)$  in the classical case – in this latter case the ribbon element is the identity natural isomorphism of the identity functor. The factorization homology of oriented surfaces with coefficients in  $\mathcal{A}$  is the canonical ‘left Kan’ extension,

$$\begin{array}{ccc} \text{Disk}_{\text{or}}^2 & \xrightarrow{\mathcal{A}} & \mathbf{Pr} \\ & \searrow & \nearrow \scriptstyle S \mapsto \int_S \mathcal{A} \\ & \mathbf{Mfld}_{\text{or}}^2 & \end{array}$$

For any surface  $S$ , the embedding of the empty surface into  $S$  induces a distinguished object  $\text{Dist}_S$  in  $\int_S \mathcal{A}$ . In the case  $\mathcal{A} = \text{Rep}(G)$ ,  $\text{Dist}_S$  is simply the structure sheaf,  $\mathcal{O}(\text{Ch}_G^{\text{fr}}(S))$ , of the character stack (recall that we regard quasi-coherent sheaves on a quotient stack as  $G$ -equivariant objects on the framed character variety). Following [10], in the case  $\mathcal{A} = \text{Rep}_q(G)$ , we call  $\text{Dist}_S$  the *quantum structure sheaf*. To alleviate notation, we will call the classical distinguished object  $\text{Dist}_S^{\text{cl}}$ , and the quantum distinguished object  $\text{Dist}_S$ .

If  $S^\circ$  is a surface with boundary, the designation of an interval on the boundary determines on  $\int_{S^\circ} \mathcal{A}$  the structure of an  $\mathcal{A}$ -module category, and allows us in particular to define internal homomorphisms. We denote

$$A_{S^\circ} = \underline{\text{End}}(\mathcal{O}_{S^\circ}^q) \in \mathcal{A},$$

where  $\underline{\text{End}}$  denotes the internal endomorphism algebra. The algebra  $A_{S^\circ}$  is a deformation quantization of the coordinate algebra of the framed character variety; it was computed in [10] in terms of Alekseev–Grosse–Schomerus algebras [1]. We will denote by  $A_{S^\circ}^{\mathcal{R}}$  the  $\mathcal{R}$ -algebra obtained by replacing  $\text{Rep}_q(G)$  by  $\text{Rep}_{\mathcal{R}}(G)$  in the above construction.

If  $S$  is a closed surface, we consider the quantum character variety to be  $\text{End}(\text{Dist}_S)$ , the global algebra of endomorphisms in  $\int_S \text{Rep}_q(G)$ .

#### 4.1. The Frobenius Poisson order

By its construction as a left Kan extension, factorization homology is functorial both in the surface  $S$  (under embeddings), and in the braided tensor category  $\mathcal{A}$  (under ribbon tensor functors). Hence the quantum Frobenius functor induces a further functor

$$\int_S \mathrm{Fr}^*: \int_S \mathrm{Rep}(G) \rightarrow \int_S \mathrm{Rep}_q(G),$$

for any surface  $S$ . Moreover, this functor is pointed with respect to the distinguished objects  $\mathrm{Dist}_S$ : it means that we obtain a canonical isomorphism in  $\int_S \mathrm{Rep}_q(S)$ ,

$$\left( \int_S \mathrm{Fr}^* \right) (\mathrm{Dist}_S^{\mathrm{cl}}) \cong \mathrm{Dist}_S.$$

For the remainder of this section, let us adopt the following convention:  $S$  will henceforth denote a closed connected surface of genus  $g$ , and  $S^\circ = S \setminus D$  will denote the surface with boundary obtained by removing a disk  $D$  from  $S$ .

By the above discussion,  $\int_{S^\circ} \mathrm{Rep}(G)$  is a  $\mathrm{Rep}(G)$ -module category while  $\int_{S^\circ} \mathrm{Rep}_q(G)$  is a  $\mathrm{Rep}_q(G)$ -module category. In addition, we obtain an embedding of algebra objects in  $\mathrm{Rep}_q(G)$ ,

$$\mathrm{Fr}^*(\mathcal{O}(\mathrm{Ch}_G^{\mathrm{fr}}(S^\circ))) \rightarrow A_{S^\circ}. \quad (4.1)$$

It follows from the fact that  $\mathrm{Rep}(G)$  lies in the Müger center of  $\mathrm{Rep}_q(G)$  that the embedding (4.1) is central. Note that we may identify

$$\mathcal{O}(\mathrm{Ch}_G^{\mathrm{fr}}(S^\circ)) \cong \mathcal{O}(G^{2g})$$

as a  $G$ -representation, and by [10, Theorem 5.14] we have

$$A_{S^\circ}^{\mathcal{R}} \cong \mathcal{O}_{\mathcal{R}}(G)^{\otimes 2g}$$

as an object of  $\mathrm{Rep}_{\mathcal{R}}(G)$ . In particular,  $A_{S^\circ}^{\mathcal{R}}$  is flat over  $\mathcal{R}$ .

**Lemma 4.1.**  *$A_{S^\circ}$  is a finitely generated  $\mathcal{O}(\mathrm{Ch}_G^{\mathrm{fr}}(S^\circ))$ -module.*

*Proof.* By [24, Theorem 7.2],  $\mathcal{O}_q(G)$  is finitely generated over  $\mathcal{O}(G)$ . But  $A_{S^\circ} \cong \mathcal{O}_q(G)^{\otimes 2g}$  considered as an  $\mathcal{O}(G)^{\otimes 2g}$ -module is a tensor product of finitely generated modules, so it is also finitely generated.  $\blacksquare$

**Lemma 4.2.** *The fiber of the algebra  $A_{S^\circ}$  at the trivial local system on  $\mathrm{Ch}_G^{\mathrm{fr}}(S)$  is a matrix algebra.*

*Proof.* The main observation is that the fiber of  $A_{S^\circ}$  at the identity is itself an instance of a distinguished object in factorization homology, namely it is the algebra  $A_{S^\circ}^{\mathrm{sm}}$ , the internal endomorphism algebra of the distinguished object in the factorization homology category of  $S^\circ$  with coefficients in the braided tensor category  $\mathrm{Vect} \otimes_{\mathrm{Rep}(G)} \mathrm{Rep}_q(G)$ . By [7] this category is equivalent to  $\mathrm{Rep}(u_q(\mathfrak{g}))$ .



Recall the notion of an elliptic double  $E_H$  from [17] associated to a quasi-triangular Hopf algebra  $H$ . The results of [10] therefore give an isomorphism

$$A_{S^\circ}^{\text{sm}} \cong E_{u_q(\mathfrak{g})}^{\tilde{\otimes} g},$$

the braided tensor product in  $\text{Rep}(u_q(\mathfrak{g}))$  of the elliptic double  $E_{u_q(\mathfrak{g})}$ . By [17, Theorem 5.6] the elliptic double  $E_{u_q(\mathfrak{g})}$  is isomorphic to the Heisenberg double since  $u_q(\mathfrak{g})$  is factorizable. By the definition of the Heisenberg double,  $E_{u_q(\mathfrak{g})}$  acts faithfully on  $u_q(\mathfrak{g})$  and hence (comparing dimensions) we have an isomorphism

$$E_{u_q(\mathfrak{g})} \cong \text{End}(u_q(\mathfrak{g})).$$

But the braided tensor product of matrix algebras is a matrix algebra by Proposition 3.30. ■

In particular, by Lemma 3.31 we conclude that  $\mathcal{O}(Ch_G^{\text{fr}}(S^\circ))$  coincides with the center of  $A_{S^\circ}$ . But then by Proposition 3.23, we obtain the structure of a Frobenius Poisson order on the pair  $(A_{S^\circ}, Ch_G^{\text{fr}}(S^\circ))$ .

**Proposition 4.3.** *The Poisson  $G$ -variety  $Ch_G^{\text{fr}}(S^\circ)$  is non-degenerate.*

*Proof.* Recall from [28] that  $Ch_G^{\text{fr}}(S^\circ)$  carries a natural Poisson structure coming from a classical  $r$ -matrix on  $\mathfrak{g}$ . Using Proposition 3.22, one can generalize [10, Theorem 7.3] to show that the Poisson structure on  $Ch_G^{\text{fr}}(S^\circ)$  obtained by Proposition 3.23 coincides with the Fock–Rosly Poisson structure associated to the classical  $r$ -matrix in Proposition 3.22.

We may construct the surface  $S^\circ$  by fusing together cylinders. By Proposition 3.25 this means the Poisson structure on  $Ch_G^{\text{fr}}(S^\circ)$  is given by the fusion of the framed character varieties for a cylinder.

The Fock–Rosly Poisson structure on the cylinder is twist equivalent to the quasi-Poisson  $(G \times G)$ -variety  $D(G)$  (see [3, Example 5.4]), which is non-degenerate by [3, Example 10.5]. ■

#### 4.2. The Frobenius quantum moment map

Let us now recall how to compute the quantum character variety of a closed surface in terms of quantum Hamiltonian reduction following [11]. Let  $\text{Ann}$  be the annulus and consider the embedding  $\text{Ann} \subset S^\circ$  as a closed neighborhood of the boundary. We may identify  $\int_{\text{Ann}} \text{Rep}_{\mathcal{R}}(G)$  with the category of  $\mathcal{O}_{\mathcal{R}}(G)$ -modules in  $\text{Rep}_{\mathcal{R}}(G)$ . Moreover,  $\int_{\text{Ann}} \text{Rep}_{\mathcal{R}}(G)$  carries a natural monoidal structure coming from stacking of annuli. Moreover, embedding of the disk into the annulus gives a monoidal functor  $\text{Rep}_{\mathcal{R}}(G) \rightarrow \int_{\text{Ann}} \text{Rep}_{\mathcal{R}}(G)$ .

By functoriality of factorization homology, we get the structure of a  $\int_{\text{Ann}} \text{Rep}_{\mathcal{R}}(G)$ -module category on  $\int_{S^\circ} \text{Rep}_{\mathcal{R}}(G)$ . In particular, taking the internal endomorphism algebras of the distinguished objects gives a map

$$\mu_{\mathcal{R}}: \mathcal{O}_{\mathcal{R}}(G) \rightarrow A_{S^\circ}^{\mathcal{R}}$$

of algebras in  $\text{Rep}_{\mathcal{R}}(G)$ . By [11, Proposition 4.2] and [64, Proposition 3.6] it satisfies the quantum moment map equation (3.2).

Finally, by [11, Theorem 5.4] we may identify the quantum character variety of the closed surface as a quantum Hamiltonian reduction:

$$\text{End}(\text{Dist}_S^{\mathcal{R}}) \cong A_{S^\circ}^{\mathcal{R}} // U_{\mathcal{R}}^L(\mathfrak{g}),$$

and similarly for its specialization at  $t = q$ .

Combining Lemma 3.27 and Proposition 3.26 we find that  $(A_{S^\circ}, \text{Ch}_G^{\text{fr}}(S^\circ))$  is a weakly  $G$ -Hamiltonian Frobenius Poisson order. In particular, there is a classical moment map  $\mu: \text{Ch}_G^{\text{fr}}(S^\circ) \rightarrow G$ . We will now identify this moment map explicitly.

**Proposition 4.4.** *The classical moment map  $\mu: \text{Ch}_G^{\text{fr}}(S^\circ) \cong G^{2g} \rightarrow G$  is given by*

$$\mu(x_1, y_1, \dots, x_g, y_g) = \prod_{i=1}^g [x_i, y_i].$$

*Proof.* Consider the category  $\Delta^1$  consisting of two objects and a single morphism between them. The  $(2, 1)$ -category  $\text{Fun}(\Delta^1, \text{Pr})$  carries a natural symmetric monoidal structure given by the pointwise tensor product. Then the braided tensor functor  $(\text{Rep}(G) \rightarrow \text{Rep}_q(G))$  defines a functor  $\text{Disk}_{\text{or}}^2 \rightarrow \text{Fun}(\Delta^1, \text{Pr})$ . Since colimits in  $\text{Fun}(\Delta^1, \text{Pr})$  are computed pointwise, the factorization homology is also computed pointwise.

By functoriality of factorization homology we get a morphism

$$\int_{\text{Ann}} (\text{Rep}(G) \rightarrow \text{Rep}_q(G)) \rightarrow \int_{S^\circ} (\text{Rep}(G) \rightarrow \text{Rep}_q(G))$$

in  $\text{Fun}(\Delta^1, \text{Pr})$ , i.e., a commutative diagram

$$\begin{array}{ccc} \int_{\text{Ann}} \text{Rep}_q(G) & \longrightarrow & \int_{S^\circ} \text{Rep}_q(G) \\ \uparrow & & \uparrow \\ \int_{\text{Ann}} \text{Rep}(G) & \longrightarrow & \int_{S^\circ} \text{Rep}(G) \end{array}$$

of pointed categories. Computing the internal endomorphisms of the distinguished objects, we get the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_q(G) & \xrightarrow{\mu_q} & A_{S^\circ} \\ \uparrow & & \uparrow \\ \mathcal{O}(G) & \longrightarrow & \mathcal{O}(G^{2g}) \end{array}$$

Since the map  $\mathcal{O}(G) \hookrightarrow \mathcal{O}_q(G)$  is injective, the classical moment map is given by the bottom morphism. But it is induced by the homomorphism  $\pi_1(\text{Ann}) \rightarrow \pi_1(S^\circ)$  given by monodromy around the boundary. ■

### 4.3. Proof of Theorem 1.1

Consider the open subvariety  $Ch_G^{\text{fr}, \text{good}}(S) \subset Ch_G^{\text{fr}}(S)$  consisting of points whose  $G$ -orbit is closed and whose  $G$ -stabilizer is the center (see e.g. [67]). This locus is empty when  $g = 0, 1$ , but otherwise non-empty. Denote by  $Ch_G^{\text{good}}(S) \subset Ch_G(S)$  its image under the quotient map.

We are now ready to prove our main theorem about character varieties, Theorem 1.1, which we recall here:

**Theorem 4.5** (Theorem 1.1). *Let  $G$  be a connected reductive group and  $q$  a primitive  $\ell$ -th root of unity, which together satisfy Assumption 3.1. Let  $S$  be a closed topological surface of genus  $g$ , and denote by  $S^\circ$  the surface obtained by removing some open disk from  $S$ .*

- (1) *The moduli algebra  $A_{S^\circ}$  is finitely generated over its center, which is isomorphic to the coordinate ring of the classical framed  $G$ -character variety  $Ch_G^{\text{fr}}(S^\circ) \cong G^{2g}$ .*
- (2) *The Azumaya locus of the moduli algebra  $A_{S^\circ}$  coincides with the preimage of the open cell  $G^\circ \subset G$  under the classical moment map  $Ch_G^{\text{fr}}(S^\circ) \rightarrow G$ .*
- (3) *The quantized character variety of the closed surface  $S$  is finitely generated over its center, which is isomorphic to the coordinate ring of the classical character variety. It may be constructed as a Frobenius quantum Hamiltonian reduction of  $A_{S^\circ}$ .*
- (4) *The quantized character variety of the closed surface  $S$  is Azumaya over the entire ‘good locus’  $Ch_G^{\text{good}}(S) \subset Ch_G(S)$ .*

*Proof.* The first statement is established in Lemma 4.1. The containment of the Azumaya locus within  $\mu^{-1}(G^\circ)$  is Lemma 3.33. To confirm the remaining statements, we will use Theorem 3.32, and hence we need only confirm the assumptions of that theorem.

We take for  $A_{\mathcal{R}}$  the algebra  $A_{S^\circ}^{\mathcal{R}}$ , and for  $\xi_q$  the counit  $\varepsilon_q$  of  $\mathcal{O}_q(G)$ , lying over the identity element  $e \in G$  viewed as a character on  $\mathcal{O}(G)$ . For  $Z$  we take  $\mathcal{O}(Ch_G^{\text{fr}}(S^\circ))$ . For  $G_{\text{stab}}$  we take the center  $Z(G) \subset G$ .

The moment map  $\mu: Ch_G^{\text{fr}}(S^\circ) \rightarrow G$  is a product of commutators, so it lands in the derived subgroup  $G_{\text{der}} = G_0 \subset G$ . Note that  $G_{\text{der}} \subset G \rightarrow G/Z(G)$  is étale.

The classical character variety  $Ch_G^{\text{fr}}(S^\circ)$  is isomorphic to  $G^{2g}$ , so it is smooth and connected. Moreover, by Proposition 4.3 it is a non-degenerate Poisson  $G$ -variety, which proves assumption (1) of Theorem 3.32.

For  $U$  we take the locus of ‘good representations’  $Ch_G^{\text{fr}, \text{good}}(S) \subset \mu^{-1}(e)$ . Note that the center  $Z(G) = G_{\text{stab}}$  acts trivially on the whole character variety  $Ch_G^{\text{fr}}(S^\circ)$ . By Remark 3.6 the weights of  $\mathcal{O}_q(G)$  lie in the root lattice. Since the quantum algebra  $\mathcal{A}_{S^\circ}$  is a tensor product of the algebras  $\mathcal{O}_q(G)$ , the weights of  $\mathcal{A}_{S^\circ}$  also lie in the root lattice. Therefore, the weights of  $\mathcal{A}_{S^\circ}|_{\mu^{-1}(e)} // u_q(\mathfrak{g})$  also lie in the root lattice of  $G$ , so  $Z(G)$  acts trivially, which proves assumption (2) of Theorem 3.32.

The surjectivity criterion, assumption (3) of Theorem 3.32, can be checked using the theory of good filtrations. Namely,  $\mathcal{O}_{\mathcal{R}}(G)$  admits a good filtration by Theorem 3.7. Since

$\mathcal{A}_{S^\circ}^{\mathcal{R}}$  is a tensor product of  $\mathcal{O}_{\mathcal{R}}(G)$ , by Proposition 3.4 it also admits a good filtration and hence satisfies the surjectivity criterion. ■

**Remark 4.6.** We note that the proof of Theorem 4.5 gives a mechanism to compute the fibers of the algebras  $\mathcal{A}_{S^\circ}$  and its Hamiltonian reduction over the good locus, even when  $q$  and  $G$  do not satisfy Assumption 3.1, i.e.,  $\text{Rep}(u_q(\mathfrak{g}))$  is not factorizable. We hope to return to this in future work.

The results are, in a sense, optimal when  $G = \text{GL}_n$  or  $\text{SL}_n$  [9, Lemma 2.4].

**Proposition 4.7.** *Suppose  $G = \text{GL}_n$  or  $\text{SL}_n$  and  $g \geq 2$ . Then  $\text{Ch}_G^{\text{good}}(S)$  coincides with the smooth locus of  $\text{Ch}_G(S)$ .*

#### 4.4. Kauffman bracket skein algebras and the Unicity Conjecture

Let  $S$  be an oriented surface and let  $K_A(S)$  be the Kauffman bracket skein algebra which is a  $\mathbb{C}[A, A^{-1}]$ -algebra (see e.g. [29, Section 3] for recollections). The following is first proved in [60] (see also [29, Theorem 3.1]).

**Theorem 4.8.**  *$K_A(S)$  is free as a  $\mathbb{C}[A, A^{-1}]$ -module, with basis given by the set of isotopy classes of simple multi-curves on  $S$ .*

Furthermore, we have the following result.

**Theorem 4.9.** *Each spin structure on  $S$  induces an isomorphism between the Poisson algebra  $K_1(S)$  equipped with its natural Poisson bracket and the Poisson algebra  $\mathcal{O}(\text{Ch}_{\text{SL}_2}(S))$  equipped with its Fock–Rosly (equivalently, Goldman) Poisson structure.*

*Proof.* It is shown in [20, 69] that the Poisson algebra  $K_{-1}(S)$  is isomorphic to  $\mathcal{O}(\text{Ch}_{\text{SL}_2}(S))$  equipped with the Fock–Rosly Poisson structure.

In addition, Barrett [8, Theorem 1] constructs an isomorphism of  $\mathbb{C}[A, A^{-1}]$ -algebras

$$K_A(S) \cong K_{-A}(S)$$

using a spin structure on  $S$ .

At  $A = 1$  both algebras become commutative. So, by functoriality of the Poisson bracket induced on the center (see Proposition 3.23), we obtain an isomorphism of Poisson algebras

$$K_1(S) \cong K_{-1}(S). \quad \blacksquare$$

We will now show that the same Poisson structure appears naturally when we degenerate to roots of unity. Fix an odd number  $\ell > 1$  and suppose  $\zeta$  is a primitive  $\ell$ -th root of unity. Denote

$$K_\zeta(S) = K_A(S)/(A - \zeta).$$

A fundamental construction in the study of Kauffman bracket skein modules at root of unity parameters is the following statement [14].

**Theorem 4.10.** *There exist injective homomorphisms*

$$\mathrm{Fr}^{(\ell)}: K_1(S) \rightarrow Z(K_\zeta(S)),$$

*natural for surface embeddings.*

In [29, Theorem 4.1], the quantum Frobenius map  $\mathrm{Fr}^{(\ell)}$  was proved to be an isomorphism onto the center in many cases. The particular case of their theorem we will require is as follows:

**Theorem 4.11.** *Suppose the surface  $S$  is closed. Then  $\mathrm{Fr}^{(\ell)}$  is an isomorphism.*

Therefore, Proposition 3.23 provides a Poisson order structure on  $(K_\zeta(S), K_1(S))$ . So, for every  $\ell$  we obtain a Poisson structure on the same algebra  $K_1(S)$ .

**Lemma 4.12.** *Let  $S^\circ$  be a surface obtained by removing some disk from a closed surface  $S$ . Then  $(K_\zeta(S^\circ), K_1(S^\circ))$  is a Poisson order.*

*Proof.* We would like to apply Proposition 3.23 (3) directly to conclude that  $K_1(S)$  is closed under the Poisson bracket, but because  $S^\circ$  is not closed,  $\mathrm{Fr}^{(\ell)}(K_1(S^\circ)) \neq Z(K_\zeta(S^\circ))$ , so the criterion does not apply. Neither will it suffice to embed  $S^\circ$  back into the closed surface  $S$ , because the induced map on skein algebras has a kernel. Instead, let us embed  $S^\circ$  into a surface  $T$  of genus 1 greater than  $S$ , so that by Theorem 4.8 the induced map  $K_A(S^\circ) \rightarrow K_A(T)$  is injective. Since  $T$  is closed and  $\mathrm{Fr}^{(\ell)}$  is natural for surface embeddings, we may indeed apply Proposition 3.23 (3), to conclude that

$$\{\mathrm{Fr}^{(\ell)}(K_1(S^\circ)), \mathrm{Fr}^{(\ell)}(K_1(S^\circ))\} \subset \mathrm{Fr}^{(\ell)}(K_1(T)) \cap K_\zeta(S^\circ) = \mathrm{Fr}^{(\ell)}(K_1(S^\circ)),$$

as desired. ■

**Proposition 4.13.** *The Poisson structure on  $K_1(S)$  in the Poisson order  $(K_\zeta(S), K_1(S))$  is independent, up to a factor, of the order  $\ell$  of the root of unity.*

*Proof.* The claim is trivial if  $S$  has genus 0, so we will assume that it has genus  $\geq 1$ . Let us first denote by  $S^\circ$  the surface obtained from  $S$  by removing some disk.

Choose  $\omega$  such that  $\omega^{-2} = \zeta$ . By [13, Theorem 1] (see also [55]), any suitable triangulation  $\Delta$  of  $S^\circ$  determines an algebra embedding

$$\mathrm{Tr}_\Delta^\omega: K_\zeta(S^\circ) \rightarrow \mathcal{T}_\zeta^\Delta,$$

of the Kauffman bracket skein algebra into a quantum torus  $\mathcal{T}_\zeta^\Delta$ . Here by quantum torus we mean that  $\mathcal{T}_\zeta^\Delta$  is presented with invertible generators  $X_1, \dots, X_r$ , for some  $r$ , with relations  $X_i X_j = \zeta^{n_{ij}} X_j X_i$ , for some skew-symmetric integer matrix  $(n_{ij})$ .

On the quantum torus we have a simple quantum Frobenius homomorphism,

$$\mathrm{Fr}^{(\ell)}: \mathcal{T}_1^\Delta \rightarrow Z(\mathcal{T}_\zeta^\Delta),$$

which sends each generator  $X_i$  of the ring  $\mathcal{T}_1^\Delta$  to the  $\ell$ -th power of the corresponding generator in  $\mathcal{T}_\zeta^\Delta$ . In particular, by Proposition 3.23 we obtain the structure of a Poisson

order on  $(\mathcal{T}_\xi^\Delta, \mathcal{T}_1^\Delta)$ . It is straightforward to see that the induced Poisson bracket on  $\mathcal{T}_1^\Delta$  depends on  $\ell$  only up to an overall scalar factor.

By [14, Theorem 21] we have a map of Poisson orders

$$(\mathrm{Tr}_\Delta^\omega, \mathrm{Tr}_\Delta^{\omega^{\ell^2}}): (K_\xi(S^\circ), K_1(S^\circ)) \hookrightarrow (\mathcal{T}_\xi^\Delta, \mathcal{T}_1^\Delta).$$

Since any skein in  $S$  is isotopic to one missing the disk, the map  $p: K_A(S^\circ) \rightarrow K_A(S)$  is surjective. By naturality of  $\mathrm{Fr}^{(\ell)}$  under surface embeddings (Theorem 4.10) we have a map of Poisson orders

$$p: (K_\xi(S^\circ), K_1(S^\circ)) \twoheadrightarrow (K_\xi(S), K_1(S)).$$

Consider a pair of elements  $f, g \in K_1(S)$ . Our goal will be to show that  $\{f, g\}$  is independent of  $\ell$  up to a factor. Since  $p: K_1(S^\circ) \rightarrow K_1(S)$  is surjective, we may find the lifts  $\tilde{f}, \tilde{g} \in K_1(S^\circ)$  of  $f, g$  which we assume are independent of  $\ell$ .

The Poisson bracket  $\{\mathrm{Tr}_\Delta^{\omega^{\ell^2}}(\tilde{f}), \mathrm{Tr}_\Delta^{\omega^{\ell^2}}(\tilde{g})\}$  is independent of  $\ell$  up to a factor. Since  $\mathrm{Tr}_\Delta^{\omega^{\ell^2}}: K_1(S^\circ) \hookrightarrow \mathcal{T}_1^\Delta$  is Poisson, the same claim holds for  $\{\tilde{f}, \tilde{g}\}$ . Since  $p: K_1(S^\circ) \rightarrow K_1(S)$  is Poisson, the same claim also holds for  $\{f, g\}$ . ■

**Remark 4.14.** In a previous version of this paper, Proposition 4.13 was incorrectly attributed to the literature. We are grateful to Thang Le for bringing this to our attention, and suggesting that we add a complete proof.

**Theorem 4.15** (Theorem 1.2). *The skein algebra  $K_\xi(S)$  is Azumaya over the whole smooth locus of  $Ch_{\mathrm{SL}_2}(S)$ .*

*Proof.* It is shown in [29, Theorem 2] that  $K_\xi(S)$  is generically Azumaya over  $Ch_{\mathrm{SL}_2}(S)$ . Consider the open symplectic leaf in  $Ch_{\mathrm{SL}_2}(S)$  (which is, in particular, smooth). Since  $(K_\xi(S), \mathcal{O}(Ch_{\mathrm{SL}_2}(S)))$  is a Poisson order, by Theorem 3.16,  $K_\xi(S)$  is Azumaya over the open symplectic leaf in  $Ch_{\mathrm{SL}_2}(S)$ . To prove the claim, we have to establish that the open symplectic leaf coincides with the smooth locus.

The case  $g = 0$  is trivial.

Suppose  $g = 1$ . We have an isomorphism  $Ch_{\mathrm{SL}_2}(S) \cong (\mathbb{C}^\times \times \mathbb{C}^\times) / \mathbb{Z}_2$  where  $\mathbb{Z}_2$  acts on each factor by  $z_i \mapsto z_i^{-1}$ . In particular, the smooth locus is the locus of pairs  $(z_1, z_2)$  where  $z_i \neq \pm 1$ . The Poisson structure on  $Ch_{\mathrm{SL}_2}(S)$  comes from the standard symplectic structure on  $\mathbb{C}^\times \times \mathbb{C}^\times$ . Since the map  $\mathbb{C}^\times \times \mathbb{C}^\times \rightarrow Ch_{\mathrm{SL}_2}(S)$  restricts to a  $\mathbb{Z}_2$ -torsor over the smooth locus, we deduce that the Poisson structure on the smooth locus is symplectic.

Suppose  $g \geq 2$ . Then by Proposition 4.7 the open symplectic leaf coincides with the smooth locus. ■

## 5. Quantized multiplicative quiver varieties

We now recall the definition of the multiplicative quiver variety, following [23, 76], and its quantization, following [37]. We subsequently implement the program of Section 3,

in order to deduce that quantized multiplicative varieties define sheaves of Azumaya algebras over classical multiplicative quiver varieties. Throughout this section,  $q$  denotes a primitive root of unity of order  $\ell$ , where  $\ell > 1$  is odd.

### 5.1. The classical multiplicative quiver variety

Let  $Q = (E, V)$  be a finite quiver. The doubled quiver  $\bar{Q}$  has the same vertex set  $\bar{V} = V$  but with an additional ‘dual’ edge  $e^\vee : j \rightarrow i$ , for each edge  $e : i \rightarrow j$  of  $Q$ , so  $\bar{E} = E \sqcup E^\vee$ . For  $e \in \bar{E}$ , set  $\varepsilon(e)$  to be equal to 1 if  $e \in E$  and  $-1$  if  $e \in E^\vee$ , and write  $\alpha = \alpha(e) \in V$  and  $\beta = \beta(e) \in V$  for the source and target of  $e$ . Fix a dimension vector  $\mathbf{d} = (\mathbf{d}_v)_{v \in V} \in (\mathbb{Z}_{\geq 0})^V$ .

A framed representation of  $\bar{Q}$  with dimension vector  $\mathbf{d}$  is an assignment of a linear map  $\mathbb{C}^{\mathbf{d}_\alpha} \rightarrow \mathbb{C}^{\mathbf{d}_\beta}$  to each edge  $e \in \bar{E}$ . The moduli space  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  of framed representations of  $\bar{Q}$  is therefore a Cartesian product of spaces of matrices,

$$\mathcal{M}_{\text{fr}}(Q, \mathbf{d}) = \prod_{e \in E} \text{Mat}(e) \times \text{Mat}(e^\vee), \quad \text{where} \quad \text{Mat}(e) = \text{Hom}_{\mathbb{C}}(\mathbb{C}^{\mathbf{d}_\alpha}, \mathbb{C}^{\mathbf{d}_\beta}).$$

The group  $G = \text{GL}_{\mathbf{d}} = \prod_v \text{GL}_{d_v}$  acts on  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  by change of basis at each vertex.

Let  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ$  denote the Zariski open locus of  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  on which the determinants of the matrices  $\text{Id}_\alpha + X_{e^\vee} X_e$  are non-vanishing for all  $e \in \bar{E}$ . We have a *multiplicative moment map*

$$\tilde{\mu} : \mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ \rightarrow \text{GL}_{\mathbf{d}}, \quad X \mapsto \prod_{e \in \bar{E}} (\text{Id}_\alpha + X_{e^\vee} X_e)^{\varepsilon(e)}. \quad (5.1)$$

Fix a function  $\xi : V \rightarrow \mathbb{C}^\times$ , satisfying  $\prod_v \xi_v = 1$ , which we regard as a collection  $\xi = (\xi_v)_{v \in V} \in \text{GL}_{\mathbf{d}}$  of scalar matrices. Fix a character  $\theta : G \rightarrow \mathbb{C}^\times$ . Denote by  $\mathbb{C}_\theta \in \text{Rep}(G)$  the corresponding one-dimensional representation.

**Definition 5.1.** The *multiplicative quiver variety* is the GIT Hamiltonian reduction

$$\begin{aligned} \mathcal{M}(Q, \mathbf{d}, \xi, \theta) &= \mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ //_{\theta} \text{GL}_{\mathbf{d}} \\ &= \tilde{\mu}^{-1}(\xi) /_{\theta} \text{GL}_{\mathbf{d}} = \text{Proj} \left( \bigoplus_{m=0}^{\infty} \mathcal{O}(\tilde{\mu}^{-1}(\xi)) \otimes \mathbb{C}_{\theta^{-m}} \right)^{\text{GL}_{\mathbf{d}}}. \end{aligned}$$

Recall the notion of  $\theta$ -semistable and  $\theta$ -stable points of the  $G$ -variety  $\tilde{\mu}^{-1}(\xi)$  (see [56, Chapter 1] and [40, Section 2]). Then  $\tilde{\mu}^{-1}(\xi) /_{\theta} G$  may be identified with the quotient of the open subset of  $\tilde{\mu}^{-1}(\xi)$  of  $\theta$ -semistable points by a certain equivalence relation. In particular, we obtain a surjective morphism

$$\pi : \tilde{\mu}^{-1}(\xi)^{\theta\text{-ss}} \rightarrow \tilde{\mu}^{-1}(\xi) /_{\theta} G.$$

**Definition 5.2.** The *stable multiplicative quiver variety*  $\mathcal{M}^s(Q, \mathbf{d}, \xi, \theta) \subset \mathcal{M}(Q, \mathbf{d}, \xi, \theta)$  is the image of the  $\theta$ -stable points of  $\tilde{\mu}^{-1}(\xi)$ .

Note that by construction we have a projective surjection  $\tilde{\mu}^{-1}(\xi)/_{\theta}G \rightarrow \tilde{\mu}^{-1}(\xi)/G$ , which is an isomorphism for  $\theta$  trivial. The construction of a Poisson structure on  $\tilde{\mu}^{-1}(\xi)/_{\theta}G$  mimics the construction of the Poisson structure on  $\tilde{\mu}^{-1}(\xi)/G$  given by Proposition 2.9.

**Remark 5.3.** The multiplicative quiver variety was constructed first in [23] as a moduli space of representations of a multiplicative preprojective algebra. The construction as a multiplicative Hamiltonian reduction was accomplished in [71, 72], using the formalism of quasi-Hamiltonian moment maps. See Remark 5.24 for more details.

**Example 5.4.** An important special case is when  $\theta = 0$ . Then it is well-known (see e.g. [31]) that every representation of  $Q$  is semistable, and that only the simple representations are stable. For general  $\theta$ , A. King [40] gave a purely algebraic description of  $\theta$ -(semi)stability in terms of excluded ‘slopes’ of subrepresentations.

**Remark 5.5.** It can often happen that  $\mathcal{M}^s(Q, \mathbf{d}, \xi, \theta)$  is empty altogether – for instance when  $\theta = 0$ , not every dimension vector supports a simple representation of the preprojective algebra. At the other extreme, it can happen that the semistable and stable loci coincide. In this case  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$  defines a symplectic resolution of  $\mathcal{M}(Q, \mathbf{d}, \xi, 0)$ , by results of [65]. Crawley-Boevey and Shaw [23] have given an explicit characterization for when the stable locus is non-empty in terms of a certain root datum on the quiver; we refer to [65] for the complete statement.

## 5.2. The quantization of $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$

We recall the quantizations of the framed moduli space and the multiplicative quiver variety, following [37, Section 3], and we describe a straightforward extension of the construction to non-trivial GIT quotients. We begin by recalling the quantization of the framed moduli space.

**Definition 5.6** ([37, Example 4.8]). The algebra  $\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet})$  is the algebra generated over  $\mathbb{C}$  by elements  $x_j^i$  and  $\partial_k^l$ , for  $1 \leq i, k \leq M$  and  $1 \leq j, l \leq N$ , organized into an  $N \times M$  matrix  $X$  and an  $M \times N$  matrix  $D$ , with relations given by the following matrix equations:

$$\begin{aligned} RX_2X_1 &= X_1X_2R_{21}, \\ RD_2D_1 &= D_1D_2R_{21}, \\ D_2R^{-1}X_1 &= X_1RD_2 + \Omega, \quad \text{where} \quad \Omega = \sum E_j^i \otimes E_i^j. \end{aligned}$$

Define  $\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet})^{\circ}$  as the non-commutative localization at the quantum determinants  $\det_q$  of the following matrices:

$$\begin{aligned} g^{\alpha} &:= \text{Id} + (q - q^{-1})DX, \\ g^{\beta} &:= \text{Id} + (q - q^{-1})XD. \end{aligned}$$



**Lemma 5.7.** *In coordinates,  $x_j^i$  and  $\partial_l^k$  satisfy*

$$\begin{aligned}
 x_m^i x_n^j &= q^{\delta_{mn}} x_n^j x_m^i + \theta(n-m)(q-q^{-1})x_m^j x_n^i & (i > j), \\
 x_m^i x_n^i &= q^{-1} x_n^i x_m^i & (m > n), \\
 \partial_m^i \partial_n^j &= q^{\delta_{mn}} \partial_n^j \partial_m^i + \theta(n-m)(q-q^{-1})\partial_m^j \partial_n^i & (i > j), \\
 \partial_m^i \partial_n^i &= q^{-1} \partial_n^i \partial_m^i & (m > n), \\
 \partial_m^i x_n^j &= q^{\delta_{in} + \delta_{jm}} x_n^j \partial_m^i + \delta_{in} q^{\delta_{jm}} (q-q^{-1}) \sum_{p>i} x_p^j \partial_m^p \\
 &\quad + \delta_{jm} (q^2 - 1) \sum_{p<j} \partial_p^i x_n^p + q \delta_{in} \delta_{jm}.
 \end{aligned}$$

We recall the following PBW basis theorem for the algebra  $\mathcal{D}_q(\bullet \rightarrow \bullet)$ .

**Theorem 5.8** ([37]). *The ordered monomials (in any fixed ordering) of the form*

$$(x_{j_1}^{i_1})^{k_1} \dots (x_{j_m}^{i_m})^{k_m} \cdot (\partial_{s_1}^{r_1})^{t_1} \dots (x_{s_n}^{r_n})^{t_n}$$

*define a basis of the algebra  $\mathcal{D}_q(\bullet \rightarrow \bullet)$ .*

**Definition 5.9.** The algebra  $\mathcal{D}_q(\bullet \curvearrowright \bullet)$  is generated over  $\mathbb{C}$  by elements  $a_i^j$  and  $d_i^j$ , organized into  $N \times N$  matrices  $A$  and  $D$ , with relations given by the following matrix equations:

$$\begin{aligned}
 R_{21} A_1 R D_2 &= A_2 R_{21} A_1 R, & R_{21} D_1 R D_2 &= D_2 R_{21} D_1, \\
 R_{21} D_1 R A_2 &= A_2 R_{21} D_1 (R_{21})^{-1}.
 \end{aligned}$$

The algebra  $\mathcal{D}_q(\bullet \curvearrowright \bullet)^\circ$  is the localization at the element  $\det_q(A)\det_q(D)$ , which is a  $q$ -central element.

**Remark 5.10.** An important observation is that  $\mathcal{D}_q(\bullet \curvearrowright \bullet)^\circ$  is precisely the algebra  $A_{T^2 \setminus D^2}$  for the group  $\mathrm{GL}_N$ .

**Definition 5.11.** Let  $Q$  be a quiver equipped with a dimension vector  $\mathbf{d}$ . Let  $e$  be an edge of  $Q$  with  $d_\alpha = N$  and  $d_\beta = M$ . Set

$$\mathcal{D}_q(\mathrm{Mat}(e)) := \begin{cases} \mathcal{D}_q(\bullet \rightarrow \bullet) & \text{if } e \text{ is not a loop,} \\ \mathcal{D}_q(\bullet \curvearrowright \bullet) & \text{if } e \text{ is a loop (so } d_\alpha = d_\beta = N). \end{cases}$$

**Definition 5.12.** Let  $\mathrm{Rep}_q(\mathrm{GL}_{\mathbf{d}})$  denote the Deligne–Kelly tensor product of the categories  $\mathrm{Rep}_q(\mathrm{GL}_{d_v})$ ,

$$\mathrm{Rep}_q(\mathrm{GL}_{\mathbf{d}}) = \boxtimes_{v \in V} \mathrm{Rep}_q(\mathrm{GL}_{d_v}).$$

We regard  $\mathrm{Rep}_q(\mathrm{GL}_{\mathbf{d}})$  as a braided tensor category with the product braiding. The tensor product of Frobenius homomorphisms gives functors

$$\mathrm{Fr}^*: \mathrm{Rep}(\mathrm{GL}_{\mathbf{d}}) \rightarrow \mathrm{Rep}_q(\mathrm{GL}_{\mathbf{d}}), \quad \mathrm{Fr}_*: \mathrm{Rep}_q(\mathrm{GL}_{\mathbf{d}}) \rightarrow \mathrm{Rep}(\mathrm{GL}_{\mathbf{d}}).$$

**Remark 5.13.** In other words, an object of  $\text{Rep}_q(\text{GL}_{\mathbf{d}})$  is a vector space equipped with commuting actions of  $U_q^L(\mathfrak{gl}_{\mathbf{d}_v})$  for each  $v \in V$ . The tensor product and braiding on each such module is induced componentwise, meaning in particular that the universal  $R$ -matrix is simply the product of those for each  $v$ .

The quantized edge algebras  $\mathcal{D}_q(\bullet \xrightarrow{N} \bullet)$  and  $\mathcal{D}_q(\bullet \curvearrowright \bullet)$  were each defined invariantly as an algebra in  $\text{Rep}_q(\text{GL}_{\mathbf{d}})$  as a quotient of a tensor algebra by the image of a certain morphism mimicking the Weyl algebra relations (see [37, Section 3.2]). We now recall the definition of the quantization of the moduli space  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  of framed representations of  $\bar{Q}$ .

**Definition 5.14** ([37]). Let  $Q$  be a quiver with dimension vector  $\mathbf{d}$ . The algebra  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  (respectively  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ$ ) is the braided tensor product in  $\text{Rep}_q(\text{GL}_{\mathbf{d}})$  of the corresponding edge algebras:

$$\mathcal{D}_q(\text{Mat}(Q, \mathbf{d})) = \bigotimes_{e \in E} \mathcal{D}_q(\text{Mat}(e)), \quad \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ = \bigotimes_{e \in E} \mathcal{D}_q(\text{Mat}(e))^\circ.$$

**Definition 5.15.** The *quantum moment map*

$$\mu_q: \mathcal{O}_q(\text{GL}_{\mathbf{d}}) \rightarrow \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ$$

is the braided tensor product of moment maps  $\mathcal{O}_q(\text{GL}_{d_\alpha} \times \text{GL}_{d_\beta}) \rightarrow \mathcal{D}_q(\text{Mat}(e))$ , each given by the formulas

$$\mu_q(a_j^i \otimes a_r^s) = \left( \delta_j^i + (q - q^{-1}) \sum_k \partial_k^i x_j^k \right)^{-1} \left( \delta_r^s + (q - q^{-1}) \sum_k x_k^s \partial_r^k \right).$$

It is proved in [37, Propositions 7.11 and 7.12] that each  $\mu_q: \mathcal{O}_q(\text{GL}_{d_\alpha} \times \text{GL}_{d_\beta}) \rightarrow \mathcal{D}_q(\text{Mat}(e))$  defines a quantum moment map in the sense of Definition 3.8, so by Proposition 3.11,  $\mu_q: \mathcal{O}_q(\text{GL}_{\mathbf{d}}) \rightarrow \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ$  is also a quantum moment map.

**Remark 5.16.** A priori, the definitions of  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  and  $\mu_q$  depend on an orientation of  $\bar{Q}$ , as well as a total ordering of the edges of  $\bar{Q}$ ; however, in [37], canonical isomorphisms are constructed identifying the different choices compatibly.

### 5.3. The Frobenius Poisson order on $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$

We now turn to the construction of the Frobenius Poisson order structure on  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$ .

**Lemma 5.17.** Fix  $i, j, n$ , and  $m$ . For  $r \geq 1$ , define the following quantities:

$$\begin{aligned} a_r &= \delta_{in} q^{\delta_{jm} r} (q^r - q^{-r}), & b_r &= \delta_{jm} q^{\delta_{in}(r-1)+1} (q^r - q^{-r}), \\ c_r &= \delta_{in} \delta_{jm} (q^{2r} - 1)(1 - q^{-2(r-1)}), & d_r &= \delta_{in} \delta_{jm} \frac{q^{2r} - 1}{q - q^{-1}}, \\ t_r &= \theta(n - m)(q^{2r-1} - q^{-1}). \end{aligned}$$

For  $r, s \geq 1$ , the following identities hold in  $\mathcal{D}_q(\bullet \rightarrow \bullet)$ :

$$\begin{aligned}
x_m^i (x_n^j)^r &= q^{r\delta_{mn}} (x_n^j)^r x_m^i + t_r (x_n^j)^{r-1} x_m^j x_n^i \quad (i > j), \\
(x_m^i)^r x_n^j &= q^{r\delta_{mn}} x_n^j (x_m^i)^r + t_r x_m^j x_n^i (x_m^i)^{r-1} \quad (i > j), \\
\partial_m^i (\partial_n^j)^r &= q^{r\delta_{mn}} (\partial_n^j)^r \partial_m^i + t_r (\partial_n^j)^{r-1} \partial_m^j \partial_n^i \quad (i > j), \\
(\partial_m^i)^r \partial_n^j &= q^{r\delta_{mn}} \partial_n^j (\partial_m^i)^r + t_r \partial_m^j \partial_n^i (\partial_m^i)^{r-1} \quad (i > j), \\
(x_m^i)^r (x_n^i)^s &= q^{-rs} (x_n^i)^s (x_m^i)^r \quad (m > n), \\
(\partial_m^i)^r (\partial_n^i)^s &= q^{-rs} (\partial_n^i)^s (\partial_m^i)^r \quad (m > n), \\
\partial_m^i (x_n^j)^r &= q^{(\delta_{in} + \delta_{jm})r} (x_n^j)^r \partial_m^i + a_r \sum_{p>i} (x_n^j)^{r-1} x_p^m \partial_m^p + b_r \sum_{p'<j} (x_n^j)^{r-1} \partial_{p'}^i x_n^{p'} \\
&\quad + c_r \sum_{p>n, p'<m} (x_n^m)^{r-2} x_p^m \partial_{p'}^p x_n^{p'} + d_r (x_n^j)^{r-1} \\
(\partial_m^i)^r x_n^j &= q^{(\delta_{in} + \delta_{jm})r} x_n^j (\partial_m^i)^r + a_r \sum_{p>i} x_p^m \partial_m^p (\partial_m^i)^{r-1} + b_r \sum_{p'<j} \partial_{p'}^i x_n^{p'} (\partial_m^i)^{r-1} \\
&\quad + c_r \sum_{p>n, p'<m} \partial_{p'}^i x_p^{p'} \partial_m^p (\partial_m^i)^{r-2} + d_r (\partial_m^i)^{r-1}.
\end{aligned}$$

*Proof.* We give a proof of the identity involving  $\partial_m^i (x_n^j)^r$ ; the justification of the other identities is similar or easier. We proceed by induction on  $r$ . The base case  $r = 1$  follows from definitions. Now, if  $p > i = n$ , then

$$x_p^j \partial_j^p x_n^j = q^{\delta_{jm}-1} x_n^j x_p^j \partial_j^p + \delta_{jm} (q^2 - 1) \sum_{p'<j} x_p^j \partial_{p'}^p x_n^{p'},$$

and if  $p' < j = m$ , then

$$\partial_{p'}^i x_i^{p'} x_n^j = q^{\delta_{in}-1} x_n^j \partial_{p'}^i x_i^{p'} + \delta_{in} (1 - q^{-2}) \sum_{p>i} x_p^j \partial_{p'}^p x_i^{p'}.$$

Finally, if  $p > i = n$  and  $p' < j = m$ , then  $x_p^j \partial_{p'}^p x_i^{p'} x_n^j = q^{-2} x_i^j x_p^j \partial_{p'}^p x_i^{p'}$ . We write

$$\begin{aligned}
\partial_m^i (x_n^j)^r &= q^{(\delta_{in} + \delta_{jm})r} (x_n^j)^r \partial_m^i + a_r \sum_{p>i} (x_n^j)^{r-1} x_p^m \partial_m^p \\
&\quad + b_r \sum_{p'<j} (x_n^j)^{r-1} \partial_{p'}^i x_n^{p'} + c_r \sum_{p>n, p'<m} (x_n^j)^{r-2} x_p^m \partial_{p'}^p x_n^{p'} + d_r (x_n^j)^{r-1}
\end{aligned}$$

for some  $a_r, b_r, c_r, d_r$  in  $\mathbb{C}$ . Straightforward computations imply the recursive relations

$$\begin{aligned}
a_{r+1} &= q^{\delta_{jm}-1} a_r + \delta_{in} q^{(1+\delta_{jm})r + \delta_{jm}} (q - q^{-1}), \\
b_{r+1} &= q^{\delta_{in}-1} b_r + \delta_{jm} q^{(1+\delta_{in})r} (q^2 - 1), \\
c_{r+1} &= \delta_{in} \delta_{jm} (q^2 - 1) a_r + (1 - q^{-2}) b_r + q^{-2} c_r, \\
d_{r+1} &= d_r + \delta_{in} \delta_{jm} q^{2r+1}.
\end{aligned}$$

From the initial conditions  $a_1 = \delta_{in} q^{\delta_{jm}} (q - q^{-1})$ ,  $b_1 = \delta_{jm} (q^2 - 1)$ ,  $c_1 = 0$ , and  $d_1 = q$ , one deduces the formulas in the statement. ■

**Corollary 5.18.** *As an object of  $\text{Rep}_q(\text{GL}_M)$ ,  $\mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$  is isomorphic to the tensor product*

$$(\text{Sym}_{q^{-1}}((\mathbb{C}^M)^*))^{\otimes N},$$

*while as an object of  $\text{Rep}_q(\text{GL}_N)$ ,  $\mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$  is isomorphic to the tensor product*

$$(\text{Sym}_q(\mathbb{C}^N))^{\otimes M},$$

*where we denote*

$$\text{Sym}_q(V) = T(V)/(\sigma_{V,V} - q), \quad \text{Sym}_{q^{-1}}(V) = T(V)/(\sigma_{V,V} - q^{-1}).$$

*Proof.* We prove the first statement, the second being identical. For  $i = 1, \dots, N$ , let  $\mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)_{(i)}$  denote the subalgebra generated by the  $x_i^j$  for  $j = 1, \dots, M$ . Inspection of the defining relations gives an isomorphism

$$\mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)_{(i)} \cong \text{Sym}_{q^{-1}}((\mathbb{C}^M)^*).$$

The PBW basis Theorem 5.8 implies an isomorphism of objects

$$\mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right) = \mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)_{(1)} \otimes \dots \otimes \mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)_{(N)}. \quad \blacksquare$$

**Proposition 5.19.** *The algebras  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$ ,  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rhd \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)^\circ$ ,  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  admit good filtrations.*

*Proof.* The PBW theorem gives an isomorphism  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right) \cong \mathcal{O}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right) \otimes \mathcal{O}_q\left(\begin{smallmatrix} M \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} N \\ \bullet \end{smallmatrix}\right)$ . By the preceding proposition, we need only construct a good filtration of the  $q$ -symmetric algebras. We note that each summand  $\text{Sym}_q^k(V)$  identifies with the coinvariants for the finite Hecke algebra action on  $V^{\otimes k}$ , hence it is dual to the invariants  $\Delta(k\omega_{N-1})$  of the finite Hecke algebra action on  $V^{*\otimes k}$ , and hence it is isomorphic precisely to a dual-Weyl module  $\nabla(k\omega_1)$ . The claim for  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rhd \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)^\circ$  follows similarly from the tensor decomposition  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rhd \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)^\circ \cong \mathcal{O}_q(\text{GL}_N) \otimes \mathcal{O}_q(\text{GL}_N)$ , and then application of Theorem 3.7. Finally, the good filtration on each edge algebra tensors to give a good filtration on  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$ , by applying Proposition 3.4. ■

**Lemma 5.20.** *Suppose  $q$  is a primitive  $\ell$ -th root of unity with  $\ell > 1$  odd. Then the Frobenius pushforward of the representation  $\text{Sym}_q(\mathbb{C}^N)$  of  $U_q^L(\mathfrak{gl}_N)$  is naturally identified with the symmetric algebra of the defining representation  $\mathbb{C}^N$  of  $\mathfrak{gl}_N$ . That is,*

$$\text{Fr}_*(\text{Sym}_q(\mathbb{C}^N)) = \text{Sym}(\mathbb{C}^N).$$

*Proof.* The Frobenius pushforward functor is the functor of taking small quantum group invariants. We show that the small quantum group invariants of  $\text{Sym}_q(\mathbb{C}^N)$  can be identified with the subspace  $S_\ell$  consisting of polynomials in the  $\ell$ -th powers  $x_1^\ell, \dots, x_N^\ell$ . This subspace can be naturally identified with  $\text{Sym}(\mathbb{C}^N)$ , and the induced action of the classical enveloping algebra on  $S_\ell$  matches the usual one on  $\text{Sym}(\mathbb{C}^N)$ .

By definition we have the following formulas for the divided power generators of  $U_q^L(\mathfrak{g})$  on the generators  $x_n^{s_n}$  of  $\text{Sym}_q(\mathbb{C}^N)$ :

$$\begin{aligned} K_j x_n^{s_n} &= \begin{cases} q^{s_n} x_n^{s_n} & \text{if } n = j, \\ q^{-s_n} x_n^{s_n} & \text{if } n = j - 1, \\ x_n^{s_n} & \text{otherwise,} \end{cases} \\ E_i^{(r)} x_n^{s_n} &= \begin{cases} \begin{bmatrix} s_n \\ r \end{bmatrix}_q x_{n-1}^r x_n^{s_n-r} & \text{if } i = n - 1, \\ 0 & \text{otherwise,} \end{cases} \\ F_i^{(r)} x_n^{s_n} &= \begin{cases} \begin{bmatrix} s_n \\ r \end{bmatrix}_q x_n^{s_n-r} x_{n+1}^r & \text{if } i = n + 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here,  $n, j \in \{1, \dots, N\}$ ,  $s_n \in \mathbb{Z}_{\geq 0}$ , and  $i \in \{1, \dots, N - 1\}$ . We note the identity

$$\begin{bmatrix} r \\ s \end{bmatrix}_q = \begin{bmatrix} r_0 \\ s_0 \end{bmatrix}_q \begin{pmatrix} r_1 \\ s_1 \end{pmatrix},$$

where  $r = r_0 + r_1 \ell$  and  $s = s_0 + s_1 \ell$  for  $0 \leq r_0, s_0 \leq \ell - 1$ . Thus, a polynomial is invariant for  $K_j$  if and only if each of its monomial summands is invariant for  $K_j$ . From the identities above for the action of  $K_j$ , we see that the condition that all  $K_j$  act on a monomial  $\prod_{n=1}^N x_n^{s_n}$  by the scalar 1 is equivalent to the condition that each  $s_n$  is divisible by  $\ell$ . Thus, the invariants are contained in  $S_\ell$ . Similarly, the formulas for the action of the divided powers show that  $S_\ell$  is indeed invariant for the  $E_i$  and  $F_i$ . As is well-known, the classical enveloping algebra  $U(\mathfrak{gl}_N)$  has a presentation in terms of Serre generators  $\bar{E}_i, \bar{F}_i$ , and  $H_j$  for  $i = 1, \dots, N - 1$  and  $j = 1, \dots, N$ . We have  $\text{Fr}(E_i^{(r)}) = \bar{E}_i$  and  $\text{Fr}(F_i^{(r)}) = \bar{F}_i$ . Thus, the induced action of  $\mathfrak{gl}_N$  on  $S_\ell$  is the same as the usual action of  $\mathfrak{gl}_N$  on  $\text{Sym}(\mathbb{C}^N)$ . ■

We are now ready to state the main result of this section:

**Theorem 5.21.** *Let  $q$  be a primitive  $\ell$ -th root of unity, where  $\ell > 1$  is odd.*

(1) *We have a central,  $U_q^L(\mathfrak{gl}_N \times \mathfrak{gl}_M)$ -equivariant embedding*

$$\text{Fr}^*(\mathcal{O}(\text{Mat}(e) \times \text{Mat}(e^\vee))) \hookrightarrow \mathcal{D}_q \left( \begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix} \right).$$

(2) We have a central  $U_q^L(\mathfrak{gl}_N)$ -equivariant embedding

$$\mathrm{Fr}^*(\mathcal{O}(\mathrm{GL}_N \times \mathrm{GL}_N)) \hookrightarrow \mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightrightarrows \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)^\circ,$$

(3) We have a central,  $U_q^L(\mathfrak{gl}_{\mathbf{d}})$ -equivariant embedding

$$\mathrm{Fr}^*(\mathcal{O}(\mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d}))) \hookrightarrow \mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d})),$$

(4) We have a Frobenius Poisson order structure on  $(\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right), \mathcal{M}_{\mathrm{fr}}(Q, \mathbf{d}))$ .

*Proof.* For the central subalgebra asserted in (1), we let  $Z \subset \mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$  be the subalgebra generated by the  $\ell$ -th powers  $(x_j^i)^\ell$ ,  $(\partial_l^k)^\ell$  of all generators. Inspecting formulas (5.17), we note that commutators with all  $\ell$ -th powers of generators vanish when  $q^\ell = 1$ , i.e., the subalgebra  $Z$  is central. Clearly it is isomorphic as an algebra to

$$\mathcal{O}(\mathrm{Mat}(e) \times \mathrm{Mat}(e^\vee)).$$

Hence to complete the proof of (1), it remains to prove equivariance for the action of the restricted quantum group. The algebra  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$  is isomorphic as a  $U_q^L(\mathfrak{gl}_N)$ -module (that is, upon forgetting the  $U_q^L(\mathfrak{gl}_M)$ -action) to a tensor product of  $2M$  copies of the  $q$ -symmetric algebra  $\mathrm{Sym}_q(\mathbb{C}^N)$ , where each subalgebra is determined by fixing the lower indices. Hence the equivariance of the inclusion follows immediately from Lemma 5.20.

Finally, to conclude the proof of (1), it follows from Lemma 5.20 that the small quantum group invariants contain the  $\ell$ -th powers of the generators and the induced action of the classical enveloping algebra  $U(\mathfrak{gl}_N)$  on the copy of  $\mathcal{O}(\mathrm{Mat}(N, M) \times \mathrm{Mat}(M, N))$  they generate coincides with the action induced from  $\mathrm{GL}_N$  acting on the space  $\mathrm{Mat}(N, M)$  by multiplication. A similar observation holds for the action of  $U_q^L(\mathfrak{gl}_M)$  (here  $q$  is replaced by  $q^{-1}$ , because the defining representation is replaced by its dual), and we conclude that the embeddings in the statement of the theorem are equivariant for the restricted quantum group.

Claim (2) has already been proved in Section 4, because the algebra  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightrightarrows \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)^\circ$  is isomorphic (as an algebra object of  $\mathrm{Rep}_q(G)$ ) to the framed quantum character variety of the punctured torus,  $A_{T^2 \setminus D^2}$ , for  $\mathrm{GL}_N$ .

Claim (3) now follows from the fact that  $\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$  is a braided tensor product of its edge algebras; this implies easily that the central subalgebras of each edge algebra is  $U_q^L(\mathfrak{gl}_{\mathbf{d}})$ -equivariant, and  $u_q(\mathfrak{gl}_{\mathbf{d}})$ -invariant. This further implies that the central subalgebra on each edge is in fact central in the whole algebra, as the braided tensor product commutativity relations become trivial on  $u_q(\mathfrak{gl}_{\mathbf{d}})$ -invariant subalgebras.

Finally, to prove claim (4), we appeal to Proposition 3.24. Combining Proposition 5.19 and Proposition 3.4 establishes the required surjectivity, so we need only show that the resulting derivation  $D$  on  $\mathcal{D}_q(\mathrm{Mat}(Q, \mathbf{d}))$  preserves  $Z$ . For this, we can compute directly with the relations of Lemma 5.17 to see that, on each edge algebra  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$ , the

derivation of Proposition 3.23 restricts on  $Z$  to give the Poisson bivector

$$\begin{aligned} \pi = & \sum_{i>j;n,m} (\delta_{mn} y_n^j y_m^i + 2\theta(n-m) y_m^j y_n^i) \frac{\partial}{\partial y_m^i} \wedge \frac{\partial}{\partial y_n^j} - \sum_{m>n} y_n^i y_m^i \frac{\partial}{\partial y_m^i} \wedge \frac{\partial}{\partial y_n^i} \\ & + \sum_{i>j;n,m} (\delta_{mn} z_n^j z_m^i + 2\theta(n-m) z_m^j z_n^i) \frac{\partial}{\partial z_m^i} \wedge \frac{\partial}{\partial z_n^j} - \sum_{m>n} z_n^i z_m^i \frac{\partial}{\partial z_m^i} \wedge \frac{\partial}{\partial z_n^i} \\ & + 2 \sum_{i,j,m,n} \left( (\delta_{in} + \delta_{jm}) y_n^j z_m^i + \delta_{in} \sum_{p>i} y_p^j z_m^p \right. \\ & \quad \left. + \delta_{jm} \sum_{p<j} y_n^p z_p^i + \frac{\delta_{in} \delta_{jm}}{(q^2 - 1)^\ell} \right) \frac{\partial}{\partial z_m^i} \wedge \frac{\partial}{\partial y_n^j}. \quad (5.2) \end{aligned}$$

Meanwhile, we know that the derivation preserves the central subalgebra on loop edges, as is proved in Section 4.1. And now once again the claim for  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  follows by considering braided tensor products. ■

**Remark 5.22.** The direct check that  $D(Z)(Z) \subset Z$  given here can be avoided: Theorem 5.25 below gives the existence of a single Azumaya point, and  $Z = \text{Spec}(Z)$  is smooth and connected. Hence Lemma 3.31 implies that  $Z$  is the center of  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$ , and then the desired containment follows by Proposition 3.23.

#### 5.4. Non-degeneracy of the Poisson $G$ -variety $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$

We are now going to analyze the non-degeneracy of the Poisson  $\text{GL}_{\mathbf{d}}$ -variety structure on  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  given by Theorem 5.21.

Consider the quiver  $(Q, \mathbf{d}) = (\bullet \xrightarrow{N} \bullet)$ . From formula (5.2) it is clear that the Poisson structure is independent of  $\ell$  up to scale. So, it is enough to analyze the non-degeneracy of the Poisson structure for  $q \rightarrow 1$ . Let  $r_M$  and  $r_N$  be the classical  $r$ -matrices on  $\mathfrak{gl}_M$  and  $\mathfrak{gl}_N$  respectively. By our convention the symmetric parts of  $r_M$  and  $r_N$  are given by  $P/2$ , where  $P: \mathbb{C}^M \otimes \mathbb{C}^M \rightarrow \mathbb{C}^M \otimes \mathbb{C}^M$  is the flip  $v \otimes w \mapsto w \otimes v$  and similarly for  $\mathbb{C}^N$ .

Then the  $q \rightarrow 1$  limit of the relations in Definition 5.6 gives the following Poisson structure on  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$ :

$$\begin{aligned} \{X_1, X_2\} &= r_N X_1 X_2 - X_1 X_2 (r_M)_{21}, \\ \{D_1, D_2\} &= r_M D_1 D_2 - D_1 D_2 (r_N)_{21}, \\ \{X_1, D_2\} &= -D_2 r_N X_1 - X_1 r_M D_2 - \Omega. \end{aligned}$$

Here the relations hold in  $\mathcal{O}(\mathcal{M}_{\text{fr}}(Q, \mathbf{d}))$  tensored with  $\text{Hom}(\mathbb{C}^M \otimes \mathbb{C}^M, \mathbb{C}^N \otimes \mathbb{C}^N)$  in the first line,  $\text{Hom}(\mathbb{C}^N \otimes \mathbb{C}^N, \mathbb{C}^M \otimes \mathbb{C}^M)$  in the second line, and  $\text{Hom}(\mathbb{C}^M \otimes \mathbb{C}^N, \mathbb{C}^M \otimes \mathbb{C}^N)$  in the last line, and  $\Omega: \mathbb{C}^M \otimes \mathbb{C}^N \rightarrow \mathbb{C}^M \otimes \mathbb{C}^N$  simply denotes the identity map.

**Theorem 5.23.** *Let  $Q$  be an arbitrary quiver. The Poisson  $\text{GL}_{\mathbf{d}}$ -variety  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ$  is non-degenerate.*

*Proof.* By Definition 5.14 and Proposition 3.25,  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ$  is obtained by the fusion of the corresponding varieties for a single edge. But by Proposition 2.14 fusion preserves non-degeneracy, so it is enough to prove non-degeneracy of  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ$  when  $Q$  is a single edge. When  $Q$  is a single loop, the result follows from Proposition 4.3. So, we just have to analyze the case  $(Q, \mathbf{d}) = (\bullet \xrightarrow{N} \bullet)$ .

Let  $a: \mathfrak{gl}_N \oplus \mathfrak{gl}_M \rightarrow \Gamma(\mathcal{M}_{\text{fr}}(Q, \mathbf{d}), T_{\mathcal{M}_{\text{fr}}(Q, \mathbf{d})})$  be the infinitesimal action map and consider the new bivector  $\tilde{\pi} = \pi + a(r_M + r_N) \in \Gamma(\mathcal{M}_{\text{fr}}(Q, \mathbf{d}), T_{\mathcal{M}_{\text{fr}}(Q, \mathbf{d})}^{\otimes 2})$  (note that it is no longer antisymmetric). The corresponding biderivation  $\{-, -\}'$  is given by the following formulas:

$$\begin{aligned} \{X_1, X_2\}' &= -X_1 X_2 P, \\ \{D_1, D_2\}' &= -D_1 D_2 P, \\ \{X_1, D_2\}' &= -\Omega, \\ \{D_2, X_1\}' &= D_2 P X_1 + X_1 P D_2 + \Omega. \end{aligned}$$

By Proposition 2.13 the non-degeneracy of the Poisson  $G$ -variety  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  is equivalent to the condition that this matrix, which we will denote by  $M$ , is invertible. Writing it out in coordinates, we have

$$\begin{aligned} \{y_i^j, y_k^l\}' &= -y_k^j y_i^l = (M_{XX})_{ik}^{jl}, \\ \{z_i^j, z_k^l\}' &= -z_k^j z_i^l = (M_{DD})_{ik}^{jl}, \\ \{y_i^j, z_k^l\}' &= -\delta_i^l \delta_k^j = (M_{XD})_{ik}^{jl}, \\ \{z_k^l, y_i^j\}' &= \sum_m z_m^l y_i^m \delta_k^j + \sum_m z_k^m y_m^j \delta_i^l + \delta_k^j \delta_i^l = (M_{DX})_{ki}^{lj}. \end{aligned}$$

We have a block form

$$M = \begin{pmatrix} M_{XX} & M_{XD} \\ M_{DX} & M_{DD} \end{pmatrix}.$$

Since  $M_{XD}$  is a scalar matrix, we have  $\det(M) = \det(M_{DD} M_{XX} - M_{XD} M_{DX})$ . We get

$$\begin{aligned} (M_{DD} M_{XX})_{ik}^{jl} &= (DX)_k^j (XD)_i^l, \\ (M_{XD} M_{DX})_{ik}^{jl} &= -(DX)_k^j \delta_i^l - (XD)_i^l \delta_k^j - \delta_i^l \delta_k^j, \end{aligned}$$

and hence

$$(M_{DD} M_{XX} - M_{XD} M_{DX})_{ik}^{jl} = ((DX)_k^j + \delta_k^j)((XD)_i^l + \delta_i^l).$$

In other words,  $M_{DD} M_{XX} - M_{XD} M_{DX}: \mathbb{C}^M \otimes \mathbb{C}^N \rightarrow \mathbb{C}^M \otimes \mathbb{C}^N$  is the tensor product of  $1 + DX: \mathbb{C}^M \rightarrow \mathbb{C}^M$  and  $1 + XD: \mathbb{C}^N \rightarrow \mathbb{C}^N$ . We have  $\det(1 + DX) = \det(1 + XD)$ , so

$$\det(M) = \det(1 + DX)^{N+M}.$$

In particular, it is invertible on the locus  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ \subset \mathcal{M}_{\text{fr}}(Q, \mathbf{d})$ . ■



**Remark 5.24.** Van den Bergh [71] defined a natural quasi-Poisson structure on  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  using the theory of double quasi-Poisson structures. In particular, he shows that his quasi-Poisson structure is non-degenerate on  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ \subset \mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  (see [72, Proposition 8.3.1]). We expect that the twist of his quasi-Poisson structure with respect to the anti-symmetric part of the  $r$ -matrix coincides with the Poisson structure studied in this paper which is obtained by degenerating the quantization given in [37].

### 5.5. An Azumaya point on $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$

Let  $Q_{\text{Kr}}$  be the Kronecker quiver with dimension vector  $\mathbf{d} = (M, N)$ . Recall from Definition 5.6 the algebra  $\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet})$ , which has generators  $x_j^i$  and  $\partial_p^r$  for  $1 \leq i, p \leq M$  and  $1 \leq j, r \leq N$ . Recall also that we organize these generators into an  $M \times N$  matrix  $X$  and an  $N \times M$  matrix  $D$ . Similarly, set  $X^{[\ell]}$  to be the  $M \times N$  matrix with entries  $(x_n^i)^\ell$  and  $D^{[\ell]}$  to be the  $N \times M$  matrix with entries  $(\partial_i^n)^\ell$ . We fix  $q$  to be a primitive  $\ell$ -th root of unity, where  $\ell > 1$  is odd. By Theorem 5.21, the entries in  $X^{[\ell]}$  and  $D^{[\ell]}$  generate a central subalgebra  $Z_\ell$  isomorphic to the coordinate algebra of  $\text{Mat}(M, N) \times \text{Mat}(N, M)$ . The quotients

$$A = \mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet}) / (D^{[\ell]} = 0) \quad \text{and} \quad \mathcal{M} = \mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet}) / (D = 0)$$

of  $\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet})$  by the ideal generated by the entries of  $D^{[\ell]}$  and the ideal generated by the  $\partial$ 's are each modules for the quotient  $Z_\ell / (D^{[\ell]} = 0)$ . The former is in fact an algebra over  $Z_\ell / (D^{[\ell]} = 0)$ , and by the following theorem, it is the endomorphism algebra of the latter:

**Theorem 5.25.** (1) *The left multiplication action induces isomorphisms*

$$\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet}) / (D^{[\ell]} = 0) \xrightarrow{\sim} \text{End}_{Z_\ell / (D^{[\ell]} = 0)}(\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet}) / (D = 0)),$$

*Therefore, the fiber  $\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet}) / (D^{[\ell]} = 0, X^{[\ell]} = 0)$  of the sheaf defined by  $\mathcal{D}_q(\overset{N}{\bullet} \rightarrow \overset{M}{\bullet})$  over zero in  $\text{Mat}(M, N) \times \text{Mat}(N, M)$  is a matrix algebra.*

(2) *The fiber of  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  over the zero representation is a matrix algebra.*

*Proof.* For (1) we observe that the module  $\mathcal{M}$  is of rank  $\ell^{MN}$  over  $\tilde{Z} := Z_\ell / (D^{[\ell]} = 0)$  with basis given by the (ordered) monomials

$$\bar{x}(\bar{r}) := (x_N^1)^{r_N^1} (x_{N-1}^1)^{r_{N-1}^1} \dots (x_1^M)^{r_1^M}$$

where  $\bar{r} = (r_N^1, r_{N-1}^1, \dots, r_1^1, r_N^2, \dots, r_1^M)$  ranges over the elements in  $\{0, \dots, \ell - 1\} \times \{0, \dots, \ell - 1\}$ . Clearly the element 1 is a cyclic generator for the module; hence our strategy is to show that we can reach 1 from an arbitrary element of  $\mathcal{M}$  using the  $A$ -action, and hence conclude that  $\mathcal{M}$  is an irreducible representation of  $A$  of the correct dimension.

Note that  $\bar{r}$  is constantly zero if and only if  $\bar{x}(\bar{r}) = 1$ . An arbitrary element of  $\mathcal{M}$  can be expressed as

$$\bar{y} = \sum_{\bar{r}} z_{\bar{r}} \bar{x}(\bar{r})$$

for some  $z_{\bar{r}} \in \tilde{Z}$ . Define a total order on the set  $S = ([1, M] \times [1, N]) \sqcup \{0\}$  by

$$(j, m) \succ 0, \quad (i, n) \succ (j, m) \quad \text{if} \quad \begin{cases} n > m \text{ or} \\ n = m \text{ and } i < j, \end{cases}$$

for any  $i, j \in [1, N]$  and any  $m, n \in [1, M]$ . The leading index of the monomial  $\bar{x}(\bar{r})$  is defined as

$$\text{LI}(\bar{x}(\bar{r})) = \max \{(i, n) \mid r_n^i \neq 0\} \in S$$

if  $x \neq 1$ , and  $\text{LI}(1) = 0 \in S$ . The leading index of an element  $\bar{y} = \sum_{\bar{r}} z_{\bar{r}} \bar{x}(\bar{r})$  is defined as

$$\text{LI}(\bar{y}) = \max \{\text{LI}(\bar{x}(\bar{r})) \mid z_{\bar{r}} \neq 0\} \in S$$

if  $\bar{y} \neq 0$ , and  $\text{LI}(0) = 0$ .

**Lemma 5.26.** *For any  $p$ ,  $1 \leq p \leq M$ , the element*

$$\partial_p^i x_n^p - \delta_{in} q^{2p+1} \in \mathcal{D}_q(\bullet \rightarrow \bullet)$$

*can be expressed as a sum of elements, each divisible on the right by some  $\partial_{p'}^s$ , where  $(p', s) \succeq (p, i)$ .*

*Proof.* By the defining relations of the algebra, we have

$$\partial_p^i x_n^p - \delta_{in} q = q^{1+\delta_{in}} x_n^p \partial_p^i + \delta_{in} (q^2 - 1) \sum_{s>i} x_s^p \partial_p^s + (q^2 - 1) \sum_{p'<p} \partial_{p'}^i x_n^{p'}.$$

We proceed by induction on  $p$ . From the above expression, the base case  $p = 1$  is clear. If  $p > 1$ , then by the inductive hypothesis, all elements on the RHS are of the desired form except for  $(q^2 - 1)\delta_{in} \sum_{p'<p} q^{2p'+1}$ . Adding this to the LHS, we obtain  $\delta_{in} q^{2p+1}$ . ■

**Lemma 5.27.** *Suppose  $y$  is the form  $y = (x_n^j)^r x$ , where  $\text{LI}(x) \prec (j, n)$  and  $1 \leq r \leq \ell - 1$ . Then*

$$\text{LI}(\partial_m^i \triangleright (y)) = 0 \quad \text{for any } (m, i) \text{ with } (m, i) \succ (j, n).$$

*Proof.* The condition  $(m, i) \succ (j, n)$  implies that we cannot have both  $i = n$  and  $j = m$ , i.e.,  $\delta_{in}\delta_{jm} = 0$ . Using Lemma 5.17, we compute

$$\begin{aligned} \partial_m^i \triangleright y &= q^{(\delta_{in} + \delta_{jm})r} (x_n^j)^r (\partial_m^i \triangleright x) \\ &\quad + a_r \sum_{p>i} (x_n^j)^{r-1} [x_p^m \partial_m^p \triangleright x] + b_r \sum_{p'<j} (x_n^j)^{r-1} [\partial_{p'}^i x_n^{p'} \triangleright x], \end{aligned}$$

noting that  $c_r$  and  $d_r$  vanish since they are divisible by  $\delta_{in}\delta_{jm}$ .

We now prove the claim by induction on  $(j, n)$  in the totally ordered set  $S$ . The base case is  $(j, n) = (1, N)$ . In this case,  $y = (x_N^1)^r$  for some  $1 \leq r \leq \ell - 1$ , and there are no  $p$  or  $p'$  with  $p > N$  and  $p' < 1$ . Thus,

$$\partial_m^i \triangleright y = q^{(\delta_{in} + \delta_{jm})r} (x_N^1)^r (\partial_m^i \triangleright 1) = 0.$$

For the induction step, let  $y = (x_n^j)^r x$ , where  $\text{LI}(x) < (j, n)$  and  $1 \leq r \leq \ell - 1$ . Now,

$$\begin{aligned} \partial_m^i \triangleright y &= q^{(\delta_{in} + \delta_{jm})r} (x_n^j)^r (\partial_m^i \triangleright x) \\ &\quad + a_r \sum_{p>i} (x_n^j)^{r-1} [x_p^m \partial_m^p \triangleright x] + b_r \sum_{p'<j} (x_n^j)^{r-1} [\partial_{p'}^i x_n^{p'} \triangleright x]. \end{aligned}$$

The inductive hypothesis implies that  $\partial_m^i \triangleright x = 0$  and  $\partial_m^p \triangleright x = 0$  since  $(m, p) > (m, i)$ . Suppose  $m = j$ , so that  $b_r$  is non-zero. Then necessarily  $i \neq n$ . By Lemma 5.26,  $\partial_{p'}^i x_n^{p'}$  can be expressed as a sum of elements, each divisible on the right by some  $\partial_{p''}^s$ , where  $(p'', s) \succeq (p', i) > (m, i)$ . Consequently, this element acts on  $x$  as zero, and the inductive step follows. ■

**Lemma 5.28.** Suppose  $y = (x_n^j)^r x$ , where  $\text{LI}(x) < (j, n)$  and  $1 \leq r \leq \ell - 1$ . Then

$$0 \neq \text{LI}(\partial_j^n \triangleright y) \preceq (j, n).$$

*Proof.* We record the following computation:

$$\begin{aligned} \partial_j^n \triangleright (x_n^j)^r x &= q^{2r} (x_n^j)^r (\partial_j^n \triangleright x) \\ &\quad + a_r \sum_{p>i} (x_n^j)^{r-1} [x_p^j \partial_j^p \triangleright x] + b_r \sum_{p'<j} (x_n^j)^{r-1} [\partial_{p'}^n x_n^{p'} \triangleright x] \\ &\quad + c_r \sum_{p>n, p'<j} (x_n^j)^{r-2} [x_p^j \partial_{p'}^p x_n^{p'} \triangleright x] + d_r (x_n^j)^{r-1} x. \end{aligned}$$

We now proceed by induction on  $(j, n)$  in the totally ordered set  $S$ . The base case is  $(j, n) = (1, N)$ . In this case,  $y = (x_N^1)^r$  for some  $1 \leq r \leq \ell - 1$ , and there are no  $p$  or  $p'$  with  $p > N$  and  $p' < 1$ . Thus,

$$\partial_1^N \triangleright y = q^{2r} (x_N^1)^r (\partial_1^N \triangleright 1) + d_r (x_N^1)^{r-1} = \frac{q^{2r} - 1}{q - q^{-1}} (x_N^1)^{r-1},$$

which is non-zero.

For the induction step, let  $y = (x_n^j)^r x$ , where  $\text{LI}(x) < (j, n)$  and  $1 \leq r \leq \ell - 1$ . Lemma 5.27 implies that  $\partial_j^n \triangleright x = 0$ . Now fix  $p > i$  and  $p' < j$ . Then Lemma 5.27 also implies that  $x_p^j \partial_j^p \triangleright x = 0$ . By Lemma 5.26, for  $p' < j$ ,  $\partial_{p'}^n x_n^{p'} - q^{2p'+1}$  can be expressed as a sum of elements, each divisible on the right by some  $\partial_{p''}^s$ , where  $(p'', s) \succeq (p', n) > (j, n)$ . Thus, this element acts on  $x$  as  $q^{2p'-1}$ . Finally,  $\text{LI}(x_n^{p'} x) < (p', p)$ , so  $\partial_{p'}^p \triangleright (x_n^{p'} x) = 0$ . What remains in the expression at the beginning of this proof is

$$\partial_m^i \triangleright (x_n^j)^r x = \left( (q^{2r} - 1) \sum_{p'<j} q^{2p'+1} + \frac{q^{2r} - 1}{q - q^{-1}} \right) (x_n^j)^{r-1} x = \frac{q^{2j} (q^{2r} - 1)}{q - q^{-1}} (x_n^j)^{r-1} x,$$

which is non-zero and of leading index at most  $(j, n)$ . ■

**Lemma 5.29.** *Suppose  $\text{LI}(y) = (j, n)$ . Then, for some  $r$ ,  $1 \leq r \leq \ell - 1$ ,*

$$0 \neq \text{LI}((\partial_j^n)^r \triangleright y) \prec (j, n).$$

*Proof.* We can write

$$y = (x_n^i)^r x^{(r)} + (x_n^i)^{(r-1)} x^{(r-1)} + \cdots + x_n^i x^{(1)} + x^{(0)}$$

for some  $x^{(\bullet)} \in \mathcal{M}$  with  $\text{LI}(x^{(\bullet)}) \prec (j, n)$  and  $1 \leq r \leq \ell - 1$ . Then, iterating the computation in the proof of the previous lemma, we obtain

$$(\partial_j^n)^r \triangleright y = \left( \frac{q^{2j}}{q - q^{-1}} \right)^r \prod_{s=1}^r (q^{2s} - 1) x^{(r)},$$

which is non-zero. ■

**Lemma 5.30.** *For any  $y \in \mathcal{M}$  non-zero, there is an element  $d$  of  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$  such that  $d \triangleright y$  is a non-zero multiple of 1, and  $\text{LI}(y) = (j, n)$ .*

*Proof.* Proceed by induction on  $\text{LI}(\bar{y}) \in S$ . If  $\text{LI}(\bar{y}) = 0$ , there is nothing to show. Otherwise, let  $(j, n) = \text{LI}(\bar{y})$ . By the previous corollary we can lower the leading index using an element of  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)$ . ■

To conclude the proof of (1) of Theorem 5.25: we have shown that under the action of  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)/(D^{[\ell]} = 0)$  on  $\mathcal{M}$  any element lies in the orbit of 1, and conversely that 1 lies in the orbit of any non-zero element of  $\mathcal{M}$ . Hence the representation is irreducible. Since it is a representation whose rank over  $\tilde{Z}$  is equal to the square root of the rank of  $\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)/(D^{[\ell]} = 0)$  as a module over  $\tilde{Z}$ , we conclude that the action map

$$\mathcal{D}_q\left(\begin{smallmatrix} N \\ \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} M \\ \bullet \end{smallmatrix}\right)/(D^{[\ell]} = 0) \rightarrow \text{End}_{\tilde{Z}}(\mathcal{M})$$

is an isomorphism.

For (2), we once again appeal to the construction of  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  via braided tensor products, and the result follows from Proposition 3.30. ■

## 5.6. The Frobenius quantum moment map

The quantum moment map

$$\mu_q: \mathcal{O}_q(\text{GL}_{\mathbf{d}}) \rightarrow \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ$$

from Definition 5.15 is valid for  $q$  a root of unity when working in  $\text{Rep}_q(\text{GL}_{\mathbf{d}})$ . We also have the group-valued moment map

$$\tilde{\mu}: \mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ \rightarrow \text{GL}_{\mathbf{d}}$$

defined on a Zariski open subset  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ$  of  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  (see Section 5.1). These two moment maps fit into a Frobenius moment map structure, as explained in the following key theorem.

**Theorem 5.31.** *We have a central,  $U_q^L(\mathfrak{gl}_{\mathbf{d}})$ -equivariant embedding*

$$\mathcal{O}(\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ) \hookrightarrow \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ$$

which fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ) & \longrightarrow & \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ \\ \tilde{\mu}^\# \uparrow & & \uparrow \mu_q \\ \mathcal{O}(\text{GL}_{\mathbf{d}}) & \xrightarrow{i_{\mathbf{d}}} & \mathcal{O}_q(\text{GL}_{\mathbf{d}}) \end{array}$$

where  $\tilde{\mu}^\#$  is pullback along the group-valued moment map  $\tilde{\mu}$ . Thus,

$$(\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))^\circ, \mathcal{O}(\mathcal{M}_{\text{fr}}(Q, \mathbf{d})^\circ))$$

is a  $G$ -Hamiltonian Frobenius Poisson order.

*Proof.* We may apply Proposition 3.28 to conclude that there exists some classical multiplicative moment map  $\tilde{\mu}^\#$  making the diagram commute. By Proposition 2.18, the set of possible multiplicative moment maps associated to our fixed  $\text{GL}_{\mathbf{d}}$ -action is a torsor over the center of  $\text{GL}_{\mathbf{d}}$ , so we may assume without loss of generality that it recovers the classical moment map from equation (5.1) exactly. ■

### 5.7. The quantized multiplicative quiver variety at a root of unity

First, we recall the definition of the quantized multiplicative quiver variety with trivial GIT character:

**Definition 5.32** ([37, Definition 8.1]). Fix an algebra homomorphism  $\xi_q: \mathcal{O}_q(G) \rightarrow \mathbb{C}$  in  $\text{Rep}_q(G)$ , and let  $\mathcal{I}_{\xi_q}$  denote its kernel. The *quantized multiplicative quiver variety* corresponding to the above data is defined as the quantum Hamiltonian reduction of  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  at the ideal  $\mathcal{I}_{\xi_q}$ :

$$\mathcal{A}^\xi(Q, \mathbf{d}) := \text{Hom}_{\text{Rep}_q(\text{GL}_{\mathbf{d}})}(\mathbb{C}, \mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))/\mathcal{I}_{\xi_q}).$$

We now generalize this to non-trivial GIT characters, to construct a sheaf of algebras on the multiplicative quiver variety, via a version of quantum Hamiltonian reduction for non-affine GIT quotients.

First suppose  $(A, Z)$  is a  $G$ -Hamiltonian Frobenius Poisson order. Denote as before by  $I \subset Z$  the classical moment map ideal and by  $I_q \subset A$  the quantum moment map ideal. Recall that  $(A/I_q)^{u_q(\mathfrak{g})}$  is an algebra with a central subalgebra  $Z/I$ . Therefore,

$$\left( \bigoplus_{m=0}^{\infty} (A/I_q)^{u_q(\mathfrak{g})} \otimes \mathbb{C}_{\theta-m} \right)^G$$

is a graded algebra with a central subalgebra

$$\left( \bigoplus_{m=0}^{\infty} Z/I \otimes \mathbb{C}_{\theta-m} \right)^G.$$

**Definition 5.33.** We denote by  $\mathcal{A}_\theta^\xi(Q, \mathbf{d})$  the sheaf of algebras over  $\mathbb{Z} //_\theta G$  so constructed.

We will say a sheaf of algebras  $\mathcal{A}$  over a scheme  $X$  is a *Poisson order* if  $X$  is a Poisson scheme and on each open subset of  $X$  we obtain the structure of a Poisson order. Then we get the following variant of Proposition 3.21.

**Proposition 5.34.** *Suppose  $(A_q, Z)$  is a  $G$ -Hamiltonian Frobenius Poisson order where  $Z$  is Noetherian. Then  $(\mathcal{A}_\theta^\xi, \mathbb{Z} //_\theta G)$  is a Poisson order.*

### 5.8. Proof of Theorem 1.4

We are now ready to prove the main result of this section, that the sheaf of algebras constructed in the previous section is Azumaya over the entire smooth locus of  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$ .

**Theorem 5.35** (Theorem 1.4). *Let  $\ell > 1$  be an odd integer, and  $q$  a primitive  $\ell$ -th root of unity.*

- (1) *The algebra  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  is finitely generated over its center, which is isomorphic to the coordinate ring  $\mathcal{O}(\mathcal{M}_{\text{fr}}(Q, \mathbf{d}))$  of the classical framed multiplicative quiver variety.*
- (2)  *$\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  is Azumaya over the preimage in  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d})$  of the big cell  $G^\circ \subset G$  under the multiplicative moment map  $\mathcal{M}_{\text{fr}}(Q, \mathbf{d}) \rightarrow G$ .*
- (3) *Frobenius quantum Hamiltonian reduction defines a coherent sheaf of algebras over the classical multiplicative quiver variety  $\mathcal{M}(Q, \mathbf{d}, \xi, \theta)$ , which is Azumaya over the locus  $\mathcal{M}^s(Q, \mathbf{d}, \xi, \theta)$  of  $\theta$ -stable representations.*

*Proof.* We follow the template of Section 3.8. Hence, we let  $A_{\mathcal{R}}$  be the algebra  $\mathcal{D}_{\mathcal{R}}(\text{Mat}(Q, \mathbf{d}))$ , and  $\mu_{\mathcal{R}}$  be the quantum moment map of Definition 5.15. For  $Z$  we take the central subalgebra constructed in Theorem 5.21. For the Azumaya point  $p$  required in Lemma 3.31, we take the zero representation, as in Theorem 5.25. The first claim of the theorem now follows from Lemma 3.31.

We prove the remaining two claims together by applying Theorem 3.32. For  $G_{\text{stab}}$  we take the copy of  $\mathbb{C}^\times$  embedded diagonally in  $\text{GL}_{\mathbf{d}}$ . For  $G_0$  we take the subgroup of  $\text{GL}_{\mathbf{d}}$  on which the product of all determinants is 1. We choose for  $\xi_q$  the character  $(\xi_v \varepsilon_v)$ . We take for the open subset  $U$  the entire stable locus (which we assume is non-empty), on which  $G$  acts with common stabilizer  $G_{\text{stab}}$ .

Then assumption (1) of Theorem 3.32 is confirmed in Theorem 5.23, assumption (2) is clear, since the entire algebra  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$  lies in total degree zero for the grading given by summing the degrees at all vertices, hence  $G_{\text{stab}}$  acts trivially on  $\mathcal{D}_q(\text{Mat}(Q, \mathbf{d}))$ . Assumption (3) is clear, since  $\xi_q$  lives over the identity element in  $G$ , which clearly lies in  $G^\circ$ . ■

*Acknowledgments.* We would like to thank David Ben-Zvi and Kobi Kremnitzer for their guidance and encouragement throughout our work, Nicholas Cooney for many helpful conversations, Michael

Gröchenig for discussions about the Langlands dual case, Pavel Etingof for remarks about the small quantum group, Florian Naef for helpful remarks about non-degenerate quasi-Poisson structures, and Thang Le for assistance in navigating skein theory literature.

*Funding.* The work of I.G. was supported by the Advanced Grant ‘Arithmetic and Physics of Higgs moduli spaces’ No. 320593 of the European Research Council. The work of D.J. was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme [grant agreement no. 637618]. The work of P.S. was supported by the NCCR SwissMAP grant of the Swiss National Science Foundation.

## References

- [1] Alekseev, A. Y., Grosse, H., Schomerus, V.: [Combinatorial quantization of the Hamiltonian Chern–Simons theory](#). II. Comm. Math. Phys. **174**, 561–604 (1996) Zbl [0842.58038](#) MR [1370081](#)
- [2] Alekseev, A., Kosmann-Schwarzbach, Y.: [Manin pairs and moment maps](#). J. Differential Geom. **56**, 133–165 (2000) Zbl [1046.53055](#) MR [1863024](#)
- [3] Alekseev, A., Kosmann-Schwarzbach, Y., Meinrenken, E.: [Quasi-Poisson manifolds](#). Canad. J. Math. **54**, 3–29 (2002) Zbl [1006.53072](#) MR [1880957](#)
- [4] Alekseev, A. Y., Malkin, A. Z.: [Symplectic structures associated to Lie–Poisson groups](#). Comm. Math. Phys. **162**, 147–173 (1994) Zbl [0797.58020](#) MR [1272770](#)
- [5] Alekseev, A., Malkin, A., Meinrenken, E.: [Lie group valued moment maps](#). J. Differential Geom. **48**, 445–495 (1998) Zbl [0948.53045](#) MR [1638045](#)
- [6] Angiono, I., Galindo, C., Pereira, M.: [De-equivariantization of Hopf algebras](#). Algebr. Represent. Theory **17**, 161–180 (2014) Zbl [1303.16032](#) MR [3160718](#)
- [7] Arkhipov, S., Gaiitsgory, D.: [Another realization of the category of modules over the small quantum group](#). Adv. Math. **173**, 114–143 (2003) Zbl [1025.17004](#) MR [1954457](#)
- [8] Barrett, J. W.: [Skein spaces and spin structures](#). Math. Proc. Cambridge Philos. Soc. **126**, 267–275 (1999) Zbl [0939.57019](#) MR [1670233](#)
- [9] Bellamy, G., Schedler, T.: [Symplectic resolutions of character varieties](#). Geom. Topol. **27**, 51–86 (2023) Zbl [1514.14010](#) MR [4584260](#)
- [10] Ben-Zvi, D., Brochier, A., Jordan, D.: [Integrating quantum groups over surfaces](#). J. Topol. **11**, 874–917 (2018) Zbl [1409.14028](#) MR [3847209](#)
- [11] Ben-Zvi, D., Brochier, A., Jordan, D.: [Quantum character varieties and braided module categories](#). Selecta Math. (N.S.) **24**, 4711–4748 (2018) Zbl [1456.17010](#) MR [3874702](#)
- [12] Bezrukavnikov, R., Finkelberg, M., Ginzburg, V.: [Cherednik algebras and Hilbert schemes in characteristic  \$p\$](#) . Represent. Theory **10**, 254–298 (2006) Zbl [1130.14005](#) MR [2219114](#)
- [13] Bonahon, F., Wong, H.: [Quantum traces for representations of surface groups in  \$SL\_2\(\mathbb{C}\)\$](#) . Geom. Topol. **15**, 1569–1615 (2011) Zbl [1227.57003](#) MR [2851072](#)
- [14] Bonahon, F., Wong, H.: [Representations of the Kauffman bracket skein algebra I: invariants and miraculous cancellations](#). Invent. Math. **204**, 195–243 (2016) Zbl [1383.57015](#) MR [3480556](#)
- [15] Bonahon, F., Wong, H.: [Representations of the Kauffman bracket skein algebra II: Punctured surfaces](#). Algebr. Geom. Topol. **17**, 3399–3434 (2017) Zbl [1422.57032](#) MR [3709650](#)
- [16] Bonahon, F., Wong, H.: [Representations of the Kauffman bracket skein algebra III: closed surfaces and naturality](#). Quantum Topol. **10**, 325–398 (2019) Zbl [1447.57017](#) MR [3950651](#)
- [17] Brochier, A., Jordan, D.: [Fourier transform for quantum  \$D\$ -modules via the punctured torus mapping class group](#). Quantum Topol. **8**, 361–379 (2017) Zbl [1382.16028](#) MR [3659493](#)
- [18] Brown, K. A., Gordon, I.: [Poisson orders, symplectic reflection algebras and representation theory](#). J. Reine Angew. Math. **559**, 193–216 (2003) Zbl [1025.17007](#) MR [1989650](#)

- [19] Brown, K. A., Yakimov, M. T.: [Azumaya loci and discriminant ideals of PI algebras](#). Adv. Math. **340**, 1219–1255 (2018) Zbl [1433.16016](#) MR [3886192](#)
- [20] Bullock, D., Frohman, C., Kania-Bartoszyńska, J.: [Understanding the Kauffman bracket skein module](#). J. Knot Theory Ramifications **8**, 265–277 (1999) Zbl [0932.57015](#) MR [1691437](#)
- [21] Chari, V., Pressley, A.: A guide to quantum groups. Cambridge University Press, Cambridge (1995) Zbl [0839.17010](#) MR [1358358](#)
- [22] Cooney, N., Ganev, I., Jordan, D.: [Quantum Weyl algebras and reflection equation algebras at a root of unity](#). J. Pure Appl. Algebra **224**, article no. 106440, 15 pp. (2020) Zbl [07299911](#) MR [4102177](#)
- [23] Crawley-Boevey, W., Shaw, P.: [Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem](#). Adv. Math. **201**, 180–208 (2006) Zbl [1095.15014](#) MR [2204754](#)
- [24] De Concini, C., Lyubashenko, V.: [Quantum function algebra at roots of 1](#). Adv. Math. **108**, 205–262 (1994) Zbl [0846.17008](#) MR [1296515](#)
- [25] Donin, J., Kulish, P. P., Mudrov, A. I.: [On a universal solution to the reflection equation](#). Lett. Math. Phys. **63**, 179–194 (2003) Zbl [1042.17011](#) MR [1992884](#)
- [26] Drézet, J.-M.: Luna’s slice theorem and applications. In: Algebraic group actions and quotients, Hindawi, Cairo, 39–89 (2004) Zbl [1109.14307](#) MR [2210794](#)
- [27] Etingof, P., Schiffmann, O.: Lectures on quantum groups. 2nd ed., Lectures in Mathematical Physics, International Press, Somerville, MA (2002) Zbl [1106.17015](#) MR [2284964](#)
- [28] Fock, V. V., Rosly, A. A.: [Poisson structure on moduli of flat connections on Riemann surfaces and the  \$r\$ -matrix](#). In: Moscow Seminar in Mathematical Physics, American Mathematical Society Transl. (2) 191, American Mathematical Society, Providence, RI, 67–86 (1999) Zbl [0945.53050](#) MR [1730456](#)
- [29] Frohman, C., Kania-Bartoszyńska, J., Lê, T.: [Unicity for representations of the Kauffman bracket skein algebra](#). Invent. Math. **215**, 609–650 (2019) Zbl [1491.57014](#) MR [3910071](#)
- [30] Ganev, I.: [Quantizations of multiplicative hypertoric varieties at a root of unity](#). J. Algebra **506**, 92–128 (2018) Zbl [1400.14134](#) MR [3800073](#)
- [31] Ginzburg, V.: Lectures on Nakajima’s quiver varieties. In: Geometric methods in representation theory. I, Séminaires et Congrès 24, Société Mathématique de France, Paris, 145–219 (2012) Zbl [1305.16009](#) MR [3202703](#)
- [32] Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. Inst. Hautes Études Sci. Publ. Math. **24**, 1–231 (1965) Zbl [0135.39701](#)
- [33] Gwilliam, O., Scheimbauer, C.: Duals and adjoints in higher Morita categories. arXiv:[1804.10924v2](#) (2018)
- [34] Hausel, T., Rodriguez-Villegas, F.: [Mixed Hodge polynomials of character varieties](#). Invent. Math. **174**, 555–624 (2008) Zbl [1213.14020](#) MR [2453601](#)
- [35] Hayashi, T.: [Sugawara operators and Kac–Kazhdan conjecture](#). Invent. Math. **94**, 13–52 (1988) Zbl [0674.17005](#) MR [958588](#)
- [36] Jantzen, J. C.: [Representations of algebraic groups](#). 2nd ed., Mathematical Surveys and Monographs 107, American Mathematical Society, Providence, RI (2003) Zbl [1034.20041](#) MR [2015057](#)
- [37] Jordan, D.: [Quantized multiplicative quiver varieties](#). Adv. Math. **250**, 420–466 (2014) Zbl [1337.16012](#) MR [3122173](#)
- [38] Jordan, D., White, N.: [The center of the reflection equation algebra via quantum minors](#). J. Algebra **542**, 308–342 (2020) Zbl [1461.17018](#) MR [4019788](#)
- [39] Joseph, A., Letzter, G.: [Local finiteness of the adjoint action for quantized enveloping algebras](#). J. Algebra **153**, 289–318 (1992) Zbl [0779.17012](#) MR [1198203](#)
- [40] King, A. D.: [Moduli of representations of finite-dimensional algebras](#). Quart. J. Math. Oxford Ser. (2) **45**, 515–530 (1994) Zbl [0837.16005](#) MR [1315461](#)



- [41] Kirillov, A. N., Reshetikhin, N.:  *$q$ -Weyl group and a multiplicative formula for universal  $R$ -matrices*. Comm. Math. Phys. **134**, 421–431 (1990) Zbl 0723.17014 MR 1081014
- [42] Klimyk, A., Schmüdgen, K.: *Quantum groups and their representations*. Texts and Monographs in Physics, Springer, Berlin (1997) Zbl 0891.17010 MR 1492989
- [43] Kolb, S., Stokman, J. V.: *Reflection equation algebras, coideal subalgebras, and their centres*. Selecta Math. (N.S.) **15**, 621–664 (2009) Zbl 1236.17023 MR 2565052
- [44] Kremnizer, K.: Proof of the De Concini–Kac–Procesi conjecture. arXiv:math/0611236v1 (2006)
- [45] Lentner, S.: *A Frobenius homomorphism for Lusztig’s quantum groups for arbitrary roots of unity*. Commun. Contemp. Math. **18**, article no. 1550040, 42 pp. (2016) Zbl 1394.17038 MR 3477402
- [46] Lentner, S., Ohrmann, T.: *Factorizable  $R$ -matrices for small quantum groups*. SIGMA Symmetry Integrability Geom. Methods Appl. **13**, article no. 076, 25 pp. (2017) Zbl 1394.17039 MR 3705499
- [47] Levendorskiĭ, S. Z., Soĭbel’man, Y. S.: *Some applications of the quantum Weyl groups*. J. Geom. Phys. **7**, 241–254 (1990) Zbl 0729.17009 MR 1120927
- [48] Li-Bland, D., Ševera, P.: *Quasi-Hamiltonian groupoids and multiplicative Manin pairs*. Int. Math. Res. Notices **2011**, 2295–2350 (2011) Zbl 1218.53083 MR 2806566
- [49] Lu, J.-H.: *Momentum mappings and reduction of Poisson actions*. In: Symplectic geometry, groupoids, and integrable systems (Berkeley, CA, 1989), Mathematical Sciences Research Institute Publications 20, Springer, New York, 209–226 (1991) Zbl 0735.58004 MR 1104930
- [50] Lu, J.-H.: *Moment maps at the quantum level*. Comm. Math. Phys. **157**, 389–404 (1993) Zbl 0801.17019 MR 1244874
- [51] Lusztig, G.: *Quantum deformations of certain simple modules over enveloping algebras*. Adv. Math. **70**, 237–249 (1988) Zbl 0651.17007 MR 954661
- [52] Lusztig, G.: *Quantum groups at roots of 1*. Geom. Dedicata **35**, 89–113 (1990) Zbl 0714.17013 MR 1066560
- [53] Lusztig, G.: *Introduction to quantum groups*. Modern Birkhäuser Classics, Birkhäuser/Springer, New York (2010) Zbl 1246.17018 MR 2759715
- [54] Lyubashenko, V., Majid, S.: *Braided groups and quantum Fourier transform*. J. Algebra **166**, 506–528 (1994) Zbl 0810.17006 MR 1280590
- [55] Muller, G.: *Skein and cluster algebras of marked surfaces*. Quantum Topol. **7**, 435–503 (2016) Zbl 1375.13038 MR 3551171
- [56] Mumford, D., Fogarty, J., Kirwan, F.: *Geometric invariant theory*. 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34, Springer, Berlin (1994) Zbl 0797.14004 MR 1304906
- [57] Negron, C.: *Log-modular quantum groups at even roots of unity and the quantum Frobenius I*. Comm. Math. Phys. **382**, 773–814 (2021) Zbl 1486.17027 MR 4227163
- [58] Nguyen, B., Trampel, K., Yakimov, M.: *Noncommutative discriminants via Poisson primes*. Adv. Math. **322**, 269–307 (2017) Zbl 1403.17020 MR 3720799
- [59] Paradowski, J.: *Filtrations of modules over the quantum algebra*. In: Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), Proceedings of Symposia in Pure Mathematics 56, American Mathematical Society, Providence, RI, 93–108 (1994) Zbl 0831.20059 MR 1278729
- [60] Przytycki, J. H., Sikora, A. S.: *On skein algebras and  $Sl_2(\mathbb{C})$ -character varieties*. Topology **39**, 115–148 (2000) Zbl 0958.57011 MR 1710996
- [61] Rosso, M.: *Analogues de la forme de Killing et du théorème d’Harish-Chandra pour les groupes quantiques*. Ann. Sci. École Norm. Sup. (4) **23**, 445–467 (1990) Zbl 0721.17012 MR 1055444

- [62] Rosso, M.: [Quantum groups at a root of 1 and tangle invariants](#). *Int. J. Modern Phys. B* **7**, 3715–3726 (1993) Zbl [0806.17013](#) MR [1241466](#)
- [63] Safronov, P.: [Quasi-Hamiltonian reduction via classical Chern–Simons theory](#). *Adv. Math.* **287**, 733–773 (2016) Zbl [1440.53096](#) MR [3422691](#)
- [64] Safronov, P.: A categorical approach to quantum moment maps. *Theory Appl. Categ.* **37**, 818–862 (2021) Zbl [1468.18019](#) MR [4276764](#)
- [65] Schedler, T., Tirelli, A.: [Symplectic resolutions for multiplicative quiver varieties and character varieties for punctured surfaces](#). In: Representation theory and algebraic geometry—a conference celebrating the birthdays of Sasha Beilinson and Victor Ginzburg, *Trends in Mathematics*, Birkhäuser/Springer, Cham, 393–459 (2022) Zbl [1498.53107](#) MR [4486921](#)
- [66] Semenov-Tian-Shansky, M. A.: [Dressing transformations and Poisson group actions](#). *Publ. RIMS Kyoto Univ.* **21**, 1237–1260 (1985) Zbl [0674.58038](#) MR [842417](#)
- [67] Sikora, A. S.: [Character varieties](#). *Trans. Amer. Math. Soc.* **364**, 5173–5208 (2012) Zbl [1291.14022](#) MR [2931326](#)
- [68] Stadnik, T. J., Jr.: [Étale splittings of certain Azumaya algebras on toric and hypertoric varieties in positive characteristic](#). *Adv. Math.* **233**, 268–290 (2013) Zbl [1295.14019](#) MR [2995672](#)
- [69] Turaev, V. G.: [Skein quantization of Poisson algebras of loops on surfaces](#). *Ann. Sci. École Norm. Sup. (4)* **24**, 635–704 (1991) Zbl [0758.57011](#) MR [1142906](#)
- [70] Turaev, V. G.: [Quantum invariants of knots and 3-manifolds](#). *De Gruyter Studies in Mathematics* 18, De Gruyter, Berlin (2016) Zbl [1346.57002](#) MR [3617439](#)
- [71] Van den Bergh, M.: [Double Poisson algebras](#). *Trans. Amer. Math. Soc.* **360**, 5711–5769 (2008) Zbl [1157.53046](#) MR [2425689](#)
- [72] Van den Bergh, M.: [Non-commutative quasi-Hamiltonian spaces](#). In: Poisson geometry in mathematics and physics, *Contemporary Mathematics* 450, American Mathematical Society, Providence, RI, 273–299 (2008) Zbl [1236.58019](#) MR [2397630](#)
- [73] Van Oystaeyen, F., Zhang, Y.: [The Brauer group of a braided monoidal category](#). *J. Algebra* **202**, 96–128 (1998) Zbl [0909.18005](#) MR [1614178](#)
- [74] Varagnolo, M., Vasserot, E.: [Double affine Hecke algebras at roots of unity](#). *Represent. Theory* **14**, 510–600 (2010) Zbl [1280.20005](#) MR [2672950](#)
- [75] Walton, C., Wang, X., Yakimov, M.: [Poisson geometry of PI three-dimensional Sklyanin algebras](#). *Proc. London Math. Soc. (3)* **118**, 1471–1500 (2019) Zbl [1418.17049](#) MR [3957827](#)
- [76] Yamakawa, D.: [Geometry of multiplicative preprojective algebra](#). *Int. Math. Res. Papers* **2008**, article no. rpn008, 77 pp. Zbl [1197.16019](#) MR [2470573](#)