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On the continuity of the growth rate on the space of Coxeter systems

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Abstract. Floyd showed that if a sequence of compact hyperbolic Coxeter polygons converges, then so does the sequence of the growth rates of the Coxeter groups associated with the polygons. For the case of the hyperbolic 3-space, Kolpakov discovered the same phenomena for specific convergent sequences of hyperbolic Coxeter polyhedra. In this paper, we show that the growth rate is a continuous function on the space of Coxeter systems. This is an extension of the results due to Floyd and Kolpakov since the convergent sequences of Coxeter polyhedra give rise to that of Coxeter systems in the space of marked groups.

1. Introduction

Let G be a group and $S = (s_1, \dots, s_n)$ be an ordered generating set of G. The pair (G, S) is called an *n-marked group*. Two n-marked groups (G, S) and (H, T) are isomorphic if the bijection $\phi\colon S \to T$ which maps s_i to t_i extends to a group isomorphism from G to H. The set $\mathcal{G}_n$ of all isomorphism classes of n-marked groups is called the *space of n-marked groups*. The set $\mathcal{G}_n$ has a natural topology which makes $\mathcal{G}_n$ a compact totally disconnected space [12]. The word length $|x|_S$ of an element $x \in G$ is defined by

$$|x|_S = \min\{n \geq 1 \mid x = s_1 \dots s_n,\ s_i \in S \cup S^{-1}\},$$

and this gives rise to the word metric d_S on G. We adopt the convention that $|1_G|_S = 0$, where 1_G is the identity element of G. We denote by $B_{(G,S)}(m)$ the ball of radius m centered at 1_G. Then the *growth function* of (G, S), denoted by $s_{(G,S)}(m)$, is defined to be the number of the elements of $B_{(G,S)}(m)$, that is,

$$s_{(G,S)}(m) = \#\{x \in G \mid |x|_S \leq m\}.$$

The *growth rate* $\omega(G, S)$ is given by $\omega(G, S) = \lim_{m\to\infty} \sqrt[m]{s_{(G,S)}(m)}$.

A group G is called a *Coxeter group of rank n* if there exists an ordered generating set $S = (s_1, \dots, s_n)$ such that G has the following presentation:

$$G = \langle s_1, \dots, s_n \mid s_1^2 = \dots = s_n^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \text{ for } 1 \leq i \neq j \leq n\rangle.$$

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Such a generating set S is called a *Coxeter generating set of* G. We call the pair (G, S) a *Coxeter system of rank* n.

In the hyperbolic geometry, Coxeter groups arise from the geometry of polyhedra as follows. Let $\mathbb{H}^d$ denote the hyperbolic d-space. A *hyperbolic polyhedron* $P \subset \mathbb{H}^d$ is the intersection of finitely many closed half-spaces. If dihedral angles of P are of the form π/m ($m \geq 2$), then P is called a *hyperbolic Coxeter polyhedron*. The group generated by the reflections in the bounding hyperplanes of P, denoted by G_P, is a Coxeter group. We call G_P the *hyperbolic Coxeter group associated with* P.

Concerning the study of the growth rates of hyperbolic Coxeter groups, we mention the results due to Floyd and Kolpakov. Let us write $\Delta(a_1, \dots, a_n)$ for a hyperbolic Coxeter n-gon whose interior angles are $\pi/a_1, \dots, \pi/a_n$. We say that $\Delta_k = \Delta(a_1(k), \dots, a_n(k))$ converges to $\Delta = \Delta(a_1, \dots, a_n)$ as $k \to \infty$ if $\lim_{k\to\infty} a_i(k) = a_i$ for any i. In [9], Floyd showed that $\lim_{k\to\infty} \omega(G_k) = \omega(G)$ if $\lim_{k\to\infty} \Delta_k = \Delta$, where G_k and G are the hyperbolic Coxeter groups associated with Δ_k and Δ, respectively. For the case of the hyperbolic 3-space, Kolpakov investigated a convergence of hyperbolic Coxeter polyhedra in [13] and proved that if a sequence P_k of hyperbolic Coxeter polyhedra converges, then so does the sequence $\omega(G_k)$ of the growth rates, where G_k is the hyperbolic Coxeter group associated with P_k.

The set of all Coxeter systems of rank n, denoted by $\mathcal{C}_n$, is called *the space of Coxeter systems of rank* n. We consider the space $\mathcal{C}_n$ as a subspace of $\mathcal{G}_n$ (see Definition 3.3). In this paper, we show the following theorem as an extension of the results due to Floyd and Kolpakov (see Sections 3 and 4).

Theorem. *The growth rate* $\omega\colon \mathcal{C}_n \to \mathbb{R}$ *is a continuous function.*

We mention that for hyperbolic groups, Fujiwara and Sela studied growth rates with respect to all of its finite generating sets and showed the continuity of the growth rates (see [10]).

2. The space of marked groups and the growth rates

In this section, we recall the space of marked groups (see [5, 12] for more details). For readability, we give proofs for some known facts.

Definition 2.1. For a group G and its ordered generating set $S = (s_1, \dots, s_n)$, the pair (G, S) is called an n-marked group. Two n-marked groups (G, S) and (H, T) are said to be isomorphic if the bijection from S to T that maps s_i to t_i extends to a group isomorphism. The set of all isomorphism classes of n-marked groups is denoted by $\mathcal{G}_n$, and is called the space of n-marked groups.

In the sequel, we fix an ordered generating set $X = (x_1, \dots, x_n)$ of the free group $\mathbb{F}_n$ of rank n. For any n-marked group (G, S), let us denote the epimorphism from $\mathbb{F}_n$ onto G

that maps x_i to s_i by $\pi_{(G,S)}$. It is easy to see that two n-marked groups (G,S) and (H,T) are isomorphic if and only if $\operatorname{Ker}\pi_{(G,S)} = \operatorname{Ker}\pi_{(H,T)}$.

Definition 2.2. Let (G,S) be an n-marked group. For any $x \in G$, the word length of x with respect to S, denoted by $|x|_S$, is defined by

$$|x|_S = \min\{n \geq 0 \mid x = s_1 \dots s_n,\ s_i \in S \cup S^{-1}\}.$$

We denote by $B_{(G,S)}(R)$ the ball of radius R centered at 1_G, that is,

$$B_{(G,S)}(R) = \{x \in G \mid |x|_S \leq R\}.$$

For abbreviation, we write $B(R)$ instead of $B_{(\mathbb{F}_n,X)}(R)$. We define a metric d on $\mathcal{G}_n$ as follows. For two n-marked groups (G,S) and (H,T), let us denote by $v((G,S),(H,T))$ the maximum radius of the balls centered at $1_{\mathbb{F}_n}$ where $\operatorname{Ker}\pi_{(G,S)} = \operatorname{Ker}\pi_{(H,T)}$, that is,

$$v((G,S),(H,T)) = \max\{R \geq 0 \mid B(R) \cap \operatorname{Ker}\pi_{(G,S)} = B(R) \cap \operatorname{Ker}\pi_{(H,T)}\}.$$

Then the metric d on $\mathcal{G}_n$ is defined by

$$d((G,S),(H,T)) = e^{-v((G,S),(H,T))}.$$

It is known that the metric space $(\mathcal{G}_n, d)$ is compact.

The *Cayley graph* $\operatorname{Cay}(G,S)$ of an n-marked group (G,S) is an edge-labeled directed graph defined as follows. The vertex set is G, and two vertices x, y are joined by an oriented edge from x to y labeled with i if $x^{-1}y = s_i$. We consider the ball $B_{(G,S)}(R)$ as a subgraph of $\operatorname{Cay}(G,S)$.

Lemma 2.3. *For two n-marked groups $(G,S),(H,T) \in \mathcal{G}_n$ and $R \geq 1$, the distance satisfies that $d((G,S),(H,T)) \leq e^{-(2R+1)}$ if and only if there exists an orientation- and label-preserving graph isomorphism from $B_{(G,S)}(R)$ to $B_{(H,T)}(R)$ that maps the identity element of G to that of H.*

Proof. Suppose that $d((G,S),(H,T)) \leq e^{-(2R+1)}$. By the definition of the metric on $\mathcal{G}_n$, we have

$$B(2R+1) \cap \operatorname{Ker}\pi_{(G,S)} = B(2R+1) \cap \operatorname{Ker}\pi_{(H,T)}. \tag{2.1}$$

We define a map $\alpha\colon B_{(G,S)}(R) \to B_{(H,T)}(R)$ as follows. For $g = s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k}$, $k \leq R$,

$$\alpha(s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k}) = t_{i_1}^{\varepsilon_1} \dots t_{i_k}^{\varepsilon_k}.$$

First we show that the map α is well defined. For that, suppose that an element $g \in B_{(G,S)}(R)$ has two expressions as follows:

$$g = s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k} = s_{j_1}^{\delta_1} \dots s_{j_l}^{\delta_l},$$

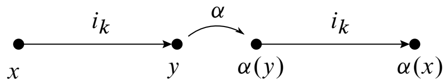


Figure 1. The oriented edge from x to y labeled with i_k.

where $0 \leq k,\ l \leq R$. Then the element $w = x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k} x_{j_l}^{-\delta_l} \dots x_{j_1}^{-\delta_1}$ of $\mathbb{F}_n$ belongs to $B(2R+1) \cap \operatorname{Ker} \pi_{(G,S)}$. By (2.1), we see that $\pi_{(H,T)}(w) = 1_H$, and hence the map α is well defined. It is easy to see that the map α is an orientation- and label-preserving graph homomorphism. In the same manner, the map $\beta\colon B_{(H,T)}(R) \to B_{(G,S)}(R)$ is defined by

$$\beta(t_{i_1}^{\varepsilon_1} \dots t_{i_k}^{\varepsilon_k}) = s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k}.$$

Since $\beta = \alpha^{-1}$, α is the desired graph isomorphism.

Conversely, suppose that there exists an orientation- and label-preserving graph isomorphism $\alpha\colon B_{(G,S)}(R) \to B_{(H,T)}(R)$ such that $\alpha(1_G) = 1_H$. We claim that

$$\alpha(s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k}) = t_{i_1}^{\varepsilon_1} \dots t_{i_k}^{\varepsilon_k}. \tag{2.2}$$

The proof is carried out by induction on k. It is clear for the case $k = 0$. Set $x = s_{i_1}^{\varepsilon_1} \dots s_{i_{k-1}}^{\varepsilon_{k-1}}$ and $y = x s_{i_k}$. Then the vertices x and y are joined by the oriented edge from x to y labeled with i_k; so are the vertices $\alpha(x)$ and $\alpha(y)$ (see Figure 1).

By the inductive hypothesis, we have $\alpha(x) = t_{i_1}^{\varepsilon_1} \dots t_{i_{k-1}}^{\varepsilon_{k-1}}$. Since the terminal vertex of the oriented edge labeled with i_k emanating from $t_{i_1}^{\varepsilon_1} \dots t_{i_{k-1}}^{\varepsilon_{k-1}}$ is $t_{i_1}^{\varepsilon_1} \dots t_{i_{k-1}}^{\varepsilon_{k-1}} t_{i_k}$, we obtain $\alpha(y) = t_{i_1}^{\varepsilon_1} \dots t_{i_{k-1}}^{\varepsilon_{k-1}} t_{i_k}$. By applying similar arguments to the case where $x = s_{i_1}^{\varepsilon_1} \dots s_{i_{k-1}}^{\varepsilon_{k-1}} s_{i_k}^{-1}$ and $y = s_{i_1}^{\varepsilon_1} \dots s_{i_{k-1}}^{\varepsilon_{k-1}}$, we have $\alpha(s_{i_1}^{\varepsilon_1} \dots s_{i_{k-1}}^{\varepsilon_{k-1}} s_{i_k}^{-1}) = t_{i_1}^{\varepsilon_1} \dots t_{i_{k-1}}^{\varepsilon_{k-1}} t_{i_k}^{-1}$. Therefore, we obtain (2.2). Fix an element $w = x_{i_1}^{\varepsilon_1} \dots x_{i_k}^{\varepsilon_k} \in B(2R+1) \cap \operatorname{Ker} \pi_{(G,S)}$. Let us denote by p_w the closed path in Cay (G, S) corresponding to w. We write u for the farthest vertex of p_w from 1_G. If $|u|_S \geq R+1$, then the length of p_w must be greater than or equal to $2(R+1)$. Therefore, the closed path p_w belongs to $B_{(G,S)}(R)$. By equality (2.2), the closed path of Cay (H, T) corresponding to w is $\alpha(p_w)$, and this implies $(B(2R+1) \cap \operatorname{Ker} \pi_{(G,S)}) \subset (B(2R+1) \cap \operatorname{Ker} \pi_{(H,T)})$. Replacing α by α^{-1}, we obtain $(B(2R+1) \cap \operatorname{Ker} \pi_{(H,T)}) \subset (B(2R+1) \cap \operatorname{Ker} \pi_{(G,S)})$. ■

For any n-marked group $(G, S) \in \mathcal{G}_n$, the *growth function* $s_{(G,S)}(m)$ is defined to be the number of elements of G whose lengths are at most m, that is,

$$s_{(G,S)}(m) = \#\{x \in G \mid |x|_S \leq m\}.$$

Since $s_{(G,S)}(m)$ is submultiplicative, we have

$$\lim_{m \to \infty} \sqrt[m]{s_{(G,S)}(m)} = \inf_{m \geq 0} \sqrt[m]{s_{(G,S)}(m)}.$$

We define the *growth rate* $\omega(G, S)$ to be $\lim_{m \to \infty} \sqrt[m]{s_{(G,S)}(m)}$. We say that an n-marked group (G, S) has *exponential growth rate* if $\omega(G, S) > 1$.

3. The space of Coxeter systems

In this section, we give the definitions of Coxeter groups, Coxeter matrices, and the space $\mathcal{C}_n$ of Coxeter systems of rank n. Then we prove that $\mathcal{C}_n$ is a compact subspace of $\mathcal{G}_n$. The general reference here is [4].

Definition 3.1. Let us denote the set of the natural numbers with ∞ by $\widehat{\mathbb{N}}$. An $n \times n$ matrix $M = (m_{ij})$ over $\widehat{\mathbb{N}}$ is called a Coxeter matrix if it is symmetric and satisfies the condition $m_{ij} = 1$ if and only if $i = j$. We denote the set of all Coxeter matrices of $n \times n$ size by $\mathcal{CM}_n$, and define a metric D on $\mathcal{CM}_n$ as follows:

$$D(M, M') = \max_{1 \le i, j \le n} \left| \frac{1}{e^{m_{ij}}} - \frac{1}{e^{m'_{ij}}} \right|,$$

where $M = (m_{ij})$, $M' = (m'_{ij})$.

Since $\mathcal{CM}_n$ is closed subset of the product space $\widehat{\mathbb{N}}^{n^2}$, $\mathcal{CM}_n$ is compact. It is easy to see that any convergent sequence $\{m(k)\}_{k \ge 1}$ of $\widehat{\mathbb{N}}$ is either eventually constant or converges to ∞. Thus we have the following.

Lemma 3.2. *Let $\{M_k = (m_{ij}(k))\}_{k \ge 1}$ be a sequence of Coxeter matrices. Then M_k converges to a Coxeter matrix $M = (m_{ij})$ if and only if for any positive integer L, there exists $K_L \ge 0$ such that for $k \ge K_L$,*

$$\begin{cases} m_{ij}(k) \ge L & \text{if } m_{ij} = \infty, \\ m_{ij}(k) = m_{ij} & \text{if } m_{ij} < \infty. \end{cases}$$

Definition 3.3. Any $n \times n$ size Coxeter matrix $M = (m_{ij})$ determines a group $G(M)$ with the presentation

$$G(M) = \langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \text{ for } 1 \le i \le j \le n \rangle.$$

A group G having such a presentation is called a Coxeter group and the pair (G, S) is called a Coxeter system. The cardinality of the generating set S is called the rank of (G, S). We denote the set of all Coxeter systems of rank n by $\mathcal{C}_n \subset \mathcal{G}_n$. The subspace $\mathcal{C}_n$ is called the space of Coxeter systems of rank n.

In order to prove that $\mathcal{CM}_n$ and $\mathcal{C}_n$ are homeomorphic, we will show that $\mathcal{C}_n$ is closed in $\mathcal{G}_n$.

Theorem 3.4 ([4, p. 18, Theorem 1.5.1]). *Let G be a group and S be its generating set. Suppose that every element of S is of order 2. Then the followings are equivalent:*

(i) *The pair (G, S) is a Coxeter system.*

(ii) *Let $x = s_{i_1} \dots s_{i_l}$ be a reduced word in G, and let $s \in S$. If $|sx|_S < |x|_S$, then $sx = s_{i_1} \dots \widehat{s}_{i_m} \dots s_{i_l}$.*

Lemma 3.5. *The space of Coxeter systems $\mathcal{C}_n$ is closed in $\mathcal{G}_n$.*

Proof. Suppose that a sequence $\{(G_k, S_k)\}_{k\geq 1}$ of Coxeter systems of rank n converges to an n-marked group (G, S). We prove that (G, S) is a Coxeter system of rank n. By Lemma 2.3 and the assumption $\lim_{k\to\infty}(G_k, S_k) = (G, S)$, for any $R \geq 0$, there exists an integer $k = k(R) \geq 1$ such that the balls $B_{(G_k,S_k)}(R)$ and $B_{(G,S)}(R)$ are isomorphic. By taking $R = 2$, we see that every element of S is of order 2. Fix a reduced word $x = s_{i_1} \dots s_{i_l}$ in G and $s \in S$. Then the expression $s_{i_1} \dots s_{i_l}$ gives rise to the shortest path from 1_G to x. Fix an integer $R \geq 0$ such that $x, sx \in B_{(G,S)}(R)$. Let us denote the graph isomorphism from $B_{(G,S)}(R)$ to $B_{(G_k,S_k)}(R)$ by α. The assumption that $x = s_{i_1} \dots s_{i_l}$ is a reduced word implies that $\alpha(x) = \alpha(s_{i_1}) \dots \alpha(s_{i_l})$ is a reduced word in G_k. Since G_k is a Coxeter group, by Theorem 3.4, if $|sx|_S < |x|_S$, then

$$\alpha(sx) = \alpha(s)\alpha(x) = \alpha(s_{i_1}) \dots \widehat{\alpha(s_{i_m})} \dots \alpha(s_{i_l}) = \alpha(s_{i_1} \dots \widehat{s}_{i_m} \dots s_{i_l}).$$

Therefore, we obtain $sx = s_{i_1} \dots \widehat{s}_{i_m} \dots s_{i_l}$. ∎

Theorem 3.6. *The map $\Phi\colon \mathcal{CM}_n \to \mathcal{C}_n$ that maps M to $G(M)$ is a homeomorphism.*

Proof. It is trivial that Φ is a bijection. If we prove that Φ^{-1} is continuous, the fact that $\mathcal{G}_n$ is compact together with Lemma 3.5 implies that Φ^{-1} is a homeomorphism, since any continuous bijection from a compact space to a Hausdorff space is homeomorphism. Let (G_k, S_k) and (G, S) be Coxeter systems of rank n and write $M_k = (m_{ij}(k))$ and $M = (m_{ij})$ for the Coxeter matrices corresponding to (G_k, S_k) and (G, S), respectively. Suppose that $\lim_{k\to\infty}(G_k, S_k) = (G, S)$. Fix a defining relation $r = (s_i s_j)^{m_{ij}}$ of G. The relation r corresponds to the cycle of length $2m_{ij}$ in $\mathrm{Cay}\,(G, S)$ labeled with i and j alternately. For the case $m_{ij} < \infty$, by Lemma 2.3, the ball $B_{(G_k,S_k)}(m_{ij})$ must contain the cycle of length $2m_{ij}$ labeled with i and j alternately for sufficiently large k. Therefore, $\lim_{k\to\infty} m_{ij}(k) = m_{ij}$. Consider the case $m_{ij} = \infty$. In order to obtain a contradiction, suppose that there exists an integer $R \geq 0$ such that $m_{ij}(k) \leq R$ for any k. Since $B_{(G,S)}(2R)$ does not contain the cycle labeled with i and j alternately, we see that the balls $B_{(G,S)}(2R)$ and $B_{(G_k,S_k)}(2R)$ are not isomorphic for any k. This contradicts to Lemma 2.3. Therefore, we obtain $\lim_{k\to\infty} m_{ij}(k) = m_{ij}$ for any i, j, that is, the map Φ^{-1} is continuous. ∎

Definition 3.7. Let (G, S) be a Coxeter system of rank n and $M = (m_{ij})$ be the Coxeter matrix corresponding to (G, S). The Gram matrix $\mathrm{Gram}\,(G, S) = (g_{ij})$ is a symmetric matrix of $n \times n$ size defined as follows:

$$g_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\cos\frac{\pi}{m_{ij}} & \text{if } i \neq j. \end{cases}$$

If $\mathrm{Gram}\,(G, S)$ is positive definite (resp. positive semidefinite), the Coxeter system (G, S) is said to be elliptic (resp. affine) (see [19] for more details). We call (G, S) a *non-affine* Coxeter system if (G, S) is neither elliptic nor affine.

Lemma 3.8. *The set of all elliptic or affine Coxeter systems of rank n is closed in $\mathcal{C}_n$.*

Proof. Suppose that a sequence $\{(G_k, S_k)\}_{k\geq 1}$ of elliptic or affine Coxeter systems of rank n converges to a Coxeter system (G, S) of rank n. Let us write $M_k = (m_{ij}(k))$ and $M = (m_{ij})$ for the Coxeter matrices corresponding to (G_k, S_k) and (G, S), respectively. By Theorem 3.6, we have $\lim_{k\to\infty} g_{ij}(k) = g_{ij}$, where $g_{ij}(k)$ and g_{ij} denote the (i, j)-th entries of $\mathrm{Gram}\,(G_k, S_k)$ and $\mathrm{Gram}\,(G, S)$, respectively. Therefore, the eigenvalues $\lambda_1(k), \dots, \lambda_n(k)$ of $\mathrm{Gram}\,(G_k, S_k)$ converge to the eigenvalues $\lambda_1, \dots, \lambda_n$ of $\mathrm{Gram}\,(G, S)$ (see [3, Section VI]), which proves the assertion. ■

The growth type of a Coxeter system (G, S) is as follows:

(i) If (G, S) is elliptic, then G is finite, so $\omega(G, S) = 1$ (see [6, 19]).

(ii) If (G, S) is affine, then G must contain a free abelian subgroup of finite index, so $\omega(G, S) = 1$ (see [6, 19]).

(iii) If (G, S) is non-affine, then G must contain a free subgroup of rank at least 2, so $\omega(G, S) > 1$ (see [7]).

Corollary 3.9. *A Coxeter system (G, S) is non-affine if and only if $\omega(G, S) > 1$.*

There is a partial ordering $\prec$ on $\mathcal{C}_n$ defined in [14] as follows:

$$(G, S) \prec (G', S') \Leftrightarrow m_{ij} \leq m'_{ij} \quad \text{for } 1 \leq i, j \leq n.$$

Since any convergent sequence of $\widehat{\mathbb{N}}$ is eventually constant or converges to ∞, we obtain the following.

Lemma 3.10. *Let (G_k, S_k) be a Coxeter system of rank n that converges to (G, S). We write $M_k = (m_{ij}(k))$ and $M = (m_{ij})$ for the Coxeter matrices corresponding to (G_k, S_k) and (G, S), respectively. Suppose that the sequence (G_k, S_k) is not eventually constant. Then some of m_{ij}'s are infinity and the sequence contains an increasing subsequence.*

The following theorem due to Terragni is of fundamental importance for this paper. For an n-marked group (G, S), we define the function $a_{(G,S)}(m)$ to be the number of elements of G whose word lengths with respect to S are equal to m, that is,

$$a_{(G,S)}(m) = \#\{x \in G \mid |x|_S = m\}.$$

We also call the function $a_{(G,S)}(m)$ the *growth function* of (G, S).

Theorem 3.11 ([18, p. 607, Theorem A]). *Let (G, S) and (G', S') be Coxeter systems of rank n. If $(G, S) \prec (G', S')$, then $a_{(G,S)}(m) \leq a_{(G',S')}(m)$ for $m \geq 0$.*

Since $s_{(G,S)}(m) = a_{(G,S)}(0) + \dots + a_{(G,S)}(m)$ if Coxeter systems (G, S) and (G', S') satisfy $(G, S) \prec (G', S')$, we have $s_{(G,S)}(m) \leq s_{(G',S')}(m)$ for any $m \geq 0$.

Corollary 3.12. *The set of all non-affine Coxeter systems of rank n is closed in $\mathcal{C}_n$.*

Proof. Suppose that a sequence $\{(G_k, S_k)\}_{k\geq 1}$ of non-affine Coxeter systems of rank n converges to a Coxeter system (G, S). We prove that (G, S) is non-affine. It is trivial for the case that the sequence is eventually constant. For the other case, by Lemma 3.10, there exists an increasing subsequence $\{(G_{k_l}, S_{k_l})\}_{l\geq 1}$ of non-affine Coxeter systems of rank n. Theorem 3.11 implies that

$$\omega(G, S) \geq \omega(G_{k_1}, S_{k_1}) > 1,$$

and hence G is non-affine by Corollary 3.9. ■

Steinberg's formula is fundamental to compute the growth rates of Coxeter systems.

Definition 3.13. For a Coxeter system (G, S) and a subset $T \subset S$, the subgroup generated by T, denoted by G_T, is called a *parabolic subgroup* of G. The pair (G_T, T) itself is a Coxeter system. We denote by $\mathcal{F}(G, S)$ the set of all subsets of S generating finite parabolic subgroups, that is,

$$\mathcal{F}(G, S) = \{T \subset S \mid \#G_T < \infty\}.$$

The growth series $f_{(G,S)}(z)$ is defined by

$$f_{(G,S)}(z) = \sum_{m=0}^{\infty} a_{(G,S)}(m) z^m.$$

Let us now recall the classification of finite Coxeter groups. In order to do that, we need the notion of *Coxeter diagram*.

Definition 3.14. For a Coxeter system (G, S), the *Coxeter diagram* $X(G, S)$ of (G, S) is constructed as follows: its vertex set is S. If $m_{ij} \geq 4$, we join the pair of vertices by an edge and label it with m_{ij}. If $m_{ij} = 3$, we join the pair of vertices by an edge without any label. For any $L \in \widehat{\mathbb{N}}$, we denote by $\mathcal{F}_L(G, S)$ the set of all elements of $\mathcal{F}(G, S)$ whose Coxeter diagrams have edges labeled with L, that is,

$$\mathcal{F}_L(G, S) = \{T \in \mathcal{F}(G, S) \mid X(G_T, T) \text{ has edges labeled with } L\}.$$

A Coxeter system (G, S) is said to be *irreducible* if the Coxeter diagram $X(G, S)$ is connected. Irreducible finite Coxeter systems of rank n are classified as in Table 1 (see [4]). For integers $m_1, \ldots, m_k \geq 1$, the polynomial $[m_1; \ldots; m_k]$ is defined by

$$[m_1; \ldots; m_k] = (1 + \cdots + z^{m_1 - 1}) \cdots (1 + \cdots + z^{m_k - 1}).$$

The growth series of irreducible finite Coxeter systems are determined by Solomon's formula (see [16] for details). Table 1 shows the list of the growth series of irreducible finite Coxeter systems of rank n. Note that the growth series of irreducible finite Coxeter systems are products of cyclotomic polynomials.

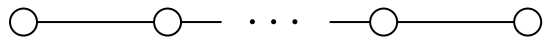
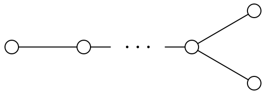
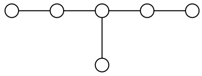
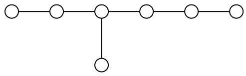
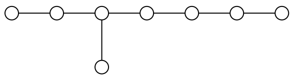
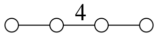
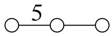
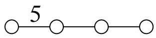
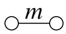
Coxeter group	Diagram	Growth series
A_n		$[2; 3; \dots; n+1]$
B_n		$[2; 4; \dots; 2n]$
D_n		$[2; 4; \dots; 2n-2; n]$
E_6		$[2; 5; 6; 8; 9; 12]$
E_7		$[2; 6; 8; 10; 12; 14; 18]$
E_8		$[2; 8; 12; 14; 18; 20; 24; 30]$
F_4		$[2; 6; 8; 12]$
H_3		$[2; 6; 10]$
H_4		$[2; 12; 20; 30]$
$I_2(m)$		$[2; m]$

Table 1. The classification of irreducible finite Coxeter systems of rank n and their growth series.

For a Coxeter system (G, S), we denote the vertex sets of connected components of $X(G, S)$ by $S_1, \dots, S_k$. Let $G_1, \dots, G_k$ be the Coxeter groups generated by $S_1, \dots, S_k$. The following hold for (G, S):

$$G = G_1 \times \cdots \times G_k, \quad S = S_1 \sqcup \cdots \sqcup S_k.$$

Then we say that (G, S) is the product of $(G_1, S_1), \dots, (G_k, S_k)$.

Lemma 3.15. *Let (G, S) be a Coxeter system. If (G, S) is the product of irreducible Coxeter systems $(G_1, S_1), \dots, (G_k, S_k)$, then the growth series satisfies*

$$f_{(G,S)}(z) = f_{(G_1,S_1)}(z) \cdots f_{(G_k,S_k)}(z).$$

By Lemma 3.15 and the fact that the growth series of irreducible finite Coxeter systems are products of cyclotomic polynomials, we see that the growth series $f_{(G,S)}(z)$ of a finite Coxeter system (G, S) has the following property:

$$z^{\deg f_{(G,S)}} \cdot f_{(G,S)}(z^{-1}) = f_{(G,S)}(z). \tag{3.1}$$

The growth series of infinite Coxeter systems are written as rational functions.

Theorem 3.16 (Steinberg's formula [17]). *Let $G = (G, S)$ be an infinite Coxeter system. The following identity holds for the growth series $f_{(G,S)}(z)$:*

$$\frac{1}{f_{(G,S)}(z^{-1})} = \sum_{T \in \mathcal{F}(G,S)} \frac{(-1)^{\#T}}{f_{(G_T,T)}(z)}. \tag{3.2}$$

Substituting (3.1) into Steinberg's formula (3.2), we obtain

$$\frac{1}{f_{(G,S)}(z)} = \sum_{T \in \mathcal{F}(G,S)} (-1)^{\#T} \frac{z^{d_T}}{f_{(G_T,T)}(z)}, \tag{3.3}$$

where $d_T = \deg f_{(G_T,T)}(z)$ for $T \in \mathcal{F}(G, S)$. Set $F_{(G,S)}(z) = 1/(f_{(G,S)}(z))$. Since the growth series $f_{(G_T,T)}(z)$ is a product of cyclotomic polynomials, $F_{(G,S)}(z)$ is a rational function and all of those poles lie on the unit circle. Therefore, $F_{(G,S)}(z)$ is holomorphic on the unit open disk.

Lemma 3.17. *Let (G, S) be an infinite Coxeter system. The reciprocal of the growth rate $\omega(G, S)$ is the zero of $F_{(G,S)}(z)$ whose modulus is minimum among all zeros of $F_{(G,S)}(z)$.*

Proof. Since G is infinite, we have $\omega(G, S) = \limsup_{m\to\infty} \sqrt[m]{a_{(G,S)}(m)}$ (see [8, Section VI.C, p. 182]). Let us denote the radius of convergence of the series $f_{(G,S)}(z)$ by R. By the Cauchy–Hadamard formula,

$$R = \frac{1}{\limsup_{m\to\infty} \sqrt[m]{a_{(G,S)}(m)}} = \omega(G, S)^{-1}.$$

Since $\inf_{m\geq 0} \sqrt[m]{a_{(G,S)}(m)} = \omega(G, S)$, we obtain $a_{(G,S)}(m) \geq \omega(G, S)^m$ (see [8, p. 183, Proposition 56]), and hence the series $f_{(G,S)}(\omega(G, S)^{-1})$ diverges, which proves our assertion. ∎

Proposition 3.18. *Let (G, S) be a non-affine Coxeter system of rank n and write $M = (m_{ij})$ for the Coxeter matrix corresponding to (G, S). For any $l \geq 6$, define the Coxeter matrix $M(l) = (m_{ij}(l))$ of $n \times n$ size as follows:*

$$m_{ij}(l) = \begin{cases} l & \text{if } m_{ij} = \infty, \\ m_{ij} & \text{if } m_{ij} < \infty. \end{cases}$$

The Coxeter system defined by $M(l)$ is denoted by $(G(l), S(l))$. Then the meromorphic function $F_{(G(l),S(l))}(z)$ converges normally to $F_{(G,S)}(z)$ on the unit open disk.

Proof. By Lemma 3.8, the set of non-affine Coxeter systems is open in $\mathcal{C}_n$. Therefore, the Coxeter system $(G(l), S(l))$ is non-affine for sufficiently large l. From now on, we assume that $(G(l), S(l))$ is non-affine. Since the ordering on the generating set $S(l)$ of $G(l)$ is defined by the Coxeter matrix $M(l)$, we identify $S(l)$ with S by the correspondence $s_i(l) \mapsto s_i$. For example, we may consider that any subset T of S does not only generate

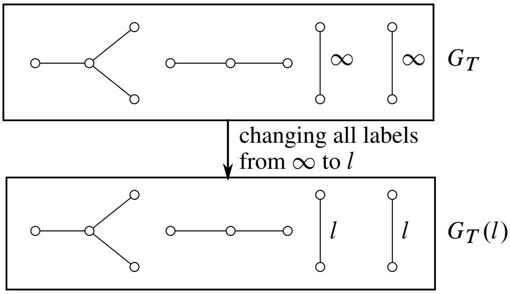


Figure 2. Coxeter diagrams $X(G_T, T)$ and $X(G(l)_T, T)$.

the parabolic subgroup G_T of G but also generates the parabolic subgroup $G(l)_T$ of $G(l)$. For any $T \subset S$, the Coxeter diagram $X(G(l)_T, T)$ is obtained from the Coxeter diagram $X(G_T, T)$ by changing all labels from ∞ to l (see Figure 2 for an example). Therefore, the underlying graph structure of $X(G(l)_T, T)$ is the same as that of $X(G_T, T)$. In this proof, we use the following notations:

$$\begin{aligned}
\mathcal{F} &= \mathcal{F}(G, S) = \{T \subset S \mid \#G_T < \infty\},\\
\mathcal{F}_L &= \mathcal{F}_L(G, S) = \{T \in \mathcal{F} \mid X(G_T, T) \text{ has edges labeled with } L\},\\
\mathcal{F}(l) &= \mathcal{F}(G(l), S) = \{T \subset S \mid \#G(l)_T < \infty\},\\
\mathcal{F}_L(l) &= \mathcal{F}_L(G(l), S) = \{T \in \mathcal{F}(l) \mid X(G(l)_T, T) \text{ has edges labeled with } L\}.
\end{aligned}$$

First, we show that $\mathcal{F}_l(l) = \mathcal{F}_{l'}(l')$ for any $l, l' \geq 6$. The classification of irreducible finite Coxeter systems implies that if $T \in \mathcal{F}_l(l)$, then every component of the Coxeter diagram $X(G(l)_T, T)$ having at least one edge labeled with l must be $I_2(l)$. Hence $G(l)_T$ generated by $T \in \mathcal{F}_l(l)$ can be expressed as the product of the finite Coxeter group H_T and $I_2(l)$'s, where the Coxeter diagram of H_T has no edges labeled with l (see Figure 3 for an example). Since the Coxeter diagram $X(G(l')_T, T)$ is obtained from $X(G(l)_T, T)$ by changing the labels l into l', $G(l')_T$ is the product of H_T and $I_2(l')'s$. Therefore, $T \in \mathcal{F}_{l'}(l')$ for any $T \in \mathcal{F}_l(l)$. By interchanging the roles of l and l', we have $\mathcal{F}_l(l) = \mathcal{F}_{l'}(l')$.

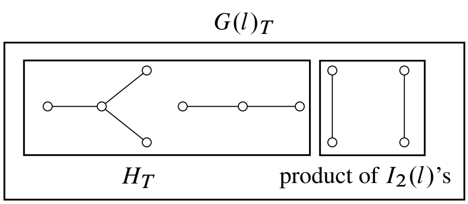


Figure 3. Decomposition of $X(G(l)_T, T)$ into H_T and $I_2(l)$'s.

Given $T \in \mathcal{F}_l(l)$, we write k_T and $I_2(l)_T$ for the number of the edges of $X(G(l)_T, T)$ labeled with l and the parabolic subgroup of $G(l)_T$ generated by $T \setminus H_T$, respectively. Note that the number k_T does not depend on $l \geq 6$. By Lemma 3.15 and the fact that $(G(l)_T, T) = (H_T, T \cap H_T) \times (I_2(l)_T, I_2(l)_T \cap T)$, we have the following equality for $T \in \mathcal{F}_l(l)$:

$$f_{(G(l)_T,T)}(z) = f_{(H_T,T\cap H_T)}(z) f_{(I_2(l)_T,I_2(l)_T\cap T)}(z) = f_{(H_T,T\cap H_T)}(z)[2;l]^{k_T}. \quad (3.4)$$

For $l \geq 6$ and $T \in \mathcal{F}_l(l)$, let us denote the degrees of the polynomials $f_{(G(l)_T,T)}$ and $f_{(H_T,T\cap H_T)}$ by $d_T(l)$ and $h_T(l)$, respectively. By equality (3.4),

$$d_T(l) = h_T(l) + k_T l.$$

Then the identity (3.3) for $F_{(G(l),S)}(z)$ is rewritten as

$$\begin{aligned}
&F_{(G(l),S)}(z) \\
&\quad= \sum_{T\in\mathcal{F}(l)} (-1)^{\#T} \frac{z^{d_T(l)}}{f_{(G(l)_T,T)}(z)} \\
&\quad= \sum_{T\in\mathcal{F}(l)\setminus\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{d_T(l)}}{f_{(G(l)_T,T)}(z)} + \sum_{T\in\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{d_T(l)}}{f_{(G(l)_T,T)}(z)} \\
&\quad= \sum_{T\in\mathcal{F}(l)\setminus\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{d_T(l)}}{f_{(G(l)_T,T)}(z)} + \sum_{T\in\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{h_T(l)+k_T l}}{f_{(H_T,T\cap H_T)}(z)\cdot[2;l]^{k_T}} \\
&\quad= \sum_{T\in\mathcal{F}(l)\setminus\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{d_T(l)}}{f_{(G(l)_T,T)}(z)} + \sum_{T\in\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{h_T(l)}}{f_{(H_T,T\cap H_T)}(z)} \cdot \frac{z^{k_T l}}{[2;l]^{k_T}}.
\end{aligned}$$

If l tends to ∞, any parabolic subgroup $G(l)_T$ generated by $T \in \mathcal{F}_l(l)$ becomes an infinite Coxeter group. This observation implies

$$\mathcal{F} = \mathcal{F}(l) \setminus \mathcal{F}_l(l).$$

Since the degree $d_T(l) = \deg f_{(G(l)_T,T)}(z)$ does not depend on l for $T \in \mathcal{F}(l) \setminus \mathcal{F}_l(l)$, we obtain

$$\sum_{T\in\mathcal{F}(l)\setminus\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{d_T(l)}}{f_{(G(l)_T,T)}(z)} = \sum_{T\in\mathcal{F}} (-1)^{\#T} \frac{z^{d_T}}{f_{(G(l)_T,T)}(z)} = F_{(G,S)}(z),$$

and hence

$$F_{(G(l),S)}(z) = F_{(G,S)}(z) + \sum_{T\in\mathcal{F}_l(l)} (-1)^{\#T} \frac{z^{h_T(l)}}{f_{(H_T,T\cap H_T)}(z)} \cdot \frac{z^{k_T l}}{[2;l]^{k_T}}. \quad (3.5)$$

Now we are in a position to show that $F_{(G(l),S)}(z)$ converges normally to $F_{(G,S)}(z)$ on the unit open disk. Let us regard $0 < \rho < 1$ as fixed and write D_ρ for the closed disk of radius ρ centered at 0. Since $f_{(H_T, T\cap H_T)}(z)$ is a product of cyclotomic polynomials, there are no zeros in D_ρ, and hence $z^{h_T(l)}/(f_{(H_T,T\cap H_T)}(z))$ is continuous on the compact set D_ρ. Since $\mathcal{F}_l(l) = \mathcal{F}_{l'}(l')$ for $l, l' \geq 6$, then we may take a positive constant M large enough that for any $l \geq 6$, $T \in \mathcal{F}_l(l)$ and $z \in D_\rho$,

$$\left|\frac{z^{h_T(l)}}{f_{(H_T,T\cap H_T)}(z)}\right| < M. \tag{3.6}$$

By multiplying $(1-z)^{k_T}$ to the denominator and numerator of $z^{k_T l}/([2;l]^{k_T})$, we have

$$\frac{z^{k_T l}}{[2;l]^{k_T}} = \left(\frac{1-z}{1+z}\right)^{k_T} \cdot \frac{z^{k_T l}}{(1-z^l)^{k_T}}. \tag{3.7}$$

Since the function $(1-z)/(1+z)$ is continuous on D_ρ, there exists a positive constant M' such that for any $l \geq 6$, $T \in \mathcal{F}_l(l)$ and $z \in D_\rho$,

$$\left|\left(\frac{1-z}{1+z}\right)^{k_T}\right| < M'. \tag{3.8}$$

By the triangle inequality,

$$|1-z^l| \geq 1-\rho^l \quad \text{for } l \geq 6,\ z \in D_\rho.$$

This observation together with the equality and inequalities (3.5), (3.6), (3.7), and (3.8) give the following inequality for $z \in D_\rho$:

$$|F_{(G(l),S)}(z) - F_{(G,S)}(z)| \leq \sum_{T\in\mathcal{F}_l(l)} MM' \frac{\rho^L}{(1-\rho^L)^{k_T}}.$$

Since the cardinality of the set $\mathcal{F}_l(l)$ is constant and finite for $l \geq 6$,

$$\lim_{l\to\infty} \sup_{z\in D_\rho} |F_{(G(l),S)}(z) - F_{(G,S)}(z)| = 0,$$

and this is precisely the assertion of Proposition 3.18. ∎

Corollary 3.19. *Under the same assumption as in Proposition 3.18, the growth rate $\omega(G(l), S(l))$ converges to $\omega(G, S)$.*

Proof. Since the Coxeter system $(G(l), S(l))$ is non-affine for sufficiently large l, we may assume that $(G(l), S(l))$ is non-affine. In order to obtain a contradiction, suppose that $\omega(G(l), S(l))$ does not converge to $\omega(G, S)$. Fix $\varepsilon > 0$ such that the closed disk $\overline{D(\omega(G,S)^{-1}, \varepsilon)}$ of radius ε centered at $\omega(G,S)^{-1}$ does not contain $\omega(G(l), S(l))^{-1}$ for any l. Since $\omega(G,S)^{-1}$ is a zero of $F_{(G,S)}(z)$, by Proposition 3.18 and Hurwitz's theorem (see [11, p. 231, Theorem]), the disk $D(\omega(G,S)^{-1}, \varepsilon)$ contains at least one zero z_l of $F_{(G(l),S(l))}(z)$ for sufficiently large l. By the triangle inequality,

$$|z_l| \leq |z_l - \omega(G,S)^{-1}| + |\omega(G,S)^{-1}| < \varepsilon + \omega(G,S)^{-1}. \tag{3.9}$$

By Theorem 3.11, $\omega(G,S)^{-1} < \omega(G(l),S(l))^{-1}$ for any l. The assumption

$$\omega(G(l),S(l))^{-1} \notin \overline{D(\omega(G,S)^{-1},\varepsilon)}$$

implies

$$\omega(G(l),S(l))^{-1} > \omega(G,S)^{-1} + \varepsilon. \tag{3.10}$$

By inequalities (3.9) and (3.10), z_l is a zero of $F_{(G(l),S(l))}(z)$ whose modulus is smaller than $\omega(G(l),S(l))^{-1}$. This contradicts to Lemma 3.17. ∎

Theorem 3.20. *The growth rate $\omega\colon \mathcal{C}_n \to \mathbb{R}_{\geq 1}$ is a continuous function.*

Proof. Let $\{(G_k,S_K)\}_{k\geq 1}$ be a convergent sequence of Coxeter systems of rank n and write (G,S) for the limit. We shall show that $\lim_{k\to\infty}\omega(G_k,S_k) = \omega(G,S)$. The proof is divided into two cases; one is the case that (G,S) is either elliptic or affine, and the other is the case that (G,S) is non-affine.

Suppose that (G,S) is either elliptic or affine. By Corollary 3.12, the set of all elliptic or affine Coxeter systems is open in $\mathcal{C}_n$. This implies that (G_k,S_k) is either elliptic or affine for all but finitely many k, and hence

$$\lim_{k\to\infty}\omega(G_k,S_k) = 1 = \omega(G,S).$$

Consider the case that (G,S) is non-affine. Let us denote the Coxeter matrices corresponding to (G_k,S_k) and (G,S) by $M_k = (m_{ij}(k))$ and $M = (m_{ij})$, respectively. The assertion is trivial for the case that the sequence (G_k,S_k) is eventually constant. We assume that (G_k,S_k) is not eventually constant. We denote by $(G(l),S(l))$ the Coxeter system defined by the following Coxeter matrix $M(l) = (m_{ij}(l))$ for $l \geq 0$:

$$m_{ij}(l) = \begin{cases} l & \text{if } m_{ij} = \infty, \\ m_{ij} & \text{if } m_{ij} < \infty. \end{cases}$$

By Theorem 3.11 and Corollary 3.19, for any $\varepsilon > 0$, there exists $L \geq 0$ such that

$$\omega(G,S) - \omega(G(L),S(L)) < \varepsilon.$$

By Lemma 3.2 and Theorem 3.6, there exists $K_L \in \mathbb{N}$ such that for $k \geq K_L$,

$$\begin{cases} m_{ij}(k) \geq L & \text{if } m_{ij} = \infty, \\ m_{ij}(k) = m_{ij} & \text{if } m_{ij} < \infty. \end{cases}$$

Theorem 3.11 implies that for $k \geq K_L$,

$$\omega(G,S) - \omega(G_k,S_k) \leq \omega(G,S) - \omega(G(L),S(L)) < \varepsilon,$$

and this proves our assertion. ∎

4. An application to hyperbolic geometry

4.1. The growth rates of hyperbolic Coxeter polygons of finite volume

We recall the result due to Floyd. Let Δ be a hyperbolic n-gon and $v_1, \dots, v_n$ be the vertices of Δ in cyclic order. We call Δ a *hyperbolic Coxeter n-gon* if the interior angles of Δ are of the form π/a, $a \geq 2$. If Δ is a hyperbolic Coxeter n-gon, then the n reflections along the edges of Δ generates the Coxeter group $G(\Delta)$ of rank n. We write $\Delta(a_1, \dots, a_n)$ for a hyperbolic Coxeter n-gon whose interior angle at v_i equals π/a_i, $a_i \geq 2$. A sequence $\{\Delta(a_1(k), \dots, a_n(k))\}_{k \geq 1}$ of hyperbolic Coxeter n-gons *converges* to $\Delta(a_1, \dots, a_n)$ if $\lim_{k \to \infty} a_i(k) = a_i$ for any i.

Theorem 4.1 ([9, p. 476, Theorem]). *Let a sequence $\{\Delta(a_1(k), \dots, a_n(k))\}_{k \geq 1}$ of hyperbolic Coxeter n-gons converge to $\Delta(a_1, \dots, a_n)$. Then*

$$\lim_{k \to \infty} \omega(a_1(k), \dots, a_n(k)) = \omega(a_1, \dots, a_n),$$

where $\omega(a_1(k), \dots, a_n(k))$ and $\omega(a_1, \dots, a_n)$ are the growth rates of the Coxeter groups associated with $\Delta(a_1(k), \dots, a_n(k))$ and $\Delta(a_1, \dots, a_n)$, respectively.

We give another proof of Theorem 4.1.

Proof. Let us denote the Coxeter group associated with $\Delta(a_1, \dots, a_n)$ by $G(a_1, \dots, a_n)$, and write $M(a_1, \dots, a_n)$ for the Coxeter matrix of $G(a_1, \dots, a_n)$. By the definition of the convergence of hyperbolic Coxeter n-gons, we have

$$\lim_{k \to \infty} M(a_1(k), \dots, a_n(k)) = M(a_1, \dots, a_n).$$

Therefore, the assertion follows from Theorem 3.20. ∎

4.2. The growth rates of hyperbolic Coxeter polyhedra of finite volume

We recall the edge contraction of hyperbolic Coxeter polyhedra (see [13] for more details). A hyperbolic Coxeter polyhedron P is called a hyperbolic *Coxeter* polyhedron if all dihedral angles are of the form π/m, $m \geq 2$. Let P be a hyperbolic Coxeter polyhedron of finite volume. For any edge e whose the endpoints are trivalent vertices, we call e an edge of *type* $\langle k_1, k_2, n, l_1, l_2 \rangle$ if the edges incident to e has dihedral angles as in Figure 4.

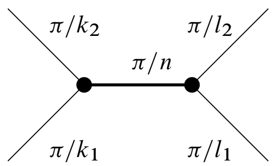


Figure 4. The picture of an edge e whose the endpoints are trivalent vertices.

Suppose that a hyperbolic Coxeter polyhedron P of finite volume has an edge e of type $\langle 2, 2, N, 2, 2\rangle$. By Andreev's theorem [1] and its application [13], there exist a hyperbolic Coxeter polytopes P_k for $N \leq k \leq \infty$ of finite volume satisfying the followings:

(i) For $N \leq m < \infty$, P_m has the same combinatorial type as P.

(ii) The polytope P_m has the same dihedral angles as P except for the edge e and the dihedral angle π/m at e.

(iii) The combinatorial type of P_∞ is obtained from P by contracting the edge e of P to a four-valent vertex.

(iv) The polytope P_∞ has the same dihedral angles as P except for the edge e.

This gives us the sequence $\{P_m\}_{m \geq N}$ of hyperbolic Coxeter polyhedra converging to P_∞ (see Figure 5).

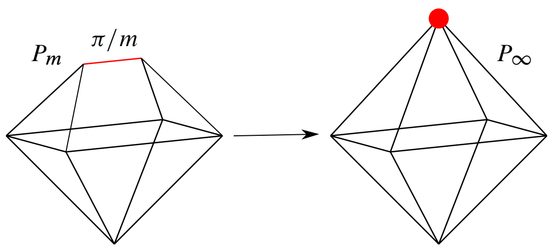


Figure 5. An example of a convergent sequence of hyperbolic Coxeter polyhedra of finite-volume. The red colored edge of P_m is contracted to the red colored vertex of P_∞ as $m \to \infty$.

We call an edge P of type $\langle 2, 2, N, 2, 2\rangle$ a *contractible edge*.

Theorem 4.2 ([13, p. 1717, Proposition 3]). *Let P be a hyperbolic Coxeter polyhedron of finite volume and e be an edge of P of type $\langle 2, 2, N, 2, 2\rangle$. Then*

$$\lim_{m\to\infty} \omega(G_m) = \omega(G_\infty),$$

where G_m and G are the Coxeter groups associated with P_m $(m \geq N)$ and P_∞.

We give another proof of Theorem 4.2.

Proof. Let us denote by M_m and M the Coxeter matrices of G_m and G, respectively. Since the dihedral angle π/m converges to 0 and the other dihedral angles are constant, we have

$$\lim_{m\to\infty} M_m = M.$$

Therefore, the assertion follows from Theorem 3.20. ∎

4.3. The arithmetic nature of the limiting growth rates

Let us recall Salem numbers and Pisot numbers (see [2] for details). A real algebraic integer α bigger than 1 is called a *Salem number* if α^{-1} is a Galois conjugate of α and all other Galois conjugates lie on the unit circle. The set of all Salem numbers is denoted by $\mathcal{T}$. Parry showed that the growth rates of the Coxeter groups associated with compact hyperbolic Coxeter polygons and polyhedra are Salem numbers [15]. A real algebraic integer α bigger than 1 is called a *Pisot number* if all Galois conjugates of α have modulus less than 1. The set of all Pisot numbers is denoted by $\mathcal{S}$. Salem numbers and Pisot numbers are closely related as follows.

Theorem 4.3 ([2, p. 111, Theorem 6.4.1]). *The set $\mathcal{S}$ is contained in the closure of $\mathcal{T}$ in $\mathbb{R}$.*

Floyd showed that the growth rates of the Coxeter groups associated with hyperbolic Coxeter polygons of finite volume are Pisot numbers [9], and Kolpakov proved that the growth rates of the Coxeter groups associated with hyperbolic Coxeter polyhedra with single four-valent ideal vertex are Pisot numbers [13]. The fact that a hyperbolic Coxeter polygon and polyhedron of finite-volume are the limits of compact hyperbolic Coxeter polygons and polyhedra is of fundamental importance for their proofs.

Theorem 4.4. *Let P be a hyperbolic Coxeter polyhedron of finite volume whose the ideal vertices are valency* 4. *Then the growth rate $\omega(G)$ of the Coxeter group G associated with P is a Pisot number.*

Note that Kolpakov proved Theorem 4.4 for the case that P has only one ideal vertex.

Proof. Suppose that P has N ideal vertices. By opening the ideal vertices to edges and giving sufficiently small dihedral angles π/a_i, $1 \leq i \leq N$, we construct a compact hyperbolic Coxeter polyhedron $P(a_1, \dots, a_N)$ whose the dihedral angles are the same as P other than $\pi/a_1, \dots, \pi/a_N$. By the result due to Kolpakov (see [13, p. 1721, Theorem 5]), the growth rate $\omega(\infty, a_2, \dots, a_N)$ is a Pisot number. Since the set $\mathcal{S}$ is closed (see [2, p. 102, Theorem 6.1]) and $\omega(\infty, a_2, \dots, a_N)$ converges to $\omega(\infty, \infty, a_3, \dots, a_N)$ by Theorem 3.20, the growth rate $\omega(\infty, \infty, a_3, \dots, a_N)$ is a Pisot number. Repeating this argument together with the equation $\omega(G) = \omega(\infty, \dots, \infty)$ leads to our assertion. ∎

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