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A free discontinuity approach to optimal profiles in Stokes flows

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Abstract. In this paper we study obstacles immersed in a Stokes flow with Navier boundary conditions. We prove the existence and regularity of an obstacle with minimal drag, among all shapes of prescribed volume and controlled surface area, taking into account that these shapes may naturally develop geometric features of codimension 1. The existence is carried out in the framework of free discontinuity problems and leads to a relaxed solution in the space of special functions of bounded deformation (SBD). In dimension two, we prove that the solution is classical.

Contents

1. Introduction

Consider an obstacle $E \subset \mathbb{R}^d$ ($d = 2, 3$ in real applications) contained in a (finite) channel Ω in which a fluid with viscosity coefficient $\mu > 0$ is flowing. Assume that the flow is stationary and incompressible, and that the associated velocity field u is equal to a constant vector V_∞ on the walls of the channel. The obstacle E experiences a force, whose component in the direction of V_∞ will be denoted by $\mathrm{Drag}(E)$ and is usually called the *drag force*. If we further assume that the velocity of the fluid satisfies the Stokes equation

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in $\Omega \setminus E$ and obeys the *Navier boundary conditions* on ∂E, the expression of the drag force turns out to be given (up to a multiplicative constant) by

$$\mathrm{Drag}(E) = 2\mu \int_{\Omega\setminus E} |e(u)|^2\, dx + \beta \int_{\partial E} |u|^2\, d\mathcal{H}^{d-1}, \tag{1.1}$$

where $e(u) := \frac{1}{2}(Du + (Du)^*)$ denotes the symmetrized gradient of u and $\beta > 0$ is the friction coefficient (we refer to Section 3.2 for details).

We are interested in minimizing the drag force among all obstacles E with a prescribed volume and controlled surface area. Precisely, we look for the existence of such an optimal obstacle and for its qualitative properties. The existence question is not very relevant as soon as one imposes strong geometric constraints on the admissible obstacles (e.g. convexity and uniform cone conditions) since this may hide some specific features which would naturally occur. Indeed, letting the geometry of the obstacle be completely free, some qualitative behavior (blocked by rigid geometric constraints) can be observed. This is the case for our problem, where the optimal obstacle (that we prove to exist without imposing any geometric or topological constraint) may be composed, roughly speaking, as the union of a body with the prescribed volume and pieces of surface of dimension $d-1$. These surfaces do not have volume, but count for the total surface area $\mathcal{H}^{d-1}(\partial E)$ and of course have a strong influence on the flow.

Penalizing the surface area and the volume, the model problem we are interested in can be written as

$$\min_E \left\{ \mathrm{Drag}(E) + c\mathcal{H}^{d-1}(\partial E) + f(|E|) \right\},$$

where $c > 0$ and $f\colon (0, |\Omega|) \to \mathbb{R} \cup \{+\infty\}$ is a lower-semicontinuous function. Roughly speaking, the terms involving perimeter and volume can be thought as the price to pay to build the obstacle E, and we can give the two relevant choices of function f:

$$f(m) = +\infty 1_{\{m \neq m_0\}} \text{ for some } m_0 \in (0, |\Omega|), \quad \text{or } f(m) = -\lambda m \text{ for some } \lambda > 0.$$

Many similar optimization problems have been considered under the "no-slip" boundary condition, meaning flows for which $u = 0$ at ∂E. Under volume constraint and an a priori symmetry hypothesis around an axis parallel to the flow, the minimal drag question has been studied in [36] on smooth surfaces. In [31], still under symmetry hypotheses, it was conjectured that the optimal profile in three dimensions is a prolate spheroid with sharp ends of angle 120 degrees. In the same symmetry context, let us also mention the slender body approximation of [35]. We also refer the reader to Šverák [32] who, in two dimensions, proves the existence of an optimal obstacle under topological hypotheses, namely that the obstacle has at most a given number of connected components (in particular this number can be equal to 1). The proof is genuinely two-dimensional and cannot be extended to higher dimensions.

The Navier boundary condition gives many new challenges, namely the possible appearance of lower-dimensional structures in the obstacle that minimize the drag, something which was absent under the no-slip condition. The Navier boundary condition may be seen as a partial adherence to the boundary of the obstacle, and it may be asymptotically obtained as a limit of flows with *perfect slip* on an obstacle with rough boundary. More precisely, a periodic microstructure with the right scaling on the boundary is modeled at the limit by a Navier boundary condition, as was proved in [14]. In dimensions higher than two it is also necessary to take into account more complex geometries for the microstructure, which at the limit produce an anisotropic factor that favors certain directions of the flow. Moreover, infinitesimal boundary perturbations can dramatically modify the solution of the Stokes equation with Navier boundary conditions, while in the presence of no-slip boundary conditions the solution remains stable. We refer the reader to [8] for an analysis of these phenomena and for a discussion of the pertinence of the Navier boundary conditions in physical models.

For a fixed obstacle E, the minimization of the drag with respect to the friction parameter β of the Navier conditions (meaning, from a physical point of view, with respect to the microstructure on the boundary) has been studied in [5], for both Stokes and Navier–Stokes flows. While for Stokes flows the drag is increasing with the friction parameter, an important observation which occurs for the Navier–Stokes equation is that the monotonicity of the drag with respect to the parameter β does not hold. This is a reason for which the results we give in this paper for the Stokes flows are not expected to hold, as such, for the Navier–Stokes equation.

Since the stationary velocity field associated to a Lipschitz obstacle E turns out to be characterized variationally as the minimizer of the right hand side of (1.1) in the class of admissible velocities

$$\mathcal{V}^{\mathrm{reg}}_{E,V_\infty}(\Omega) = \left\{u \in H^1(\Omega \setminus E; \mathbb{R}^d) : \operatorname{div} u = 0,\ u_{|\partial E} \cdot \nu_E = 0,\ u_{|\partial\Omega} = V_\infty\right\}$$

(see (3.4) in Section 3.1 for more details), we can conveniently rephrase the minimization problem by also letting the velocity fields intervene explicitly in the form

$$\min_{E, u \in \mathcal{V}^{\mathrm{reg}}_{E,V_\infty}(\Omega)} \left\{2\mu \int_{\Omega\setminus E} |e(u)|^2\,dx + \beta \int_{\partial E} |u|^2\,d\mathcal{H}^{d-1} + c\,\mathcal{H}^{d-1}(\partial E) + f(|E|)\right\}. \quad (1.2)$$

The first main goal of the paper is to find suitable relaxations of problem (1.2) for which we can prove the existence of minimizers without any a priori constraint on the regularity or the topology of the sets E.

In order to avoid unnatural geometric restrictions on the obstacle E, it is natural in view of the third term appearing in (1.2) to let it vary within the class of sets of *finite perimeter* (see Section 2.2), and replace the topological boundary with a reduced one $\partial^* E$.

In order to properly describe obstacles with very narrow spikes which in the limit degenerate to $(d-1)$-surfaces and that cannot be taken into account through the reduced boundary, it is convenient to consider admissible velocity fields which can be *discontinuous* outside E (see Section 3.3). Since the symmetrized gradient $e(u)$ is involved explicitly in (1.2), a natural family for the admissible velocities is given by the space of *functions* *of bounded deformation* SBD. The natural *relaxation* of the energy takes the form (see Remark 4.11 for further comments)

$$\begin{aligned}\mathcal{J}(E,u) := 2\mu \int_{\Omega\setminus E} |e(u)|^2\,dx + \beta \int_{\partial^* E} |u^+|^2\,d\mathcal{H}^{d-1} \\ + \beta \int_{J_u\setminus\partial^* E} [|u^+|^2 + |u^-|^2]\,d\mathcal{H}^{d-1} \\ + c\mathcal{H}^{d-1}(\partial^* E) + 2c\mathcal{H}^{d-1}(J_u \setminus \partial^* E) + f(|E|),\end{aligned} \tag{1.3}$$

where u is set equal to zero a.e. in E, while J_u denotes the discontinuity set of u and $u^\pm$ are the traces of u on $\partial^* E$ and J_u (the trace u^- vanishes on $\partial^* E$ by the choice of orientation, while u^+ is on the outward side).

Within this framework the *global* obstacle is given by $E \cup J_u$, so that it also contains *lower-dimensional parts*, namely $J_u \setminus \partial^* E$: roughly speaking, for the optimal velocity these discontinuous regions generate $(d-1)$-surfaces which correspond to volumeless, lower-dimensional subsets of the optimal obstacle.

Admissible velocities must be tangent to the obstacles, meaning that not only is u tangent to $\partial^* E$, but also the two traces $u^\pm$ are orthogonal to the normal ν_u along the jump set J_u. The contribution of the *Navier surface term* naturally takes into account the contribution from both sides given by $u^\pm$. Concerning the perimeter term, we count the lower-dimensional parts *twice* because we see the relaxed obstacle as a limit of regular obstacles, such that points of $J_u \setminus \partial^* E$ correspond to thin parts of the regular obstacle that collapse on a lower-dimensional structure. We could also see the perimeter term as the price to pay to construct the obstacle and just keep $\mathcal{H}^{d-1}(\partial^* E \cup J_u)$ instead, and the main results of the paper would not be affected.

The relaxed optimization problem can be seen as a minimization problem on the pairs (E, u) which has the features of classical geometrical problems for E coupled with a *free discontinuity problem* for u, with a surface term depending on the traces which are subject to suitable tangency constraints and boundary conditions.

The first main result of the paper (Theorem 4.8) concerns the existence of minimizers for the relaxed functional $\mathcal{J}$ in (1.3) among the class of admissible configurations (see Definition 4.1 for the precise definition).

The main difficulties we have to face in order to prove that the problem is well posed are the following:

(a) the closure of the *non-penetration* constraint for the velocity on $\partial^* E \cup J_u$ under the natural weak convergence of the problem;

(b) the lower semicontinuity of energies of the form

$$\int_{J_u} [|u^+|^2 + |u^-|^2]\,d\mathcal{H}^{d-1} \tag{1.4}$$

associated to the Navier conditions.

Point (a) is a consequence of a lower-semicontinuity result for the energy

$$\int_{J_u} [|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|] \, d\mathcal{H}^{d-1},$$

which is proved in Theorem 5.2 by resorting to recent lower-semicontinuity results for functionals on SBD by Friedrich, Perugini and Solombrino [28].

The energy of point (b) naturally appears in a scalar setting when dealing with shape optimization problems involving *Robin boundary conditions* (see e.g. [10–13]), and it is easily seen to enjoy lower-semicontinuity properties by working with sections. The lower-semicontinuity result in the vectorial SBD setting is given by Theorem 5.4 and cannot rely on an easy argument by sections, which instead would yield the lower semicontinuity of an energy of the form

$$\int_{J_u} [|u^+ \cdot \xi|^2 + |u^- \cdot \xi|^2] |\xi \cdot \nu_u| \, d\mathcal{H}^{d-1}$$

with $\xi \in \mathbb{R}^d$ with $|\xi| = 1$: the optimization in ξ in order to recover (1.4) does not seem feasible in dimension $d \geq 3$. We thus follow a different strategy based on a blow-up argument in which we reconstruct the vector quantities $u^\pm$ by controlling them in a sufficiently high number of directions (see Section 5.3 for details): in this way we can deal with more general energy densities of the form $\phi(u^+) + \phi(u^-)$, where ϕ is a lower-semicontinuous function.

The second main result of the paper (see Theorem 4.10) concerns the regularity of the relaxed minimizers of (1.3). Provided that the volume penalization function f is Lipschitz and that we are in two dimensions, we prove that for a minimizer (E, u) of $\mathcal{J}$, the optimal obstacle $E \cup J_u$ is a closed set, while the optimal velocity u is a smooth Sobolev function outside the obstacle, recovering somehow the classical setting of the problem. More precisely, we show that

$$\mathcal{H}^1(\Omega \cap \overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0, \tag{1.5}$$

so that the optimal obstacle can be described as the closed set obtained by the complement of the connected components of $\Omega \setminus \overline{\partial^* E \cup J_u}$ on which u does not vanish identically.

The technical ideas to prove (1.5) stem from the pioneering result of De Giorgi, Carriero and Leaci on the Mumford–Shah problem [24, 30], where the key of the proof is a decay estimate obtained by a contradiction/compactness argument. For vectorial problems, a similar strategy, but definitely more involved, was used for the Griffith fracture problem in [19] (for the two-dimensional case) and in [16] (for higher dimensions). In the fracture problem, the key compactness result relies on the possibility of approximating a field $u \in \mathrm{SBD}([-1, 1]^d)$ with a small jump set by a Sobolev function which is locally controlled in H^1 (via the classical Korn inequality).

In our case, we follow a similar approximation procedure, but we have to handle two additional constraints: incompressibility and non-penetration at the jumps. From a technical point of view, this is problematic since the bound in [19] is not strong enough to stay in divergence-free vector fields and the method in [16] creates new jumps on which the non-penetration constraint is not a priori verified. However, when restricted to two dimensions, the method of [16] leads to a stronger result, so that both constraints can be handled.

The paper is organized as follows. In Section 2 we fix the notation and recall some basic facts concerning sets of finite perimeter, functions of bounded deformation and Hausdorff convergence of compact sets. Section 3 is devoted to the precise exposition of the drag optimization problem. In Section 4 we detail the relaxation of the problem in the family of obstacles of finite perimeter and with velocities of bounded deformation, and formulate the main results of the paper concerning the existence of minimizers (in any dimension) and their regularity in dimension two. The proof of the existence of minimizers is given in Section 6, and it is based on some technical results for SBD functions collected in Section 5, while the regularity result is proved in Section 7.

2. Notation and preliminaries

2.1. Basic notation

If $E \subseteq \mathbb{R}^d$, we denote by $|E|$ its d-dimensional Lebesgue measure, and by $\mathcal{H}^{d-1}(E)$ its $(d-1)$-dimensional Hausdorff measure: we refer to [25, Chapter 2] for a precise definition, recalling that for sufficiently regular sets, $\mathcal{H}^{d-1}$ coincides with the usual area measure. Moreover, we denote by E^c the complementary set of E, and by 1_E its characteristic function, i.e. $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ otherwise. In addition, we say that $E_1 \Subset E_2$ if $\overline{E_1} \subset E_2$. Finally, we denote by $Q_{x,r} \subseteq \mathbb{R}^d$ the cube of center x and side r: when $x = 0$, we simply write Q_r.

If $A \subseteq \mathbb{R}^d$ is open and $1 \leq p \leq +\infty$, we denote by $L^p(A)$ the usual space of p-summable functions on A with norm indicated by $\|\cdot\|_p$. We denote by $W^{1,p}(A)$ the Sobolev space of functions in $L^p(A)$ whose gradient in the sense of distributions belongs to $L^p(A;\mathbb{R}^d)$. Finally, given a finite-dimensional unitary space Y, we denote by $\mathcal{M}_b(A;Y)$ the space of Y-valued Radon measures on A, which can be identified with the dual of Y-valued continuous functions on A vanishing at the boundary.

We denote by $M^{d\times m}$ the set of $d \times m$ matrices with values in $\mathbb{R}$: when $d = m$ we denote by $\mathrm{M}^{d\times d}_{\mathrm{sym}}$ the subspace of $d \times d$ symmetric matrices. For $a \in \mathbb{R}^d$ and $b \in \mathbb{R}^m$ we denote by $a \otimes b$ the element of $M^{d\times m}$ such that

$$(a \otimes b)_{ij} = a_i b_j,$$

while if $a, b \in \mathbb{R}^d$ we denote by $a \odot b$ the matrix in $\mathrm{M}^{d\times d}_{\mathrm{sym}}$ such that

$$(a \odot b)_{ij} = \frac{a_i b_j + a_j b_i}{2}.$$

Given $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, we denote by $\xi^\perp$ the hyperplane through the origin orthogonal to ξ. If $E \subseteq \mathbb{R}^d$, we set

$$E^\xi := \pi_{\xi^\perp}(E), \tag{2.1}$$

where π denotes the orthogonal projection, and for $y \in \xi^\perp$ we set

$$E_y^\xi := \{t \in \mathbb{R} : y + t\xi \in E\}.$$

2.2. Functions of bounded variation and sets of finite perimeter

If $\Omega \subseteq \mathbb{R}^d$ is open, we say that $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ if $u \in L^1(\Omega; \mathbb{R}^m)$ and its derivative in the sense of distributions is a finite Radon measure on Ω, i.e. $Du \in \mathcal{M}_b(\Omega; M^{d\times m})$. The space $\mathrm{BV}(\Omega; \mathbb{R}^m)$ is called the space of *functions of bounded variation* on Ω with values in $\mathbb{R}^m$ and it is a Banach space under the norm $\|u\|_{\mathrm{BV}(\Omega;\mathbb{R}^m)} := \|u\|_{L^1(\Omega;\mathbb{R}^m)} + \|Du\|_{\mathcal{M}_b(\Omega;M^{d\times m})}$. We call $|Du|(\Omega) := \|Du\|_{\mathcal{M}_b(\Omega;M^{d\times m})}$ the *total variation* of u. We refer the reader to [2] for an exhaustive treatment of the space BV.

We say that $u \in \mathrm{SBV}(\Omega; \mathbb{R}^m)$ if $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ and its distributional derivative can be written in the form

$$Du = \nabla u\, dx + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{d-1} \lfloor J_u,$$

where $\nabla u \in L^1(\Omega; M^{d\times m})$ denotes the approximate gradient of u, J_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on J_u, and $\nu_u(x)$ is the normal to J_u at x.

Note that if $u \in \mathrm{SBV}(\Omega; \mathbb{R}^m)$, then the singular part of Du is concentrated on J_u which is a countably $\mathcal{H}^{d-1}$-rectifiable set: there exists a set E with $\mathcal{H}^{d-1}(E) = 0$ and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^d$ such that $J_u \subseteq E \cup \bigcup_{i\in\mathbb{N}} M_i$.

We say that $E \subseteq \mathbb{R}^d$, with $|E| < +\infty$, has finite perimeter if $1_E \in \mathrm{BV}(\mathbb{R}^d)$. The perimeter of E is defined as

$$\mathrm{Per}(E) = |D1_E|(\mathbb{R}^d).$$

It turns out that

$$D1_E = \nu_E \mathcal{H}^{d-1} \lfloor \partial^* E, \quad \mathrm{Per}(E) = \mathcal{H}^{d-1}(\partial^* E),$$

where $\partial^* E$ is called the *reduced boundary* of E, and ν_E is the associated inner approximate normal (see [2, Section 3.5]). We have that $\partial^* E \subseteq \partial E$, but the topological boundary can in general be much larger than the reduced one. If $A \subseteq \mathbb{R}^d$ is open and bounded with $\mathcal{H}^{d-1}(A) < +\infty$, then A has finite perimeter with $\mathrm{Per}(A) \le \mathcal{H}^{d-1}(\partial A)$.

2.3. Functions of bounded deformation

If $\Omega \subseteq \mathbb{R}^d$ is open, we say that $u \in \mathrm{BD}(\Omega)$ if $u \in L^1(\Omega; \mathbb{R}^d)$ and its symmetric gradient $Eu := \frac{Du+(Du)^*}{2}$ in the sense of distributions is a finite Radon measure on Ω, i.e. $Eu \in \mathcal{M}_b(\Omega; \mathrm{M}^{d\times d}_{\mathrm{sym}})$. The space $\mathrm{BD}(\Omega)$ is called the space of *functions of bounded deformation* on Ω. We refer the reader to [33, 34] for the main properties of the space BD.

We make use of a subspace of $\mathrm{BD}(\Omega)$ called the space of *special functions of bounded deformation* introduced in [1]. We say that $u \in \mathrm{SBD}(\Omega)$ if $u \in \mathrm{BD}(\Omega)$ and its symmetrized distributional derivative can be written in the form

$$Eu = e(u)\,dx + (u^+ - u^-) \odot \nu_u \mathcal{H}^{d-1} \lfloor J_u,$$

where $e(u) \in L^1(\Omega; \mathrm{M}^{d\times d}_{\mathrm{sym}})$ denotes the approximate symmetrized gradient of u, J_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on J_u, and $\nu_u(x)$ is the normal to J_u at x. As in the case of functions of bounded variation, J_u is an $\mathcal{H}^{d-1}$-countably rectifiable set.

We use the following compactness and lower-semicontinuity result proved in [3].

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and with a Lipschitz boundary, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in* $\mathrm{SBD}(\Omega)$ *such that*

$$\sup_n \left[|Eu_n|(\Omega) + \|u_n\|_{L^1(\Omega;\mathbb{R}^d)} + \|e(u_n)\|_{L^p(\Omega;\mathrm{M}^{d\times d}_{\mathrm{sym}})} + \mathcal{H}^{d-1}(J_{u_n}) \right] < +\infty$$

for some $p > 1$. Then there exists $u \in \mathrm{SBD}(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$\begin{aligned} &u_{n_k} \to u && \text{strongly in } L^1(\Omega;\mathbb{R}^d),\\ &e(u_{n_k}) \rightharpoonup e(u) && \text{weakly in } L^p(\Omega;\mathrm{M}^{d\times d}_{\mathrm{sym}}),\\ &\mathcal{H}^{d-1}(J_u) \le \liminf_{k\to+\infty} \mathcal{H}^{d-1}(J_{u_{n_k}}). \end{aligned}$$

We will need also some properties of the *sections* of SBD-functions. If $\Omega \subseteq \mathbb{R}^d$ is open and $u \in \mathrm{SBD}(\Omega)$, let us consider the scalar function on Ω^ξ_y given by

$$\hat{u}^\xi_y(t) := u(y + t\xi) \cdot \xi$$

and the set

$$J^\xi_u := \left\{ x \in J_u : (u^+(x) - u^-(x)) \cdot \xi \neq 0 \right\}. \tag{2.2}$$

The following result holds true (see [1]).

Theorem 2.2 (One-dimensional sections of SBD-functions). *Let $\Omega \subseteq \mathbb{R}^d$ be open, $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ and let $u \in \mathrm{SBD}(\Omega)$. Then for $\mathcal{H}^{d-1}$-a.e. $y \in \Omega^\xi$ we have*

$$\hat{u}^\xi_y \in \mathrm{SBV}(\Omega^\xi_y),$$

with

$$(\hat{u}^\xi_y)'(t) = (e(u)\xi \cdot \xi)(y + t\xi) \quad \text{for a.e. } t \in \Omega^\xi_y$$

and

$$J_{\hat{u}^\xi_y} = (J^\xi_u)^\xi_y.$$

3. Obstacles in Stokes fluids and drag minimization

In this section we explain the drag problem for an obstacle immersed in a stationary flow.

3.1. The flow around the obstacle

Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary, and let $V \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ be a divergence-free vector field. Given $E \Subset \Omega$ open and with a Lipschitz boundary, let us consider the stationary flow for a viscous incompressible fluid around E with boundary conditions on $\partial\Omega$ given by V, and with Navier boundary conditions on ∂E. More precisely, if $u\colon \Omega \setminus E \to \mathbb{R}^d$ is the velocity field, we require that the following items hold true:

(a) *Incompressibility:* $\operatorname{div} u = 0$ in $\Omega \setminus E$.

(b) *Boundary conditions:* we have

$$u = V \text{ on } \partial\Omega \quad \text{and} \quad \text{the non-penetration condition } u \cdot \nu = 0 \text{ on } \partial E,$$

where ν denotes the exterior normal to E.

(c) *Equilibrium:* considering the stress

$$\sigma := -pI_d + 2\mu e(u), \tag{3.1}$$

where $\mu > 0$ is a viscosity parameter, $e(u)$ is the symmetrized gradient of u (also denoted by $D(u)$) and p is the pressure, we require

$$\operatorname{div} \sigma = 0 \quad \text{in } \Omega \setminus E. \tag{3.2}$$

(d) *Navier conditions on the obstacle:* we have

$$(\sigma\nu)_\tau = \beta u \quad \text{on } \partial E,$$

where $\beta > 0$ is a friction parameter, and $(\sigma\nu)_\tau$ denotes the tangential component of force $\sigma\nu$.

Stationary flow has the following variational characterization: u is the minimizer of the energy

$$\mathcal{E}(u) := 2\mu \int_{\Omega\setminus E} |e(u)|^2 \, dx + \beta \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1} \tag{3.3}$$

among the class of (sufficiently regular) admissible fields

$$\mathcal{V}^{\text{reg}}_{E,V}(\Omega) := \big\{ v \in H^1(\Omega \setminus E;\mathbb{R}^d) : v \text{ satisfies points (a) and (b)} \big\}, \tag{3.4}$$

where $\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure, which reduces to the area measure on sufficiently regular sets. Indeed, if u is a minimizer, and φ is an admissible variation, so that $\varphi = 0$ on $\partial\Omega$, we get

$$\begin{aligned} 0 &= 2\mu \int_{\Omega\setminus E} e(u) : e(\varphi)\, dx + \beta \int_{\partial E} u\cdot\varphi\, d\mathcal{H}^{d-1} \\ &= 2\mu \int_{\Omega\setminus E} e(u) : \nabla\varphi\, dx + \beta \int_{\partial E} u\cdot\varphi\, d\mathcal{H}^{d-1} \\ &= -2\mu \int_{\Omega\setminus E} \operatorname{div} e(u)\cdot\varphi\, dx + \int_{\partial E} [-2\mu e(u)\nu + \beta u]\cdot\varphi\, d\mathcal{H}^{d-1}. \end{aligned}$$

In particular, choosing φ with compact support in $\Omega\setminus E$ we have

$$2\mu \operatorname{div} e(u) = \nabla p$$

for some pressure field p: as a consequence $\sigma := -pI_d + 2\mu e(u)$ satisfies (3.2) of condition (c).

Since the admissible functions φ are tangent to ∂E, the optimality condition reduces to

$$0 = \int_{\partial E} [-2\mu e(u)\nu + \beta u]\cdot\varphi\, d\mathcal{H}^{d-1} = \int_{\partial E} [-\sigma\nu + \beta u]\cdot\varphi\, d\mathcal{H}^{d-1}. \tag{3.5}$$

Notice that every tangential vector field η on ∂E can be extended to a divergence-free vector field on $\Omega\setminus E$ which vanishes on $\partial\Omega$, hence it is the trace of an admissible variation φ: indeed, any extension W which vanishes on $\partial\Omega$ has a divergence with zero mean, so that considering W_1 with $\operatorname{div} W_1 = \operatorname{div} W$ with $W_1 = 0$ on $\partial\Omega$ and on ∂E (whose existence is guaranteed, for example by [6, Theorem IV.3.1])), the required extension is given by $W - W_1$. We conclude that the optimality condition (3.5) yields the Navier condition of point (b).

3.2. The drag force

Assume now that the external vector field V is equal to a constant $V_\infty \in \mathbb{R}^d\setminus\{0\}$, i.e. the obstacle E is immersed in a uniform flow. The flow is perturbed near E, assuming the value u, and the obstacle experiences a force which has a component in the direction V_∞ which is given by

$$\operatorname{Drag}(E) := \int_{\partial E} \sigma\nu\cdot\frac{V_\infty}{|V_\infty|}\, d\mathcal{H}^{d-1},$$

which is called the *drag force* on E in the direction of the flow.

We claim that

$$\operatorname{Drag}(E) = \frac{1}{|V_\infty|}\mathcal{E}(u), \tag{3.6}$$

where $\mathcal{E}(u)$ is the energy defined in (3.3). Using the facts that σ is symmetric and with zero divergence (so that also the vector field σV_∞ is divergence-free), and that $u = V_\infty$

on $\partial\Omega$, we may write

$$\begin{aligned}\int_{\partial E} \sigma\nu \cdot V_\infty \, d\mathcal{H}^{d-1} &= \int_{\partial E} \sigma V_\infty \cdot \nu \, d\mathcal{H}^{d-1} = \int_{\partial\Omega} \sigma V_\infty \cdot \nu \, d\mathcal{H}^{d-1} \\ &= \int_{\partial\Omega} \sigma u \cdot \nu \, d\mathcal{H}^{d-1} \\ &= \int_{\Omega\setminus E} \operatorname{div}(\sigma u) \, dx + \int_{\partial E} \sigma u \cdot \nu \, d\mathcal{H}^{d-1} \\ &= \int_{\Omega\setminus E} \sigma : \nabla u \, dx + \int_{\partial E} \sigma\nu \cdot u \, d\mathcal{H}^{d-1}. \end{aligned} \tag{3.7}$$

Using again that σ is symmetric and that u is divergence-free, together with the constitutive equation (3.1), we have

$$\begin{aligned}\int_{\Omega\setminus E} \sigma : \nabla u \, dx &= \int_{\Omega\setminus E} \sigma : e(u) \, dx = \int_{\Omega\setminus E} (-pI_d + 2\mu e(u)) : e(u) \, dx \\ &= \int_{\Omega\setminus E} (-p \operatorname{div} u + 2\mu |e(u)|^2) \, dx = 2\mu \int_{\Omega\setminus E} |e(u)|^2 \, dx,\end{aligned}$$

while in view of the Navier conditions on ∂E and the fact that u is tangent to the obstacle,

$$\int_{\partial E} \sigma\nu \cdot u \, d\mathcal{H}^{d-1} = \int_{\partial E} (\sigma\nu)_\tau \cdot u \, d\mathcal{H}^{d-1} = \beta \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1}.$$

Inserting into (3.7), we get that (3.6) follows.

3.3. The optimization problem

Let $c > 0$ and let $f : (0, |\Omega|) \to \mathbb{R} \cup \{+\infty\}$ be a lower-semicontinuous function that is not identically equal to $+\infty$. We are concerned with the following optimization problem:

$$\min_E \bigl\{ \mathrm{Drag}(E) + c\mathcal{H}^{d-1}(\partial E) + f(|E|) \bigr\}.$$

We are thus interested in finding the optimal shape of an obstacle which minimizes the drag force, under a penalization involving its perimeter and its volume.

In view of the energetic characterization of the drag force established in Section 3.2, we can formulate the problem as a minimization problem among the pairs (E, u), where u is a velocity field belonging to the family $\mathcal{V}^{\mathrm{reg}}_{E,V_\infty}(\Omega)$ defined in (3.4):

$$\min_{E, u \in \mathcal{V}^{\mathrm{reg}}_{E,V_\infty}(\Omega)} \left\{ \frac{2\mu}{|V_\infty|} \int_{\Omega\setminus E} |e(u)|^2 \, dx + \frac{\beta}{|V_\infty|} \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1} + c\mathcal{H}^{d-1}(\partial E) + f(|E|) \right\}.$$

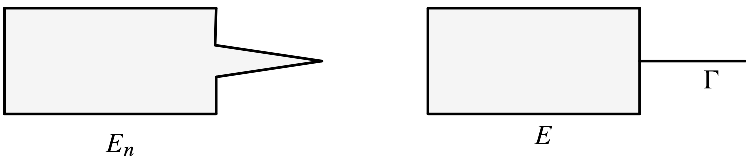
Setting all the constants equal to 1, and replacing V_∞ by a given divergence-free velocity vector field V as in Section 3.1, the drag minimization problem above is a particular case of the following shape optimization problem:

$$\min_{E, u \in \mathcal{V}^{\mathrm{reg}}_{E,V}(\Omega)} \left\{ \int_{\Omega\setminus E} |e(u)|^2 \, dx + \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}(\partial E) + f(|E|) \right\}. \tag{3.8}$$

If we want to apply the direct method of the calculus of variations to the problem, i.e. if we want to recover a minimizer by looking at minimizing sequences $(E_n, u_n)_{n\in\mathbb{N}}$, the following considerations are quite natural:

(a) Since the problem involves the perimeter of E, the sequence $(E_n)_{n\in\mathbb{N}}$ is relatively compact in the family of *sets of finite perimeter* (see Section 2).

(b) Concerning the velocities, it turns out naturally that it is convenient to also consider *discontinuous* vector fields. Indeed, if $u_n \to u$ in some sense, and ∂E_n collapses in some parts generating a surface Γ *outside* the limit set E, the limit velocity field u can present, in general, discontinuities across Γ.



We thus expect an extra term in the surface integral related to the Navier conditions, which amounts at least to

$$\int_{\Gamma\setminus\partial E} [|u^+|^2 + |u^-|^2]\, d\mathcal{H}^{d-1},$$

where $u^\pm$ are the two traces from both sides of Γ.

The previous considerations yield a relaxed version of problem (3.8) in which E varies among the family of sets of finite perimeter contained in Ω, while the family of associated admissible velocity fields u is naturally contained in the space of *special functions of bounded deformation* $\mathrm{SBD}(\Omega)$ (see Section 2).

In Section 4 we will give a precise formulation of the problem in this weak setting, which guarantees existence of optimal solutions, describing in particular how the boundary conditions on $\partial\Omega$ and on the obstacle have to be rephrased in this context.

4. A relaxed formulation of the shape optimization problem and statements of the main results

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and with a Lipschitz boundary, and let $V \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ be a divergence-free vector field. In order to deal conveniently with the boundary conditions, let us consider $\Omega' \subseteq \mathbb{R}^d$ open and bounded such that $\Omega \Subset \Omega'$.

The following definition deals with the family of admissible configurations in the relaxed setting.

Definition 4.1 (The class $\mathcal{A}(V)$ of admissible obstacle-velocity configurations). We say that (E, u) is an admissible configuration for the external velocity V, and we write $(E, u) \in \mathcal{A}(V)$, if $E \subseteq \Omega$ is a set of finite perimeter, while

$$u \in \mathrm{SBD}(\Omega') \cap L^2(\Omega'; \mathbb{R}^d)$$

is such that $u = 0$ a.e. on E and the following conditions are satisfied:

(a) The flow is divergence-free: $\operatorname{div} u = 0$ in the sense of distributions in Ω'.

(b) External boundary conditions: $u = V$ a.e. on $\Omega' \setminus \bar{\Omega}$.

(c) Non-penetration condition on the obstacle:

$$u^{\pm} \cdot \nu = 0 \quad \text{on } \partial^* E \cup J_u,$$

where ν denotes the normal to the rectifiable set $\partial^* E \cup J_u$.

Remark 4.2. The crucial difference between admissible velocities in the present framework and those of the family $\mathcal{V}^{\mathrm{reg}}_{E,V}(\Omega)$ introduced before (see (3.4)) is that they may have discontinuities outside E. Within the new setting, the global obstacle is given by

$$E \cup J_u,$$

i.e. it may contain $(d-1)$-dimensional parts.

Given $(E, u) \in \mathcal{A}(V)$, concerning the traces of u on $\partial^* E$, we denote by u^+ the trace in the direction of the external normal ν_E, so that $u^- = 0$ $\mathcal{H}^{d-1}$-a.e. on $\partial^* E$.

Concerning the non-penetration constraint, notice that it suffices to require it only on J_u, since it is then automatically verified also on $\partial^* E$. Indeed, for $\mathcal{H}^{d-1}$-a.e. $x \in \partial^* E \setminus J_u$, we have $u^-(x) = u^+(x) = 0$ and the constraint is verified, while for $\mathcal{H}^{d-1}$-a.e. $x \in J_u \cap \partial^* E$, the two rectifiable sets J_u and $\partial^* E$ share the same normal vector.

Remark 4.3. The space $\mathrm{SBD}(\Omega')$ is naturally a subspace of $L^1(\Omega'; \mathbb{R}^d)$: we require for admissibility that $u \in L^2(\Omega'; \mathbb{R}^d)$ to ensure that the velocity field has finite kinetic energy. It will turn out that velocities in $\mathrm{SBD}(\Omega')$ which are interesting for our problem (i.e. with finite energy) are automatically elements of $L^2(\Omega'; \mathbb{R}^d)$ (see Theorem 5.1).

Remark 4.4 (On the boundary condition). If $(E, u) \in \mathcal{A}(V)$, then $u \in \mathrm{SBD}(\Omega')$ with $u = V$ a.e. on $\Omega' \setminus \bar{\Omega}$, so that

$$J_u \cap \partial\Omega = \big\{x \in \partial\Omega : \gamma(u)(x) \neq V(x)\big\},$$

where $\gamma(u)$ is the trace of u on $\partial\Omega$ coming from Ω (i.e. the usual trace of u seen as an element of $\mathrm{SBD}(\Omega)$). We conclude that within the present framework, the boundary condition is somehow relaxed: a possible mismatch between u and V on $\partial\Omega$ is admitted, but then the zone is counted as a jump part of the velocity field, and consequently as a part of the obstacle $\partial^* E \cup J_u$, and will carry a contribution for the energy (see (4.2) below). Such a relaxation of the boundary condition is a feature which is common to several applications of functions of bounded variation to problems in continuum mechanics (see for example [23, 27] in connection with fracture mechanics or [22] for problems in plasticity).

Remark 4.5. Given $(E,u) \in \mathcal{A}(V)$, the obstacle $E \cup J_u$ may touch $\partial\Omega$ only on those parts where V is tangent to Ω: this is due to the fact that on $(\partial^* E \cup J_u) \cap \partial\Omega$, the two sets share $\mathcal{H}^{d-1}$-a.e. the same normal, and $u^+ = V$ (if the orientation is suitably chosen).

Remark 4.6. Let $E \Subset \Omega$ be open and with a Lipschitz boundary. Then we can find $W \in H^1(\Omega \setminus E; \mathbb{R}^d)$ such that $W = V$ on $\partial\Omega$, $W = 0$ on ∂E and $\operatorname{div} W = 0$. Indeed, if $\varphi \in C^\infty(\mathbb{R}^d)$ is such that $\varphi = 1$ on a neighborhood of $\mathbb{R}^d \setminus \Omega$ and $\varphi = 0$ on a neighborhood of E, we can consider the vector field $V_1 := \varphi V$, whose divergence has zero mean on $\Omega \setminus E$ (by the Gauss theorem). Then we can find $V_2 \in H^1_0(\Omega \setminus E; \mathbb{R}^d)$ such that $\operatorname{div} V = \operatorname{div} V_1$ (see [6, Theorem IV.3.1]), so that the field $W := V_1 - V_2$ is an admissible choice. In particular, we get that $(E, W) \in \mathcal{A}(V)$, so that the class of admissible configurations is not empty.

Let

$$f : [0, |\Omega|] \to \mathbb{R} \cup \{+\infty\} \text{ be lower semicontinuous, not identically equal to } +\infty. \tag{4.1}$$

For every $(E,u) \in \mathcal{A}(V)$, let us set (normalizing to 1 the constants involved in the drag force problem)

$$\begin{aligned} \mathcal{J}(E,u) := & \int_{\Omega'} |e(u)|^2 \, dx + \int_{\partial^* E} |u^+|^2 \, d\mathcal{H}^{d-1} \\ & + \int_{J_u \setminus \partial^* E} [|u^+|^2 + |u^-|^2] \, d\mathcal{H}^{d-1} \\ & + \mathcal{H}^{d-1}(\partial^* E) + 2\mathcal{H}^{d-1}(J_u \setminus \partial^* E) + f(|E|). \end{aligned} \tag{4.2}$$

Remark 4.7. Concerning the volume integral in $\mathcal{J}(E,u)$, the density $e(u)$ is equal to $e(V)$ a.e. on $\Omega' \setminus \bar{\Omega}$ and equal to 0 a.e. on E: as a consequence we could replace it with an integral on $\Omega \setminus E$ without affecting the minimization of $\mathcal{J}$.

Concerning the *Navier energy* and the surface penalization for $\partial^* E \cup J_u$, notice that it counts also for the possible mismatch at the boundary between u and V as pointed out in Remark 4.4: the mismatch is thus "penalized" by the energy of the problem.

The previous observations show that the larger domain Ω' plays only an instrumental role in the problem, as it can be replaced by any open domain strictly containing Ω.

The first main result of the paper is the following.

Theorem 4.8 (Existence of optimal obstacles). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, $V \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ a divergence-free vector field, and f a function satisfying (4.1). Let the family of admissible configurations $\mathcal{A}(V)$ be given by Definition 4.1 and let $\mathcal{J}$ be the functional defined in (4.2). Then the problem*

$$\min_{(E,u) \in \mathcal{A}(V)} \mathcal{J}(E,u) \tag{4.3}$$

admits a solution.

Remark 4.9. We recover the original drag minimization problem when V is a constant non-zero vector V_∞, and in the functional we properly restore the physical constants μ and β, together with the perimeter penalization constant c.

The second main result of the paper concerns the regularity of minimizers in the two-dimensional setting.

Theorem 4.10 (Regularity in dimension two). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, $V \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ a divergence-free vector field and $f\colon [0, |\Omega|] \to [0,+\infty[$ a Lipschitz function. Let $(E,u) \in \mathcal{A}(V)$ be a solution to* (4.3) *according to Theorem* 4.8*. Then*

$$\mathcal{H}^1\big(\Omega \cap (\overline{J_u \cup \partial^* E} \setminus (J_u \cup \partial^* E))\big) = 0,$$

and $u \in C^\infty(\Omega \setminus \overline{J_u \cup \partial^* E};\mathbb{R}^2)$.

Theorem 4.8 will be proved in Section 6, on the basis of some technical results established in Section 5. The proof of Theorem 4.10 will be addressed in Section 7.

Remark 4.11. In order to prove that the functional $\mathcal{J}(E,u)$ is the relaxation of the energy appearing in the original problem (3.8) in the sense of *the lower-semicontinuous envelope of the calculus of variations*, we need to approximate in energy any $(E,u) \in \mathcal{A}(V)$ through "regular" configurations $(E_n,u_n) \in \mathcal{A}(V)$, where E_n has Lipschitz boundary and $u_n \in H^1(\Omega \setminus E_n;\mathbb{R}^d)$. This resembles the situation studied in [7], which can be extended to the case of energies involving only the symmetrized gradient, like in the study of material voids in linearly elastic materials (in this direction, see for example [20]). However, the constraints of our problem make the analysis very hard to carry out: more specifically, admissibility requires the divergence-free condition $\operatorname{div} u_n = 0$ and the tangency constraint $u_n \perp \partial E_n$, and it is not clear how to enforce them within the currently available approximation procedures.

5. Some technical results in SBD

In this section we collect some technical properties concerning the space SBD that will be fundamental in the proof of Theorem 4.8. In particular, in Theorem 5.1 we will prove that admissible velocity vector fields enjoy higher summability properties (indeed they belong to $L^{\frac{2d}{d-1}}$). In Theorem 5.3 we will prove that velocity fields u with $u^\pm$ tangent to the discontinuity set J_u form a closed set under the natural convergence of minimizing sequences for the main optimization problem. Finally, in Theorem 5.4 we will prove a lower-semicontinuity result for surface energies depending on the traces, which entails in particular the lower semicontinuity of the term associated to the Navier conditions.

5.1. An immersion result

The following embedding result holds true.

Theorem 5.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $u \in \mathrm{SBD}(\mathbb{R}^d)$ be supported in Ω such that*

$$\mathcal{E}(u) := \int_\Omega |e(u)|^2\,dx + \int_{J_u} [|u^+|^2 + |u^-|^2]\,d\mathcal{H}^{d-1} < +\infty.$$

Then $u \in L^{\frac{2d}{d-1}}(\Omega)$ with

$$\|u\|_{\frac{2d}{d-1}} \le C\sqrt{\mathcal{E}(u)},$$

where C depends on d and $\mathrm{diam}(\Omega)$ *only.*

Proof. It suffices to follow the strategy of the proof of the classical embedding of BD into $L^{d/d-1}$ explained in [33], but concentrating on the square of the components.

Let us consider the unit vector

$$\xi := \frac{1}{\sqrt{d}}(1,1,\dots,1) \in \mathbb{R}^d.$$

Employing the characterization by sections recalled in Section 2, for $\mathcal{H}^{d-1}$-a.e. $y \in \xi^\perp$ we have

$$\hat{u}^\xi_y \in \mathrm{SBV}(\Omega^\xi_y)$$

with

$$\int_{\Omega^\xi_y} |(\hat{u}^\xi_y)'|^2\,dt + \sum_{t\in J_{\hat{u}^\xi_y}} [|(\hat{u}^\xi_y)^+(t)|^2 + |(\hat{u}^\xi_y)^-(t)|^2] < +\infty.$$

Then we can write for a.e. $t \in \mathbb{R}$,

$$\begin{aligned}
\|\hat{u}^\xi_y\|^2_{L^\infty(\Omega^\xi_y)} &\le |D(\hat{u}^\xi_y)^2|(\Omega^\xi_y)\\
&= \int_{\Omega^\xi_y} 2|\hat{u}^\xi_y(\hat{u}^\xi_y)'|\,dt + \sum_{t\in J_{\hat{u}^\xi_y}} \left||(\hat{u}^\xi_y)^+(t)|^2 - |(\hat{u}^\xi_y)^-(t)|^2\right|\\
&\le \frac{1}{2}\|\hat{u}^\xi_y\|^2_{L^\infty(\Omega^\xi_y)} + 2|\Omega^\xi_y|\int_{\Omega^\xi_y} |(\hat{u}^\xi_y)'|^2\,dt\\
&\quad + \sum_{t\in J_{\hat{u}^\xi_y}} \left(|(\hat{u}^\xi_y)^+(t)|^2 + |(\hat{u}^\xi_y)^-(t)|^2\right).
\end{aligned} \tag{5.1}$$

Let us set

$$g_\xi(x) := \int_{\Omega^\xi_y} |(\hat{u}^\xi_y)'|^2\,dt + \sum_{t\in J_{\hat{u}^\xi_y}} [|(\hat{u}^\xi_y)^+(t)|^2 + |(\hat{u}^\xi_y)^-(t)|^2],$$

where $y := \pi_{\xi^\perp}(x)$, i.e. the projection of x on the hyperplane $\xi^\perp$, and $g_\xi(x)$ only depends on the projection of x on $\xi^\perp$ and

$$\begin{aligned}
\int_{\xi^\perp} g_\xi\,d\mathcal{H}^{d-1} &= \int_\Omega |e(u)\xi\cdot\xi|^2\,dx + \int_{J_u} [|u^+|^2 + |u^-|^2]|\xi\cdot\nu|\,d\mathcal{H}^{d-1}\\
&\le C\left[\int_\Omega |e(u)|^2\,dx + \int_{J_u} [|u^+|^2 + |u^-|^2]\,d\mathcal{H}^{d-1}\right]
\end{aligned}$$

where C depends only on d. Thanks to (5.1) we have

$$|\xi \cdot u|^2 \leq C g_\xi \quad \text{a.e. on } \Omega, \tag{5.2}$$

where C depends on the diameter of Ω, and from now on all the constants C that appear depend on n, $\mathrm{diam}(\Omega)$. For every $k = 1, \dots, d-1$, we can write

$$\xi = \frac{1}{\sqrt{d}} e_k + \sqrt{\frac{d-1}{d}} h_k,$$

where e_k is the kth vector of the canonical base, and h_k is the unit vector in the direction $\sqrt{d}\xi - e_k$. Reasoning as above on the decomposition

$$\xi \cdot u = \sqrt{\frac{d-1}{d}} h_k \cdot u + \frac{1}{\sqrt{d}} e_k \cdot u,$$

we obtain a similar estimate

$$|\xi \cdot u|^2 \leq C(g_{h_k} + g_{e_k}). \tag{5.3}$$

Multiplying inequality (5.2) with inequalities (5.3) for $k = 1, \dots, d-1$, we obtain, reasoning as in [33, Chapter II, Theorem 1.2],

$$\|(\xi \cdot u)^2\|_{\frac{d}{d-1}} \leq C\left[\int_\Omega |e(u)|^2\, dx + \int_{J_u} [|u^+|^2 + |u^-|^2]\, d\mathcal{H}^{d-1}\right].$$

Since this estimate does not depend on the particular choice of the basis and hence holds for any ξ with norm 1, the theorem is proved. ∎

5.2. Closure of the non-penetration constraint

In the context of equi-Lipschitz boundaries, the preservation of the non-penetration property for a sequence of Sobolev functions converging weakly, comes rather directly via the divergence theorem (we refer the reader, for instance, to [8]). However, in the case of collapsing boundaries, so that the limit function lives on both sides of a surface and in the absence of any smoothness of the limit set, this technique does not work. The proof of the non-penetration preservation requires different technical arguments that we handle in the SBD context.

Let us start with the following lower-semicontinuity result.

Theorem 5.2. *Let* $\Omega \subseteq \mathbb{R}^d$ *be a bounded open set, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{SBD}(\Omega)$ *such that*

$$\sup_n \left[\int_\Omega |e(u_n)|^2\, dx + \mathcal{H}^{d-1}(J_{u_n})\right] < +\infty$$

with

$$u_n \to u \quad \textit{in measure}$$

for some $u \in \mathrm{SBD}(\Omega)$. *Then*

$$\int_{J_u} [|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|] \, d\mathcal{H}^{d-1} \le \liminf_{n\to+\infty} \int_{J_{u_n}} [|u_n^+ \cdot \nu_{u_n}| + |u^- \cdot \nu_{u_n}|] \, d\mathcal{H}^{d-1}.$$

Proof. Let us consider a countable set of functions $\{\varphi_h : h \in \mathbb{N}\}$ which is dense with respect to $\|\cdot\|_\infty$ inside the set

$$\left\{ f \in C_c^0(]0,+\infty[) : \int_0^{+\infty} f \, \mathrm{d}t = 0 \text{ and } \|f\|_\infty \le 1 \right\}.$$

Given $\varepsilon > 0$, let us consider

$$g_{h,k}(x) := \int_0^{\frac{1}{2}|x-x_k|^2} \varphi_h(t)\, dt,$$

where $\{x_k : k \in \mathbb{N}\}$ is a countable and dense set in $B_\varepsilon(0) \subset \mathbb{R}^d$ with $x_0 = 0$. Clearly, $g_{h,k} \in C_c^1(\mathbb{R}^d)$ with

$$G_{h,k}(x) := \nabla g_{h,k}(x) = \varphi_h\left(\frac{1}{2}|x-x_k|^2\right)(x - x_k).$$

We have that $G_{h,k}$ is a continuous conservative vector field with compact support on $\mathbb{R}^d$.

Let us set for $(i, j) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\nu \in \mathbb{R}^d$ with $|\nu| = 1$,

$$f_\varepsilon(i, j, \nu) := \sup_{h,k} (G_{h,k}(i) - G_{h,k}(j)) \cdot \nu.$$

By construction, f_ε is a *symmetric jointly convex function* according to [28, Definition 3.1]. We claim that for $i \ne j$,

$$|i \cdot \nu| + |j \cdot \nu| \le f_\varepsilon(i, j, \nu) \le |i \cdot \nu| + |j \cdot \nu| + 2\varepsilon. \tag{5.4}$$

In view of the lower-semicontinuity result [28, Theorem 5.1] we have

$$\liminf_{n\to+\infty} \int_{J_{u_n}} f_\varepsilon(u_n^+, u_n^-, \nu_{u_n}) \, d\mathcal{H}^{d-1} \ge \int_{J_u} f_\varepsilon(u^+, u^-, \nu_u) \, d\mathcal{H}^{d-1}.$$

We can thus write

$$\begin{aligned}
&\liminf_{n\to+\infty} \left[\int_{J_{u_n}} [|u_n^+ \cdot \nu_{u_n}| + |u_n^- \cdot \nu_{u_n}|] \, d\mathcal{H}^{d-1} + 2\varepsilon \mathcal{H}^{d-1}(J_{u_n}) \right] \\
&\quad \ge \liminf_{n\to+\infty} \int_{J_{u_n}} f_\varepsilon(u_n^+, u_n^-, \nu_{u_n}) \, d\mathcal{H}^{d-1} \ge \int_{J_u} f_\varepsilon(u^+, u^-, \nu_u) \, d\mathcal{H}^{d-1} \\
&\quad \ge \int_{J_u} [|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|] \, d\mathcal{H}^{d-1},
\end{aligned}$$

so that the result follows taking into account the bound on $\mathcal{H}^{d-1}(J_{u_n})$ and letting $\varepsilon \to 0$.

In order to complete the proof, we need to show claim (5.4). The estimate from above follows from

$$[G_{h,k}(i) - G_{h,k}(j)] \cdot \nu \leq |(i - x_k) \cdot \nu| + |(j - x_k) \cdot \nu| \leq |i \cdot \nu| + |j \cdot \nu| + 2\varepsilon$$

since $\|\varphi_h\|_\infty \leq 1$ and $|x_k| < \varepsilon$. Let us prove the estimate from below. We select $x_{k_n} \to 0$ such that $|i - x_{k_n}| \neq |j - x_{k_n}|$ (which is always possible in view of the density of $\{x_k : k \in \mathbb{N}\}$ inside $B_\varepsilon(0)$ and since $i \neq j$) and then φ_{h_n} such that for $n \to +\infty$,

$$\varphi_{h_n}\Big(\frac{1}{2}|i - x_{k_n}|^2\Big) \to \frac{i \cdot \nu}{|i \cdot \nu| + \eta} \quad \text{and} \quad \varphi_{h_n}\Big(\frac{1}{2}|j - x_{k_n}|^2\Big) \to -\frac{j \cdot \nu}{|j \cdot \nu| + \eta},$$

where $\eta > 0$. By definition of f_ε we infer that

$$f_\varepsilon(i, j, \nu) \geq |i \cdot \nu| + |j \cdot \nu| - 2\eta,$$

so that the estimate from below follows by sending $\eta \to 0$. ■

We are now in a position to prove the main result of the section.

Theorem 5.3 (Closure of the non-penetration constraint on the jump set). *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in* SBD(Ω) *such that*

$$\sup_n \Big[\int_\Omega |e(u_n)|^2\, dx + \mathcal{H}^{d-1}(J_{u_n})\Big] < +\infty$$

and

$$u_n \to u \quad \textit{in measure}$$

for some $u \in$ SBD(Ω). *If*

$$u_n^\pm \cdot \nu_{u_n} = 0 \quad \mathcal{H}^{d-1}\textit{-a.e. on } J_{u_n},$$

then

$$u^\pm \cdot \nu_u = 0 \quad \mathcal{H}^{d-1}\textit{-a.e. on } J_u.$$

Proof. By Theorem 5.2 we may write

$$\int_{J_u} [|u^+ \cdot \nu_u| + |u^- \cdot \nu_u|]\, d\mathcal{H}^{d-1} \leq \liminf_{n\to+\infty} \int_{J_{u_n}} [|u_n^+ \cdot \nu_{u_n}| + |u^- \cdot \nu_{u_n}|]\, d\mathcal{H}^{d-1} = 0,$$

so that the result follows. ■

5.3. A lower-semicontinuity result for surface energies in SBD

In this section we deal with the lower semicontinuity of the surface term of the functional J in (4.2) connected with the Navier conditions on the obstacle. The following lower-semicontinuity result holds true.

Theorem 5.4. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $u_n, u \in \mathrm{SBD}(\Omega)$ such that*

$$u_n \to u \quad \textit{strongly in } L^1(\Omega;\mathbb{R}^d)$$

and

$$\sup_n \left[\int_\Omega |e(u_n)|^2\,dx + \mathcal{H}^{d-1}(J_{u_n}) \right] < +\infty.$$

Then if $\phi\colon \mathbb{R}^d \to [0,+\infty]$ is a lower-semicontinuous function, we have

$$\int_{J_u} [\phi(u^+) + \phi(u^-)]\,d\mathcal{H}^{d-1} \le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\,d\mathcal{H}^{d-1}.$$

This applies in particular to $\phi(u) = |u|^2$ and $\phi(u) = 1_{\{u\neq 0\}}$, which will be of interest to us.

Proof of Theorem 5.4. Notice first that ϕ may be supposed to be continuous. Indeed, for any lower-semicontinuous non-negative ϕ, by considering a sequence of continuous non-negative functions $\phi_k \nearrow \phi$ we get

$$\begin{aligned}\int_{J_u} [\phi(u^+) + \phi(u^-)]\,d\mathcal{H}^{d-1} &= \sup_k \int_{J_u} [\phi_k(u^+) + \phi_k(u^-)]\,d\mathcal{H}^{d-1}\\ &\le \sup_k \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi_k(u_n^+) + \phi_k(u_n^-)]\,d\mathcal{H}^{d-1}\\ &\le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\,d\mathcal{H}^{d-1}.\end{aligned}$$

Through a now-standard blow-up argument (see Remark 5.6), we can reduce the problem to the following lower-semicontinuity result. Let $Q_1 \subseteq \mathbb{R}^d$ be the unit square centered at 0, and let us set

$$H := Q_1 \cap \{x_d = 0\} \quad \text{and} \quad Q_1^\pm := Q_1 \cap \{x_d \gtrless 0\}.$$

Given $u^\pm \in \mathbb{R}^d$ with $u^+ \neq u^-$ and $u_n \in \mathrm{SBD}(Q_1)$ with

$$u_n \to u := u^+ 1_{Q_1^+} + u^- 1_{Q_1^-} \quad \text{strongly in } L^1(Q_1;\mathbb{R}^d), \tag{5.5}$$

$$\sup_n \mathcal{H}^{d-1}(J_{u_n}) < +\infty \tag{5.6}$$

and

$$e(u_n) \to 0 \quad \text{strongly in } L^1(Q_1; M^{d\times d}_{\mathrm{sym}}), \tag{5.7}$$

then

$$\phi(u^+) + \phi(u^-) \le \liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\,d\mathcal{H}^{d-1}. \tag{5.8}$$

We now divide the proof into several steps, and we employ the characterization by sections of SBD functions explained in Section 2.

Step 1. Let $\varepsilon > 0$ be given. We fix $\delta > 0$ and $N \in \mathbb{N}$ with $N > d$: these numbers will be subject to several constraints that will appear during the proof.

Let us fix N unit vectors $\{\xi_i\}_{1 \le i \le N}$ such that

$$|e_d \cdot \xi_i - 1| < \delta \tag{5.9}$$

and such that any subset of d of them forms a basis of $\mathbb{R}^d$. Moreover, we may assume in addition that

$$(u^+ - u^-) \cdot \xi_i \neq 0 \tag{5.10}$$

for every $i = 1, \dots, N$.

Thanks to (5.5) and (5.6), we can fix $a > 0$ small such that setting $H^\pm := H \times \{\pm a\} = H \pm a e_d$, we have

$$(u_n)_{|H^\pm} \to u^\pm \quad \text{strongly in } L^1(H^\pm; \mathbb{R}^d)$$

and

$$\forall n \in \mathbb{N} : \mathcal{H}^{d-1}(J_{u_n} \cap H^\pm) = 0. \tag{5.11}$$

Step 2. We claim that, up to a subsequence, we can find $H_\varepsilon^- \subset H^-$ with

$$\mathcal{H}^{d-1}(H^- \setminus H_\varepsilon^-) < \varepsilon \tag{5.12}$$

such that for every $i = 1, \dots, N$, for every $y \in H_\varepsilon^-$ and for every $n \in \mathbb{N}$,

$$H_\varepsilon^- \cap J_{u_n} = \emptyset, \tag{5.13}$$

and

$$\mathcal{H}^0((J_{u_n})_y^{\xi_i}) < +\infty, \quad \mathcal{H}^0((J_{u_n})_y^{\xi_i} \cap \mathbb{R}_+) \ge 1. \tag{5.14}$$

Moreover, setting

$$\widehat{(u_n)}_y^{\xi_i} := u_n(y + t\xi_i) \cdot \xi_i,$$

for every $y \in H_\varepsilon^-$ we have

$$\begin{aligned} &\widehat{(u_n)}_y^{\xi_i} \in SBV((Q_1)_y^{\xi_i}), \\ &J_{\widehat{(u_n)}_y^{\xi_i}} = (J_{u_n})_y^{\xi_i} \end{aligned} \tag{5.15}$$

(cf. notation (2.2)),

$$\|[\widehat{(u_n)}_y^{\xi_i}]'\|_{L^1} \to 0 \quad \text{uniformly for } y \in H_\varepsilon^-, \tag{5.16}$$

and

$$(u_n)_{|H^-} \to u^- \quad \text{uniformly on } H_\varepsilon^-. \tag{5.17}$$

Indeed, if the number δ appearing in (5.9) is small enough, we can find $A_\varepsilon^- \subseteq H^-$ with

$$\mathcal{H}^{d-1}(H^- \setminus A_\varepsilon^-) < \frac{\varepsilon}{2} \tag{5.18}$$

and such that for every $y \in A_\varepsilon^-$ the lines $\{y + t\xi_i : t \in \mathbb{R}\}$ intersect H^+ for every $i = 1, \dots, N$. In view of (5.5), (5.6) and (5.7), and since pointwise convergence implies almost uniform convergence, we can find $N_\varepsilon \subset A_\varepsilon^-$ with

$$\mathcal{H}^{d-1}(N_\varepsilon) < \frac{\varepsilon}{2} \tag{5.19}$$

and such that, up to a subsequence,

$$\|\widehat{(u_n)}_y^{\xi_i} - \hat{u}_y^{\xi_i}\|_{L^1} \to 0 \quad \text{uniformly for } y \in A_\varepsilon^- \setminus N_\varepsilon, \tag{5.20}$$

$$\|[\widehat{(u_n)}_y^{\xi_i}]'\|_{L^1} \to 0 \quad \text{uniformly for } y \in A_\varepsilon^- \setminus N_\varepsilon, \tag{5.21}$$

$$(u_n)_{|H^-} \to u^- \quad \text{uniformly on } A_\varepsilon^- \setminus N_\varepsilon, \tag{5.22}$$

and for every $y \in A_\varepsilon^- \setminus N_\varepsilon$,

$$\mathcal{H}^0((J_{u_n})_y^{\xi_i}) < +\infty. \tag{5.23}$$

Notice that for n large enough and for every $y \in A_\varepsilon^- \setminus N_\varepsilon$ we have

$$(J_{u_n})_y^{\xi_i} \neq \emptyset. \tag{5.24}$$

Indeed, otherwise, we would get for $n_k \to +\infty$ the existence of $y_k \in A_\varepsilon^- \setminus N_\varepsilon$ with $\widehat{(u_{n_k})}_{y_k}^{\xi_i} \in W^{1,1}((Q_1)_{y_k}^{\xi_i})$, and (5.22) together with (5.21) would yield

$$\|\widehat{(u_{n_k})}_{y_k}^{\xi_i} - u^-\|_1 \to 0$$

against (5.20) (recall that by the choice (5.10) of the ξ_i, the functions $\hat{u}_y^{\xi_i}$ have a jump). The claim follows by setting

$$H_\varepsilon^- := A_\varepsilon \setminus \left[N_\varepsilon \cup \bigcup_n (J_{u_n} \cap H^-) \right].$$

Indeed, (5.12) follows from (5.18), (5.19) and (5.11), while (5.13) is clearly satisfied. Relation (5.14) follows by (5.23) and (5.24), while relation (5.16) follows from (5.21). Finally, relation (5.17) follows from (5.22).

Step 3. For every $i = 1, \dots, N$, let us consider the set $J_n^{i,-}$ given by the first point of intersection (with $t > 0$) of the line $\{y + t\xi^i : t \in \mathbb{R}\}$ with the jump set J_{u_n} as y varies in the set H_ε^- defined in Step 2 (recall (5.14) and (5.15)). In view of (5.16) and (5.17), we can find $\eta_n \to 0$ such that for every $x \in J_n^{i,-}$ with $\nu_{u_n} \cdot \xi_i > 0$,

$$|u_n^-(x) \cdot \xi_i - u^- \cdot \xi_i| < \eta_n. \tag{5.25}$$

Step 4. We claim that, for δ small enough and N large enough, up to a subsequence, we can find $\tilde{J}_n^- \subseteq J_{u_n}$ with

$$\mathcal{H}^{d-1}(\tilde{J}_n^-) \geq 1 - c_\varepsilon, \tag{5.26}$$

where $c_\varepsilon \to 0$ as $\varepsilon \to 0$, and such that for every $x \in \tilde{J}_n^-$,

$$x \in J_n^{i,-} \quad \text{for } d \text{ different indices } i \in \{1, \dots, N\},$$

where $J_n^{i,-}$ is defined in Step 3. Moreover, we can orient ν_{u_n} on $\tilde{J}_n^-$ in such a way that

$$e_d \cdot \nu_{u_n} > 0 \quad \text{and} \quad \xi_i \cdot \nu_{u_n} > 0 \ \text{ for every } i = 1, \dots, N. \tag{5.27}$$

Intuitively speaking, the points in $\tilde{J}_n^-$ are seen from H_ε^- under d different directions: moreover, the associated lines cut the jump transversally, from the "lower" to the "upper" part.

Indeed, in view of the definition of ξ_i (which forms a very small angle with e_d as $\delta \to 0$) and of the area formula (cf. for instance [26, Sec. 3.2]), we can assume that δ is so small that for every $i = 1, \dots, N$,

$$\mathcal{H}^{d-1}(J_n^{i,-}) \geq \int_{J_n^{i,-}} |\nu_{u_n} \cdot \xi_i| \, d\mathcal{H}^{d-1} = \mathcal{H}^{d-1}((H_\varepsilon^-)^{\xi_i}) = \frac{1}{1+\hat{c}_\delta} \mathcal{H}^{d-1}(H_\varepsilon^-), \tag{5.28}$$

where the notation $(H_\varepsilon^-)^{\xi_i}$ is defined in (2.1) and where $\hat{c}_\delta \to 0$, so that, taking into account (5.12), for small δ we have

$$\mathcal{H}^{d-1}(J_n^{i,-}) \geq 1 - 2\varepsilon. \tag{5.29}$$

By Lemma 5.5 below (with $X = J_{u_n}$, $\mu = \mathcal{H}^{d-1}$ and $\mathcal{M}$ given by the family of Borel sets), if N is large enough we can find an index $\bar{i}$ such that

$$\mathcal{H}^{d-1}\Bigg(J_n^{\bar{i},-} \setminus \bigcup_{\substack{i_1<i_2<\cdots<i_d \\ i_h=1,\dots,N}} (J_n^{i_1,-} \cap J_n^{i_2,-} \cap \cdots \cap J_n^{i_d,-})\Bigg) < \varepsilon. \tag{5.30}$$

Intuitively speaking, most of the points in $J_n^{\bar{i},-}$ are seen from H_ε^- at least under d different directions: we call this set $\tilde{J}_n^-$, i.e.

$$\tilde{J}_n^- := J_n^{\bar{i},-} \cap \bigcup_{\substack{i_1<i_2<\cdots<i_d \\ i_h=1,\dots,N}} (J_n^{i_1,-} \cap J_n^{i_2,-} \cap \cdots \cap J_n^{i_d,-}). \tag{5.31}$$

In view of (5.29) and (5.30) we get

$$\mathcal{H}^{d-1}(\tilde{J}_n^-) \geq 1 - 3\varepsilon. \tag{5.32}$$

Finally, if we set

$$G_{n,\varepsilon} := \left\{x \in \tilde{J}_n^- : |\nu_{u_n} \cdot \xi_{\bar{i}}| > \varepsilon\right\} \quad \text{and} \quad B_{n,\varepsilon} := \tilde{J}_n^- \setminus G_{n,\varepsilon},$$

coming back to (5.28) we have

$$\mathcal{H}^{d-1}(G_{n,\varepsilon}) + \varepsilon^2 \mathcal{H}^{d-1}(B_{n,\varepsilon}) > 1 - 3\varepsilon,$$

so that

$$\mathcal{H}^{d-1}(G_{n,\varepsilon}) > 1 - 3\varepsilon - \varepsilon^2 C,$$

where $C := \sup_n \mathcal{H}^{d-1}(J_{u_n}) < +\infty$. Finally, we orient the normal ν_{u_n} on $G_{n,\varepsilon}$ in such a way that

$$\nu_{u_n} \cdot \xi_{\bar{i}} > \varepsilon.$$

The inequalities (5.27) then also hold true on $G_{n,\varepsilon}$ if δ is small enough thanks to (5.9). Reducing $\tilde{J}_n^-$ to $G_{n,\varepsilon}$ if necessary, the full claim follows, taking into account (5.31) and (5.32).

Step 5. Let $\tilde{J}_n^- \subseteq J_{u_n}$ be the set given by Step 4. Since the points of this set are seen from H_ε^- under d different directions, in view of (5.25) we infer that there exists $\tilde{\eta}_n \to 0$ such that for every $x \in \tilde{J}_n^-$,

$$|u_n^-(x) - u^-| < \tilde{\eta}_n.$$

Reasoning in a similar way starting from the upper part H_ε^+, and employing the opposite directions $\{-\xi_i : i = 1, \dots, N\}$, we can construct $\tilde{J}_n^+ \subseteq J_{u_n}$ with ν_{u_n} oriented such that again

$$e_d \cdot \nu_{u_n} > 0 \quad \text{and} \quad \xi_i \cdot \nu_{u_n} > 0 \ \text{ for every } i = 1, \dots, N,$$

such that

$$\mathcal{H}^{d-1}(\tilde{J}_n^+) \geq 1 - c_\varepsilon \tag{5.33}$$

with $c_\varepsilon \to 0$ as $\varepsilon \to 0$, and such that for every $x \in \tilde{J}_n^+$,

$$|u_n^+(x) - u^+| < \tilde{\eta}_n.$$

Notice that for $x \in \tilde{J}_n^- \cap \tilde{J}_n^+$, the orientation chosen is compatible with that of (5.27), so that indeed $u_n^-(x)$ and $u_n^+(x)$ are the two traces of u_n at x.

We can thus write, in view of the continuity of ϕ,

$$\begin{aligned}
\int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)] \, d\mathcal{H}^{d-1} &\geq \int_{\tilde{J}_n^+ \cap \tilde{J}_n^-} [\phi(u_n^+) + \phi(u_n^-)] \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\tilde{J}_n^+ \Delta \tilde{J}_n^-} [\phi(u_n^+) + \phi(u_n^-)] \, d\mathcal{H}^{d-1} \\
&\geq \int_{\tilde{J}_n^+ \cap \tilde{J}_n^-} [\phi(u_n^+) + \phi(u_n^-)] \, d\mathcal{H}^{d-1} \\
&\quad + \int_{\tilde{J}_n^+ \setminus \tilde{J}_n^-} \phi(u_n^+) \, d\mathcal{H}^{d-1} + \int_{\tilde{J}_n^- \setminus \tilde{J}_n^+} \phi(u_n^-) \, d\mathcal{H}^{d-1} \\
&\geq \int_{\tilde{J}_n^+} \phi(u_n^+) \, d\mathcal{H}^{d-1} + \int_{\tilde{J}_n^-} \phi(u_n^-) \, d\mathcal{H}^{d-1} \\
&\geq [\phi(u^+) - \tilde{\tilde{\eta}}_n] \mathcal{H}^{d-1}(\tilde{J}_n^+) + [\phi(u^-) - \tilde{\tilde{\eta}}_n] \mathcal{H}^{d-1}(\tilde{J}_n^-),
\end{aligned}$$

where $\tilde{\tilde{\eta}}_n \to 0$, so that, taking into account (5.26) and (5.33),

$$\liminf_{n\to+\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)]\, d\mathcal{H}^{d-1} \geq [\phi(u^+) + \phi(u^-)](1 - 2c_\varepsilon).$$

The conclusion follows by letting $\varepsilon \to 0$. ∎

In the proof of Theorem 5.4 we made use of the following abstract lemma.

Lemma 5.5. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\varepsilon > 0$ and $d \geq 2$. Then there exists $N \in \mathbb{N}$ that only depends on $\mu(X)$, ε, d such that if $\{E_i\}_{i=1,\dots,N}$ is a family of sets in $\mathcal{M}$, we can find $\bar{i}$ such that*

$$\mu\Big(E_{\bar{i}} \setminus \bigcup_{j_1<j_2<\dots<j_d} (E_{j_1} \cap E_{j_2} \cap \dots \cap E_{j_d})\Big) < \varepsilon.$$

Proof. Up to dividing ε by $\mu(X)$ we suppose without loss of generality that $\mu(X) = 1$. It is enough to prove that for any $d \geq 2$, $\varepsilon > 0$, there is some $N(d, \varepsilon) \geq 1$ such that any family of $N \geq N(d, \varepsilon)$ of sets $(E_i)_{1\leq i\leq N}$ there is some i that verifies

$$\mu\Big(E_i \setminus \bigcup_{\substack{J\subset[1,N]\setminus\{i\} \\ |J|=d-1}} \bigcap_{j\in J} E_j\Big) < \varepsilon,$$

meaning that there is some i such that every point of E_i outside a set of measure less than ε is in (at least) $d-1$ other sets E_j (for $j \neq i$).

We prove it by recursion. If $d = 2$, let $N := [\frac{1}{\varepsilon}]$, where $[\cdot]$ denotes the integer part. Given $(E_i)_{1\leq i\leq N}$, let us consider the sets $(E_i \setminus \bigcup_{1\leq j\leq N, j\neq i} E_j)_{1\leq i\leq N}$. These are disjoint and $\mu(X) = 1$, so there is some i such that

$$\mu\Big(E_i \setminus \bigcup_{1\leq j\leq N, j\neq i} E_j\Big) \leq \frac{1}{N} \leq \varepsilon,$$

which proves the initialization.

Assume now that the result is true for d and let us check it for $d+1$. Let

$$N := N\Big(d, \frac{\varepsilon}{2}\Big) \quad \text{and} \quad M := \Big[\frac{2}{\varepsilon}\Big],$$

and let us consider $N \times M$ sets that we classify into N groups of M sets, written $(E_{k,i})_{1\leq k\leq N, 1\leq i\leq M}$. For every $k \in [1, N]$, the sets $(E_{k,i} \setminus \bigcup_{1\leq j\leq M, j\neq i} E_{k,j})_{1\leq i\leq M}$ are disjoint, so there is some i_k such that

$$\mu\Big(E_{k,i_k} \setminus \bigcup_{1\leq i\leq M, i\neq i_k} E_{k,i}\Big) \leq \frac{1}{M} \leq \frac{\varepsilon}{2}.$$

Considering the sets $(E_{k,i_k})_{1\le k\le N}$, since $N = N(d, \frac{\varepsilon}{2})$ we find some $\bar{k}$ such that

$$\mu\Big(E_{\bar{k},i_{\bar{k}}} \setminus \bigcup_{\substack{K\subset[1,N]\setminus\{\bar{k}\}\\ |K|=d-1}} \bigcap_{k\in K} E_{k,i_k}\Big) \le \frac{\varepsilon}{2}.$$

This means that outside a set of measure at most $\frac{\varepsilon}{2}$, every point of $E_{\bar{k},i_{\bar{k}}}$ is in $d-1$ sets of the form E_{k,i_k} for $k \ne \bar{k}$, and similarly every point outside a set of measure at most $\frac{\varepsilon}{2}$ is also in one set of the form $E_{\bar{k},i}$ for some $i \ne i_{\bar{k}}$. We conclude that outside a set of measure at most ε, every point of $E_{\bar{k},i_{\bar{k}}}$ belongs to d other sets, meaning $N(d+1,\varepsilon)$ is well defined and $N(d+1,\varepsilon) \le N(d, \frac{\varepsilon}{2})[\frac{2}{\varepsilon}]$. ∎

Remark 5.6. Let us detail the blow-up argument used in the proof of Theorem 5.4. If we set

$$\mu_n := [\phi(u_n^+) + \phi(u_n^-)]\mathcal{H}^{d-1}\lfloor J_{u_n}$$

and assume that (up to a subsequence)

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{weakly* in } \mathcal{M}_b(\Omega)$$

for some Radon measure μ on Ω, the conclusion follows if we show that

$$\mu \ge [\phi(u^+) + \phi(u^-)]\mathcal{H}^{d-1}\lfloor J_u \quad \text{as measures on } \Omega.$$

With this aim it is sufficient to show that

$$\frac{d\mu}{d\mathcal{H}^{d-1}}(x) \ge [\phi(u^+(x)) + \phi(u^-(x))] \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in J_u, \tag{5.34}$$

where $\frac{d\mu}{d\mathcal{H}^{d-1}}$ denotes the Radon–Nikodym derivative of μ with respect to $\mathcal{H}^{d-1}$ (restricted to J_u).

Let us assume (up to subsequences) that

$$\lambda_n := \mathcal{H}^{d-1}\lfloor J_{u_n} \overset{*}{\rightharpoonup} \lambda \quad \text{weakly* in } \mathcal{M}_b(\Omega),$$

and that

$$|e(u_n)|\,dx \overset{*}{\rightharpoonup} f\,dx \quad \text{weakly* in } \mathcal{M}_b(\Omega),$$

where $f \in L^1(\Omega)$ (this is possible since $(e(u_n))_{n\in\mathbb{N}}$ is bounded in L^2).

Let $x \in J_u$ be such that

$$\frac{d\mu}{d\mathcal{H}^{d-1}}(x) = \lim_{r\to 0}\frac{\mu(Q_{x,r})}{r^{d-1}}, \quad \lim_{r\to 0}\frac{\lambda(Q_{x,r})}{r^{d-1}} < +\infty, \quad \lim_{r\to 0}\frac{1}{r^{d-1}}\int_{Q_r(x)}|f|\,dx = 0,$$

and (having chosen the axis so that $\nu_u(x) = e_d$), for $r \to 0^+$,

$$u(x + r\cdot) \to u^+(x)1_{Q_1^+} + u^-(x)1_{Q_1^-} \quad \text{strongly in } L^1(Q_1;\mathbb{R}^d).$$

Since $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ satisfies these properties, it suffices to concentrate on such points to prove inequality (5.34).

Let $r_k \to 0$ be such that

$$\mu(\partial Q_{x,r_k}) = \lambda(\partial Q_{x,r_k}) = 0.$$

Since by weak convergence and the relation above we have $\mu_n(Q_{x,r_k}) \to \mu(Q_{x,r_k})$, and similarly for λ, we can choose $n_k \nearrow +\infty$ such that

$$\mu(Q_{x,r_k}) \geq \mu_{n_k}(Q_{x,r_k}) - \frac{r_k^{d-1}}{k}, \quad \lambda(Q_{x,r_k}) \geq \lambda_{n_k}(Q_{x,r_k}) - \frac{r_k^{d-1}}{k}$$

and

$$\int_{Q_{x,r_k}} |f|\,dx \geq \int_{Q_{x,r_k}} |e(u_{n_k})|\,dx - \frac{r_k^{d-1}}{k}.$$

Moreover, setting $v_k(y) := u_{n_k}(x + r_k y)$ we can also assume

$$v_k \to u^+(x)1_{Q_1^+} + u^-(x)1_{Q_1^-} \quad \text{strongly in } L^1(Q_1;\mathbb{R}^d).$$

We get

$$\int_{Q_1} |e(v_k)|\,dx = \frac{1}{r_k^{d-1}} \int_{Q_{x,r_k}} |e(u_{n_k})|\,dx \leq \frac{1}{r_k^{d-1}} \int_{Q_{x,r_k}} |f|\,dx + \frac{1}{k} \to 0$$

and

$$\begin{aligned}
\mathcal{H}^{d-1}(J_{v_k}) = \frac{1}{r_k^{d-1}} \mathcal{H}^{d-1}(J_{u_{n_k}} \cap Q_{x,r_k}) &= \frac{\lambda_{n_k}(Q_{x,r_k})}{r_k^{d-1}} \\
&\leq \frac{\lambda(Q_{x,r_k})}{r_k^{d-1}} + \frac{1}{k} \to c < +\infty,
\end{aligned}$$

so that, using the lower semicontinuity (5.8) concerning functions on the unit square (and to which the proof of the theorem has been reduced),

$$\begin{aligned}
\frac{d\mu}{d\mathcal{H}^{d-1}}(x) = \lim_{k\to+\infty} \frac{\mu(Q_{x,r_k})}{r_k^{d-1}} &\geq \liminf_{k\to+\infty} \frac{\mu_{n_k}(Q_{x,r_k})}{r_k^{d-1}} \\
&= \liminf_{k\to+\infty} \int_{J_{v_k}} [\phi(v_k^+) + \phi(v_k^-)]\,d\mathcal{H}^{d-1} \geq \phi(u^+(x)) + \phi(u^-(x))
\end{aligned}$$

and (5.34) follows.

6. Existence of minimizers: Proof of Theorem 4.8

We are now in a position to prove the first main result of the paper.

Proof of Theorem 4.8. Let $(E_n, u_n)_{n\in\mathbb{N}}$ be a minimizing sequence: since the function f is not identically equal to $+\infty$, and in view of Remark 4.6, there exists $C > 0$ such that

$$\mathcal{J}(E_n, u_n) \leq C.$$

Since $u_n = 0$ a.e. on E_n we may write

$$\int_{\partial^* E_n} |u_n^+|^2 \, d\mathcal{H}^{d-1} + \int_{J_{u_n}\setminus\partial^* E_n} [|u_n^+|^2 + |u_n^-|^2] \, d\mathcal{H}^{d-1} = \int_{J_{u_n}} [|u_n^+|^2 + |u_n^-|^2] \, d\mathcal{H}^{d-1}$$

so that we infer

$$\mathcal{H}^{d-1}(\partial^* E_n) \leq C$$

and

$$\int_\Omega |e(u_n)|^2 \, dx + \mathcal{H}^{d-1}(J_{u_n}) + \int_{J_{u_n}} [|u_n^+|^2 + |u_n^-|^2] \, d\mathcal{H}^{d-1} \leq C.$$

Notice that

$$\begin{aligned} |E(u_n)|(\Omega') &= \int_{\Omega'} |e(u_n)| \, dx + \int_{J_{u_n}} |u_n^+ - u_n^-| \, d\mathcal{H}^{d-1} \\ &\leq \int_{\Omega'\setminus\Omega} |e(V)| \, dx + \int_\Omega |e(u_n)| \, dx + \int_{J_{u_n}} [|u_n^+| + |u_n^-|] \, d\mathcal{H}^{d-1} \\ &\leq \int_{\Omega'\setminus\Omega} |e(V)| \, dx + \frac{1}{2}\Big[|\Omega| + \int_\Omega |e(u_n)|^2 \, dx + 2\mathcal{H}^{d-1}(J_{u_n}) \\ &\qquad + \int_{J_{u_n}} [|u_n^+|^2 + |u_n^-|^2] \, d\mathcal{H}^{d-1}\Big] \leq \tilde{C}, \end{aligned}$$

for some $\tilde{C} > 0$. Moreover, thanks to Theorem 5.1 applied to $u - V$ we may assume also that

$$\|u_n\|_{L^{\frac{2d}{d-1}}(\Omega')} \leq \tilde{C}. \tag{6.1}$$

By the compactness result in SBD (see Theorem 2.1), there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u \in \mathrm{SBD}(\Omega')$ with $u = V$ on $\Omega' \setminus \Omega$ and such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(\Omega'; \mathbb{R}^d), \tag{6.2}$$

$$e(u_{n_k}) \rightharpoonup e(u) \qquad \text{weakly in } L^2(\Omega'; M^{d\times d}_{\mathrm{sym}}), \tag{6.3}$$

$$\mathcal{H}^{d-1}(J_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{d-1}(J_{u_{n_k}}).$$

Concerning the sets E_{n_k}, we may assume, up to a further subsequence if necessary, that there exists a set of finite perimeter $E \subseteq \Omega$ such that

$$1_{E_{n_k}} \to 1_E \qquad \text{strongly in } L^1(\mathbb{R}^d) \tag{6.4}$$

with

$$\mathcal{H}^{d-1}(\partial^* E) \le \liminf_{k\to+\infty} \mathcal{H}^{d-1}(\partial^* E_{n_k}).$$

In particular, we get

$$f(|E|) \le \liminf_{n\to+\infty} f(|E_n|). \tag{6.5}$$

Let us prove that

$$(E, u) \in \mathcal{A}(V). \tag{6.6}$$

In view of (6.1) we infer that $u \in L^{\frac{2d}{d-1}}(\Omega'; \mathbb{R}^d)$ so that in particular $u \in L^2(\Omega'; \mathbb{R}^d)$. Moreover, $u = V$ on $\Omega' \setminus \Omega$, while $u = 0$ a.e. on E thanks to (6.2) and (6.4).

Since the divergence constraint is intended in the sense of distributions on Ω, this passes easily to the limit thanks to (6.2). Moreover, in view of Theorem 5.3 we deduce

$$u^{\pm} \perp \nu_u \quad \text{on } J_u.$$

In particular, this entails

$$u^+ \perp \nu_E \quad \text{on } \partial^* E \cap \Omega,$$

since for $x \in \partial^* E$ we have either $x \in J_u$ or $u^+(x) = 0$. We conclude that the non-penetration constraint for the velocity field holds on $\partial^* E$ and on $J_u \setminus \partial^* E$, so that (6.6) holds true.

Let us prove the pair (E, u) is a minimizer for the problem. Thanks to (6.3) we get

$$\int_{\Omega'} |e(u)|^2\, dx \le \liminf_{k\to+\infty} \int_{\Omega'} |e(u_{n_k})|^2\, dx,$$

while in view of Theorem 5.4 we have

$$\int_{J_u} [|u^+|^2 + |u^-|^2]\, d\mathcal{H}^{d-1} \le \liminf_{k\to+\infty} \int_{J_{u_{n_k}}} [|u^+_{n_k}|^2 + |u^-_{n_k}|^2]\, d\mathcal{H}^{d-1},$$

which entails

$$\begin{aligned} &\int_{\partial^* E} |u^+|^2\, d\mathcal{H}^{d-1} + \int_{J_u \setminus \partial^* E} [|u^+|^2 + |u^-|^2]\, d\mathcal{H}^{d-1} \\ &\qquad \le \liminf_{k\to+\infty} \left[\int_{\partial^* E_{n_k}} |u^+_{n_k}|^2\, d\mathcal{H}^{d-1} + \int_{J_{u_{n_k}} \setminus \partial^* E_{n_k}} [|u^+_{n_k}|^2 + |u^-_{n_k}|^2]\, d\mathcal{H}^{d-1} \right] \end{aligned} \tag{6.7}$$

since $u = 0$ a.e. on E and $u_{n_k} = 0$ a.e. on E_{n_k}.

Let us prove that

$$\begin{aligned} &2\mathcal{H}^{d-1}(J_u \setminus \partial^* E) + \mathcal{H}^{d-1}(\partial^* E) \\ &\qquad \le \liminf_{k\to+\infty} (2\mathcal{H}^{d-1}(J_{u_{n_k}} \setminus \partial^* E_{n_k}) + \mathcal{H}^{d-1}(\partial^* E_{n_k})). \end{aligned} \tag{6.8}$$

Let us choose $h \in \mathbb{R}^d$ such that

$$\begin{aligned}\mathcal{H}^{d-1}\big(\{x \in \partial^* E \cup J_u : u^+(x) = h\}\big) &= \mathcal{H}^{d-1}\big(\{x \in \partial^* E \cup J_u : u^-(x) = h\}\big)\\ &= \mathcal{H}^{d-1}\big(\{x \in \partial^* E_{n_k} \cup J_{u_{n_k}} : u^+_{n_k}(x) = h\}\big)\\ &= \mathcal{H}^{d-1}\big(\{x \in \partial^* E_{n_k} \cup J_{u_{n_k}} : u^-_{n_k}(x) = h\}\big) = 0.\end{aligned}$$

This is possible because, for example, the sets $\{x \in \partial^* E \cup J_u : u^+(x) = h\}$ are disjoint as h varies, and similarly for the other sets. In particular, setting

$$v^h := u + h 1_E \quad \text{and} \quad v^h_{n_k} := u_{n_k} + h 1_{E_{n_k}}$$

we have

$$J_{v^h} = J_u \cup J_{1_E} = \partial^* E \cup J_u \quad \text{and} \quad J_{v^h_{n_k}} = J_{u_{n_k}} \cup J_{1_{E_{n_k}}} = \partial^* E_{n_k} \cup J_{u_{n_k}}$$

up to $\mathcal{H}^{d-1}$-negligible sets. If we apply Theorem 5.4 with the choice $\phi_h(s) = 1_{\{s \neq h\}}$ to the sequence $(v^h_{n_k})_{k\in\mathbb{N}}$ we get

$$\begin{aligned}\mathcal{H}^{d-1}(\partial^* E) + 2\mathcal{H}^{d-1}(J_u \setminus \partial^* E) &= \int_{J_{v^h}} [\phi_h((v^h)^+) + \phi_h((v^h)^-)]\, d\mathcal{H}^{d-1}\\ &\le \liminf_{k\to+\infty} \int_{J_{v^h_{n_k}}} [\phi_h((v^h_{n_k})^+) + \phi_h((v^h_{n_k})^-)]\, d\mathcal{H}^{d-1}\\ &= \liminf_{k\to+\infty} [\mathcal{H}^{d-1}(\partial^* E_{n_k}) + 2\mathcal{H}^{d-1}(J_{u_{n_k}} \setminus \partial^* E_{n_k})]\end{aligned}$$

so that (6.8) holds true.

Gathering (6.3), (6.7), (6.5) and (6.8), we deduce

$$\mathcal{J}(E, u) \le \liminf_{k\to+\infty} \mathcal{J}(E_{n_k}, u_{n_k})$$

so that, taking into account (6.6), the pair (E, u) is a minimizer of the main problem (4.3), and the proof is concluded. ∎

7. Regularity of two-dimensional minimizers: Proof of Theorem 4.10

This section is devoted to the proof of Theorem 4.10 concerning the regularity of minimizers in dimension two.

As mentioned in the introduction, the general strategy used by De Giorgi, Carriero and Leaci for the Mumford–Shah problem in [24] faces the new difficulties given by the vectorial context, considered in [16, 19] in connection with the Griffith fracture problem, and also by extra conditions proper to our problem, that is, incompressibility and non-penetration for the velocity fields. We follow the main lines of [16, 19]: however, technical difficulties allow us to deal only with dimension two (see point (a) below).

Since our drag problem involves pairs (E, u) as admissible configurations, and some points of $\partial^* E$ may not be jump points of u, it will be useful to deal with pairs (J, u), where J is a rectifiable set and u is a function whose jumps are contained (up to $\mathcal{H}^1$-negligible sets) in J and satisfy the constraints of zero divergence and non-penetration. More precisely, we formulate the following definition.

Definition 7.1 (The class $\mathcal{V}$). Let $\Omega \subseteq \mathbb{R}^2$ be an open set. We say that $(J, u) \in \mathcal{V}(\Omega)$ if $J \subseteq \Omega$ is a rectifiable set, and $u \in \mathrm{SBD}(\Omega)$ is such that $\operatorname{div} u = 0$ in the sense of distributions in Ω, $\mathcal{H}^1(J_u \setminus J) = 0$ and $u^\pm_{|J} \cdot \nu_J = 0$ $\mathcal{H}^1$-a.e. on J.

The structure of the section is the following:

(a) In Section 7.1 we prove a fundamental approximation lemma (smoothing Lemma 7.2), which allows us to approximate every $(J, u) \in \mathcal{V}(Q_1)$ with $\mathcal{H}^1(J)$ small by a configuration $(J \setminus Q_r, v) \in \mathcal{V}(Q_1)$, where v is a Sobolev function in the slightly smaller square Q_r with a control on the energy. The idea is that the jumps of u in Q_r are "smoothed out", giving rise to the function v which preserves the divergence-free constraint together with the non-penetration condition. This result is inspired by [16], and it is here that the dimension two is fundamental.

(b) In Section 7.2 we prove regularity for local minimizers of a Griffith functional

$$G(J, u) := \int_\Omega |e(u)|^2 \, dx + \mathcal{H}^1(J),$$

defined on pairs $(J, u) \in \mathcal{V}(\Omega)$. The kind of local minimality considered is very weak, and inspired by the kind of competitors that can be constructed thanks to the smoothing Lemma 7.2. The key result to get regularity is given by the decay estimate contained in Proposition 7.7.

Regularity for minimizers of the Griffith energy is then used in Section 7.3 to prove Theorem 4.10, that is, to show the regularity of minimizers of the drag problem.

(c) Finally, motivated by the regularity result of Theorem 4.10, in Section 7.4 we describe a different relaxation of the drag problem which involves topologically closed obstacles and Sobolev velocities: the regularity result can be used to prove that such a formulation is well posed in dimension two.

7.1. The smoothing lemma

We fix a standard radial, smooth, non-negative mollifier ρ with support in a disc of radius $1/8$ and denote

$$\rho_\delta(x) := \delta^{-2} \rho\Big(\frac{x}{\delta}\Big).$$

The main result of the section is the following smoothing lemma which is in the spirit of [16].

Lemma 7.2 (Smoothing lemma). *There exist $C, \eta > 0$ such that for any $(J,u) \in \mathcal{V}(Q_1)$ with $\mathcal{H}^1(J) < \eta$, then letting $\delta := \mathcal{H}^1(J)^{\frac{1}{2}}$ there exist $r \in]1-\delta^{\frac{1}{2}}, 1[$ and $v \in \mathrm{SBD}(Q_1) \cap H^1(Q_r)$ such that the following items hold true:*

(a) *$\mathcal{H}^0(J \cap \partial Q_r) = 0$ and for every $0 < s < r$,*

$$\mathcal{H}^1(J \cap (Q_r \setminus Q_{r-s})) \le C\delta^{\frac{3}{2}} s.$$

(b) *$\{v \neq u\} \subseteq Q_r$ and $(J \setminus Q_r, v) \in \mathcal{V}(Q_1)$.*

(c) *It holds that*

$$\|e(v)\|_{L^2(Q_1)} \le (1 + C\delta^{\frac{1}{6}})\|e(u)\|_{L^2(Q_1)}.$$

(d) *There exists a cut-off function $\varphi \in C^\infty(Q_r, [0,1])$ with $\varphi = 0$ on $Q_r \setminus Q_{r-\delta}$, $\varphi = 1$ on $Q_{r-4\delta}$, and such that*

$$\|e(v) - \varphi\rho_\delta * e(u)\|_{L^2(Q_r)} \le C\delta^{\frac{1}{6}} \|e(u)\|_{L^2(Q_1)}.$$

Proof. The proof follows the strategy introduced in [16], and some parts will be referred directly to that paper. However, since our conclusion is slightly different, we prefer to develop some computations in detail. We use the notation $a \lesssim b$ when $a \le Cb$ for some dimensional constant C.

We divide the proof into several steps.

Step 1: Subdivision into small squares. Let us set

$$N := 1 + [\mathcal{H}^1(J)^{-\frac{1}{2}}],$$

where $[\cdot]$ denotes the integer part. In the following we assume that $\mathcal{H}^1(J)$ is arbitrarily small, so that N is arbitrarily large. For convenience in the construction, we set $\delta = 1/N \le \mathcal{H}^1(J)^{\frac{1}{2}}$, which (mildly) differs from the choice of the statement: yet since δ is asymptotically equivalent to $\mathcal{H}^1(J)^{\frac{1}{2}}$, the mismatch does not affect the validity of the conclusion.

For $r \in]1-\delta^{\frac{1}{2}}, 1[$ and each $k \ge -2$, let us set

$$\delta_k := \frac{\delta r}{2^k} \quad \text{and} \quad r_k = \Big(N - \frac{1}{2^k}\Big)\delta.$$

Then we consider a partition (up to a negligible set) of Q_r into cubes obtained by filling Q_{r_0} with cubes of side δ_0 and denoted by $(\tilde{q}_{0,j})_j$, and then each $Q_{r_k} \setminus Q_{r_{k-1}}$ with cubes of side δ_k and denoted $(\tilde{q}_{k,j})_j$ (note that there is only one way to do this).

For any square $q = z + [-t,t]^2$, we write

$$q' := z + \Big[-\frac{8}{7}t, \frac{8}{7}t\Big]^2 \quad \text{and} \quad q'' := (q')'.$$

We set

$$q_{k,j} := (\tilde{q}_{k,j})'.$$

We may notice that with our choices,

$$\forall k \geq 1 : q''_{k,j} \Subset Q_{r_{k+1}} \setminus Q_{r_{k-2}}, \tag{7.1}$$

and $\{q''_{k,j}\}_{k,j}$ is a covering of Q_r with a fixed finite number of overlaps: indeed, each $q''_{k,j}$ meets at most 8 neighbors $q''_{p,i}$, and they all verify $|k-p| \leq 1$, meaning $\delta_k/\delta_p \in \{\frac{1}{2}, 1, 2\}$. This is because the factor $\frac{8}{7}$ above is chosen such that $(\frac{8}{7})^3 < \frac{3}{2}$.

Step 2: Choice of the square Q_r. We now make a convenient choice of r such that the density of J near ∂Q_r is small, following an approach similar to [18, Theorem 2.1].

We claim that there exist $C, \eta > 0$ such that for $\delta < \eta$ we can choose $r \in]1-\sqrt{\delta}, 1[$ with $\mathcal{H}^0(J \cap \partial Q_r) = 0$,

$$\forall\, s \in]0, r[\colon \mathcal{H}^1(J \cap (Q_r \setminus Q_{r-s})) \leq C\delta^{\frac{3}{2}} s \tag{7.2}$$

and

$$\int_{Q_r \setminus Q_{r-2}} |e(u)|^2\, dx < C\delta^{\frac{1}{2}} \int_{Q_1} |e(u)|^2\, dx. \tag{7.3}$$

Consider indeed the measure μ on $[0, 1]$ defined as

$$\mu(E) := \frac{\mathcal{H}^1(J \cap Q_E)}{\mathcal{H}^1(J)} + \frac{\int_{Q_E} |e(u)|^2\, dx}{\int_{Q_1} |e(u)|^2\, dx},$$

where $Q_E := \bigcup_{r \in E} \partial Q_r$ is the cubic shell associated to $E \subset [0, 1]$. It suffices to prove that we can find $r \in]1-\delta^{\frac{1}{2}}, 1[$ such that

$$\mathcal{H}^0(J \cap \partial Q_r) = 0, \tag{7.4}$$

and, denoting $I_r^s := [r-s, r[$ for $0 < s < r$,

$$\mu(I_r^s) \leq \widehat{C}\delta^{-\frac{1}{2}} s, \tag{7.5}$$

where $\widehat{C} > 0$ is a suitable constant which we fix below. Indeed, if δ is small enough this implies that (recall that $\mathcal{H}^1(J)$ behaves like δ^2)

$$\mathcal{H}^1(J \cap (Q_r \setminus Q_{r-s})) \leq \mathcal{H}^1(J)\mu(I_r^s) \leq \widehat{C}\delta^{\frac{3}{2}} s$$

and

$$\int_{Q_r \setminus Q_{r-4\delta r}} |e(u)|^2\, dx \leq \widehat{C}\delta^{-\frac{1}{2}}(4\delta r) \int_{Q_1} |e(u)|^2\, dx \leq 4\widehat{C}\delta^{\frac{1}{2}} \int_{Q_1} |e(u)|^2\, dx,$$

so that (7.2) and (7.3) follow by choosing $C := 4\widehat{C}$.

Let I_1 be the union of all intervals that do not satisfy (7.5). If $(I_{r_i}^{s_i})$ is a *Vitali covering* of I, then

$$2 = \mu([0,1]) \geq \sum_i \mu(I_{r_i}^{s_i}) > \widehat{C}\delta^{-\frac{1}{2}} \sum_i |I_{r_i}^{s_i}| = \frac{\widehat{C}\delta^{-\frac{1}{2}}}{5} \sum_i |5I_{r_i}^{s_i}| \geq \frac{\widehat{C}\delta^{-\frac{1}{2}}}{5} |I_1|,$$

hence $|I_1| < \frac{10}{\widehat{C}}\delta^{\frac{1}{2}}$.

Let $I_2 := \pi_x(J) \cup \pi_y(J)$, where π_x, π_y denote the projection on the coordinate axis: we have asymptotically $|I_2| \leq 2\delta^2$. If $C > 10$, this implies that for δ small enough,

$$]1-\sqrt{\delta}, 1[\setminus (I_1 \cup I_2) \neq \emptyset,$$

which yields the existence of r, which verifies claims (7.4) and (7.5).

Step 3: A first approximation. In view of (7.2) and of (7.1), for every $k \geq 1$ we have

$$\mathcal{H}^1(J_u \cap q''_{k,j}) \lesssim \delta^{\frac{3}{2}} \delta_k,$$

while if δ is small enough (recall that $\mathcal{H}^1(J)$ behaves like δ^2 and $r \in]1-\delta^{\frac{1}{2}}, 1[$)

$$\mathcal{H}^1(J_u \cap q''_{0,j}) \leq \mathcal{H}^1(J_u) \lesssim \delta\delta_0.$$

This means that the jump set of u in every cube of the constructed subdivision is arbitrarily small compared to its sides.

Thanks to [15, Proposition 3], and taking into account the preceding inequalities , for every (k, j) there is a set $\omega_{k,j} \subset q'_{k,j}$ and an affine function $a_{k,j}$ with $e(a_{k,j}) = 0$, such that

$$|\omega_{k,j}| \lesssim \delta_k \mathcal{H}^1(J_u \cap q''_{k,j}) \lesssim \delta\delta_k^2, \tag{7.6}$$

$$\int_{q'_{k,j}\setminus\omega_{k,j}} |u - a_{k,j}|^4 \, dx \lesssim \left(\delta_k \int_{q''_{k,j}} |e(u)|^2 \, dx\right)^2, \tag{7.7}$$

and the function $v_{k,j} := u + (a_{k,j} - u)1_{\omega_{k,j}}$ verifies

$$\begin{aligned}\int_{q_{k,j}} |e(\rho_{\delta_k} * v_{k,j}) - \rho_{\delta_k} * e(u)|^2 \, dx &\lesssim \left(\frac{\mathcal{H}^1(J_u \cap q''_{k,j})}{\delta_k}\right)^{\frac{1}{3}} \int_{q''_{k,j}} |e(u)|^2 \, dx \\ &\lesssim \delta^{\frac{1}{3}} \int_{q''_{j,k}} |e(u)|^2 \, dx\end{aligned} \tag{7.8}$$

(see [15, p. 1389]), where ρ is the mollifier defined at the beginning of the section.

Notice that in view of our construction (namely the choice of r), we have

$$|\omega_{k,j}| \ll |q_{k,j}|, \tag{7.9}$$

and this is where we most use the fact that we are in two dimensions.

We now let $(\varphi_{k,j})$ be a partition of unity associated to the covering $(q_{k,j})$ of Q_r and such that $|\nabla\varphi_{k,j}| \lesssim \frac{1}{\delta_k}$. Let us set

$$w := 1_{Q_1\setminus Q_r} u + 1_{Q_r} \sum_{k,j} \varphi_{k,j} w_{k,j}, \quad \text{where } w_{k,j} := \rho_{\delta_k} * v_{k,j}.$$

We claim that

$$w \in \mathrm{SBD}(Q_1) \cap H^1(Q_r), \quad \{w \neq u\} \subset Q_r, \quad \mathcal{H}^1(J_w \setminus J) = 0, \tag{7.10}$$

$$\Big\| e(w) - \sum_{k,j} \varphi_{k,j} \rho_{\delta_k} * e(u) \Big\|_{L^2(Q_r)} \lesssim \delta^{\frac{1}{6}} \|e(u)\|_{L^2(Q_1)} \tag{7.11}$$

and

$$\text{the trace of } w \text{ and } u \text{ on } \partial Q_r \text{ coincide.} \tag{7.12}$$

We postpone the proof of these claims to Step 5.

Let us set

$$\varphi := \sum_{(0,j)\in\mathcal{K}} \varphi_{0,j},$$

where $\mathcal{K}$ denotes the set of indices such that $q_{0,j}$ has a distance greater than $2\delta r$ from ∂Q_r. Since $r \in]1-\delta^{\frac{1}{2}}, 1[$, in view of the definition of the set of indices $\mathcal{K}$, we get that the function φ vanishes on $Q \setminus Q_{r-\delta}$ and it is equal to 1 on $Q_{r-4\delta}$.

We can write

$$e(w) - \sum_{k,j} \varphi_{k,j} \rho_{\delta_k} * e(u) = [e(w) - \varphi\rho_\delta * e(u)] - \sum_{(k,j)\notin\mathcal{K}} \varphi_{k,j} \rho_{\delta_k} * e(u).$$

Thanks to (7.3) we have

$$\begin{aligned}
\Big\| \sum_{(k,j)\notin\mathcal{K}} \varphi_{k,j} \rho_{\delta_k} * e(u) \Big\|^2_{L^2(Q_r)} &= \Big\| \sum_{(k,j)\notin\mathcal{K}} \varphi_{k,j} \rho_{\delta_k} * e(u) \Big\|^2_{L^2(Q_r\setminus Q_{r-2\delta r})} \\
&\lesssim \sum_{(k,j)\notin\mathcal{K}} \|\varphi_{j,k} \rho_{\delta_k} * e(u)\|^2_{L^2(Q_1\setminus Q_{r-2\delta r})} \\
&\lesssim \|e(u)\|^2_{L^2(Q_1\setminus Q_{r-3\delta r})} \lesssim \delta^{\frac{1}{2}} \|e(u)\|^2_{L^2(Q_1)},
\end{aligned}$$

so that in view of (7.11) we conclude

$$\|e(w) - \varphi\rho_\delta * e(u)\|_{L^2(Q_r)} \lesssim \delta^{\frac{1}{6}} \|e(u)\|_{L^2(Q_1)}. \tag{7.13}$$

Moreover, we may write

$$\begin{aligned}
\|e(w)\|_{L^2(Q_r)} &\leq \|\varphi\rho_\delta * e(u)\|_{L^2(Q_r)} + \|e(w) - \varphi\rho_\delta * e(u)\|_{L^2(Q_r)} \\
&= \|\varphi\rho_\delta * e(u)\|_{L^2(Q_{r-\delta})} + \|e(w) - \varphi\rho_\delta * e(u)\|_{L^2(Q_r)} \\
&\leq \|e(u)\|_{L^2(Q_1)} + \|e(w) - \varphi\rho_\delta * e(u)\|_{L^2(Q_r)},
\end{aligned}$$

so that taking into account (7.13) we deduce

$$\|e(w)\|_{L^2(Q_1)} \le (1 + C\delta^{\frac{1}{6}})\|e(u)\|_{L^2(Q_1)}, \tag{7.14}$$

where $C > 0$.

Step 4: Enforcing the divergence-free constraint. By admissibility, u is divergence-free in the sense of distributions in Q_1, so that the trace of $e(u)$ is zero in Q_1, while

$$\int_{\partial Q_r} u \cdot \nu \, d\mathcal{H}^1 = 0, \tag{7.15}$$

where ν is the outward normal vector of Q_r, and u denotes the trace on ∂Q_r (J does not intersect ∂Q_r by construction).

Recalling that $w \in H^1(Q_r)$, we may write thanks to (7.13),

$$\|\operatorname{div} w\|_{L^2(Q_r)} = \|\operatorname{Tr}(e(w))\|_{L^2(Q_r)} = \|\operatorname{Tr}(e(w) - \varphi\rho_\delta * e(u))\|_{L^2(Q_r)} \lesssim \delta^{\frac{1}{6}} \|e(u)\|_{L^2(Q_1)}.$$

By (7.12) the trace of u on ∂Q_r coincides with that of w, so that from (7.15) we deduce

$$\int_{Q_r} \operatorname{div} w \, dx = 0.$$

Using a classical result (recorded at the end of this proof in Lemma 7.3), there exists a vector field $q \in H^1_0(Q_r)$ such that

$$\operatorname{div} q = \operatorname{div} w \quad \text{and} \quad \|\nabla q\|_{L^2(Q_r)} \lesssim \|\operatorname{div} w\|_{L^2(Q_r)} \lesssim \delta^{\frac{1}{6}} \|e(u)\|_{L^2(Q_1)}. \tag{7.16}$$

Let

$$v := \begin{cases} w - q & \text{in } Q_r, \\ u & \text{in } Q_1 \setminus Q_r, \end{cases}$$

and let us check that v satisfies the conclusions of the lemma.

The choice of r given by Step 2 immediately yields point (a). Clearly, $v \in SBD(Q_1) \cap H^1(Q_r)$ with $\{v \neq u\} \subseteq Q_r$. Moreover, since the trace of $w - q$ and u coincide on ∂Q_r, we get $\operatorname{div} v = 0$ in the sense of distributions in Q_1, so that point (b) is proved. Points (c) and (d) follow from the corresponding properties for w (see (7.13) and (7.14)) taking into account that the correction term q has a small gradient norm of the order $\delta^{\frac{1}{6}}$ as estimated in (7.16).

Step 5: Proof of the claims (7.10), (7.11) and (7.12). In order to conclude the proof, we need to check the claims on the function w contained in Step 3.

Let us start by noticing that the oscillation of the maps $a_{k,j}$ on intersecting squares can be estimated. Indeed, as soon as $q_{k,j}$ and $q_{p,i}$ intersect, then

$$|q_{k,j} \cap q_{p,i}| \gtrsim \max(|q_{k,j}|, |q_{p,i}|),$$

and since (see (7.9))

$$|(q'_{k,j} \cap q'_{p,i}) \cap (\omega_{k,j} \cup \omega_{p,i})| \ll |q'_{k,j} \cap q'_{p,i}|$$

and $a_{j,k}$, $a_{i,p}$ are affine, then using [16, Lemma 3.4] and (7.7) we deduce

$$\begin{aligned}\|a_{k,j} - a_{p,i}\|_{L^4(q'_{k,j} \cap q'_{p,i})} &\lesssim \|a_{k,j} - a_{p,i}\|_{L^4((q'_{k,j} \cap q'_{p,i}) \setminus (\omega_{k,j} \cup \omega_{p,i}))} \\ &\leq \|a_{k,j} - u\|_{L^4(q'_{k,j} \setminus \omega_{k,j})} + \|a_{p,i} - u\|_{L^4(q'_{p,i} \setminus \omega_{p,i})} \\ &\lesssim \delta_k^{\frac{1}{2}} \|e(u)\|_{L^2(q''_{k,j})} + \delta_p^{\frac{1}{2}} \|e(u)\|_{L^2(q''_{p,i})} \\ &\lesssim \delta_k^{\frac{1}{2}} \|e(u)\|_{L^2(q''_{k,j} \cup q''_{p,i})}, \end{aligned} \tag{7.17}$$

as δ_k and δ_p are comparable.

Let us come to the claims. Clearly,

$$e(w) = \sum_{k,j} \varphi_{k,j} e(w_{k,j}) + \sum_{k,j} \nabla \varphi_{k,j} \odot w_{k,j},$$

so that

$$\begin{aligned} e(w) - \sum_{k,j} \varphi_{k,j} \rho_{\delta_k} * e(u) \\ = \sum_{k,j} \varphi_{k,j} [e(w_{k,j}) - \rho_{\delta_k} * e(u)] + \sum_{k,j} \nabla \varphi_{k,j} \odot w_{k,j}. \end{aligned} \tag{7.18}$$

For the first term of the right-hand side, we have, thanks to (7.8),

$$\begin{aligned} &\left\| \sum_{k,j} \varphi_{k,j} [e(w_{k,j}) - \rho_{\delta_k} * e(u)] \right\|^2_{L^2(Q_r)} \\ &\quad \lesssim \sum_{k,j} \left\| \varphi_{k,j} [e(w_{k,j}) - \rho_{\delta_k} * e(u)] \right\|^2_{L^2(Q_r)} \\ &\quad \leq \sum_{k,j} \|e(w_{k,j}) - \rho_{\delta_k} * e(u)\|^2_{L^2(q_{k,j})} \\ &\quad \leq \delta^{\frac{1}{3}} \sum_{k,j} \|e(u)\|^2_{L^2(q''_{k,j})} \lesssim \delta^{\frac{1}{3}} \|e(u)\|^2_{L^2(Q_r)}, \end{aligned} \tag{7.19}$$

where we used the finite overlapping of the squares $q''_{k,j}$ for the first and last estimates.

Let us estimate the second term on the right-hand side of (7.18). Notice that we may write

$$\sum_{k,j} \nabla \varphi_{k,j} \odot w_{k,j} = \sum_{q_{k,j} \cap q_{p,i} \neq \emptyset} \nabla \varphi_{k,j} \odot (w_{k,j} - w_{p,i}) \quad \text{on } q_{p,i}$$

since $\sum_{k,j} \nabla \varphi_{k,j} = 0$.

(a1) If $q''_{p,i} \Subset Q_{r_{-1}}$, then $q_{j,k} \cap q_{i,p} \neq \emptyset$ means that $\delta_k = \delta_p = \delta$, $k = p = 0$, and we may rewrite the term as

$$\sum_{q_{0,j} \cap q_{0,i} \neq \emptyset} \nabla\varphi_{0,j} \odot (w_{0,j} - w_{0,i}).$$

We get

$$\begin{aligned} \Big\| \sum_{q_{0,j} \cap q_{0,i} \neq \emptyset} \nabla\varphi_{0,j} \odot (w_{0,j} - w_{0,i}) \Big\|^2_{L^2(q_{0,i})} \\ \lesssim \sum_{q_{0,j} \cap q_{0,i} \neq \emptyset} \frac{1}{\delta^2} \|w_{0,j} - w_{0,i}\|^2_{L^2(q_{0,j} \cap q_{0,i})}. \end{aligned} \tag{7.20}$$

Now

$$\begin{aligned} \|w_{0,j} - w_{0,i}\|_{L^2(q_{o,j} \cap q_{0,i})} &= \|\rho_\delta * (v_{0,j} - v_{0,i})\|_{L^2(q_{0,j} \cap q_{0,i})} \\ &\leq \|v_{0,j} - v_{0,i}\|_{L^2(q'_{0,j} \cap q'_{0,i})}. \end{aligned}$$

Since

$$\begin{aligned} \|v_{0,j} - v_{0,i}\|_{L^2(q'_{0,j} \cap q'_{0,i})} &\leq \|(a_{0,j} - a_{0,i}) 1_{\omega_{0,j} \cup \omega_{0,i}}\|_{L^2(q'_{0,j} \cap q'_{0,i})} \\ &\quad + \|(u - a_{0,j}) 1_{\omega_{0,i}}\|_{L^2(q'_{0,j} \setminus \omega_{0,j})} \\ &\quad + \|(u - a_{0,i}) 1_{\omega_{0,j}}\|_{L^2(q'_{0,i} \setminus \omega_{0,i})} \\ &\leq \|(a_{0,j} - a_{0,i})\|_{L^4(q'_{0,j} \cap q'_{0,i})} |\omega_{0,j} \cup \omega_{0,i}|^{\frac{1}{4}} \\ &\quad + \|(u - a_{0,j})\|_{L^4(q'_{0,j} \setminus \omega_{0,j})} |\omega_{0,i}|^{\frac{1}{4}} \\ &\quad + \|(u - a_{0,i})\|_{L^4(q'_{0,i} \setminus \omega_{0,i})} |\omega_{0,j}|^{\frac{1}{4}}, \end{aligned}$$

recalling (7.6), (7.7) and (7.17) we get

$$\|w_{0,j} - w_{0,i}\|_{L^2(q_{o,j} \cap q_{0,i})} \leq \|v_{0,j} - v_{0,i}\|_{L^2(q'_{0,j} \cap q'_{0,i})} \leq \delta^{1 + \frac{1}{4}} \|e(u)\|_{L^2(q''_{0,j} \cup q''_{0,i})}.$$

Coming back to (7.20) we infer

$$\begin{aligned} \Big\| \sum_{k,j} \nabla\varphi_{k,j} \odot w_{k,j} \Big\|^2_{L^2(q_{0,i})} &\leq \Big\| \sum_{q_{0,j} \cap q_{0,i} \neq \emptyset} \nabla\varphi_{0,j} \odot (w_{0,j} - w_{0,i}) \Big\|^2_{L^2(q_{0,i})} \\ &\lesssim \delta^{\frac{1}{2}} \sum_{q_{0,j} \cap q_{0,i} \neq \emptyset} \|e(u)\|^2_{L^2(q''_{0,j} \cup q''_{0,i})}. \end{aligned} \tag{7.21}$$

(a2) If $q_{p,i} \not\Subset Q_{r_{-1}}$, then for $q_{k,j} \cap q_{p,i} \neq \emptyset$, we decompose

$$w_{p,i} - w_{k,j} = \rho_{\delta_p} * (v_{p,i} - a_{p,i}) - \rho_{\delta_k} * (w_{k,j} - a_{k,j}) + (a_{p,i} - a_{k,j}).$$

Notice the crucial step that $\rho_{\delta_k} * a_{k,j} = a_{k,j}$ due to the fact that $a_{k,j}$ is harmonic (since it is affine). Then we have, thanks to (7.7) and (7.17),

$$\|\rho_{\delta_k} * (v_{p,i} - a_{p,i})\|_{L^2(q_{k,j}\cap q_{p,i})} \leq \|v_{p,i} - a_{p,i}\|_{L^2(q'_{p,i})} \lesssim \delta_p \|e(u)\|_{L^2(q''_{i,p})},$$

$$\|\rho_{\delta_k} * (v_{k,j} - a_{k,j})\|_{L^2(q_{k,j}\cap q_{p,i})} \leq \|v_{k,j} - a_{k,j}\|_{L^2(q'_{k,j})} \lesssim \delta_p \|e(u)\|_{L^2(q''_{k,j})},$$

$$\|a_{p,i} - a_{k,j}\|_{L^2(q_{k,j}\cap q_{p,i})} \lesssim \delta_p^{1+\frac{1}{4}} \|e(u)\|_{L^2(q''_{k,j}\cup q''_{p,i})},$$

where we also used the fact that δ_p and δ_k differ by at most a factor 2. And so, with the same computations as in the previous point, we obtain

$$\Big\|\sum_{k,j} \nabla\varphi_{k,j} \odot w_{k,j}\Big\|^2_{L^2(q_{p,i})} \leq \sum_{q_{k,j}\cap q_{p,i}\neq\emptyset} \|e(u)\|^2_{L^2(q''_{k,j}\cup q''_{p,i})}. \tag{7.22}$$

Gathering (7.21) and (7.22), and in view of the choice of r which satisfies (7.3), we deduce

$$\begin{aligned}\Big\|\sum_{k,j} \nabla\varphi_{k,j} \odot w_{k,j}\Big\|^2_{L^2(Q_r)} &\leq \sum_{p,i}\Big\|\sum_{k,j} \nabla\varphi_{k,j} \odot w_{k,j}\Big\|^2_{L^2(q_{p,i})} \\ &\lesssim \delta^{\frac{1}{2}} \|e(u)\|^2_{L^2(Q_{r_1})} + \|e(u)\|^2_{L^2(Q_r\setminus Q_{r_{-2}})} \\ &\lesssim \delta^{\frac{1}{2}} \|e(u)\|^2_{L^2(Q_1)}. \end{aligned} \tag{7.23}$$

Coming back to (7.18), in view of (7.19) and (7.23) we deduce that

$$\Big\|e(w) - \sum_{k,j} \varphi_{k,j}\rho_{\delta_k} * e(u)\Big\|_{L^2(Q_r)} \lesssim \delta^{\frac{1}{6}} \|e(u)\|_{L^2(Q_1)},$$

so that claim (7.11) follows.

In particular, we get also that $w \in H^1(Q_r)$. Claim (7.12) concerning the traces follows by the construction which involves convolutions whose radii become finer and finer as we approach ∂Q_r as detailed in [16]. Finally, we deduce that $w \in \mathrm{SBD}(Q_1)$, and that claim (7.10) holds true. ∎

In the proof of Proposition 7.2 we made use of the following lemma due to Nečas (see [6, Theorem IV.3.1], or also [4]).

Lemma 7.3. *Let Ω be a bounded, connected open set with Lipschitz boundary, and let $L^2_0(\Omega)$ be the set of zero-average L^2-functions. Then there is a continuous linear map $\Phi\colon L^2_0(\Omega) \to H^1_0(\Omega;\mathbb{R}^d)$ such that* $\operatorname{div}\circ\Phi = \mathrm{Id}_{L^2_0(\Omega)}$.

7.2. Regularity for quasi minimizers of the Griffith energy

Let $\Omega \subseteq \mathbb{R}^2$ be an open set. In all the following, we will consider the *Griffith functional*

$$G(J,u,B) := \int_B |e(u)|^2\,dx + \mathcal{H}^1(J\cap B),$$

where $B \subseteq \Omega$ is a Borel set.

We consider the following (*very* weak) notion of local minimality.

Definition 7.4 (Quasi minimizers). Let $\Lambda, \bar{r} > 0$. We say that $(J, u) \in \mathcal{V}(\Omega)$ (recall Definition 7.1) is a $(\Lambda, \bar{r})$ quasi minimizer of G on $\mathcal{V}(\Omega)$ if $G(J, u, \omega) < +\infty$ for any open set $\omega \Subset \Omega$, and for any square $Q_{x,r} \Subset \Omega$ with $r \in (0, \bar{r})$, $\mathcal{H}^0(J \cap \partial Q_{x,r}) = 0$ and

$$\limsup_{s \to 0^+} \frac{1}{s} \mathcal{H}^1(J \cap (Q_{x,r} \setminus Q_{x,r-s})) < 1,$$

and for any function $v \in H^1(Q_{x,r}; \mathbb{R}^2)$ with $\operatorname{div} v = 0$ and $v = u$ on $\partial Q_{x,r}$, we have

$$\int_{Q_{x,r}} |e(u)|^2 \, dx + \mathcal{H}^1(J \cap Q_{x,r}) \leq \int_{Q_{x,r}} |e(v)|^2 \, dx + \Lambda r^2. \tag{7.24}$$

Remark 7.5. Notice that under the assumption of the previous definition, we have $(J \setminus Q_{x,r}, v) \in \mathcal{V}(\Omega)$, where we extended v to the entire Ω by setting $v = u$ in $\Omega \setminus Q_{x,r}$, and inequality (7.24) may be written as

$$G(J, u, Q_{x,r}) \leq G(J \setminus Q_{x,r}, v, Q_{x,r}) + \Lambda r^2.$$

The local minimality property involves thus a comparison between (J, u) and very special competitors: the Sobolev function v is obtained by "smoothing out" the jumps of u inside suitable squares $Q_{x,r}$, so that it can be paired with the rectifiable set $J \setminus Q_{x,r}$, yielding the admissible pair $(J \setminus Q_{x,r}, v)$. Such competitors are provided by the smoothing Lemma 7.2, for which the dimension two is essential. A somehow related weak notion of minimality involving Sobolev competitors, still in dimension two, has been investigated in [9] (*minimality with respect to its own jump set*) for the (scalar) Mumford–Shah functional.

Remark 7.6. The notion of minimality is weak enough to include any local minimizer of a functional of the form

$$F(u, A) := \int_A |e(u)|^2 \, dx + \int_{J_u \cap A} \Theta(\nu_u, u^+, u^-) \, d\mathcal{H}^1,$$

where Θ is a measurable function such that $\inf(\Theta) \geq 1$ (or $\inf(\Theta) > 0$ up to scaling).

The following result is the key ingredient for obtaining regularity.

Proposition 7.7 (Decay estimate). *Let $\Lambda > 0$. There exists a universal constant $\bar{\tau} \in (0, 1)$ such that for every $\tau \in (0, \bar{\tau})$ there exist $\varepsilon = \varepsilon(\tau)$ and $\bar{r} = \bar{r}(\tau)$ with the property that for any $(\Lambda, \bar{r})$-quasi minimizer (J, u) of G on $\mathcal{V}(\Omega)$, if for $r < \bar{r}$,*

$$G(J, u, Q_r) \geq r^{3/2} \quad \text{and} \quad \mathcal{H}^1(J \cap Q_r) \leq \varepsilon r,$$

then

$$G(J, u, Q_{\tau r}) \leq \tau^{3/2} G(J, u, Q_r).$$

Proof. By contradiction assume that for τ sufficiently small there exist $\varepsilon_n \to 0$, $\bar{r}_n \to 0$, $0 < r_n < \bar{r}_n$, and a sequence (K_n, w_n) of $(\Lambda, \bar{r}_n)$-minimizers such that for every n,

$$G(K_n, w_n, Q_{r_n}) \geq r_n^{3/2}, \quad \mathcal{H}^1(K_n \cap Q_{r_n}) \leq \varepsilon_n r_n,$$
$$G(K_n, w_n, Q_{\tau r_n}) > \tau^{3/2} G(K_n, w_n, Q_{r_n}).$$

Let

$$g_n := G(K_n, w_n, Q_{r_n}), \quad J_n := \frac{K_n}{r_n} \quad \text{and} \quad u_n(x) := \frac{w_n(r_n x)}{\sqrt{g_n}}.$$

Then (J_n, u_n) is a $(\Lambda\sqrt{r_n}, 1)$-minimizer of $G_n(\cdot, \cdot, Q_1)$, where

$$G_n(J, u, A) := \int_A |e(u)|^2 \, dx + \frac{r_n}{g_n} \mathcal{H}^1(J \cap A),$$

with

$$G_n(J_n, u_n, Q_1) = 1, \quad G_n(J_n, u_n, Q_\tau) > \tau^{3/2} \quad \text{and} \quad \mathcal{H}^1(J_n \cap Q_1) = \varepsilon_n. \tag{7.25}$$

Let us apply the smoothing Lemma 7.2: if $\delta_n = \varepsilon_n^{\frac{1}{2}}$, let Q_{s_n} with $1 - \delta_n^{\frac{1}{2}} < s_n < 1$ be the square on which the jumps of u_n are smoothed out giving rise to the function v_n, associated to an admissible pair $(J \setminus Q_{s_n}, v_n) \in \mathcal{V}(Q_1)$. In particular,

$$\|e(v_n)\|_{L^2(Q_1)} \leq (1 + C\delta_n^{\frac{1}{6}}) \|e(u_n)\|_{L^2(Q_1)} \quad \text{with } \|e(u_n)\|_{L^2(Q_1)} \leq 1, \tag{7.26}$$

and

$$\|e(v_n) - \varphi_n \rho_{\delta_n} * e(u)\|_{L^2(Q_{s_n})} \leq C\delta_n^{\frac{1}{6}} \|e(u_n)\|_{L^2(Q_1)}, \tag{7.27}$$

where $C > 0$ is independent of n and $\varphi_n \in C^\infty(Q_{s_n}, [0, 1])$ is such that $\varphi_n = 0$ on $Q_{s_n} \setminus Q_{s_n - \delta_n}$, $\varphi_n = 1$ on $Q_{s_n - 4\delta_n}$. Since v_n is divergence-free and Sobolev on Q_{s_n} we have

$$\int_{\partial Q_{s_n}} v_n \cdot \nu \, d\mathcal{H}^1 = 0. \tag{7.28}$$

By the classical Korn inequality on Q_{s_n} there is an antisymmetric affine function a_n such that $\int_{Q_{s_n}} (v_n - a_n) \, dx = 0$ and

$$\int_{Q_{s_n}} |\nabla(v_n - a_n)|^2 \, dx \leq C_1 \int_{Q_{s_n}} |e(v_n)|^2 \, dx$$

for some $C_1 > 0$ independent of n. We infer that $(v_n - a_n)$ is bounded in $H^1(Q_{s_n})$. Since $s_n \to 1$, we can assume, up to extracting a further subsequence,

$$v_n - a_n \rightharpoonup w \quad \text{weakly in } H^1_{\text{loc}}(Q_1; \mathbb{R}^2) \tag{7.29}$$

for some $w \in H^1(Q_1)$. Since every $v_n - a_n$ has zero divergence, then so does w. Moreover, $\|e(w)\|_{L^2(Q_1)} \leq 1$.

Let $\psi \in C_c^\infty(Q_1;\mathbb{R}^2)$ have zero divergence, and let $\eta \in C_c^\infty(Q_1,[0,1])$ be a cut-off function such that $\{\psi \neq 0\} \Subset \{\eta = 1\}$. Let us consider

$$z_n := \begin{cases} \mathbb{P}_{Q_{s_n}}[(1-\eta)v_n + \eta(a_n + w + \psi)] & \text{in } Q_{s_n}, \\ u_n & \text{in } Q_1 \setminus Q_{s_n}, \end{cases}$$

where $\mathbb{P}_{Q_{s_n}}$ denotes the projection on divergence-free $H^1(Q_{s_n})$ vector fields which preserves the trace obtained according to Lemma 7.3 by considering

$$\mathbb{P}_{Q_{s_n}}(u) := u - \Phi_{Q_{s_n}}(\operatorname{div} u)$$

for any $u \in H^1(Q_{s_n};\mathbb{R}^2)$ with a zero mean divergence. Note that z_n is well defined as

$$(1-\eta)v_n + \eta(a_n + w + \psi) = v_n \quad \text{on } \partial Q_{s_n}$$

for n large enough, and so its divergence has zero mean thanks to (7.28).

Since $(J_n \setminus \partial Q_{s_n}, z_n)$ is an admissible competitor for (J_n, u_n) according to Definition 7.4, we obtain

$$\begin{aligned} G_n(J_n, u_n, Q_{s_n}) &\le \|e(\mathbb{P}_{Q_{s_n}}[(1-\eta)v_n + \eta(a_n + w + \psi)])\|^2_{L^2(Q_{s_n})} + \Lambda\sqrt{r_n} \\ &\le \big(\|e((1-\eta)v_n + \eta(a_n + w + \psi))\|_{L^2(Q_{s_n})} \\ &\qquad + C\|\operatorname{div}((1-\eta)v_n + \eta(a_n + w + \psi))\|_{L^2(Q_{s_n})}\big)^2 + \Lambda\sqrt{r_n} \\ &\le \big(\|e((1-\eta)v_n + \eta(a_n + w + \psi))\|_{L^2(Q_{s_n})} \\ &\qquad + C\|\nabla\eta \cdot (w + a_n - v_n)\|_{L^2(Q_{s_n})}\big)^2 + \Lambda\sqrt{r_n}. \end{aligned}$$

Since

$$\|\nabla\eta \cdot (w + a_n - v_n)\|_{L^2(Q_{s_n})} \to 0$$

and (recall that $\{\psi \neq 0\} \Subset \{\eta = 1\}$)

$$\begin{aligned} &\|e((1-\eta)v_n + \eta(a_n + w + \psi))\|_{L^2(Q_{s_n})} \\ &\quad = \|(1-\eta)e(v_n) + \eta e(w + \psi) + \nabla\eta \odot (w + a_n - v_n)\|_{L^2(Q_{s_n})} \\ &\quad \le \|(1-\eta)e(v_n) + \eta e(w + \psi)\|_{L^2(Q_{s_n})} + o_n, \end{aligned}$$

where $o_n \to 0$, we infer thanks to (7.26) (and since $e(a_n) = 0$),

$$G_n(J_n, u_n, Q_{s_n}) \le \|(1-\eta)e(v_n - a_n) + \eta e(w + \psi)\|^2_{L^2(Q_{s_n})} + o_n. \tag{7.30}$$

Now, still using (7.26) we may write

$$\begin{aligned} \int_{Q_{s_n}} |e(v_n - a_n)|^2\,dx &\le (1 + C\delta_n^{\frac{1}{6}})^2 \int_{Q_{s_n}} |e(u_n)|^2\,dx \\ &\le (1 + C\delta_n^{\frac{1}{6}})^2 G_n(J_n, u_n, Q_{s_n}) + o_n, \end{aligned}$$

and so coming back to (7.30) we deduce

$$\|e(v_n - a_n)\|^2_{L^2(Q_{s_n})} \le \|(1-\eta)e(v_n - a_n) + \eta e(w+\psi)\|^2_{L^2(Q_{s_n})} + o_n.$$

This yields

$$\begin{aligned}\int_{Q_{s_n}} &(1-(1-\eta)^2)|e(v_n - a_n)|^2\,dx \\ &\le \int_{Q_{s_n}} (2\eta(1-\eta)e(v_n-a_n) : e(w) + \eta^2|e(w+\psi)|^2)\,dx + o_n,\end{aligned}$$

so that in view of (7.29),

$$\begin{aligned}\int_{Q_1} (1-(1-\eta)^2)|e(w)|^2\,dx &\le \limsup_{n\to\infty} \int_{Q_1} (1-(1-\eta)^2)|e(v_n - a_n)|^2\,dx \\ &\le \int_{Q_1} (2\eta(1-\eta)e(w) : e(w) + \eta^2|e(w+\psi)|^2)\,dx.\end{aligned}$$

Notice that by choosing $\psi = 0$ and letting η localize on characteristic functions of open sets, we infer that

$$e(v_n - a_n) \to e(w) \quad \text{strongly in } L^2_{\text{loc}}(Q_1; M^{2\times 2}_{\text{sym}}). \tag{7.31}$$

In particular, we get

$$\int_{Q_1} |e(w)|^2\,dx \le \int_{Q_1} |e(w+\psi)|^2\,dx,$$

which means that w is a local minimizer of the energy $z \mapsto \|e(z)\|^2_{L^2(Q_1)}$ on H^1 functions with zero divergence. This yields $\Delta w = \nabla p$ for some $p \in L^2(Q_1)$. Using Lemma 7.8 below, we have

$$\int_{Q_\tau} |e(w)|^2\,dx \le \frac{1}{2}\tau^{\frac{3}{2}} \int_{Q_1} |e(w)|^2\,dx \le \frac{1}{2}\tau^{\frac{3}{2}}.$$

Taking into account (7.31) we deduce

$$\|e(v_n)\|^2_{L^2(Q_{\tau+\delta_n})} \le \frac{1}{2}\tau^{\frac{3}{2}} + o_n. \tag{7.32}$$

By minimality we have

$$\begin{aligned}G(u_n, J_n, Q_{s_n}) &\le \|e(v_n)\|^2_{L^2(Q_{s_n})} + \Lambda\sqrt{r_n} \\ &= \|e(v_n)\|^2_{L^2(Q_{\tau+\delta_n})} + \|e(v_n)\|^2_{L^2(Q_{s_n}\setminus Q_{\tau+\delta_n})} + \Lambda\sqrt{r_n},\end{aligned}$$

while thanks to (7.27),

$$\begin{aligned}\|e(v_n)\|_{L^2(Q_{s_n}\setminus Q_{\tau+\delta_n})} &\le \|e(v_n) - \varphi_n\rho_{\delta_n} * e(u_n)\|_{L^2(Q_{s_n}\setminus Q_{\tau+\delta_n})} \\ &\quad + \|\varphi_n\rho_{\delta_n} * e(u_n)\|_{L^2(Q_{s_n}\setminus Q_{\tau+\delta_n})} \\ &\le o_n + \|\rho_{\delta_n} * e(u_n)\|_{L^2(Q_{s_n-\delta_n}\setminus Q_{\tau+\delta_n})} \\ &\le o_n + \|e(u_n)\|_{L^2(Q_{s_n}\setminus Q_\tau)}.\end{aligned}$$

In view of (7.32) we infer

$$G(u_n, J_n, Q_{s_n}) \leq \|e(v_n)\|^2_{L^2(Q_{\tau+\delta_n})} + [o_n + \|e(u_n)\|_{L^2(Q_{s_n}\setminus Q_\tau)}]^2 + \Lambda\sqrt{r_n}$$
$$\leq \frac{1}{2}\tau^{\frac{3}{2}} + \tilde{o}_n + \|e(u_n)\|^2_{L^2(Q_{s_n}\setminus Q_\tau)}$$

so that

$$G(u_n, J_n, Q_\tau) \leq \frac{1}{2}\tau^{\frac{3}{2}} + \tilde{o}_n.$$

In conclusion, taking into account (7.25), if n is large enough we get

$$\tau^{\frac{3}{2}} < G(u_n, J_n, Q_\tau) \leq \frac{1}{2}\tau^{\frac{3}{2}} + \tilde{o}_n,$$

which is a contradiction. ∎

In the preceding proof, we made use of the following result.

Lemma 7.8. *There exists a constant $C_0 > 0$ such that for any divergence-free vector field $u \in H^1(Q_1;\mathbb{R}^2)$ such that $\Delta u = \nabla p$ for some pressure $p \in L^2(Q_1)$, we have*

$$\forall \tau \in (0, 1/2]: \quad \int_{Q_\tau} |e(u)|^2\,dx \leq C_0\tau^2 \int_{Q_1} |e(u)|^2\,dx.$$

In particular, for any $0 < \tau \leq \bar{\tau} := \frac{1}{4C_0^2} \wedge \frac{1}{2}$ we have

$$\int_{Q_\tau} |e(u)|^2\,dx \leq \frac{1}{2}\tau^{3/2} \int_{Q_1} |e(u)|^2\,dx.$$

Proof. Notice that $e(u)$ is invariant by the addition of an asymmetric affine function a. Up to a translation by such a function, Korn's inequality tells us that

$$\int_{Q_1} u^2\,dx \leq C \int_{Q_1} |e(u)|^2\,dx.$$

The equations verified by u are equivalent to the existence of $\varphi \in H^2(Q_1)$ such that $\varphi(0) = 0$, $u = \nabla^\perp\varphi$, and $\Delta^2\varphi = 0$. By elliptic regularity there is a constant C' such that

$$\sup_{Q_{1/2}} |\nabla^2\varphi|^2 \leq C' \int_{Q_1} |\nabla\varphi|^2\,dx$$

and so for any $\tau \leq 1/2$,

$$\int_{Q_\tau} |e(u)|^2\,dx \leq 4|Q_\tau| \sup_{Q_{1/2}} |\nabla^2\varphi|^2 \leq 4CC'|Q_1|\tau^2 \int_{Q_1} |e(u)|^2\,dx. \quad ∎$$

The decay estimate can be iterated as follows.

Lemma 7.9 (Iteration of the decay). *Let $\Lambda > 0$ and, according to Proposition 7.7, let τ_0 be small enough such that the decay estimate applies with $\varepsilon_0 = \varepsilon(\tau_0)$ and $\bar{r}_0 = \bar{r}(\tau_0)$, and let $\tau_1 \in (0, \varepsilon_0^2)$ be small enough that the decay property applies with ε_1, $\bar{r}_1$. Finally, let*

$$\bar{r} := \min\Big(\bar{r}_0, \bar{r}_1, \varepsilon_0^2\tau_1^2, \frac{\varepsilon_0^2\tau_0^3}{\tau_1}\Big).$$

Suppose that (J, u) is a $(\Lambda, \bar{r})$-quasi minimizer of G on $\mathcal{V}(\Omega)$ and $G(J, u, Q_{x,r}) \leq \varepsilon_1 r$ for some $r \in (0, \bar{r})$. Then for all $k \in \mathbb{N}$,

$$G(J, u, Q_{x,\tau_0^k\tau_1 r}) \leq \varepsilon_0\tau_0^{\frac{3}{2}k}\tau_1 r.$$

Proof. Let us prove the statement by induction on k. In the following, we write $g(r) = G(J, u, Q_{x,r})$, so that we need to check that if $g(r) \leq \varepsilon_1 r$, then for every $k \in \mathbb{N}$,

$$g(\tau_0^k\tau_1 r) \leq \varepsilon_0\tau_0^{\frac{3}{2}k}\tau_1 r. \tag{7.33}$$

The inequality is true for $k = 0$. Indeed, we have the following alternatives:

(a) If $g(r) > r^{3/2}$ then $g(\tau_1 r) \leq \tau_1^{3/2} g(r) \leq \sqrt{\tau_1}\tau_1\varepsilon_1 r \leq \varepsilon_0\tau_1 r$ by definition of τ_1.

(b) If $g(r) \leq r^{3/2}$, then $g(\tau_1 r) \leq g(r) \leq r^{3/2} \leq \varepsilon_0\tau_1 r$ by definition of $\bar{r}$.

Assume now that (7.33) holds. Notice that by definition of G we have (since $\tau_0 < 1$)

$$\mathcal{H}^1(J \cap Q_{\tau_0^k\tau_1 r}) \leq g(\tau_0^k\tau_1 r) \leq \varepsilon_0\tau_0^{\frac{3}{2}k}\tau_1 r \leq \varepsilon_0\tau_0^k\tau_1 r,$$

so the decay property of Proposition 7.7 may be applied. Again we have two alternatives:

(a) If $g(\tau_0^k\tau_1 r) > (\tau_0^k\tau_1 r)^{3/2}$, by the decay property we have, using (7.33),

$$g(\tau_0^{k+1}\tau_1 r) \leq \tau_0^{3/2} g(\tau_0^k\tau_1 r) \leq \varepsilon_0\tau_0^{\frac{3}{2}(k+1)}\tau_1 r.$$

(b) If $g(\tau_0^k\tau_1 r) \leq (\tau_0^k\tau_1 r)^{3/2}$ then by the definition of $\bar{r}$,

$$g(\tau_0^{k+1}\tau_1 r) \leq g(\tau_0^k\tau_1 r) \leq \sqrt{\frac{\tau_1 r}{\varepsilon_0^2\tau_0^3}}\varepsilon_0\tau_0^{\frac{3}{2}(k+1)}\tau_1 r \leq \varepsilon_0\tau_0^{\frac{3}{2}(k+1)}\tau_1 r.$$

In both cases, (7.33) follows for the choice $k + 1$, so that the induction step is proved. ■

If we want to draw some conclusions on the regularity of quasi minimizers (J, u), we need somehow to bound the freedom connected to the choice of J: notice indeed that any pair $(J\Delta N, u)$ with $\mathcal{H}^1(N) = 0$ is essentially equivalent to (J, u), where $A\Delta B$ denotes the symmetric difference of sets.

We set

$$J^+ := \Big\{x \in \Omega : \limsup_{r\to 0} \frac{\mathcal{H}^1(J\cap Q_{x,r})}{r} > 0\Big\}, \tag{7.34}$$

where J^+ is a sort of normalized version of J, where points of density zero have been erased.

By standard properties of rectifiable sets we have

$$\mathcal{H}^1(J \Delta J^+) = 0.$$

As a consequence if $(J, u) \in \mathcal{V}(\Omega)$, then also $(J^+, u) \in \mathcal{V}(\Omega)$ with $G(J, u, A) = G(J^+, u, A)$ for every Borel set $A \subseteq \Omega$.

Proposition 7.10. *Given $\Lambda > 0$, there exist $\varepsilon, \bar{r} > 0$ such that of any $(\Lambda, \bar{r})$-quasi minimizer (J, u) of G on $\mathcal{V}(\Omega)$, if $G(J, u, Q_{x,r}) \leq \varepsilon r$ for some $Q_{x,r} \Subset \Omega$ with $r < \bar{r}$, then $J^+ \cap Q_{x,\frac{r}{2}} = \emptyset$.*

Proof. Let $\varepsilon_0, \varepsilon_1, \tau_1, \tau_2, \bar{r}$ be given according to Lemma 7.9. Notice that if $G(J, u, Q_{x,r}) \leq \varepsilon_1 r$ with $r < \bar{r}$, then for any $\rho \in (0, r)$,

$$G(J, u, Q_{x,\rho}) \leq C_0 r^{-\frac{1}{2}} \rho^{\frac{3}{2}}, \quad \text{where } C_0 := \max\{\varepsilon_1 \tau_1^{-\frac{3}{2}}, \varepsilon_0 \tau_0^{-\frac{1}{2}} \tau_1^{-\frac{1}{2}}\}. \tag{7.35}$$

Let us set $\varepsilon := \frac{1}{2}\varepsilon_1$, and assume $G(J, u, Q_{x,r}) \leq \varepsilon r$. Notice that for any $y \in Q_{x,\frac{r}{2}}$, we have

$$G(J, u, Q_{y,\frac{r}{2}}) \leq G(J, u, Q_{x,r}) \leq \varepsilon r = \varepsilon_1 \frac{r}{2},$$

so that from (7.35),

$$0 = \lim_{\rho \to 0^+} \frac{G(J, u, Q_{y,\rho})}{\rho} \geq \limsup_{\rho \to 0^+} \frac{\mathcal{H}^1(J \cap Q_{y,\rho})}{\rho},$$

which yields $J^+ \cap Q_{x,\frac{r}{2}} = \emptyset$. ∎

Proposition 7.11 (Regularity for quasi minimizers). *Let $\Lambda, \bar{r} > 0$. Then for any $(\Lambda, \bar{r})$-quasi minimizer (J, u) of G on $\mathcal{V}(\Omega)$ we have that J^+ (see (7.34)) is essentially closed in Ω, i.e.*

$$\mathcal{H}^1(\Omega \cap (\overline{J^+} \setminus J^+)) = 0,$$

while $u \in C^\infty(\Omega \setminus \overline{J^+})$.

Proof. Since the functional G coincides with a volume integral outside J, there exists an $\mathcal{H}^1$-negligible set $N \subset \Omega \setminus J$ such that for every $x \in \Omega \setminus (J \cup N)$ we have

$$\lim_{\rho \to 0} \frac{G(J, u, Q_{x,\rho})}{\rho} = 0.$$

Thanks to Proposition 7.10 we infer

$$\Omega \cap \overline{J^+} \subset J \cup N \subset J^+ \cup (J \setminus J^+) \cup N.$$

Since the last two sets are $\mathcal{H}^1$-negligible, we infer $\mathcal{H}^1(\Omega \cap (\overline{J^+} \setminus J^+)) = 0$.

Since

$$\mathcal{H}^1(J_u \setminus \overline{J^+}) \leq \mathcal{H}^1(J \setminus \overline{J^+}) = 0,$$

we get that u is locally H^1 on $\Omega \setminus \overline{J^+}$ (thanks to Korn's inequality): smoothness then follows from the regularity theory for solutions to Stokes equation (see e.g. [6, Theorem IV.5.8]). ∎

7.3. Proof of Theorem 4.10

We are now in a position to prove the regularity result given by Theorem 4.10.

Let (E,u) be a minimizer of $\mathcal{J}$ and let us set

$$\Lambda := 4\,\mathrm{Lip}(f) \quad \text{and} \quad J := J_u \cup \partial^* E.$$

We also assume (up to multiplying u by $c^{-\frac{1}{2}}$) that the constant c of (4.2) is 1.

We first prove that (J,u) is a $(\Lambda,1)$ quasi minimizer of the Griffith functional G on $\mathcal{V}(\Omega)$ according to Definition 7.4. Indeed, let $Q_{x,r} \Subset \Omega$ with $r < 1$ be a square as in Definition 7.4, with associated competitor $(J \setminus Q_{x,r}, v)$. We claim that either

$$\mathcal{H}^1(\partial Q_{x,r} \setminus E^{(1)}) = 0 \quad \text{or} \quad \mathcal{H}^1(\partial Q_{x,r} \setminus E^{(0)}) = 0. \tag{7.36}$$

In the first case, from the minimality inequality

$$\mathcal{J}(E,u) \le \mathcal{J}(E, u 1_{\Omega\setminus Q_{x,r}})$$

we deduce $u = 0$ a.e. on $Q_{x,r}$ and $\mathcal{H}^1(\partial^* E \cap Q_{x,r}) = 0$, so that the inequality to check for quasi minimality is trivially satisfied. Notice that admissibility of $(E, u1_{\Omega\setminus Q_{x,r}})$ for the main problem follows from the fact that the trace of u on $\partial Q_{x,r}$ is zero, that boundary being composed of points of density 1 of the set E on which u vanishes.

If the second possibility in (7.36) holds true, then the relations (see [29, Theorem 16.3] and recall that $\mathcal{H}^0(\partial Q_{x,r} \cap \partial^* E) = 0$ by the properties of $Q_{x,r}$)

$$\mathcal{J}(E,u) \le \mathcal{J}(E \setminus Q_{x,r}, v) \quad \text{and} \quad \partial^*(E \setminus Q_{x,r}) = \partial^* E \setminus \overline{Q_{x,r}} = \partial^* E \setminus Q_{x,r}$$

yield in particular

$$\int_{Q_{x,r}} |e(u)|^2\,dx + \mathcal{H}^1(J \cap Q_{x,r}) \le \int_{Q_{x,r}} |e(v)|^2\,dx + \Lambda r^2,$$

so that the quasi minimality of (J,u) follows.

By Proposition 7.11, we get that the normalized set J^+ (see (7.34)) is essentially closed in Ω, i.e.

$$\mathcal{H}^1(\Omega \cap (\overline{J^+} \setminus J^+)) = 0,$$

and u is smooth on $\Omega \setminus \overline{J^+}$, so that

$$\Omega \cap J_u \subseteq \overline{J^+}.$$

On the other hand, in view of the general properties of the reduced boundary of sets of finite perimeter (see [2, Theorem 3.59] or [29, Theorem 15.5]) we have $\partial^* E \subseteq J^+$. Taking into account that $\mathcal{H}^1(J^+ \Delta J) = 0$ (where Δ denotes the symmetric difference of sets) we infer

$$\begin{aligned}\mathcal{H}^1(\Omega \cap \overline{J_u \cup \partial^* E} \setminus (J_u \cup \partial^* E)) &\le \mathcal{H}^1(\Omega \cap \overline{J^+} \setminus J) \\ &\le \mathcal{H}^1(\Omega \cap \overline{J^+} \setminus J^+) + \mathcal{H}^1(J^+ \setminus J) = 0,\end{aligned}$$

so that the conclusion follows.

In order to complete the proof, we need to check claim (7.36).

Assume by contradiction that the claim is false. Then there exists $p \in E^{(1)} \cap \partial Q_{x,r}$ and $q \in E^{(0)} \cap \partial Q_{x,r}$ that are not in one of the corners. Without loss of generality we suppose $p, q \in \{x - re_2 + \mathbb{R}e_1\}$, with $p_1 < q_1$, the case when both are in different sides being analogous. For $s > 0$ small, we let

- $C_p := p + [-s, 0] \times [0, s]$ and $C_q := q + [0, s]^2$,
- $g_s \colon [p_1 - s, q_1 + s] \to [0, 1]$ be zero at the extremes, affine on $[p_1 - s, p_1]$ and $[q_1, q_1 + s]$ and equal to 1 on $[p_1, q_1]$,
- $f_s \in C_c^1(]0, s[)$ with $0 \leq f_s \leq 1$,
- $\varphi_s(x) = g_s(x_1) f_s(x_2 + r)$.

Then

$$\begin{aligned}\mathcal{H}^1(J \cap (Q_{x,r} \setminus Q_{x,r-s})) &\geq \int_{\partial^* E} \varphi_s (\nu_E)_1 \, d\mathcal{H}^1 = \int_E \partial_1 \varphi_s \, d\mathcal{H}^1 \\ &= \frac{1}{s} \int_{E \cap C_p} f_s(y_2 + r) \, dy - \frac{1}{s} \int_{E \cap C_q} f_s(y_2 + r) \, dy,\end{aligned}$$

so that, letting $f_s \nearrow 1$ we get

$$\frac{\mathcal{H}^1(J \cap (Q_{x,r} \setminus Q_{x,r-s}))}{s} \geq \frac{|E \cap C_p|}{|C_p|} - \frac{|E \cap C_q|}{|C_q|}.$$

Since as $s \to 0^+$, by assumption on the density properties of p and q, we have

$$\frac{|E \cap C_p|}{|C_p|} \to 1 \quad \text{and} \quad \frac{|E \cap C_q|}{|C_q|} \to 0,$$

we infer

$$\limsup_{s \to 0} \frac{\mathcal{H}^1(J \cap (Q_{x,r} \setminus Q_{x,r-s}))}{s} \geq 1,$$

which is against the assumption on r in Definition 7.4 of quasi minimality. The proof is thus concluded.

7.4. Some remarks on a "strong" formulation of the problem

In this section we elaborate on a different relaxation of the drag minimization problem which involves topologically closed (but not necessarily regular) obstacles F in the channel Ω and velocity vector fields which are H^1_{loc} on $\Omega \setminus F$.

Within this perspective, given $\Omega \subset \mathbb{R}^d$ open and bounded, it is natural to start with pairs (F, u) such that

$$F \subseteq \Omega \text{ is relatively closed}, \quad \Omega \cap \partial F \text{ is rectifiable}, \quad \mathcal{H}^{d-1}(\Omega \cap \partial F) < +\infty \tag{7.37}$$

and

$$u \in H^1_{\mathrm{loc}}(\Omega \setminus F; \mathbb{R}^d), \quad \operatorname{div} u = 0 \text{ in } \Omega \setminus F, \quad e(u) \in L^2(\Omega \setminus F; M^{d \times d}_{\mathrm{sym}}). \tag{7.38}$$

Notice that, as for the relaxation studied in the previous sections, ∂F may contain "lower-dimensional" parts. The set $\Omega \setminus F$ is open, so that the space $H^1_{\mathrm{loc}}(\Omega \setminus F; \mathbb{R}^d)$ is well defined.

It is not clear how to talk about traces on $\partial(\Omega \setminus F)$, which are fundamental in formulating the tangency constraint, as the set is in general not regular. It turns out that velocities admit a well-defined trace at $\mathcal{H}^{d-1}$-almost every point of ∂F, even if this set is not assumed to be only rectifiable and not regular. This is a consequence of the following result, which involves the space GSBD of *generalized functions of bounded deformations* introduced in [21]. Let us set

$$\tilde{u} := \begin{cases} u & \text{in } \Omega \setminus F, \\ 0 & \text{in } F. \end{cases} \tag{7.39}$$

Lemma 7.12. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and assume that the pair (F, u) satisfies* (7.37) *and* (7.38). *Then $\tilde{u} \in \mathrm{GSBD}(\Omega)$ with $\mathcal{H}^{d-1}(J_{\tilde{u}} \setminus \partial F) = 0$.*

Proof. Since $\mathcal{H}^{d-1}(\Omega \cap \partial F) < \infty$, for every $\varepsilon > 0$ we may find some covering of ∂F through a finite union of balls of radius less than ε, denoted $(B_i^\varepsilon)_{1 \le i \le N^\varepsilon}$, such that

$$\sum_{i=1}^{N^\varepsilon} \left(\frac{\mathrm{diam}(B_i^\varepsilon)}{2} \right)^{d-1} \le C$$

for some $C > 0$ that does not depend on ε. Let B^ε be the union of these balls – which is a Lipschitz set up to a small perturbation of the radii – and let $u^\varepsilon := u 1_{\Omega \setminus B^\varepsilon}$. Then $u^\varepsilon \in \mathrm{SBD}(\Omega)$ with

$$Eu^\varepsilon = e(u)\, dx \lfloor (\Omega \setminus (F \cup B^\varepsilon)) + u \mathcal{H}^{d-1} \lfloor \partial B^\varepsilon.$$

Moreover,

$$u^\varepsilon \to \tilde{u} \quad \text{a.e. in } \Omega$$

with

$$\limsup_{\varepsilon \to 0} \int_\Omega |e(u^\varepsilon)|^2\, dx + \mathcal{H}^{d-1}(J_{u^\varepsilon}) < +\infty.$$

We apply [17, Theorem 1.1] to (u^ε): since $\tilde{u}$ is finite almost everywhere, we directly identify $\tilde{u}$ with the limit that is obtained, and we infer $\tilde{u} \in \mathrm{GSBD}(\Omega)$; moreover, up to an $\mathcal{H}^{d-1}$-negligible set, $J_{\tilde{u}} \subset \partial F$ by construction, and the result follows. ∎

Coming back to configurations (F, u) satisfying (7.37) and (7.38), up to a choice of orientation of the rectifiable set $\Omega \cap \partial F$, there is no ambiguity in defining the traces $u^\pm_{|\partial F}$ of u on $\mathcal{H}^{d-1}$-almost all points of $\Omega \cap \partial F$.

In addition to the previous items, we thus also require for (F, u) the non-penetration condition

$$u^\pm_{|\partial F} \cdot \nu_{\partial F} = 0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \Omega \cap \partial F. \tag{7.40}$$

Given an admissible configuration (F,u), we can consider the following energy (all the constants have been normalized to 1):

$$\begin{aligned} J(F,u) := &\int_{\Omega\setminus F} |e(u)|^2\,dx + \int_{\Omega\cap\partial^e F} |u^+|^2\,d\mathcal{H}^{d-1} \\ &+ \int_{\Omega\cap F^{(0)}\cap F} [|u^+|^2 + |u^-|^2]\,d\mathcal{H}^{d-1} \\ &+ \mathcal{H}^{d-1}(\Omega\cap\partial^e F) + 2\mathcal{H}^{d-1}(\Omega\cap F^{(0)}\cap F) + f(|F|), \end{aligned}$$

where $\partial^e F$ denotes the *measure-theoretical boundary* of F, and f is the penalization function introduced in the previous sections (see (4.1)).

Configurations with finite energy are linked to admissible configurations of our main relaxed problem by the following result.

Lemma 7.13. *Let $\Omega\subseteq\mathbb{R}^d$ be open and bounded, and let (F,u) satisfy (7.37) and (7.38) with $J(F,u)<+\infty$. Then the function $\tilde u$ defined in (7.39) is such that $\tilde u\in \mathrm{SBD}(\Omega)$.*

Proof. It suffices to note that for every direction $\xi\in\mathbb{S}^{d-1}$ we have

$$J(F,u)\geq \int_{\xi^\perp}\left[\int_{\Omega_y^\xi} |(\tilde u_y^\xi)'(t)|^2\,dt + \sum_{t\in J_{\tilde u_y^\xi}} (1+|(\tilde u_y^\xi)^+(t)|^2 + |(\tilde u_y^\xi)^-(t)|^2)\right] d\mathcal{H}^1(y). \quad\blacksquare$$

Dealing with boundary conditions yields the same problem highlighted in our main relaxation. Assume Ω has a Lipschitz boundary, and let us simply write u in place of $\tilde u$. We have that $u\in\mathrm{SBD}(\Omega)$ so that the trace on $\partial\Omega$ is well defined. Given a divergence-free vector field $V\in C^1(\mathbb{R}^d;\mathbb{R}^d)$, we can deal with the relaxation of the boundary condition by considering the set

$$\Gamma_{u,V} := \{x\in\partial\Omega : u(x)\neq V(x)\},$$

and enforcing the non-penetration constraint leading to

$$u\cdot\nu_{\partial\Omega} = 0 \quad\text{and}\quad V\cdot\nu_{\partial\Omega}=0 \quad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_{V,\partial\Omega}. \tag{7.41}$$

So for a configuration (F,u) satisfying (7.37), (7.38), (7.40) and (7.41), we can consider the energy

$$\mathcal{J}^{\text{strong}}(F,u) := J(F,u) + 2\mathcal{H}^{d-1}(\Gamma_{u,V}) + \int_{\Gamma_{u,V}} [|V|^2+|u|^2]\,d\mathcal{H}^{d-1}.$$

The minimization of $\mathcal{J}^{\text{strong}}$ on admissible configurations is a different possible relaxation of the original drag minimization problem. We clearly have

$$\min_{(E,u)\in\mathcal{A}_V(\Omega)} \mathcal{J}(E,u) \leq \inf_{(F,u)} \mathcal{J}^{\text{strong}}(F,u).$$

Equality is reached in dimension two thanks to the regularity result given by Theorem 4.10. Indeed, if (E,u) is a minimizer for $\mathcal{J}$, we know that $\partial^*E\cup J_u$ is essentially closed, so that an admissible relatively closed set F arises by considering the complement of the union of the connected components of $\Omega \setminus \overline{\partial^* E \cup J_u}$ on which u does not vanish identically. The function u is smooth outside F, so that the pair (F, u) is strongly admissible with $\mathcal{J}^{\text{strong}}(F, u) = \mathcal{J}(E, u)$. As a consequence, in dimension two the relaxed problem

$$\min_{(F,u)} \mathcal{J}^{\text{strong}}(F, u)$$

is indeed well posed.

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