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Large deviations of multichordal SLE_{0+} , real rational functions, and zeta-regularized determinants of Laplacians

Received June 21, 2021

Abstract. We prove a strong large deviation principle (LDP) for multiple chordal SLE_{0+} curves with respect to the Hausdorff metric. In the single-chord case, this result strengthens an earlier partial result by the second author. We also introduce a Loewner potential, which in the smooth case has a simple expression in terms of zeta-regularized determinants of Laplacians. This potential differs from the LDP rate function by an additive constant depending only on the boundary data, which satisfies PDEs arising as a semiclassical limit of the Belavin–Polyakov–Zamolodchikov equations of level 2 in conformal field theory with central charge $c \rightarrow -\infty$.

Furthermore, we show that every multichord minimizing the potential in the upper half-plane for given boundary data is the real locus of a rational function and is unique, thus coinciding with the $\kappa \rightarrow 0+$ limit of the multiple SLE_κ . As a by-product, we provide an analytic proof of the Shapiro conjecture in real enumerative geometry, first proved by Eremenko and Gabrielov: if all critical points of a rational function are real, then the function is real up to post-composition with a Möbius transformation.

Keywords. Schramm–Loewner evolution (SLE), large deviations, enumeration of real rational functions, determinants of Laplacians, BPZ partial differential equations, semiclassical limit of conformal field theory

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Mathematics Subject Classification (2020): Primary 30C55; Secondary 60J67, 60F10

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1. Introduction

The Schramm–Loewner evolution (SLE) is a model for random conformally invariant fractal curves in the plane, introduced by Schramm by combining Loewner’s classical theory for evolution of planar slit domains with stochastic analysis [68]. Schramm’s SLE is a one-parameter family of probability measures on non-self-crossing curves, indexed by $\kappa \geq 0$, and thus denoted by SLE_κ . SLEs play a central role in 2D random conformal geometry. For instance, they describe interfaces in conformally invariant systems arising from statistical physics, which was also Schramm’s original motivation; see, e.g., [48, 69–71]. Through their relationship with critical statistical physics models, SLEs are also closely related to conformal field theory (CFT); see, e.g., [9, 16, 19, 23, 24, 30, 31, 39, 60]. The parameter κ reflects the roughness of these fractal curves, and it also determines the central charge $c(\kappa) = (3\kappa - 8)(6 - \kappa)/2\kappa$ of the associated CFT. Renormalization group arguments [15] suggest that κ encodes the universality classes of the models.

In this article, we consider the chordal case, where SLEs are families of random curves (multichords) connecting pairwise distinct boundary points of some planar domain. Throughout, we let D be a simply connected Jordan domain of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We include the marked distinct boundary points to the domain data $(D; x_1, \dots, x_{2n})$, assuming that they appear in counterclockwise order along the boundary ∂D . We also assume that ∂D is smooth in a neighborhood of the marked points,

unless stated otherwise.¹ Due to the planarity, there exist C_n different possible pairwise non-crossing connections for the curves, where

$$C_n = \frac{1}{n+1} \binom{2n}{n} \quad (1.1)$$

is the n th Catalan number. We enumerate them in terms of n -link patterns

$$\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}\}, \quad (1.2)$$

that is, planar pair partitions of $\{1, \dots, 2n\}$ giving a pairing of the marked points.

We investigate the asymptotic behavior of multichordal SLE_κ as $\kappa \rightarrow 0+$. The case of a single SLE_{0+} was studied in [75] (see also [21, Sec. 9.3]), where the second author introduced the Loewner energy (for a single chord) and studied large deviations of left-right passing events. We strengthen and generalize this result, establishing a strong large deviation principle for chordal and multichordal SLE_{0+} with respect to the Hausdorff metric. The rate function is given by the multichordal Loewner energy, which we introduce shortly.

We also introduce a Loewner potential, which differs from the Loewner energy by a function (the minimal potential) only depending on the boundary data $(x_1, \dots, x_{2n}; \alpha)$. We show that the Loewner potential has a simple expression in terms of zeta-regularized determinants of Laplacians – similar relations between SLEs and determinants of Laplacians have been observed in, e.g., [22, 24, 30, 39, 76]. In particular, the minimal potential is directly linked to SLE partition functions studied in [10, 11, 20, 21, 40, 41, 45, 62] and correlation functions and conformal blocks in boundary CFT with $c \leq 1$ (see, e.g., [24, 37, 60] for details and discussion of further literature). Indeed, we show that the minimal potential can be seen as the semiclassical $c \rightarrow -\infty$ limit of certain CFT correlation functions.

Rather surprisingly, we classify all minimizers of the potential by showing that, for the upper half-plane \mathbb{H} with marked points $x_1 < \dots < x_{2n}$, any minimizer must be the real locus of a rational function. Applying an algebraic geometry result by Goldberg [33], we show that for each $2n$ -tuple $x_1 < \dots < x_{2n}$, there are exactly the n th Catalan number C_n of different real loci of rational functions, each corresponding to a different n -link pattern α for the multichord. Our large deviation result then implies that the multichordal SLE_κ measure, given α , converges to the minimizer of the multichordal Loewner potential. We thereby establish a previously unknown link between SLEs and real enumerative geometry, and as a by-product of this, we obtain a new proof for the so-called Shapiro conjecture [25, 72].

Next, we summarize our main findings and discuss the organization of the article. For the readers' convenience, we first briefly recall chordal SLE and multichordal SLE. Then, we discuss the main results of the present article divided into four subtopics.

¹Some of these regularity assumptions could be relaxed, but we do not gain anything by doing so.

Reminder: Schramm–Loewner evolutions

By a *chord*, we refer to a simple curve connecting two distinct boundary points $x, y \in \partial D$ in D touching ∂D only at its endpoints. Since we are interested in the asymptotic behavior of SLE_κ as $\kappa \rightarrow 0+$, we shall assume throughout that $\kappa \in [0, 8/3)$, so that $c(\kappa) < 0$ and chordal SLE_κ in $(D; x, y)$ is a random chord in D from x to y [66]. Furthermore, by its reversibility property [81], the laws of SLE_κ in $(D; x, y)$ and in $(D; y, x)$ are the same as unparametrized curves, so the order of the endpoints is insignificant.

In fact, the law of chordal SLE_κ can be uniquely characterized by its *conformal invariance* and *domain Markov property*. The former asserts that, for any conformal map (i.e., biholomorphic function) $\varphi: D \rightarrow D'$, the law of SLE_κ in $(D'; \varphi(x), \varphi(y))$ is just the pushforward law of SLE_κ in $(D; x, y)$ by φ . The latter Markovian property for the growth of the SLE curve is very natural from the point of view of statistical mechanics systems. For details, we refer to the textbooks [14, 38, 44, 80], which provide detailed introductions to SLEs and applications, all from slightly different perspectives.

In this article, we denote by $\mathcal{X}(\mathbb{H}; 0, \infty)$ the family of unparametrized chords in $(\mathbb{H}; 0, \infty)$ with infinite total half-plane capacity, and by $\mathcal{X}(D; x, y)$ the image of this family under a conformal map sending $(\mathbb{H}; 0, \infty)$ to $(D; x, y)$. For each $n \geq 1$, domain data $(D; x_1, \dots, x_{2n})$, and n -link pattern α as in (1.2), we let $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ denote the set of *multichords* $\bar{\gamma} = (\gamma_1, \dots, \gamma_n)$ consisting of pairwise disjoint chords where $\gamma_j \in \mathcal{X}(D; x_{a_j}, x_{b_j})$ for each $j \in \{1, \dots, n\}$. *Multichordal* SLE_κ is a probability measure on $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$. It can be defined by the property that, for each j , given the other curves $\{\gamma_i \mid i \neq j\}$, the conditional law of the random curve γ_j is the single chordal SLE_κ in the connected component \hat{D}_j of $D \setminus \bigcup_{i \neq j} \gamma_i$ containing γ_j , highlighted in grey in Figure 1.

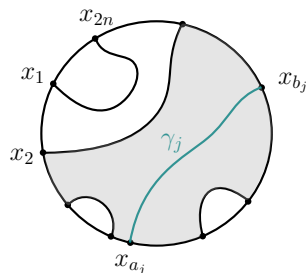


Fig. 1. Illustration of a multichord and the domain \hat{D}_j containing γ_j .

Constructions for multiple chordal SLEs have been obtained in many works [11, 12, 16, 21, 40, 45, 55, 56, 62, 79]. The above definition of multichordal SLE_κ appeared explicitly in, e.g., [12], where the authors proved that multichordal SLE_κ is the unique stationary measure of a Markov chain on $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ defined by re-sampling the curves from their conditional laws. However, this idea was already introduced and used earlier in [55, 56], where Miller & Sheffield studied interacting SLE curves coupled with the Gaussian free field (GFF) in the framework of “imaginary geometry”. See also the recent [57, App. A].

Concretely, the multichordal SLE_κ measures can be constructed from independent chordal SLE_κ measures by weighting by a suitable Radon–Nikodym derivative. However, when $\kappa = 0$, the measures must be singular to each other, so this method does not apply.

1.1. Real rational functions from geodesic multichords

Multichordal SLE_0 in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ is a deterministic multichord $\bar{\eta} = (\eta_1, \dots, \eta_n)$ with the property that each η_j is SLE_0 in its own component $(\hat{D}_j; x_{a_j}, x_{b_j})$. In other words, each η_j is the hyperbolic geodesic in $(\hat{D}_j; x_{a_j}, x_{b_j})$, i.e., the image of $i\mathbb{R}_+$ by a uniformizing conformal map $\varphi: \mathbb{H} \rightarrow \hat{D}_j$ such that $\varphi(0) = x_{a_j}$ and $\varphi(\infty) = x_{b_j}$.

We call a multichord with this property a *geodesic multichord*. As far as we are aware, the existence of such objects for $n \geq 3$ has not been explicitly stated, and their uniqueness has been unknown. Combining an algebraic geometry result of Goldberg [33] with analytic techniques, we obtain

Theorem 1.1. *There exists a unique geodesic multichord in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ for each α .*

The existence of geodesic multichords for each α follows by characterizing them as minimizers of a lower semicontinuous Loewner energy (or potential), to be discussed shortly (see Corollary 1.6). The uniqueness is a consequence of the following algebraic result. (Note that, by the conformal invariance of the geodesic property, we may assume that $D = \mathbb{H}$.)

A *rational function* is an analytic branched covering h of $\hat{\mathbb{C}}$ over $\hat{\mathbb{C}}$ (or equivalently a ratio of two polynomials). The *real locus* of h is the preimage of $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ by h . A *real rational function* is a ratio of two real polynomials. We discuss rational functions in Section 4.1, where we give a constructive proof for the next theorem (see Proposition 4.1).

Theorem 1.2. *Let $\bar{\eta}$ be a geodesic multichord in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2n})$. The union of $\bar{\eta}$, its complex conjugate, and the real line is the real locus of a real rational function of degree $n + 1$ with critical points $\{x_1, \dots, x_{2n}\}$.*

By the Riemann–Hurwitz formula, a rational function of degree $n + 1$ has $2n$ critical points if and only if all of its critical points are of index 2 (namely, the function is locally a two-to-one branched cover near the critical points). To complete the proof of Theorem 1.1, it thus suffices to classify such rational functions with a given set of $2n$ critical points. Goldberg showed in [33] that there are at most C_n of them, up to post-composition with a Möbius transformation of $\hat{\mathbb{C}}$. Since C_n is also the number of n -link patterns, this yields the uniqueness in Theorem 1.1.

In 1995, B. Shapiro and M. Shapiro made a remarkable conjecture concerning real solutions to enumerative geometric problems on the Grassmannian; see [72]. In [25], Eremenko and Gabrielov proved this conjecture for the Grassmannian of 2-planes in n -space, in which case the conjecture is equivalent to the following statement (Corollary 1.3). We obtain this result easily from Theorem 1.1 when the critical points are all of index 2. (The case of degenerating critical points follows similarly to [25], and we do not attempt to provide a new proof.)

Corollary 1.3. *If all critical points of a rational function are real, then it is a real rational function up to post-composition with a Möbius transformation of $\hat{\mathbb{C}}$.*

Conversely, the proof in [25] also implies Theorem 1.1 via Theorem 1.2.

1.2. Large deviations of SLEs

Chordal SLE_κ in $(\mathbb{H}; 0, \infty)$ can be constructed as a random Loewner evolution with driving function $W := \sqrt{\kappa} B$, where B is the standard one-dimensional Brownian motion (see Section 2.2). In this description, SLE_κ is a dynamically growing random curve.

In general, Loewner’s theorem asserts that any simple curve γ in $(\mathbb{H}; 0, \infty)$ can be encoded in a Loewner evolution with some real-valued continuous driving function $t \mapsto W_t$ with $W_0 = 0$. It can then be pulled back to any domain $(D; x, y)$ via a uniformizing conformal map $\varphi: D \rightarrow \mathbb{H}$ sending x and y respectively to 0 and ∞ . The conformally invariant *Loewner energy* of a chord γ in $(D; x, y)$ was introduced in [75] as

$$I_{D;x,y}(\gamma) := I_{\mathbb{H};0,\infty}(\varphi(\gamma)) := \frac{1}{2} \int_0^\infty \left| \frac{dW_t}{dt} \right|^2 dt, \tag{1.3}$$

where W is the Loewner driving function of $\varphi(\gamma)$ and the right-hand side is the Dirichlet energy of W . The Loewner energy of a chord is not always finite, but a driving function W of finite Dirichlet energy such that $W_0 = 0$ always generates a chord in $(\mathbb{H}; 0, \infty)$. Notice also that γ has zero energy in $(D; x, y)$ if and only if $W \equiv 0$ (which generates SLE_0), or equivalently, γ is the hyperbolic geodesic in $(D; x, y)$.

The above definition of the Loewner energy was motivated by large deviations of chordal SLE_κ as $\kappa \rightarrow 0+$. In [75], a weak large deviation result following from Schilder’s theorem for Brownian paths was proved: loosely speaking, we have

$$\mathbb{P}[SLE_\kappa \text{ in } (D; x, y) \text{ stays close to } \gamma] \stackrel{\kappa \rightarrow 0+}{\approx} \exp\left(-\frac{I_{D;x,y}(\gamma)}{\kappa}\right).$$

This result was stated with a rather complicated notion of “staying close” in terms of events of passing to the right or left of a collection of points. It was sufficient to deduce the reversibility of the Loewner energy based on that of SLE_κ , namely, the property that switching the endpoints of the chord does not matter, $I_{D;x,y} = I_{D;y,x}$. We thus consider the chords as being unoriented and sometimes omit the endpoints x, y from the notation.

In the present article, we strengthen the above result to a (strong) large deviation principle (LDP) for SLE_{0+} with respect to the Hausdorff metric. This is a special case of the general LDP for multichordal SLE_{0+} . To state it, for each link pattern α as in (1.2), we endow the curve space $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n}) \subset \prod_j \mathcal{X}(D; x_{a_j}, x_{b_j})$ with the product topology induced by the Hausdorff metric in the closed unit disc \mathbb{D} via a uniformizing² conformal map $\mathbb{D} \rightarrow D$ (see Section 2.1).

²The uniformizing map is not unique, and different maps do not induce the same Hausdorff metric. The induced topologies are still equivalent, as conformal automorphisms of \mathbb{D} are uniformly continuous on \mathbb{D} .

To describe the rate function, we first introduce the *Loewner potential*

$$\mathcal{H}_D(\bar{\gamma}) := \frac{1}{12} \sum_{j=1}^n I_D(\gamma_j) + m_D(\bar{\gamma}) - \frac{1}{4} \sum_{j=1}^n \log P_{D;x_{a_j},x_{b_j}}, \tag{1.4}$$

where the factor $m_D(\bar{\gamma})$ is expressed in terms of the so-called Brownian loop measure introduced by Lawler, Schramm & Werner [47, 49] (see (3.1)) and $P_{D;x,y}$ is the Poisson excursion kernel (see Sections 2.4 and 3.1). Intuitively, the factor $m_D(\bar{\gamma})$ represents interaction of the chords, and the other terms in (1.4) constitute their individual potentials. Alternatively, in Section 1.4 below we will write the potential (for smooth chords) in terms of zeta-regularized determinants of Laplacians (see (1.13)).

Taking the infimum over all multichords $\bar{\gamma} \in \mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$, we denote the minimal Loewner potential by

$$\mathcal{M}_D^\alpha(x_1, \dots, x_{2n}) := \inf_{\bar{\gamma}} \mathcal{H}_D(\bar{\gamma}). \tag{1.5}$$

Then, we define the *multichordal Loewner energy*

$$I_D^\alpha(\bar{\gamma}) := I_{D;x_1,\dots,x_{2n}}^\alpha(\bar{\gamma}) := 12(\mathcal{H}_D(\bar{\gamma}) - \mathcal{M}_D^\alpha(x_1, \dots, x_{2n})). \tag{1.6}$$

Remark 1.4. The multichordal Loewner energy is a non-negative functional on multichords with *fixed* boundary data $(x_1, \dots, x_{2n}; \alpha)$. However, the Loewner potential, from which we can deduce the Loewner energy, is more general since it takes into account different boundary data and can take negative values. The minimal potential (1.5) is also a meaningful function of the boundary data, as we will see in Section 1.3.

Theorem 1.5. *The family $(\mathbb{P}_\alpha^\kappa)_{\kappa>0}$ of laws of the multichordal SLE_κ curves $\bar{\gamma}^\kappa$ satisfies the following LDP in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ with good rate function I_D^α :*

For any closed subset F and open subset O of $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$, we have

$$\begin{aligned} \overline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}_\alpha^\kappa[\bar{\gamma}^\kappa \in F] &\leq - \inf_{\bar{\gamma} \in F} I_D^\alpha(\bar{\gamma}), \\ \underline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}_\alpha^\kappa[\bar{\gamma}^\kappa \in O] &\geq - \inf_{\bar{\gamma} \in O} I_D^\alpha(\bar{\gamma}), \end{aligned}$$

I_D^α is lower semicontinuous, and its level set $(I_D^\alpha)^{-1}[0, c]$ is compact for any $c \geq 0$.

Throughout the present work, “an LDP” shall refer to a large deviation principle of the above type. The reader should mind carefully the topology involved, which can be a subtle point. Also, by a “good rate function” we refer to a rate function whose level sets are compact (which also implies its lower semicontinuity).

To show the compactness of the level sets in Theorem 1.5, we generalize the result [75, Prop. 2.1] for the single-chord Loewner energy, by proving that any finite-energy multichord $\bar{\gamma}$ is the image of a smooth reference multichord in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ under a K -quasiconformal self-map of D , where the constant K only depends on the energy (see Proposition 3.13). From this, we obtain the required precompactness in order to find convergent subsequences, which also shows that the infimum in (1.5) is achieved.

To establish the LDP, we first prove it for the single-chord SLE_{0+} (Section 5.2) via Schilder’s theorem as in [75].³ Note that the contraction principle does not apply naively, since the Loewner transform is not continuous from the space of driving functions to the curve space endowed with the Hausdorff metric, and Schilder’s theorem applies for Brownian paths over a finite time interval. We deal with these technical subtleties by identifying the discontinuity points of the Loewner transform (Lemma 2.4) and by introducing an appropriate truncation (Sections 5.1–5.2). The general LDP for multichordal SLE_{0+} is then obtained via its Radon–Nikodym derivative⁴

$$\frac{1}{Z_\alpha} \exp\left(\frac{1}{2}c(\kappa) m_D(\bar{\gamma})\right) \tag{1.7}$$

with respect to the product measure of n independent SLE_κ chords with the same boundary data (Section 5.3), where $Z_\alpha = Z_\alpha(D; x_1, \dots, x_{2n})$ is the normalizing factor making this a probability measure. The passage from the single-chord case to the multichord case is an application of Varadhan’s lemma (Lemma G), and the constant 12 in the energy (1.6) emerges from the asymptotic behavior of the central charge: $c(\kappa)/2 \sim -12/\kappa$ as $\kappa \rightarrow 0+$.

We also show that any energy-minimizing multichord is a geodesic multichord (Corollary 3.9). The uniqueness of this geodesic multichord (Theorem 1.1) thus implies

Corollary 1.6. *As $\kappa \rightarrow 0+$, multichordal SLE_κ in $(D; x_1, \dots, x_{2n})$ associated to the n -link pattern α converges in probability to the unique minimizer of the potential in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$. This minimizer is the unique geodesic multichord associated to α .*

1.3. Semiclassical limit of CFT null-state equations

To define the energy I_D^α , one could have added to the potential \mathcal{H}_D an arbitrary constant that depends only on the boundary data $(x_1, \dots, x_{2n}; \alpha)$. However, with the precise form (1.4) of our definition, the Loewner potential \mathcal{H}_D not only captures the rate function of multichordal SLE_{0+} , but it also relates multichords of different boundary data. This becomes interesting when $n \geq 2$, as the moduli space of the boundary data is non-trivial.

For instance, \mathcal{H}_D encodes a system of Loewner differential equations which generates the geodesic multichord by varying the marked points (Proposition 1.7). Moreover, it relates to the more general framework of defining a global notion of Loewner energy (cf. Theorem 1.9 and (1.15)), to be discussed shortly. Lastly, the minimal Loewner potential \mathcal{M}_D^α satisfies PDEs arising as a semiclassical limit of the Belavin–Polyakov–Zamolodchikov (BPZ) equations of level 2 in CFT with central charge $c \rightarrow -\infty$ (Proposition 1.8).

³However, we consider general closed and open sets as opposed to just left-right passing events.

⁴When $c(\kappa) > 0$, one has to multiply the exponent by the indicator function $\mathbf{1}\{\bar{\gamma} \text{ is disjoint}\}$. As we are anyway concerned with the limit $\kappa \rightarrow 0+$, we shall restrict our discussion to $\kappa < 8/3$ throughout.

We state below the results for $D = \mathbb{H}$ and $x_1 < \dots < x_{2n}$. We also fix the n -link pattern α and denote $\mathcal{U} := 12\mathcal{M}_{\mathbb{H}}^\alpha$. We pick one chord η_j going from x_{a_j} to x_{b_j} (by reversibility, the role of x_{a_j} to x_{b_j} could be interchanged), and we consider the Loewner flow generated by η_j (parametrized by capacity) while the starting points of the other chords are evolving according to the Loewner equation.

Proposition 1.7. *Let $\bar{\eta}$ be the minimizer of $\mathcal{H}_{\mathbb{H}}$ in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2n})$. Then, for each $j \in \{1, \dots, n\}$, the Loewner driving function W of the chord η_j from x_{a_j} to x_{b_j} and the time evolutions $V_t^i = g_t(x_i)$ of the other marked points satisfy the differential equations*

$$\begin{cases} \frac{dW_t}{dt} = -\partial_{a_j} \mathcal{U}(V_t^1, \dots, V_t^{a_j-1}, W_t, V_t^{a_j+1}, \dots, V_t^{2n}), & W_0 = x_{a_j}, \\ \frac{dV_t^i}{dt} = \frac{2}{V_t^i - W_t}, & V_0^i = x_i, \text{ for } i \neq a_j, \end{cases} \quad (1.8)$$

for $0 \leq t < T$, where T is the lifetime of the solution and $(g_t)_{t \in [0, T]}$ is the Loewner flow generated by η_j .

Proposition 1.8. *For each $j \in \{1, \dots, 2n\}$, we have*

$$\frac{1}{2}(\partial_j \mathcal{U}(x_1, \dots, x_{2n}))^2 - \sum_{i \neq j} \frac{2}{x_i - x_j} \partial_i \mathcal{U}(x_1, \dots, x_{2n}) = \sum_{i \neq j} \frac{6}{(x_i - x_j)^2}. \quad (1.9)$$

We prove Proposition 1.7 in Section 4.2 and Proposition 1.8 in Section 4.3. Both proofs are based on direct analysis of the minimal potential $\mathcal{M}_{\mathbb{H}}^\alpha$ and the associated self-similar Loewner flow, using the fact that the geodesic property of a multichord is preserved under the Loewner flow generated by any of its chords.

We now relate these results to SLE partition functions (see, e.g., [22, Def. 3.1]). For each n -link pattern α , one associates to the multichordal SLE $_\kappa$ a (pure) partition function \mathcal{Z}_α defined in terms of the total mass of the multichordal SLE $_\kappa$ measure in (1.7), i.e.,

$$\mathcal{Z}_\alpha(\mathbb{H}; x_1, \dots, x_{2n}) := \left(\prod_{j=1}^n P_{\mathbb{H}; x_{a_j}, x_{b_j}} \right)^{(6-\kappa)/2\kappa} \times Z_\alpha(\mathbb{H}; x_1, \dots, x_{2n}).$$

As a consequence of the proof of Theorem 1.5, the minimal potential can be viewed as a semiclassical limit of \mathcal{Z}_α (Corollary 5.12), in the sense that

$$\kappa \log \mathcal{Z}_\alpha(\mathbb{H}; x_1, \dots, x_{2n}) \xrightarrow{\kappa \rightarrow 0} -12\mathcal{M}_{\mathbb{H}}^\alpha(x_1, \dots, x_{2n}) = -\mathcal{U}(x_1, \dots, x_{2n}). \quad (1.10)$$

Equation (1.8) is analogous to the marginal law of one chord in the multichordal SLE $_\kappa$, given by a stochastic Loewner equation derived from \mathcal{Z}_α (explicitly, see [62, (4.10)]).

It is well known that the partition functions \mathcal{Z}_α can also be seen as correlation functions associated to conformal fields with degeneracy at level 2 in CFTs of central charge

$c(\kappa) \in (-\infty, 1]$ (see, e.g., [24, 60] and references therein for more discussion). In particular, the correlators \mathcal{Z}_α labeled by all n -link patterns α form a basis for a solution space of the following system of null-state BPZ PDEs (predicted in [13]):

$$\left(\frac{\kappa}{2} \partial_j^2 + \sum_{i \neq j} \left(\frac{2}{x_i - x_j} \partial_i - \frac{(6 - \kappa)/\kappa}{(x_i - x_j)^2} \right) \right) \mathcal{Z}(x_1, \dots, x_{2n}) = 0 \quad (1.11)$$

for all $j \in \{1, \dots, 2n\}$. Here, the number $(6 - \kappa)/\kappa$ is twice the conformal weight $h_{1,2}(\kappa)$ of a conformal primary field associated to a particular representation of the Virasoro algebra (labeled by the so-called Kac weights $h_{r,s}(\kappa)$); correlation functions $\mathcal{Z}(x_1, \dots, x_{2n})$ involving fields of weight $h_{1,2}(\kappa)$ satisfy null-state PDEs of the form (1.11).

Heuristically, plugging the expression $\mathcal{Z} = \exp(-\mathcal{U}/\kappa)$ into the PDE system (1.11) gives exactly the asymptotic form (1.9) as $\kappa \rightarrow 0$. In the physics literature, this appears as a system of Hamilton–Jacobi type equations (also related to Painlevé VI) associated to the semiclassical conformal blocks; see [53] and references therein, as also briefly pointed out in [11, Sec. 4.5]. A semiclassical limit of the dual version of these PDEs (associated to the dual fields of conformal weight $h_{2,1}$ in the Liouville CFT of central charge $c \geq 25$) were used to give a probabilistic proof for the Takhtajan–Zograf theorem relating Poincaré’s accessory parameters to the classical Liouville action; see [42, (1.11) and Sec. 4.4]. Let us also remark that in the recent preprint [1], the evolution of the poles and critical points of the rational function in our Theorem 1.2 was investigated, and related to a particular Calogero–Moser integrable system. A closed form expression of the minimal potential in terms of the rational function is also given in [1, Thm. 2.3].

1.4. Zeta-regularized determinants of Laplacians

Our definition of the Loewner potential is not only natural from the point of view of SLE partition functions and CFT, but also guided by the more general quest for defining a general, *global* notion of Loewner energy and potential for curves on Riemannian surfaces. Indeed, the local growth definition (1.3) of the energy via the Loewner evolution is only well-adapted to the case of a single chord, or a rooted Jordan curve (i.e., *Loewner loop energy* introduced in [67]), and does not explain the presence of many symmetries such as reversibility and root-invariance. In contrast, these symmetries are apparent from the global definitions, which can also be naturally extended to other scenarios: e.g.,

- multichords as in the present work,
- radial or multi-radial curves going from the boundary towards interior points,
- collections of loops embedded in a (closed) surface of higher genus,
- chords joining two boundary components of an annulus, etc.

As a first step towards such a general notion, we establish a relation between the Loewner potential \mathcal{H} and zeta-regularized determinants of Laplacians (denoted by $\det_\zeta \Delta$). Our sign convention is that the Laplacian $\Delta_{D;g}$ on D with Dirichlet boundary

conditions is a positive operator:

$$\Delta_{D;g} := -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^2 \partial_i \sqrt{\det(g)} g^{ij} \partial_j, \tag{1.12}$$

where g is a Riemannian metric on D . If the metric is not mentioned, g is taken by default to be the Euclidean metric dz^2 , in which case we write $\Delta_{D;dz^2} = \Delta_D = -\partial_1^2 - \partial_2^2$. Throughout, we only consider the Laplacian with Dirichlet boundary conditions. Also, we only⁵ consider a bounded smooth D , for then the Laplacians appearing in the following theorem have a discrete spectrum, and their determinants are well-defined (see Section 6.1).

Theorem 1.9. *For any smooth multichord $\bar{\gamma}$ in a bounded smooth domain D , we have*

$$\mathcal{H}_D(\bar{\gamma}) = \log \det_{\zeta} \Delta_D - \sum_C \log \det_{\zeta} \Delta_C - n\lambda, \tag{1.13}$$

where the sum is taken over all connected components C of $D \setminus \bigcup_i \gamma_i$, and $\lambda = \frac{1}{2} \log \pi \approx 0.5724$.

In Theorem 1.9, we consider the Euclidean metric to simplify notation – the result can, however, be easily generalized to arbitrary smooth Riemannian metrics and the associated Laplace–Beltrami operators (in which case \mathcal{H}_D will depend on the metric, and D can be unbounded if the metric tensor decays fast enough).

Theorem 1.9 provides another reason for why it is natural to include Poisson kernels in the definition (1.4) of \mathcal{H} . Since I_D and m_D are conformally invariant quantities, only the Poisson kernel terms contribute to the change of \mathcal{H}_D under conformal maps $\varphi: D \rightarrow D'$:

$$\mathcal{H}_D(\gamma_1, \dots, \gamma_n) = \mathcal{H}_{D'}(\varphi(\gamma_1), \dots, \varphi(\gamma_n)) - \frac{1}{4} \sum_{j=1}^{2n} \log |\varphi'(x_j)|. \tag{1.14}$$

The expression (1.13) captures this conformal covariance more intrinsically. In fact, finite-energy multichords always meet the boundary ∂D perpendicularly (see Lemma B), and $\log \det_{\zeta} \Delta$ vary under a conformal change of metric according to the Polyakov–Alvarez anomaly formula [3] for domains with piecewise smooth boundary, allowing corners (see Theorem H). Importantly, the corners have a non-trivial global contribution to the variation of $\log \det_{\zeta} \Delta$, which coincides with the contribution from the Poisson kernels. This fact is also instrumental in our proof of Theorem 1.9 presented in Section 6.2.

Generally speaking, consider a pair $(\bar{\gamma}, \mathcal{S})$ where $\bar{\gamma}$ is a collection of simple curves and \mathcal{S} is an ambient Riemannian surface endowed with a metric g . In light of our results, the following definition (up to additive constants) for the Loewner potential seems to be the most natural one to consider:

$$\mathcal{H}(\bar{\gamma}, \mathcal{S}) := \log \det_{\zeta} \Delta_{\mathcal{S};g} - \sum_C \log \det_{\zeta} \Delta_{C;g}, \tag{1.15}$$

where the sum is taken over all connected components C of $\mathcal{S} \setminus \bar{\gamma}$.

⁵In fact, one could also allow ∂D to have finitely many corners; cf. Definition 6.1.

For the Loewner loop energy, the identity (1.15) was proved in [76], where the loop energy was further connected to the Kähler potential on the Weil–Petersson Teichmüller space, also of independent interest. Our proposed definition (1.15) agrees with our Theorems 1.5 and 1.9. Furthermore, when γ is a simple loop embedded in a Riemann surface \mathcal{S} (in this case, $\mathcal{H}(\gamma, \mathcal{S})$ is conformally invariant, so we can view \mathcal{S} as a Riemann surface), Equation (1.15) is consistent with the axiomatization of the Malliavin–Kontsevich–Suhov loop measure [39] and Dubédat’s SLE/GFF partition function couplings [22]. In turn, the λ -SAW introduced by Kozdron & Lawler [40] is closely related to these ideas on the discrete lattice model side.

Finally, let us point out that $-\log \det_{\zeta} \Delta_D$ can be formally identified with the total mass of Brownian loop measure in D , as was first observed in [22, 51]. Theorem 1.9 then suggests that $-\mathcal{H}_D(\bar{\gamma})$ equals the total mass of Brownian loops touching the multichord $\bar{\gamma}$. However, computing naively, Brownian loop measure has an infinite total mass coming from the small loops, so a renormalization procedure is needed (as for determinants of Laplacians). The recent [6] shows how one can make sense of the identity by cutting out the small loops (i.e., introducing a UV-cutoff). Using the same idea, we obtain

Theorem 1.10. *Let $\bar{\gamma}$ be a smooth finite-energy multichord in D . The total mass of loops in D touching $\bar{\gamma}$ with quadratic variation greater than 4δ under Brownian loop measure has the expansion*

$$\frac{l(\bar{\gamma})}{2\sqrt{\pi\delta}} - \mathcal{H}_D(\bar{\gamma}) - n\lambda + \frac{n}{4}(\log \delta + \gamma) + O(\delta^{1/2} \log \delta) \quad \text{as } \delta \rightarrow 0+,$$

where $l(\bar{\gamma})$ is the total arclength of $\bar{\gamma}$, λ is the universal constant from Theorem 1.9, and $\gamma \approx 0.5772$ is the Euler–Mascheroni constant.

The asserted expansion follows rather directly from the short-time expansion of the heat trace (Theorem I) and the Mellin transform. We give the proof in Section 6.3.

Structure of the article. In Section 2, we recall definitions and collect necessary preliminary results. Then, in Section 3 we define the multichordal Loewner potential and energy, and derive their first properties. Sections 4, 5, and 6 correspond respectively to our results presented above in Sections 1.1 & 1.3, 1.2, and 1.4. These three sections can be read independently. We also include two appendices: Appendix A concerns a technical SLE estimate needed for Theorem 1.5, and in Appendix B we discuss the Polyakov–Alvarez formula for curvilinear domains, used for Theorem 1.9.

Conventions. All probability spaces are assumed to be completed, and we only consider completed Borel σ -algebras. We label all known results by letters, and results derived in the present work by numbers.

2. Preliminaries

This section is devoted to fixing notation and stating preliminary results. Throughout this article, D will denote a simply connected Jordan domain of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $x, y \in \partial D$ or $x_1, \dots, x_{2n} \in \partial D$ distinct boundary points on smooth boundary segments ordered counterclockwise. Common reference domains are the *unit disc* $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and the *upper half-plane* $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$.

2.1. Hausdorff and Carathéodory topologies

To begin, we discuss two different topologies considered in this work.

Definition 2.1. The *Hausdorff distance* d_h of two compact sets $F_1, F_2 \subset \bar{\mathbb{D}}$ is defined as

$$d_h(F_1, F_2) := \inf \left\{ \varepsilon \geq 0 \mid F_1 \subset \bigcup_{x \in F_2} \bar{B}_\varepsilon(x) \text{ and } F_2 \subset \bigcup_{x \in F_1} \bar{B}_\varepsilon(x) \right\},$$

where $B_\varepsilon(x)$ denotes the Euclidean ball around $x \in \bar{\mathbb{D}}$ of radius ε . Using this, we define the Hausdorff distance for closed subsets of D via the pullback by a conformal map $D \rightarrow \mathbb{D}$. Although the Hausdorff distance depends on the choice of this conformal map, the topology induced by d_h is canonical, as (continuous extensions of) conformal automorphisms of \mathbb{D} are uniformly continuous on $\bar{\mathbb{D}}$. We endow the curve spaces $\mathcal{X}(D; x; y)$ (resp. $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$) with the relative topology (resp. product topology) induced by d_h .

When $D = \mathbb{H}$, we fix throughout this article a conformal map $\vartheta: \mathbb{H} \rightarrow \mathbb{D}$ and we let \mathcal{C} denote the space of non-empty closed subsets of $\bar{\mathbb{H}}$ endowed with the Hausdorff distance defined by pullback by ϑ . In particular, \mathcal{C} is a compact metric space. We say that $K \in \mathcal{C}$ is *bounded* if it is bounded for the Euclidean metric in $\bar{\mathbb{H}}$, i.e., there exists $R > 0$ such that $|z| \leq R$ for all $z \in K$.

We also consider the set of $\bar{\mathbb{H}}$ -*hulls* (or simply, *hulls*), denoted by

$$\mathcal{K} := \{K \in \mathcal{C} \mid K \text{ is bounded, } \mathbb{H} \setminus K \text{ is simply connected, and } \overline{K \cap \mathbb{H}} = K\},$$

where $\overline{K \cap \mathbb{H}}$ is the closure of $K \cap \mathbb{H}$ in \mathbb{C} . For each hull $K \in \mathcal{K}$, there exists a unique uniformizing conformal map $g_K: \mathbb{H} \setminus K \rightarrow \mathbb{H}$, referred to as the *mapping-out function of K* , such that $g_K(z) - z \rightarrow 0$ as $|z| \rightarrow \infty$. Via Schwarz reflection, g_K always extends to a conformal map on the open set $\hat{\mathbb{C}} \setminus (K \cup K^*)$, where K^* is the complex conjugate of K . In particular, g_K is also well-defined on $\mathbb{R} \setminus K$.

It has the expansion

$$g_K(z) = z + \frac{\text{hcap}(K)}{z} + O(|z|^{-2}), \quad |z| \rightarrow \infty, \tag{2.1}$$

where the non-negative constant $\text{hcap}(K) \geq 0$ is called the (*half-plane*) *capacity* of the hull K . The capacity is an increasing function: for all $K, \tilde{K} \in \mathcal{K}$,

$$K \subset \tilde{K} \implies \text{hcap}(K) \leq \text{hcap}(\tilde{K}). \tag{2.2}$$

For fixed $T \in (0, \infty)$, we set

$$\mathcal{K}_T := \{K \in \mathcal{K} \mid \text{hcap}(K_T) = 2T\}.$$

The following topology on \mathcal{K} plays well with Loewner theory (see, e.g., [14, Sec. 3.3] for a detailed discussion of the Carathéodory convergence, and [44, Sec. 3.6] and [38, Sec. 6] for its applications to Loewner theory).

Definition 2.2. We say that a sequence $(K^k)_{k \in \mathbb{N}}$ of hulls converges in the *Carathéodory topology* to $K \in \mathcal{K}$ if the functions $(g_{K^k}^{-1})_{k \geq 0}$ converge uniformly away from⁶ \mathbb{R} to g_K^{-1} . Geometrically, this is equivalent to the *Carathéodory kernel convergence* of the complementary domains $D^k := \mathbb{H} \setminus K^k$ to $D := \mathbb{H} \setminus K$ with respect to ∞ : for any subsequence $(D^{k_j})_{j \in \mathbb{N}}$, we have $D = \bigcup_{j \geq 1} V_j$, where V_j is the unbounded connected component of the interior of $\bigcap_{i \geq j} D^{k_i}$.

Note that the Hausdorff convergence and Carathéodory convergence of hulls are different, and none implies the other. For example, the arcs $K^k = \{e^{i\theta} \mid \theta \in [1/k, \pi]\}$ converge to the half-disc $K = \bar{\mathbb{D}} \cap \bar{\mathbb{H}}$ in the Carathéodory topology, but to the semicircle $\partial\mathbb{D} \cap \bar{\mathbb{H}}$ for the Hausdorff metric.

Lemma 2.3. *If a sequence $(K^k)_{k \in \mathbb{N}}$ of hulls converges to $K \in \mathcal{K}$ in the Carathéodory topology and it also converges to a bounded set $\tilde{K} \in \mathcal{C}$ for the Hausdorff metric, then $\mathbb{H} \setminus K$ coincides with the unbounded connected component of $\mathbb{H} \setminus \tilde{K}$. In particular, $\tilde{K} \cap \mathbb{H} = K \cap \mathbb{H}$ if and only if $\mathbb{H} \setminus \tilde{K}$ is connected.*

Proof. Set $D^k := \mathbb{H} \setminus K^k$ and for $j \geq 1$, let V_j (resp. \tilde{D}) be the unbounded connected component of the interior of $\bigcap_{k \geq j} D^k$ (resp. $\mathbb{H} \setminus \tilde{K}$). It suffices to show that $\tilde{D} = \bigcup_{j \geq 1} V_j$.

By definition, any $z \in V_j$ has a neighborhood contained in all D^k for $k \geq j$. On the one hand, as K^k converges in the Hausdorff metric to \tilde{K} , we see that $z \notin \tilde{K}$, so $V_j \subset \mathbb{H} \setminus \tilde{K}$. Hence, as each V_j is connected and unbounded, we deduce that $\bigcup_{j \geq 1} V_j \subset \tilde{D}$. On the other hand, for $z \in \tilde{D}$, let Γ_z be a path connecting z to ∞ at a positive distance from \tilde{K} . The Hausdorff convergence of K^k implies that Γ_z has a neighborhood which belongs to only finitely many K^k . This implies that $z \in \bigcup_{j \geq 1} V_j$, so $\tilde{D} \subset \bigcup_{j \geq 1} V_j$, finishing the proof. ■

2.2. Loewner chains, driving functions, and SLE

We denote by $C^0[0, \infty)$ the space of continuous, real-valued functions $t \mapsto W_t$ such that $W_0 = 0$. A *chordal Loewner chain* driven by $W \in C^0[0, \infty)$ is a family $(g_t)_{t \geq 0}$

⁶That is, for any $\varepsilon > 0$, we have $g_{K^k}^{-1} \rightarrow g_K^{-1}$ uniformly on $\{z \in \mathbb{C} \mid \text{Im } z \geq \varepsilon\}$; cf. [44, Sec. 3.6].

of conformal maps associated with a collection $(K_t)_{t \geq 0}$ of growing hulls obtained by solving the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \quad \text{with initial condition} \quad g_0(z) = z \tag{2.3}$$

for each $z \in \bar{\mathbb{H}}$. The solution $t \mapsto g_t(z)$ to (2.3) is defined up to the swallowing time of z ,

$$\tau(z) := \sup \left\{ t \geq 0 \mid \inf_{s \in [0, t]} |g_s(z) - W_s| > 0 \right\}, \quad \tau(0) := 0,$$

and the increasing family of sets $K_t := \{z \in \bar{\mathbb{H}} \mid \tau(z) \leq t\}$ are $\bar{\mathbb{H}}$ -hulls. In particular, $K_t = \bar{K}_t \cap \bar{\mathbb{H}}$. Moreover, $g_t = g_{K_t}$ are the mapping-out functions of K_t , and the hulls $(K_t)_{t \geq 0}$ are parametrized by capacity, i.e., $K_t \in \mathcal{K}_t$ for all $t \geq 0$.

Throughout, we endow the space $C^0[0, T]$ of continuous functions $t \mapsto W_t$ on $[0, T]$ such that $W_0 = 0$ with the topology induced by the uniform norm $\|W\|_\infty := \sup_{t \in [0, T]} |W_t|$, and $C^0[0, \infty)$ with the topology of uniform convergence on all compact subsets of $[0, \infty)$.

Note that, for fixed $T > 0$, the hull K_T generated by W depends only on $W_{[0, T]}$, that is, W restricted on $[0, T]$. Moreover, the *Loewner transform*

$$\mathcal{L}_T: C^0[0, T] \rightarrow \mathcal{K}_T, \quad W_{[0, T]} \mapsto K_T,$$

is continuous when \mathcal{K}_T is equipped with the Carathéodory topology (see [38, Prop. 6.1]). However, this is not true for the Hausdorff metric. When dealing with the LDP of SLE curves with respect to the Hausdorff metric, we will use the following observation.

We say that F (resp. O) is a *Hausdorff-closed* (resp. *Hausdorff-open*) set in \mathcal{K}_T if it is closed (resp. open) for the relative topology on \mathcal{K}_T induced by the Hausdorff metric.

Lemma 2.4. *Let F be a Hausdorff-closed set in \mathcal{K}_T . If $W \in \overline{\mathcal{L}_T^{-1}(F)} \setminus \mathcal{L}_T^{-1}(F)$, then the hull $\mathcal{L}_T(W)$ has a non-empty interior.*

Proof. Take a sequence $(W^k)_{k \in \mathbb{N}}$ of driving functions such that $W^k \in \mathcal{L}_T^{-1}(F)$ and W^k converges to $W \notin \mathcal{L}_T^{-1}(F)$ in $C^0[0, T]$. Then the corresponding hulls $K^k := \mathcal{L}_T(W^k)$ converge to $K := \mathcal{L}_T(W) \in \mathcal{K}_T$ in the Carathéodory topology. Also, by the compactness of \mathcal{C} , there exists a subsequence K^{k_j} that converges to some $\tilde{K} \in \bar{F}$ for the Hausdorff metric, where \bar{F} is the closure of F in \mathcal{C} . We will first show that \tilde{K} is bounded and then apply Lemma 2.3 to compare \tilde{K} with the Carathéodory limit K of K^k . Indeed, by considering the real part and the imaginary part of the Loewner equation (2.3), we see that for all $z \in K^k$, we have $|\operatorname{Re} z| \leq \|W^k\|_\infty$ and $\operatorname{Im} z \leq 2\sqrt{T}$. Since the sequence $(W^k)_{k \in \mathbb{N}}$ is uniformly bounded in $C^0[0, T]$, the sequence $(K^k)_{k \in \mathbb{N}}$ is uniformly bounded in $\bar{\mathbb{H}}$ for the Euclidean distance, which implies that \tilde{K} is also bounded.

If $K \cap \mathbb{H} \neq \tilde{K} \cap \mathbb{H}$, Lemma 2.3 shows that $\mathbb{H} \setminus \tilde{K}$ has several connected components, so K has a non-empty interior. If $K \cap \mathbb{H} = \tilde{K} \cap \mathbb{H}$, we claim that $\tilde{K} = K$. Indeed,

$$K = \overline{K \cap \mathbb{H}} = \overline{\tilde{K} \cap \mathbb{H}} \subset \tilde{K}. \tag{2.4}$$

Now let $x \in \mathbb{R} \setminus K$ and $\varepsilon > 0$ be such that $\text{dist}(x, K) > 2\varepsilon$. Since W^k converges uniformly to W , the Loewner equation (2.3) shows that for large enough k , $\text{dist}(x, K^k) > \varepsilon$. Hence, $x \notin \tilde{K}$ and $K = \tilde{K}$. From (2.4), we obtain $K = \tilde{K} \in \bar{F} \cap \mathcal{K}_T = F$, which contradicts the assumption that $W \notin \mathcal{L}_T^{-1}(F)$. This finishes the proof. ■

We say that a function $W \in C^0[0, \infty)$ generates a chord γ in $\mathcal{X}(\mathbb{H}; 0, \infty)$ if the map $t \mapsto \gamma_t$ is a continuous injection from $[0, \infty)$ to $\bar{\mathbb{H}}$ such that $\gamma_0 = 0$, $\gamma_{(0, \infty)} \subset \mathbb{H}$, $|\gamma_t| \rightarrow \infty$ as $t \rightarrow \infty$, and the image $\gamma_{[0, t]}$ equals $\mathcal{L}_t(W_{[0, t]})$ for all $t \geq 0$. We also say that γ is capacity-parametrized if $\gamma_{[0, t]} \in \mathcal{K}_t$ for all $t \geq 0$. Conversely, any chord $\gamma \in \mathcal{X}(\mathbb{H}; 0, \infty)$ can be endowed with capacity parametrization and is generated by the driving function $W \in C^0[0, \infty)$ given by the image of its tip γ_t by the mapping-out function of $\gamma_{[0, t]}$, i.e.,

$$W_t := g_{\gamma_{[0, t]}}(\gamma_t) \quad \text{for all } t \geq 0. \tag{2.5}$$

We define the Dirichlet energies for $W \in C^0[0, \infty)$ as

$$I(W) := \frac{1}{2} \int_0^\infty \left| \frac{dW_t}{dt} \right|^2 dt \quad \text{and} \quad I_T(W) := \frac{1}{2} \int_0^T \left| \frac{dW_t}{dt} \right|^2 dt \quad \text{for } T > 0 \tag{2.6}$$

if W is absolutely continuous, and set them equal to ∞ otherwise. When $I(W) < \infty$, the function W always generates a chord in $(\mathbb{H}; 0, \infty)$ by [75, Prop. 2.1].

Definition 2.5. SLE_κ in $(\mathbb{H}; 0, \infty)$ is the Loewner chain driven by $W = \sqrt{\kappa} B$, where $B = (B_t)_{t \geq 0}$ is the standard Brownian motion and $\kappa \geq 0$ is the diffusivity parameter.

In the present article, we are only interested in the behavior of SLE_κ when $\kappa \rightarrow 0+$. Thus, for convenience, we assume throughout that $\kappa < 8/3$, so the associated central charge $c(\kappa) = (3\kappa - 8)(6 - \kappa)/2\kappa$ is negative, and SLE_κ is almost surely generated by a simple chord.⁷ The latter fact was proved by Rohde & Schramm [66]. Then, SLE_κ in $(D; x, y)$ is defined as the pullback of SLE_κ in $(\mathbb{H}; 0, \infty)$ by any conformal map φ from D to \mathbb{H} sending x to 0 and y to ∞ . We remark that another choice $\tilde{\varphi}$ of the conformal map differs only by a scaling factor, i.e., $\tilde{\varphi} = c\varphi$ for some $c > 0$. However, the driving function transforms from W to $t \mapsto cW_{c^{-2}t}$ under the scaling $z \mapsto cz$, so in particular the law of Brownian motion is preserved. Thus, SLE_κ in $(D; x, y)$ is well-defined and conformally invariant.

2.3. Chordal Loewner energy

The Loewner energy of a chord $\gamma \in \mathcal{X}(D; x, y)$ is defined as the Dirichlet energy (2.6) of its driving function,

$$I_D(\gamma) = I_{D; x, y}(\gamma) := I_{\mathbb{H}; 0, \infty}(\varphi(\gamma)) := I(W), \tag{2.7}$$

⁷ SLE_κ is generated by a simple chord whenever $\kappa \leq 4$, but the central charge is positive if $\kappa \in (8/3, 4]$.

where φ is any conformal map from D to \mathbb{H} such that $\varphi(x) = 0$ and $\varphi(y) = \infty$, and W is the driving function of $\varphi(\gamma)$. Note that the definition of $I_D(\gamma)$ does not depend on the choice of φ , since $I(t \mapsto W_t) = I(t \mapsto cW_{c^{-2}t})$ as we discussed after Definition 2.5.

The Loewner energy $I_{D;x,y}(\gamma)$ is non-negative and it is minimized by the *hyperbolic geodesic* $\eta = \eta_{D;x,y}$, that is, the preimage of $i\mathbb{R}_+$ under φ . Indeed, the driving function of $\varphi(\eta)$ is just the constant function $W \equiv 0$, so $I_D(\eta) = 0$.

The Loewner energy was introduced in [75], where some of its basic properties were also studied. A key property of finite-energy chords is that they are images of analytic curves by quasiconformal maps. We briefly recall the results needed in the present work, and refer the readers to [75] for more details and to, e.g., [50, Ch. 1] for basics on quasiconformal maps.

Lemma 2.6. *If $I_{D;x,y}(\gamma) < \infty$, then there exists a constant $K \in [1, \infty)$ depending only on $I_{D;x,y}(\gamma)$ such that γ is the image of η by a K -quasiconformal map φ with $\varphi(D) = D$, and φ extends continuously to \bar{D} and equals the identity function on ∂D .*

Proof. We prove the assertion first for $(\mathbb{H}; 0, \infty)$. According to [75, Prop. 2.1], there exists a K -quasiconformal map $\tilde{\varphi}: \mathbb{H} \rightarrow \mathbb{H}$ such that $\tilde{\varphi}(0) = 0$, $\tilde{\varphi}(\infty) = \infty$, and $\tilde{\varphi}(i\mathbb{R}_+) = \gamma$. Here, $K \geq 1$ denotes a constant depending only on $I_{\mathbb{H};0,\infty}(\gamma)$ and may change from line to line. Because $\tilde{\varphi}$ fixes ∞ , we know from [50, Thm. 5.1] that $\tilde{\varphi}$ extends to a homeomorphism on $\bar{\mathbb{H}}$, whose restriction $\tilde{\varphi}|_{\mathbb{R}}$ is moreover a $\lambda(K)$ -quasisymmetric homeomorphism of \mathbb{R} for some distortion function $\lambda(K)$.

Now, note that $\tilde{\varphi}|_{\mathbb{R}}$ is not yet the identity function. To fix this issue, from the Jerison–Kenig extension theorem (Theorem A below), we know that $\tilde{\varphi}|_{\mathbb{R}}^{-1}$ can be extended to a K -quasiconformal map ψ on \mathbb{H} with $i\mathbb{R}_+$ fixed. Hence, the map $\varphi := \tilde{\varphi} \circ \psi$ is the desired quasiconformal map.

Finally, for a general Jordan domain $(D; x, y)$, the result follows by conjugating φ by a uniformizing conformal map from D to \mathbb{H} . Indeed, recall that pre-composition or post-composition of a K -quasiconformal map with a conformal map is again K -quasiconformal. Moreover, Carathéodory’s theorem [64, Thm. 2.6] shows that the uniformizing map extends to a homeomorphism from \bar{D} to $\bar{\mathbb{H}}$. Therefore, the conjugated quasiconformal map from D to D also has the asserted boundary values, being the identity function on ∂D . ■

Theorem A (Jerison–Kenig extension). *There exists a function $K(\lambda)$ such that every λ -quasisymmetric function h on \mathbb{R} with $h(0) = 0$ can be extended to a $K(\lambda)$ -quasiconformal map on \mathbb{H} with $i\mathbb{R}_+$ fixed.*

Proof. The extension is constructed in [7, Thm. 5.8.1], following the original proof from [35]. The fact that the extension fixes $i\mathbb{R}_+$ follows from the construction. ■

The next lemma shows that the Loewner energy is good as a rate function.

Lemma 2.7. *The map $I_{D;x,y}$ from $\mathcal{X}(D; x, y)$ to $[0, \infty]$ is lower semicontinuous, and furthermore, its level set $\{\gamma \in \mathcal{X}(D; x, y) \mid I_{D;x,y}(\gamma) \leq c\}$ is compact for any $c \geq 0$.*

Proof. Note that compactness of all level sets implies that they are closed, which is equivalent to the lower semicontinuity. Also, by conformal invariance, we may take $D = \mathbb{D}$ without loss of generality. Therefore, it suffices to show that for any $c \geq 0$, the set $\{\gamma \in \mathcal{X}(\mathbb{D}; x, y) \mid I_{\mathbb{D};x,y}(\gamma) \leq c\}$ is compact. For this, let $(\gamma^k)_{k \in \mathbb{N}}$ be a sequence of chords in $\mathcal{X}(\mathbb{D}; x, y)$ with $I_{\mathbb{D}}(\gamma^k) \leq c$. As \mathcal{C} is compact, there exists a convergent subsequence in \mathcal{C} (still denoted by γ^k). We prove that the limit is a curve $\gamma \in \mathcal{X}(\mathbb{D}; x, y)$ with $I_{\mathbb{D}}(\gamma) \leq c$.

We first show that the convergence of γ^k (along a further subsequence) also holds uniformly on compact subsets of $[0, \infty)$ as capacity-parametrized curves. For this purpose, for each k , we let $\phi^k: \mathbb{D} \rightarrow \mathbb{D}$ be a K -quasiconformal map as in Lemma 2.6, so that $\phi^k(\eta_{\mathbb{D};x,y}) = \gamma^k$. Note that K depends on the Loewner energy of γ^k only, and since the energies are uniformly bounded by c , we can take K independent of k . Since $\phi^k|_{\partial\mathbb{D}}$ is the identity function, we can extend ϕ^k by reflection to a K -quasiconformal map on $\hat{\mathbb{C}}$, and they form a normal family (see [50, Thm. 2.1]). In particular, along a subsequence, ϕ^{k_j} converges uniformly on $\overline{\mathbb{D}}$ to a quasiconformal map ϕ . This shows that the limit of γ^k in \mathcal{C} is the curve $\gamma := \phi(\eta_{\mathbb{D};x,y})$. Furthermore, by [52, proof of Lem. 4.1], the capacity parametrization of a K -quasiconformal curve has modulus of continuity depending only on K , so the Arzelà–Ascoli theorem implies that, as capacity-parametrized curves, the sequence γ^{k_j} converges to γ uniformly on compact subsets of $[0, \infty)$.

It remains to prove the bound $I_{\mathbb{D}}(\gamma) \leq c$ for the limit curve. Since the corresponding level set $\{W \in C^0[0, t] \mid I_t(W) \leq c\}$ of the Dirichlet energy is compact for all $t > 0$, we see that along a subsequence, the driving function W^k of γ^k converges uniformly on compact subsets of $[0, \infty)$ to some function $W \in C^0[0, \infty)$. Furthermore, [52, Lem. 4.2] and the uniform convergence of γ^k imply that W is the driving function of γ . The lower semicontinuity of the Dirichlet energy then gives

$$I_{\mathbb{D}}(\gamma) = I(W) \leq \liminf_{k \rightarrow \infty} I(W^k) = \liminf_{k \rightarrow \infty} I_{\mathbb{D}}(\gamma^k) \leq c,$$

which concludes the proof. ■

The above proof also shows that I_D is a good rate function on the space of capacity-parametrized curves with the topology of uniform convergence on compact subsets.

Lastly, we recall that any finite-energy chord meets the domain’s boundary perpendicularly. By the conformal invariance of I_D and the assumption that ∂D is smooth in neighborhoods of the marked points, we may assume that $(D; x, y) = (\mathbb{H}; 0, \infty)$. For any angle $\theta \in (0, \pi/2)$, we define $\text{Cone}(\theta) := \{z \in \mathbb{H} \mid \theta < \arg z < \pi - \theta\}$.

Lemma B. *We have $\inf I_{\mathbb{H};0,\infty}(\gamma) = -8 \log \sin(\theta)$, where the infimum is taken over all chords $\gamma \in \mathcal{X}(\mathbb{H}; 0, \infty)$ which ever exit $\text{Cone}(\theta)$. In particular, if $\gamma \in \mathcal{X}(\mathbb{H}; 0, \infty)$ has finite energy, then for all $\theta \in (0, \pi/2)$, there exists $\delta > 0$ such that $\gamma_{[0,\delta]} \subset \overline{\text{Cone}(\theta)}$.*

Proof. The asserted identity was shown in [75, Prop. 3.1]. Let W be a driving function of a chord γ with $I(W) < \infty$ and $\theta \in (0, \pi/2)$. Then there exists $\delta > 0$ such that $I_{\delta}(W) < -8 \log \sin(\theta)$, so the curve generated by $t \mapsto W_{\min(t,\delta)}$ is contained in $\overline{\text{Cone}(\theta)}$ and coincides with γ up to capacity 2δ . This proves the lemma. ■

Let us remark here on a subtle point in the definition of Loewner energy. We require from the definition of $\mathcal{X}(\mathbb{H}; 0, \infty)$ that the chords in this set have infinite total half-plane capacity. A priori, one can define Loewner energy for any chord using $I_T(W)$ in (2.6), without assuming that its total half-plane capacity $2T$ is infinite. However, Lemma B then implies that if the chord has finite energy, then it is contained in $\text{Cone}(\theta)$ for some $\theta \in (0, \pi/2)$, which implies by [43, Thm. 1] that the chord must have infinite total capacity and thus belongs to $\mathcal{X}(\mathbb{H}; 0, \infty)$. In other words, chords with finite total capacity always have infinite energy.

2.4. Brownian loop measure

To define the multichordal Loewner energy and potential in Section 3, we use the *Brownian loop measure* [44, 47, 49]. In short, it is a σ -finite measure on planar unrooted Brownian loops. We refer to [49, Secs. 3–4] for the properties of this measure, and only briefly recall below its features important to the present work.

Fix $z \in D$ and $t > 0$. Consider the (subprobability) measure \mathbb{W}_z^t on Brownian paths started from z on the time interval $[0, t]$ and killed upon hitting the boundary of D (run at speed 2, namely, with diffusion generator $-\Delta_D$ with Dirichlet boundary conditions). The disintegration of this measure with respect to the endpoint w gives the (subprobability) measures $\mathbb{W}_{z \rightarrow w}^t$ on Brownian paths from z to w such that

$$\mathbb{W}_z^t = \int_D \mathbb{W}_{z \rightarrow w}^t \, dw^2.$$

The total mass of $\mathbb{W}_{z \rightarrow w}^t$ is the heat kernel $p_t(z, w) = e^{-t\Delta_D}(z, w)$. The *Brownian loop measure* on D is defined as

$$\mu_D^{\text{loop}} := \int_0^\infty \frac{dt}{t} \int_D \mathbb{W}_{z \rightarrow z}^t \, dz^2.$$

Upon forgetting the root z , this yields a measure on the set of unrooted and unparametrized loops in D (so we distinguish loops only by their trace).

It has the following properties:

- *Restriction property:* If $U \subset D$, then $d\mu_U^{\text{loop}}(\ell) = \mathbf{1}\{\ell \subset U\} d\mu_D^{\text{loop}}(\ell)$.
- *Conformal invariance:* If $\varphi: D \rightarrow D'$ is a conformal map, then $\mu_{D'}^{\text{loop}} = \varphi_*(\mu_D^{\text{loop}})$.

The total mass of μ_D^{loop} is infinite (e.g., because of small loops) but when considering only loops that intersect two macroscopic disjoint sets, the measure is finite: if K_1, K_2 are two disjoint compact subsets of \bar{D} , the total mass of Brownian loops that stay in D and intersect both K_1 and K_2 is finite. More generally, if K_1, \dots, K_m are disjoint compact subsets of \bar{D} , we denote by

$$\mathcal{B}_D(K_1, \dots, K_m) := \mu_D^{\text{loop}}(\{\ell \mid \ell \cap K_j \neq \emptyset \text{ for all } j = 1, \dots, m\})$$

the mass of the Brownian loops in D which intersect all of K_1, \dots, K_m . This quantity is positive, finite, and conformally invariant.

We also often use the *Poisson excursion kernel* $P_{D;x,y}$, defined via

$$P_{D;x,y} := |\varphi'(x)| |\varphi'(y)| P_{\mathbb{H};\varphi(x),\varphi(y)}, \quad \text{where } P_{\mathbb{H};x,y} := |y - x|^{-2}, \quad (2.8)$$

and where $\varphi: D \rightarrow \mathbb{H}$ is a conformal map. (Recall that ∂D is smooth in neighborhoods of x and y so that $\varphi'(x)$ and $\varphi'(y)$ are defined.)

The following conformal restriction formula for the Loewner energy, expressed in terms of Brownian loop measure and Poisson excursion kernels, is crucial.

Lemma C ([77, Prop. 3.1]). *Let $U \subset D$ be a simply connected subdomain which coincides with D in neighborhoods of x and y . For each $\gamma \in \mathcal{X}(U; x, y)$, we also have $\gamma \in \mathcal{X}(D; x, y)$ and its Loewner energies in U and in D differ by*

$$I_U(\gamma) - I_D(\gamma) = 12\mathcal{B}_D(D \setminus U, \gamma) + 3 \log|\varphi'(x)\varphi'(y)|, \quad (2.9)$$

where φ is a conformal map from U to D fixing x and y .

Applying (2.9) to the hyperbolic geodesic η in $(U; x, y)$, we get the following relation.

Corollary 2.8. *If η is the hyperbolic geodesic in $(U; x, y)$, then*

$$\log P_{U;x,y} - \log P_{D;x,y} = \log |\varphi'(x)\varphi'(y)| = -\frac{1}{3}I_D(\eta) - 4\mathcal{B}_D(D \setminus U, \eta) \leq 0.$$

Virág [74] showed that $|\varphi'(x)\varphi'(y)|$ equals the probability that Brownian excursion in D from x to y avoids $D \setminus U$, which also implies that $\log |\varphi'(x)\varphi'(y)| \leq 0$.

3. Multichordal Loewner potential

In this section, we introduce the Loewner potential \mathcal{H} and Loewner energy I for multichords. They are defined using Brownian loop measure (cf. Section 2.4) and the single-chord Loewner energy (2.7) (cf. Section 2.3). The quantities $12\mathcal{H}$ and I only differ by an additive constant that is a function of the boundary data $(x_1, \dots, x_{2n}; \alpha)$, in particular, independent of the multichord. Salient properties of \mathcal{H} and I include conformal covariance (Lemma 3.7), the fact that all finite-energy multichords are obtained as quasi-conformal images of analytic multichords (Proposition 3.12), and the fact that the multichordal Loewner energy is a good rate function (Proposition 3.13), important to the LDP in Section 5.

3.1. A loop measure

The following loop measure m_D will be used to define the multichordal Loewner potential and energy. The setup and notation is illustrated in Figure 1. Throughout, we fix an n -link pattern

$$\alpha = \{\{a_1, b_1\}, \{a_2, b_2\}, \dots, \{a_n, b_n\}\}.$$

As in [45], for a multichord $\bar{\gamma} = (\gamma_1, \dots, \gamma_n) \in \mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$, we set

$$\begin{aligned}
 m_D(\bar{\gamma}) &:= \sum_{p=2}^n \mu_D^{\text{loop}}(\{\ell \mid \ell \cap \gamma_i \neq \emptyset \text{ for at least } p \text{ of the } i \in \{1, \dots, n\}\}) \\
 &= \int \max(\#\{\text{chords hit by } \ell\} - 1, 0) \, d\mu_D^{\text{loop}}(\ell)
 \end{aligned}
 \tag{3.1}$$

and $m_D(\gamma) := 0$ if $n = 1$. See also [20, 21, 40, 62] for alternative forms.

Lemma 3.1. *For each $j \in \{1, \dots, n\}$, we have*

$$m_D(\bar{\gamma}) = \mathcal{B}_D(D \setminus \hat{D}_j, \gamma_j) + m_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n)$$

(recall that \hat{D}_j is the connected component of $D \setminus \bigcup_{i \neq j} \gamma_i$ containing the chord γ_j).

Proof. This follows immediately from the definition of m_D . ■

Lemma 3.2. *The map m_D is continuous from $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ to $[0, \infty)$.*

Proof. By conformal invariance, we may take $D = \mathbb{D}$ without loss of generality. The non-negativity of $m_{\mathbb{D}}(\bar{\gamma})$ follows from its definition (3.1). For the continuity, let $\bar{\gamma}^k$ be a sequence converging to $\bar{\gamma}$ as $k \rightarrow \infty$ in $\mathcal{X}_\alpha(\mathbb{D}; x_1, \dots, x_{2n})$. Then each γ_j^k converges to γ_j in $\mathcal{X}(\mathbb{D}; x_{a_j}, x_{b_j})$. Now, it suffices to show that

$$\mu_{\mathbb{D}}^{\text{loop}}(A_p^k) \xrightarrow{k \rightarrow \infty} \mu_{\mathbb{D}}^{\text{loop}}(A_p) \quad \text{for each } p = 2, \dots, n,$$

where

$$\begin{aligned}
 A_p^k &:= \{\ell \mid \ell \cap \gamma_i^k \neq \emptyset \text{ for at least } p \text{ of the } i \in \{1, \dots, n\}\}, \\
 A_p &:= \{\ell \mid \ell \cap \gamma_i \neq \emptyset \text{ for at least } p \text{ of the } i \in \{1, \dots, n\}\}.
 \end{aligned}$$

For this, it suffices to show that for each $p = 2, \dots, n$, we have

$$|m_{\mathbb{D}}(\bar{\gamma}) - m_{\mathbb{D}}(\bar{\gamma}^k)| \leq \mu_{\mathbb{D}}^{\text{loop}}(A_p \Delta A_p^k) \xrightarrow{k \rightarrow \infty} 0, \tag{3.2}$$

where $A_p \Delta A_p^k := (A_p \setminus A_p^k) \cup (A_p^k \setminus A_p)$ is the symmetric difference. To prove (3.2), we fix $p \in \{2, \dots, n\}$ and consider a Brownian loop $\ell \in A_p \Delta A_p^k$. Then either ℓ intersects less than p of the chords $\bar{\gamma}^k$ and at least p of the chords in $\bar{\gamma}$, or vice versa. Without loss of generality, let us consider the former case. Then, since $p \geq 2$, ℓ intersects at least two the chords in $\bar{\gamma}$, so in particular ℓ is a macroscopic loop. Furthermore, there exists an index $j \in \{1, \dots, n\}$ such that ℓ intersects γ_j but not γ_j^k . On the other hand, as γ_j^k converges to γ_j , we see that when k is large enough, the Hausdorff distance of γ_j^k and γ_j is small, so both γ_j^k and γ_j belong to a narrow tube. But then the total mass of ℓ intersecting γ_j but avoiding γ_j^k tends to zero when $k \rightarrow \infty$. This proves (3.2) and concludes the proof. ■

3.2. Definition of potential and energy

The loop measure term $m_D(\bar{\gamma})$ represents interaction of the chords. We now use it to define the multichordal Loewner potential \mathcal{H} and energy I .

Definition 3.3. For $n \geq 1$, we define the *Loewner potential*

$$\mathcal{H}_D(\bar{\gamma}) := \frac{1}{12} \sum_{j=1}^n I_{D;x_{a_j},x_{b_j}}(\gamma_j) + m_D(\bar{\gamma}) - \frac{1}{4} \sum_{j=1}^n \log P_{D;x_{a_j},x_{b_j}}, \quad (3.3)$$

where x_{a_j}, x_{b_j} are the endpoints of the chord γ_j for each $j \in \{1, \dots, n\}$. Note that if $n = 1$, then the loop measure term is zero, so

$$\mathcal{H}_D(\gamma) := \frac{1}{12} I_{D;x,y}(\gamma) - \frac{1}{4} \log P_{D;x,y}. \quad (3.4)$$

Since the first two terms in (3.3) are non-negative, we can also define

$$\mathcal{M}_D^\alpha(x_1, \dots, x_{2n}) := \inf_{\bar{\gamma}} \mathcal{H}_D(\bar{\gamma}) > -\infty,$$

where the infimum is taken over all $\bar{\gamma} \in \mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$. Note that the infimum depends on the marked points $x_1, \dots, x_{2n} \in \partial D$ as well as on the link pattern α .

Definition 3.4. The *multichordal Loewner energy* of $\bar{\gamma} \in \mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ is

$$I_D^\alpha(\bar{\gamma}) := 12(\mathcal{H}_D(\bar{\gamma}) - \mathcal{M}_D^\alpha(x_1, \dots, x_{2n})) \geq 0.$$

We say that the multichord $\bar{\gamma}$ in D has *finite energy* if $I_D^\alpha(\bar{\gamma}) < \infty$.

Remark 3.5. The third term in the Loewner potential (3.3) only depends on the boundary data, so it does not affect the Loewner energy. However, as we will see, this term is relevant when we compare multichords of different boundary data (Section 4). This term also encodes a conformal covariance property (Lemma 3.7) that is natural in light of the conformal restriction properties satisfied by SLEs, as studied in [47]. Moreover, in the $\kappa \rightarrow 0+$ limit, the SLE $_\kappa$ partition function gives the minimal Loewner potential, which can be seen as the semiclassical limit of certain CFT correlation functions (cf. Proposition 1.8).

Note that $\bar{\gamma}$ having finite energy is equivalent to it having finite potential.

Lemma 3.6. *The multichord $\bar{\gamma}$ has finite energy in D if and only if $I_D(\gamma_j) < \infty$ for all j .*

Proof. The third term in Definition 3.3 of \mathcal{H}_D is finite, as the boundary points are distinct. The second term is finite, as $\gamma_1, \dots, \gamma_n$ are pairwise disjoint by the definition of multichords. The asserted equivalence thus follows from the non-negativity of Loewner energy. ■

Lemma 3.7 (Conformal covariance). *If $\varphi: D \rightarrow D'$ is a conformal map, then*

$$\mathcal{H}_D(\gamma_1, \dots, \gamma_n) = \mathcal{H}_{D'}(\varphi(\gamma_1), \dots, \varphi(\gamma_n)) - \frac{1}{4} \sum_{j=1}^{2n} \log |\varphi'(x_j)|,$$

$$I_D^\alpha(\gamma_1, \dots, \gamma_n) = I_{D'}^\alpha(\varphi(\gamma_1), \dots, \varphi(\gamma_n)).$$

Proof. The Poisson kernel transforms under the conformal map φ as

$$P_{D;x,y} = |\varphi'(x)| |\varphi'(y)| P_{D',\varphi(x),\varphi(y)}$$

by definition (2.8), and the single-chord Loewner energy and Brownian loop measure are conformally invariant. Both assertions thus follow from Definitions 3.3 and 3.4. ■

3.3. Further properties

Next, we show that minimizers of the Loewner potential are geodesic multichords (Corollary 3.9); finite-energy multichords are quasiconformal images of analytic multichords (Proposition 3.12); the potential is lower semicontinuous with compact level sets; and potential minimizers exist (both in Proposition 3.13).

Lemma 3.8 (Cascade relation of \mathcal{H}). *For each $j \in \{1, \dots, n\}$, we have*

$$\mathcal{H}_D(\bar{\gamma}) = \mathcal{H}_{\hat{D}_j}(\gamma_j) + \mathcal{H}_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n). \tag{3.5}$$

Proof. By Definition 3.3 and Lemma 3.1, the left-hand side of (3.5) reads

$$\begin{aligned} \mathcal{H}_D(\bar{\gamma}) &= \frac{1}{12} \sum_{j=1}^n I_{D;x_{a_j},x_{b_j}}(\gamma_j) - \frac{1}{4} \sum_{j=1}^n \log P_{D;x_{a_j},x_{b_j}} \\ &\quad + \mathcal{B}_D(D \setminus \hat{D}_j, \gamma_j) + m_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n). \end{aligned}$$

On the other hand, by Definition 3.3, the right-hand side of (3.5) reads

$$\begin{aligned} &\mathcal{H}_{\hat{D}_j}(\gamma_j) + \mathcal{H}_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n) \\ &= \frac{1}{12} I_{\hat{D}_j;x_{a_j},x_{b_j}}(\gamma_j) - \frac{1}{4} \log P_{\hat{D}_j;x_{a_j},x_{b_j}} + \frac{1}{12} \sum_{i \neq j} I_{D;x_{a_i},x_{b_i}}(\gamma_i) \\ &\quad + m_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_n) - \frac{1}{4} \sum_{i \neq j} \log P_{D;x_{a_i},x_{b_i}}. \end{aligned}$$

After using the conformal restriction formula (2.9) with $(U; x, y) = (\hat{D}_j; x_{a_j}, x_{b_j})$, we see that the left-hand and right-hand sides of the asserted formula (3.5) agree. ■

Corollary 3.9. *Any minimizer of \mathcal{H}_D in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ is a geodesic multichord.*

We will show in Proposition 3.13 that there actually exists a minimizer and in Corollary 4.2 that the minimizer is unique. Thus, in particular, for each link pattern α there exists a unique geodesic multichord.

Proof of Corollary 3.9. Let $\bar{\eta} = (\eta_1, \dots, \eta_n)$ be a minimizer of the Loewner potential \mathcal{H}_D in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$. Then, by Lemma 3.8, for each j , the chord η_j minimizes $\mathcal{H}_{\hat{D}_j}$ and thus $I_{\hat{D}_j}$ among all chords in \hat{D}_j with the same boundary points. Hence, each η_j is the hyperbolic geodesic in its component \hat{D}_j , i.e., $\bar{\eta}$ is a geodesic multichord. ■

Remark 3.10. Note that the minimizers of \mathcal{H}_D and I_D^α in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ are the same. In particular, $\bar{\eta}$ is a minimizer of \mathcal{H}_D in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ if and only if $I_D^\alpha(\bar{\eta}) = 0$.

The following estimates will be useful later.

Lemma 3.11. *For each $j \in \{1, \dots, n\}$, the following hold:*

(i) *We have*

$$\mathcal{H}_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_{n-1}) \leq \mathcal{H}_D(\bar{\gamma}) + \frac{1}{4} \log P_{D, x_{a_j}, x_{b_j}}.$$

(ii) *For each n -link pattern α , we have*

$$I_{\hat{D}_j}(\gamma_j) \leq I_D^\alpha(\bar{\gamma}).$$

Proof. For (i), Lemma 3.8 and formula (3.4) imply that

$$\begin{aligned} \mathcal{H}_D(\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_{n-1}) &= \mathcal{H}_D(\bar{\gamma}) - \mathcal{H}_{\hat{D}_j}(\gamma_j) \\ &= \mathcal{H}_D(\bar{\gamma}) - \frac{1}{12} I_{\hat{D}_j}(\gamma_j) + \frac{1}{4} \log P_{\hat{D}_j, x_{a_j}, x_{b_j}} \\ &\leq \mathcal{H}_D(\bar{\gamma}) + \frac{1}{4} \log P_{D, x_{a_j}, x_{b_j}}, \end{aligned}$$

where the last inequality follows from the non-negativity of the Loewner energy and the domain monotonicity of the Poisson kernel (Corollary 2.8 with $U = \hat{D}_j$).

To prove (ii), without loss of generality, we may assume that $j = 1$. Let η be the hyperbolic geodesic in \hat{D}_1 with the same endpoints as γ_1 . Because $I_{\hat{D}_1}(\eta) = 0$, Definition 3.3 and Lemma 3.8 show that

$$\begin{aligned} I_{\hat{D}_1}(\gamma_1) &= I_{\hat{D}_1}(\gamma_1) - I_{\hat{D}_1}(\eta) = 12(\mathcal{H}_{\hat{D}_1}(\gamma_1) - \mathcal{H}_{\hat{D}_1}(\eta)) \\ &= 12(\mathcal{H}_D(\bar{\gamma}) - \mathcal{H}_D(\eta, \gamma_2, \dots, \gamma_n)) \\ &\leq 12(\mathcal{H}_D(\bar{\gamma}) - \mathcal{M}_D^\alpha(x_1, \dots, x_{2n})) = I_D^\alpha(\bar{\gamma}). \end{aligned} \quad \blacksquare$$

Next, in Proposition 3.12 we show that any finite-energy multichord is a quasiconformal image of a certain smooth reference multichord, where the quasiconformal constant only depends on the domain data, link pattern, and the Loewner energy of the multichord. Similarly to Lemma 2.7 in Section 2.3, this property provides the required precompactness in order to find a subsequence converging to a potential (and energy) minimizer. This allows us to conclude this section with Proposition 3.13, comprising the most important properties of the Loewner potential.

We fix an (arbitrary) ordering $(\{a_1, b_1\}, \dots, \{a_n, b_n\})$ of the pairs in each n -link pattern α . Then we define the *reference multichord* (ρ_1, \dots, ρ_n) associated to α and $(D; x_1, \dots, x_{2n})$ as follows. We first take ρ_1 to be the hyperbolic geodesic in $(D; x_{a_1}, x_{b_1})$, and next, for each $j = 2, \dots, n$, we let ρ_j be the hyperbolic geodesic from x_{a_j} to x_{b_j} in the appropriate connected component of $D \setminus \bigcup_{i < j} \rho_i$. (We note, as a word of caution, that this object is not a geodesic multichord unless $n = 1$.)

Proposition 3.12. *Fix domain data $(D; x_1, \dots, x_{2n})$ and a link pattern α . If $I_D^\alpha(\bar{\gamma}) < \infty$, then there exists $K \in [1, \infty)$, depending only on $I_D^\alpha(\bar{\gamma})$, and a K -quasiconformal map φ such that $\gamma_j = \varphi(\rho_j)$ for all $j \in \{1, \dots, n\}$, $\varphi(D) = D$, and φ extends continuously to \bar{D} and equals the identity function on ∂D .*

Proof. We construct the desired K -quasiconformal map by induction on $n \geq 1$. The constant K may change from line to line, as long as it depends only on the domain data $(D; x_1, \dots, x_{2n})$, link pattern α , and energy $I_D^\alpha(\bar{\gamma})$. The initial case $n = 1$ is covered by Lemma 2.6. For the induction step, we assume that the assertion holds for the $(n - 1)$ -link pattern $\hat{\alpha} = (\{a_1, b_1\}, \dots, \{a_{n-1}, b_{n-1}\})$ on the same marked domain after omitting the markings at x_{a_n} and x_{b_n} . It follows immediately from the construction that $(\rho_1, \dots, \rho_{n-1})$ is the reference multichord associated to $\hat{\alpha}$.

Lemma 3.11 (ii) and Definition 3.4 imply that $I_D^{\hat{\alpha}}(\gamma_1, \dots, \gamma_{n-1})$ is uniformly bounded by a constant that only depends on $(D; x_1, \dots, x_{2n})$, α , and $I_D^\alpha(\gamma_1, \dots, \gamma_n)$. Therefore, by the induction hypothesis, there exists a K -quasiconformal map $\hat{\varphi}: D \rightarrow D$ equaling the identity function on ∂D and sending ρ_j to γ_j for all $j \leq n - 1$. Note that $\hat{\varphi}(\hat{D}_\rho) = \hat{D}_n$, where \hat{D}_ρ and \hat{D}_n are respectively the connected components containing ρ_n and γ_n .

We will construct the desired map φ by modifying $\hat{\varphi}$ in \hat{D}_ρ in such a way that the modified map sends ρ_n to γ_n while remaining K -quasiconformal. For this purpose, we let $f_\gamma: \hat{D}_n \rightarrow \mathbb{H}$ (resp. $f_\rho: \hat{D}_\rho \rightarrow \mathbb{H}$) be a uniformizing conformal map sending x_{a_n} to 0 and x_{b_n} to ∞ . Then $f_\gamma \circ \hat{\varphi} \circ f_\rho^{-1}$ is a K -quasiconformal map from \mathbb{H} to itself, so its boundary values define a K -quasisymmetric homeomorphism of \mathbb{R} . Let $\psi: \mathbb{H} \rightarrow \mathbb{H}$ be its Jerison–Kenig extension: a K -quasiconformal map that fixes $i\mathbb{R}_+$ with boundary values

$$\psi|_{\mathbb{R}} = f_\gamma \circ \hat{\varphi} \circ f_\rho^{-1}.$$

Lemma 3.11 (ii) implies that $I_{\mathbb{H}}(f_\gamma(\gamma_n)) = I_{\hat{D}_n}(\gamma_n) \leq I_D^\alpha(\bar{\gamma})$, so we may apply Lemma 2.6 to find a K -quasiconformal map $\psi_n: \mathbb{H} \rightarrow \mathbb{H}$ that equals the identity function on \mathbb{R} and maps $i\mathbb{R}_+$ to $f_\gamma(\gamma_n)$. In conclusion, the composition

$$f_\gamma^{-1} \circ \psi_n \circ \psi \circ f_\rho: \hat{D}_\rho \rightarrow \hat{D}_n$$

coincides with $\hat{\varphi}$ on $\partial \hat{D}_\rho$ and sends ρ_n to γ_n . Hence, we can define the sought K -quasiconformal map $\varphi: D \rightarrow D$ as

$$\varphi(z) := \begin{cases} \hat{\varphi}(z), & z \in D \setminus \hat{D}_\rho, \\ (f_\gamma^{-1} \circ \psi_n \circ \psi \circ f_\rho)(z), & z \in \hat{D}_\rho, \end{cases}$$

so $\gamma_j = \varphi(\rho_j)$ for all j , and φ extends to the identity function on ∂D . ■

Proposition 3.13. *The Loewner potential has the following properties:*

- (1) \mathcal{H}_D is lower semicontinuous from $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ to $[\mathcal{M}_D^\alpha(x_1, \dots, x_{2n}), \infty]$.
- (2) The level set $\{\bar{\gamma} \in \mathcal{X}_\alpha(D; x_1, \dots, x_{2n}) \mid \mathcal{H}_D(\bar{\gamma}) \leq c\}$ is compact for any $c \in \mathbb{R}$.
- (3) There exists a multichord minimizing \mathcal{H}_D in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$.

Proof. The lower semicontinuity follows from Definition 3.3: $\gamma_j \mapsto I_{D;x_{a_j},x_{b_j}}(\gamma_j)$ is lower semicontinuous for each j by Lemma 2.7, the quantity $\log P_{D;x_{a_j},x_{b_j}}$ does not depend on the multichords, and the map $\bar{\gamma} \mapsto m_D(\bar{\gamma})$ is continuous by Lemma 3.2.

The compactness of level sets follows from similar arguments as Lemma 2.7. Indeed, for a sequence $(\bar{\gamma}^k)_{k \in \mathbb{N}}$ of multichords in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ with $\mathcal{H}_D(\bar{\gamma}^k) \leq c$, Proposition 3.12 gives $K(c)$ -quasiconformal maps $\varphi^k: D \rightarrow D$ sending the reference multichord $\bar{\rho} := (\rho_1, \dots, \rho_n)$ to $\bar{\gamma}^k$ and such that $\varphi^k|_{\partial D}$ is the identity function for each k . Along a subsequence, φ^k converges uniformly on \bar{D} to a quasiconformal map φ . Hence, along this subsequence, $\bar{\gamma}^k$ converges to the multichord $\bar{\gamma} := \varphi(\bar{\rho})$ for the Hausdorff metric. The lower semicontinuity (property (1)) of \mathcal{H}_D then shows that $\mathcal{H}_D(\bar{\gamma}) \leq c$.

The existence of a minimizer is an immediate consequence of the compactness of the level sets by considering a minimizing sequence. This finishes the proof. ■

4. Minimizers of the potential

In this section, we investigate minimizers of the Loewner potential (and energy). Relying on the conformal covariance of \mathcal{H} (Lemma 3.7), we shall assume that $D = \mathbb{H}$. We show that any geodesic multichord gives rise to a rational function with prescribed set of critical points (Proposition 4.1). This result proves Theorem 1.2. Furthermore, by classifying such rational functions, we show that for each n -link pattern α and domain data $(\mathbb{H}; x_1, \dots, x_{2n})$, there exists a unique geodesic multichord (Corollary 4.2). Theorem 1.1 is a consequence of this result and the fact that all minimizers of the potential are geodesic (Corollary 3.9). Finally, in Section 4.3 we derive the Loewner flows of the minimizer (Proposition 1.7) and the semiclassical null-state PDEs for the minimal potential (Proposition 1.8).

4.1. Geodesic multichords and rational functions

To begin, we study geodesic multichords $\bar{\eta} := (\eta_1, \dots, \eta_n)$ in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2n})$, where $x_1 < \dots < x_{2n}$. By considering the union of $\bar{\eta}$, its complex conjugate $\bar{\eta}^*$, and the real line \mathbb{R} , we can regard them as graphs embedded in $\hat{\mathbb{C}}$. Precisely, to each geodesic multichord $\bar{\eta}$ we associate a graph $G_{\bar{\eta}}$ with vertices $\{x_1, \dots, x_{2n}\}$ and edges

$$E = \{\eta_1, \eta_1^*, \dots, \eta_n, \eta_n^*, [x_1, x_2], \dots, [x_{2n-1}, x_{2n}], [x_{2n}, x_1]\},$$

where $[x_{2n}, x_1]$ denotes the segment $[x_{2n}, +\infty) \cup [-\infty, x_1]$ in $\hat{\mathbb{C}}$. We call the connected components of $\hat{\mathbb{C}} \setminus \bigcup_{e \in E} e$ the *faces* of $G_{\bar{\eta}}$. By symmetry, $G_{\bar{\eta}}$ is again a graph with the

geodesic property, that is, each edge $e \in E$ is the hyperbolic geodesic in the domain \hat{D}_e formed by the union of the two faces adjacent to e . In fact, to each $\bar{\eta}$ and thus to G_η , we can associate a unique rational function h_η whose critical points are given by the endpoints of the chords η_1, \dots, η_n (see Proposition 4.1).

We first recall some terminology. A *rational function* is an analytic branched covering h of $\hat{\mathbb{C}}$ over $\hat{\mathbb{C}}$, or equivalently, the ratio $h = P/Q$ of two polynomials $P, Q \in \mathbb{C}[X]$. A point $x_0 \in \hat{\mathbb{C}}$ is a *critical point* (equivalently, a branched point) with *index* $k \geq 2$ if

$$h(x) = h(x_0) + C(x - x_0)^k + O((x - x_0)^{k+1})$$

for some constant $C \neq 0$ in local coordinates of $\hat{\mathbb{C}}$. A point $y \in \hat{\mathbb{C}}$ is a *regular value* of h if y is not the image by h of any critical point. The *degree* of h is the number of preimages of any regular value. We call $h^{-1}(\hat{\mathbb{R}})$ the *real locus* of h . Further, h is a *real rational function* if P and Q can be chosen from $\mathbb{R}[X]$, or equivalently, $h(\hat{\mathbb{R}}) \subset \hat{\mathbb{R}}$.

By the Riemann–Hurwitz formula for Euler characteristics, a rational function of degree $n + 1$ has $2n$ critical points if and only if each of them has index 2 (in this case, we say that the critical points of h are distinct):

$$(n + 1)\chi(\hat{\mathbb{C}}) - 2n(2 - 1) = 2n + 2 - 2n = 2 = \chi(\hat{\mathbb{C}}). \tag{4.1}$$

Proposition 4.1. *Fix $x_1 < \dots < x_{2n}$. For each geodesic multichord $\bar{\eta}$ in $(\mathbb{H}; x_1, \dots, x_{2n})$, there exists a unique rational function $h_\eta: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of degree $n + 1$, up to post-composition with elements of $\text{PSL}(2, \mathbb{R})$ and with $\iota: z \mapsto -z$, such that the set of critical points of h_η is exactly $\{x_1, \dots, x_{2n}\}$. Moreover, the graph G_η is given by the real locus of h_η , i.e.,*

$$h_\eta^{-1}(\hat{\mathbb{R}}) = \bigcup_{e \in E} e.$$

We note that since h_η preserves $\hat{\mathbb{R}}$, it is actually a *real* rational function. This result proves Theorem 1.2.

Proof of Proposition 4.1. The complement $\mathbb{H} \setminus \bar{\eta}$ has $n + 1$ faces. Pick one face F and consider a uniformizing conformal map h_η from F onto \mathbb{H} (unique up to post-composition with elements of $\text{PSL}(2, \mathbb{R})$) or onto \mathbb{H}^* (by replacing h_η with $\iota \circ h_\eta$). Without loss of generality, we consider the former case. Also, we suppose that η_1 is adjacent to F and another face F' . Then, since η_1 is a hyperbolic geodesic in \hat{D}_{η_1} , the map h_η extends by reflection to a conformal map on \hat{D}_{η_1} . In particular, this extension of h_η maps F' conformally onto \mathbb{H}^* . By iterating these analytic continuations across all of the chords η_k , we obtain a meromorphic function $h_\eta: \mathbb{H} \rightarrow \hat{\mathbb{C}}$. Furthermore, h_η also extends to \mathbb{H} , and its restriction $h_\eta|_{\hat{\mathbb{R}}}$ takes values in $\hat{\mathbb{R}}$. Hence, via Schwarz reflection, we extend h_η to $\hat{\mathbb{C}}$ via $h_\eta(z) := h_\eta(z^*)^*$ for all $z \in \mathbb{H}^*$.

Now, it follows from the construction that $h_\eta^{-1}(\hat{\mathbb{R}}) = \bigcup_{e \in E} e$. Moreover, h_η is a rational function of degree $n + 1$, as exactly $n + 1$ faces are mapped to \mathbb{H} and $n + 1$ faces to \mathbb{H}^* . Finally, another choice of initial face F yields the same function up to post-composition with elements of $\text{PSL}(2, \mathbb{R})$ or with ι . This concludes the proof. ■

Now, thanks to Proposition 4.1, in order to classify geodesic multichords it suffices to classify equivalence classes of rational functions with prescribed critical points. Since post-composition with elements of $\text{PSL}(2, \mathbb{C})$ (i.e., Möbius transformations of $\hat{\mathbb{C}}$) does not change the critical points, we may consider the equivalence classes Rat_d of rational functions of degree $d = n + 1 \geq 2$ modulo post-composition with elements of $\text{PSL}(2, \mathbb{C})$. We obtain information from them using Goldberg’s results from [33], which we now briefly recall.

Let Poly_d denote the space of polynomials of degree at most d . If h is a rational function of degree d , then we can write $h = P/Q$ for two linearly independent polynomials $P, Q \in \text{Poly}_d$. In particular, $V_h := \text{span}_{\mathbb{C}}(P, Q)$ lies in the Grassmann manifold $G_2(\text{Poly}_d)$ of two-dimensional subspaces of Poly_d . The group $\text{PSL}(2, \mathbb{C})$ acts on h by post-composition,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot h = \frac{aP + bQ}{cP + dQ},$$

and the image generates the same element in $G_2(\text{Poly}_d)$ as h . Conversely, any basis $\{P, Q\}$ of V_h gives a rational function P/Q which equals a post-composition of h with an element of $\text{PSL}(2, \mathbb{C})$. Therefore, we identify $\text{Rat}_d \simeq G_2(\text{Poly}_d)$ by $[h] \mapsto V_h$. Furthermore, the *Wroński map*

$$\Phi_d([h]) := [P'Q - Q'P]/\mathbb{C}^* \quad \text{with} \quad [h] \simeq V_h = \text{span}_{\mathbb{C}}(P, Q),$$

from Rat_d to the projective space $\text{Poly}_{2d-2}^*/\mathbb{C}^* \simeq \mathbb{P}^{2d-2}$ is well-defined, because

$$\begin{aligned} \left(\frac{P}{Q}\right)' &= \frac{P'Q - Q'P}{Q^2} \neq 0, \\ \Phi_d\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot h\right] &= [(ad - bc)(P'Q - Q'P)]/\mathbb{C}^* = \Phi_d([h]). \end{aligned}$$

Moreover, the zeros of $\Phi_d([h])$ are exactly the critical points of h . We therefore view Φ_d as a map from a rational function in Rat_d to the set of critical points counted with multiplicity.

Goldberg’s result below is crucial to our classification of geodesic multichords. Recall from (1.1) that C_n denotes the n th Catalan number.

Theorem D ([33, Prop. 2.3 and Thm. 3.4]). *The function Φ_d is a complex analytic map of degree C_{d-1} .*

In particular, with $d = n + 1$, we have

Theorem E ([33, Thm. 1.3]). *A set of $2n$ distinct points is the set of critical points of at most C_n rational functions of degree $n + 1$ that are not $\text{PSL}(2, \mathbb{C})$ -equivalent.*

A combination of the above theorems with Proposition 4.1 and the existence of the geodesic multichord for any given boundary data $(x_1, \dots, x_{2n}; \alpha)$ from Proposition 3.13 and Corollary 3.9 shows that the maximal number is achieved when the critical points are $2n$ distinct real numbers.

Corollary 4.2. *There are exactly C_n preimages of $\{x_1 < \dots < x_{2n}\}$ by Φ_{n+1} . In particular, for given boundary data $(x_1, \dots, x_{2n}; \alpha)$, there exists a unique geodesic multichord (which is also the unique potential minimizer in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2n})$).*

This result implies Theorem 1.1.

Proof of Corollary 4.2. To each geodesic multichord $\bar{\eta}$ in $(\mathbb{H}; x_1, \dots, x_{2n})$ we associate a rational function h_η as in Proposition 4.1. Let $\bar{\eta}$ and $\bar{\eta}'$ be two geodesic multichords. If there exists $A \in \text{PSL}(2, \mathbb{C})$ such that $h_\eta = A \circ h_{\eta'}$, then we have

$$(A \circ h_{\eta'})([x_1, x_2]) = h_\eta([x_1, x_2]) \subset \mathbb{R},$$

which implies that either A or $\iota \circ A$ belongs to $\text{PSL}(2, \mathbb{R})$, so

$$h_\eta^{-1}(\hat{\mathbb{R}}) = h_{\eta'}^{-1}(\hat{\mathbb{R}}).$$

Hence, $\bar{\eta} = \bar{\eta}'$ and the map $\eta \rightarrow [h_\eta]$ is injective. Theorem D implies that there exist at most C_n geodesic multichords in $(\mathbb{H}; x_1, \dots, x_{2n})$. On the other hand, C_n also equals the number of n -link patterns α . By Proposition 3.13, for any α there exists at least one minimizer of the potential, which is a geodesic multichord. This proves the corollary. ■

As a by-product, we obtain an analytic proof of the following equivalent form of the Shapiro conjecture for the Grassmannian of 2-planes (see [72] for the general conjecture and [25, 26] for other proofs).

Corollary 1.3. *If all critical points of a rational function are real, then it is a real rational function up to post-composition with a Möbius transformation of $\hat{\mathbb{C}}$.*

Proof. If a rational function of degree $n + 1$ has $2n$ distinct real critical points, then it is $\text{PSL}(2, \mathbb{C})$ -equivalent to h_η associated to a geodesic multichord $\bar{\eta}$ via Proposition 4.1. In particular, h_η maps the real line to the real line and has real coefficients. The general case follows by deforming the polynomial $\Phi_{n+1}([h])$ in $\text{Poly}_{2n}^*/\mathbb{C}^*$ to those with simple zeros; see [25, Sec. 7]. ■

Corollary 4.3. *The class $[h_\eta] \in \text{Rat}_{n+1}$ associated to the unique geodesic multichord $\bar{\eta}$ in \mathbb{H} with boundary data $(x_1, \dots, x_{2n}; \alpha)$ depends analytically on $x_1 < \dots < x_{2n}$.*

Proof. From Theorem D and Corollary 4.2, we see that $\{x_1 < \dots < x_{2n}\}$ is a regular value of Φ_{n+1} , which is locally an analytic diffeomorphism in a neighborhood of any of its preimages (where the link pattern of its real locus is constant). ■

4.2. Characterization by Loewner differential equations

Next, we derive the Loewner flow for the geodesic multichord. For this purpose, we first need to show that the minimal potential $\mathcal{M}_{\mathbb{H}}^\alpha$ is differentiable with respect to variation of the marked points $x_1 < \dots < x_{2n}$, for each link pattern α . This is a consequence of the analyticity of the Wronski map (cf. Corollary 4.3).

Proposition 4.4. *Let $\bar{\eta}$ be the unique geodesic multichord in \mathbb{H} associated to the boundary data $(x_1, \dots, x_{2n}; \alpha)$. For each $i \in \{1, \dots, 2n\}$, the function $\mathcal{M}_{\mathbb{H}}^\alpha(x_1, \dots, x_{2n})$ is differentiable in x_i . Moreover, for each $j \in \{1, \dots, 2n\}$, we have*

$$\partial_{a_j} \mathcal{M}_{\mathbb{H}}^\alpha(x_1, \dots, x_{2n}) = \partial_1 \mathcal{M}_{\hat{\mathbb{H}}_j}(x_{a_j}, x_{b_j}),$$

where x_{a_j} and x_{b_j} are the endpoints of η_j , and $\hat{\mathbb{H}}_j$ is its component in $\mathbb{H} \setminus \bigcup_{i \neq j} \eta_i$.

In the case $n = 1$, the minimal potential has an explicit formula obtained from (1.5) and (2.8), which is obviously differentiable:

$$\mathcal{M}_{\mathbb{H}}(x_1, x_2) = \frac{1}{2} \log |x_2 - x_1| \implies \partial_1 \mathcal{M}_{\mathbb{H}}(x_1, x_2) = \frac{1}{2(x_1 - x_2)}. \tag{4.2}$$

Proof of Proposition 4.4. For simplicity of notation and without loss of generality, we assume that $i = 1$ and $\{1, 2\} \in \alpha$, and we let η_1 be the chord connecting x_1 and x_2 . In particular, we keep the variables x_2, \dots, x_{2n} fixed. To vary the position of x_1 , for all $y < x_2$, we let $(\eta_1^y, \dots, \eta_n^y)$ denote the geodesic multichord in \mathbb{H} associated to α with boundary points $\{y < x_2 < \dots < x_{2n}\}$. Also, we let $\hat{\mathbb{H}}_1(y)$ denote the connected component of $\mathbb{H} \setminus \{\eta_2^y, \dots, \eta_n^y\}$ containing y and x_2 on its boundary. Note that $\hat{\mathbb{H}}_1(y)$ depends on y . Now, recalling the cascade relation from Lemma 3.8, we set

$$\psi_y(x) := \mathcal{H}_{\mathbb{H}}(\eta^x, \eta_2^y, \dots, \eta_n^y) = \mathcal{M}_{\hat{\mathbb{H}}_1(y)}(x, x_2) + \mathcal{H}_{\mathbb{H}}(\eta_2^y, \dots, \eta_n^y),$$

where η^x is the hyperbolic geodesic in $(\hat{\mathbb{H}}_1(y); x, x_2)$. Then, at $x = y$, we have $\eta^y = \eta_1^y$ and $\psi_y(y) = \mathcal{M}_{\mathbb{H}}^\alpha(y, x_2, \dots, x_{2n})$. We will argue that the function

$$y \mapsto \psi'_y(y) = \partial_1 \mathcal{M}_{\hat{\mathbb{H}}_1(y)}(y, x_2)$$

is continuous. Indeed, if φ_y is the conformal map from $\hat{\mathbb{H}}_1(y)$ onto \mathbb{H} fixing x_2 and such that $\varphi'_y(x_2) = 1$ and $\varphi''_y(x_2) = 0$, then by Lemma 3.7, we have

$$\mathcal{M}_{\hat{\mathbb{H}}_1(y)}(\cdot, x_2) = \mathcal{M}_{\mathbb{H}}(\varphi_y(\cdot), x_2) - \frac{\log \varphi'_y(\cdot)}{4}.$$

Hence, using Corollary 4.3 and the explicit formula (4.2), we see that the derivative

$$y \mapsto \psi'_y(y) = \frac{\varphi'_y(y)}{2(\varphi_y(y) - x_2)} - \frac{\varphi''_y(y)}{4\varphi'_y(y)}$$

is continuous in y . With these preparations, we are ready to show the differentiability of $\mathcal{M}_{\mathbb{H}}^\alpha$ in its first variable. For y in a small neighborhood $B_\varepsilon(x_1)$ of x_1 , we have

$$\begin{aligned} \mathcal{M}_{\mathbb{H}}^\alpha(y, x_2, \dots, x_{2n}) - \mathcal{M}_{\mathbb{H}}^\alpha(x_1, x_2, \dots, x_{2n}) &= \psi_y(y) - \psi_{x_1}(x_1) \\ &= \psi_y(y) - \psi_y(x_1) + \psi_y(x_1) - \psi_{x_1}(x_1) \\ &\geq \psi_y(y) - \psi_y(x_1) = (y - x_1)\psi'_y(y) - R_y(y, x_1), \end{aligned}$$

for some remainder $R_y(y, x_1)$. In fact, the remainder is bounded since $|R_y(y, x_1)| \leq c|y - x_1|^2$, where $c \in (0, \infty)$ is uniform over all $y \in B_\varepsilon(x_1)$, because c depends on ψ''_y in $B_\varepsilon(x_1)$, which depends smoothly on y by Corollary 4.3. Similarly, after inverting the roles of y and x_1 , we have

$$\mathcal{M}_{\mathbb{H}}^\alpha(y, x_2, \dots, x_{2n}) - \mathcal{M}_{\mathbb{H}}^\alpha(x_1, x_2, \dots, x_{2n}) \leq (y - x_1)\psi'_{x_1}(x_1) + R_{x_1}(x_1, y).$$

Now, $\psi'_y(y) \rightarrow \psi'_{x_1}(x_1)$ as $y \rightarrow x_1$, so we obtain the differentiability of $\mathcal{M}_{\mathbb{H}}^\alpha$ in the first variable, and we also see that $\partial_1 \mathcal{M}_{\mathbb{H}}^\alpha(x_1, x_2) = \partial_1 \mathcal{M}_{\mathbb{H}}^\alpha(x_1, \dots, x_{2n})$. ■

Remark 4.5. Compared to the differentiability of SLE_κ partition functions with respect to the boundary points in [34], the above proof is different, relying on the analytic dependence of the geodesic multichord on the marked points, which we get from the associated rational function.

Now, we derive the Loewner flow for the potential-minimizing geodesic multichord. Recall from the introduction that we wrote $\mathcal{U} := 12\mathcal{M}_{\mathbb{H}}^\alpha$ for notational simplicity.

Proposition 1.7. *Let $\bar{\eta}$ be the minimizer of $\mathcal{H}_{\mathbb{H}}$ in $\mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2n})$. Then, for each $j \in \{1, \dots, n\}$, the Loewner driving function W of the chord η_j from x_{a_j} to x_{b_j} and the time evolutions $V_t^i = g_t(x_i)$ of the other marked points satisfy the differential equations*

$$\begin{cases} \frac{dW_t}{dt} = -\partial_{a_j} \mathcal{U}(V_t^1, \dots, V_t^{a_j-1}, W_t, V_t^{a_j+1}, \dots, V_t^{2n}), & W_0 = x_{a_j}, \\ \frac{dV_t^i}{dt} = \frac{2}{V_t^i - W_t}, & V_0^i = x_i, \text{ for } i \neq a_j, \end{cases} \tag{1.8}$$

for $0 \leq t < T$, where T is the lifetime of the solution and $(g_t)_{t \in [0, T]}$ is the Loewner flow generated by η_j .

For the proof, we first recall a conformal distortion formula of Loewner driving functions. Let $\varphi: U \rightarrow \tilde{U}$ be a conformal map between two neighborhoods U and \tilde{U} in \mathbb{H} of $x \in \mathbb{R}$ such that $\varphi(x) = x$. Let $\gamma \in \mathcal{X}(\mathbb{H}; x, y)$ and $\tilde{\gamma} := \varphi(\gamma \cap U)$. Let $t \mapsto W_t$ (resp. $s \mapsto \tilde{W}_s$) be the driving function of γ (resp. $\tilde{\gamma}$), defined in a neighborhood of 0. Then by [67, (11)] we know that W is right-differentiable at 0 if and only if \tilde{W} is right-differentiable at 0. Moreover, we have

$$\varphi'(x) \frac{d\tilde{W}_s}{ds} \Big|_{s=0} = \frac{dW_t}{dt} \Big|_{t=0} - 3 \frac{\varphi''(x)}{\varphi'(x)}. \tag{4.3}$$

Proof of Proposition 1.7. We first consider the single-chord case $n = 1$. In this case, the minimizer of $\mathcal{H}_{\mathbb{H}}$ in $\mathcal{X}(\mathbb{H}; x_1, x_2)$ is given by the hyperbolic geodesic in \mathbb{H} , namely, the semicircle η with endpoints x_1 and x_2 . Let $\varphi: \mathbb{H} \rightarrow \mathbb{H}$ be a Möbius transformation with $\varphi(x_1) = x_1$ and $\varphi(x_2) = \infty$. Then the driving function of $\varphi(\eta)$ is the constant function $\tilde{W}_s \equiv x_1$ and we have $\varphi''(x_1)/\varphi'(x_1) = 2/(x_2 - x_1)$. Hence, (4.3) gives

$$\frac{dW_t}{dt} \Big|_{t=0} = \frac{6}{x_2 - x_1}.$$

Thus, since hyperbolic geodesics are preserved under their own Loewner flow, we obtain

$$\begin{cases} \frac{dW_t}{dt} = \frac{6}{V_t - W_t}, & W_0 = x_1, \\ \frac{dV_t}{dt} = \frac{2}{V_t - W_t}, & V_0 = x_2, \end{cases}$$

where $V_t := V_t^2$ is the Loewner flow of x_2 . By (4.2), this is exactly (1.8).

For the general case, note that under the Loewner flow starting from any point x_a , the resulting multichord is still the minimizer for the corresponding boundary point and link pattern. Therefore, we only need to prove the asserted equations (1.8) at $t = 0$. For simplicity of notation, we assume that $x_a := x_{a_1}$ and $x_b := x_{b_1}$ are the endpoints of η_1 . If $\psi: \hat{\mathbb{H}}_1 \rightarrow \mathbb{H}$ is a conformal map, then since $\psi(\eta_1)$ minimizes $\mathcal{H}_{\mathbb{H}}$ among all chords in $(\mathbb{H}; \psi(x_a), \psi(x_b))$, the driving function \tilde{W} of $\psi(\eta_1)$ satisfies $(d\tilde{W}_t/dt)|_{t=0} = -12\partial_1 \mathcal{M}_{\mathbb{H}}(\psi(x_a), \psi(x_b))$. Therefore,

$$\begin{aligned} -12\partial_{x_a} \mathcal{M}_{\mathbb{H}}(\psi(x_a), \psi(x_b)) &= -12\psi'(x_a) \partial_1 \mathcal{M}_{\mathbb{H}}(\psi(x_a), \psi(x_b)) \\ &= \psi'(x_a) \frac{d\tilde{W}_t}{dt} \Big|_{t=0} = \frac{dW_t}{dt} \Big|_{t=0} - 3 \frac{\psi''(x_a)}{\psi'(x_a)}, \end{aligned}$$

where the last equality follows from (4.3). On the other hand, Lemma 3.7 gives

$$-\mathcal{M}_{\hat{\mathbb{H}}_1}(x_a, x_b) = \frac{1}{4} \log |\psi'(x_a)\psi'(x_b)| - \mathcal{M}_{\mathbb{H}}(\psi(x_a), \psi(x_b)),$$

so we deduce that

$$-12\partial_1 \mathcal{M}_{\hat{\mathbb{H}}_1}(x_a, x_b) = 3 \frac{\psi''(x_a)}{\psi'(x_a)} - 12\partial_{x_a} \mathcal{M}_{\mathbb{H}}(\psi(x_a), \psi(x_b)) = \frac{dW_t}{dt} \Big|_{t=0}.$$

From Proposition 4.4, we now obtain the time-evolution of W in (1.8) with $\mathcal{U} = 12\mathcal{M}_{\mathbb{H}}^\alpha$. The time-evolutions of V^i for $i \neq a$ follow from the Loewner equation (2.3) with $V_t^i = g_t(x_i)$. ■

4.3. Semiclassical null-state PDEs

In this section, we derive the system of semiclassical null-state PDEs for the minimal potential $\mathcal{M}_{\mathbb{H}}^\alpha$. Later, in Corollary 5.12 we will show that $\mathcal{M}_{\mathbb{H}}^\alpha$ describes the semiclassical limit of the SLE $_\kappa$ partition functions \mathcal{Z}_α , which are solutions to the level two null-state PDEs (1.11).

Proposition 1.8. *For each $j \in \{1, \dots, 2n\}$, we have*

$$\frac{1}{2}(\partial_j \mathcal{U}(x_1, \dots, x_{2n}))^2 - \sum_{i \neq j} \frac{2}{x_i - x_j} \partial_i \mathcal{U}(x_1, \dots, x_{2n}) = \sum_{i \neq j} \frac{6}{(x_i - x_j)^2}. \quad (1.9)$$

Proof. Let $\bar{\eta}$ be the unique geodesic multichord with boundary data $(x_1, \dots, x_{2n}; \alpha)$. Then $\mathcal{U} := 12\mathcal{M}_{\mathbb{H}}^\alpha = 12\mathcal{H}_{\mathbb{H}}(\bar{\eta})$ by definition. Without loss of generality, we prove (1.9) for $j = 1$, and we assume that η_1 is the chord connecting x_1 to x_b , for $\{1, b\} \in \alpha$.

We consider the Loewner flow $(g_t)_{t \in [0, T]}$ associated to the driving function W of η_1 , satisfying (1.8) with $j = 1$. We write $\bar{\eta}^t = (\eta_1^t, \dots, \eta_n^t)$ for the image multichord under the flow g_t . Since $\bar{\eta}^t$ is still a geodesic multichord, we have

$$12\mathcal{H}_{\mathbb{H}}(\bar{\eta}^t) = \mathcal{U}(W_t, g_t(x_2), g_t(x_3), \dots, g_t(x_{2n})) \tag{4.4}$$

for any $t \in [0, T)$. We will take the time-derivative of (4.4) at $t = 0$ in two ways, whose equality yields the asserted PDE (1.9). First, we have

$$\begin{aligned} 12 \frac{d}{dt} \mathcal{H}_{\mathbb{H}}(\bar{\eta}^t) \Big|_{t=0} &= \frac{dW_t}{dt} \Big|_{t=0} \partial_1 \mathcal{U}(x_1, \dots, x_{2n}) + \sum_{i \neq 1} \frac{2\partial_i \mathcal{U}(x_1, \dots, x_{2n})}{x_i - x_1} \\ &= -(\partial_1 \mathcal{U}(x_1, \dots, x_{2n}))^2 + \sum_{i \neq 1} \frac{2\partial_i \mathcal{U}(x_1, \dots, x_{2n})}{x_i - x_1}. \end{aligned} \tag{4.5}$$

On the other hand, let \mathbb{H}^L and \mathbb{H}^R be the two connected components of $\mathbb{H} \setminus \eta_1$, and $\bar{\eta}^L$ and $\bar{\eta}^R$ the submultichords of $\bar{\eta}$ contained in \mathbb{H}^L and \mathbb{H}^R . Also, for each $t \in [0, T)$, denote their images under the conformal map g_t by $\mathbb{H}^{L,t} := g_t(\mathbb{H}^L)$, $\mathbb{H}^{R,t} := g_t(\mathbb{H}^R)$, $\bar{\eta}^{L,t} := g_t(\bar{\eta}^L)$, and $\bar{\eta}^{R,t} := g_t(\bar{\eta}^R)$. We claim that the following decomposition holds:

$$12\mathcal{H}_{\mathbb{H}}(\bar{\eta}^t) = 12\mathcal{H}_{\mathbb{H}}(\eta_1^t) + 12\mathcal{H}_{\mathbb{H}^{L,t}}(\bar{\eta}^{L,t}) + 12\mathcal{H}_{\mathbb{H}^{R,t}}(\bar{\eta}^{R,t}). \tag{4.6}$$

Indeed, this follows by applying Lemma 3.8 successively $n - 1$ times to the chords $\eta_2^t, \dots, \eta_n^t$ on both sides of (4.6) (or alternatively, using the conformal covariance of \mathcal{H} from Lemma 3.7 and the determinant expression from Theorem 1.9, whose proof is independent). Now, from Lemma 3.7 and the Loewner flow (1.8), we obtain

$$\begin{aligned} \frac{d}{dt} \left[12\mathcal{H}_{\mathbb{H}^{L,t}}(\bar{\eta}^{L,t}) + 12\mathcal{H}_{\mathbb{H}^{R,t}}(\bar{\eta}^{R,t}) \right] \Big|_{t=0} &= \frac{d}{dt} \left[12\mathcal{H}_{\mathbb{H}^L}(\bar{\eta}^L) + 12\mathcal{H}_{\mathbb{H}^R}(\bar{\eta}^R) + 3 \sum_{i \neq 1, b} \log |g'_t(x_i)| \right] \Big|_{t=0} \\ &= - \sum_{i \neq 1, b} \frac{6}{(x_i - x_1)^2}. \end{aligned}$$

Also, using the definition (2.6) of the Loewner energy $I_{\mathbb{H}; 0, \infty}$, we compute

$$\begin{aligned} 12 \frac{d}{dt} \mathcal{H}_{\mathbb{H}}(\eta_1^t) \Big|_{t=0} &= \frac{d}{dt} \left[I_{\mathbb{H}; W_t, g_t(x_b)}(\eta_1^t) + 6 \log |g_t(x_b) - W_t| \right] \Big|_{t=0} \\ &= -\frac{1}{2} \left(\varphi'(x_1) \frac{d\tilde{W}_s}{ds} \right) \Big|_{s=0} - \frac{6}{x_b - x_1} \frac{dW_t}{dt} \Big|_{t=0} + \frac{12}{(x_b - x_1)^2}, \end{aligned}$$

where $(\varphi'(x_1))^2$ is the scaling factor of the capacity parametrization, and $\varphi: \mathbb{H} \rightarrow \mathbb{H}$ is a Möbius transformation with $\varphi(x_1) = x_1$ and $\varphi(x_b) = \infty$. Now, (4.3) gives

$$\varphi'(x_1) \frac{d\tilde{W}_s}{ds} \Big|_{s=0} = \frac{dW_t}{dt} \Big|_{t=0} - 3 \frac{\varphi''(x_1)}{\varphi'(x_1)} = -\partial_1 \mathcal{U}(x_1, \dots, x_{2n}) - \frac{6}{x_b - x_1}.$$

After combining the computations above, we finally obtain

$$\begin{aligned} 12 \frac{d}{dt} \mathcal{H}_{\mathbb{H}}(\bar{\eta}^t) \Big|_{t=0} &= -\frac{1}{2} \left(\partial_1 \mathcal{U} + \frac{6}{x_b - x_1} \right)^2 + \frac{6 \partial_1 \mathcal{U}}{x_b - x_1} + \frac{12}{(x_b - x_1)^2} - \sum_{i \neq 1, b} \frac{6}{(x_i - x_1)^2} \\ &= -\frac{1}{2} (\partial_1 \mathcal{U})^2 - \sum_{i \neq 1} \frac{6}{(x_i - x_1)^2}, \end{aligned}$$

and by equating this with the right-hand side of (4.5), we obtain the asserted PDE (1.9) with $j = 1$. The other PDEs follow by symmetry. ■

Remark 4.6. Like the BPZ PDEs (1.11), the semiclassical PDE system (1.9) does not depend on the link pattern α . Therefore, by Proposition 1.8, we have already found C_n solutions to it. From analogy with conformal blocks in boundary CFT, we believe that the total number of solutions should be given by counting more general planar link patterns, as in [61, Secs. 2.5 and 3.1]:

$$\sum_{s \in \{0, 2, 4, \dots, 2n\}} \frac{s + 1}{n + s/2 + 1} \binom{2n}{n + s/2}, \tag{4.7}$$

where each summand is the number of link patterns with $2n$ indices and s “defects” (i.e., lines going to infinity), using the terminology in [61]. Note also that the s th summand in (4.7) is the Kostka number of $(s, 1, 1, \dots, 1)$, also used in [26, 27] to enumerate nets.

5. Large deviations of SLEs

The purpose of this section is to prove the LDP Theorem 1.5 for multichordal SLE_{0+} . For this, we consider multichordal SLE_{κ} as a probability measure on $\mathcal{X}_{\alpha}(D; x_1, \dots, x_{2n})$ endowed with the Hausdorff metric from Definition 2.1. Since the Loewner energy, the topology, and the SLE measures are conformally invariant, we use $D = \mathbb{H}$ throughout as the reference domain (so we have a simple description of driving functions). We also assume throughout that $\kappa < 8/3$ for convenience and without loss of generality.

5.1. Large deviations of single-chord SLE: finite time

Note that if a Loewner driving function $W \in C^0[0, T]$ has finite Dirichlet energy (2.6), $I_T(W) < \infty$, then the extension of W by a constant function generates a chord (by Lemma 2.6) in $\mathcal{X}(\mathbb{H}; 0, \infty)$ which has finite Loewner energy. In particular, $\mathcal{L}_T(W)$ is necessarily a simple curve in \mathbb{H} starting from 0 and bounded away from ∞ . Moreover, any capacity-parametrized bounded simple curve $\gamma_{[0, T]}$ in \mathbb{H} starting from 0 determines a unique driving function $W \in C^0[0, T]$ such that $\gamma_{[0, T]} = \mathcal{L}_T(W)$ according to (2.5). With a slight abuse of notation, we define the finite-time Loewner energy

$$I_T(\gamma_{[0, T]}) := I_T(W) \in [0, \infty], \quad \text{and} \quad I_T(K_T) := \infty = \inf_{W \in \mathcal{L}_T^{-1}(\{K_T\})} I_T(W)$$

for all sets $K_T \in \mathcal{K}_T$ that are not bounded simple curves.

Throughout this section, we endow the space $\mathcal{K}_T \subset \mathcal{C}$ with the topology induced from the Hausdorff metric (Definition 2.1) and we endow the space of driving functions $C^0[0, T]$ with the uniform norm. Now, recall that the Loewner transform \mathcal{L}_T is continuous for the Carathéodory topology (Definition 2.2), but not for the Hausdorff metric. However, thanks to Lemma 2.4, when considering the infimum of the Loewner energy, this is not an issue because driving functions at which \mathcal{L}_T is not continuous do not generate simple curves, thus having infinite energy. More precisely, we have the following lemma.

Lemma 5.1. *For any Hausdorff-closed subset F and Hausdorff-open subset O of \mathcal{K}_T ,*

$$\begin{aligned} \inf_{W \in \overline{\mathcal{L}_T^{-1}(F)}} I_T(W) &= \inf_{W \in \mathcal{L}_T^{-1}(F)} I_T(W), \\ \inf_{W \in \mathcal{L}_T^{-1}(O)^\circ} I_T(W) &= \inf_{W \in \mathcal{L}_T^{-1}(O)} I_T(W), \end{aligned}$$

where $\overline{\mathcal{L}_T^{-1}(F)}$ (resp. $\mathcal{L}_T^{-1}(O)^\circ$) denotes the closure (resp. interior) of $\mathcal{L}_T^{-1}(F)$ (resp. $\mathcal{L}_T^{-1}(O)$) in $C^0[0, T]$ for the uniform norm.

Proof. We first prove the assertion for the closed set F . By Lemma 2.4, we know that for any driving function $W \in \overline{\mathcal{L}_T^{-1}(F)} \setminus \mathcal{L}_T^{-1}(F)$, the corresponding hull $\mathcal{L}_T(W)$ has a non-empty interior, which has infinite Loewner energy. This proves the assertion for F . Next, the complement $\tilde{F} := \mathcal{K}_T \setminus O$ of the open set O is closed, and we have

$$C^0[0, T] \setminus \mathcal{L}_T^{-1}(O) = \mathcal{L}_T^{-1}(\tilde{F}) \quad \text{and} \quad \mathcal{L}_T^{-1}(O) \setminus \mathcal{L}_T^{-1}(O)^\circ = \overline{\mathcal{L}_T^{-1}(\tilde{F})} \setminus \mathcal{L}_T^{-1}(\tilde{F}).$$

The previous argument then shows that $I_T(W) = \infty$ for $W \in \mathcal{L}_T^{-1}(O) \setminus \mathcal{L}_T^{-1}(O)^\circ$. ■

The next lemma holds for SLE_κ with $\kappa \in (0, 4]$, although in the present article we are only concerned with $\kappa \rightarrow 0+$.

Lemma 5.2. *For any Hausdorff-closed subset F and Hausdorff-open subset O of \mathcal{K}_T ,*

$$\begin{aligned} \mathbb{P}[\sqrt{\kappa} B_{[0, T]} \in \overline{\mathcal{L}_T^{-1}(F)}] &= \mathbb{P}[\sqrt{\kappa} B_{[0, T]} \in \mathcal{L}_T^{-1}(F)], \\ \mathbb{P}[\sqrt{\kappa} B_{[0, T]} \in \mathcal{L}_T^{-1}(O)^\circ] &= \mathbb{P}[\sqrt{\kappa} B_{[0, T]} \in \mathcal{L}_T^{-1}(O)]. \end{aligned}$$

Proof. As SLE_κ is almost surely generated by a chord, similar arguments to those for Lemma 5.1 give

$$\mathbb{P}[\sqrt{\kappa} B_{[0, T]} \in \overline{\mathcal{L}_T^{-1}(F)} \setminus \mathcal{L}_T^{-1}(F)] = \mathbb{P}[\sqrt{\kappa} B_{[0, T]} \in \mathcal{L}_T^{-1}(O) \setminus \mathcal{L}_T^{-1}(O)^\circ] = 0. \quad \blacksquare$$

We denote by \mathbb{P}^κ the SLE_κ probability measure on $\mathcal{X}(\mathbb{H}; 0, \infty)$, and by \mathbb{P} the standard Wiener measure. By collecting the results from the previous lemmas, we obtain a LDP for these curves from Schilder’s theorem on Brownian paths:

Theorem F (Schilder; see, e.g., [17, Ch. 5.2]). *Fix $T \in (0, \infty)$. The process $(\sqrt{\kappa} B_t)_{t \in [0, T]}$ satisfies the following LDP in $C^0[0, T]$ with good rate function I_T :*

For any closed subset F and open subset O of $C^0[0, T]$, we have

$$\begin{aligned} \overline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}[\sqrt{\kappa} B_{[0,T]} \in F] &\leq - \inf_{W \in F} I_T(W), \\ \underline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}[\sqrt{\kappa} B_{[0,T]} \in O] &\geq - \inf_{W \in O} I_T(W). \end{aligned}$$

Proposition 5.3. Fix $T \in (0, \infty)$. The initial segments $\gamma_{[0,T]}^\kappa \in \mathcal{K}_T$ of chordal SLE_κ curves satisfy the following LDP in \mathcal{K}_T with good rate function I_T :

For any Hausdorff-closed subset F and Hausdorff-open subset O of \mathcal{K}_T , we have

$$\begin{aligned} \overline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \in F] &\leq - \inf_{K_T \in F} I_T(K_T), \\ \underline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \in O] &\geq - \inf_{K_T \in O} I_T(K_T). \end{aligned} \tag{5.1}$$

Proof. By Schilder’s theorem (Theorem F) and Lemma 5.1, we know that

$$\begin{aligned} \overline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}[\sqrt{\kappa} B_{[0,T]} \in \overline{\mathcal{L}_T^{-1}(F)}] &\leq - \inf_{W \in \overline{\mathcal{L}_T^{-1}(F)}} I_T(W) = - \inf_{W \in \mathcal{L}_T^{-1}(F)} I_T(W), \\ \underline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}[\sqrt{\kappa} B_{[0,T]} \in \mathcal{L}_T^{-1}(O)^\circ] &\geq - \inf_{W \in \mathcal{L}_T^{-1}(O)^\circ} I_T(W) = - \inf_{W \in \mathcal{L}_T^{-1}(O)} I_T(W). \end{aligned}$$

So (5.1) follows from Lemma 5.2. The lower semicontinuity and compactness of the level sets of I_T follow as in the proof of Lemma 2.7, using the fact that a finite-energy curve $\gamma_{[0,T]}$ is the image of the interval $[0, i]$ under a quasiconformal self-map of \mathbb{H} . ■

5.2. Large deviations of single-chord SLE: infinite time

The goal of this section is to establish the LDP for chordal SLE_κ all the way to the target point. The idea is to replace the event of the whole SLE_κ curve being close to a given chord with the event that a “truncated” SLE_κ is close to the truncated curve, and then apply the finite-time LDP from Proposition 5.3 combined with a suitable estimate for the error made in this truncation.

For brevity, we denote the curve space by $\mathcal{X} := \mathcal{X}(\mathbb{H}; 0, \infty)$, and we denote $I := I_{\mathbb{H}; 0, \infty}$. In fact, we will prove the LDP for the SLE_κ probability measures on the compact space \mathcal{C} introduced in Section 2.1. As SLE_κ curves for small κ belong almost surely to \mathcal{X} , the following statement is exactly the same as for \mathcal{X} endowed with the relative topology induced from \mathcal{C} . The reason for using \mathcal{C} instead is purely topological: the space \mathcal{C} is compact whereas \mathcal{X} is not. Note also that a set is compact in \mathcal{X} if and only if it is compact in \mathcal{C} .

To state our result, we extend the definition of the Loewner energy to all elements of \mathcal{C} by defining $I(K) := \infty$ if $K \in \mathcal{C} \setminus \mathcal{X}$.

Theorem 5.4. The family $(\mathbb{P}^\kappa)_{\kappa > 0}$ of probability measures of the chordal SLE_κ curve γ^κ satisfies the following LDP in \mathcal{C} with good rate function I :

For any closed subset F and open subset O of \mathcal{C} , we have

$$\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in F] \leq - \inf_{K \in F} I(K), \tag{5.2}$$

$$\underline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in O] \geq - \inf_{K \in O} I(K). \tag{5.3}$$

Remark 5.5. The compactness of the level sets of I is already proved in Lemma 2.7.

In order to apply the finite-time LDP from Proposition 5.3, we introduce a truncated Loewner energy for all sets $K \in \mathcal{C}$ (which do not necessarily have driving functions): we set

$$\tilde{I}(K) := \inf_{K \subset \gamma \in \mathcal{X}} I(\gamma) \in [0, \infty], \tag{5.4}$$

with the convention that the infimum of an empty set is ∞ . Note that a generic set $K \in \mathcal{C}$ can be disconnected and it does not have to contain 0 nor ∞ , in which case $I(K)$ is infinite while $\tilde{I}(K)$ can be finite. The lower semicontinuity of I implies that \tilde{I} is lower semicontinuous on \mathcal{C} . Note also that this definition coincides with our original definition of the Loewner energy in the following cases:

- for $K \in \mathcal{C}$ containing a simple path γ in $\bar{\mathbb{H}}$ connecting 0 to ∞ (and which may touch \mathbb{R}), we have $\tilde{I}(K) = I(\gamma)$ if $K = \gamma \in \mathcal{X}$, and $\tilde{I}(K) = \infty = I(K)$ otherwise;
- for $K_T \in \mathcal{K}_T$, we have $I_T(K_T) = \tilde{I}(K_T)$.

To prove Theorem 5.4, we use the following key Lemmas 5.6–5.9. We denote the Euclidean closed half-disc and semicircle by

$$\bar{D}_R := \{z \in \bar{\mathbb{H}} \mid |z| \leq R\} \quad \text{and} \quad S_R := \{z \in \bar{\mathbb{H}} \mid |z| = R\}.$$

We also denote by $\mathcal{B}_\varepsilon^h(K)$ the ε -Hausdorff-neighborhood of $K \in \mathcal{C}$, that is,

$$\mathcal{B}_\varepsilon^h(K) := \{\tilde{K} \in \mathcal{C} \mid d_h(K, \tilde{K}) < \varepsilon\}, \quad \bar{\mathcal{B}}_\varepsilon^h(K) := \{\tilde{K} \in \mathcal{C} \mid d_h(K, \tilde{K}) \leq \varepsilon\}.$$

Lemma 5.6. *Let $0 \leq M < \infty$. If $K \in \mathcal{C}$ satisfies $\tilde{I}(K) \in [M, \infty]$, then for all $\delta > 0$, there exist $r > 0$ and $\varepsilon > 0$ such that*

$$\tilde{I}(\tilde{K}) \geq M - \delta \quad \text{for all } \tilde{K} \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r). \tag{5.5}$$

Proof. Note that $K \cap \bar{D}_r$ converges to K in \mathcal{C} as $r \rightarrow \infty$. Hence, by the lower semicontinuity of \tilde{I} and the assumption $\tilde{I}(K) \geq M$, there exists $r > 0$ such that $\tilde{I}(K \cap \bar{D}_r) \geq M - \delta/2$, and the lower semicontinuity then again gives $\varepsilon > 0$ such that (5.5) holds. ■

For a curve $\gamma \in \mathcal{X}$ endowed with capacity parametrization and a radius $R > 0$, we define $\tau_R \in (0, \infty)$ to be the hitting time of γ to S_R . We also set $T_R := \text{hcap}(\bar{D}_R)/2$. Then, by the monotonicity of the capacity (2.2), we have $\tau_R \leq T_R$. Note also that if $\gamma = \gamma^\kappa \sim \mathbb{P}^\kappa$ is the chordal SLE_κ curve, then τ_R is a stopping time. The following result controls both the decay rate of the probability of the SLE_κ curve γ^κ and the minimal energy needed for any curve γ to come back to \bar{D}_r after hitting a large radius R .

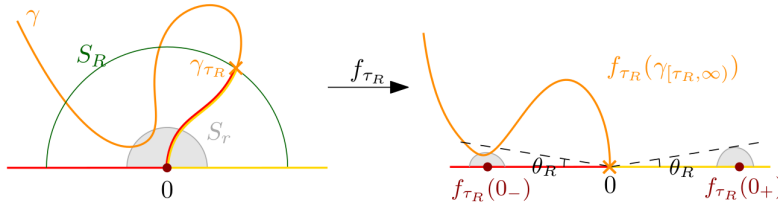


Fig. 2. Illustration of the conformal map f_{τ_R} , the angle θ_R , the left boundary ∂_- (red) and the right boundary ∂_+ (gold) used to derive (5.6) in the proof of Proposition 5.7.

Proposition 5.7. For each $r > 0$ and for any $M \in [0, \infty)$, there exists $R > r$ such that

- (i) $\inf\{I(\gamma) \mid \gamma \in \mathcal{X}, \gamma_{[\tau_R, \infty)} \cap S_r \neq \emptyset\} \geq M$;
- (ii) $\overline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma_{[\tau_R, \infty)}^\kappa \cap S_r \neq \emptyset] \leq -M$.

Proof. Claim (ii) is treated in Corollary A.4 in Appendix A. To study (i), as illustrated in Figure 2, we let $\theta_R(\gamma)$ be the infimum of $\theta \in [0, \pi/2]$ such that

$$f_{\tau_R}(S_r \setminus \gamma_{[0, \tau_R]}) \subset \overline{\mathbb{H}} \setminus \overline{\text{Cone}(\theta)} = \{z \in \overline{\mathbb{H}} \setminus \{0\} \mid \arg z < \theta \text{ or } \arg z > \pi - \theta\},$$

where $f_{\tau_R}: \mathbb{H} \setminus \gamma_{[0, \tau_R]} \rightarrow \mathbb{H}$ is a conformal map fixing ∞ such that $f_{\tau_R}(\gamma_{\tau_R}) = 0$. Let $\omega_{\gamma, z}$ be the harmonic measure on the boundary of $\mathbb{H} \setminus \gamma_{[0, \tau_R]}$ seen from a point $z \in S_r \setminus \gamma_{[0, \tau_R]}$. Let ∂_- (resp. ∂_+) be the union of \mathbb{R}_- (resp. \mathbb{R}_+) with the left side (resp. right side) of $\gamma_{[0, \tau_R]}$. Let \mathbb{P}_z be the law of a planar Brownian motion $B^{(2)}$ starting from z . Then

$$\max\{\omega_{\gamma, z}(\partial_-), \omega_{\gamma, z}(\partial_+)\} \geq \mathbb{P}_z[B^{(2)} \text{ hits } \mathbb{R} \text{ before } S_R],$$

which implies that

$$\inf_{\gamma \in \mathcal{X}} \inf_{z \in S_r \setminus \gamma_{[0, \tau_R]}} \max\{\omega_{\gamma, z}(\partial_-), \omega_{\gamma, z}(\partial_+)\} \xrightarrow{R \rightarrow \infty} 1.$$

Since harmonic measure is preserved under conformal maps, $\arg f_{\tau_R}(z)$ is determined by $\omega_{\gamma, z}(\partial_-)$, and we obtain

$$\lim_{R \rightarrow \infty} \sup_{\gamma \in \mathcal{X}} \theta_R(\gamma) = 0. \tag{5.6}$$

Now, note that if $\gamma_{[\tau_R, \infty)} \cap S_r \neq \emptyset$, then $f_{\tau_R}(\gamma)$ exits $\text{Cone}(\theta_R(\gamma))$ (see Figure 2). In particular, Lemma B from Section 2.3 shows that $I(f_{\tau_R}(\gamma_{[\tau_R, \infty)})) \geq -8 \log \sin(\theta_R(\gamma))$. The limit (5.6) then implies that

$$\lim_{R \rightarrow \infty} \inf_{\gamma} I(\gamma) \geq \lim_{R \rightarrow \infty} \inf_{\gamma} I(f_{\tau_R}(\gamma_{[\tau_R, \infty)})) = \infty,$$

where the infimums are taken over all $\gamma \in \mathcal{X}$ such that $\gamma_{[\tau_R, \infty)} \cap S_r \neq \emptyset$. Therefore, we can find $R > 0$ large enough satisfying both claims (i) and (ii) in the statement. ■

Now we state the main lemma that is crucial for the upper bound (5.2).

Lemma 5.8. *Let $0 \leq M < \infty$ and $\delta > 0$. Let $K \in \mathcal{C}$ such that $\tilde{I}(K) \geq M$. Let r and ε be chosen as in Lemma 5.6. Then*

$$\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)] \leq -M + \delta. \tag{5.7}$$

Proof. First, we choose R as in Proposition 5.7. For notational simplicity, we denote $T := T_R$, and we recall that $T \geq \tau_R$. We have

$$\mathbb{P}^\kappa[\gamma^\kappa \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)] \leq \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)] + \mathbb{P}^\kappa[\gamma_{[\tau_R,\infty)}^\kappa \cap S_r \neq \emptyset].$$

Thus, the left-hand side of (5.7) is bounded from above by the maximum of the two terms

$$\begin{aligned} &\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)], \\ &\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma_{[\tau_R,\infty)}^\kappa \cap S_r \neq \emptyset] \leq -M, \end{aligned}$$

where we have used Proposition 5.7 (ii) to immediately bound the second term. It remains to bound the first term. Since the restriction map $K \mapsto K \cap \bar{D}_r$ is continuous $\mathcal{C} \rightarrow \mathcal{C}$, the set

$$F_T := \{K_T \in \mathcal{K}_T \mid K_T \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)\}$$

is Hausdorff-closed in \mathcal{K}_T . Hence, Proposition 5.3 shows that

$$\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)] \leq - \inf_{K_T \in F_T} I_T(K_T). \tag{5.8}$$

Now, we show that $I_T(K_T) \geq M - \delta$ for all $K_T \in F_T$. In fact, if there exists $K_T = \gamma_{[0,T]} \in F_T$ such that $I_T(\gamma_{[0,T]}) < M$, then by Proposition 5.7, we know that γ does not come back to S_r after hitting S_R . Let W be the driving function of $\gamma_{[0,T]}$ and let $\hat{\gamma} \in \mathcal{X}$ be the chord driven by $t \mapsto W_{\min(t, \tau_R)}$, namely, the one coinciding with γ up to the hitting time τ_R and then continued with the hyperbolic geodesic in the slit domain $(\mathbb{H} \setminus \gamma_{[0, \tau_R]}; \gamma_{\tau_R}, \infty)$. As $\hat{\gamma}$ does not come back to \bar{D}_r either, we still have $\hat{\gamma}_{[0,T]} \in F_T$, so Lemma 5.6 shows that

$$M - \delta \leq \tilde{I}(\hat{\gamma}_{[0,T]}) = I(\hat{\gamma}) = I_{\tau_R}(\gamma) \leq I_T(\gamma).$$

We conclude that (5.8) is bounded from above by $-M + \delta$. This finishes the proof. ■

We now state the main lemma for the lower bound (5.3).

Lemma 5.9. *Let $\gamma \in \mathcal{X}$ be such that $I(\gamma) = M < \infty$. Then, for all $\varepsilon > 0$, we have*

$$\underline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in \mathcal{B}_\varepsilon^h(\gamma)] \geq -M. \tag{5.9}$$

Proof. According to Definition 2.1, for any $\delta > 0$, $N_\delta(\gamma) := \{z \in \mathbb{H} \mid \text{dist}_{\mathbb{D}}(\vartheta(z), \vartheta(\gamma)) \geq \delta\}$ is a bounded set in \mathbb{H} , where $\text{dist}_{\mathbb{D}}$ is the Euclidean distance and $\vartheta: \mathbb{H} \rightarrow \mathbb{D}$ is the fixed uniformizing conformal map from Section 2.1. We claim that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\tilde{\gamma} \in \mathcal{X}$ with $\tilde{\gamma} \cap N_\delta(\gamma) = \emptyset$, we have $\tilde{\gamma} \in \mathcal{B}_\varepsilon^h(\gamma)$. Indeed, assuming

the opposite, let $\varepsilon > 0$ and $\delta_j \rightarrow 0$ as $j \rightarrow \infty$, and let $\gamma^j \in \mathcal{X}$ be such that $\gamma^j \cap N_{\delta_j}(\gamma) = \emptyset$ while remaining at ε -Hausdorff distance away from γ . Since \mathcal{C} is a compact space, we can extract a subsequence of γ^j that converges to a limit \tilde{K} in \mathcal{C} . Furthermore, since $\delta_j \rightarrow 0$, we have $\tilde{K} \subset \gamma$. But \tilde{K} also connects 0 to ∞ and γ is a chord, so we actually have $\tilde{K} = \gamma$, contradicting the assumption that $\tilde{K} \notin \mathcal{B}_\varepsilon^h(\gamma)$.

Now, fix $\delta = \delta(\varepsilon)$ as above, and let $r > 0$ be such that $N_\delta(\gamma) \subset \bar{D}_r$. Choose $R > r$ as in Proposition 5.7 corresponding to $M + 1$, and let $T > T_R \geq \tau_R$. Then

$$\begin{aligned} \mathbb{P}^\kappa[\gamma^\kappa \in \mathcal{B}_\varepsilon^h(\gamma)] &\geq \mathbb{P}^\kappa[\gamma^\kappa \cap N_\delta(\gamma) = \emptyset] \\ &\geq \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \subset \bar{\mathbb{H}} \setminus N_\delta(\gamma)] - \mathbb{P}^\kappa[\gamma_{[\tau_R,\infty)}^\kappa \cap N_\delta(\gamma) \neq \emptyset]. \end{aligned}$$

Now, Proposition 5.3 shows that

$$\varliminf_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma_{[0,T]}^\kappa \subset \bar{\mathbb{H}} \setminus N_\delta(\gamma)] \geq -\inf_{K_T} I_T(K_T) \geq -I_T(\gamma) \geq -M, \tag{5.10}$$

where the infimum is taken over $\{K_T \in \mathcal{K}_T \mid K_T \subset \bar{\mathbb{H}} \setminus N_\delta(\gamma)\}$. Furthermore, Proposition 5.7 shows that

$$\varlimsup_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma_{[\tau_R,\infty)}^\kappa \cap N_\delta(\gamma) \neq \emptyset] \leq -M - 1. \tag{5.11}$$

The bounds (5.10) and (5.11) together imply that the left-hand side of (5.9) is bounded by $-\min(I_T(\gamma), M)$ from below, which tends to $-M$ as $T \rightarrow \infty$. ■

With these preparations, we are now ready to prove the main result of this section.

Proof of Theorem 5.4. First, we prove the lower bound (5.3). Without loss of generality, we assume that $M_O := \inf_{K \in O} I(K) < \infty$. For $\delta > 0$, take a chord $\gamma \in O \cap \mathcal{X}$ such that $I(\gamma) \leq M_O + \delta$. Since O is open, there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon^h(\gamma) \subset O$. Lemma 5.9 then shows that

$$\varliminf_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in O] \geq \varliminf_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in \mathcal{B}_\varepsilon^h(\gamma)] \geq -M_O - \delta,$$

and we obtain the lower bound (5.3) after letting $\delta \searrow 0$.

Second, we prove the upper bound (5.2). Because SLE $_\kappa$ curves belong almost surely to \mathcal{X} , we assume without loss of generality that F lies in the closure of \mathcal{X} in \mathcal{C} . Then every element of F connects 0 to ∞ in $\bar{\mathbb{H}}$, so

$$M_F := \inf_{K \in F} I(K) = \inf_{K \in F} \tilde{I}(K) \in [0, \infty].$$

Now, let $M < M_F$ and $\delta > 0$. Since for each $K \in F$, we have $\tilde{I}(K) > M$, we may choose a neighborhood $\mathcal{B}_\varepsilon^h(K)$ for K and $r > 0$ according to Lemma 5.8 for the given values of M and δ . In particular, we have

$$\begin{aligned} \varlimsup_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in \mathcal{B}_\varepsilon^h(K)] &\leq \varlimsup_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \cap \bar{D}_r \in \bar{\mathcal{B}}_\varepsilon^h(K \cap \bar{D}_r)] \\ &\leq -M + \delta. \end{aligned}$$

Since F is compact in \mathcal{C} and can be covered by finitely many such balls, we see that

$$\overline{\lim}_{\kappa \rightarrow 0^+} \kappa \log \mathbb{P}^\kappa[\gamma^\kappa \in F] \leq -M + \delta.$$

Finally, after letting $\delta \searrow 0$ and then $M \nearrow M_F$, we obtain the upper bound (5.2).

Remark 5.5 gives the goodness of the rate function, which concludes the proof. ■

5.3. Large deviations of multichordal SLE

In this section, we prove the LDP for multiple chordal SLE _{κ} curves (Theorem 1.5) in \mathbb{H} . We fix an n -link pattern α and boundary points $x_1 < \dots < x_{2n}$. For brevity, we denote the curve space by $\mathcal{X}_\alpha := \mathcal{X}_\alpha(\mathbb{H}; x_1, \dots, x_{2n})$. Recall that this is the space of disjoint chords in \mathbb{H} which connect the marked boundary points x_{a_j} and x_{b_j} for all j according to the link pattern α . As before, we use the shorthand notation $\bar{\gamma} := (\gamma_1, \dots, \gamma_n)$ for a multichord, and similarly $\bar{\eta}$ for the unique geodesic multichord in \mathcal{X}_α (cf. Corollary 4.2).

Let \mathbb{Q}_α^κ denote the product measure of n independent chordal SLE _{κ} curves associated to the link pattern α , (i.e., connecting the marked boundary points according to α , but not necessarily disjoint). On the product space $\prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j})$ endowed with the product topology induced from the Hausdorff metric on \mathcal{C} , the probability measures $(\mathbb{Q}_\alpha^\kappa)_{\kappa > 0}$ satisfy a LDP with rate function just the sum of the independent rate functions,

$$I_0^\alpha(\bar{\gamma}) := \sum_{j=1}^n I_{\mathbb{H}; x_{a_j}, x_{b_j}}(\gamma_j).$$

Lemma 5.10. *The family $(\mathbb{Q}_\alpha^\kappa)_{\kappa > 0}$ of laws of n independent chordal SLE _{κ} curves $\bar{\gamma}^\kappa$ satisfies the LDP in $\prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j})$ with good rate function I_0^α .*

Proof. This follows from Theorem 5.4, as $\gamma_1^\kappa, \dots, \gamma_n^\kappa$ are independent. ■

Multichordal SLE _{κ} is a family of SLE _{κ} curves with interaction. In this section, we shall derive a LDP for them with rate function including contribution from the interaction, namely from the loop measure term introduced in Section 3.1. This will be a simple consequence of Varadhan’s lemma (stated as Lemma G below).

We denote by \mathbb{E}_α^κ the expectation with respect to \mathbb{Q}_α^κ . Then, as proved, e.g., in [12, Thm. 1.2], the multichordal SLE _{κ} probability measure \mathbb{P}_α^κ on \mathcal{X}_α can be obtained by weighting \mathbb{Q}_α^κ with the Radon–Nikodym derivative⁸

$$R_\alpha^\kappa(\bar{\gamma}^\kappa) = \frac{d\mathbb{P}_\alpha^\kappa}{d\mathbb{Q}_\alpha^\kappa}(\bar{\gamma}^\kappa) := \frac{\exp(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma}^\kappa))}{\mathbb{E}_\alpha^\kappa[\exp(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma}^\kappa))]}, \quad \text{where} \quad \Phi_\kappa(\bar{\gamma}) := \frac{\kappa}{2} c(\kappa) m_{\mathbb{H}}(\bar{\gamma}) \tag{5.12}$$

and $c(\kappa) = (3\kappa - 8)(6 - \kappa)/2\kappa$. Note that when $\kappa < 8/3$, we have $c(\kappa) < 0$, so the normalization factor (total mass) is clearly finite. We also set $\Phi_\kappa(\bar{\gamma}) := -\infty$ if $\bar{\gamma} \notin \mathcal{X}_\alpha$.

⁸The difference of $1/2$ compared to [12, 62] is due to normalization conventions for Brownian loop measure.

For all $\bar{\gamma} \in \mathcal{X}_\alpha$, as $\kappa \searrow 0$, we have

$$\Phi_\kappa(\bar{\gamma}) = -\frac{c(\kappa)\kappa}{24}\Phi_0(\bar{\gamma}) \searrow \Phi_0(\bar{\gamma}) = -12m_{\mathbb{H}}(\bar{\gamma}) < 0, \tag{5.13}$$

since $-c(\kappa)\kappa \nearrow 24$. This factor gives the additional contribution to the LDP rate function.

Lemma G (Varadhan’s lemma [17, Lemmas 4.3.4 and 4.3.6]). *Suppose that the probability measures $(\mathbb{Q}^\kappa)_{\kappa>0}$ satisfy a LDP with good rate function I_0^α . Let $\Phi: \prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j}) \rightarrow \mathbb{R}$ be a function bounded from above. Then the following hold:*

(1) *If Φ is upper semicontinuous, then for any closed subset F of $\prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j})$,*

$$\overline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa}\Phi(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in F\} \right] \leq - \inf_{\bar{\gamma} \in F} (I_0^\alpha(\bar{\gamma}) - \Phi(\bar{\gamma})).$$

(2) *If Φ is lower semicontinuous, then for any open subset O of $\prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j})$,*

$$\underline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa}\Phi(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in O\} \right] \geq - \inf_{\bar{\gamma} \in O} (I_0^\alpha(\bar{\gamma}) - \Phi(\bar{\gamma})).$$

Using these results, it is straightforward to derive the LDP for multichordal SLE_{0+} . Recalling Definition 3.4, we denote the multichordal Loewner energy in $(\mathbb{H}; x_1, \dots, x_{2n})$ by

$$I^\alpha(\bar{\gamma}) := I_{\mathbb{H}}^\alpha(\bar{\gamma}) = 12(\mathcal{H}_{\mathbb{H}}(\bar{\gamma}) - \mathcal{M}_{\mathbb{H}}^\alpha(x_1, \dots, x_{2n})) = I_0^\alpha(\bar{\gamma}) - \Phi_0(\bar{\gamma}) - L,$$

where

$$L := I_0^\alpha(\bar{\eta}) - \Phi_0(\bar{\eta}) = \inf_{\bar{\gamma} \in \mathcal{X}_\alpha} (I_0^\alpha(\bar{\gamma}) - \Phi_0(\bar{\gamma})).$$

Theorem 1.5 follows from the next result.

Theorem 5.11. *The family $(\mathbb{P}_\alpha^\kappa)_{\kappa>0}$ of laws of multichordal SLE_κ curves $\bar{\gamma}^\kappa$ satisfies the following LDP in \mathcal{X}_α with good rate function I^α :*

For any closed subset F and open subset O of \mathcal{X}_α , we have

$$\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}_\alpha^\kappa[\bar{\gamma}^\kappa \in F] \leq - \inf_{\bar{\gamma} \in F} I^\alpha(\bar{\gamma}), \tag{5.14}$$

$$\underline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}_\alpha^\kappa[\bar{\gamma}^\kappa \in O] \geq - \inf_{\bar{\gamma} \in O} I^\alpha(\bar{\gamma}). \tag{5.15}$$

Furthermore,

$$\lim_{\kappa \rightarrow 0+} \kappa \log \mathbb{E}_\alpha^\kappa \left[\exp\left(\frac{1}{2}c(\kappa)m_{\mathbb{H}}(\bar{\gamma}^\kappa)\right) \right] = -L. \tag{5.16}$$

Proof. The lower semicontinuity and compactness of the level sets of I^α follow from Proposition 3.13. Recall that the multichordal SLE_κ takes values in \mathcal{X}_α almost surely. Using the Radon–Nikodym derivative (5.12), for any Borel set $B \subset \mathcal{X}_\alpha$, we have

$$\begin{aligned} & \kappa \log \mathbb{P}_\alpha^\kappa[\bar{\gamma}^\kappa \in B] \\ &= \kappa \log \mathbb{E}_\alpha^\kappa \left[\exp\left(\frac{1}{\kappa}\Phi_\kappa(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in B\} \right] - \kappa \log \mathbb{E}_\alpha^\kappa \left[\exp\left(\frac{1}{\kappa}\Phi_\kappa(\bar{\gamma}^\kappa)\right) \right]. \end{aligned}$$

Equation (5.16) asserts that the second term on the right-hand side has the limit L as $\kappa \rightarrow 0+$. We will establish this together with (5.14)–(5.15).

Upper bound. Let \tilde{F} be the closure of F in $\prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j})$. Let $\varepsilon, M > 0$. By (5.13), there exists $\kappa_0 \in (0, 8/3)$ such that $\Phi_\kappa(\bar{\gamma}) < (1 - \varepsilon)\Phi_0(\bar{\gamma})$ for all $\bar{\gamma} \in \tilde{F}$ and for all $\kappa \in [0, \kappa_0)$. Applying Varadhan’s Lemma G (1) to \tilde{F} and the continuous function $\Phi^{M,\varepsilon}(\bar{\gamma}) := \max\{(1 - \varepsilon)\Phi_0(\bar{\gamma}), -M\}$ (which is continuous by Lemma 3.2), we obtain

$$\begin{aligned} \overline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in F\} \right] &\leq \overline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in \tilde{F}\} \right] \\ &\leq \overline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi^{M,\varepsilon}(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in \tilde{F}\} \right] \\ &\leq - \inf_{\bar{\gamma} \in \tilde{F}} (I_0^\alpha(\bar{\gamma}) - \Phi^{M,\varepsilon}(\bar{\gamma})) \xrightarrow[\varepsilon \searrow 0]{M \nearrow \infty} - \inf_{\bar{\gamma} \in F} (I_0^\alpha(\bar{\gamma}) - \Phi_0(\bar{\gamma})), \end{aligned}$$

since $-\Phi_0(\bar{\gamma}) = \infty$ for $\bar{\gamma} \in \tilde{F} \setminus F$. Also, taking $F = \mathcal{X}_\alpha$ shows that

$$\overline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma})\right) \right] \leq -L. \tag{5.17}$$

Lower bound. Without loss of generality, we assume $M_O := \inf_{\bar{\gamma} \in O} (I_0^\alpha(\bar{\gamma}) - \Phi_0(\bar{\gamma})) < \infty$. Let $\varepsilon > 0$ and $\bar{\gamma}_\varepsilon \in O$ be such that $I_0^\alpha(\bar{\gamma}_\varepsilon) - \Phi_0(\bar{\gamma}_\varepsilon) \leq M_O + \varepsilon$, in particular, $\Phi_0(\bar{\gamma}_\varepsilon) > -\infty$. By continuity of Φ_0 , let $\mathcal{B} \subset O$ be an open neighborhood of $\bar{\gamma}_\varepsilon$ in $\prod_j \mathcal{X}(\mathbb{H}; x_{a_j}, x_{b_j})$ such that $\Phi_0 \geq \Phi_0(\bar{\gamma}_\varepsilon) - 1$ on \mathcal{B} . Let Φ be the lower semicontinuous function that equals Φ_0 on \mathcal{B} and $\Phi_0(\bar{\gamma}_\varepsilon) - 2$ otherwise. Since $\Phi_0 \leq \Phi_\kappa \leq 0$, applying Varadhan’s Lemma G (2) to \mathcal{B} and the lower semicontinuous function Φ , we obtain

$$\begin{aligned} \underline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in O\} \right] &\geq \underline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi_0(\bar{\gamma}^\kappa)\right) \mathbf{1}\{\bar{\gamma}^\kappa \in \mathcal{B}\} \right] \\ &\geq - \inf_{\bar{\gamma} \in \mathcal{B}} (I_0^\alpha(\bar{\gamma}) - \Phi(\bar{\gamma})) \geq -(I_0^\alpha(\bar{\gamma}_\varepsilon) - \Phi_0(\bar{\gamma}_\varepsilon)) \\ &\geq -M_O - \varepsilon \xrightarrow{\varepsilon \searrow 0} -M_O. \end{aligned}$$

Taking $O = \mathcal{X}_\alpha$ shows that

$$\underline{\lim}_{\kappa \rightarrow 0} \kappa \log \mathbb{E}^\kappa \left[\exp\left(\frac{1}{\kappa} \Phi_\kappa(\bar{\gamma}^\kappa)\right) \right] \geq -L. \tag{5.18}$$

Combining (5.17)–(5.18) yields (5.16). The above upper bound and lower bound then give (5.14) and (5.15), since $I_0^\alpha(\bar{\gamma}) - \Phi_0(\bar{\gamma}) - L = I^\alpha(\bar{\gamma})$. This concludes the proof. ■

The LDP immediately implies the following corollary (which we state in the general domain D as in the introduction).

Corollary 1.6. *As $\kappa \rightarrow 0+$, multichordal SLE_κ in $(D; x_1, \dots, x_{2n})$ associated to the n -link pattern α converges in probability to the unique minimizer of the potential in $\mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$. This minimizer is the unique geodesic multichord associated to α .*

Proof. Recall from Corollary 4.2 that the geodesic multichord $\bar{\eta}$ is the unique minimizer of I_D^α . Let $\mathcal{B}_\varepsilon^h(\bar{\eta}) \subset \mathcal{X}_\alpha(D) := \mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ be a Hausdorff-open ball of radius ε around $\bar{\eta}$. Then

$$\lim_{\kappa \rightarrow 0+} \overline{\kappa \log \mathbb{P}^\kappa[\bar{\gamma}^\kappa \in \mathcal{X}_\alpha(D) \setminus \mathcal{B}_\varepsilon^h(\bar{\eta})]} \leq - \inf_{\bar{\gamma} \in \mathcal{X}_\alpha(D) \setminus \mathcal{B}_\varepsilon^h(\bar{\eta})} I_D^\alpha(\bar{\gamma}) < 0,$$

by Theorem 5.11. This proves the corollary. ■

Lastly, we state another corollary. The normalization factor in (5.12) determines the multichordal SLE_κ partition function \mathcal{Z}_α associated to α ,

$$\mathcal{Z}_\alpha(D; x_1, \dots, x_{2n}) := \left(\prod_{j=1}^n P_{D; x_{a_j}, x_{b_j}} \right)^{(6-\kappa)/2\kappa} \mathbb{E}_\alpha^\kappa \left[\exp \left(\frac{1}{2} c(\kappa) m_D(\bar{\gamma}^\kappa) \right) \right]. \quad (5.19)$$

Note that \mathcal{Z}_α is a function of the boundary points x_1, \dots, x_{2n} and the domain D . Equation (5.16) implies that the minimal potential \mathcal{M}_D^α can be regarded as a semiclassical limit of the multichordal SLE_κ partition function \mathcal{Z}_α in the following sense:

Corollary 5.12. *We have*

$$- \lim_{\kappa \rightarrow 0+} \kappa \log \mathcal{Z}_\alpha(D; x_1, \dots, x_{2n}) = 12 \mathcal{M}_D^\alpha(x_1, \dots, x_{2n}).$$

Proof. This follows immediately from (5.16) and definitions (1.5) and (5.19). ■

6. Determinants of Laplacians and the Loewner potential

In this section, we show Theorem 1.9: the identity between the Loewner potential \mathcal{H} and zeta-regularized determinants of Laplacians. This relies on the Polyakov–Alvarez conformal anomaly formula for domains with corners (see Theorem H and Appendix B for a more detailed discussion), and is applicable only to smooth multichords. Throughout this section, we consider bounded curvilinear polygonal domains with piecewise smooth boundary:

Definition 6.1. We say that $D \subsetneq \mathbb{C}$ is a *curvilinear polygonal domain* if its boundary ∂D is a piecewise smooth Jordan curve with finitely many corners $z_1, \dots, z_m \in \partial D$ with opening interior angles $\pi\beta_1, \dots, \pi\beta_m$ such that

- (1) the boundary ∂D in a neighborhood of each corner z_j is defined by a continuous curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow \mathbb{C}$ for some $\varepsilon > 0$ such that $\gamma(0) = z_j$, the curve γ is smooth on $(-\varepsilon, 0]$ and $[0, \varepsilon)$ with $|\gamma'(t)| = 1$ for all $t \in (-\varepsilon, \varepsilon)$, and

$$\lim_{t \rightarrow 0-} \gamma'(t) \quad \text{and} \quad \lim_{t \rightarrow 0+} \gamma'(t)$$

are tangent vectors at $\gamma(0) = z_j$;

(2) for each $j \in \{1, \dots, n\}$, the opening angle $\pi\beta_j$ at the corner at z_j is the interior angle between the tangent vectors $\lim_{t \rightarrow 0^-} \gamma'(t)$ and $\lim_{t \rightarrow 0^+} \gamma'(t)$; so $\beta_j \in (0, 1) \cup (1, 2)$. In other words, ∂D is asymptotically straight on each side of the corner point.

We call two metrics g and g_0 on D *Weyl-equivalent* if g is a *Weyl-scaling* of g_0 , i.e., $g = e^{2\sigma} g_0$ for some $\sigma \in C^\infty(\overline{D})$. Beware that our notion of Weyl-equivalence is not the same as conformal invariance: especially, we do not allow any log-singularities of σ , which would change the opening angles of the boundary. Throughout this section, we only consider metrics that are Weyl-equivalent to the Euclidean metric dz^2 , so that the opening angle on the boundary is the same as for the Euclidean metric. We use the following notation with respect to the metric g :

- $\Delta_g := \Delta_{D;g}$ is the (positive) Laplacian on D (see (1.12)) with Dirichlet boundary conditions;
- $d\text{vol}_g$ is the area measure;
- dl_g is the arc-length measure on the boundary;
- K_g is the Gauss curvature in the interior of D ;
- k_g is the geodesic curvature on the boundary ∂D .

6.1. Determinants of Laplacians

The purpose of this section is to give a brief summary of zeta-regularized determinants of Laplacians and to state the Polyakov–Alvarez conformal anomaly formula. Let $D \subsetneq \mathbb{C}$ be a curvilinear polygonal domain having corners at $z_1, \dots, z_m \in \partial D$ with opening interior angles $\pi\beta_1, \dots, \pi\beta_m$.

The Dirichlet Laplacian Δ_g on D has a discrete spectrum. We order its eigenvalues as $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and recall that the heat kernel in D is represented as the series

$$p_t(z, w) := \sum_{j=1}^{\infty} e^{-\lambda_j t} u_j(z) u_j(w), \tag{6.1}$$

where u_j are the eigenfunctions of Δ_g corresponding to λ_j , forming an orthonormal basis for $L^2(D)$. Following Ray & Singer [65], a notion of determinant for the Laplace operator Δ_g makes sense using its *spectral zeta function*, defined in terms of the heat trace

$$\sum_{j \geq 1} e^{-\lambda_j t} = \text{Tr}(e^{-t\Delta_g}) = \text{Tr}(p_t) = \int_D p_t(z, z) d\text{vol}_g(z).$$

For $\text{Re } s > 1$, the spectral zeta function is defined as

$$\zeta_{\Delta_g}(s) := \sum_{\lambda_j > 0} \lambda_j^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(p_t) dt, \tag{6.2}$$

where $\Gamma(\cdot)$ is the Gamma function. The spectral zeta function $\zeta_{\Delta_g}(s)$ is a holomorphic function on $\{s \in \mathbb{C} \mid \text{Re } s > 1\}$. Using fine estimates of the short-time expansion of the

heat kernel in curvilinear domains, only established very recently in [58, Thm. 1.4], one shows that the analytic continuation of ζ_{Δ_g} is holomorphic in a neighborhood of 0. The *zeta-regularized determinant* of Δ_g is then defined as

$$\det_\zeta \Delta_g := \exp(-\zeta'_{\Delta_g}(0)).$$

In fact, the determinant of an operator is formally the product of its eigenvalues:

$$\zeta'_{\Delta_g}(s) = \sum_{\lambda_j > 0} \log(\lambda_j) \lambda_j^{-s} \implies \zeta'_{\Delta_g}(0) = -\log \prod_{\lambda_j > 0} \lambda_j = -\log \det \Delta_g.$$

Importantly, the zeta-regularized determinant of the Laplacian depends on the metric. The change of the determinant under a Weyl-scaling is given by the Polyakov–Alvarez conformal anomaly formula. For curvilinear polygonal domains, this formula was proved recently in [3].

Theorem H (Generalized Polyakov–Alvarez conformal anomaly formula [3, Thm. 3]). *Consider a metric $g = e^{2\sigma} g_0$ on a curvilinear polygonal domain D which is Weyl-equivalent to a reference metric g_0 . Then*

$$\begin{aligned} & \log \det_\zeta \Delta_0 - \log \det_\zeta \Delta_g \\ &= \frac{1}{6\pi} \left[\frac{1}{2} \int_D |\nabla_0 \sigma|^2 \, \text{dvol}_0 + \int_D K_0 \sigma \, \text{dvol}_0 + \int_{\partial D \setminus \{z_1, \dots, z_m\}} k_0 \sigma \, \text{dl}_0 \right] \\ & \quad + \frac{1}{4\pi} \int_{\partial D \setminus \{z_1, \dots, z_m\}} \partial_{v_0} \sigma \, \text{dl}_0 + \frac{1}{12} \sum_{j=1}^m \left(\frac{1}{\beta_j} - \beta_j \right) \sigma(z_j), \end{aligned} \tag{6.3}$$

where ∂_{v_0} is the outward normal derivative with respect to the metric g_0 , and for notational simplicity, we replace the subscripts “ g_0 ” by “0”.

For the readers’ convenience, we outline the key steps in the proof and discuss some heuristics about this formula in Appendix B.

6.2. Identity with the Loewner potential

In this section, we consider a curvilinear polygonal domain D with marked boundary points $x_1, \dots, x_{2n} \in \partial D$ on smooth boundary segments. We also fix an n -link pattern α throughout. We say that a multichord $\bar{\gamma} \in \mathcal{X}_\alpha(D; x_1, \dots, x_{2n})$ is *smooth* if each γ_j is the image of an injective C^∞ -function on $[-1, 1]$, with $j = 1, \dots, n$. We let $\mathcal{X}_\alpha^\infty(D) := \mathcal{X}_\alpha^\infty(D; x_1, \dots, x_{2n})$ be the space of smooth finite-energy multichords (dropping the notation α when $n = 1$). According to Lemmas B and 3.6, each chord of $\bar{\gamma} \in \mathcal{X}_\alpha^\infty(D)$ meets ∂D perpendicularly, and the connected components of $D \setminus \bar{\gamma}$ are curvilinear polygonal domains. We define

$$\tilde{\mathcal{H}}_D(\bar{\gamma}; g) := \log \det_\zeta \Delta_{D;g} - \sum_C \log \det_\zeta \Delta_{C;g} \tag{6.4}$$

for each $\bar{\gamma} \in \mathcal{X}_\alpha^\infty(D)$ and for each metric g on D , where the sum is taken over all connected components C of $D \setminus \bigcup_i \gamma_i$. We also define

$$\tilde{\mathcal{M}}_D^\alpha(x_1, \dots, x_{2n}; g) := \inf_{\bar{\gamma}} \tilde{\mathcal{H}}_D(\bar{\gamma}; g),$$

where the infimum is taken over all $\bar{\gamma} \in \mathcal{X}_\alpha^\infty(D; x_1, \dots, x_{2n})$.

The goal of this section is to prove the following result, equivalent to Theorem 1.9.

Theorem 6.2. *There exists a universal constant $\lambda \in \mathbb{R}$ such that for all $n \geq 1$ and for all $\bar{\gamma} \in \mathcal{X}_\alpha^\infty(D)$, we have*

$$\tilde{\mathcal{H}}_D(\bar{\gamma}; dz^2) = \mathcal{H}_D(\bar{\gamma}) + n\lambda.$$

Remark 6.3. To determine the numerical value of λ , we apply Theorem 6.2 to the chord $[-1, 1]$ in \mathbb{D} . The determinant of the Laplacian on the flat unit disc with Dirichlet boundary conditions was evaluated in [78, (28)]:

$$\log \det_\xi \Delta_{\mathbb{D}; dz^2} = -\frac{1}{6} \log 2 - \frac{1}{2} \log \pi - 2\zeta'_R(-1) - \frac{5}{12},$$

where ζ_R is the Riemann zeta function. From [3, Sect. 3], we obtain the determinant of the Laplacian on the flat half-unit disc $\mathbb{D}_+ := \{z \in \mathbb{D} \mid \text{Im } z > 0\}$ with Dirichlet boundary conditions:⁹

$$\log \det_\xi \Delta_{\mathbb{D}_+; dz^2} = -\frac{5}{24} - \zeta'_R(-1) - \frac{1}{3} \log 2 - \frac{1}{2} \log \pi.$$

Using these formulas, we obtain

$$\tilde{\mathcal{H}}_{\mathbb{D}}([-1, 1], dz^2) = \log \det_\xi \Delta_{\mathbb{D}; dz^2} - 2 \log \det_\xi \Delta_{\mathbb{D}_+; dz^2} = \frac{1}{2} \log 2 + \frac{1}{2} \log \pi.$$

Also, using the conformal map $\varphi(z) = i(i - z)/(i + z)$ from \mathbb{D} to \mathbb{H} , with $|\varphi'(-1)| = |\varphi'(1)| = 1$, we see that

$$\mathcal{H}_{\mathbb{D}}([-1, 1]) = -\frac{1}{4} \log P_{\mathbb{D}; -1, 1} = -\frac{1}{4} \log \frac{|\varphi'(1)\varphi'(-1)|}{(\varphi(1) - \varphi(-1))^2} = \frac{1}{2} \log 2,$$

and combining everything, we evaluate the constant λ as

$$\lambda = \tilde{\mathcal{H}}_{\mathbb{D}}([-1, 1], dz^2) - \mathcal{H}_{\mathbb{D}}([-1, 1]) = \frac{1}{2} \log \pi \approx 0.5724.$$

The constant λ is also given by Corollary 6.5.

To prove Theorem 6.2, we first consider the case of a single chord, and define, for $\gamma \in \mathcal{X}^\infty(D; x, y)$,

$$J_D(\gamma; g) := 12(\tilde{\mathcal{H}}_D(\gamma; g) - \tilde{\mathcal{M}}_D^\alpha(x, y; g)). \tag{6.5}$$

Recall that here $(D; x, y)$ is a curvilinear domain with boundary points $x, y \in \partial D$ on smooth boundary segments, that is, there are smooth neighborhoods $U_x \ni x$ and $U_y \ni y$

⁹This appears to disagree with [18, (28)], claiming that $\log \det_\xi \Delta_{\mathbb{D}_+; dz^2} = \frac{1}{12} - \zeta'_R(-1) - \frac{2}{3} \log 2 - \frac{1}{2} \log \pi$.

on ∂D . If $(D'; x', y')$ is another such domain, then a conformal map φ from $(D; x, y)$ to $(D'; x', y')$ may not map corners of D to corners of D' . Therefore, $\sigma = \log |\varphi'|$ is in general only smooth on $D \cup U_x \cup U_y$ with possibly logarithmic singularities elsewhere on ∂D .

Lemma 6.4. *Let $\sigma: \bar{D} \rightarrow \mathbb{R}$ be a smooth function and $g = e^{2\sigma} dz^2$ a metric Weyl-equivalent to dz^2 . Then, for all $\gamma \in \mathcal{X}^\infty(D; x, y)$, we have*

$$\tilde{\mathcal{H}}_D(\gamma; g) - \tilde{\mathcal{H}}_D(\gamma; dz^2) = \frac{1}{4}(\sigma(x) + \sigma(y)). \tag{6.6}$$

In particular, J is Weyl-invariant, i.e., $J_D(\gamma; g) = J_D(\gamma; dz^2)$.

Moreover, if φ is a conformal map from $(D; x, y)$ to $(D'; x', y')$ and $\sigma = \log |\varphi'|$, then (6.6) still holds. In particular,

$$J_{D'}(\varphi(\gamma); dz^2) = J_D(\gamma; dz^2), \tag{6.7}$$

i.e., J is invariant under φ .

By Weyl-invariance, we will write $J_D(\gamma)$, omitting the notation for the metric g when it is Weyl-equivalent to the Euclidean metric dz^2 .

Proof of Lemma 6.4. Note that (6.6) shows that the variation of the functional $\tilde{\mathcal{H}}_D(\gamma; g)$ under a Weyl-scaling of the metric is independent of the curve γ , so J is Weyl-invariant. We first prove the asserted identity (6.6) when $\sigma \in C^\infty(\bar{D})$ and ∂D is smooth. Let D^L and D^R be the two connected components of the complement $D \setminus \gamma$. First, applying Theorem H with $g_0 = dz^2$, we have

$$\begin{aligned} \log \det_\zeta \Delta_{D;0} - \log \det_\zeta \Delta_{D;g} &= \frac{1}{6\pi} \left[\frac{1}{2} \int_D |\nabla_0 \sigma|^2 \, d\text{vol}_0 + \int_{\partial D} k_0 \sigma \, dl_0 \right] + \frac{1}{4\pi} \int_{\partial D} \partial_{\nu_0} \sigma \, dl_0, \end{aligned}$$

since $K_0 \equiv 0$. Second, applying Theorem H to D^L , which has two corners at x and y both with an opening angle of $\pi/2$, we obtain

$$\begin{aligned} \log \det_\zeta \Delta_{D^L;0} - \log \det_\zeta \Delta_{D^L;g} &= \frac{1}{6\pi} \left[\frac{1}{2} \int_{D^L} |\nabla_0 \sigma|^2 \, d\text{vol}_0 + \int_{\partial D^L \setminus \{x,y\}} k_0 \sigma \, dl_0 \right] \\ &\quad + \frac{1}{4\pi} \int_{\partial D^L \setminus \{x,y\}} \partial_{\nu_0} \sigma \, dl_0 + \frac{1}{8}(\sigma(x) + \sigma(y)), \end{aligned}$$

and similarly for D^R . Hence, we see that in the difference $\tilde{\mathcal{H}}_D(\gamma; g) - \tilde{\mathcal{H}}_D(\gamma; dz^2)$ all terms except the corner contributions (6.6) cancel, since σ is continuous across γ .

Next, we consider the case where $\sigma = \log |\varphi'|$, which is smooth on $D \cup U_x \cup U_y$. Let $\tilde{D} \subset D$ be a smooth domain such that $U_x \cup U_y \subset \partial \tilde{D}$ and $\gamma \subset \tilde{D}$. Using the Brownian loop measure interpretation of $\log \det_\zeta \Delta$ from Proposition 6.10 and letting $\delta \rightarrow 0$, we see that

$$\tilde{\mathcal{H}}_{\tilde{D}}(\gamma; g) - \tilde{\mathcal{H}}_D(\gamma; g) = \tilde{\mathcal{H}}_{\tilde{D}}(\gamma; dz^2) - \tilde{\mathcal{H}}_D(\gamma; dz^2),$$

where both sides also equal the total mass of Brownian loops in D intersecting both $D \setminus \tilde{D}$ and γ (which is finite and conformally invariant). Now, (6.6) with $\sigma = \log |\varphi'|$ follows from the identity (6.6) applied to \tilde{D} . Furthermore, the φ -invariance then follows:

$$J_{D'}(\varphi(\gamma); dz^2) = J_D(\gamma; g) = J_D(\gamma; dz^2). \quad \blacksquare$$

Corollary 6.5. $\tilde{\mathcal{M}}_D(x, y; dz^2) + \frac{1}{4} \log P_{D;x,y} =: \lambda \in \mathbb{R}$ is a universal constant.

Proof. Let φ be a conformal map between $(D; x, y)$ and $(D'; \varphi(x), \varphi(y))$. Then, for any $\gamma \in \mathcal{X}^\infty(D; x, y)$, we have

$$\tilde{\mathcal{H}}_{D'}(\varphi(\gamma); dz^2) = \tilde{\mathcal{H}}_D(\gamma; e^{2\sigma(z)} dz^2), \quad \text{where } \sigma(z) = \log |\varphi'(z)|.$$

Therefore, we deduce from Lemma 6.4 and (2.8) that

$$\begin{aligned} \tilde{\mathcal{M}}_{D'}(\varphi(x), \varphi(y); dz^2) - \tilde{\mathcal{M}}_D(x, y; dz^2) &= \tilde{\mathcal{M}}_D(x, y; e^{2\sigma(z)} dz^2) - \tilde{\mathcal{M}}_D(x, y; dz^2) \\ &= \frac{1}{4}(\sigma(x) + \sigma(y)) = \frac{1}{4} \log |\varphi'(x)\varphi'(y)| \\ &= \frac{1}{4}(\log P_{D;x,y} - \log P_{D';\varphi(x),\varphi(y)}), \end{aligned} \quad (6.8)$$

which implies the claim. \blacksquare

In Proposition 6.9, we will show that J_D coincides with the single-chord Loewner energy (2.7) on smooth chords, i.e., $J_D(\gamma) = I_D(\gamma)$ for all $\gamma \in \mathcal{X}^\infty(D; x, y)$. Assuming this fact, we now prove the main result of this section.

Proof of Theorem 6.2. For $n = 1$, the assertion follows immediately from the definition (6.5) of J , Proposition 6.9, Corollary 6.5, and the definition (3.4) of \mathcal{H}_D :

$$\begin{aligned} \tilde{\mathcal{H}}_D(\gamma; dz^2) &= \frac{1}{12} J_D(\gamma) + \tilde{\mathcal{M}}_D(x, y; dz^2) = \frac{1}{12} I_D(\gamma) - \frac{1}{4} \log P_{D;x,y} + \lambda \\ &= \mathcal{H}_D(\gamma) + \lambda. \end{aligned}$$

The general case follows by induction on $n \geq 2$. We use the Euclidean metric below and omit it from the notation. Let D^L and D^R be the two connected components of $D \setminus \bigcup_i \gamma_i$ adjacent to the chord γ_1 , so that $\hat{D}_1 = D^L \cup D^R \cup \gamma_1$ (recall Figure 1). Then

$$\begin{aligned} \tilde{\mathcal{H}}_D(\bar{\gamma}) &= \log \det_\xi \Delta_D - \sum_C \log \det_\xi \Delta_C \\ &= \log \det_\xi \Delta_D - \left(\sum_{C \neq D^L, D^R} \log \det_\xi \Delta_C + \log \det_\xi \Delta_{\hat{D}_1} \right) \\ &\quad + \log \det_\xi \Delta_{\hat{D}_1} - \log \det_\xi \Delta_{D^L} - \log \det_\xi \Delta_{D^R} \\ &= \tilde{\mathcal{H}}_D(\gamma_2, \dots, \gamma_n) + \tilde{\mathcal{H}}_{\hat{D}_1}(\gamma_1). \end{aligned}$$

Hence, $\tilde{\mathcal{H}}$ and \mathcal{H} satisfy the same cascade relation (cf. Lemma 3.8), so it follows by induction that they are equal up to the additive constant $n\lambda$. This proves Theorem 6.2. \blacksquare

The rest of this section is devoted to showing that $I_D = J_D$ (Proposition 6.9).

Remark 6.6. When $(D; x, y) = (\mathbb{C} \setminus \mathbb{R}_+; 0, \infty)$, we deduce from [76, Thm. 7.3] that

$$I_{\mathbb{C} \setminus \mathbb{R}_+}(\gamma) = J_{\mathbb{C} \setminus \mathbb{R}_+; g}(\gamma), \tag{6.9}$$

where g is Weyl-equivalent to the spherical metric (here, $\mathbb{C} \setminus \mathbb{R}_+$ is unbounded). Indeed, because for a chord γ in $(\mathbb{C} \setminus \mathbb{R}_+; 0, \infty)$, the chordal energy is the same as the loop energy I^L of $\gamma \cup \mathbb{R}_+$, [76, Thm. 7.3] shows that

$$\begin{aligned} I_{\mathbb{C} \setminus \mathbb{R}_+}(\gamma) &= I^L(\gamma \cup \mathbb{R}_+) = 12\tilde{\mathcal{H}}_{\hat{\mathbb{C}}; g}(\gamma \cup \mathbb{R}_+) - 12\tilde{\mathcal{H}}_{\hat{\mathbb{C}}; g}(\mathbb{R}) \\ &= 12\tilde{\mathcal{H}}_{\mathbb{C} \setminus \mathbb{R}_+; g}(\gamma) - 12\tilde{\mathcal{H}}_{\mathbb{C} \setminus \mathbb{R}_+; g}(\mathbb{R}_-), \end{aligned}$$

which yields (6.9) since $I_{\mathbb{C} \setminus \mathbb{R}_+}(\cdot) \geq 0$, and consequently \mathbb{R}_- minimizes $\tilde{\mathcal{H}}_{\mathbb{C} \setminus \mathbb{R}_+}(\cdot, g)$ in $\mathcal{X}^\infty(\mathbb{C} \setminus \mathbb{R}_+; 0, \infty)$.

Since the Loewner energy I is conformally invariant, if we could show the full conformal invariance of J , then from (6.9) we would deduce that $I = J$ holds for all chords in any domain D . However, in order to establish this, we should understand in general how $\det_\xi \Delta_D$ changes when varying opening angles of corners. (For instance, when ∂D is smooth near the marked points, D^L makes right angles at those points, whereas when $D = \mathbb{C} \setminus \mathbb{R}_+$, D^L has no corners.) Although the variation of $\det_\xi \Delta_D$ while changing the angles is computable [3], this is rather technical, so we restrict ourselves to Weyl-scalings only.

Now, recall that the chordal Loewner energy I satisfies a conformal restriction formula, Lemma C. We next check that the same formula holds for J . Let $U \subset D$ be a curvilinear subdomain which coincides with D in neighborhoods of x and y . Suppose that $\gamma \in \mathcal{X}^\infty(U; x, y)$ and let $\varphi: U \rightarrow D$ be a conformal map fixing x and y .

Proposition 6.7. *The functional J satisfies the same conformal restriction formula as I :*

$$J_U(\gamma) - J_D(\gamma) = 3 \log |\varphi'(x)\varphi'(y)| + 12\mathcal{B}_D(\gamma, D \setminus U).$$

Proof. According to [22, Prop. 2.1], or Proposition 6.10, the Brownian loop measure $\mathcal{B}_D(\gamma, D \setminus U)$ can be written in terms of $\tilde{\mathcal{H}}$ as

$$\mathcal{B}_D(\gamma, D \setminus U) = \tilde{\mathcal{H}}_U(\gamma; dz^2) - \tilde{\mathcal{H}}_D(\gamma; dz^2). \tag{6.10}$$

This formula gives

$$\begin{aligned} J_U(\gamma) - J_D(\gamma) &= 12(\tilde{\mathcal{H}}_U(\gamma; dz^2) - \tilde{\mathcal{H}}_D(\gamma; dz^2)) \\ &\quad + 12(\tilde{\mathcal{M}}_D(x, y; dz^2) - \tilde{\mathcal{M}}_U(x, y; dz^2)) \\ &= 12\mathcal{B}_D(\gamma, D \setminus U) + 3 \log |\varphi'(x)\varphi'(y)|, \end{aligned}$$

where the second equality follows from (6.10) and (6.8). ■

Corollary 6.8. *Let D' be a curvilinear polygonal domain such that the boundaries ∂D and $\partial D'$ coincide in neighborhoods of x and y . Suppose $\gamma \subset D \cap D'$. Then*

$$I_{D'}(\gamma) - I_D(\gamma) = J_{D'}(\gamma) - J_D(\gamma). \tag{6.11}$$

Proof. The conformal restriction formula (Lemma C) applied to a domain $U \subset D \cap D'$ containing γ gives

$$I_{D'}(\gamma) - I_D(\gamma) = 3 \log |\varphi'(x)\varphi'(y)| + 12\mathcal{B}_D(\gamma, D \setminus D') - 12\mathcal{B}_D(\gamma, D' \setminus D), \tag{6.12}$$

where φ is a conformal map from D' to D fixing x and y . The same argument (with Proposition 6.7) also shows that (6.12) holds for J when $\gamma \in \mathcal{X}^\infty(D; x, y)$. ■

Now we are ready to show that $I_D = J_D$, using the conformal restriction formula and approximation of smooth chords by analytic chords.

Proposition 6.9. *The functional J_D attains its infimum in $\mathcal{X}^\infty(D; x, y)$ and its unique minimizer is the hyperbolic geodesic η . In particular, $J_D(\gamma) = I_D(\gamma)$ for all smooth chords $\gamma \in \mathcal{X}^\infty(D; x, y)$.*

Proof. We assume without loss of generality (by Lemma 6.4) that $(D; x, y) = (\mathbb{D}; -1, 1)$. We call γ an *analytic curve* in $\mathcal{X}^\infty := \mathcal{X}^\infty(\mathbb{D}; -1, 1)$ if there exists a neighborhood U of γ with smooth boundary which coincides with \mathbb{D} near -1 and 1 , and a conformal map $\varphi: \mathbb{D} \rightarrow U$ such that $\varphi([-1, 1]) = \gamma$. We denote the family of such analytic curves by \mathcal{X}^ω . We first show that the segment $[-1, 1]$ minimizes J_D in \mathcal{X}^ω . Corollary 6.8 gives

$$I_{\mathbb{D}}(\gamma) = I_{\mathbb{D}}(\gamma) - I_U(\gamma) = J_D(\gamma) - J_U(\gamma) = J_{\mathbb{D}}(\gamma) - J_{\mathbb{D}}([-1, 1]), \tag{6.13}$$

using also the facts that $I_U(\gamma) = I_{\mathbb{D}}([-1, 1]) = 0$ and $J_{\mathbb{D}}([-1, 1]) = J_U(\gamma)$, thanks to the Weyl-invariance of J (Lemma 6.4), equivalent to the conformal invariance here, for both D and U have smooth boundary. Since $I_{\mathbb{D}}(\gamma) \geq 0$, we have indeed

$$J_{\mathbb{D}}(\gamma) \geq J_{\mathbb{D}}([-1, 1]) \quad \text{for all } \gamma \in \mathcal{X}^\omega. \tag{6.14}$$

Now we claim that, for every $\gamma \in \mathcal{X}^\infty$ and an approximating sequence $\gamma^k \in \mathcal{X}^\omega$ for the C^3 -norm, as $k \rightarrow \infty$, the functionals $J_{\mathbb{D}}(\gamma^k)$ and $I_{\mathbb{D}}(\gamma^k)$ converge respectively to $J_{\mathbb{D}}(\gamma)$ and $I_{\mathbb{D}}(\gamma)$. (Since γ is the image of $[-1, 1]$ by a C^∞ -function, such an approximation always exists by Taylor expansion.) To prove this claim, let D_k^L and D_k^R (resp. D^L and D^R) be the two connected components of $\mathbb{D} \setminus \gamma^k$ (resp. $\mathbb{D} \setminus \gamma$) containing respectively i and $-i$ on their boundary. Let $\mathbb{D}_+ := \{z \in \mathbb{D} \mid \text{Im } z > 0\}$. Note that

$$\log \det_\xi \Delta_{D_k^L, dz^2} = \log \det_\xi \Delta_{\mathbb{D}_+, e^{2\sigma_k} dz^2}, \quad \text{where } \sigma_k = \log |(\varphi^k)'|,$$

and where $\varphi^k: \mathbb{D}_+ \rightarrow D_k^L$ is a conformal map fixing the corners -1 and 1 . From Theorem H, it is not hard to check that $\log \det_\xi \Delta_{D_k^L}$ converges to $\log \det_\xi \Delta_{D^L}$, since all terms in the conformal anomaly formula (6.3) converge. We note that to properly deal with the corners, one extends φ^k by Schwarz reflection across $S_+ := \{z \in S^1 \mid \text{Im } z \geq 0\} \subset \partial D_k^L$, and then the $C^{2,1-\varepsilon}$ -regularity of φ^k up to the boundary follows from Kellogg's theorem [32, Thm. II.4.3]. Combining this with a similar analysis for $\log \det_\xi \Delta_{D_k^R}$, we establish that $J_{\mathbb{D}}(\gamma^k) \rightarrow J_{\mathbb{D}}(\gamma)$ as $k \rightarrow \infty$. Similarly, using (6.9) and the conformal invariance of $I_{\mathbb{D}}$, the above argument also shows the convergence of $I_{\mathbb{D}}(\gamma^k)$ to $I_{\mathbb{D}}(\gamma)$.

Now, (6.14) combined with approximation of smooth chords by analytic chords shows that $[-1, 1]$ also minimizes $J_{\mathbb{D}}$ in \mathcal{X}^∞ . Therefore, we conclude that $J_{\mathbb{D}}([-1, 1]) = 0$ and $J_{\mathbb{D}}(\gamma) = I_{\mathbb{D}}(\gamma)$ for all $\gamma \in \mathcal{X}^\omega$ by (6.13). The same approximation argument again allows us to finally deduce the equality $I_{\mathbb{D}} = J_{\mathbb{D}}$ for all $\gamma \in \mathcal{X}^\infty$. ■

6.3. UV-cutoff for Brownian loop measure

In this section, we explain how Theorem 1.9 can be directly related to Brownian loop measure. This follows rather directly from the short-time expansion of the heat trace (Theorem I). Brownian loop measure $\mu_{D;g}^{\text{loop}}$ on D with respect to a Riemannian metric g is defined as in Section 2.4, by taking the diffusion generator to be $-\Delta_g$.

Proposition 6.10 (UV-cutoff of Brownian loop measure). *For a curvilinear polygonal domain $(D; g)$, the total mass of loops in D with quadratic variation greater than 4δ under Brownian loop measure has the expansion*

$$\frac{\text{vol}_g(D)}{4\pi\delta} - \frac{1_g(\partial D)}{4\sqrt{\pi\delta}} - \log \det_\zeta \Delta_g - \left(\frac{1}{6} + \frac{1}{24} \sum_{j=1}^m \left(\beta_j - 2 + \frac{1}{\beta_j} \right) \right) (\log \delta + \gamma) + O(\delta^{1/2} \log \delta)$$

as $\delta \rightarrow 0+$, where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant.

Our proof follows the same lines as in [6]. First, let us make a few remarks.

- 4δ is the quadratic variation of a two-dimensional Brownian path run at speed 2 until time δ (as its generator is $-\Delta_g$ instead of $-\Delta_g/2$). We will write $[\ell]$ for the quadratic variation of ℓ .
- By the same proof, after replacing the term $1/6$ by $\chi(D)/6$, Proposition 6.10 holds in a more general setup of (not necessarily planar or simply connected) curvilinear domains on Riemannian surfaces.

Proof of Proposition 6.10. Recall that Brownian loop measure on D is defined as

$$\mu_{D;g}^{\text{loop}} := \int_0^\infty \frac{dt}{t} \int_D \mathbb{W}_{z \rightarrow z}^t \, d\text{vol}_g(z),$$

where, in particular, $\mathbb{W}_{z \rightarrow z}^t$ has total mass $p_t(z, z)$. Therefore, we have

$$\mu_{D;g}^{\text{loop}}(\{\ell \mid [\ell] \geq 4\delta\}) = \int_\delta^\infty t^{-1} \text{Tr}(e^{-t\Delta_g}) \, dt.$$

As $t \rightarrow 0+$, $\text{Tr}(e^{-t\Delta_g})$ has the following expansion (see Theorem I with $\sigma \equiv 1$):

$$\text{Tr}(e^{-t\Delta_g}) = \frac{\text{vol}_g(D)}{4\pi t} - \frac{1_g(\partial D)}{8\sqrt{\pi t}} + a_2 + O(t^{1/2} \log t) \tag{6.15}$$

where

$$\begin{aligned}
 a_2 := a_2(g, 1) &= \frac{1}{12\pi} \int_D K_g \, \text{dvol}_g + \frac{1}{12\pi} \int_{\partial D \setminus \{z_1, \dots, z_m\}} k_g \, \text{dl}_g + \frac{1}{24} \sum_{j=1}^m \left(\frac{1}{\beta_j} - \beta_j \right) \\
 &= \frac{1}{6} + \frac{1}{24} \sum_{j=1}^m \left(\frac{1}{\beta_j} - 2 + \beta_j \right),
 \end{aligned}$$

and where the last equality follows from the Gauss–Bonnet theorem:

$$\int_D K_g \, \text{dvol}_g + \int_{\partial D \setminus \{z_1, \dots, z_m\}} k_g \, \text{dl}_g + \sum_{j=1}^m \pi(1 - \beta_j) = 2\pi\chi(D) = 2\pi.$$

The spectral zeta function can be computed for $\text{Re}(s) > 1$ as

$$\begin{aligned}
 \zeta_{\Delta_g}(s) &= \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} t^{s-1} \text{Tr}(e^{-t\Delta_g}) \, dt + \frac{1}{\Gamma(s)} \int_0^{\delta} O(t^{s-1/2} \log t) \, dt \\
 &\quad + \frac{1}{\Gamma(s)} \int_0^{\delta} t^{s-2} \frac{\text{vol}_g(D)}{4\pi} \, dt - \frac{1}{\Gamma(s)} \int_0^{\delta} t^{s-3/2} \frac{l_g(\partial D)}{8\sqrt{\pi}} \, dt \\
 &\quad + \frac{1}{\Gamma(s)} \int_0^{\delta} t^{s-1} a_2 \, dt \\
 &= \frac{1}{\Gamma(s)} \int_{\delta}^{\infty} t^{s-1} \text{Tr}(e^{-t\Delta_g}) \, dt + \frac{1}{\Gamma(s)} \int_0^{\delta} O(t^{s-1/2} \log t) \, dt \\
 &\quad + \frac{\text{vol}_g(D)}{4\pi} \frac{\delta^{s-1}}{(s-1)\Gamma(s)} - \frac{l_g(\partial D)}{8\sqrt{\pi}} \frac{\delta^{s-1/2}}{(s-1/2)\Gamma(s)} + a_2 \frac{\delta^s}{\Gamma(s+1)} \\
 &=: I_1(s) + I_2(s) + I_3(s) + I_4(s) + I_5(s),
 \end{aligned}$$

with obvious notation for the five terms, respectively. Taking the derivative of the analytic continuation (powers of s) at $s = 0$ and using the formulas

$$\lim_{s \rightarrow 0} s\Gamma(s) = 1 \quad \text{and} \quad \left. \frac{d}{ds} \frac{1}{\Gamma(s+1)} \right|_{s=0} = \gamma,$$

we get

$$\begin{aligned}
 I'_1(0) &= \int_{\delta}^{\infty} t^{-1} \text{Tr}(e^{-t\Delta_g}) \, dt = \mu_{D;g}^{\text{loop}}(\{\ell \mid [\ell] \geq 4\delta\}), \\
 I'_2(0) &= O(\sqrt{\delta} \log \delta), \quad I'_3(0) = -\frac{\text{vol}_g(D)}{4\pi\delta}, \quad I'_4(0) = \frac{l_g(\partial D)}{4\sqrt{\pi}\delta}, \\
 I'_5(0) &= a_2(\log \delta + \gamma).
 \end{aligned}$$

We now obtain the claimed expansion from the definition $\zeta'_{\Delta_g}(0) = -\log \det_{\zeta} \Delta_g$. ■

Next, we consider a smooth Jordan domain D with metric $g = dz^2$ (omitting the metric from the notation). It follows from Theorem 1.9 that the Loewner potential of $\bar{\gamma} \in \mathcal{X}^{\infty}(D; x_1, \dots, x_{2n})$ can be interpreted as the constant term in the expansion of the mass of Brownian loops touching $\bar{\gamma}$.

Theorem 1.10. *Let $\bar{\gamma}$ be a smooth finite-energy multichord in D . The total mass of loops in D touching $\bar{\gamma}$ with quadratic variation greater than 4δ under Brownian loop measure has the expansion*

$$\frac{l(\bar{\gamma})}{2\sqrt{\pi\delta}} - \mathcal{H}_D(\bar{\gamma}) - n\lambda + \frac{n}{4}(\log \delta + \gamma) + O(\delta^{1/2} \log \delta) \quad \text{as } \delta \rightarrow 0+,$$

where $l(\bar{\gamma})$ is the total arclength of $\bar{\gamma}$, λ is the universal constant from Theorem 1.9, and $\gamma \approx 0.5772$ is the Euler–Mascheroni constant.

Proof. Applying Proposition 6.10 to D and to all of the $n + 1$ connected components C of $D \setminus \bar{\gamma}$, we obtain

$$\begin{aligned} &\mu_D^{\text{loop}}(\{\ell \mid [\ell] \geq 4\delta \text{ and } \ell \cap \bar{\gamma} \neq \emptyset\}) \\ &= \mu_D^{\text{loop}}(\{\ell \mid [\ell] \geq 4\delta \text{ and } \ell \cap \bar{\gamma} \neq \emptyset\}) - \sum_C \mu_C^{\text{loop}}(\{\ell \mid [\ell] \geq 4\delta \text{ and } \ell \cap \bar{\gamma} \neq \emptyset\}) \\ &= \frac{l(\bar{\gamma})}{2\sqrt{\pi\delta}} - (\mathcal{H}_D(\bar{\gamma}) + n\lambda) - \left(-\frac{n}{6} - \frac{1}{24} \cdot 4n \cdot \frac{1}{2}\right)(\log \delta + \gamma) + O(\delta^{1/2} \log \delta) \\ &= \frac{l(\bar{\gamma})}{2\sqrt{\pi\delta}} - \mathcal{H}_D(\bar{\gamma}) - n\lambda + \frac{n}{4}(\log \delta + \gamma) + O(\delta^{1/2} \log \delta) \quad \text{as } \delta \rightarrow 0+, \end{aligned}$$

where the volume terms cancel out, the boundary length terms count each chord in $\bar{\gamma}$ twice, and there are $4n$ corners of opening angle $\pi/2$ contributing to the constant term. ■

We can interpret the above result heuristically as

$$\mathcal{H}_D(\bar{\gamma}) \text{ “=” } -\mu_D^{\text{loop}}(\{\ell \mid \ell \cap \bar{\gamma} \neq \emptyset\}),$$

after renormalizing by taking out small loops and the divergent term proportional to the total length of $\bar{\gamma}$. The rest of the mass then only depends on the number of chords. In fact, this interpretation is consistent with Definition 3.3 thanks to (3.1):

$$\begin{aligned} \mathcal{H}_D(\bar{\gamma}) &= \sum_{j=1}^n \mathcal{H}_D(\gamma_j) + m_D(\bar{\gamma}) \\ \text{“=”} &- \sum_{j=1}^n \mu_D^{\text{loop}}(\{\ell \mid \ell \cap \gamma_j \neq \emptyset\}) + \mu_D^{\text{loop}}((N(\ell) - 1)\mathbf{1}\{N(\ell) \neq 0\}) \\ \text{“=”} &- \mu_D^{\text{loop}}(\{\ell \mid \ell \cap \bar{\gamma} \neq \emptyset\}), \end{aligned}$$

where $N(\ell)$ is the number of chords in $\{\gamma_1, \dots, \gamma_n\}$ that ℓ intersects.

Appendix A. Refined estimate of SLE return probability

The goal of this appendix is to prove claim (ii) of Proposition 5.7, given in Corollary A.4.

Theorem A.1. *Let $\kappa \in (0, 4]$. Let $\tilde{S}_r = S_r + 1$ be a semicircle of radius $r \in (0, 1)$ centered at $1 \in \mathbb{R}$, and let γ^κ be SLE_κ in $(\mathbb{H}; 0, \infty)$. Then*

$$\mathbb{P}^\kappa[\gamma^\kappa \cap \tilde{S}_r \neq \emptyset] \leq c''_\kappa r^{8/\kappa-1}, \quad \text{where } c''_\kappa = \frac{\Gamma(12/\kappa)}{\Gamma(8/\kappa)\Gamma(4/\kappa + 1)},$$

and where $\Gamma(z) := \int_0^\infty x^{z-1} e^{-x} dx$ is the Gamma function.

Versions of this result have appeared, e.g., in [2, Thm. 3.2] and in [29, Thm. 1.1], but these theorems did not control the multiplicative κ -dependent constant in the estimate. Because we must keep track of the behavior of the constant c''_κ as $\kappa \rightarrow 0$, we present a sharper argument, using ideas of Greg Lawler [46].

From Stirling’s formula, we find that

$$\lim_{\kappa \rightarrow 0+} \kappa \log c''_\kappa = C'' := 12 \log 3 - 8 \log 2 \approx 7, 63817. \tag{A.1}$$

Proof of Theorem A.1. The proof follows [46, Sec. 2] very closely. It is convenient to measure the distance of the SLE_κ curve to the point $1 \in \mathbb{R}$ in terms of the conformal radius. Because the point of interest is on the boundary of the domain \mathbb{H} , we first define the reflected domain including the positive real line,

$$D_t := H_t \cup \{z^* \mid z \in H_t\} \cup \mathbb{R}_+,$$

where $H_t = \mathbb{H} \setminus \gamma^\kappa_{[0,t]}$. Then, for each $z \in D_t$, we set

$$\Upsilon_t(z) := \frac{1}{4} \text{crad}_{D_t}(z),$$

where $\text{crad}_{D_t}(z)$ denotes the conformal radius of D_t seen from z . We consider the process $\Upsilon = (\Upsilon_t)_{t \geq 0} := (\Upsilon_t(1))_{t \geq 0}$. By the Koebe 1/4 theorem, we have $\Upsilon_t \leq \text{dist}(1, \partial D_t)$, so upon taking $t \rightarrow \infty$, we see that $\Upsilon_\infty \leq \text{dist}(1, \gamma^\kappa)$. Hence,

$$\mathbb{P}^\kappa[\gamma^\kappa \cap \tilde{S}_r \neq \emptyset] = \mathbb{P}^\kappa[\text{dist}(1, \gamma^\kappa) \leq r] \leq \mathbb{P}^\kappa[\Upsilon_\infty \leq r],$$

and it suffices to show that

$$\mathbb{P}^\kappa[\Upsilon_\infty \leq r] \leq c''_\kappa r^{8/\kappa-1}. \tag{A.2}$$

To estimate this, we tilt the SLE_κ measure by a suitable (local) martingale. We consider the Loewner flow of γ^κ parametrized as follows: $g_t: H_t \rightarrow \mathbb{H}$ is the uniformizing map normalized at ∞ and satisfying

$$\frac{dg_t(z)}{dt} = \frac{2/\kappa}{g_t(z) + B_t}, \quad g_0(z) = z,$$

where B is the standard Brownian motion.¹⁰

¹⁰The sign of the Brownian motion here is chosen for convenience. Note that the Loewner flow here is *not* normalized as in Section 2 but differs only by a time reparametrization. The choice is made to be consistent with [46] and to simplify the computations.

We set $X_t := g_t(1) + B_t$ and $O_t := g_t(0_+) + B_t$, where $g_t(0_+)$ is the image of the prime end at 0 approached from the right. Then $X_0 = 1$ and $O_0 = 0$, and

$$dX_t = \frac{2/\kappa}{X_t} dt + dB_t, \quad dO_t = \frac{2/\kappa}{O_t} dt + dB_t,$$

that is, both are Bessel processes of effective dimension $4/\kappa + 1 \geq 2$ (so in particular $X_t > 0$ and $O_t > 0$ for all $t > 0$ almost surely). We also set

$$Y_t := X_t - O_t \geq 0 \quad \text{and} \quad J_t := \frac{Y_t}{X_t} \in [0, 1].$$

Then it can be verified by a calculation that the following equations hold [46, Sec. 2]:

$$\frac{d}{dt} Y_t = -\frac{2}{\kappa} \frac{Y_t}{X_t^2 (1 - J_t)}, \quad \frac{d}{dt} \Upsilon_t = -\frac{2}{\kappa} \frac{\Upsilon_t J_t}{X_t^2 (1 - J_t)},$$

and in particular

$$M_t = \left(\frac{g'_t(1)}{X_t} \right)^{8/\kappa - 1}, \quad dM_t = \left(1 - \frac{8}{\kappa} \right) \frac{M_t}{X_t} dB_t,$$

is a (local) martingale with initial value $M_0 = 1$. We consider the measure $\tilde{\mathbb{P}}^\kappa$ obtained from \mathbb{P}^κ via tilting by M . Then Girsanov’s theorem gives a standard Brownian motion \tilde{B} with respect to $\tilde{\mathbb{P}}^\kappa$ so that the change of measure between B and \tilde{B} is given by

$$dB_t = \left(1 - \frac{8}{\kappa} \right) \frac{dt}{X_t} + d\tilde{B}_t.$$

With this change of measure and the following time-change, we can analyze Υ_∞ : we set

$$\sigma(t) := \inf \{s \geq 0 \mid \Upsilon_s = e^{-2t/\kappa}\}$$

and on the event $\{\sigma(t) < \infty\}$, we define the time-changed processes

$$\hat{\Upsilon}_t := \Upsilon_{\sigma(t)} = e^{-2t/\kappa}, \quad \hat{X}_t := X_{\sigma(t)}, \quad \hat{J}_t := J_{\sigma(t)}, \quad \hat{M}_t := M_{\sigma(t)},$$

so that after a calculation [46, Sec. 2], we have

$$\hat{M}_t = (e^{2t/\kappa} \hat{J}_t)^{8/\kappa - 1}, \quad d\hat{J}_t = \left(\frac{4}{\kappa} - \frac{6}{\kappa} \hat{J}_t \right) dt + \sqrt{\hat{J}_t(1 - \hat{J}_t)} d\hat{B}_t,$$

where \hat{B} is another standard Brownian motion. Now, we have

$$\begin{aligned} \mathbb{P}^\kappa[\Upsilon_\infty \leq e^{-2t/\kappa}] &= (e^{-2t/\kappa})^{8/\kappa - 1} \mathbb{E}^\kappa[\hat{M}_t \hat{J}_t^{1 - 8/\kappa} \mathbf{1}\{\Upsilon_\infty \leq e^{-2t/\kappa}\}] \\ &= (e^{-2t/\kappa})^{8/\kappa - 1} \tilde{\mathbb{E}}^\kappa[\hat{J}_t^{1 - 8/\kappa}], \end{aligned}$$

so writing $r = e^{-2t/\kappa} \in (0, 1)$, we see that the probability of interest is

$$\mathbb{P}^\kappa[\Upsilon_\infty \leq r] = r^{8/\kappa - 1} \tilde{\mathbb{E}}^\kappa[\hat{J}_t^{1 - 8/\kappa}].$$

Thus, to establish (A.2) it remains to prove the estimate

$$\tilde{\mathbb{E}}^\kappa[\hat{J}_t^{1-8/\kappa}] \leq c''_\kappa. \tag{A.3}$$

It is shown in [46] that the process \hat{J} has the explicit invariant distribution

$$u \mapsto \frac{\Gamma(12/\kappa)}{\Gamma(8/\kappa)\Gamma(4/\kappa)} u^{8/\kappa-1} (1-u)^{4/\kappa-1}.$$

Now, note that the process \hat{J} starts at $\hat{J}_0 = 1$, and we have $\hat{J}_t \in [0, 1]$ for all t almost surely. Let \hat{J}_t^i be an independent process starting at the invariant distribution. Using Doebelin’s coalescing coupling, we couple \hat{J}^i and \hat{J} so that they evolve independently until they collide, and after that they evolve together. Then we have

$$\hat{J}_t^i \leq \hat{J}_t \quad \text{for all } t \geq 0.$$

Thus, using the fact that $1 - 8/\kappa < 0$, we see that

$$\begin{aligned} \tilde{\mathbb{E}}^\kappa[\hat{J}_t^{1-8/\kappa}] &\leq \tilde{\mathbb{E}}^\kappa[(\hat{J}_t^i)^{1-8/\kappa}] = \frac{\Gamma(12/\kappa)}{\Gamma(8/\kappa)\Gamma(4/\kappa)} \int_0^1 u^{1-8/\kappa} u^{8/\kappa-1} (1-u)^{4/\kappa-1} du \\ &= \frac{\Gamma(12/\kappa)}{\Gamma(8/\kappa)\Gamma(4/\kappa + 1)} = c''_\kappa. \end{aligned}$$

This shows (A.3) and concludes the proof. ■

Suppose D has a smooth boundary. If $A_1, A_2 \subset \partial D$ are two disjoint subsets of the boundary, the *Brownian excursion measure* between A_1, A_2 in D is

$$\mathcal{E}_D(A_1, A_2) := \int_{A_1} \int_{A_2} P_{D;x,y} |dx| |dy|.$$

If A_1, A_2 are smooth disjoint chords in D , we also write $\mathcal{E}_D(A_1, A_2)$ for $\mathcal{E}_C(A_1, A_2)$, where C is the connected component of $D \setminus (A_1 \cup A_2)$ such that $A_1, A_2 \subset \partial C$.

Because Brownian excursion measure is conformally invariant (this can be checked using the conformal covariance of the Poisson excursion kernel; see, e.g., [44, Prop. 5.8]), it is also well-defined in the case when $A_1, A_2 \subset \partial D$ are not smooth. The domain monotonicity of the Poisson kernel (Corollary 2.8) implies that \mathcal{E}_D has the same property: if $U \subset D$ is a simply connected subdomain which coincides with D in neighborhoods of A_1 and A_2 , then $\mathcal{E}_U(A_1, A_2) \leq \mathcal{E}_D(A_1, A_2)$.

Lemma A.2 (Variant of [29, Prop. 3.1]). *Let $\kappa \in (0, 4]$. There exist constants*

$$c'_\kappa \in (0, \infty) \quad \text{such that} \quad \lim_{\kappa \rightarrow 0+} \kappa \log c'_\kappa = C' \in (-\infty, \infty), \tag{A.4}$$

and the following holds. Let D be a simply connected domain and $x, y \in \partial D$ two distinct boundary points. Let γ be a chord from x to y in D , and let β be a chord (with arbitrary endpoints) in D disjoint from γ . Finally, let γ^κ be SLE_κ in $(D; x, y)$. Then

$$\mathbb{P}^\kappa[\gamma^\kappa \cap \beta \neq \emptyset] \leq c'_\kappa \mathcal{E}_D(\beta, \gamma)^{8/\kappa-1}. \tag{A.5}$$

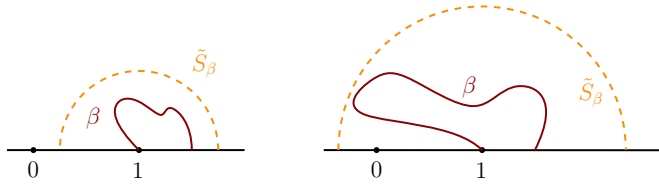


Fig. 3. Illustration of the semicircle \tilde{S}_β and the chord β in the proof of Lemma A.2. The left endpoint of the chord β is at $1 \in \mathbb{R}$. The left figure depicts the case when $\text{diam}(\beta) < \text{dist}(\beta, 0) \leq 1$, and the right figure depicts the case when $\text{diam}(\beta) \geq \text{dist}(\beta, 0)$.

Note that chords are also called crosscuts in the literature.

Proof. Without loss of generality (by conformal invariance and symmetry), we assume that $(D; x, y) = (\mathbb{H}; 0, \infty)$ and β has its endpoints on the positive real axis, with its left endpoint at $1 \in \mathbb{R}$ (see Figure 3). Let $\tilde{S}_\beta = S_{R_\beta} + 1$ be a semicircle of radius $\text{diam}(\beta)$ centered at 1, thus encircling β . Then Theorem A.1 implies that

$$\mathbb{P}^\kappa[\gamma^\kappa \cap \beta \neq \emptyset] \leq \mathbb{P}^\kappa[\gamma^\kappa \cap \tilde{S}_\beta \neq \emptyset] \leq c''_\kappa \left(\min \left\{ \frac{\text{diam}(\beta)}{\text{dist}(\beta, 0)}, 1 \right\} \right)^{8/\kappa-1}, \tag{A.6}$$

where the constant satisfies (because c''_κ is decreasing in $\kappa \in (0, 4]$ and $c''_4 = 2$)

$$c''_\kappa = \frac{\Gamma(12/\kappa)}{\Gamma(8/\kappa)\Gamma(4/\kappa + 1)} \geq 2,$$

and (A.6) holds trivially when $\text{diam}(\beta) \geq \text{dist}(\beta, 0)$. Also, by Stirling’s formula, we have

$$\lim_{\kappa \rightarrow 0^+} \kappa \log c''_\kappa = C'' := 2 \log(3^6/2^4) \approx 7,63817.$$

On the other hand, by [29, Cor. 5.2] there exists a constant $c \in (0, \infty)$ such that

$$\mathcal{E}_\mathbb{H}(\beta, \gamma) \geq \mathcal{E}_\mathbb{H}(\beta, (-\infty, 0)) \geq \frac{1}{c} \min \left\{ \frac{\text{diam}(\beta)}{\text{dist}(\beta, 0)}, 1 \right\}.$$

Combining this with (A.6), we conclude that

$$\begin{aligned} \mathbb{P}^\kappa[\gamma^\kappa \cap \beta \neq \emptyset] &\leq c''_\kappa \left(\min \left\{ \frac{\text{diam}(\beta)}{\text{dist}(\beta, 0)}, 1 \right\} \right)^{8/\kappa-1} \\ &\leq c'_\kappa \mathcal{E}_\mathbb{H}(\beta, \gamma)^{8/\kappa-1}, \end{aligned}$$

where $c'_\kappa := c''_\kappa c^{8/\kappa-1}$, and in particular

$$\lim_{\kappa \rightarrow 0^+} \kappa \log c'_\kappa = C' + 8 \log c =: C' \in (-\infty, \infty),$$

so the constant satisfies the asserted property (A.4). ■

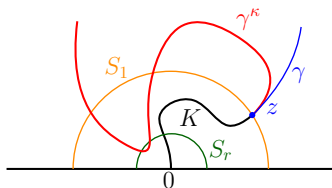


Fig. 4. Illustration of setup in the proof of Proposition A.3. The curve γ^κ is SLE $_\kappa$ in $(\mathbb{H} \setminus K; z, \infty)$, for which we estimate the probability to hit S_r , and γ is an arbitrary simple curve from z to ∞ in \mathbb{H} that intersects S_1 only at z .

Proposition A.3 (Variant of [29, Prop. 3.4]). *Let $\kappa \in (0, 4]$. There exist constants*

$$c_\kappa \in (0, \infty) \quad \text{such that} \quad \lim_{\kappa \rightarrow 0^+} \kappa \log c_\kappa = C \in (-\infty, \infty), \tag{A.7}$$

and the following holds. Let $K \subset \overline{\mathbb{H}}$ be a hull such that $K \cap (\mathbb{H} \setminus \mathbb{D}) = \{z\}$, and let γ^κ be SLE $_\kappa$ in $(\mathbb{H} \setminus K; z, \infty)$. Then, for any $r \in (0, 1/3)$, we have

$$\mathbb{P}^\kappa[\gamma^\kappa \cap S_r \neq \emptyset] \leq c_\kappa r^{8/\kappa-1}. \tag{A.8}$$

The setup is illustrated in Figure 4.

Proof. Write $D = \mathbb{H} \setminus K$ and $D \cap S_r = \bigcup_{j=1}^\infty \beta_j$ as a union of chords in D . Then

$$\begin{aligned} \mathbb{P}^\kappa[\gamma^\kappa \cap S_r \neq \emptyset] &= \mathbb{P}^\kappa\left[\gamma^\kappa \cap \bigcup_{j=1}^\infty \beta_j \neq \emptyset\right] \\ &\leq c'_\kappa \sum_{j=1}^\infty \mathcal{E}_D(\beta_j, \gamma)^{8/\kappa-1} \quad \text{[by Lemma A.2]} \\ &\leq c'_\kappa \left(\sum_{j=1}^\infty \mathcal{E}_D(\beta_j, \gamma)\right)^{8/\kappa-1} \quad \text{[by } L^p\text{-norm monotonicity, since } \kappa \leq 4\text{],} \end{aligned}$$

for any simple curve γ from z to ∞ in \mathbb{H} such that $\gamma \cap S_1 = \{z\}$, where the constant $c'_\kappa \in (0, \infty)$ satisfies (A.4). It remains to estimate the sum of the Brownian excursion measures $\mathcal{E}_D(\beta_j, \gamma)$. To this end, we note that

$$\begin{aligned} \sum_{j=1}^\infty \mathcal{E}_D(\beta_j, \gamma) &\leq \sum_{j=1}^\infty \mathcal{E}_D(\beta_j, S_1) \quad \text{[by monotonicity of } \mathcal{E}\text{]} \\ &\leq 2\mathcal{E}_D(S_r, S_1) \quad \text{[by [29, Lem. 3.3]]} \\ &\leq 2\mathcal{E}_\mathbb{H}(S_r, S_1) \quad \text{[by domain monotonicity of } \mathcal{E}\text{].} \end{aligned}$$

By [44, Ex. 5.10, (5.11)] there exists a constant $c \in (0, \infty)$ such that $\mathcal{E}_\mathbb{H}(S_r, S_1) \leq cr$, since the semi-annulus bounded by $S_r, S_1, [r, 1]$ and $[-1, -r]$ is conformally equivalent to

the rectangle $[0, -\log r] \times [0, \pi]$, where S_r and S_1 are both mapped to a side of length π . Thus, we obtain

$$\mathbb{P}^\kappa[\gamma^\kappa \cap S_r \neq \emptyset] \leq c'_\kappa \left(\sum_{j=1}^\infty \mathcal{E}_D(\beta_j, \gamma) \right)^{8/\kappa-1} \leq c_\kappa r^{8/\kappa-1},$$

where $c_\kappa := c'_\kappa (2c)^{8/\kappa-1}$. The asserted property (A.7) follows from property (A.4). ■

Corollary A.4. *For each $r > 0$ and for any $M \in [0, \infty)$, there exists $R > r$ such that*

$$\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log \mathbb{P}^\kappa[\gamma_{[\tau_R, \infty)}^\kappa \cap S_r \neq \emptyset] \leq -M. \tag{A.9}$$

Proof. Proposition A.3 with $K = \gamma_{[0, \tau_R]}^\kappa$, combined with scale-invariance and the strong Markov property of SLE_κ , gives the estimate

$$\mathbb{P}^\kappa[\gamma_{[\tau_R, \infty)}^\kappa \cap S_r \neq \emptyset] \leq c_\kappa (r/R)^{8/\kappa-1}, \tag{A.10}$$

for any $R > 3r$, where the constant c_κ satisfies (A.7). We furthermore choose R large enough such that

$$\overline{\lim}_{\kappa \rightarrow 0+} \kappa \log c_\kappa + 8 \log(r/R) = C + 8 \log(r/R) \leq -M$$

from (A.7). This proves (A.9). ■

Appendix B. Polyakov–Alvarez conformal anomaly formula

The *Polyakov–Alvarez conformal anomaly formula* gives the change of the determinant of the Laplacian under a Weyl-scaling. The case of closed surfaces goes back to Polyakov [63] and the case of compact surfaces with smooth boundary to Alvarez [5]. Osgood, Phillips & Sarnak [59, Sec. 1] give a straightforward derivation of this formula for compact surfaces without boundary. (We have not found an explicit derivation for the case with boundary, although it is considered well-known.) When the surface has conical singularities, the classical derivation [5, 59, 63] is not directly applicable. One of the difficulties is to derive rigorously the trace of the heat kernel multiplied by the variation of the log-conformal factor (Theorem I).

For curvilinear polygonal domains, it is relatively straightforward to find the appropriate contribution to the heat trace from the corners, as has been observed in many works [4, 8, 28, 36, 54, 73]. However, the rigorous proof of the short-time expansion of the heat kernel including the constant term was only established very recently in [58], which is then used to prove the trace expansion Theorem I. Finally, the Polyakov–Alvarez formula proved in [3] follows as we explain briefly below. We use the notations from Section 6.

Theorem H (Generalized Polyakov–Alvarez conformal anomaly formula [3, Thm. 3]). *Consider a metric $g = e^{2\sigma} g_0$ on a curvilinear polygonal domain D which is Weyl-equivalent to a reference metric g_0 . Then*

$$\begin{aligned} & \log \det_{\zeta} \Delta_0 - \log \det_{\zeta} \Delta_g \\ &= \frac{1}{6\pi} \left[\frac{1}{2} \int_D |\nabla_0 \sigma|^2 \, \text{dvol}_0 + \int_D K_0 \sigma \, \text{dvol}_0 + \int_{\partial D \setminus \{z_1, \dots, z_m\}} k_0 \sigma \, \text{dl}_0 \right] \\ &+ \frac{1}{4\pi} \int_{\partial D \setminus \{z_1, \dots, z_m\}} \partial_{\nu_0} \sigma \, \text{dl}_0 + \frac{1}{12} \sum_{j=1}^m \left(\frac{1}{\beta_j} - \beta_j \right) \sigma(z_j), \end{aligned} \tag{6.3}$$

where ∂_{ν_0} is the outward normal derivative with respect to the metric g_0 , and for notational simplicity, we replace the subscripts “ g_0 ” by “0”.

Recall that $\log \det_{\zeta} \Delta_g = -\zeta'_{\Delta_g}(0)$. We compare $\zeta'_{\Delta_0}(0)$ and $\zeta'_{\Delta_g}(0)$ by a variational computation. For this purpose, we define $g_u := e^{2u\sigma} g_0$ for $u \in [0, 1]$, so $g = g_1$. Theorem I and Proposition B.1 together give the variation $\partial_u \zeta'_{\Delta_{g_u}}(0)$, which is integrated over $u \in [0, 1]$ to prove Theorem H.

The first step of the proof of Theorem H is the following short-time expansion of the trace $\text{Tr}(\sigma e^{-t\Delta_{g_u}})$ of the heat kernel multiplied by the log-conformal factor σ .

Theorem I. *The operator $\sigma e^{-t\Delta_{g_u}}$ is trace class and we have*

$$\text{Tr}(\sigma e^{-t\Delta_{g_u}}) = \frac{a_0(g_u, \sigma)}{t} + \frac{a_1(g_u, \sigma)}{\sqrt{t}} + a_2(g_u, \sigma) + O(t^{1/2} \log t), \quad t \rightarrow 0+, \tag{B.1}$$

where

$$\begin{aligned} a_0(g_u, \sigma) &= \frac{1}{4\pi} \int_D \sigma \, \text{dvol}_{g_u}, \\ a_1(g_u, \sigma) &= -\frac{1}{8\sqrt{\pi t}} \int_{\partial D \setminus \{z_1, \dots, z_m\}} \sigma \, \text{dl}_{g_u}, \\ a_2(g_u, \sigma) &= \frac{1}{12\pi} \int_D \sigma K_{g_u} \, \text{dvol}_{g_u} + \frac{1}{12\pi} \int_{\partial D \setminus \{z_1, \dots, z_m\}} \sigma k_{g_u} \, \text{dl}_{g_u} \\ &+ \frac{1}{8\pi} \int_{\partial D \setminus \{z_1, \dots, z_m\}} \partial_{\nu_u} \sigma \, \text{dl}_{g_u} + \frac{1}{24} \sum_{j=1}^m \left(\frac{1}{\beta_j} - \beta_j \right) \sigma(z_j), \end{aligned}$$

and $\partial_{\nu_u} \sigma$ is the outward normal derivative with respect to the metric g_u .

Proof idea. Because σ is bounded, $\sigma e^{-t\Delta_{g_u}}$ is trace class. To obtain the asserted expansion heuristically, one can approximate the heat kernel in the bulk by the explicit heat kernel of the whole space; near smooth boundary by the explicit heat kernel of the half-space; and near a corner by the Sommerfeld–Carslaw heat kernel formula as in [8, 36], or alternatively, using the Kontorovich–Lebedev transform of the Green’s function as in [28, 73]. On a heuristic level, the so-called *Kac’s locality principle* [36] (i.e., asymptotic

locality of the heat kernel at short times) suggests that the error made in this approximation is negligible. (Intuitively, Brownian motion does not reach far from its starting point in a short time, thus only feels the local geometry of the domain.) After calculating all of these different local contributions, one patches them together to obtain (B.1). However, such a patchwork a priori works rigorously only when the local geometries coincide *exactly*, for instance, for polygonal domains (when the edges are not curved). The rigorous treatment for curvilinear domains is rather subtle and technical, and we refer to the recent literature [3, 4, 58] for this. ■

Proposition B.1. *We have*

$$\partial_u \zeta'_{\Delta_{g_u}}(0) := \partial_\varepsilon \zeta'_{\Delta_{g_u+\varepsilon}}(0)|_{\varepsilon=0} = 2a_2(g_u, \sigma). \tag{B.2}$$

Proof. Using $\Delta_{g_u+\delta u} = e^{-2(\delta u)\sigma} \Delta_{g_u}$, we have

$$\begin{aligned} \delta \operatorname{Tr}(e^{-t\Delta_{g_u}}) &= \operatorname{Tr}(\delta e^{-t\Delta_{g_u}}) = \operatorname{Tr}(-t(\delta\Delta_{g_u})e^{-t\Delta_{g_u}}) = 2t \operatorname{Tr}(\sigma\Delta_{g_u}e^{-t\Delta_{g_u}})\delta u \\ &= -2t \frac{d}{dt} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}})\delta u. \end{aligned}$$

For $\operatorname{Re} s > 1$, the variation of the zeta function is

$$\begin{aligned} \delta \zeta_{\Delta_{g_u}}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \delta \operatorname{Tr}(e^{-t\Delta_{g_u}}) dt = -\frac{2\delta u}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt \\ &= \delta u \frac{2s}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt, \end{aligned}$$

where we have used integration by parts. Splitting the integral into \int_0^1 and \int_1^∞ , we have

$$\frac{\delta \zeta_{\Delta_{g_u}}(s)}{\delta u} = \frac{2s}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt + \frac{2s}{\Gamma(s)} \int_1^\infty t^{s-1} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt, \tag{B.3}$$

where the second term is holomorphic on $\{s \in \mathbb{C} \mid \operatorname{Re} s > -1\}$. Now, using Theorem I, the integral in the first term in (B.3) can be written in the form

$$\int_0^1 t^{s-1} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt = \frac{a_0(g_u, \sigma)}{s-1} + \frac{a_1(g_u, \sigma)}{s-1/2} + \frac{a_2(g_u, \sigma)}{s} + \int_0^1 O(t^{s-1/2}) dt,$$

which shows that

$$\frac{\delta \zeta_{\Delta_{g_u}}(0)}{\delta u} = \frac{2s}{\Gamma(s)} \left(\int_0^1 t^{s-1} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt + \int_1^\infty t^{s-1} \operatorname{Tr}(\sigma e^{-t\Delta_{g_u}}) dt \right) \Big|_{s=0} = 0,$$

because $2s/\Gamma(s) = 2s^2 + O(s^3)$. Therefore, as $|s| \rightarrow 0$, we can expand

$$\begin{aligned} \delta(\Gamma(s)\zeta_{\Delta_{g_u}}(s)) &= \Gamma(s)(\delta\zeta_{\Delta_{g_u}}(0) + s\delta\zeta'_{\Delta_{g_u}}(0) + O(s^2)) \\ &= \Gamma(s)(s\delta\zeta'_{\Delta_{g_u}}(0) + O(s^2)). \end{aligned}$$

Dividing by $s\Gamma(s)$, we obtain

$$\begin{aligned} \delta\zeta'_{\Delta_{g_u}}(0) &= \frac{1}{s\Gamma(s)} \delta(\Gamma(s)\zeta_{\Delta_{g_u}}(s)) \Big|_{s=0} \\ &= \delta u(2s + O(s^2)) \left(\int_0^\infty t^{s-1} \text{Tr}(\sigma e^{-t\Delta_{g_u}}) dt \right) \Big|_{s=0} = 2a_2(g_u, \sigma) \delta u, \end{aligned}$$

which is the sought variation of the derivative of the spectral zeta function. ■

Proof of Theorem H. By Proposition B.1, we have

$$\log \det_\zeta \Delta_0 - \log \det_\zeta \Delta_g = \int_0^1 \partial_u \zeta'_{\Delta_{g_u}}(0) du = 2 \int_0^1 a_2(g_u, \sigma) du,$$

where the coefficient $a_2(g_u, \sigma)$ of interest is given in Theorem I. To write this in different form, we use the following simple transformation rules with $g_u = e^{2u\sigma} g_0$,

$$\begin{aligned} \Delta_{g_u} &= e^{-2u\sigma} \Delta_0, & \text{dvol}_{g_u} &= e^{2u\sigma} \text{dvol}_0, & K_{g_u} &= e^{-2u\sigma} (K_0 + u\Delta_0\sigma), \\ \partial_{v_u}\sigma &= e^{-u\sigma} \partial_{v_0}\sigma, & dl_{g_u} &= e^{u\sigma} dl_0, & k_{g_u} &= e^{-u\sigma} (k_0 + u\partial_{v_0}\sigma), \end{aligned}$$

to obtain

$$\begin{aligned} a_2(g_u, \sigma) &= \frac{1}{12\pi} \left(\int_D \sigma(K_0 + u\Delta_0\sigma) \text{dvol}_0 + \int_{\partial D \setminus \{z_1, \dots, z_m\}} \sigma(k_0 + u\partial_{v_0}\sigma) dl_0 \right) \\ &\quad + \frac{1}{8\pi} \int_{\partial D \setminus \{z_1, \dots, z_m\}} \partial_{v_0}\sigma dl_0 + \frac{1}{24} \sum_{j=1}^m \left(\frac{1}{\beta_j} - \beta_j \right) \sigma(z_j). \end{aligned}$$

We finally obtain the asserted formula after using Stokes' formula

$$\int_D \sigma \Delta_0 \sigma \text{dvol}_0 + \int_{\partial D \setminus \{z_1, \dots, z_m\}} \sigma \partial_{v_0} \sigma dl_0 = \int_D |\nabla_0 \sigma|^2 \text{dvol}_0$$

and integrating over $u \in [0, 1]$. ■

Acknowledgments. We thank Dapeng Zhan and an anonymous referee for pointing out a gap in the application of [29, Thm. 1] in an earlier draft, and Greg Lawler for helpful correspondence on filling this gap in Appendix A. We thank Osama Abuzaid, Alexandre Eremenko, Ellen Krusell, Julie Rowlett, Vincent Vargas, Fredrik Viklund, and Wendelin Werner for useful comments and discussions, and the referee for very careful reading of our manuscript and many constructive comments. Y.W. thanks Wendelin Werner for the suggestion to look into rational functions.

Funding. Y.W. and is funded by the NSF grant DMS-1953945. E.P. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - GZ 2047/1, Projekt-ID 390685813, and by the Academy of Finland grant number 340461 "Conformal invariance in planar random geometry". We would like to thank the Mathematisches Forschungsinstitut Oberwolfach (MFO) for supporting us through the Research in Pairs program, where this project was initiated.

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