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Splittings of triangle Artin groups

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Abstract. We show that a triangle Artin group Art_{MNP}, where $M \le N \le P$, splits as an amalgamated product or an HNN extension of finite rank free groups, provided that either $M > 2$ or $N > 3$. We also prove that all even 3-generator Artin groups are residually finite.

A triangle Artin group is given by the presentation

$$\mathrm{Art}_{MNP} = \langle a, b, c \mid (a,b)_M = (b,a)_M, (b,c)_N = (c,b)_N, (c,a)_P = (a,c)_P \rangle,$$

where $(a,b)_M$ denote the alternating word $aba\ldots$ of length M. Squier showed that the Euclidean triangle Artin group, i.e., Art_{236}, Art_{244} and Art_{333}, split as amalgamated product or an HNN extension of finite rank free groups along finite index subgroups [25]. We generalize that result to other triangle Artin groups.

Theorem A. *Suppose that $M \le N \le P$, where either $M > 2$ or $N > 3$. Then the Artin group Art_{MNP} splits as an amalgamated product or an HNN extension of finite rank free groups.*

The assumptions of the above theorem are satisfied for all triples of numbers except for $(2,2,P)$ and $(2,3,P)$. An Artin group is *spherical*, if the associated Coxeter group is finite. A 3-generator Artin group Art_{MNP} is spherical exactly when $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} > 1$, i.e., $(M,N,P) = (2,2,P)$ or $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. None of 3-generator spherical Artin groups splits as a graph of finite rank free groups (see Proposition 2.10). The remaining cases are $(2,3,P)$, where $P \ge 6$. The above theorem holds for triple $(2,3,6)$ by [25]. It remains unknown for $(2,3,P)$ with $P \ge 7$. The cases where $M > 2$ were considered in [18, Theorem B], and it was proven that they all split as amalgamated products of finite rank free groups.

Graphs of free groups form an important family of examples in geometric group theory. Graphs of free groups with cyclic edge groups that contain no Baumslag–Solitar subgroups are virtually special [15], and contain quasiconvex surface subgroups [27]. Graphs of free groups with arbitrary edge groups can exhibit various behaviors. For example, an amalgamated product $A *_C B$ of finite rank free groups, where C is malnormal

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in A and B, is hyperbolic [1], and virtually special [16]. On the other hand, there are examples of amalgamated products of finite rank free groups that are not residually finite [2,28], and even simple [8]. The last two arise as lattices in the automorphism group of a product of two trees.

By further analysis of the splitting, we are also able to show that some of the considered Artin groups are residually finite.

Theorem B. *The Artin group* Art_{2MN}*, where* $M, N \geq 4$ *and at least one of* M*,* N *is even, is residually finite.*

An Artin group Art_{MNP} is *even* if all M, N, P are even. The above theorem combined with our result in [18] (and the fact that $\mathrm{Art}_{22P} = \mathbb{Z} \times \mathrm{Art}_P$ is linear) gives us the following.

Corollary C. *All even Artin groups on three generators are residually finite.*

All linear groups are residually finite [21], so residual finiteness can be viewed as testing for linearity. Spherical Artin groups are known to be linear ([3, 19] for braid groups, and [9, 11] for other spherical Artin groups). The right-angled Artin groups are also well known to be linear, but not much more is known about linearity of Artin groups. In last years, a successful approach in proving that groups are linear is by showing that they are virtually special. Artin groups whose defining graphs are forests are the fundamental groups of graph manifolds with boundary [7, 14], and so they are virtually special [20, 23]. Many Artin groups in certain classes (including 2-dimensional, or 3-generator) are not cocompactly cubulated even virtually, unless they are sufficiently similar to RAAGs [12, 17]. In particular, the only (virtually) cocompactly cubulated 3-generator Artin groups are $\mathrm{Art}_{22M} = \mathbb{Z} \times \mathrm{Art}_M$, $\mathrm{Art}_{MN\infty}$, where M and N are both even, $\mathrm{Art}_{M\infty\infty} = \mathbb{Z} * \mathrm{Art}_M$, and $\mathrm{Art}_{\infty\infty\infty} = F_3$. Some triangle-free Artin groups act properly but not cocompactly on locally finite, finite-dimensional CAT(0) cube complexes [13].

In [18], we showed that Art_{MNP} are residually finite when $M, N, P \geq 3$, except for the cases where $(M, N, P) = (3, 3, 2p + 1)$ with $p \geq 2$. Few more families of Artin groups are known to be residually finite, e.g., even FC type Artin groups [5], and certain triangle-free Artin groups [4].

Organization

In Section 1, we provide some background. In Section 2, we prove Theorem A as Proposition 2.5 and Corollary 2.9. We also show that the only irreducible spherical Artin groups splitting as graph of finite rank free groups are dihedral. In Section 3, we recall a criterion for residual finiteness of amalgamated products and HNN extensions of free groups from [18] and prove Theorem B.

1. Background

1.1. Graphs

Let X be a finite graph with directed edges. We denote the vertex set of X by $V(X)$ and the edge set of X by $E(X)$. The vertices incident to an edge e are denoted by e^+ and e^-. A *map of graphs* $f: X_1 \to X_2$ sends vertices to vertices, and edges to concatenations of edges. A map f is a *combinatorial map* if single edges are mapped to single edges. A combinatorial map f is a *combinatorial immersion* if given two edges e_1, e_2 such that $e_1^- = e_2^-$, we have $f(e_1) = f(e_2)$ (as oriented edges) if and only if $e_1 = e_2$. Consider two edges e_1, e_2 with $e_1^- = e_2^-$. A *fold* is the natural combinatorial map $X \to \bar{X}$, where

$$V(\bar{X}) = V(X)/e_1^+ \sim e_2^+ \quad \text{and} \quad E(\bar{X})/e_1 \sim e_2.$$

Stallings showed that every combinatorial map $X \to X'$ factors as $X \to \bar{X} \to X'$, where $X \to \bar{X}$ is a composition of finitely many folds, and $\bar{X} \to X'$ is a combinatorial immersion [26]. We refer to $X \to \bar{X}$ as a *folding map*.

1.2. Maps between free groups

Let H, G be finite rank free groups. Let Y be a bouquet of $n = \operatorname{rk} G$ circles. We can identify $\pi_1 Y \simeq F_n$ with G by orienting and labeling edges of Y with the generators of G. Every homomorphism $\varphi: H \to G$ can be represented by a combinatorial immersion of graphs. Indeed, start with a map of graphs $X \to Y$, where X is a bouquet of $m = \operatorname{rk} H$ circles. We think of each circle in X as subdivided with edges oriented and labeled by the generators of G, so that each circle is labeled by a word from a generating set of H. By Stallings, the map $X \to Y$ factors as $X \to \bar{X} \to Y$, where $X \to \bar{X}$ is a folding map, and $\bar{X} \to Y$ is a combinatorial immersion. Indeed, $\bar{X}$ is obtained by identifying two edges with the same orientation and label that share an endpoint.

Note that the rank of $\varphi(H)$ is equal to $\operatorname{rk} \pi_1 \bar{X} = 1 - \chi(\bar{X})$, where χ denotes the Euler characteristic. In particular, a homomorphism φ is injective if and only if the folding map $X \to \bar{X}$ is a homotopy equivalence. In that case, $\bar{X}$ is a precover of Y which can be completed to a cover of Y corresponding to the subgroup H of G via the Galois correspondence. In particular, every subgroup of G is uniquely represented by a combinatorial immersion $(X, x) \to (Y, y)$, where y is the unique vertex of Y, and X is a folded graph with basepoint x. We refer to [26] for more details.

1.3. Intersections of subgroups of a free group

Let Y be a graph, and let $\rho_i: (X_i, x_i) \to (Y, y)$ be a combinatorial immersion for $i = 1, 2$. The *fiber product of* X_1 *and* X_2 *over* Y, denoted by $X_1 \otimes_Y X_2$ is a graph with the vertex set

$$V(X_1 \otimes_Y X_2) = \{(v_1, v_2) \in V(X_1) \times V(X_2): \rho_1(v_1) = \rho_2(v_2)\},$$

and the edge set

$$E(X_1 \otimes_Y X_2) = \{(e_1, e_2) \in E(X_1) \times E(X_2) \colon \rho_1(e_1) = \rho_2(e_2)\}.$$

The graph $X_1 \otimes_Y X_2$ often has several connected components. There is a natural combinatorial immersion $X_1 \otimes_Y X_2 \to Y$, and it induces an embedding $\pi_1(X_1 \otimes_Y X_2, (x_1, x_2)) \to \pi_1(Y, y)$. We have the following.

Theorem 1.1 ([26, Theorem 5.5]). *Let H_1, H_2 be two subgroups of $G = \pi_1 Y$, and for $i = 1, 2$, let $(X_i, x_i) \to (Y, y)$ be a combinatorial immersion of graphs inducing the inclusion $H_i \hookrightarrow G$. The intersection $H_1 \cap H_2$ is represented by a combinatorial immersion $(X_1 \otimes_Y X_2, (x_1, x_2)) \to (Y, y)$.*

In particular, when Y is a bouquet of circles with $\pi_1 Y = G$, and $(X, x) \to (Y, y)$ is a combinatorial immersion inducing $H = \pi_1 X \hookrightarrow G$, then for every pair of (not necessarily distinct) vertices $x_1, x_2 \in X$, the group $\pi_1(X \otimes_Y X, (x_1, x_2))$ is an intersection $H^{g_1} \cap H^{g_2}$ for some $g_1, g_2 \in G$. In fact, every non-trivial intersection $H \cap H^g$ is equal to $\pi_1(X \otimes_Y X, (x_1, x_2))$, where $x_1 = x$, and x_2 is some (possibly the same) vertex in X. The connected component of $X \otimes_Y X$ containing (x, x) is a copy of X, which we refer to as a *diagonal component*. The group $\pi_1(X \otimes_Y X, (x, x))$ is the intersection $H \cap H^g = H$, i.e., where $g \in H$. A connected component of $X \otimes_Y X$ that has no edges is called *trivial*.

1.4. Graph of groups and spaces

We recall the definitions of a graph of groups and a graph of spaces, following [24].

A *graph of spaces* consists of

- a graph Γ, called the *underlying graph*,
- a collection of CW-complexes X_v for each $v \in V(\Gamma)$, called *vertex spaces*,
- a collection of CW-complexes X_e for each $e \in E(\Gamma)$, called *edge spaces*,
- a collection of continuous π_1-injective maps $f_{(e,\pm)} \colon X_e \to X_{e^\pm}$ for each $e \in E(\Gamma)$.

The *total space* $X(\Gamma)$ is defined as

$$X(\Gamma) = \bigsqcup_{v \in V(\Gamma)} X_v \sqcup \bigsqcup_{e \in E(\Gamma)} X_e \times [-1, 1]/\sim,$$

where $(x, \pm 1) \sim f_{(e,\pm)}(x)$ for $x \in X_e$.

Similarly, a *graph of groups* consists of

- the *underlying graph* Γ,
- a collection of *vertex groups* G_v for each $v \in V(\Gamma)$,
- a collection of *edge groups* G_e for each $e \in E(\Gamma)$,
- a collection of injective homomorphisms $\varphi_{(e,\pm)} \colon G_e \to G_{e^\pm}$ for each $e \in E(\Gamma)$.

The *fundamental group of a graph of groups* is defined as the fundamental group of the graph of spaces $X(\Gamma)$, where $X_v = K(G_v, 1)$ for each $v \in V(\Gamma)$, $X_e = K(G_e, 1)$ for each edge $e \in E(\Gamma)$, and $f_{(e,\pm)}$ induces homomorphism $\varphi_{(e,\pm)}$ on the fundamental groups. Note that the fundamental group $\pi_1\Gamma$ is a subgroup of G.

1.5. HNN extensions and doubles

We will denote the *HNN extension of A relative to $\beta\colon B \to A$*, where $B \subseteq A$, by $A*_{B,\beta}$, that is,

$$A*_{B,\beta} = \langle A, t \mid t^{-1}xt = \beta(x) \text{ for all } x \in B \rangle.$$

The generator t is called the *stable letter*. Note that $A*_{B,\beta}$ can be viewed as a graph of group $G(\Gamma)$, where Γ is a single vertex v with a single loop e, $G_v = A$, $G_e = B$, $\varphi_{(e,-)}$ is the inclusion of B in A, and $\varphi_{(e,+)} = \beta$.

A *double of A along C twisted by an automorphism $\beta\colon C \to C$*, denoted by $D(A, C, \beta)$, is an amalgamated product $A *_C A$, where C is mapped to the first factor via the standard inclusion, and to the second via the standard inclusion precomposed with β. As usual, $D(A, C, \beta)$ depends only on the outer automorphism class of β, and not a particular representative. A double $D(A, C, \beta)$ can be viewed as a graph of groups $G(\Gamma)$, where Γ is a single edge e with distinct endpoints, $G_{e^\pm} = A$, $G_e = C$, and $\varphi_{(e,-)}$ is the inclusion of C in A, and $\varphi_{(e,+)}$ is the inclusion precomposed with β. Note that an amalgamated product $A *_C B$ with $[B : C] = 2$ has an index 2 subgroup $D(A, C, \beta)$. The homomorphism $\beta\colon C \to C$ is conjugation by some (any) representative $g \in B$ of the non-trivial coset of B/C.

In both situations where there is unique edge in the underlying graph Γ, we will skip the label e and denote $\varphi_{(e,\pm)}$ simply by $\varphi_\pm$. Similarly, in a graph of spaces with a unique edge e, we will write $f_\pm$ instead of $f_{(e,\pm)}$.

1.6. Triangle groups

A *triangle (Coxeter) group* is given by the presentation

$$W_{MNP} = \langle a, b, c \mid a^2, b^2, c^2, (ab)^M, (bc)^N, (ca)^P \rangle.$$

The group W_{MNP} acts as a reflection group on

- the sphere if $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} > 1$,
- the Euclidean plane if $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} = 1$,
- the hyperbolic plane if $\frac{1}{M} + \frac{1}{N} + \frac{1}{P} < 1$.

Hyperbolic triangle groups are commensurable with the fundamental groups of negatively curved surfaces, and therefore they are locally quasiconvex, virtually special, and Gromov-hyperbolic. A *von Dyck triangle group* is an index 2 subgroup of W_{MNP} with the presentation

$$\langle x, y \mid x^M, y^N, (x^{-1}y)^P \rangle$$

obtained by setting $x = ba$ and $y = bc$.

2. Splittings

The goal of this section is to prove Theorem A. In [18], we proved the following.

Theorem 2.1 ([18, Theorem B]). *The Artin group* Art_{MNP} *with* $M, N, P \geq 3$ *splits as an amalgamated product or an HNN extension of finite rank free groups.*

The remaining cases in Theorem A are Art_{2MN}, where $M, N \geq 4$. The case where M, N are both even is Proposition 2.5, and the other case is Corollary 2.9. We start with considering a non-standard presentation of Art_{2MN}. In the next two subsections, we construct the splittings. Then we show that the only spherical Artin groups that split as graphs of free groups are dihedral Artin groups. Spherical Artin groups on three generators Art_{MNP} correspond to one of the following triples: $(2, 2, P)$ for any $P \geq 2$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$. For completeness, in the last subsection we include the proof that the 3-generator Artin groups with at least one ∞ label admit splittings as HNN extensions or amalgamated products of finite rank free groups.

The Artin group Art_{236} splits as $F_3 *_{F_7} F_4$ by Squier [25]. The only remaining 3-generator Artin groups are Art_{23M}, where $M \geq 7$, and the following remains unanswered.

Question 2.2. *Does the Artin group* Art_{23M}*, where* $M \geq 7$*, splits as a graph of finite rank free groups?*

We conjecture that the answer is positive. More generally, we ask the following.

Question 2.3. *Do all 2-dimensional Artin groups split as a graph of finite rank free groups?*

2.1. Presentations of Art_{2MN}

Here is the standard presentation of Artin group Art_{2MN}:

$$\langle a, b, c \mid (a, b)_M = (b, a)_M, (b, c)_N = (c, b)_N, ac = ca\rangle.$$

Let $x = ab$ and $y = cb$, and consider a new presentation of Art_{2MN} with generators b, x, y. The relation $(a, b)_M = (b, a)_M$ is replaced by $bx^m b^{-1} = x^m$ when $M = 2m$, and by $bx^m b = x^{m+1}$ when $M = 2m + 1$. We denote this relation by $r_M(b, x)$. Note that $yx^{-1} = ca^{-1}$, so relation $ac = ca$ can be replaced by $yx^{-1} = bx^{-1}yb^{-1}$. See Figure 1(a).

This gives us the following presentation:

$$\mathrm{Art}_{2MN} = \langle b, x, y \mid r_M(b, x), r_N(b, y), bx^{-1}yb^{-1} = yx^{-1}\rangle. \qquad (*)$$

Let X_{2MN} be the presentation complex associated to presentation $(*)$. Let X_A be the bouquet of two loops labeled by x and y. The complex X_{2MN} can be viewed as a union of the graph X_A and for each relation in $(*)$, a cylinder (for relations $bx^{-1}yb^{-1} = yx^{-1}$ and each $r_M(b, x)$ with M even) or a Möbius strip (for each relation $r_M(b, x)$ with M odd) with boundary cycles are glued to X_A. We can metrize them so that the height of each cylinder/Möbius strip equals 2.

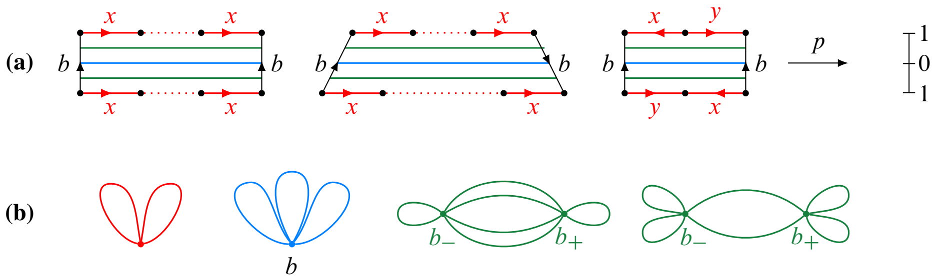

Figure 1. (a) The relation $R_M(b,x)$, where M is even (left) and odd (middle), and the relation $bx^{-1}yb^{-1} = yx^{-1}$ (right) with the projection p of the cells onto the interval $[0,1]$. The horizontal graphs X_A, X_B, X_C are the preimages $p^{-1}(1)$, $p^{-1}(0)$, $p^{-1}(\frac{1}{2})$, respectively. (b) Graphs X_A, X_B and two versions of X_C depending on whether M, N are both odd (left green), or one of M, N is even (right green).

We now define a map $p\colon X_{2MN} \to [0,1]$ by describing the restriction of p to each cylinder/Möbius strip of X_{2MN}. Each point of the cylinder/Möbius strip is mapped to its distance from the center circle of that cylinder/Möbius strip. In particular, the center circle of each cylinder or Möbius strip is mapped to 0, and the boundary circles of the cylinder or Möbius strip are mapped to 1. See Figure 1 (a).

We can identify X_A with the preimage $p^{-1}(1)$. We define a graph X_B as the union of all the center circles, i.e., the preimage $p^{-1}(0)$. We also define a subgraph X_C of X_{2MN} as the preimage $p^{-1}(\frac{1}{2})$. The graphs X_A, X_B and X_C are illustrated in Figure 1(b). The graph X_C has two vertices, which are its intersections with the edge b. We denote them by b_-, b_+, so that b_-, the midpoint of the edge b, b_+ are ordered consistently with the orientation of the edge b. When M, N are both even, then the graph X_C is not connected. Indeed, each of its connected components is a copy of X_B. We denote the connected component containing the vertex b_- by X_B^-, and the component containing the vertex b_+ by X_B^+. Otherwise, if at least one M, N is odd, then X_C is a connected double cover of X_B. In the next two sections, we describe the graph of spaces decomposition of X_{2MN} associated to the map p, and the induced graph of groups decomposition of Art_{2MN}. We consider separately the case where M, N are both even, and the case where at least one of them is odd.

2.2. Both even

In the case where both M, N are even and ≥ 4, presentation $(*)$ of Art_{2MN} is the standard presentation of an HNN-extension.

Proposition 2.4. *Let $M = 2m$, $N = 2n$ and both ≥ 4. Then Art_{2MN} splits as an HNN-extension $A*_{B,\beta}$, where $A = \langle x, y\rangle \simeq F_2$ and $B = \langle x^m, y^n, x^{-1}y\rangle \simeq F_3$, and $\beta\colon B \to A$ is an injective homomorphism given by $\beta(x^m) = x^m$, $\beta(y^n) = y^n$ and $\beta(x^{-1}y) = yx^{-1}$.*

Proposition 2.4 follows from the following.

Proposition 2.5. *Let $M = 2m$, $N = 2n$ and both ≥ 4. Then X_{2MN} is a graph of spaces $X(\Gamma)$, where Γ is a single vertex with a single loop. The vertex space is the graph X_A, the edge space is the graph X_B, and the two maps $X_B \to X_A$ are given in the Figure 2.*

Proof. Indeed, the map p factors as

$$X_{2MN} \xrightarrow{\tilde{p}} S^1 = [-1, 1]/(-1 \sim 1) \to [0, 1],$$

where the second map is the absolute value, and where $\tilde{p}$ sends the loop b isometrically onto S^1 and is extended linearly. By construction, the preimage $\tilde{p}^{-1}(t)$ is homeomorphic to X_B when $t \in (-1, 1)$, and to X_A when $t = 1$. In particular, X_{2MN} can be expressed as a graph of spaces, induced by $\tilde{p}$, where the cellular structure of S^1 consists of a single vertex $v = 1$ and a single edge e. Indeed,

$$X_{2MN} = X_A \cup X_B \times [-1, 1]/(x, -1) \sim f_-(x), (x, 1) \sim f_+(x),$$

where $f_-, f_+ \colon X_B \to X_A$ are the two maps obtained by "pushing" the graph X_B in Figure 1 (a) "upwards" and "downwards", respectively. See Figure 2 for f_-, f_+ expressed as a composition of Stallings fold and a combinatorial immersion. ■

Remark 2.6. The subgroups B and $\beta(B)$ are conjugate. See Figure 2. Indeed, the graphs $\bar{X}_B^-$ and $\bar{X}_B^+$ are identical (but have different basepoints).

Example 2.7 (Group Art_{244}). In the case where $M = N = 4$, Proposition 2.5 provides the splitting of $\mathrm{Art}_{244} = A*_{B,\beta}$, where

$$A = \langle x, y \rangle \quad \text{and} \quad B = \langle x^2, y^2, x^{-1}y \rangle,$$

and $\beta \colon B \to B$ is given by $\beta(x^2) = x^2$, $\beta(y^2) = y^2$, $\beta(x^-1y) = yx^{-1}$. In particular, B has index 2 in A. This splitting was first proven by Squier [25].

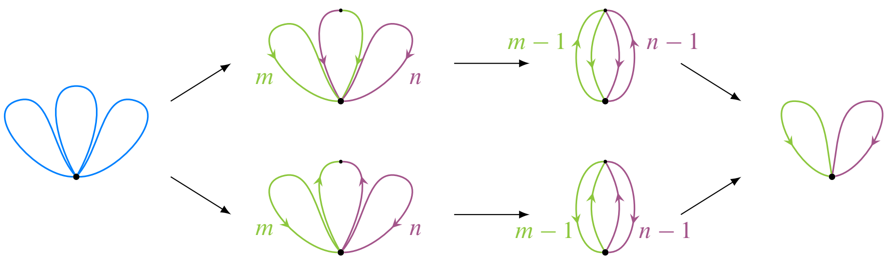

Figure 2. The map $f_- \colon X_B \to X_B^- \to \bar{X}_B^- \to X_A$ (top), and the map $f_+ \colon X_B \to X_B^+ \to \bar{X}_B^+ \to X_A$ (bottom).

2.3. At least one odd

We now assume that at least one M, N is odd. We have the following description of the complex X_{2MN}.

Proposition 2.8. *The complex X_{2MN} is a graph of spaces whose underlying graph is an interval, the vertex spaces are graphs X_A and X_B, and the edge space is X_C. The attaching map $X_C \to X_B$ is a double cover, and the attaching map $X_C \to X_A$ factors as $X_C \to \overline{X}_C \to X_A$, illustrated in Figure 3, where the first map is a homotopy equivalence and the second map is a combinatorial immersion.*

Proof. Indeed, X_{2MN} can be obtained as a union of X_A, X_B and $X_C \times [0, 1]$, where $X_C \times \{1\}$ is glued to X_A, and $X_C \times \{0\}$ is glued to X_B. Note that the preimage $p^{-1}([0, \frac{1}{2}])$ is a union of cylinders and Möbius strips, and its boundary is the graph X_C. The projection onto the center circle of the boundary of each cylinder or Möbius strip is a (connected or not) double cover of the center circle. It follows that $X_C \to X_B$ is a double cover. The map $X_C \to X_A$ is induced by "pushing" X_C "downwards" and "upwards" onto X_A, and it can be described by the labeling of the right graphs in Figure 3. The factorization $X_C \to \overline{X}_C$ is obtained by performing Stallings folds. Note that the middle graphs in Figure 3 are fully folded, provided that $m - 1, n - 1 > 0$, which is equivalent to the condition that $M, N \geq 4$. It follows that the map $\overline{X}_C \to X_A$ is a combinatorial immersion. Since the ranks of $\pi_1 X_C$ and $\pi_1 \overline{X}_C$ are both equal to 5, the folding map is a homotopy equivalence. ∎

Corollary 2.9. *Suppose at most one of M, N is even and $M, N \geq 4$. Then Art_{2MN} splits as a free product with amalgamation $A *_C B$, where $A = F_2$ and $B = F_3$, and $C = F_5$.*

Proof. This directly follows from Proposition 2.8. We get that

$$\mathrm{Art}_{2MN} = \pi_1 X_{2MN} = A *_C B,$$

where $A = \pi_1 X_A$, $B = \pi_1 X_B$ and $C = \pi_1 X_C$. From Figure 1, we see that $\mathrm{rk}\, A = 2$, $\mathrm{rk}\, B = 3$ and $\mathrm{rk}\, C = 5$. ∎

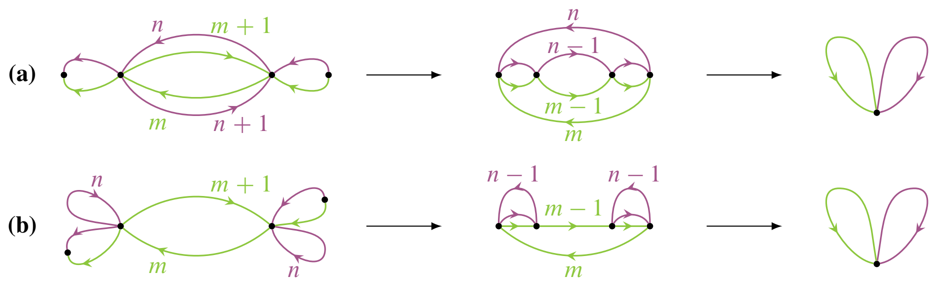

Figure 3. The maps $X_C \to \overline{X}_C \to X_A$ when (a): $M = 2m + 1$, $N = 2n + 1$; and (b): $M = 2m + 1$, $N = 2n$.

2.4. Splittings of spherical Artin groups

All spherical Artin groups have non-trivial center and their cohomological dimension is equal to the number of standard generators [6, 10]. We now give a characterization of graph of finite rank free groups with non-trivial center. This allows us to deduce that the only spherical Artin groups that split as graphs of finite rank free groups are the dihedral Artin groups (i.e., on two generators).

Proposition 2.10. *The only irreducible spherical Artin groups that split as non-trivial graphs of free groups are the dihedral Artin groups and* $\mathbb{Z}$*.*

Proof. Clearly, $\mathbb{Z}$ is an HNN-extension of a trivial group. Let Art_M be a dihedral Artin group with the presentation $\langle a, b \mid (a,b)_M = (b,a)_M \rangle$. Let $M = 2m$, and let $x = ab$. Then $\mathrm{Art}_M \simeq \langle a, x \mid a x^m a^{-1} = x^m \rangle$. In particular, $\mathrm{Art}_M = \langle x \rangle *_{\langle x^m \rangle} = \mathbb{Z} *_{\mathbb{Z}}$. Now if $M = 2m+1$, let $x = ab$ and $y = (a,b)_M$. Then $\mathrm{Art}_M \simeq \langle x, y \mid x^M = y^2 \rangle$. In particular, $\mathrm{Art}_M \simeq \langle x \rangle *_{\langle x^M \rangle} \langle y \rangle = \mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$. Conversely, non-trivial graphs of free groups have cohomological dimension at most 2 since the corresponding graphs of spaces are aspherical. The only irreducible spherical Artin groups of cohomological dimension at most 2 are dihedral Artin groups and $\mathbb{Z}$. ∎

2.5. Splittings of 3-generator Artin groups with ∞ labels

To complete the picture, we prove that the remaining 3-generator Artin groups, i.e., those with at least one ∞ label, also split as graphs of finite rank free groups. The Artin group $\mathrm{Art}_{\infty\infty\infty}$ is the free group on three generators. The group $\mathrm{Art}_{M\infty\infty} = \mathrm{Art}_M * \mathbb{Z}$ can be described as

$$\langle x, c \rangle *_{x^M = y^2} \langle y \rangle = F_2 *_{\mathbb{Z}} \mathbb{Z},$$

where $x = ab$ and $y = (a,b)_M$. Finally, for the Artin group $\mathrm{Art}_{MN\infty}$, we can use presentation (∗) skipping the relation $bx^{-1}yb^{-1} = yx^{-1}$, i.e.,

$$\mathrm{Art}_{MN\infty} = \langle x, y, b \mid r_M(b,x), r_N(b,y) \rangle.$$

We get that $\mathrm{Art}_{MN\infty}$ splits as

- $A *_{B,\beta}$, where $A = \langle x, y \rangle \simeq F_2$, $B = \langle x^m, y^n \rangle \simeq F_2$, $\beta = \mathrm{id}_C$, when $M = 2m$ and $N = 2n$,
- $A *_C B$, where $A = \langle x, y \rangle \simeq F_2$, $B \simeq F_2$ and $C \simeq F_3$, when at least one of M, N is odd. The splitting is obtained in the same way as in the case of Art_{2MN}.

3. Residual finiteness

In the section, we prove Theorem B. We do so separately in the case where M, N are both even (Theorem 3.6), and where exactly one of M, N is odd (Theorem 3.7). We start with recalling a criterion for residual finiteness of amalgamated products and HNN extensions of finite rank free groups, proven in [18], which relies on Wise's result on residual finiteness of *algebraically clean* graphs of free groups [29]. In Section 3.2, we compute the intersections of conjugates of the amalgamating subgroup in the factor groups. Finally, we give proofs of the main theorems.

We start with a motivating example.

Example 3.1 (Group Art_{244}). By Example 2.7, the group Art_{244} fits in the short exact sequence of groups

$$1 \to C \to \mathrm{Art}_{244} \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z} \to 1.$$

In particular, Art_{244} is a (finite rank free group)-by-(virtually free group), and so it is virtually (finite rank free group)-by-(free group). The residual finiteness of Art_{244} follows from the fact that every split extension of a finitely generated residually finite group by residually finite group is residually finite [22].

3.1. Criteria for residual finiteness

Recall that a subgroup C is *malnormal* in a group A, if $C \cap g^{-1}Cg = \{1\}$ for every $g \in A - C$. Similarly, C is *almost malnormal* in A, if $|C \cap g^{-1}Cg| < \infty$ for every $g \in A - C$. More generally, a collection of subgroup $\{C_i\}_{i \in I}$ is an *almost malnormal* collection, if $|C_i \cap gC_jg^{-1}| < \infty$ whenever $g \notin C_i$ or $i \neq j$.

Assume that the inclusion of free groups $C \to A$ is induced by a map $f\colon X_C \to X_A$ of graphs, and the automorphism $\beta\colon C \to C$ is induced by some graph automorphism $X_C \to X_C$. The following theorem was proven in [18].

Theorem 3.2 ([18, Theorem 2.9]). *Let $\widehat{f}\colon \widehat{X}_C \to \widehat{X}_A$ be a map of 2-complexes that restricted to the 1-skeletons is equal to f, and let $\pi\colon A \to \widehat{A}$ be the natural quotient induced by the inclusion $X_A \hookrightarrow \widehat{X}_A$ of the 1-skeleton. Suppose that the following conditions hold:*

(1) *$\widehat{A}$ is a locally quasiconvex, virtually special hyperbolic group.*

(2) *$\pi(C) = \pi_1\widehat{X}_C$ and the lift of $\widehat{f}$ to the universal covers is an embedding.*

(3) *$\pi(C)$ is almost malnormal in $\widehat{A}$.*

(4) *β projects to an isomorphism $\pi(C) \to \pi(C)$.*

Then $D(A, C, \beta)$ is residually finite.

The theorem above is a combination of Theorem 2.9 and Lemma 2.6 from [18]. Condition (2) in the statement of [18, Theorem 2.9] is that $\pi(C)$ is *malnormal* in $\widehat{A}$. However, the proof is identical when we replace it with *almost malnormal*. Indeed, the Bestvina–Feighn combination theorem [1], as well as the Hsu–Wise combination theorem [16] only require almost malnormality.

We now state a version for HNN extension. Similarly as above, combining Theorem 2.12 and Lemma 2.6 from [18], we obtain the following.

Theorem 3.3 ([18, Theorem 2.12]). *Let $\widehat{f}_-, \widehat{f}_+\colon \widehat{X}_B \to \widehat{X}_A$ be two maps of 2-complexes that restricted to the 1-skeletons are equal to f_-, f_+, respectively, and let $\pi\colon A \to \widehat{A}$*

be the natural quotient induced by the inclusion $X_A \hookrightarrow \widehat{X}_A$. Suppose that the following conditions hold:

(1) *$\widehat{A}$ is a locally quasiconvex, virtually special hyperbolic group.*

(2) *$\pi(B) = \widehat{f}_{-*}(\pi_1 \widehat{X}_B)$, and $\pi(\beta(B)) = \widehat{f}_{+*}(\pi_1 \widehat{X}_B)$ and the lifts of $\widehat{f}_-$, $\widehat{f}_+$ to the universal covers are both embeddings.*

(3) *The collection $\{\pi(B), \pi(\beta(B))\}$ is almost malnormal in $\widehat{A}$.*

(4) *$\beta \colon B \to \beta(B)$ projects to an isomorphism $\pi(B) \to \pi(\beta(B))$.*

Then $A*_{B,\beta}$ is residually finite.

3.2. Intersection of the conjugates of the amalgamating subgroup

Let $G = \mathrm{Art}_{2MN}$, where $M, N \geq 4$ and at least one of them is odd. Let A, B, C be free groups of ranks 2, 3, 5, respectively, provided by Corollary 2.9. In this section, we describe intersections $C \cap g^{-1}Cg$, where $g \in A$.

Proposition 3.4. *Let $M = 2m + 1$ be odd, $N = 2n$ even, and let A, B, C be as in Corollary 2.9. Let $g \in A - C$. Then the intersection $C \cap g^{-1}Cg$ is one of the sets: $\langle x^{2m+1}, y^n, x^{-1}y\rangle$, $\langle x^{2m+1}, y^n, yx^{-1}\rangle$, $\langle x^{2m+1}, y^n\rangle$, $\langle x^{2m+1}\rangle$, $\langle y^n\rangle$.*

Proof. Let X_A, $\overline{X}_C$ be as in Figure 3 (b). By Theorem 1.1, the conjugates $C \cap g^{-1}Cg$ can be represented by the connected components of the fiber product $\overline{X}_C \otimes_{X_A} \overline{X}_C$. Let x_1, x_2, x_3, x_4 be the four vertices of valence 4 in $\overline{X}_C$ such that x_1, x_2 belong to the same y-cycle and x_3, x_4 belong to the same y-cycle, and so that the ordering of the indices is consistent with the order of the vertices on the x-cycle. Then the connected component of $\overline{X}_C \otimes_{X_A} \overline{X}_C$ containing one of (x_i, x_j) is

- the graph in Figure 4 (a), if $|i - j| = 2$, in which case $C \cap g^{-1}Cg$ is $\langle x^{2m+1}, y^n, x^{-1}y\rangle$ or $\langle x^{2m+1}, y^n, yx^{-1}\rangle$, or
- a bouquet of two circles, labeled by x^{2m+1} and y^n otherwise, in which case $C \cap g^{-1}Cg$ is $\langle x^{2m+1}, y^n\rangle$.

Every other non-diagonal connected component of $\overline{X}_C \otimes_{X_A} \overline{X}_C$ is either trivial or a single circle, which is labeled by either x^{2m+1} or y^n, in which case $C \cap g^{-1}Cg$ is $\langle x^{2m+1}\rangle$ or $\langle y^n\rangle$, respectively. ■

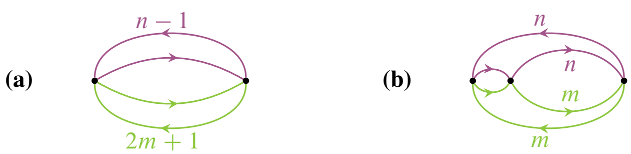

Figure 4. A connected component of $\overline{X}_C \otimes_{X_A} \overline{X}_C$ when (a): $M = 2m + 1$, $N = 2n$, and (b): $M = 2m + 1$, $N = 2n + 1$.

We finish with the following observation regarding the case where M and N are both odd.

Remark 3.5. Let $M = 2m + 1$, $N = 2n + 1$, and let A, B, C be as in Corollary 2.9. Let $g \in A - C$. Then the intersection $C \cap g^{-1}Cg$ is one of the sets: $\langle x^{2m+1}, y^n, x^{-1}y\rangle$, $\langle x^{2m+1}, y^n\rangle$, $\langle x^{2m+1}\rangle$, $\langle y^n\rangle$. Let X_A, $\overline{X}_C$ be as in Figure 3. Let x_1, x_2, x_3, x_4 be the four vertices of valence 4 in $\overline{X}_C$ ordered consistently with the orientation of the x-cycle such that x_1 and x_4 are at distance m in the x-cycle. Then the connected component of $\overline{X}_C \otimes_{X_A} \overline{X}_C$ containing vertices (x_1, x_3), (x_2, x_4), (x_4, x_1) (or vertices (x_3, x_1), (x_4, x_2), (x_1, x_4)) looks like the graph in Figure 4 (b).

3.3. Proof of residual finiteness

We now use Theorem 3.3 to prove that Art_{2MN}, where M, N are even and at least 4, is residually finite.

Theorem 3.6. *Let $M = 2m$, $N = 2n$ and both ≥ 4. Then Art_{2MN} is residually finite.*

Proof. The case where $M = N = 4$ is proven in Example 3.1, so we assume that at least one of M, N, say M, is at least 6. By Proposition 2.5, Art_{2MN} splits as an HNN-extension $A*_{B,\beta}$, where $A = \langle x, y\rangle$ and $B = \langle x^m, y^n, x^{-1}y\rangle$, and $\beta\colon B \to A$ is given by $x^m \mapsto x^m$, $y^n \mapsto y^n$ and $x^{-1}y \mapsto yx^{-1}$. We deduce residual finiteness of Art_{2MN} from Theorem 3.3. We now check that all its assumptions are satisfied.

Let $\widehat{A} = \langle x, y \mid x^m, y^n, (x^{-1}y)^p\rangle$, where $p \geq 7$, and let $\widehat{X}_A$ be the presentation complex of $\widehat{A}$. Let $\pi\colon A \to \widehat{A}$ be the natural quotient. Since $m \geq 3$, the group $\widehat{A}$ is a hyperbolic von Dyck triangle group, and in particular, condition (1) of Theorem 3.3 is satisfied.

The image $\pi(B)$ is a finite cyclic group $\mathbb{Z}/p$ of order p generated by $x^{-1}y$, and the image $\pi(\beta(B))$ is a copy of $\mathbb{Z}/p$ generated by yx^{-1}. Since $\pi(B)$, $\pi(\beta(B))$ are finite groups, they form an almost malnormal collection in $\widehat{A}$, so condition (3) in Theorem 3.3 is satisfied. Let $\widehat{X}_B$ be obtained from X_B by attaching a 2-cell to each of the left and the right loop of X_B (left and right in Figure 2) via a 1-to-1 map (corresponding to the relations x^m, y^n), and one 2-cell to the middle loop via a p-to-1 map (corresponding to the relation $(x^{-1}y)^p$). It is immediate that conditions (2) and (4) of Theorem 3.3 hold. See Figure 5. The proof is complete. ∎

Similarly, we use Theorem 3.2 to prove that Art_{2MN}, where one of M, N is odd, is residually finite.

Theorem 3.7. *Let $M = 2m + 1$ and $N = 2n$ be both ≥ 4. Then Art_{2MN} is residually finite.*

Proof. By Corollary 2.9, Art_{2MN} splits as $A *_C B$, and therefore Art_{2MN} has an index 2 subgroup $D(A, C, \beta)$. We use Theorem 3.2 to prove that $D(A, C, \beta)$ is residually finite.

Let $\widehat{A} = \langle x, y \mid x^{2m+1}, y^n, (x^{-1}y)^p\rangle$, where $p \geq 6$, is even, and let $\widehat{X}_A$ be its presentation 2-complex. Since $n \geq 2$ and $2m + 1 \geq 5$, the group $\widehat{A}$ is a hyperbolic von Dyck

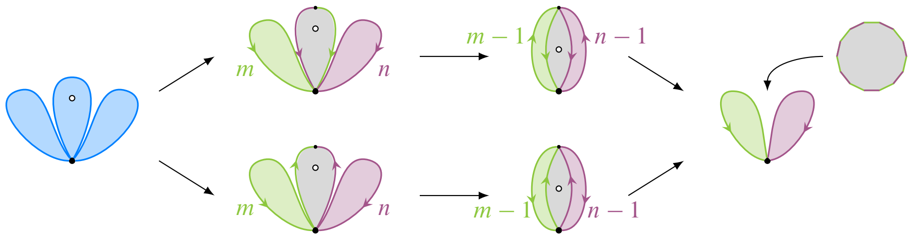

Figure 5. The maps $\widehat{f}_-\colon \widehat{X}_B \to \widehat{\widehat{X}}{}^-_B \to \widehat{X}_A$ (top) and $\widehat{f}_+\colon \widehat{X}_B \to \widehat{\widehat{X}}{}^+_B \to \widehat{X}_A$ (bottom). White nodes are contained in the 2-cells whose boundary is mapped p-to-1.

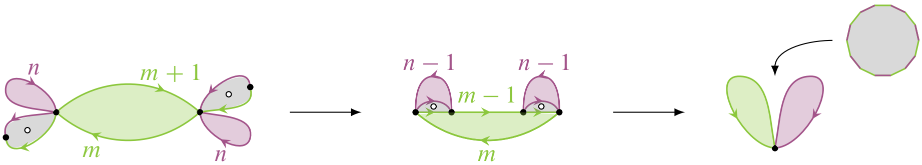

Figure 6. The maps $\widehat{X}_C \to \widehat{\widehat{X}}_C \to \widehat{X}_A$. White nodes are in the 2-cells whose boundary is mapped p-to-1.

triangle group, and in particular, $\widehat{A}$ satisfies condition (1) of Theorem 3.2. Let $\pi\colon A \to \widehat{A}$ be the natural quotient. Let $\widehat{\widehat{X}}_C$ be a 2-complex obtained from $\overline{X}_C$ by attaching five 2-cells: one along the unique cycle labeled x^{2m+1}, one along each of the two cycles labeled by y^n, and one along each of two cycles xy^{-1} via a p-to-one map. See Figure 6. In Lemma 3.8 (below), we verify that condition (2) of Theorem 3.2 is satisfied.

By Proposition 3.4, the intersection of distinct conjugates of $\pi(C)$ in $\widehat{A}$ is either $\mathbb{Z}/p$ or trivial. In particular, $\pi(C)$ is almost malnormal in $\widehat{A}$, so condition (3) of Theorem 3.2 is satisfied. Finally, we note that the 2-cells of $\widehat{\widehat{X}}_C$ can be pulled back via $X_C \to \overline{X}_C$, and are preserved under the (unique) non-trivial deck transformation of $X_C \to X_B$. See Figure 6. Condition (4) of Theorem 3.2 follows. This proves that $D(A, C, \beta)$, and therefore Art_{2MN}, is residually finite. ■

Lemma 3.8. *The image $\pi(C)$ is isomorphic to $\mathbb{Z}/p * \mathbb{Z}/p$. In particular, $\pi(C) = \pi_1 \widehat{X}_C$. Moreover, $\widehat{f}$ lifts to an embedding in the universal covers.*

Proof. For simplicity, we set $z = xy^{-1}$. The image $\pi(C)$ is generated by z, and $z' = x^m z x^{-m}$. The universal cover of the complex $\widehat{X}_A$ can be identified with the hyperbolic plane. Consider the tiling of $\mathbb{H}^2$ by a triangle with angles $\frac{\pi}{2m+1}, \frac{\pi}{n}, \frac{\pi}{p}$. Each vertex of the tiling is a fixed point of a conjugate of one of x, y, z, and the action of $\widehat{A}$ preserves the type of a vertex (i.e., whether it is fixed by a conjugate of x, y or z). We abuse the notation and identify each vertex v with the conjugate x^g, y^g or z^g which generates the stabilizer of v (where g is some element of $\widehat{A}$). The tiling is the dual of the universal cover of the complex $\widehat{X}_A$, in the way that the vertices of types x, y, z correspond to the 2-cells with boundary words x^{2m+1}, y^n, $(xy^{-1})^p$, respectively.

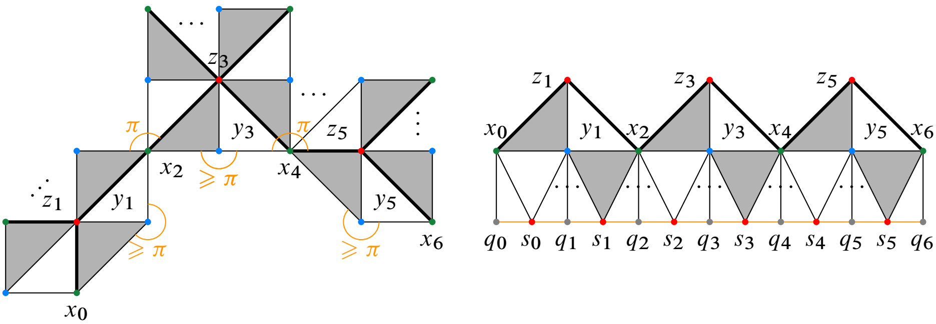

Figure 7. The vertices z_1, z_5 are $\pi(C)$-conjugates of z, and the vertex z_3 is a $\pi(C)$-conjugate of z'. Thick edges are in the image of the tree T. Odd vertices q_1, q_3, q_5 are either green or red, depending on parity of n. Even vertices q_0, q_2, q_4, q_6 are either blue or red, depending on parity of m. Note that the orange segments are not necessarily edges in the tiling.

Consider the Bass–Serre tree T of the free product $\mathbb{Z}/p * \mathbb{Z}/p$, i.e., a regular tree of valence p, where each vertex is stabilized by a conjugate of one of two $\mathbb{Z}/p$ factors. In order to prove that $\pi(C)$ splits as a free product and that $\hat{f}$ lifts to embedding of the universal covers, we show that there is $\mathbb{Z}/p * \mathbb{Z}/p$-equivariant embedding of T in $\mathbb{H}^2$ where the action of $\mathbb{Z}/p * \mathbb{Z}/p$ on $\mathbb{H}^2$ is the action of the group $\langle z, z' \rangle$.

First consider the union of the orbits of z and z' under the action of $\pi(C)$. Note that it is a collection of vertices of type z. We join z^g and $(z')^g$ by a path consisting of two edges of the tiling meeting at a vertex of type x. See Figure 7. Note that the image of the orbits of z and z' under the action of subgroup $\langle z^{p/2}, (z')^{p/2} \rangle \simeq \mathbb{Z}/2 * \mathbb{Z}/2$ is contained in a geodesic line in $\mathbb{H}^2$ (which is the line stabilized by $\langle z^{p/2}, (z')^{p/2} \rangle$).

The map from T to $\mathbb{H}^2$ is locally injective at every vertex. In order to see that T is embedded in $\mathbb{H}^2$, we verify that any bi-infinite path γ always turning rightmost in the image of T never crosses itself. Indeed, if T were not embedded, then a closed path in the image of T could be tracked by following a path turning rightmost in T. In Figure 7, a part of that path is presented as the path with vertices x_0, z_1, x_2, z_3, x_4, z_5, x_6. We will construct another path γ' (whose part is labeled by $q_0, s_0, q_1, s_1, \ldots, q_5, s_5, q_6$ in Figure 7) that stays at a finite distance from γ. We will think of γ' as an oriented path (where q_0 arises before q_1 etc.). By construction (describe below), γ' never intersects the image of T. Since the image of T contains a geodesic line and lies in the region on the left side of γ' with respect to its orientation, the region of $\mathbb{H}^2$ on the left side of γ' is unbounded. It suffices to show that the region of $\mathbb{H}^2$ on the right side of γ' is unbounded. We will do it by showing that the region on the right side of γ' is a union of halfspaces.

Let us explain how the vertices q_i, s_i are defined. The vertex s_i with odd i is the unique vertex other than z_i that forms a triangle with y_i and x_{i+1}. The vertex s_i with even i is the unique vertex other than z_{i+1} that forms a triangle with x_i and y_{i+1}. In particular, vertices s_i are always of type z. The vertices q_i with odd i are images of a z_i under the

π-rotation at y_i, i.e., the vertex y_i is a midpoint of the segment $[z_i, q_i]$. The vertices q_i with even i are chosen so that the angles $\angle s_{i-1} x_i q_i$ and $\angle s_i x_i q_i$ are equal. The path γ' is obtained by joining each pair s_{i-1}, q_i and q_i, s_i be a geodesic segment (which are not necessarily edges of the tiling).

The vertices s_{i-1}, q_i, s_i are not necessarily distinct. If $n = 2$, then $s_{i-1} = q_i = s_i$ for each odd i. Also, if $2m + 1 = 5$, then $s_{i-1} = q_i = s_i$ for $i = 4k + 2$. In that extreme case, γ' is a geodesic line, as long as $p \geq 6$. In more general case, the "left-side" angle between the segments of γ' at each vertex q_i or s_i (i.e., the angle of the sector containing y_i or x_i depending on the parity of i) is always at most π. Thus the "right-side" angle at each vertex q_i, s_i is always at least π. Consequently, the subspace of $\mathbb{H}^2$ bounded by γ' which is on the right side of γ' is a union of halfspaces. This proves our claim. ■

Our approach in the last two theorems fails in the case where $M = 2m + 1$, $N = 2n + 1$. Indeed, the fiber product $\bar{X}_C \otimes_{X_A} \bar{X}_C$ is too "large". The fiber product was computed in Remark 3.5. After attaching 2-cells along x^{2m+1}, y^{2n+1} and $(x^{-1}y)^p$, the resulting 2-complex has fundamental group $\mathbb{Z} * \mathbb{Z}/p$.

3.4. Summary of residual finiteness of 3-generator Artin groups

To summarize, the only 3-generator Artin groups that are not known to be residually finite are $\mathrm{Art}_{33(2m+1)}$ for $m \geq 2$, $\mathrm{Art}_{2(2m+1)(2n+1)}$ for $m + n \geq 4$ and $\mathrm{Art}_{23(2m)}$ for $m \geq 4$. Indeed, if at least one label is ∞, then the defining graph is a tree, and hence virtually special [7, 14, 20, 23]. Artin groups Art_{22M} for any $M \geq 2$, and Art_{23M}, where $M \in \{3, 4, 5\}$ are spherical, and so linear [9, 11]. The cases $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ follow from [25]. The cases where $M, N, P \geq 3$, except the case of $(3, 3, 2m + 1)$, were covered by [18].

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